

HLIN602 - Logique 2 - TD5

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Exercice 1

1. $(\forall x.P(x)) \Rightarrow \exists y.P(y)$
 $\rightsquigarrow \exists x.P(x) \Rightarrow \exists y.P(y)$
 $\rightsquigarrow \exists x.\exists y.P(x) \Rightarrow P(y)$
2. Une forme polie de $(\forall x.\exists y.R(x,y)) \Rightarrow \exists x.\forall y.R(x,y)$ est
 $(\forall x.\exists y.R(x,y)) \Rightarrow \exists z.\forall t.R(z,t)$
 $\rightsquigarrow \exists x.(\exists y.R(x,y)) \Rightarrow \exists z.\forall t.R(z,t)$
 $\rightsquigarrow \exists x.\forall y.R(x,y) \Rightarrow \exists z.\forall t.R(z,t)$
 $\rightsquigarrow \exists x.\forall y.\exists z.R(x,y) \Rightarrow \forall t.R(z,t)$
 $\rightsquigarrow \exists x.\forall y.\exists z.\forall t.R(x,y) \Rightarrow R(z,t)$
3. Une forme polie de $(\exists x.\forall y.R(x,y)) \Rightarrow \forall x.\exists y.R(x,y)$ est
 $(\exists x.\forall y.R(x,y)) \Rightarrow \forall z.\exists t.R(z,t)$
 $\rightsquigarrow \forall x.(\forall y.R(x,y)) \Rightarrow \forall z.\exists t.R(z,t)$
 $\rightsquigarrow \forall x.\exists y.R(x,y) \Rightarrow \forall z.\exists t.R(z,t)$
 $\rightsquigarrow \forall x.\exists y.\forall z.R(x,y) \Rightarrow \exists t.R(z,t)$
 $\rightsquigarrow \forall x.\exists y.\forall z.\exists t.R(x,y) \Rightarrow R(z,t)$
4. Une forme polie de $(P(x) \Rightarrow \forall x.Q(x)) \Rightarrow ((\exists x.P(x)) \Rightarrow \forall x.Q(x))$ est
 $(P(a) \Rightarrow \forall x.Q(x)) \Rightarrow ((\exists y.P(y)) \Rightarrow \forall z.Q(z))$
 $\rightsquigarrow (\forall x.P(a) \Rightarrow Q(x)) \Rightarrow ((\exists y.P(y)) \Rightarrow \forall z.Q(z))$
 $\rightsquigarrow \exists x.(P(a) \Rightarrow Q(x)) \Rightarrow ((\exists y.P(y)) \Rightarrow \forall z.Q(z))$
 $\rightsquigarrow \exists x.(P(a) \Rightarrow Q(x)) \Rightarrow (\forall y.P(y) \Rightarrow \forall z.Q(z))$
 $\rightsquigarrow \exists x.(P(a) \Rightarrow Q(x)) \Rightarrow \forall y.(P(y) \Rightarrow \forall z.Q(z))$
 $\rightsquigarrow \exists x.(P(a) \Rightarrow Q(x)) \Rightarrow \forall y.\forall z.(P(y) \Rightarrow Q(z))$
 $\rightsquigarrow \exists x.\forall y.\forall z.(P(a) \Rightarrow Q(x)) \Rightarrow (P(y) \Rightarrow Q(z))$
5. Une forme polie de $(\exists x.\forall y.(\exists z.S(x,y,z)) \wedge R(x,y)) \Rightarrow \exists y.(\forall x.S(x,y,z)) \wedge \exists x.R(x,y)$ est
 $(\exists x.\forall y.(\exists z.S(x,y,z)) \wedge R(x,y)) \Rightarrow \exists t.(\forall u.S(u,t,a)) \wedge \exists v.R(v,t)$
 $\rightsquigarrow (\exists x.\forall y.\exists z.(S(x,y,z) \wedge R(x,y)) \Rightarrow \exists t.(\forall u.S(u,t,a)) \wedge \exists v.R(v,t)$
 $\rightsquigarrow \forall x.\exists y.\forall z.(S(x,y,z) \wedge R(x,y)) \Rightarrow \exists t.(\forall u.S(u,t,a)) \wedge \exists v.R(v,t)$
 $\rightsquigarrow \forall x.\exists y.\forall z.(S(x,y,z) \wedge R(x,y)) \Rightarrow \exists t.\forall u.S(u,t,a) \wedge \exists v.R(v,t)$
 $\rightsquigarrow \forall x.\exists y.\forall z.(S(x,y,z) \wedge R(x,y)) \Rightarrow \exists t.\forall u.\exists v.S(u,t,a) \wedge R(v,t)$
 $\rightsquigarrow \forall x.\exists y.\forall z.\exists t.\forall u.\exists v.(S(x,y,z) \wedge R(x,y)) \Rightarrow S(u,t,a) \wedge R(v,t)$

