Imperial College London

Department of Physics

# Fifth Forces and the Vainshtein Mechanism

BSc Project Report

Word Count: 4751

*Author:*
Jean Luo

*Supervisor:*
Prof. Andrew Tolley

*CID:*
01863971

*Assessor:*
Prof. Claudia de Rham

Submitted in partial fulfillment of the requirements for the BSc degree in Physics with Theoretical Physics of Imperial College London

May 2023

## Abstract

We apply the variational approximation method to study the behaviour of the fifth force by considering the helicity-0 mode of the field in the DGP model of massive gravity for a two-body system, which has never been studied before in this context. By choosing a Gaussian-like function as the ansatz, the simplest possible function that vanishes at large distances, we show that in the one-body problem with spherical symmetry, the solutions obtained from the variational approximation method incorporate correct physics of the Vainshtein mechanism, as demonstrated by comparison with the existing exact solution. This Gaussian-like ansatz is then applied to the two-body system, from which a mathematical form of the gravitational potential energy as a function of the separation between the two masses has been determined by applying analytical and numerical techniques in the variational approximation method. This potential energy is used to form a Lagrangian describing the dynamics of a general two-body system in the context of the fifth force, which is then applied to an astrophysical-like two-body system. The study of this astrophysical-like system shows that the fifth force alone can sustain stable procession orbits of such a binary system under appropriate initial conditions. To verify the validity of the Gaussian-like ansatz, other possible ansatzes are tested in the one-body problem and then compared with the exact solution, which shows that improvements can be made in terms of the ansatz used.

# Feedback from the assessor

## Structure and Language: Exceptional first class

The structure and language of the report are flawless. The abstract provides an excellent overview of what the project is about and its context. The material presented in the introduction is captivating and that sets the context and what has been achieved in the project. The quality of the thesis then reaches an even higher level of mastery with a beautiful introduction of the DGP model and its physical implications before moving to the core of the project in chapters 3 and 4 where variational methods to infer analytic approximated solutions to the one and two body screening problem are presented. The goal, methods, assumptions, and results are structured in a very professional way.

## Clarity: Exceptional first class

The clarity of the thesis is outstanding, providing beautiful context to the work. The explanations on the DGP model were particularly well written and clear and provide a very useful guide. Chapters 3 and 4 explained the use of the variational approximation method very clearly putting in context of what is already known and how it can be used to extend the state of the art making clear what the strengths and weaknesses of these approximations are. Giving that many concepts are much deeper but the student has been able to extract the useful and relevant elements in a concise way, the overall quality of the thesis is exceptional.

## Techniques: Outstanding first class

The project has required getting familiarized with highly technical methods, well beyond what is typically thought in 4th year GR and the students has shown an outstanding mastery of these techniques.

## Scientific Achievements: Outstanding first class

Novel results have been presented and put in context of state of the art research. Establishing how kinetic screening occurs in two-body systems has been the topic of extended investigations, with many research groups focusing solely on numerical investigations which provide less insight on the nature of the screening at scales relevant for astrophysics and cosmology. On the other hand, the work achieved in this project manages to set up the first analytic insight on how screening occurs in two-body systems and shows a remarkable good agreement with numerical simulations when those are available but are applicable to much more generic situations. The scientific achievements are hence outstanding!

# Declaration of work undertaken

For this project, I mainly focused on solving the two-body problem using the Gaussian-like ansatz, whereas my partner concentrated on testing the possible alternative ansatzes (Section 3.3 is dedicated to his work). The content of Chapter 2 and Section 3.1 (and the part of Chapter 3 before Section 3.1) is split between us.

# Acknowledgments

We would like to express our gratitude towards Prof. Andrew Tolley for giving us opportunities to work on this very interesting and frontier research project and for his tremendous effort made in teaching and guiding us throughout this project.
Our thanks and appreciations also go to our friend and colleague Mr. Roger Liu for the useful discussions.

# Contents

# Chapter 1

# Introduction

## 1.1 Background

General relativity (GR) is widely considered to be the most successful modern description of the nature of gravity. Nonetheless, many alternative theories that aim to modify GR **at large cosmological scales** have been proposed since the inception of GR. These alternative theories are not only used to test GR but can also offer alternative explanations for certain observed physical phenomena that are not naturally incorporated in GR, such as the accelerating late universe expansion [1; 2; 3].

GR is known to be the only theory of gravity that satisfies the following four key physical properties by requirements of symmetries [4]: locality, Lorentz invariance, massless force carrier (graviton) and force carrier being a spin 2 particle [5; 6]. Therefore, modifying at least one of these four properties is necessary to form an alternative theory of gravity. One possible modification is to **consider gravitons as massive particles, and theories obtained in this way are known as massive gravity**. In recent years, de Rham, Gabadadze, and Tolley revived the study of massive gravity by resolving the problem of the Boulware-Deser ghost [7], an extra degree of freedom (DoF) that leads to unbounded energy, which exists in the previous models of massive gravity [8; 9]. Granting a mass to the graviton is a natural choice to modify gravity from the perspective of particle physics, as this could be explained by the Higgs mechanism [10; 11], just like the other massive force carriers in the Standard Model, namely, W and Z bosons [12].

A significant issue in the earlier development of massive gravity, such as the quadratic Fierz-Pauli theory [13; 14], is that the predictions of massive gravity do not converge to those of GR as the graviton mass $m \to 0$ and hence deviate from the standard tests of GR in the solar system, which is referred as the van Dam-Veltman-Zakharov (vDVZ) discontinuity [15; 16]. An example of these incorrect predictions is that, as $m \to 0$, the light bending by the Sun calculated using the quadratic Fierz-Pauli theory is only 3/4 of the prediction of GR [14; 17]. Later, Vainshtein discovered that **this discontinuity problem can be naturally resolved by introducing non-linear interactions** into the Fierz-Pauli theory [9], which screens the extra force contribution from massive gravity within the scale of the solar system and is known as the Vainshtein screening mechanism [16; 18].

For a massive graviton with spin $s = 2$, the number of DoFs is $2s + 1 = 5$ (or equivalently the polarisations [19]) [20], whereas there are only two DoFs in GR, where the three others are only gauge DoFs [4]. By decomposing the total field $h_{\mu\nu}$ into its helicity-2 $\hat{h}_{\mu\nu}$, helicity-1 $A_\mu$, and helicity-0 components, where **the helicity-0 mode corresponds to a scalar field** $\pi$ [4; 19], it can be shown to carry one of the five DoFs [21]. This scalar field is associated with an extra force contribution to the overall gravitational force in massive gravity, referred to as **the fifth force**, and is the force contribution screened by the Vainshtein mechanism [12; 21]. As a force screened within the solar system, the fifth force has to be hence described by nonlinear equations of motion (EOMs). This non-linearity suggests that, despite the existence of an exact one-body solution, **the principle of superposition cannot be applied to studying multiple-body systems**, which makes directly solving such EOMs very complicated if lacking enough symmetries [21]. Therefore, the aim of this project is to perform a novel study of the fifth force and the associated Vainshtein effects for a two-body system by using the variation approximation method (VAM).

