# HLIN602 - Logique 2 - TD5

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## Exercice 1

1. $(\forall x.P(x)) \Rightarrow \exists y.P(y)$
   $\rightsquigarrow \exists x.P(x) \Rightarrow \exists y.P(y)$
   $\rightsquigarrow \exists x.\exists y.P(x) \Rightarrow P(y)$

2. Une forme polie de $(\forall x.\exists y.R(x,y)) \Rightarrow \exists x.\forall y.R(x,y)$ est
   $(\forall x.\exists y.R(x,y)) \Rightarrow \exists z.\forall t.R(z,t)$
   $\rightsquigarrow \exists x.(\exists y.R(x,y)) \Rightarrow \exists z.\forall t.R(z,t)$
   $\rightsquigarrow \exists x.\forall y.R(x,y) \Rightarrow \exists z.\forall t.R(z,t)$
   $\rightsquigarrow \exists x.\forall y.\exists z.R(x,y) \Rightarrow \forall t.R(z,t)$
   $\rightsquigarrow \exists x.\forall y.\exists z.\forall t.R(x,y) \Rightarrow R(z,t)$

3. Une forme polie de $(\exists x.\forall y.R(x,y)) \Rightarrow \forall x.\exists y.R(x,y)$ est
   $(\exists x.\forall y.R(x,y)) \Rightarrow \forall z.\exists t.R(z,t)$
   $\rightsquigarrow \forall x.(\forall y.R(x,y)) \Rightarrow \forall z.\exists t.R(z,t)$
   $\rightsquigarrow \forall x.\exists y.R(x,y) \Rightarrow \forall z.\exists t.R(z,t)$
   $\rightsquigarrow \forall x.\exists y.\forall z.R(x,y) \Rightarrow \exists t.R(z,t)$
   $\rightsquigarrow \forall x.\exists y.\forall z.\exists t.R(x,y) \Rightarrow R(z,t)$

4. Une forme polie de $(P(x) \Rightarrow \forall x.Q(x)) \Rightarrow ((\exists x.P(x)) \Rightarrow \forall x.Q(x))$ est
   $(P(a) \Rightarrow \forall x.Q(x)) \Rightarrow ((\exists y.P(y)) \Rightarrow \forall z.Q(z))$
   $\rightsquigarrow (\forall x.P(a) \Rightarrow Q(x)) \Rightarrow ((\exists y.P(y)) \Rightarrow \forall z.Q(z))$
   $\rightsquigarrow \exists x.(P(a) \Rightarrow Q(x)) \Rightarrow ((\exists y.P(y)) \Rightarrow \forall z.Q(z))$
   $\rightsquigarrow \exists x.(P(a) \Rightarrow Q(x)) \Rightarrow (\forall y.P(y) \Rightarrow \forall z.Q(z))$
   $\rightsquigarrow \exists x.(P(a) \Rightarrow Q(x)) \Rightarrow (\forall y.\forall z.P(y) \Rightarrow Q(z))$
   $\rightsquigarrow \exists x.(P(a) \Rightarrow Q(x)) \Rightarrow \forall y.\forall z.(P(y) \Rightarrow Q(z))$
   $\rightsquigarrow \exists x.\forall y.\forall z.(P(a) \Rightarrow Q(x)) \Rightarrow (P(y) \Rightarrow Q(z))$