Exercice 2

1.

$$\begin{array}{c}
 \frac{\overline{\Phi \vdash \Phi', \Phi}^{ax} \quad \overline{\Phi, \Phi' \vdash \Phi'}^{ax}}{\Phi \Rightarrow \Phi', \Phi \vdash \Phi'} \Rightarrow_g \\
 \frac{\overline{(\forall x. \Phi) \Rightarrow \Phi', \Phi \vdash \Phi'}^{\forall_g}}{(\forall x. \Phi) \Rightarrow \Phi' \vdash \Phi \Rightarrow \Phi'} \Rightarrow_d \\
 \frac{\overline{(\forall x. \Phi) \Rightarrow \Phi' \vdash \exists x. \Phi \Rightarrow \Phi'}^{\exists_d}}{\vdash ((\forall x. \Phi) \Rightarrow \Phi') \Rightarrow \exists x. \Phi \Rightarrow \Phi'} \Rightarrow_d
 \end{array}$$

2.

$$\begin{array}{c}
 \frac{\overline{\Phi \vdash \Phi', \Phi}^{ax}}{\Phi \vdash \Phi', \exists x. \Phi} \exists_d \\
 \frac{\overline{\forall x. \Phi \vdash \Phi', \exists x. \Phi}^{\forall_g} \quad \overline{\forall x. \Phi, \Phi' \vdash \Phi'}^{ax}}{(\exists x. \Phi \Rightarrow \Phi'), \forall x. \Phi \vdash \Phi'} \Rightarrow_g \\
 \frac{\overline{(\exists x. \Phi \Rightarrow \Phi') \vdash (\forall x. \Phi) \Rightarrow \Phi'}^{\Rightarrow_d}}{\vdash (\exists x. \Phi \Rightarrow \Phi') \Rightarrow (\forall x. \Phi) \Rightarrow \Phi'} \Rightarrow_d
 \end{array}$$

Exercice 4

1. Une forme polie de $\forall x. P(x) \Rightarrow \exists y. \forall z. R(x, y)$ est $\forall x. P(x) \Rightarrow \exists y. \forall z. R(z, y)$

$$\begin{aligned}
 & s(\forall x. P(x) \Rightarrow \exists y. \forall z. R(z, y)) \\
 & s(P(x) \Rightarrow \exists y. \forall z. R(z, y)) \\
 & = h(P(x)) \Rightarrow s(\exists y. \forall z. R(z, y)) \\
 & = P(x) \Rightarrow s(\exists y. \forall z. R(z, y)) \\
 & = P(x) \Rightarrow s(\forall z. R(z, y))[g()/y] \\
 & = P(x) \Rightarrow s(\forall z. R(z, y))[a/y] \\
 & = P(x) \Rightarrow s(R(z, y))[a/y] \\
 & = P(x) \Rightarrow R(z, a)
 \end{aligned}$$

Formule skolemisée : $\forall x. \forall z. P(x) \Rightarrow R(z, a)$

$$\begin{aligned}
 & P(x) \Rightarrow R(z, a) \\
 & = \neg P(x) \vee R(z, a)
 \end{aligned}$$

