Chapter 3: Intro to Hilbert Space.

What is Hilbert Space?

↳ Space for allowed states for a qubit

↳ 2D vector space over a field of complex numbers.

Qubit State

↳ Can be written as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

α, β : quantum amplitudes.

$|0\rangle, |1\rangle$: basis states

Inner product / Overlap

↳ $\langle\psi'|\psi\rangle = \alpha'^* \alpha + \beta'^* \beta$

↳ Rules: $\langle\psi'|\psi\rangle = \alpha'^* \alpha + \beta'^* \beta$ → implies

$$= (\alpha' \alpha^* + \beta' \beta^*)^*$$

$$= (\langle\psi|\psi'\rangle)^*$$

$\underbrace{\langle\psi|\psi\rangle}_{\text{must be real}} = |\alpha|^2 + |\beta^2| = 1$ ← Born's law.

↳ $\langle 0|1\rangle = \langle\uparrow|\downarrow\rangle = 0$ ⇒ these basis states are orthogonal.

even though they are colinear on the Bloch sphere.

why?

$$\left.\begin{array}{l} |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \langle 0| = (1, 0). \end{array}\right\} \Rightarrow \langle 0|1\rangle = (1\ 0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \quad \text{D.}$$

3.1 linear Operators on Hilbert Space

What is a linear operator?

- Operator: maps from Hilbert space back to the same Hilbert space.
- Represented by matrices.
  - Hermitian matrices specifically.
- They can express all observable outcomes.
- Example: $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow$ "The z-comp. of the polarization vec. on Bloch sph. could turn out to be +1 or -1"

$$\sigma^z|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (+1)|0\rangle$$

$$\sigma^z|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = (-1)|1\rangle$$

$\Rightarrow$ $|0\rangle$ is an eigenstate of σ^z with eigenvalue +1
$|1\rangle$ is an eigenstate of σ^z with eigenvalue -1

$\downarrow$
all possible measurement result.

* $\sigma^z|\psi\rangle$ does not represent the state of the system after the measurement of σ^z.

Instead, the state always collapses to one of the eigenstates of the operation being measured!

The measurement result is always one of the eigenvalues.

example: $\alpha|0\rangle + \beta|1\rangle \xrightarrow{\text{measurement}} |0\rangle$.

The system has to collapse into one of its eigenstates such that when you apply measurement again, you always get the same resulr.

↳ Another justification is written in the following pages (the variance part).

The average measurement result of an observable is the expectation value of that observable (operator) in the quantum state being measured.

$$\langle\psi|\sigma^z|\psi\rangle = \underbrace{(\alpha^*\langle 0| + \beta^*\langle 1|)}_{\langle\psi}\underbrace{(\alpha|0\rangle - \beta|1\rangle)}_{\sigma^z|\psi\rangle}$$

$$= |\alpha|^2 - |\beta|^2 .$$

We determine how random the measurement will be by calculating the variance.

$$Var(\xi) = \overline{\xi^2} - \overline{\xi}^2$$

$$Var(Q) = \langle\psi|Q^2|\psi\rangle - (\langle\psi|Q|\psi\rangle)^2$$

We can only be certain about the outcome of measurement when we know that $|\psi\rangle$ is an eigenstate of Q.

if $|\psi\rangle$ is an eigenstate of Q

$$\Rightarrow Q|\psi\rangle = v_m|\psi\rangle .$$

$$Var(Q) = \langle\psi|v_m^2|\psi\rangle - (\langle\psi|v_m|\psi\rangle)^2$$

$= v_m^2 \langle \psi|\psi \rangle - (v_m \langle \psi|\psi \rangle)^2$

$= v_m^2 - v_m^2 = 0$

$\Rightarrow$ no variance $\Rightarrow$ result is certain.

↳ Hence measurement must collapse $|\psi\rangle$ into an eigenstate of Q.

Physical observables are always represented by Hermitian matrices because

① Their eigenvalues are always real (aligns with experiment: measured values of physical observables are always real)

② Their eigenvalues form a basis set that spans the Hilbert space (also aligns with experiment, we can observe $|0\rangle$ & $|1\rangle$, they form a basis for the 2D Hilbert space).

$H^\dagger$ means transpose + complex conjugation.

Demonstration that Hermitian matrices have real eigenvalues & their eigenvectors are ortho.

Consider j^{th} & k^{th} eigvec.

$H|\psi_j\rangle = \lambda_j|\psi_j\rangle$

$H|\psi_k\rangle = \lambda_k|\psi_k\rangle$ take adjoint of 2nd equation

$\langle\psi_k|H^\dagger = \lambda_k^*\langle\psi_k|$

Assuming $H^\dagger = H$ (Hermitian).

$\langle\psi_k|\underbrace{H|\psi_j\rangle} = \underbrace{\lambda_j}\langle\psi_k\underbrace{|\psi_j\rangle}$

$\langle\psi_k|H|\psi_j\rangle = \lambda_k^*\langle\psi_k|\psi_j\rangle$

Subtracting the two yields:

$$0 = (\lambda_j - \lambda_k^*)\langle\psi_k|\psi_j\rangle$$

Case 1: $j = k$: $\langle\psi_k|\psi_j\rangle = 1$

$\Rightarrow (\lambda_j - \lambda_j^*)\underbrace{\langle\psi_j|\psi_j\rangle}_{=1} = 0$

$\lambda_j = \alpha + bi$, $\lambda_j^* = \alpha - bi$

$\lambda_j - \lambda_j^* = \alpha + bi - \alpha + bi = 2bi = 0 \Rightarrow$

Hence $b=0$, $\lambda_j = \lambda_j^*$ are REAL!