5. Une forme polie de $(\exists x.\forall y.(\exists z.S(x,y,z)) \wedge R(x,y)) \Rightarrow \exists y.(\forall x.S(x,y,z)) \wedge \exists x.R(x,y)$ est
   $(\exists x.\forall y.(\exists z.S(x,y,z)) \wedge R(x,y)) \Rightarrow \exists t.(\forall u.S(u,t,a)) \wedge \exists v.R(v,t)$
   $\rightsquigarrow (\exists x.\forall y.\exists z.S(x,y,z) \wedge R(x,y)) \Rightarrow \exists t.(\forall u.S(u,t,a)) \wedge \exists v.R(v,t)$
   $\rightsquigarrow \forall x.\exists y.\forall z.(S(x,y,z) \wedge R(x,y)) \Rightarrow \exists t.(\forall u.S(u,t,a)) \wedge \exists v.R(v,t)$
   $\rightsquigarrow \forall x.\exists y.\forall z.(S(x,y,z) \wedge R(x,y)) \Rightarrow \exists t.\forall u.S(u,t,a) \wedge \exists v.R(v,t)$
   $\rightsquigarrow \forall x.\exists y.\forall z.(S(x,y,z) \wedge R(x,y)) \Rightarrow \exists t.\forall u.\exists v.S(u,t,a) \wedge R(v,t)$
   $\rightsquigarrow \forall x.\exists y.\forall z.\exists t.\forall u.\exists v.(S(x,y,z) \wedge R(x,y)) \Rightarrow S(u,t,a) \wedge R(v,t)$

## Exercice 2

1.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$ax$}
\UnaryInfC{$\Phi \vdash \Phi',\Phi$}
\RightLabel{$\forall_d$}
\UnaryInfC{$\Phi \vdash \Phi',\forall x.\Phi$}
\AxiomC{}
\RightLabel{$ax$}
\UnaryInfC{$\Phi,\Phi' \vdash \Phi'$}
\RightLabel{$\Rightarrow_g$}
\BinaryInfC{$(\forall x.\Phi) \Rightarrow \Phi',\Phi \vdash \Phi'$}
\RightLabel{$\exists_g$}
\UnaryInfC{$(\forall x.\Phi) \Rightarrow \Phi',\exists y.\Phi \vdash \Phi'$}
\RightLabel{$\Rightarrow_d$}
\UnaryInfC{$(\forall x.\Phi) \Rightarrow \Phi' \vdash \exists y.\Phi \Rightarrow \Phi'$}
\RightLabel{$\Rightarrow_d$}
\UnaryInfC{$\vdash ((\forall x.\Phi) \Rightarrow \Phi') \Rightarrow \exists y.\Phi \Rightarrow \Phi'$}
\end{prooftree}
$$

2.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$ax$}
\UnaryInfC{$\Phi \vdash \Phi',\Phi$}
\AxiomC{}
\RightLabel{$ax$}
\UnaryInfC{$\Phi,\Phi' \vdash \Phi'$}
\RightLabel{$\Rightarrow_g$}
\BinaryInfC{$\Phi \Rightarrow \Phi',\Phi \vdash \Phi'$}
\RightLabel{$\exists_g$}
\UnaryInfC{$(\exists x.\Phi \Rightarrow \Phi'),\Phi \vdash \Phi'$}
\RightLabel{$\forall_g$}
\UnaryInfC{$(\exists x.\Phi \Rightarrow \Phi'),(\forall y.\Phi) \vdash \Phi'$}
\RightLabel{$\Rightarrow_d$}
\UnaryInfC{$(\exists x.\Phi \Rightarrow \Phi') \vdash (\forall y.\Phi) \Rightarrow \Phi'$}
\RightLabel{$\Rightarrow_d$}
\UnaryInfC{$\vdash (\exists x.\Phi \Rightarrow \Phi') \Rightarrow (\forall y.\Phi) \Rightarrow \Phi'$}
\end{prooftree}
$$

## Exercice 4

1. Une forme polie de $\forall x.P(x) \Rightarrow \exists y.\forall x.R(x,y)$ est $\forall x.P(x) \Rightarrow \exists y.\forall z.R(z,y)$

   $s(\forall x.P(x) \Rightarrow \exists y.\forall z.R(z,y))$
   $s(P(x) \Rightarrow \exists y.\forall z.R(z,y))$
   $= h(P(x)) \Rightarrow s(\exists y.\forall z.R(z,y))$
   $= P(x) \Rightarrow s(\exists y.\forall z.R(z,y))$
   $= P(x) \Rightarrow s(\forall z.R(z,y))[g()/y]$
   $= P(x) \Rightarrow s(\forall z.R(z,y))[a/y]$
   $= P(x) \Rightarrow s(R(z,y))[a/y]$
   $= P(x) \Rightarrow R(z,a)$
   Formule skolémisée : $\forall x.\forall z.P(x) \Rightarrow R(z,a)$

