$$\vec{F} = m\vec{a}$$

$$R = \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underbrace{R(\vec{F})}_{F'} = m\underbrace{R(\vec{a})}_{a'}$$

$$F' = m a'$$

↑ transforms the same way as coordinate.

Covariant equation →

(every term transforms in the same way)

↳ include coordinates, velocity, force etc.

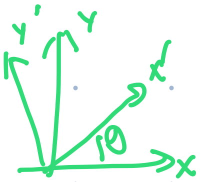
examples

$$\vec{F} = k(\Delta\vec{r})$$

$$\vec{F} = q(\vec{E} + \vec{\omega} \times \vec{B})$$

det of the rotation matrix ▷ 1.

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \Lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \Lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

↳ $\Lambda = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Yes. verified.

a vector transforms like coordinate

Math vs Physics Vector

Math → $v \in V$

↳ $v_1, v_2 \in V$ then $v_1 + v_2 \in V$

↳ $\alpha \in \mathbb{F}$, then $\alpha v_1 \in V$.

Physics → implicit transformation law under symmetry / coord. transformation.

$$\vec{v}' = R\vec{v}$$

↖ a linear transformation

↗ as read in the primed coordinate system.

$$\text{“}\begin{pmatrix} \text{mass} \\ E_x \\ B_z \end{pmatrix}\text{”}$$ not a physics vector ↘

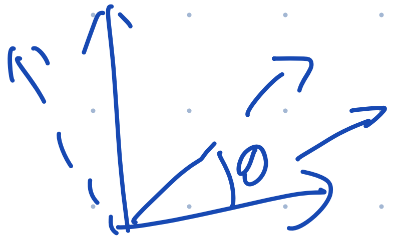
not anything can be a physics vector, they have to obey transformation rules.

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$ is a physics vector under Lorentz transformations

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}' = \Lambda \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \qquad \Lambda = \begin{pmatrix} \overset{\Lambda^0{}_0}{\gamma} & \overset{\Lambda^0{}_1}{-\gamma V} & 0 & 0 \\ -\gamma V & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \underset{\Lambda^3{}_0}{0} & 0 & 0 & 1 \end{pmatrix}$$

Indeed, here, the vector transform like the coordinate.

- Continuous rotations are not inertial: $\vec V$ is changing direction

- 

if angle stays the same ↪ inertial.

rotations

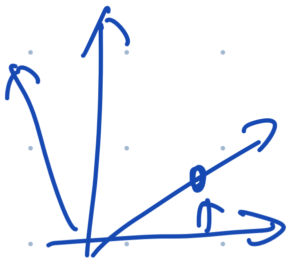
$$\vec r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r^1 \\ r^2 \\ r^3 \end{pmatrix} \qquad r^{\boxed{i}} \xrightarrow{} \text{upper index.} \quad , i = 1,2,3$$

$$\vec p = \begin{pmatrix} p^1 \\ p^2 \\ p^3 \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

$$\vec p\,' = R \vec p \qquad\qquad x' = 2x$$

$$p^i = \sum_{i=1}^{3} R^{\overset{\text{rows}}{i}}{}_{\underset{\text{columns}}{j}} \, p^j \qquad\qquad |\vec v\,'| = \frac{1}{2} V$$

$$X^\mu \Leftrightarrow \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$



$$p^\mu = \begin{pmatrix} E \\ P_x \\ P_y \\ P_z \end{pmatrix}$$

for 4 vectors.

the indexes are greek letters.

$$\underbrace{p'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu p^\nu}_{p'^\mu = \Lambda^\mu{}_\nu p^\nu}$$

the einstein sum convention repeated indices are summed over the entire range.

$\vec{p}\cdot\vec{L} \rightarrow p^i L_i \rightarrow$ you can ignore the order of things

$\vec{E}\cdot\vec{B} \rightarrow E^j B_j$

Tensors:

Tensor product

V, U are vector spaces

$v_1, v_2, v_3 \in V$

$u_1, u_2, u_3 \in U$

$$\underbrace{V \otimes U}_{\text{vector space}}$$

$$v \otimes u \in V \otimes U$$

Property:

$$(v_1 + v_2) \otimes u_1 = v_1 \otimes u_1 + v_2 \otimes u_1$$

$$v_1 \otimes (u_1 + u_2) = v_1 \otimes u_1 + v_2 \otimes u_2$$

$$\alpha (v \otimes u) = (\alpha v) \otimes u = v \otimes (\alpha u)$$

Example: V ~~is~~ are polynomials in s of order 3 or less.

U are polynomials in t of order 4 or less

$$(4+s) \otimes (7 + t + 2t^4) = 28 + 7s + 4t + st + 8t^4 + 2st^4$$

Basis on V: $\{1, s, s^2, s^3\}$

Basis on U: $\{1, t, t^2, t^3, t^4\}$

Basis on $V \otimes U$

$$= \{ 1\otimes 1, s\otimes 1, s^2\otimes 1, s^3\otimes 1, \\ 1\otimes t, s\otimes t, s^2\otimes t, s^3\otimes t \\ \cdot - - - - - - - \\ 1\otimes t^4, \cdot - - - - s^3\otimes t^4 \}.$$

$$\dim(V\otimes U) = \dim(V)\cdot\dim(U)$$

Order matters

$t\otimes s \neq s\otimes t$

$\vec{r}, \vec{p}$ two vector spaces

→ if you know how r^a, p^b transforms, you know how t^{ab} transforms.

$t^{ab} = r^a p^b$ components of $\vec{r} \otimes \vec{p}$.

$$t'^{ab} = R^a{}_c \, R^b{}_d \, r^c \, p^d \qquad R = \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$t^{01} = R^0{}_c \, R^1{}_d \, r^c p^d.$$

$$= (cr^0 + sr^1 + 0\cdot r^2)(-s\cdot p^0 + c\cdot p^1 + 0\cdot p^2)$$

$$= (cr^0 + sr^1)(-sp^0 + cp^1)$$

$$t'^{ab} = R^a{}_c \, R^b{}_d \, t^{cd}$$

$$= R^a{}_c \, t^{cd} \, R^b{}_d$$

$$= R^a{}_c \, t^{cd} \, (R^T)_d{}^b$$

$$t' = R t R^T$$

2nd index
↳ columns

$$t^{abc} = P^a \otimes E^b \otimes L^c$$

$$t'^{abc} = R^a{}_m R^b{}_n R^c{}_p \, t^{mnp}$$

→ Vector transform like coordinates

→ A tensor transform like
tensor prod. of vectors.