Case 2: $j \neq k$

if the eigvals are not degenerate $(\lambda_j \neq \lambda_k)$,
then $0 = (\lambda_j - \lambda_k^*) \langle \psi_k | \psi_j \rangle$ requires
$\langle \psi_k | \psi_j \rangle = 0 \Rightarrow$ eig vec are orthogonal.

Thus if the full spectrum is non-degenerate,
the set of eigenvectors is orthonormal. → every unit vector has length = 1 $\langle \psi_j | \psi_j \rangle = 1$.

$$\langle \psi_k | \psi_j \rangle = \delta_{kj} \quad \begin{pmatrix} = 0 & k \neq j \\ = 1 & k = j \end{pmatrix}$$

3.2 Dirac Notation For Operators

- Inner product: $\langle \Phi | \psi \rangle$: a number
- Outer product: $G = |\psi\rangle\langle \Phi|$: an operator

↳ why?

$$G|\chi\rangle = (|\psi\rangle\langle\Phi|)|\chi\rangle = (|\psi\rangle)(\langle\Phi|\chi\rangle)$$
$$= \underbrace{g}_{\text{inner product, a complex \#}} \underbrace{|\psi\rangle}_{\text{a vector.}}$$

↳ It's linear because

$$G(\alpha|\chi\rangle + \beta|\phi\rangle) = \alpha G|\chi\rangle + \beta G|\phi\rangle.$$

- In Dirac's notation,
  Hermitian operators take the form:

$$V = \sum_{j=1}^{M} v_j |\psi_j\rangle\langle\psi_j|$$

eigenvalue v_j. the j^{th} eigenvector of V.

example: $G = |0\rangle\langle 0| - |1\rangle\langle 1|$

↳ $G|0\rangle = (+1)|0\rangle$

↳ $G|1\rangle = (-1)|1\rangle$

verification (exercise 3.3).

$$V|\psi_m\rangle = \left(v_m|\psi_m\rangle\langle\psi_m|\right)|\psi_m\rangle$$

$$= v_m|\psi_m\rangle \cdot 1$$

$$= v_m|\psi_m\rangle \quad W$$

exercise 3.4.

show that

$$\sigma^x = \begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

in z-basis.

$$|+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \underline{|-z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}} \quad *$$

$$|+x\rangle = \frac{1}{\sqrt{2}}[|0\rangle + |1\rangle] = \frac{1}{\sqrt{2}}(|+z\rangle + |-z\rangle)$$

$$|-x\rangle = \frac{1}{\sqrt{2}}[|0\rangle - |1\rangle] = \frac{1}{\sqrt{2}}(|+z\rangle - |-z\rangle).$$

$$|+y\rangle = \frac{1}{\sqrt{2}}[|0\rangle + i|1\rangle] = \frac{1}{\sqrt{2}}[|+z\rangle + i|-z\rangle]$$

$$|-y\rangle = \frac{1}{\sqrt{2}}[|0\rangle - i|1\rangle] = \frac{1}{\sqrt{2}}[|+z\rangle - i|-z\rangle].$$

$$|+x\rangle\langle +x| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$|-x\rangle\langle -x| = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}$$

$$\sigma^x = (+1)\,|+x\rangle\langle +x| + (-1)\,|-x\rangle\langle -x|$$

$$= \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}0 & 2\\2 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$$

$$|+y\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + i|1\rangle\right] = \frac{1}{\sqrt{2}}\left[|+z\rangle + i|-z\rangle\right]$$

$$|-y\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle - i|1\rangle\right] = \frac{1}{\sqrt{2}}\left[|+z\rangle - i|-z\rangle\right].$$

$$|+y\rangle\langle +y| = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}$$

this is conjugate transpose

$$= \frac{1}{2}\begin{pmatrix}1 & -i\\i & +1\end{pmatrix}$$

$$|-y\rangle\langle -y| = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & +i\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1 & +i\\-i & +1\end{pmatrix}$$

$$\sigma^y = (+1)|+y\rangle\langle +y| + (-1)|-y\rangle\langle -y|$$

$$= \frac{1}{2}\left[\begin{pmatrix} 1 & -i \\ i & +1 \end{pmatrix} - \begin{pmatrix} 1 & +i \\ -i & +1 \end{pmatrix}\right]$$

$$= \frac{1}{2}\left[\begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

3.3 Orthonormal bases for qubit states.

↳ Using standard state parametrization in

$$\begin{cases} |+\hat{s}\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle \\ |-\hat{s}\rangle = \sin\frac{\theta}{2}|0\rangle - e^{i\varphi}\cos\frac{\theta}{2}|1\rangle \end{cases}$$

$$\hat{s} = (x, y, z) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

we can show that any pair of oppositely directed unit vectors $\pm\hat{n}$ on the Bloch sphere are orthogonal.

$$\langle -\hat{n}|+\hat{n}\rangle = \left(\sin\frac{\theta}{2}, \cancel{\mp} -e^{-i\varphi}\cos\frac{\theta}{2}\right)\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}$$

$$= \sin\frac{\theta}{2}\cos\frac{\theta}{2} \cancel{\mp} - \cancel{e^{-i\varphi+i\varphi}}\cos\frac{\theta}{2}\sin\frac{\theta}{2} = 0.$$

⇒ Since Hilbert space are 2D

↳ $|-\hat{n}\rangle$ and $|+\hat{n}\rangle$ are orthogonal.