   $P(x) \Rightarrow R(z,a)$
   $= \neg P(x) \vee R(z,a)$
   Ensemble des clauses : $S = \{\neg P(x) \vee R(z,a)\}$

2. Une forme polie de $(\exists x.\forall y.R(x,y)) \Rightarrow \forall y.\exists x.R(x,y)$ est $(\exists x.\forall y.R(x,y)) \Rightarrow \forall z.\exists t.R(t,z)$

$s((\exists x.\forall y.R(x,y)) \Rightarrow \forall z.\exists t.R(t,z))$
$= h(\exists x.\forall y.R(x,y)) \Rightarrow s(\forall z.\exists t.R(t,z))$
$= h(\forall y.R(x,y)) \Rightarrow s(\forall z.\exists t.R(t,z))$
$= h(R(x,y))[f(x)/y] \Rightarrow s(\forall z.\exists t.R(t,z))$
$= R(x,f(x)) \Rightarrow s(\forall z.\exists t.R(t,z))$
$= R(x,f(x)) \Rightarrow s(\exists t.R(t,z))$
$= R(x,f(x)) \Rightarrow s(R(t,z))[g(z)/t]$
$= R(x,f(x)) \Rightarrow R(g(z),z)$
Formule skolémisée et herbrandisée : $\exists x.\forall z.R(x,f(x)) \Rightarrow R(g(z),z)$

$R(x,f(x)) \Rightarrow R(g(z),z)$
$= \neg R(x,f(x)) \vee R(g(z),z)$
Ensemble des clauses : $S = \{\neg R(x,f(x)) \vee R(g(z),z)\}$

3. Une forme polie de $((\exists x.P(x) \Rightarrow Q(x)) \vee \forall y.P(y)) \wedge \forall x.\exists y.Q(y) \Rightarrow P(x)$ est
$((\exists x.P(x) \Rightarrow Q(x)) \vee \forall y.P(y)) \wedge \forall z.\exists t.Q(t) \Rightarrow P(z)$

$s(((\exists x.P(x) \Rightarrow Q(x)) \vee \forall y.P(y)) \wedge \forall z.\exists t.Q(t) \Rightarrow P(z))$
$= (s((\exists x.P(x) \Rightarrow Q(x)) \vee \forall y.P(y))) \wedge s(\forall z.\exists t.Q(t) \Rightarrow P(z))$
$= (s((P(x) \Rightarrow Q(x)) \vee \forall y.P(y))) \wedge s(\exists t.Q(t) \Rightarrow P(z))$
$= ((h(P(x)) \Rightarrow s(Q(x)) \vee s(\forall y.P(y))) \wedge s(Q(t) \Rightarrow P(z))$
$= ((P(x) \Rightarrow Q(x)) \vee s(\forall y.P(y))) \wedge h(Q(t)) \Rightarrow s(P(z))$
$= ((P(x) \Rightarrow Q(x)) \vee s(P(y))) \wedge Q(t) \Rightarrow P(z)$
$= ((P(x) \Rightarrow Q(x)) \vee P(y)) \wedge Q(t) \Rightarrow P(z)$
Formule skolémisée et herbrandisée : $\exists x.\forall y.\forall z.\exists t.((P(x) \Rightarrow Q(x)) \vee P(y)) \wedge Q(t) \Rightarrow P(z)$

$((P(x) \Rightarrow Q(x)) \vee P(y)) \wedge Q(t) \Rightarrow P(z)$
$= ((P(x) \Rightarrow Q(x)) \vee P(y)) \wedge \neg Q(t) \vee P(z)$
$= ((\neg P(x) \vee Q(x)) \vee P(y)) \wedge \neg Q(t) \vee P(z)$
$= (\neg P(x) \vee Q(x) \vee P(y)) \wedge (\neg Q(t) \vee P(z))$
Ensemble des clauses : $S = \{\neg P(x) \vee Q(x) \vee P(y), \neg Q(t) \vee P(z)\}$