## 1.2 Report structure

In this report, we aim to develop a new approach to studying multiple-body systems in the context of the Dvali-Gabadadze-Porati (DGP) model in massive gravity, with a focus on the "simplest" two-body system. Such scenarios **have never been analytically considered before** by theorists due to the complexity of the mathematics involved. Our study involves the use of **the variational approximation method**, which is typically used for the analytical study of complicated quantum systems, form which we propose a plausible solution.

In Chapter 2, we will first outline **the main features of the DGP model**, a modern theoretical model of modified gravity in higher dimensions [22; 23], and discuss **the existing exact solution** for the scalar field $\pi$ of the one-body problem, which provides valuable insights into how the Vainshtein mechanism is realised in an actual theoretical model of massive gravity [21].

In Chapter 3, we discuss the details of the variational approximation method in general for both quantum and classical scenarios. We then **applied this method to the specific two-body problem** and explain the technical details about how a plausible solution is obtained from using both analytical and numerical methods. We also discuss **the limitations of the current results** by studying plausible alternative ansatzes that can be used in the approximation.

Chapter 4 discusses **the kinematics of a two-body system** under the effects of the fifth force, using the example of an astrophysical-like binary system. Surprisingly, we found that under our approximation, **the fifth force can sustain stable procession**

**orbits**, leading to interesting observations related to different characteristic length scales of the DGP model.

Finally, in Chapter 5, we draw our conclusions and provide an outlook for the future continuation of this work. We discuss **potential improvements** that could be implemented and explore the connections between our work and astrophysical & cosmological phenomenology.

# Chapter 2

# The Dvali-Gabadadze-Porati model

## 2.1 The Lagrangian density of $\pi$

In this project we use the Dvali-Gabadadze-Porati (DGP) model, which is the first model of massive gravity to explicitly demonstrate the Vainshtein mechanism [12; 24]. This model introduces an extra spatial dimension to the usual $D = (3 + 1)$ dimensional spacetime, and therefore **describes the 4D gravity in a 5D Minkowski space** [22; 23]. In the language of brane cosmology, the DGP model is a model of gravity described in a $D = (3 + 1)$ dimensional brane embedded in a $D = (4 + 1)$ dimensional bulk, where gravitons can travel (hence gravitational waves can propagate [25]) in the extra spatial dimension and thus alter gravitational effects, as shown in Figure 2.2 [22][26].

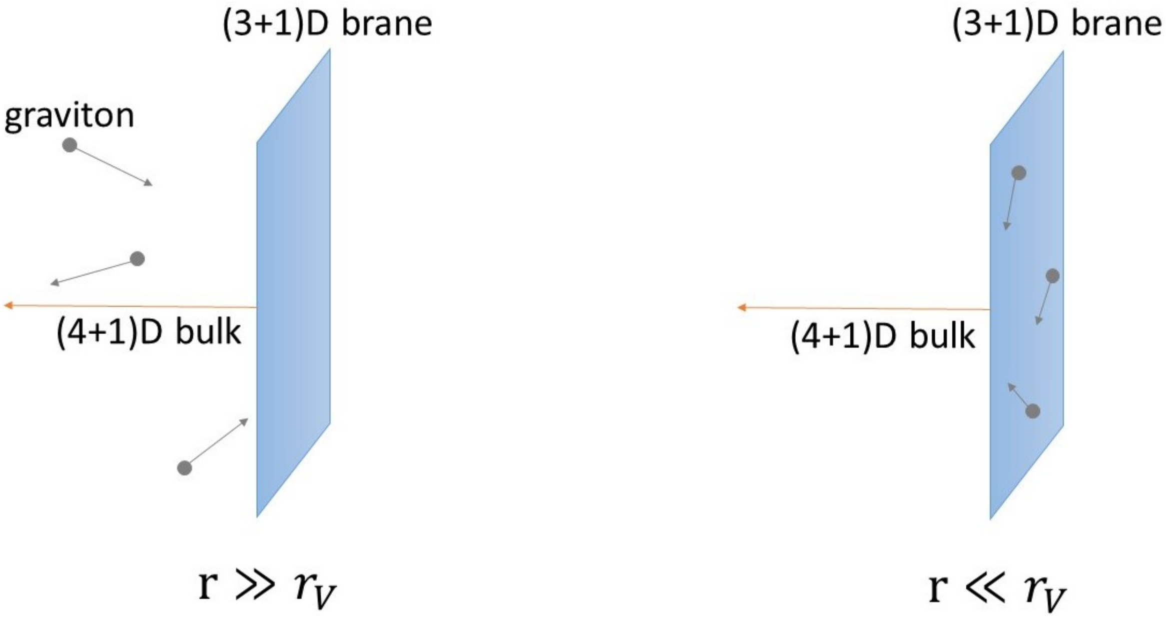


**Figure 2.1:** An illustration of how massive gravitons propagate in a $D = (4 + 1)$ dimensional bulk for $r \gg r_V$ (left) and $r \ll r_V$ (right). The gravitons, in general, can propagate freely inside the bulk, but within the Vainshtein radius $r_V$ they tend to propagate along the $D = (3 + 1)$ dimensional brane (blue plane) as in GR.

The best way to study the Vainshtein mechanism within the framework of the DGP model is by working in the so-called **decoupling limit** [21], where the strong decoupling scale $\Lambda = (m^2 M_{Pl})^{\frac{1}{3}}$ remains approximately constant whilst $m \to 0$ and $M_{Pl} \to \infty$ [27]. In this limit (along with proper gauge transformations [9]), the helicity-0 mode $\pi$ and the other modes are decoupled, which allows the helicity-0 mode to be treated mathematically on its own [9; 21]. In this context, the scalar field $\pi$ for a non-relativistic system is described by the following effective Lagrangian density, $\mathcal{L}_\pi$, in 3D [12; 21]:

$$\mathcal{L}_\pi = \frac{1}{2}(\nabla\pi)^2 - \frac{1}{\Lambda^3}(\nabla\pi)^2(\nabla^2\pi) + \frac{\pi\rho}{M_{Pl}}, \tag{2.1}$$

where $\rho$ is the density of matter, $M_{Pl}$ is the Planck mass. The first term is the usual kinetic term, whereas the non-linearity of the EOMs is from the second term of $\mathcal{L}_\pi$, the potential term, which represents a **cubic & non-linear interaction** of $\pi$. The last term describes the coupling between the field $\pi$ and the mass source $\rho$. The last two terms of e.q. 2.1 ensure that the interaction is only important for a low energy scale (i.e. large distance scale) $\Lambda \ll M_{Pl}$ [21], which indirectly infers the existence of the Vainshtein mechanism.

This description of $\pi$ is classified as a Galileon theory because $\pi$ is **invariant under the Galileon transformation** $\pi \to \pi + c + v_\mu x^\mu$, where $c/v_\mu$ is a constant scalar/four-vector [1; 21; 28]. This symmetry imposes constraints on the form of the Lagrangian density describing the dynamics of $\pi$ [1], which leads to e.q 2.1 through the introduction of the mass-field coupling.