⇒ they constitute a complete basis for expressing any vector in the Hilbert space.

Hence we can show the completeness relation: $\hat{I} = |+\hat{n}\rangle\langle+\hat{n}| + |-\hat{n}\rangle\langle-\hat{n}|$

$$|+\hat{n}\rangle\langle+\hat{n}| = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\frac{\theta}{2} & e^{i\varphi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix}$$

$$|-\hat{n}\rangle\langle-\hat{n}| = \begin{pmatrix} \sin\frac{\theta}{2} \\ -e^{i\varphi}\cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \sin\frac{\theta}{2} & -e^{-i\varphi}\cos\frac{\theta}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \sin^2\frac{\theta}{2} & -e^{i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ -e^{i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \cos^2\frac{\theta}{2} \end{pmatrix}$$

$$|+\hat{n}\rangle\langle+\hat{n}| + |-\hat{n}\rangle\langle-\hat{n}| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The completeness relation is a powerful tool that allows us to express the state $|\psi\rangle$ in any basis.

$$|\psi\rangle = I|\psi\rangle = \left(|+\vec{n}\rangle\langle+\vec{n}| + |-\vec{n}\rangle\langle-\vec{n}|\right)|\psi\rangle$$
$$= |+\vec{n}\rangle\underbrace{\left(\langle+\vec{n}|\psi\rangle\right)}_{\text{coefficient}} + |-\vec{n}\rangle\underbrace{\left(\langle-\vec{n}|\psi\rangle\right)}_{\text{coefficient}}.$$

$$= \alpha|+\vec{n}\rangle + \beta|-\vec{n}\rangle.$$

This is similar to projections

$$\vec{r} = (r_x, r_y) = \hat{x}(\hat{x}\cdot\vec{r}) + \hat{y}(\hat{y}\cdot\vec{r})$$
$$= \hat{i}(\hat{i}\cdot\vec{r}) + \hat{j}(\hat{j}\cdot\vec{r})$$

Hence we can think of the identity transformation as:

$$\hat{I} = \hat{x}\hat{x} + \hat{y}\hat{y} = \hat{i}\hat{i} + \hat{j}\hat{j}.$$

↳ We can think of $P_x = \hat{x}\hat{x}$ as a projection operator because:

$$P_x\vec{V} = (\hat{x}\hat{x})\vec{V} = \hat{x}(\hat{x}\cdot\vec{V})$$

↳ Notice that $(P_x)^2 = P_x$ (characteristic of projection operators) (once a vector is projected onto the x-axis, further projections don't do much.)

$$P_{xy} = \hat{x}\hat{x} + \hat{y}\hat{y} = I - \hat{z}\hat{z}.$$

↳ The analogous projector onto $|\hat{n}\rangle$ in the Hilbert space describing the polarization of a qubit is simply

$$P_{\hat{n}} = |\hat{n}\rangle\langle\hat{n}|$$

Example : Measurement not in the standard basis:

if we are given $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ in the standard basis, and are asked to measure a Hermitian operator

$$M = M_+ |\hat{n}\rangle\langle\hat{n}| + M_- |-\hat{n}\rangle\langle-\hat{n}|$$

in some nonstandard basis,

we want to express $|\psi\rangle$ in the eigenstates of M.

$$|\psi\rangle = \alpha' |+\hat{n}\rangle + \beta' |-\hat{n}\rangle.$$

where $\begin{cases} M_+ \text{ occurs with } P_+ = |\alpha'|^2 \\ M_- \text{ occurs with } P_- = |\beta'|^2. \end{cases}$

we can perform change of basis:

$$|\psi\rangle = \left[|+\hat{n}\rangle\langle+\hat{n}| + |-\hat{n}\rangle\langle-\hat{n}| \right] |\psi\rangle$$

$$= |+\hat{n}\rangle (\langle+\hat{n}|\psi\rangle) + |-\hat{n}\rangle (\langle-\hat{n}|\psi\rangle)$$

$$= \alpha' |+\hat{n}\rangle + \beta' |-\hat{n}\rangle$$

$$\begin{cases} \alpha' = \alpha\langle+\hat{n}|0\rangle + \beta\langle+\hat{n}|1\rangle \\ \beta' = \alpha\langle-\hat{n}|0\rangle + \beta\langle-\hat{n}|1\rangle \end{cases}$$

Exercise 3.7. show that $|\pm\hat{n}\rangle$ is an eigenvector of

$$T_{\hat{n}} = \hat{n}\cdot\vec{\sigma} = (\hat{n}\cdot\hat{x})\sigma^x + (\hat{n}\cdot\hat{y})\sigma^y + (\hat{n}\cdot\hat{z})\sigma^z$$

with eigenvalue ± 1.

