LogiqueTD 5

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Exercice 1

1) $(\forall x.\mathbf{P}(x)) \Rightarrow \exists y.\mathbf{P}(y)$ elle est polie.
Forme prénexe : $\exists x.\exists y.\mathbf{P}(x) \Rightarrow \mathbf{P}(y)$

2) $(\forall x.\exists y.\mathbf{R}(x,y)) \Rightarrow \exists x.\forall y.\mathbf{R}(x,y)$ elle n'est pas polie car x et y sont soumis à 2 quantifications.

Forme polie : $(\forall x.\exists y.\mathbf{R}(x,y)) \Rightarrow \exists a.\forall b.\mathbf{R}(a,b)$
Forme prénexe : $(\forall x.\exists y.R(x,y)) \Rightarrow \exists a.\forall b.R(a,b)$

$\exists x.(\exists y.R(x,y)) \Rightarrow \exists a.\forall b.R(a,b)$

$\exists x.\forall y.R(x,y) \Rightarrow \exists a.\forall b.R(a,b)$

$\exists x.\forall y.\exists a.R(x,y) \Rightarrow \forall b.R(a,b)$

$\exists x.\forall y.\exists a.\forall b.R(x,y) \Rightarrow R(a,b)$

3) $(\exists x.\forall y.\mathbf{R}(x,y)) \Rightarrow \forall x.\exists y.\mathbf{R}(x,y)$ elle n'est pas polie car x et y sont soumis à 2 quantifications.

Forme polie : $(\exists x.\forall y.\mathbf{R}(x,y)) \Rightarrow \forall a.\exists b.\mathbf{R}(a,b)$
Forme prénexe : $(\exists x.\forall y.R(x,y)) \Rightarrow \forall a.\exists b.R(a,b)$

$\forall x.(\forall y.R(x,y)) \Rightarrow \forall a.\exists b.R(a,b)$

$\forall x.\exists y.R(x,y) \Rightarrow \forall a.\exists b.R(a,b)$

$\forall x.\exists y.\forall a.R(x,y) \Rightarrow \exists b.R(a,b)$

$\forall x.\exists y.\forall a.\exists b.R(x,y) \Rightarrow R(a,b)$

4) $(\mathbf{P}(x) \Rightarrow \forall x.\mathbf{Q}(x)) \Rightarrow ((\exists x.\mathbf{P}(x)) \Rightarrow \forall x.\mathbf{Q}(x))$ elle n'est pas polie car x est soumis à trop de quantifications et est à la fois libre et liée.

Forme polie : $(\mathbf{P}(y) \Rightarrow \forall x.\mathbf{Q}(x)) \Rightarrow ((\exists y.\mathbf{P}(y)) \Rightarrow \forall z.\mathbf{Q}(z))$
Forme prénexe : $(P(a) \Rightarrow \forall x.Q(x)) \Rightarrow ((\exists y.P(y)) \Rightarrow \forall z.Q(z))$

$(\forall x.P(a) \Rightarrow Q(x)) \Rightarrow ((\exists y.P(y)) \Rightarrow \forall z.Q(z))$

$\exists x.(P(a) \Rightarrow Q(x)) \Rightarrow ((\exists y.P(y)) \Rightarrow \forall z.Q(z))$

$\exists x.(P(a) \Rightarrow Q(x)) \Rightarrow (\forall y.P(y) \Rightarrow \forall z.Q(z))$

$\exists x.(P(a) \Rightarrow Q(x)) \Rightarrow (\forall y.\forall z.P(y) \Rightarrow Q(z))$

$\exists x.(P(a) \Rightarrow Q(x)) \Rightarrow \forall y.\forall z.(P(y) \Rightarrow Q(z))$

$\exists x.\forall y.\forall z.(P(a) \Rightarrow Q(x)) \Rightarrow (P(y) \Rightarrow Q(z))$

5) $(\exists x.\forall y.(\exists z.\mathbf{S}(x,y,z)) \wedge \mathbf{R}(x,y)) \Rightarrow \exists y.(\forall x.\mathbf{S}(x,y,z)) \wedge \exists x.\mathbf{R}(x,y)$ elle n'est pas polie car x,y sont soumis à plusieurs quantifications et z est à la fois libre et liée.