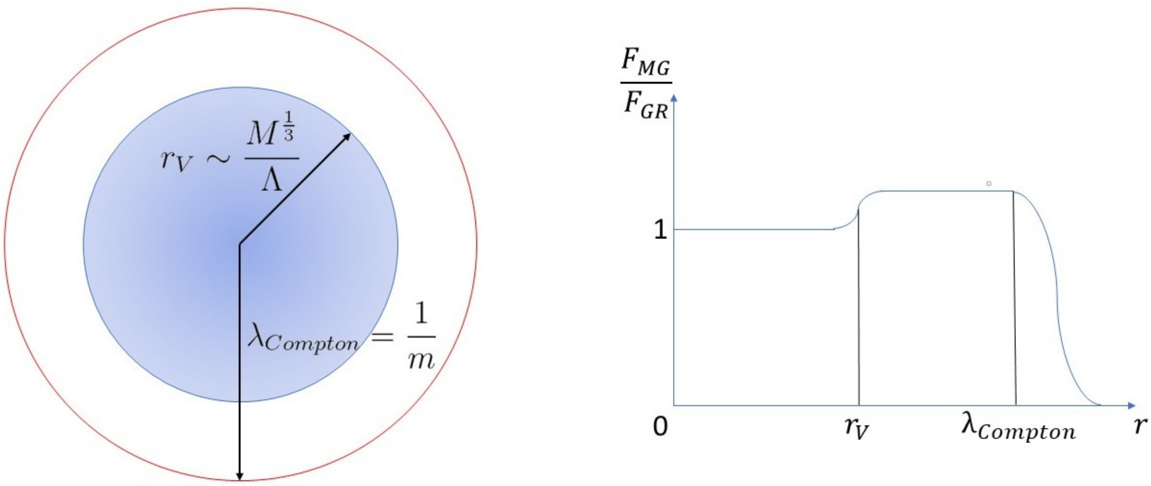


**Figure 2.2:** A plot illustrating the key length scales in massive gravity (left) and how the ratio of the force of massive gravity and the force of GR varies as a function of radial distance $r$ (right). Within the Vainshtein radius, $r_V$, massive gravity behaves almost exactly like GR, whereas outside $r_V$ the fifth force starts to manifest, which makes an extra contribution to the overall gravitational force. When reaching the Compton wavelength $\lambda_{Compton}$ of the mass graviton, the overall gravitational force of massive gravity vanishes.

## 2.2 Exact one-body solution

The static one-body problem with spherical symmetry (e.g. a point-mass $\rho = M\boldsymbol{\delta}(\boldsymbol{r})$) can be exactly solved [12; 29; 30]. First, directly varying the action formed by integrating $\mathcal{L}_\pi$ and rearrange the field equation of $\pi$ obtained [29]:

$$\nabla \cdot \left( \boldsymbol{\mathcal{E}} - \frac{1}{\Lambda^3}\nabla^2 |\boldsymbol{\mathcal{E}}|^2 + \frac{2}{\Lambda^3}\boldsymbol{\mathcal{E}}\nabla \cdot \boldsymbol{\mathcal{E}} \right) = \frac{M\delta(\boldsymbol{r})}{M_{Pl}}, \tag{2.2}$$

where a vector field is defined to simplify the equation $\boldsymbol{\mathcal{E}} = \nabla\pi$. In spherical coordinates, $\nabla \cdot \boldsymbol{\mathcal{E}} = \nabla \cdot (\mathcal{E}\hat{\boldsymbol{r}}) = \frac{1}{r^2}\frac{d}{dr}(r^2\mathcal{E})$, hence the second term in e.q 2.2 is cancelled out by part of the third term [29]. After integrating both sides over a sphere of radius $r$ and applying the divergence theorem to the left-hand side, e.q 2.2 becomes [29]:

$$r^2 \left( \mathcal{E} + \frac{4}{r\Lambda^3}\mathcal{E}^2 \right) = \frac{M}{M_{Pl}}, \tag{2.3}$$

where a factor of $4\pi$ has been ignored on purpose to make this solution consistent with the actual calculations done for this project. Solving this quadratic equation in $\boldsymbol{\mathcal{E}}$ to obtain the final solution for the fifth force $F_\pi$ [29]:

$$F_\pi = -\frac{\partial \pi}{\partial r} = -\mathcal{E} = -\frac{r\Lambda^3}{8}\left( -1 + \sqrt{1 + \frac{16M}{M_{Pl}r^3\Lambda^3}} \right), \tag{2.4}$$

where $r$ is the distance from the point source and $M$ is the mass of the point source and only the positive root is kept to avoid divergence at $r = 0$ [21].

This solution explicitly exhibits the Vainshtein mechanism: there is a length scale, referred to as the Vainshtein radius $r_V$, which has the form $r_V = \frac{1}{\Lambda}(\frac{M}{M_{Pl}})^{\frac{1}{3}}$. For $r \gg r_V$, $F_\pi = -\frac{M}{3M_{Pl}r^2}$, which recovers the Newton inverse square law [21]. For $r \ll r_V$, $F_\pi = -(\frac{M\Lambda^3}{2M_{Pl}r})^{\frac{1}{2}}$, which is much weaker than the Newtonian gravity for small $r$, hence this force is screened within $r_V$ [21]. So it can be readily seen why the Vainshtein mechanism solves the problem of vDVZ discontinuity: $r_V \to \infty$ as $m \to 0$, hence $F_\pi$ is screened over the entire universe, which recovers GR on all scales [14]. In the picture of $D = (4 + 1)$ dimensional spacetime, the gravitons tend to propagate along the brane, i.e. the $D = (3 + 1)$ dimensional spacetime for $r \ll r_V$, which agrees with GR (see Figure 2.1) [22][31]. Furthermore, another piece of information extracted from the above solution is the asymptotic behaviours of $\pi$: $\pi \propto r^{-1}$ for $r \gg r_V$, $\pi \propto r^{\frac{1}{2}}$ for $r \ll r_V$, which will be used in VAM to formulate appropriate ansatzes.

# Chapter 3

# Variational Approximation Method

Variational Approximation Method (VAM) is a mathematical method commonly used in quantum mechanics to study complicated quantum systems: an ansatz is formed for the wavefunction $\psi$ and used to calculate the expectation value of the Hamiltonian $\langle\psi|\hat{H}|\psi\rangle$, the latter is minimised w.r.t to the parameters of the ansatz. This minimised energy is then the approximated solution, or an upper bound, of the ground state energy of the quantum system [32]. A similar idea can be applied to a non-relativistic classical system. The energy density of a static system can be formed by negating the potential term of the Lagrangian density e.q 2.1, and the total energy $E$ can be hence obtained by integrating over the entire space:

$$E_{\text{before}} = \int d^3x \left[\frac{1}{2}(\nabla\pi)^2 + \frac{1}{\Lambda^3}(\nabla\pi)^2(\nabla^2\pi) + \frac{\pi\rho}{M_{Pl}}\right]. \tag{3.1}$$

An ansatz for $\pi(\boldsymbol{\lambda})$ depending on a set of $N$ parameters $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3..., \lambda_N)$ is substituted into $E_{\text{before}}$. The energy obtained after the integration, $E_{\text{after}}$, is differentiated w.r.t to every single parameter $\lambda_i$, which gives rise to a set of $N$ coupled differential equations used to extremise the energy. These differential equations are then solved either analytically or numerically, depending on how complicated they are.