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$(\hat{n}\cdot\hat{x}) = n_x, \quad (\hat{n}\cdot\hat{y}) = n_y, \quad (\hat{n}\cdot\hat{z}) = n_z.$$

$$T_{\hat{n}} = \begin{pmatrix} 0 & n_x \\ n_x & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i\cdot n_y \\ i n_y & 0 \end{pmatrix} + \begin{pmatrix} n_z & 0 \\ 0 & -n_z \end{pmatrix}$$

$$= \begin{pmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{pmatrix}$$

$$\det\begin{bmatrix} n_z - \lambda & n_x - i n_y \\ n_x + i n_y & -n_z - \lambda \end{bmatrix} = 0$$

$$-(n_z - \lambda)(n_z + \lambda) - (n_x - i n_y)(n_x + i n_y) = 0$$

$$-(n_z^2 - \lambda^2) - (n_x^2 - i^2 n_y^2) = 0$$

$$\lambda^2 - n_z^2 - n_x^2 - n_y^2 = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda = \pm 1.$$

Hence $T_{\hat{n}}|+\hat{n}\rangle = (+1)|+\hat{n}\rangle$

$$T_{\hat{n}}|-\hat{n}\rangle = (-1)|-\hat{n}\rangle$$

3.3.1 Gauge Invariance

- The states $|\pm\hat{n}\rangle$ in the Hilbert space corresponding to the Bloch sphere unit vectors $\pm\hat{n}$ are not unique.
- Each basis vector can be multiplied by a arbitrary phase factor $|V_+\rangle = e^{i\xi_+}|+\hat{n}\rangle$ and $|V_-\rangle = e^{i\xi_-}|-\hat{n}\rangle$ and we would still have an orthonormal basis obeying.

$$\langle V_+|V_+\rangle = 1 \;\; \&$$
$$\langle V_-|V_+\rangle = 0 \;\; \&$$
$$\hat{I} = |V_+\rangle\langle V_+| + |V_-\rangle\langle V_-|$$

- The choice of phase factors is called the "gauge choice."
- There is another operator (besides the identity) that is independent of the gauge choice.

↳ The diagonal operator that is measured by an apparatus that detects the $\hat{n}$ component of the qubit polarization vector is invariant. ???

$$\hat{n}\cdot\sigma = |V_+\rangle\langle V_+| - |V_-\rangle\langle V_-|$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \cancel{e^{i\xi_+}}|+\hat{n}\rangle \cancel{e^{-i\xi_+}}\langle +\hat{n}| - \cancel{e^{i\xi_-}}|-\hat{n}\rangle \cancel{e^{-i\xi_-}}\langle -\hat{n}|$$

$$= |n_+\rangle\langle n_+| - |n_-\rangle\langle n_-|.$$

the phase cancels out.

However, non-diagonal operators such as the spin flip operator are gauge dependent when expressed in the $|\pm\hat{n}\rangle$ bases.

$$\hat{Q} = |V_+\rangle\langle V_-| + |V_-\rangle\langle V_+|$$

$$= e^{i\xi_+}|+\hat{n}\rangle e^{-i\xi_-}\langle -\hat{n}| + e^{i\xi_-}|-\hat{n}\rangle e^{-i\xi_+}\langle +\hat{n}|$$

$$= e^{i(\xi_+-\xi_-)}|+\hat{n}\rangle\langle -\hat{n}| + e^{-i(\xi_+-\xi_-)}|-\hat{n}\rangle\langle +\hat{n}|.$$

The state vectors also become gauge dependent.

$$|\psi\rangle = \alpha|V_+\rangle + \beta|V_-\rangle$$

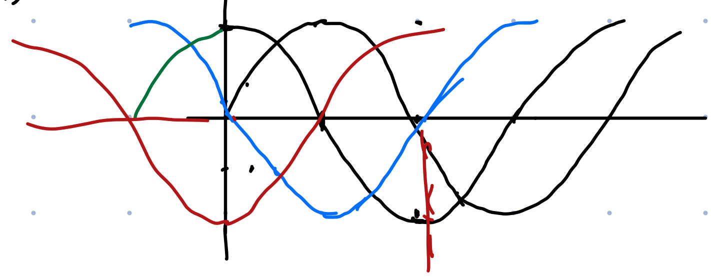
$$= \alpha e^{i\xi_+}|+\hat{n}\rangle + \beta e^{i\xi_-}|-\hat{n}\rangle.$$

so nothing changes in the physics. ??

Apparently fixed definition for $|+\hat{n}\rangle$ is ambiguous when it comes to writing down the other basis vector $|-\hat{n}\rangle$.

If $\hat{n} \equiv (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$,

we can arrive at the other basis vector $-\hat{n}$ via 2 different routes:



i) $\theta \to \theta + \pi, \ \varphi \to \varphi.$

$$|+\hat{n}\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle.$$

$$|+\hat{n}\rangle \longrightarrow \cos\left(\frac{\theta+\pi}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta+\pi}{2}\right)|1\rangle$$

$$= -\sin\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\cos\left(\frac{\theta}{2}\right)|1\rangle \equiv |-\hat{n}\rangle$$

$$= -\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

ii) $\theta \to \pi - \theta, \ \varphi \to \varphi + \pi.$

$$|+\hat{n}\rangle \longrightarrow \cos\left(\frac{\pi-\theta}{2}\right)|0\rangle + e^{i(\varphi+\pi)}\sin\left(\frac{\pi-\theta}{2}\right)|1\rangle$$

$$= \sin\frac{\theta}{2}|0\rangle - e^{-i\varphi}\cos\frac{\theta}{2}|1\rangle$$

$$= \frac{1}{\sqrt{2}}\left[|0\rangle - |1\rangle\right]$$

Method (ii) self consistently keeps θ in the range $0 \leq \theta \leq \pi$ & is the standard gauge choice.