## Exercice 6

1. $\{g(f(x),f(y)) = g(f(f(a)),f(z))\} \hookrightarrow_{decompose}$
$\{f(x) = f(f(a)), f(y) = f(z)\} \hookrightarrow_{decompose}$
$\{x = f(a), y = z\}$;
mgu$(g(f(x),f(y)),g(f(f(a)),f(z))) = [z/y,f(a)/x]$

2. $\{h(x,f(a),x) = h(h(a,b,y),f(y),h(a,b,a))\} \hookrightarrow_{decompose}$
$\{x = h(a,b,y), f(a) = f(y), x = h(a,b,a)\} \hookrightarrow_{decompose}$
$\{x = h(a,b,y), a = y, x = h(a,b,a)\} \hookrightarrow_{eliminate}$
$\{x = h(y,b,y), a = y, x = h(y,b,y)\} \hookrightarrow_{doublon}$
$\{a = y, x = h(y,b,y)\}$;
mgu$(h(x,f(a),x),h(h(a,b,y),f(y),h(a,b,a))) = [y/a,h(y,b,y)/x]$

3. $\{g(y,f(f(x))) = g(f(a),y)\} \hookrightarrow_{decompose}$
   $\{y = f(a),f(f(x)) = y\} \hookrightarrow_{eliminate}$
   $\{y = f(a),f(f(x)) = f(a)\} \hookrightarrow_{decompose}$
   $\{y = f(a),f(x) = a\} \hookrightarrow_{swap}$
   $\{y = f(a),a = f(x)\}$;
   $\mathrm{mgu}(g(y,f(f(x))),g(f(a),y)) = [f(a)/y,f(x)/a]$

4. $\{h(a,x,f(x)) = h(a,y,y)\} \hookrightarrow_{decompose}$
   $\{a = a,x = y,f(x) = y\} \hookrightarrow_{delete}$
   $\{x = y,f(x) = y\} \hookrightarrow_{swap}$
   $\{x = y,y = f(x)\} \hookrightarrow_{eliminate}$
   $\{x = f(x),y = f(x)\} \hookrightarrow_{conflict}$
   $\perp$;
   $h(a,x,f(x))$ et $h(a,y,y)$ ne sont pas unifiables.

5. $\{g(x,g(y,z)) = g(g(a,b),x),g(g(a,b),x) = g(x,g(a,z))\} \hookrightarrow_{decompose}$
   $\{x = g(a,b),g(y,z) = x,g(a,b) = x,x = g(a,z)\} \hookrightarrow_{swap}$
   $\{x = g(a,b),g(y,z) = x,x = g(a,b),x = g(a,z)\} \hookrightarrow_{doublon}$
   $\{x = g(a,b),g(y,z) = x,x = g(a,z)\} \hookrightarrow_{eliminate}$
   $\{x = g(a,b),g(y,z) = g(a,z),x = g(a,z)\} \hookrightarrow_{decompose}$
   $\{x = g(a,b),y = a,z = z,x = g(a,z)\} \hookrightarrow_{delete}$
   $\{x = g(a,b),y = a,x = g(a,z)\} \hookrightarrow_{eliminate}$
   $\{x = g(a,b),y = a,g(a,b) = g(a,z)\} \hookrightarrow_{decompose}$
   $\{x = g(a,b),y = a,a = a,b = z\} \hookrightarrow_{delete}$
   $\{x = g(a,b),y = a,b = z\} \hookrightarrow_{eliminate}$
   $\{x = g(a,z),y = a,b = z\}$;
   $\mathrm{mgu}(g(x,g(y,z)),g(g(a,b),x),g(x,g(a,z))) = [g(a,z)/x,a/y,z/b]$