The minimal requirement for an ansatz used in VAM to be valid is that it must result in **an isolated energy minimum**, i.e., at least one of the extrema found must not be a maximum nor a saddle point, which can be verified by using the Hessian matrix of $E_{\text{after}}$, a function of all $\lambda_i$'s. This $N \times N$ Hessian matrix $H_E$, in index notation, can be written as

$$(H_E)_{ij} = \frac{\partial^2 E_{\text{after}}}{\partial\lambda_i \partial\lambda_j}, \qquad i, j = 1, 2, ..., N. \tag{3.2}$$

If all eigenvalues of $H_E$ are positive at some extremum $\boldsymbol{\lambda}_0$ in the parameter space, then $H_E$ is positive-definite at $\boldsymbol{\lambda}_0$, which in turn implies that $\boldsymbol{\lambda}_0$ is an isolated minimum [33]. Otherwise, this extremum is either an isolated maximum (all eigenvalues are positive) or a saddle point (some eigenvalues are positive and some are negative) [33].

If an isolated minimum of the parameter space exists, then the minimised energy, $E_{\min}$ can be used to form a Lagrangian $L$ of the form $L = T - E_{\min}$ (the fifth force is conservative) describing the motion of the object(s) within the system, where $T$ represents the total kinetic energy of the system. Therefore, the total Lagrangian can be expressed as

$$L = \sum_j \frac{1}{2} M_j \dot{\boldsymbol{r}}_j^2 - E_{\min}, \tag{3.3}$$

where $M_j$ and $r_j$ denote, respectively, the mass and position of the j$^{\text{th}}$ object [34]. This Lagrangian can be varied w.r.t all position coordinates to obtain three EOMs per object, which can be solved to determine the dynamics of the system by applying appropriate initial conditions.

## 3.1 One-body problem: testing the VAM

Since the one-body problem can be exactly solved, applying VAM to the one-body problem can be used to test how well an ansatz behaves. The first tested ansatz is a Gaussian-like function, the simplest function that decays away at a large distance from the source, which takes the form $\pi(r; A, L) = A \exp\left(-r^2/L^2\right)$, where the two parameters $A$ and $L$ are the field amplitude and the length scale of the system, respectively. By plugging this ansatz into e.q 3.1, and then minimising $E_{\text{before}}$ to obtain $E_{\min}$, it can be shown that the length scale of the system takes the form (taking $M_{Pl} = 1$ for simplicity from now on):

$$r_V = \frac{8}{3\sqrt{3\pi}} \frac{M^{\frac{1}{3}}}{5^{\frac{1}{3}} \Lambda} \approx 0.507 \frac{M^{\frac{1}{3}}}{\Lambda}, \tag{3.4}$$

which has the correct dependence on both $M$ and $\Lambda$. It's worth noticing that the numerical factor in this $r_V$ is on the order of unity, as expected from the exact solution, which suggests that this ansatz, despite its **lack of correct asymptotic behaviours** (e.g. it doesn't vanish at $r = 0$), still incorporates the correct physics about the VM.

## 3.2 Two-body problem

Since the Gaussian-like ansatz contains the essential physics of VM, it's worth applying it to the two-body problem for its mathematical simplicity. In this two-body system in 3-dimensional Cartesian space, the two point-masses of mass $M_1$ and $M_2$ are placed, respectively, at $\boldsymbol{r_1} = (0, 0, -a/2)$ and $\boldsymbol{r_2} = (0, 0, +a/2)$, where $a$ denotes their separation distance. Hence the overlap scalar field can be expressed as

$$\pi(\boldsymbol{r}; C_1, C_2, L_1, L_2) = C_1 e^{-\frac{|\boldsymbol{r}-\boldsymbol{r}_1|^2}{L_1^2}} + C_2 e^{-\frac{|\boldsymbol{r}-\boldsymbol{r}_2|^2}{L_2^2}}, \tag{3.5}$$

where the parameters $C$'s and $L$'s are the field amplitudes and the Vainshtein length scales of each point-mass. Nonetheless, after integrating e.q 3.1 over the entire Cartesian space, the four non-linear coupled differential equations of $(C_1, C_2, L_1, L_2)$

obtained by minimising $E_{\text{before}}$ cannot be solved analytically, which suggests that numerical methods need to be introduced at this point to solve the problem.

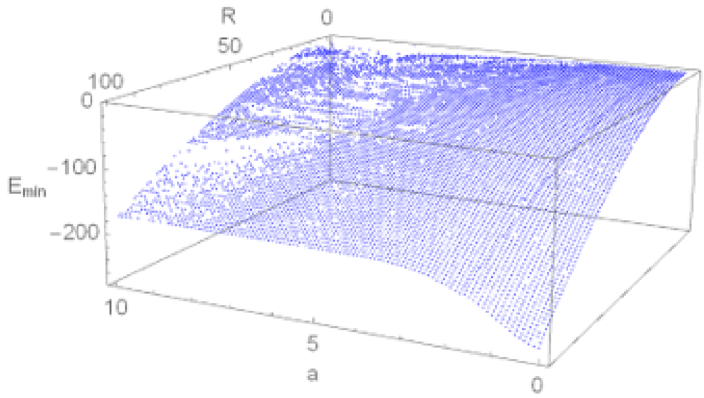


**Figure 3.1:** A 3D plot of the numerically obtained values of $E_{min}$ as a function of separation $a$ (from 0 to 10) and mass ratio $R$ (from 1 to 100), up to a constant $M_2^{\frac{5}{3}}\Lambda$.

Before solving these coupled differential equations, two useful transformations of the field and the space coordinates can be performed to greatly simplify the problem: $\pi \to M_2^{\frac{2}{3}}\Lambda\pi$ and $x_i \to M_2^{\frac{1}{3}}x_i/\Lambda$, after which e.q 3.1 can be written as the following form for the two-body problem:

$$E_{\text{before}} = M_2^{\frac{5}{3}}\Lambda \int d^3x \left[\frac{1}{2}(\nabla\pi)^2 + (\nabla\pi)^2(\nabla^2\pi) + \pi\left(R\delta(\boldsymbol{r}, \boldsymbol{r}_1(a)) + \delta(\boldsymbol{r}, \boldsymbol{r}_2(a))\right)\right], \tag{3.6}$$

where $R = M_1/M_2$ is the mass ratio. The above expression suggests that the potential energy of this two-body system, up to a constant $M_2^{\frac{5}{3}}\Lambda$, can be considered as a function of only two variables: the separation $a$ and the mass ratio $R$. Therefore, $E_{before}$ needs only to be minimised for different sets of values of $(a, R)$, which can be numerically implemented for discrete values of both variables. The results obtained in the region $0 < a < 10$, $1 < R < 100$ is shown in Figure 3.1. **It's worth noticing that all non-dimensionless physical quantities will from now on be shown in pure numerical values, but they all rely on this overall energy scale in e.q 3.6**.