3.4 Rotations in Hilbert Space

- Because every state in the single or multi-qubit Hilbert space has the same length $\langle\psi|\psi\rangle = 1$, rotations are the only operations we need to move throughout the entire Hilbert space.

$$|\Omega\rangle = \frac{1}{\sqrt{W}}\left(|\psi\rangle + |\phi\rangle\right)$$

$$\sqrt{W} = \sqrt{\left||\psi\rangle + |\phi\rangle\right|^2} = \sqrt{\left(\langle\psi| + \langle\phi|\right)\left(|\psi\rangle + |\phi\rangle\right)} = \sqrt{\langle\psi|\psi\rangle + \langle\psi|\phi\rangle + \langle\phi|\psi\rangle + \langle\phi|\phi\rangle}$$

$\hookrightarrow W = 2 + \langle\psi|\phi\rangle + \langle\phi|\psi\rangle$

$\hookrightarrow$ can be obtained from the intial state $|\psi\rangle$ via a rotation.

- Rotations in Hilbert space are executed via unitary operations.

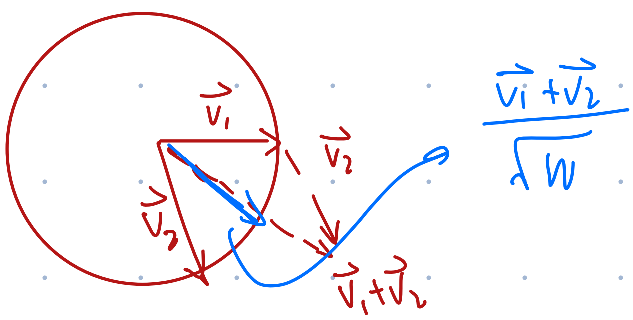
$$U^\dagger U = I \quad \xrightarrow{\text{or}} \quad U^{-1} = U^\dagger$$

$$U U^\dagger = I$$

- unitary transformations preserve the inner products between vectors in Hilbert space, just as orthogonal rotation matrices preserve the angles (dot products) between ordinary vectors.

$$(Ru)\cdot(Rv) = (Ru)^T(Rv) = u^T R^T R v = u^T v = u \cdot v$$

$$(Ux)\cdot(Uy) = x^T U^T U y = x^T U^{-1} U y = x^T y = x \cdot y$$



· If a linear operation U preserves the length of every vector in the Hilbert space, it has to be unitary $(U^\dagger U = I)$.

ψ_i

$$|\psi\rangle = \sum_{j=1}^{N} \psi_j |j\rangle$$

$$\langle\psi|\psi\rangle = \sum |\psi_j|^2 = 1$$

$$|\psi'\rangle = U|\psi\rangle = \sum_{j=1}^{N} \psi_j U|j\rangle$$

$$\langle\psi'|\psi'\rangle = \langle\psi'|U^\dagger U|\psi'\rangle$$

$$= \sum_{j,k=1}^{N} \psi_k^* \psi_j \langle k|U^\dagger U|j\rangle$$

this is 1 / this is 1

$$= \sum_{j=1}^{N} |\psi_j|^2 \langle j|U^\dagger U|j\rangle + \sum_{j\neq k}^{N} \psi_k^* \psi_j \langle k|U^\dagger U|j\rangle$$

Since U preserves the length of every vector,

$\langle\psi|U^\dagger U|\psi\rangle = \langle j|U^\dagger U|j\rangle = 1$ for every basis vector j. That means

$\sum_{j\neq k}^{N} \psi_k^* \psi_j \langle k|U^\dagger U|j\rangle = 0$, which means that

$\langle k|U^\dagger U|j\rangle = 0$ for $k \neq j$.

So $\langle k|U^\dagger U|j\rangle = \delta_{kj}$

That means $U^\dagger U = I$. $\Rightarrow$ U is unitary.

- Unitary systems also preserve the inner product between any pair of states.

$$\begin{cases} |\phi'\rangle = U|\phi\rangle \\ |\psi'\rangle = U|\psi\rangle \end{cases}$$

$$\langle\phi'|\psi'\rangle = \langle\phi|U^\dagger U|\psi\rangle = \langle\phi|\psi\rangle.$$

- To fully determine the state of a qubit, we have to measure all 3 components of the 'spin vector σ'.

↳ We can rotate the qubit around certain axes, then measure about the z-axis, to deduce the state.

↳ $|+X\rangle \rightarrow |+Z\rangle$ by a $\frac{\pi}{2}$ rotation about the y-axis.
$|-X\rangle \rightarrow |-Z\rangle$

↳ $|+Y\rangle \rightarrow |+Z\rangle$ by a $\frac{\pi}{2}$ rotation about the x-axis.
$|-Y\rangle \rightarrow |-Z\rangle$

if we only measure along z, we can deduce $|\alpha|^2$ & $|\beta|^2$, but not the relative phase between $|0\rangle$ & $|1\rangle$.

why ?

state $= \cos\frac{\theta}{2}\,\overset{|+z\rangle}{|0\rangle} + e^{i\varphi}\sin\frac{\theta}{2}\,\overset{|-z\rangle}{|1\rangle}$.