Ensemble des clauses : $S = \{\neg P(x) \vee R(z, a)\}$

2. Une forme polie de $(\exists x.\forall y.R(x,y)) \Rightarrow \forall y.\exists x.R(x,y)$ est $(\exists x.\forall y.R(x,y)) \Rightarrow \forall z.\exists t.R(t,z)$

$$\begin{aligned}
& s((\exists x.\forall y.R(x,y)) \Rightarrow \forall z.\exists t.R(t,z)) \\
& = h(\exists x.\forall y.R(x,y)) \Rightarrow s(\forall z.\exists t.R(t,z)) \\
& = h(\forall y.R(x,y)) \Rightarrow s(\forall z.\exists t.R(t,z)) \\
& = h(R(x,y))[f(x)/y] \Rightarrow s(\forall z.\exists t.R(t,z)) \\
& = R(x,f(x)) \Rightarrow s(\forall z.\exists t.R(t,z)) \\
& = R(x,f(x)) \Rightarrow s(\exists t.R(t,z)) \\
& = R(x,f(x)) \Rightarrow s(R(t,z))[g(z)/t] \\
& = R(x,f(x)) \Rightarrow R(g(z),z)
\end{aligned}$$

Formule skolémisée et herbrandisée : $\exists x.\forall z.R(x,f(x)) \Rightarrow R(g(z),z)$

$$\begin{aligned}
& R(x,f(x)) \Rightarrow R(g(z),z) \\
& = \neg R(x,f(x)) \vee R(g(z),z)
\end{aligned}$$

Ensemble des clauses : $S = \{\neg R(x,f(x)) \vee R(g(z),z)\}$

3. Une forme polie de $((\exists x.P(x) \Rightarrow Q(x)) \vee \forall y.P(y)) \wedge \forall x.\exists y.Q(y) \Rightarrow P(x)$ est $((\exists x.P(x) \Rightarrow Q(x)) \vee \forall y.P(y)) \wedge \forall z.\exists t.Q(t) \Rightarrow P(z)$

$$\begin{aligned}
& s(((\exists x.P(x) \Rightarrow Q(x)) \vee \forall y.P(y)) \wedge \forall z.\exists t.Q(t) \Rightarrow P(z)) \\
& = (s((\exists x.P(x) \Rightarrow Q(x)) \vee \forall y.P(y))) \wedge s(\forall z.\exists t.Q(t) \Rightarrow P(z)) \\
& = (s((P(x) \Rightarrow Q(x)) \vee \forall y.P(y))) \wedge s(\exists t.Q(t) \Rightarrow P(z)) \\
& = ((h(P(x)) \Rightarrow s(Q(x)) \vee s(\forall y.P(y))) \wedge s(Q(t) \Rightarrow P(z)) \\
& = ((P(x) \Rightarrow Q(x)) \vee s(\forall y.P(y))) \wedge h(Q(t)) \Rightarrow s(P(z)) \\
& = ((P(x) \Rightarrow Q(x)) \vee s(P(y))) \wedge Q(t) \Rightarrow P(z) \\
& = ((P(x) \Rightarrow Q(x)) \vee P(y)) \wedge Q(t) \Rightarrow P(z)
\end{aligned}$$

Formule skolémisée et herbrandisée : $\exists x.\forall y.\forall z.\exists t.((P(x) \Rightarrow Q(x)) \vee P(y)) \wedge Q(t) \Rightarrow P(z)$

$$\begin{aligned}
& ((P(x) \Rightarrow Q(x)) \vee P(y)) \wedge Q(t) \Rightarrow P(z) \\
& = ((P(x) \Rightarrow Q(x)) \vee P(y)) \wedge \neg Q(t) \vee P(z) \\
& = ((\neg P(x) \vee Q(x)) \vee P(y)) \wedge \neg Q(t) \vee P(z) \\
& = (\neg P(x) \vee Q(x) \vee P(y)) \wedge (\neg Q(t) \vee P(z))
\end{aligned}$$