Numerically minimising the energy using Mathematica is very similar to fitting, where the coupled differential equations serve as constraints to the parameters. Therefore, it is very important to set **appropriate boundaries** for the allowed parameter values, since these boundaries play the role of initial guesses in fitting, which would otherwise easily makes the results diverge. However, for a large number of points $(a, R)$, it is not possible to assign appropriate boundaries for each single of them, so it is necessary to empirically find an algorithm that automatically generates the boundaries for all the points, which is why in Figure 3.1 there are missing blue dots because the algorithm cannot work perfectly in all regions.

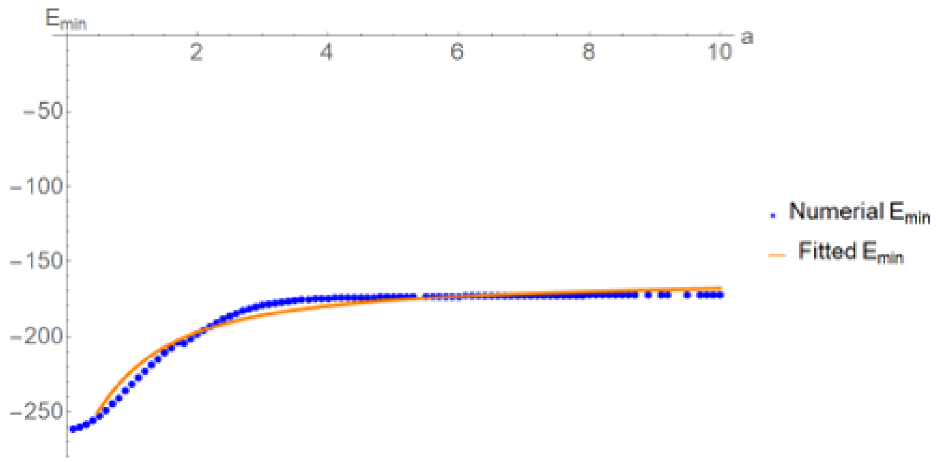


**Figure 3.2:** $E_{\rm min}$ as a function of separation $a$ (from 0 to 10) for fixed mass ratio $R = 100$, fitted using e.q 3.7 and obtained. The orange curve is the fitted curve, and the blue dots are the numerical values of $E_{\rm min}$, up to a constant $M_2^{\frac{5}{3}}\Lambda$. The fit has a normalised chi-square value of $\chi^2_{\rm normalised} = 0.64$.

As shown in Figure 3.1, for a fixed $R$, $E_{\rm min}$ increases as $a$ increases, which suggest that the associated force $F_\pi = -\frac{\partial E_{min}}{\partial a}$ is attractive. Furthermore, since $E_{\rm min}$ converges for large $a$, $F_\pi$ vanishes as $a$ goes beyond a certain length scale, as expected for a massive force carrier. An appropriate mathematical expression that describes the behaviours of $E_{\rm min}$ as a function of $a$ for a fixed $R$ is

$$E_{{\rm min},R}(a) = E_0 + E_1 \arctan(a), \tag{3.7}$$

where $E_0$ and $E_1$ are both constant parameters with the dimension of energy. This choice is mathematically natural because $\arctan$ is one of the few elementary functions that converge and the fit in Figure 3.2 confirms that this expression can indeed properly describe the numerical results obtained, which has been statistically verified by performing a normalised chi-square test, with $\chi^2_{\rm normalised} = 0.64$.

Although the solution to the force is attractive and incorporates correct behaviours at large distances, the Vainshtein effects at short distances are not accurately captured: the slope of $E_{min}$ is not flat as it theoretically should be at small $a$ as shown in Figure 3.2. This issue should be mainly due to the lack of correct asymptotes of the oversimplified ansatz used in the above calculations.

## 3.3 Alternative ansatzes

Apart from the issue with the accurate quantitative manifestation of the Vainshtein mechanism at short distances, the fact that the Gaussian-like function is a 'bad' ansatz can also be seen from the minimum energy value obtained for the one-body problem: $E_{\rm min,gaussian} = +4.21 M^{\frac{5}{3}}\Lambda$, whereas that from the exact solution is $E_{min,exact} = -1.36 M^{\frac{5}{3}}\Lambda$. $E_{\rm min,gaussian}$ and $E_{\rm min,exact}$ are not only very different in value but also in the sign of the energy, which is another main issue of this over-simplified ansatz. Thus, another crucial task is to test other possible ansatzes with better asymptotic behaviours, which might be mathematically too complicated for the two-body problem but can be analytically applied to the one-body problem just to see if they yield a better value of $E_{\text{min}}$. The main alternative ansatz used for this project is the frictional ansatz, which has the following frictional form:

$$\pi(r; A_1, A_2, B, L) = \frac{A_1 - A_2 e^{-\sqrt{\frac{r}{L}}}}{B + (\frac{r}{L})}, \tag{3.8}$$

where $A_1$ and $A_2$ are two parameters describing the amplitude of the field, $B$ is a non-zero dimensionless parameter added to avoid divergence at $r = 0$. The frictional ansatz has the correct asymptotic behaviour for both limits $r \gg r_V$ and $r \ll r_V$, which can be seen by doing appropriate expansion in each limit (also see Figure 3.3). Most importantly, this frictional ansatz indeed yields a much closer value of minimised energy to the exact solution, $E_{\text{min,fractional}} = -0.998M^{\frac{5}{3}}\Lambda$, which suggests that this ansatz well approximate the mathematical behaviours of the exact solution.

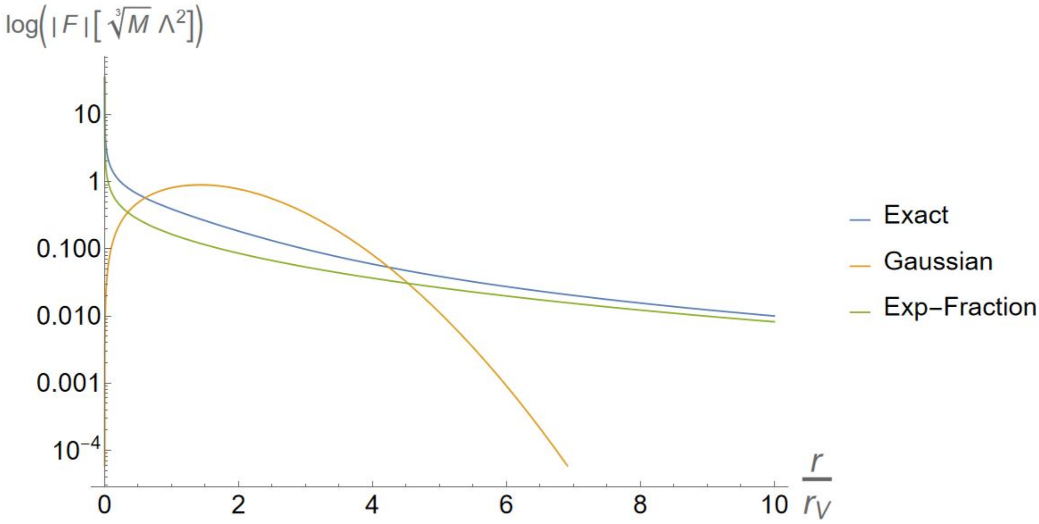


**Figure 3.3:** A plot illustrating the shape of the fifth forces $F$ as a function of the number of the Vainshtein radius $r/r_V$ obtained from both Gaussian (orange) and frictional (green) ansatzes, as well as the exact solution (blue). The vertical axis is on a logarithm scale. The Gaussian curve behaves very differently from the other two for small $r/r_V$.