Forme polie : $(\exists x.\forall y.(\exists z.S(x,y,z)) \wedge R(x,y)) \Rightarrow \exists t.(\forall u.S(u,t,a)) \wedge \exists v.R(v,t)$

Forme prénexe : $(\exists x.\forall y.(\exists z.S(x,y,z)) \wedge R(x,y)) \Rightarrow \exists t.(\forall u.S(u,t,a)) \wedge \exists v.R(v,t)$

$(\exists x.\forall y.\exists z.S(x,y,z) \wedge R(x,y)) \Rightarrow \exists t.(\forall u.S(u,t,a)) \wedge \exists v.R(v,t)$

$\forall x.\exists y.\forall z.(S(x,y,z) \wedge R(x,y)) \Rightarrow \exists t.(\forall u.S(u,t,a)) \wedge \exists v.R(v,t)$

$\forall x.\exists y.\forall z.(S(x,y,z) \wedge R(x,y)) \Rightarrow \exists t.\forall u.S(u,t,a) \wedge \exists v.R(v,t)$

$\forall x.\exists y.\forall z.(S(x,y,z) \wedge R(x,y)) \Rightarrow \exists t.\forall u.\exists v.S(u,t,a) \wedge R(v,t)$

$\forall x.\exists y.\forall z.\exists t.\forall u.\exists v.(S(x,y,z) \wedge R(x,y)) \Rightarrow S(u,t,a) \wedge R(v,t)$

Exercice 2

1) $((\forall x.\Phi) \Rightarrow \Phi') \Rightarrow \exists x.\Phi \Rightarrow \Phi'$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Phi \ \vdash \ \Phi,'\Phi \qquad \Phi, \Phi' \ \vdash \ \Phi'}{\Phi \Rightarrow \Phi', \Phi \ \vdash \ \Phi'} \Rightarrow \text{left}}{\Phi \Rightarrow \Phi' \ \vdash \ \Phi \Rightarrow \Phi'} \Rightarrow \text{right}}{\Phi \Rightarrow \Phi' \ \vdash \ \exists x.\Phi \Rightarrow \Phi'} \exists \text{ right}}{\forall x.\Phi \Rightarrow \Phi' \ \vdash \ \exists x.\Phi \Rightarrow \Phi'} \forall \text{ left}}{((\forall x.\Phi) \Rightarrow \Phi') \Rightarrow \exists x.\Phi \Rightarrow \Phi'} \Rightarrow \text{right}$$

2) $(\exists x.\Phi \Rightarrow \Phi') \Rightarrow (\forall x.\Phi) \Rightarrow \Phi'$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Phi \ \vdash \ \Phi', \Phi \qquad \Phi, \Phi' \ \vdash \ \Phi'}{\Phi \Rightarrow \Phi', \Phi \ \vdash \ \Phi'} \Rightarrow \text{left}}{\Phi \Rightarrow \Phi' \ \vdash \ \Phi \Rightarrow \Phi'} \Rightarrow \text{right}}{\vdash \ (\Phi \Rightarrow \Phi') \Rightarrow (\Phi \Rightarrow \Phi)'} \Rightarrow \text{right}}{\vdash \ (\Phi \Rightarrow \Phi') \Rightarrow (\forall x.\Phi) \Rightarrow \Phi'} \forall \text{ right}}{\vdash \ (\exists x.\Phi \Rightarrow \Phi') \Rightarrow (\forall x.\Phi) \Rightarrow \Phi'} \exists \text{ right}$$

Exercice 4

1) $\forall x.\mathbf{P}(x) \Rightarrow \exists y.\forall x.\mathbf{R}(x,y)$

Forme polie : $\forall x.\mathbf{P}(x) \Rightarrow \exists y.\forall z.\mathbf{R}(z,y)$

Skolémisation :
$s(\forall x.\mathbf{P}(x) \Rightarrow \exists y.\forall z.\mathbf{R}(z,y)) =$

$s(\mathbf{P}(x) \Rightarrow \exists y.\forall z.\mathbf{R}(z,y)) =$

$s(\mathbf{P}(x) \Rightarrow \forall z.\mathbf{R}(z,y))[f(x)/y] =$

$s(\mathbf{P}(x) \Rightarrow \mathbf{R}(z,y))[f(x)/y] =$

$\mathbf{P}(x) \Rightarrow \mathbf{R}(z,y)[f(x)/y] =$

$\mathbf{P}(x) \Rightarrow \mathbf{R}(z,f(x)) =$

On obtient : $\forall x.\forall z.\mathbf{P}(x) \Rightarrow \mathbf{R}(z,f(x))$

Finalement : $S = \{\neg\mathbf{P}(x) \vee \mathbf{R}(z,f(x))\}$

2) Forme polie $(\exists x.\forall y.\mathbf{R}(x,y)) \Rightarrow \forall y.\exists x.\mathbf{R}(x,y)$ est $(\exists x.\forall y.\mathbf{R}(x,y)) \Rightarrow \forall z.\exists t.\mathbf{R}(t,z)$