Ensemble des clauses : $S = \{\neg P(x) \vee Q(x) \vee P(y), \neg Q(t) \vee P(z)\}$

Exercice 6

1. $\{g(f(x),f(y)) = g(f(f(a)),f(z))\} \hookrightarrow_{decompose}$
 $\{f(x) = f(f(a)),f(y) = f(z)\} \hookrightarrow_{decompose}$
 $\{x = f(a),y = z\};$
 $\text{mgu}(g(f(x),f(y)),g(f(f(a)),f(z))) = [z/y,f(a)/x]$
2. $\{h(x,f(a),x) = h(h(a,b,y),f(y),h(a,b,a))\} \hookrightarrow_{decompose}$
 $\{x = h(a,b,y),f(a) = f(y),x = h(a,b,a)\} \hookrightarrow_{decompose}$
 $\{x = h(a,b,y),a = y,x = h(a,b,a)\} \hookrightarrow_{eliminate}$
 $\{x = h(y,b,y),a = y,x = h(y,b,y)\} \hookrightarrow_{doublon}$
 $\{a = y,x = h(y,b,y)\};$
 $\text{mgu}(h(x,f(a),x),h(h(a,b,y),f(y),h(a,b,a))) = [y/a,h(y,b,y)/x]$

3. $\{g(y, f(f(x))) = g(f(a), y)\} \hookrightarrow_{decompose}$
 $\{y = f(a), f(f(x)) = y\} \hookrightarrow_{eliminate}$
 $\{y = f(a), f(f(x)) = f(a)\} \hookrightarrow_{decompose}$
 $\{y = f(a), f(x) = a\} \hookrightarrow_{swap}$
 $\{y = f(a), a = f(x)\};$
 $\text{mgu}(g(y, f(f(x))), g(f(a), y)) = [f(a)/y, f(x)/a]$
4. $\{h(a, x, f(x)) = h(a, y, y)\} \hookrightarrow_{decompose}$
 $\{a = a, x = y, f(x) = y\} \hookrightarrow_{delete}$
 $\{x = y, f(x) = y\} \hookrightarrow_{swap}$
 $\{x = y, y = f(x)\} \hookrightarrow_{eliminate}$
 $\{x = f(x), y = f(x)\} \hookrightarrow_{conflict}$
 $\perp;$
 $h(a, x, f(x))$ et $h(a, y, y)$ ne sont pas unifiables.
5. $\{g(x, g(y, z)) = g(g(a, b), x), g(g(a, b), x) = g(x, g(a, z))\} \hookrightarrow_{decompose}$
 $\{x = g(a, b), g(y, z) = x, g(a, b) = x, x = g(a, z)\} \hookrightarrow_{swap}$
 $\{x = g(a, b), g(y, z) = x, x = g(a, b), x = g(a, z)\} \hookrightarrow_{doublon}$
 $\{x = g(a, b), g(y, z) = x, x = g(a, z)\} \hookrightarrow_{eliminate}$
 $\{x = g(a, b), g(y, z) = g(a, z), x = g(a, z)\} \hookrightarrow_{decompose}$
 $\{x = g(a, b), y = a, z = z, x = g(a, z)\} \hookrightarrow_{delete}$
 $\{x = g(a, b), y = a, x = g(a, z)\} \hookrightarrow_{eliminate}$
 $\{x = g(a, b), y = a, g(a, b) = g(a, z)\} \hookrightarrow_{decompose}$
 $\{x = g(a, b), y = a, a = a, b = z\} \hookrightarrow_{delete}$
 $\{x = g(a, b), y = a, b = z\} \hookrightarrow_{eliminate}$
 $\{x = g(a, z), y = a, b = z\};$
 $\text{mgu}(g(x, g(y, z)), g(g(a, b), x), g(x, g(a, z))) = [g(a, z)/x, a/y, z/b]$