Another set of ansatzes tested are variations of the Gaussian-like function, either by introducing an extra DoF carried by an extra term, e.g.,

$$\pi(r; A, B, L) = Ae^{-\frac{r^2}{L^2}} + \frac{Br^2}{L^2}e^{-\frac{r^2}{L^2}}, \tag{3.9}$$

or by deforming the original function to make its asymptotes behave more correctly:

$$\pi(r; A, L) = A\left(\frac{r}{L}\right)^{2n} e^{-\frac{r^2}{L^2}}, \tag{3.10}$$

which does equal 0 at $r = 0$. However, none of these ansatzes improves the results: the $E_{\min}$ of e.q 3.9 is found when the extra parameter $B = 0$, and $A$ and $L$ take exactly the same form as in e.q 3.4, i.e. this ansatz recovers the results of the Gaussian-like one, the extra DoF is simply ignored; the $E_{\min}$ of e.q 3.10 corresponds to $A = 0$ for any strictly positive n, i.e., as if there is no field at all.

These 'surprising' results lead to a conjecture about the form of the ansatzes used in the VAM to solve EOMs of the DGP model: if a term in the ansatz is equal to 0 at $r = 0$, then it won't make any contribution to the final results obtained from the VAM. The explanation to this conjecture is that, since the mass source considered here is a point-mass $\rho = M\delta(\boldsymbol{r})$, any contributions from such terms to the matter-field interaction in the DGP model, $\pi\rho/M_{Pl}$, will be completely **suppressed**. This kind of suppression severely restricts the possible ansatzes that can be used in VAM. Therefore, to verify whether these ansatzes could lead to untrivial solutions and hence improve the results obtained, a different form of mass source has to be chosen, e.g. a rectangular mass distribution on the radial axis, which could be a possible extension of the current work.

# Chapter 4

# The dynamics of the two-body system

## 4.1 The Lagrangian

As mentioned previously, e.q 3.7 can be used in e.q 3.3 to form a Lagrangian **(no longer a Lagrangian density)** describing the dynamics of the two-body system under the fifth force (it's important to keep in mind that the usual gravitational force from GR has not been taken into account):

$$L(\boldsymbol{r}_1, \boldsymbol{r}_2, \dot{\boldsymbol{r}}_1, \dot{\boldsymbol{r}}_2) = \frac{1}{2}M_1\dot{\boldsymbol{r}}_1^2 + \frac{1}{2}M_2\dot{\boldsymbol{r}}_2^2 - (E_0 + E_1 \arctan(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)), \tag{4.1}$$

where $\boldsymbol{r}_1 = (x_1, y_1, z_1)$ and $\boldsymbol{r}_2 = (x_2, y_2, z_2)$. The associated action is varied w.r.t to the $2 \times 3 = 6$ spatial coordinates of the two masses of the system and hence leads to six coupled non-linear differential equations that can only be solved numerically. However, unlike minimisation, to describe the dynamics of a particular two-body system, it is necessary to first set the initial conditions, like the initial position and velocity of the two masses, and then the differential equations will determine how the system evolves in time under these initial conditions.

## 4.2 Example: astrophysical-like binary system

A typical realistic two-body system that exists in the universe is a binary star, where the two stars of the system rotate about each other, so it would be physically valuable to study the dynamics of a similar system under the influence of the fifth force by using e.q 4.3. Another advantage of studying a such system is that the motion of the two masses of the system ideally remains in the same plane. Therefore, although this is a realistic system in 3-dimensional space, one of the dimensions can be safely ignored, which greatly simplifies the problem and hence makes a such system a perfect starting point for studying the dynamics of the fifth force.

For this concrete problem, the two masses initially sit on the z-axis, and their initial velocities are in -y and +y directions, respectively, these initial conditions assure that the two masses will remain in the yz-plane. The magnitudes of these quantities are vital for the existence of stable procession orbits of this binary system, which hence need to be carefully chosen. For instance, as shown in Figure 4.1, for two equal masses $M_1 = M_2 = 1$, the initial conditions are

$$\begin{aligned} \boldsymbol{r}_1(t=0) &= (0,0,-0.5); \quad & \boldsymbol{r}_2(t=0) &= (0,0,+0.5); \\ \dot{\boldsymbol{r}}_1(t=0) &= (0,-1,0); \quad & \dot{\boldsymbol{r}}_2(t=0) &= (0,+1,0), \end{aligned} \tag{4.2}$$

which results in stable orbits when applied to the EOMs obtained from e.q 4.3. On the other hand, if one mass is larger than the other, then the more massive object will need to be placed closer to the centre of the system with a slower initial speed to keep the orbits stable, e.g, for $M_1 = 5$ and $M_2 = 1$, a set of **initial conditions leading to stable orbits** are:

$$\begin{aligned} \boldsymbol{r}_1(t=0) &= (0,0,-0.1); \quad & \boldsymbol{r}_2(t=0) &= (0,0,+0.5); \\ \dot{\boldsymbol{r}}_1(t=0) &= (0,-0.2,0); \quad & \dot{\boldsymbol{r}}_2(t=0) &= (0,+1,0). \end{aligned} \tag{4.3}$$

Empirically, one kind of initial conditions that ensures the stability of the system for very large mass ratio (tested up to $R = 100000$) can be obtained as follows: divide the initial position and speed of the heavier mass by a factor $R$ relative to those of the lighter mass. Any small deviations from this kind of initial conditions could easily lead to **the disappearance of the stability** of the system.

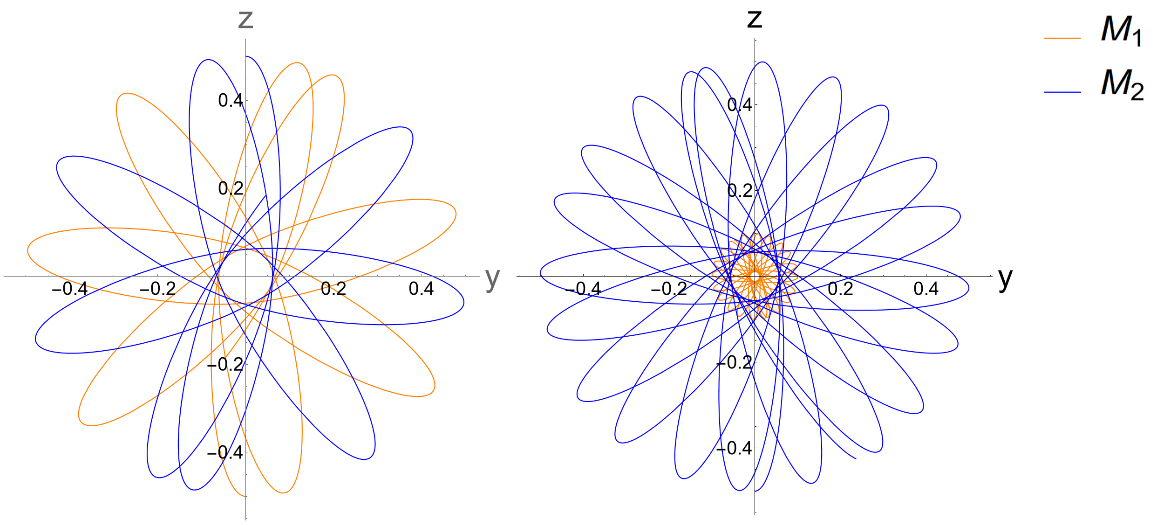


**Figure 4.1:** The stable procession orbits of an astrophysical-like binary system for different mass ratios: $M_1 = M_2 = 1$ (left), $M_1 = 5$ and $M_2 = 1$ (right). The two masses start moving from the two sides of the z-axis in opposite directions along the y-axis. The orange curve represents the orbit of $M_1$, and the blue curve represents that of $M_2$.