↳ measuring along z gives you

$$\langle\psi|\sigma^z|\psi\rangle = \left(\cos\frac{\theta}{2} \quad e^{-i\varphi}\sin\frac{\theta}{2}\right)\begin{pmatrix}\cos\frac{\theta}{2}\\ -e^{i\varphi}\sin\frac{\theta}{2}\end{pmatrix}$$

$$= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}.$$

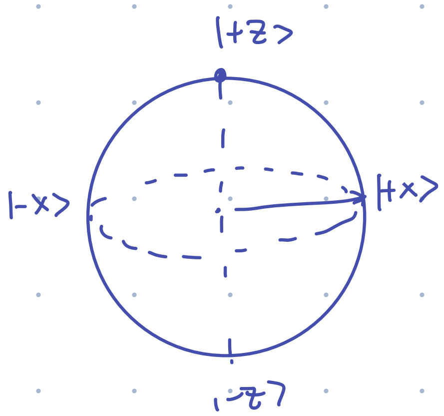
↳ measuring along x gives you.

Let's first transform the state to x.

$$|\psi\rangle = \Big[|+x\rangle\langle +x| + |-x\rangle\langle -x|\Big]\Big[\cos\frac{\theta}{2}|+z\rangle + e^{i\varphi}\sin\frac{\theta}{2}|-z\rangle\Big]$$

$$= \Big[\cos\frac{\theta}{2}\langle +x|+z\rangle\Big]|+x\rangle + \Big[e^{i\varphi}\sin\frac{\theta}{2}\langle +x|-z\rangle\Big]|+x\rangle$$

$$\Big[\cos\frac{\theta}{2}\langle -x|+z\rangle\Big]|-x\rangle + \Big[e^{i\varphi}\sin\frac{\theta}{2}\langle -x|-z\rangle\Big]|-x\rangle$$



$$= \Big[\cos\frac{\theta}{2}\left(\frac{1}{\sqrt{2}}\right) + e^{i\varphi}\sin\frac{\theta}{2}\left(\frac{1}{\sqrt{2}}\right)\Big]|+x\rangle$$

$$\Big[\cos\frac{\theta}{2}\left(\frac{1}{\sqrt{2}}\right) - e^{i\varphi}\sin\frac{\theta}{2}\left(\frac{1}{\sqrt{2}}\right)\Big]|-x\rangle$$

$$\langle\psi|\sigma^x|\psi\rangle = \frac{1}{\sqrt{2}}\Big[\cos\frac{\theta}{2} + e^{-i\varphi}\sin\frac{\theta}{2} \quad \cos\frac{\theta}{2} - e^{-i\varphi}\sin\frac{\theta}{2}\Big]\frac{1}{\sqrt{2}}\begin{bmatrix}\cos\frac{\theta}{2} + e^{i\varphi}\sin\frac{\theta}{2}\\ -\cos\frac{\theta}{2} + e^{i\varphi}\sin\frac{\theta}{2}\end{bmatrix}$$

$$= \frac{1}{2}\Big[\left(\cos\frac{\theta}{2} + e^{-i\varphi}\sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2} + e^{i\varphi}\sin\frac{\theta}{2}\right) -$$

$$\left(\cos\frac{\theta}{2} - e^{-i\varphi}\sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2} - e^{i\varphi}\sin\frac{\theta}{2}\right)\Big]$$

$$= \frac{1}{2}\Big[\Big(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} + \cos\frac{\theta}{2}\sin\frac{\theta}{2}\left[e^{-i\varphi} + e^{i\varphi}\right]\Big) -$$

$$\Big(\cos^2\frac{\theta}{2} - \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\varphi} - \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\varphi} + \sin^2\frac{\theta}{2}\Big)\Big]$$

$$= \frac{1}{2}\Big[\left(\cancel{1} + \cos\frac{\theta}{2}\sin\frac{\theta}{2}\left(e^{-i\varphi} + e^{i\varphi}\right)\right) - \left(\cancel{1} - \cos\frac{\theta}{2}\sin\frac{\theta}{2}\left[e^{i\varphi} + e^{-i\varphi}\right]\right)\Big]$$

$$|+x\rangle = \frac{1}{\sqrt{2}}\Big[|+z\rangle + |-z\rangle\Big]$$

$$|-x\rangle = \frac{1}{\sqrt{2}}\Big[|+z\rangle - |-z\rangle\Big]$$

$$\Rightarrow \langle +x| = \frac{1}{\sqrt{2}}\Big[\langle +z| + \langle -z|\Big]$$

$$= \frac{1}{\cancel{2}}\left[\cancel{2} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\left(e^{-i\varphi}+e^{i\varphi}\right)\right]$$

$$= \sin\frac{\theta}{2}\cos\frac{\theta}{2}\left(e^{-i\varphi}+e^{i\varphi}\right).$$

Now transform the state to Y ...

↳ skipped.