Skolémisation :

$s((\exists x.\forall y.\mathbf{R}(x,y)) \Rightarrow \forall z.\exists t.\mathbf{R}(t,z))$

$= h(\exists x.\forall y.\mathbf{R}(x,y)) \Rightarrow s(\forall z.\exists t.\mathbf{R}(t,z))$

$= h(\forall y.\mathbf{R}(x,y)) \Rightarrow s(\forall z.\exists t.\mathbf{R}(t,z))$

$= h(\mathbf{R}(x,y))[f(x)/y] \Rightarrow s(\forall z.\exists t.\mathbf{R}(t,z))$

$= \mathbf{R}(x,f(x)) \Rightarrow s(\forall z.\exists t.\mathbf{R}(t,z))$

$= \mathbf{R}(x,f(x)) \Rightarrow s(\exists t.\mathbf{R}(t,z))$

$= R(x,f(x)) \Rightarrow s(\mathbf{R}(t,z))[g(z)/t]$

$= \mathbf{R}(x,f(x)) \Rightarrow \mathbf{R}(g(z),z)$

Formule de skolem : $\exists x.\forall z.\mathbf{R}(x,f(x)) \Rightarrow \mathbf{R}(g(z),z)$

$\mathbf{R}(x,f(x)) \Rightarrow \mathbf{R}(g(z),z)$

$= \neg\mathbf{R}(x,f(x)) \vee \mathbf{R}(g(z),z)$

Finalement : $S = \{\neg\mathbf{R}(x,f(x)) \vee \mathbf{R}(g(z),z)\}$

3) Forme polie : $((\exists x.\mathbf{P}(x) \Rightarrow \mathbf{Q}(x)) \vee \forall y.\mathbf{P}(y)) \wedge \forall x.\exists y.\mathbf{Q}(y) \Rightarrow \mathbf{P}(x)$ est

Skolémisation :

$((\exists x.\mathbf{P}(x) \Rightarrow \mathbf{Q}(x)) \vee \forall y.\mathbf{P}(y)) \wedge \forall z.\exists t.\mathbf{Q}(t) \Rightarrow \mathbf{P}(z)$

$s(((\exists x.\mathbf{P}(x) \Rightarrow \mathbf{Q}(x)) \vee \forall y.\mathbf{P}(y)) \wedge \forall z.\exists t.\mathbf{Q}(t) \Rightarrow \mathbf{P}(z))$

$= (s((\exists x.\mathbf{P}(x) \Rightarrow \mathbf{Q}(x)) \vee \forall y.\mathbf{P}(y))) \wedge s(\forall z.\exists t.\mathbf{Q}(t) \Rightarrow \mathbf{P}(z))$

$= (s((P(x) \Rightarrow \mathbf{Q}(x)) \vee \forall y.\mathbf{P}(y))) \wedge s(\exists t.\mathbf{Q}(t) \Rightarrow \mathbf{P}(z))$

$= ((h(\mathbf{P}(x)) \Rightarrow s(\mathbf{Q}(x)) \vee s(\forall y.\mathbf{P}(y))) \wedge s(\mathbf{Q}(t) \Rightarrow \mathbf{P}(z))$

$= ((\mathbf{P}(x) \Rightarrow \mathbf{Q}(x)) \vee s(\forall y.\mathbf{P}(y))) \wedge h(\mathbf{Q}(t)) \Rightarrow s(\mathbf{P}(z))$

$= ((\mathbf{P}(x) \Rightarrow \mathbf{Q}(x)) \vee s(\mathbf{P}(y))) \wedge \mathbf{Q}(t) \Rightarrow \mathbf{P}(z)$

$= ((\mathbf{P}(x) \Rightarrow \mathbf{Q}(x)) \vee \mathbf{P}(y)) \wedge \mathbf{Q}(t) \Rightarrow \mathbf{P}(z)$

Formule de skolem : $\exists x.\forall y.\forall z.\exists t.((\mathbf{P}(x) \Rightarrow \mathbf{Q}(x)) \vee \mathbf{P}(y)) \wedge \mathbf{Q}(t) \Rightarrow \mathbf{P}(z)$

$((\mathbf{P}(x) \Rightarrow \mathbf{Q}(x)) \vee \mathbf{P}(y)) \wedge \mathbf{Q}(t) \Rightarrow \mathbf{P}(z)$

