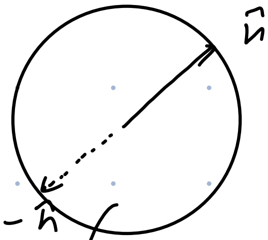
Lecture 5.

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
$= \alpha'|\hat{n}\rangle + \beta'|-\hat{n}\rangle$

$\hookrightarrow$ Figure out α':

$|\psi\rangle = \left[|+\hat{n}\rangle\langle+\hat{n}| + |-\hat{n}\rangle\langle-\hat{n}|\right]|\psi\rangle$

$= |+\hat{n}\rangle \underbrace{\langle+\hat{n}|\psi\rangle}_{\alpha'} + |-\hat{n}\rangle \underbrace{\langle-\hat{n}|\psi\rangle}_{\beta'}$



Measurement.
$\hookrightarrow$ rotate basis to the standard basis to measure.

Goal: Gate Operations

Since $\langle\psi|\psi\rangle = 1$

$\Rightarrow$ only operations are rotations in Hilbert space.

$|\psi'\rangle = U|\psi\rangle$
$\hookrightarrow$ unitary.

$\langle\psi'|\psi'\rangle = \langle\psi|U^\dagger U|\psi\rangle = 1$

must satisfy $\forall\psi$

$\Rightarrow U^\dagger U = I$.

Example: NOT gate, $\sigma^x = X$ "Pauli X-gate"

$X|0\rangle = |1\rangle$
$X|1\rangle = |0\rangle$

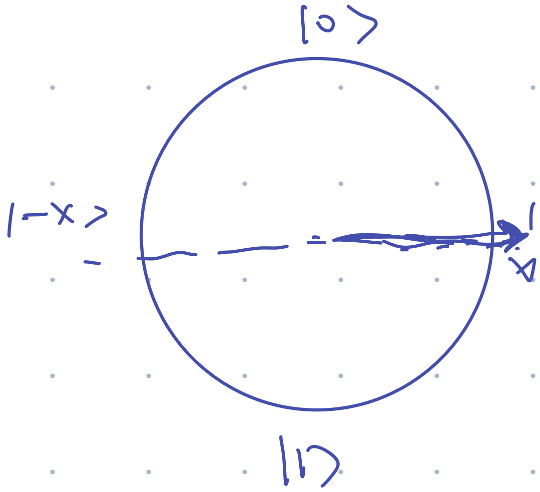
$X[\alpha|0\rangle + \beta|1\rangle] = \alpha|1\rangle + \beta|0\rangle$.

$X\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\beta\\ \alpha\end{pmatrix} \Rightarrow X = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

$X = X^\dagger$ Hermitian

$\underbrace{X^\dagger X = XX^\dagger = I}_{\text{unitary}} = X^2$

* preserves the length of vectors



is X a rotation by π about the x axis?

(then $|+x\rangle$ would not move)

Consider:

$$X|+\rangle = X\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|0\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$= \underbrace{(+1)}_{e^{i\varphi},\ \varphi=0}|+\rangle.$$

$$X|-\rangle = X\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$= \underbrace{(-1)}_{e^{i\varphi},\ \varphi=\pi.}|-\rangle$$

that's just a global phase.

Dirac Notations for linear operators

$|\psi\rangle$, $|\Phi\rangle$, inner product $\langle\Phi|\psi\rangle$.

$$|\psi\rangle = \begin{pmatrix}\psi_0 \\ \psi_1\end{pmatrix} \quad |\Phi\rangle = \begin{pmatrix}\Phi_0 \\ \Phi_1\end{pmatrix}$$

$$\langle\Phi|\psi\rangle = \begin{pmatrix}\Phi_0^* & \Phi_1^*\end{pmatrix}\begin{pmatrix}\psi_0 \\ \psi_1\end{pmatrix}$$

$$= \Phi_0^*\psi_0 + \Phi_1^*\psi_1$$

Outer Product:

$|\psi\rangle\langle\Phi|$ what kind of object is this?

$$(|\psi\rangle\langle\Phi|)|s\rangle = \underbrace{|\psi\rangle(\underbrace{\langle\Phi|s\rangle}_{\text{number}})}_{\text{vector}}$$

arbitrary state $|s\rangle$

Example: $Q = (|0\rangle\langle 1| + |1\rangle\langle 0|)$ → this is like a state filter, or a state swapper.