⇒ So the idea is, by measuring $\langle\sigma_x\rangle, \langle\sigma_y\rangle, \langle\sigma_z\rangle$, you can determine the object's state (θ, φ).

a review of rotations in ordinary space

Suppose I want to rotate a vector around the z-axis: $\theta \to \theta$, $\varphi \Rightarrow \varphi + \chi$

$$n' = \left[\sin\theta\cos(\varphi+\chi)\hat{x} + \sin\theta\sin(\varphi+\chi)\hat{y} + \cos\theta\hat{z}\right]$$

$$= \left[(\sin\theta\cos\varphi\cos\chi - \sin\theta\sin\varphi\sin\chi)\hat{x} + (\sin\theta\sin\varphi\cos\chi + \sin\theta\cos\varphi\sin\chi)\hat{y} + \cos\theta\hat{z}\right]$$

$$n' = R_z(\chi)\, n$$

$$= \begin{bmatrix} \cos\chi & -\sin\chi & 0 \\ \cos\chi & \sin\chi & 0 \\ 0 & 0 & 1 \end{bmatrix} n$$

↳ $R_z^{-1}(\chi) = R_z^T(\chi)$, & inner prod preserved: $\vec{r}_1' \cdot \vec{r}_2' = \vec{r}_1 \cdot \vec{r}_2$.

rotations in Hilbert Space

↳ Rotation around the z-axis:

$$U_z(\chi) = e^{-i\frac{\chi}{2}\sigma^z}$$

$$e^x = \sum \frac{1}{n!} x^n.$$

↳ you can interpret this as a power series expansion:

$$U_z(\chi) = \sum_{n=0}^{\infty} \frac{1}{n!}\left[-i\frac{\chi}{2}\right]^n \left[\sigma^z\right]^n.$$

↳ should converge provided $\frac{|\chi|}{2}|\sigma^z|$ lies within the radius of converge.

↳ $|\sigma^z|$ = absolute val of the largest eigval.

$= 1.$

↳ the exponential is a function

↳ it is analytic everywhere in the complex plane

⇒ has infinite radius of convergence

→ the series converge.

Can rewrite as: $\left((\sigma^z)^2 = \hat{I} \right)$

$$U_z(\chi) = e^{-i\frac{\chi}{2}\sigma^z} = \sum_{n=\text{even}} \frac{1}{n!}\left[-i\frac{\chi}{2}\right]^n \hat{I} + \sum_{n=\text{odd}} \frac{1}{n!}\left[-i\frac{\chi}{2}\right]^n \sigma^z$$

$$= \cos\left(\frac{\chi}{2}\right)\hat{I} - i\sin\left(\frac{\chi}{2}\right)\sigma^z$$

Exercise 3.8 Prove that

$$e^{-i\theta} = \cos\theta - i\sin\theta$$ without power series exp.

Hint:

Define $P(\theta) = \cos\theta - i\sin\theta$

$Q(\theta) = e^{-i\theta}$

& show:

$\frac{d}{d\theta} P(\theta) = -iP(\theta)$

$\frac{d}{d\theta} Q(\theta) = -iQ(\theta)$

they obey the same first-order diff. eq.

& $P(0) = Q(0) = 1$ have the same initial condition

$$\frac{d}{d\theta} P(\theta) = -\sin\theta - i\cos\theta$$

$$= -i(-i\sin\theta + \cos\theta) = -iP(\theta)$$

$$\frac{d}{d\theta} Q(\theta) = (-i)e^{-i\theta} = -i(e^{-i\theta}) = -iQ(\theta)$$

$P(0) = \cos(0) = 1$, $Q(0) = +e^{0} = 1$

$\Rightarrow P(\theta) = Q(\theta)$. ✓

Now, $U_z(\chi) = \cos\left(\frac{\chi}{2}\right)\hat{I} - i\sin\left(\frac{\chi}{2}\right)\sigma^z$

Using the fact that, $\begin{cases} \sigma^z|0\rangle = (+1)|0\rangle \\ \sigma^z|1\rangle = (-1)|1\rangle \end{cases}$

$$U_z(\chi)|0\rangle = \left[\cos\left(\frac{\chi}{2}\right)\hat{I} - i\sin\left(\frac{\chi}{2}\right)\sigma^z\right]|0\rangle$$

$$= \cos\left(\frac{\chi}{2}\right)\hat{I}|0\rangle - i\sin\left(\frac{\chi}{2}\right)|0\rangle$$

$$= \left(\cos\left(\frac{\chi}{2}\right)\cancel{\hat{I}} - i\sin\left(\frac{\chi}{2}\right)\right)|0\rangle$$

$$= e^{-i\frac{\chi}{2}}|0\rangle$$

$$U_z(\chi)|1\rangle = \left[\cos\left(\frac{\chi}{2}\right) + i\sin\left(\frac{\chi}{2}\right)\right]|1\rangle$$

$$= e^{i\frac{\chi}{2}}|1\rangle$$

$$U_z(\chi)|\hat{n}\rangle = U_z(\chi)\left[\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle\right]$$

$$= \left[\cos\left(\frac{\theta}{2}\right)e^{-i\frac{\chi}{2}}|0\rangle + \sin\frac{\theta}{2}e^{i\left(\varphi+\frac{\chi}{2}\right)}|1\rangle\right]$$

$$= e^{-i\frac{\chi}{2}}\left[\cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\frac{\theta}{2}e^{i(\varphi+\chi)}|1\rangle\right]$$

$$= e^{-i\frac{\chi}{2}}|R_z(\chi)\hat{n}\rangle$$

$$= e^{-i\frac{\chi}{2}}|n'\rangle$$

global phase.

Notice that $U_z(-\chi) = U_z^{-1}(\chi) = U_z^{\dagger}(\chi)$.

Hence $U_z(\chi)$ is unitary:

$$U_z^{\dagger} U_z = \hat{I}.$$

→ All rotations in Hilbert space are unitary.