$= ((\mathbf{P}(x) \Rightarrow \mathbf{Q}(x)) \vee P(y)) \wedge \neg\mathbf{Q}(t) \vee P(z)$

$= ((\neg\mathbf{P}(x) \vee \mathbf{Q}(x)) \vee \mathbf{P}(y)) \wedge \neg\mathbf{Q}(t) \vee \mathbf{P}(z)$

$= (\neg\mathbf{P}(x) \vee \mathbf{Q}(x) \vee \mathbf{P}(y)) \wedge (\neg\mathbf{Q}(t) \vee \mathbf{P}(z))$

Finalement : $S = \{\neg\mathbf{P}(x) \vee \mathbf{Q}(x) \vee \mathbf{P}(y), \neg\mathbf{Q}(t) \vee \mathbf{P}(z)\}$

Exercice 6

1) $\{g(f(x), f(y)) = g(f(f(a)), f(z))\} \hookrightarrow_{decompose}$

$\{f(x) = f(f(a)), f(y) = f(z)\} \hookrightarrow_{decompose}$

$\{x = f(a), y = z\}$;

$\text{mgu}(g(f(x), f(y)), g(f(f(a)), f(z))) = [z/y, f(a)/x]$

Fin algo.

2) $\{h(x, f(a), x) = h(h(a, b, y), f(y), h(a, b, a))\} \hookrightarrow_{decompose}$

$\{x = h(a, b, y), f(a) = f(y), x = h(a, b, a)\} \hookrightarrow_{decompose}$

$\{x = h(a, b, y), a = y, x = h(a, b, a)\} \hookrightarrow_{eliminate}$

$\{x = h(y, b, y), a = y, x = h(y, b, y)\} \hookrightarrow_{doublon}$

$\{a = y, x = h(y, b, y)\}$;

$\text{mgu}(h(x, f(a), x), h(h(a, b, y), f(y), h(a, b, a))) = [y/a, h(y, b, y)/x]$
Fin algo

3) $\{g(y, f(f(x))) = g(f(a), y)\} \hookrightarrow_{decompose}$

$\{y = f(a), f(f(x)) = y\} \hookrightarrow_{eliminate}$

$\{y = f(a), f(f(x)) = f(a)\} \hookrightarrow_{decompose}$

$\{y = f(a), f(x) = a\} \hookrightarrow_{swap}$

$\{y = f(a), a = f(x)\}$;

mgu$(g(y, f(f(x))), g(f(a), y)) = [f(a)/y, f(x)/a]$
Fin algo.

4) $\{h(a, x, f(x)) = h(a, y, y)\} \hookrightarrow_{decompose}$

$\{a = a, x = y, f(x) = y\} \hookrightarrow_{delete}$

$\{x = y, f(x) = y\} \hookrightarrow_{conflict}$
$\perp$;

$h(a, x, f(x))$ et $h(a, y, y)$ ne peuvent plus être unifiés.
Fin algo.

5) $\{g(x, g(y, z)) = g(g(a, b), x), g(x, g(y, z)) = g(x, g(a, z)), g(g(a, b), x) = g(x, g(a, z))\} \hookrightarrow_{decompose}$

$\{x = g(a, b), g(y, z) = x, x = x, g(y, z) = g(a, z), g(a, b) = x, x = g(a, z)\} \hookrightarrow_{decompose}$

$\{x = g(a, b), g(y, z) = x, x = x, y = a, z = z, g(a, b) = x, x = g(a, z)\} \hookrightarrow_{delete}$

$\{x = g(a, b), g(y, z) = x, y = a, g(a, b) = x, x = g(a, z)\} \hookrightarrow_{swap}$

$\{x = g(a, b), g(y, z) = x, y = a, x = g(a, b), x = g(a, z)\} \hookrightarrow_{doublon}$

$\{x = g(a, b), g(y, z) = x, y = a, x = g(a, z)\} \hookrightarrow_{eliminate}$

$\{x = g(a, b), g(a, z) = x, y = a, x = g(a, z)\} \hookrightarrow_{swap}$

$\{x = g(a, b), x = g(a, z), y = a, x = g(a, z)\} \hookrightarrow_{doublon}$

$\{x = g(a, b), x = g(a, z), y = a\}$;

mgu$(g(x, g(y, z)), g(g(a, b), x), g(x, g(a, z))) = [y/a, g(a, b)/x, g(a, z)/x]$
Fin algo.