$Q|0\rangle = |0\rangle\langle 1|0\rangle + |1\rangle\langle 0|0\rangle$

$= |1\rangle$

$Q|1\rangle = |0\rangle\langle 1|1\rangle + |1\rangle\cancel{\langle 0|1\rangle} = |0\rangle$

$$\boxed{Q = X}$$

The matrix representation:

$Q = |0\rangle\langle 1| + |1\rangle\langle 0|$

$= \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$X^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$

$X^2 = [|0\rangle\langle 1| + |1\rangle\langle 0|][|0\rangle\langle 1| + |1\rangle\langle 0|]$

$= [|0\rangle\langle 0| + |1\rangle\langle 1|] = I.$

$\hookrightarrow$ the completeness relation

projector on the state $|0\rangle$

projector on the state $|1\rangle$.

projector onto the entire Hilbert space

Resolution of the Identity: $I = |+\hat{n}\rangle\langle+\hat{n}| + |-\hat{n}\rangle\langle-\hat{n}|$

General single-qubit gate operation

$U = |+\hat{n}\rangle\langle 0| + |-\hat{n}\rangle\langle 1| \dashrightarrow U = e^{i\lambda}|+\hat{n}\rangle\langle 0| + e^{i\chi}|-\hat{n}\rangle\langle 1|$

"takes state $|0\rangle$ of $|\psi\rangle$ $\rightarrow$ transforms to $|+\hat{n}\rangle$: $U|0\rangle = |+\hat{n}\rangle$

takes state $|1\rangle$ of $|\psi\rangle$ $\rightarrow$ Transforms to $|-\hat{n}\rangle$: $U|1\rangle = |-\hat{n}\rangle$

Lemma: $U^\dagger U = I$. $\quad (AB)^\dagger = B^\dagger A^\dagger$

Proof: $U^\dagger = e^{-i\lambda}|0\rangle\langle+\hat{n}| + e^{-i\chi}|1\rangle\langle-\hat{n}|$

$$U^\dagger U = \left[e^{-i\lambda}|0\rangle\langle+\hat{n}| + e^{-i\chi}|1\rangle\langle-\hat{n}|\right]\left[e^{i\lambda}|+\hat{n}\rangle\langle 0| + e^{i\chi}|-\hat{n}\rangle\langle 1|\right]$$

$$= |0\rangle\langle 0| + |1\rangle\langle 1| = I$$

Similarly $UU^\dagger = I \Rightarrow$ unitary

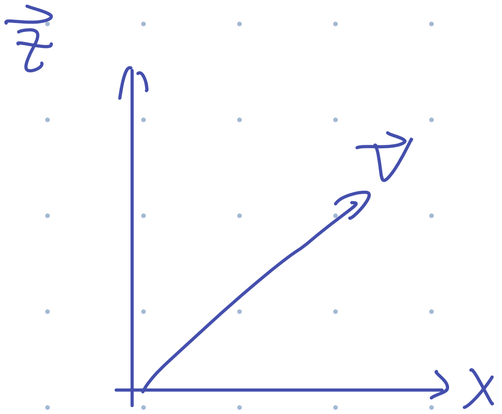
Projector: P is a linear operator obeying $P^2 = P$.

$$P_0 \equiv |0\rangle\langle 0| \qquad P_0^2 = |0\rangle\langle 0|0\rangle\langle 0| = |0\rangle\langle 0|$$

$$P_1 \equiv |1\rangle\langle 1| \qquad P_1^2 = P_1$$

$$P_0 + P_1 = I$$

→ visual representation



$$\vec{v} = (x, z)$$

$$P_z \vec{V} = (0, z)$$

so $P_z^2 \vec{V} = (0, z)$

$$P_x \vec{v} + P_z \vec{v} = \vec{V}$$

$P_0 = |0\rangle\langle 0|$ has representation $\begin{pmatrix}1\\0\end{pmatrix}(1 \ 0)^* = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$

$$P_0 \begin{pmatrix}\alpha\\\beta\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \begin{pmatrix}\alpha\\0\end{pmatrix}$$

$$P_1 = |1\rangle\langle 1| = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

$$P_1 \begin{pmatrix}\alpha\\\beta\end{pmatrix} = \begin{pmatrix}0\\\beta\end{pmatrix}$$

$$(P_0 + P_1)\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$$

Projectors and the Born rule

$$|\psi\rangle = \alpha'|+\hat{n}\rangle + \beta'|-\hat{n}\rangle$$

$$P_{+\hat{n}} = |\alpha'|^2 , \quad P_{-\hat{n}} = |\beta'|^2$$

$$\alpha' = \langle +\hat{n}|\psi\rangle , \quad \beta' = \langle -\hat{n}|\psi\rangle$$

$$|\alpha'|^2 = |\langle +\hat{n}|\psi\rangle|^2 = (\langle\psi|+\hat{n}\rangle)(\langle +\hat{n}|\psi\rangle)$$

$$= \langle\psi|\underbrace{+\hat{n}\rangle\langle +\hat{n}}_{P_{+\hat{n}}}|\psi\rangle .$$

$P_{+\hat{n}} \Rightarrow$ the projector operator

$\langle\psi|O|\psi\rangle$ "the expectation value of operation O in state $|\psi\rangle$"

Example: $P_0 = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} , \quad \langle\psi|P_0|\psi\rangle = (\alpha^* \ \beta^*)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= (\alpha^* \ \beta^*)\begin{pmatrix} \alpha \\ 0 \end{pmatrix}$$

$$= |\alpha|^2$$