To study how the solutions obtained exhibit the Vainshtein mechanism, the mass ratio is held fixed while the separation is varied. Before working with a concrete example, some qualitative definitions related to the shape of the orbits can be given: the frequency of the orbit can be defined as the time needed to go from one peak to the other, and the eccentricity of an orbit can be defined as the 'narrowness' of the latter.

The simplest example is to study two equal masses $M_1 = M_2 = 1$, which have a Vainshtein radius $\sim 0.5$ by e.q 3.4, and observe how the orbits vary as the initial separation $a$ using the kind of initial conditions defined above. The results are shown in Figure 4.2 for two very distinct separations: $a = 3$ and $a = 0.1$, from which some observations about the orbit shape can be made: the eccentricity and frequency of the procession orbits decrease as separation increases. A more important observation is that the stability of the orbits breaks down for the same kind of initial conditions when $a \gtrsim 4$, which is about one order of magnitude higher than $r_V$. This breakdown of the stability of the orbits could be a consequence of reaching **the Compton wavelength of the massive graviton**, which requires further investigations into the dynamics of this system.

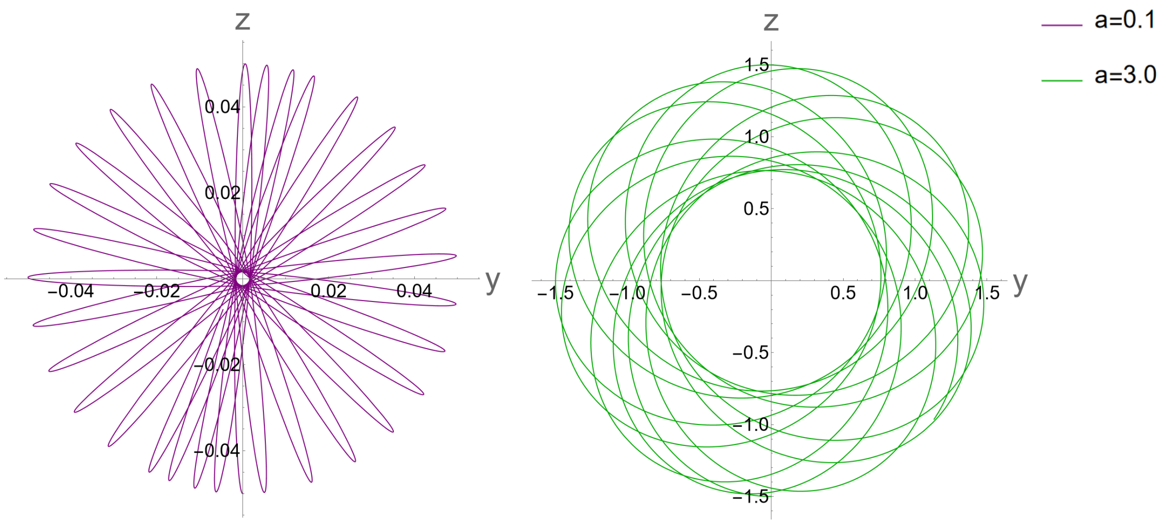


**Figure 4.2:** The stable procession orbits of an astrophysical-like binary system for same mass ratios $M_1 = M_2 = 1$ but different initial separation: $a = 0.1$ (left) and $a = 3$ (right). The orange curve represents the orbit of $M_1$, and the blue curve represents that of $M_2$. The orbits on the right took a time period of 10 times longer.

# Chapter 5

# Summary & Outlook

We have studied the analytical solutions to the dynamics of a general two-body system in the context of the DGP model by **applying the variational approximation method to the helicity-0 mode of the field**, which has never been studied analytically before. In the one-body problem, we demonstrated that a Gaussian-like ansatz yields a characteristic length scale that manifests the Vainshtein screening mechanism. Applying the same ansatz to the two-body system, we obtained **the analytical description of the minimised potential energy** $E_{\rm min}$ and hence the fifth force as a function of the separation $a$. This analytical form of $E_{\rm min}$ captures the key physics of a force mediated by a massive force carrier.

Nonetheless, the lack of correct asymptotic behaviours affects the exact behaviours of the solutions, especially at short distances where the Vainshtein mechanism is not clearly observed in the two-body system. Therefore, to evaluate the validity of these solutions obtained using the Gaussian-like ansatz, we tested other more involved ansatzes in the one-body problem and found that the correct asymptotes significantly improve the consistency with the exact solution. We also noted that choosing a point mass as the mass distribution severely limits the range of potential candidate ansatzes due to the peculiar mathematical properties of the Dirac-delta function, which could be a main focus of future work.

Finally, the analytical form of $E_{\rm min}$ has been applied to **an astrophysical-like two-body system processing in a plane**, which shows that the fifth force alone can sustain a stable two-body system that holds even for a very large mass hierarchy. Furthermore, taking the usual gravitational force into account would be a very interesting and valuable extension of the current work.

A similar study using VAM could also be applied to a more general multiple-body system with more involved ansatzes, which could be applied to a wider range of astrophysical & cosmological contexts but would be mathematically challenging due to the more complicated dependency of the minimised energy on variables relating different object in the system. The results obtained could be applied and compared to the existing study of astrophysical & cosmological phenomenology of massive gravity, like for binary pulsars [35].

# Bibliography


[1] Nicolis A, Rattazzi R, Trincherini E. Galileon as a local modification of gravity. Physical Review D. 2009 mar;79(6). doi:10.1103/physrevd.79.064036.

[2] Kaloper N, Padilla A, Saffin PM, Stefanyszyn D. Unitarity and the Vainshtein mechanism. Physical Review D. 2015 feb;91(4). doi:10.1103/physrevd.91.045017.

[3] Arkani-Hamed N, Dimopoulos S, Dvali G, Gabadadze G. Non-Local Modification of Gravity and the Cosmological Constant Problem; 2002. doi:10.48550/arXiv.hep-th.0209227.

[4] de Rham C, Gabadadze G, Tolley AJ. Helicity decomposition of ghost-free massive gravity. Journal of High Energy Physics. 2011 nov;2011(11). doi:10.1007/jhep11(2011)093.

[5] Boulware DG, Deser S. Classical general relativity derived from quantum gravity. Annals of Physics. 1975;89(1):193-240. Available from: `https://www.sciencedirect.com/science/article/pii/0003491675903024`. doi:https://doi.org/10.1016/0003-4916(75)90302-4.

[6] the Centre International de Rencontres Mathématiques. Andrew J. Tolley: A brief introduction to massive gravity;. Available from: `https://www.youtube.com/watch?v=1Jbusc4qmao`.

[7] Boulware DG, Deser S. Can Gravitation Have a Finite Range? Phys Rev D. 1972 Dec;6:3368-82. Available from: `https://link.aps.org/doi/10.1103/PhysRevD.6.3368`. doi:10.1103/PhysRevD.6.3368.

[8] de Rham C, Gabadadze G, Tolley AJ. Resummation of Massive Gravity. Phys Rev Lett. 2011 Jun;106:231101. Available from: `https://link.aps.org/doi/10.1103/PhysRevLett.106.231101`. doi:10.1103/PhysRevLett.106.231101.

[9] Babichev E, Deffayet C. An introduction to the Vainshtein mechanism. Classical and Quantum Gravity. 2013 sep;30(18):184001. doi:10.1088/0264-9381/30/18/184001.

[10] Kirsch I. Higgs mechanism for gravity. Physical Review D. 2005 jul;72(2). doi:10.1103/physrevd.72.024001.

[11] Kakushadze Z. Gravitational Higgs Mechanism and Massive Gravity. International Journal of Modern Physics A. 2008 apr;23(10):1581-91. doi:10.1142/s0217751x08039591.

[12] de Rham C. Massive Gravity. Living Reviews in Relativity. 2014 aug;17(1). doi:10.12942/lrr-2014-7.

[13] M F, Ernst PW. relativistic wave equations for particles of arbitrary spin in an electromagnetic field. Physics Letters B. 1939 Nov;A(173):211-32. doi:https://doi.org/10.1098/rspa.1939.0140.

[14] Babichev E, Deffayet C, Ziour R. Recovery of general relativity in massive gravity via the Vainshtein mechanism. Physical Review D. 2010 nov;82(10). doi:10.1103/physrevd.82.104008.

[15] Zakharov VI. Linearized Gravitation Theory and the Graviton Mass. Soviet Journal of Experimental and Theoretical Physics Letters. 1970 Jan;12:312.

[16] Falck B, Koyama K, bo Zhao G, Li B. The Vainshtein mechanism in the cosmic web. Journal of Cosmology and Astroparticle Physics. 2014 jul;2014(07):058-8. doi:10.1088/1475-7516/2014/07/058.

[17] van Dam H, Veltman M. Massive and mass-less Yang-Mills and gravitational fields. Nuclear Physics B. 1970;22(2):397-411. Available from: https://www.sciencedirect.com/science/article/pii/0550321370904165. doi:https://doi.org/10.1016/0550-3213(70)90416-5.

[18] Vainshtein AI. To the problem of nonvanishing gravitation mass. Physics Letters B. 1972;39(3):393-4. Available from: https://www.sciencedirect.com/science/article/pii/0370269372901475. doi:https://doi.org/10.1016/0370-2693(72)90147-5.

[19] Folkerts S, Pritzel A, Wintergerst N. On ghosts in theories of self-interacting massive spin-2 particles; 2011. doi:https://doi.org/10.48550/arXiv.1107.3157.

[20] Gambuti G, Maggiore N. Fierz–Pauli theory reloaded: from a theory of a symmetric tensor field to linearized massive gravity. The European Physical Journal C. 2021 feb;81(2). doi:10.1140/epjc/s10052-021-08962-8.

[21] de Rham C. Galileons in the sky. Comptes Rendus Physique. 2012 jul;13(6-7):666-81. doi:10.1016/j.crhy.2012.04.006.

[22] Dvali GR, Gabadadze G, Porrati M. 4-D gravity on a brane in 5-D Minkowski space. Phys Lett B. 2000;485:208-14. doi:10.1016/S0370-2693(00)00669-9.

[23] Deffayet C. Cosmology on a brane in Minkowski bulk. Physics Letters B. 2001 mar;502(1-4):199-208. doi:10.1016/s0370-2693(01)00160-5.

[24] Deffayet C, Dvali G, Gabadadze G, Vainshtein A. Nonperturbative continuity in graviton mass versus perturbative discontinuity. Physical Review D. 2002 jan;65(4). Available from: https://doi.org/10.1103%2Fphysrevd.65.044026. doi:10.1103/physrevd.65.044026.

[25] Khlopunov M, Gal'tsov DV. Leakage of gravitational waves into an extra dimension in the DGP model. Journal of Cosmology and Astroparticle Physics. 2022 oct;2022(10):062. Available from: https://doi.org/10.1088%2F1475-7516%2F2022%2F10%2F062. doi:10.1088/1475-7516/2022/10/062.

[26] Dvali G, Gabadadze G. Gravity on a brane in infinite-volume extra space. Physical Review D. 2001 feb;63(6). doi:10.1103/physrevd.63.065007.

[27] Luty MA, Porrati M, Rattazzi R. Strong interactions and stability in the DGP model. Journal of High Energy Physics. 2003 sep;2003(09):029-9. doi:10.1088/1126-6708/2003/09/029.

[28] Burrage C, de Rham C, Heisenberg L, Tolley AJ. Chronology protection in Galileon models and massive gravity. Journal of Cosmology and Astroparticle Physics. 2012 jul;2012(07):004-4. Available from: https://doi.org/10.1088%2F1475-7516%2F2012%2F07%2F004. doi:10.1088/1475-7516/2012/07/004.

[29] Nicolis A, Rattazzi R. Classical and Quantum Consistency of the DGP Model. Journal of High Energy Physics. 2004 jul;2004(06):059-9. doi:10.1088/1126-6708/2004/06/059.

[30] Chkareuli G, Pirtskhalava D. Vainshtein mechanism in $\Lambda_3$-theories. Physics Letters B. 2012 jun;713(2):99-103. doi:10.1016/j.physletb.2012.05.030.

[31] Dvali G, Gabadadze G, Kolanović M, Nitti F. Scales of gravity. Physical Review D. 2001 dec;65(2). Available from: https://doi.org/10.1103%2Fphysrevd.65.024031. doi:10.1103/physrevd.65.024031.

[32] Sommerfeld T. Lorentz Trial Function for the Hydrogen Atom: A Simple, Elegant Exercise. Journal of Chemical Education. 2011;88(11):1521-4. doi:10.1021/ed200040e.

[33] Binmore K, Davies J. Calculus: Concepts and Methods. Cambridge University Press; 2002. doi:10.1017/CBO9780511802997.

[34] de Rham C. Advanced Classical Physics [Lectures notes on BlackBoard]. Imperial College London: Faculty of Natural Sciences, Department of Physics; 2022 [cited 2023 Apr 11].

[35] de Rham C, Tolley AJ, Wesley DH. Vainshtein mechanism in binary pulsars. Physical Review D. 2013 feb;87(4). Available from: https://doi.org/10.1103%2Fphysrevd.87.044025. doi:10.1103/physrevd.87.044025.