<u>Exercise 3.9.</u> Prove that the Pauli matrices $\sigma^x, \sigma^y, \sigma^z$ are both Hermitian & unitary.

correspond to physical observables

corresponds to rotation operations that can be applied to quantum states.

Hermitian: $H^{\dagger} = H$.

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (\sigma^x)^{\dagger} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = (\sigma^y)^{\dagger} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = (\sigma^z)^{\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$(\sigma^x)^{\dagger} \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(\sigma^y)^{\dagger} \sigma^y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(\sigma^z)^{\dagger} \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Exercise 3.10.

a) Prove that every operator that is both Hermitian and unitary squares to the identity. Using this, prove that if such operators are traceless, their spectrum contains only +1 and -1 and has equal numbers of each.

Hermitian: $U^\dagger = U$ $\Rightarrow U^\dagger U = U^2 = I$.

Unitary: $U^\dagger U = I$.

If they are traceless, consider the 2D scenario.

$$U = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix} \Rightarrow U^2 = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix} \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix} = \begin{pmatrix} ab & 0 \\ 0 & ab \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\det \begin{pmatrix} \lambda & b \\ a & \lambda \end{pmatrix} = 0.$$

$$\lambda^2 - ab = 0.$$

$$\lambda^2 - 1 = 0$$

$$\lambda^2 = 1$$

$$\lambda = \pm 1$$

λ, the spectrum could only be ± 1.

there are 1 eigval $= 1$ & 1 eigval $= -1$ because 2 eigvecs. have to span the 2D Hilbert space.

b) Conversely, prove: every operator that both squares to the identity is Hermitian, is necessarily unitary:

Her: $U^\dagger = U$

$U^2 = I \Leftrightarrow \underbrace{(U^\dagger)U = U(U^\dagger)}_{\text{unitary.}} = I.$

Exercise 3.11 Every $N \times N$ unitary matrix can be written in the form: $U = e^{i\theta \hat{M}}$

where θ is a real parameter & $\hat{M}$ is an $N \times N$ Hermitian matrix.

↳ in sure M can be neg N?

(a) Using the fact that the eigvectors of an Hermitian matrix form a complete basis, find the spectrum (set of eigvals) and eigvectors of U in terms of the eigvalues and eigvectors in terms of $\hat{M}$.

↳ this still feels not internalized.

Making sense of exponentiating a matrix:

e^A where $A = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

$$A = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} -\theta^2 & 0 \\ 0 & -\theta^2 \end{pmatrix} = -\theta^2 I$$

$$A^3 = -\theta^2 A, \quad A^4 = \theta^4 I, \quad A^5 = \theta^4 A$$

$$\dots$$

$\begin{cases} \text{even powers} \rightarrow \text{scalar multiples of } I \\ \text{odd powers} \rightarrow \text{scaler multiples of } A \end{cases}$

$$e^A = \left(I - \frac{\theta^2}{2!} I + \frac{\theta^4}{4!} I - \dots\right) + \left(A - \frac{\theta^2}{3!} A + \frac{\theta^4}{5!} A \dots\right)$$

$$= \left(\sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!}\right) I + \left(\sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k+1)!}\right) A$$

$$= \cos\theta\, I + \frac{\sin\theta}{\theta} A$$

$$\frac{\sin\theta}{\theta} A = \frac{\sin\theta}{\theta} \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sin\theta \\ \sin\theta & 0 \end{pmatrix}$$

$$e^A = \begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{pmatrix} + \begin{pmatrix} 0 & -\sin\theta \\ \sin\theta & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

↳ "exponentiating the generator turns infinitesimal rotation into actual rotation"

↳ a generator : tells you how something changes at an instant

↳ $R(d\theta) = \begin{pmatrix} \cos d\theta & -\sin d\theta \\ \sin d\theta & \cos d\theta \end{pmatrix}$

$\cos d\theta \approx 1, \quad \sin d\theta \approx d\theta$

$$R d\theta = \begin{pmatrix} 1 & -d\theta \\ d\theta & 1 \end{pmatrix} = 1 + d\theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\underbrace{J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\text{the generator of rotations}}$$

↳ From infinitesimal to finite:

$$\left(1 + \frac{\theta}{n} J\right)^n$$

$$\lim_{n \to \infty} \left(1 + \frac{\theta}{n} J\right)^n = e^{\theta J}$$

a) Power expansion of

$$U = e^{i\theta\hat{m}}$$

$$= \sum_{n=1}^{\infty} \frac{[i\theta\hat{m}]^n}{n!}$$

Let's say eigenvectors of $\hat{m}$ are V_1, V_2,

with eigenvalues λ_1, λ_2. Does $M^n v_1 = \lambda_1 v_1$.

$$\hat{M} v_1 = \lambda_1 v_1, \quad \hat{M} v_2 = \lambda_2 v_2.$$

$$U v_1 = \sum_{n=1}^{\infty} \frac{[i\theta]^n}{n!} M^n v_1 = \lambda_1^n \sum_{n=1}^{\infty} \frac{[i\theta]^n}{n!} v_1 = e^{i\theta\lambda_1} v_1$$

$$U v_2 = \sum_{n=1}^{\infty} \frac{[i\theta]^n}{n!} M^n v_2 = \lambda_2^n \sum_{n=1}^{\infty} \frac{[i\theta]^n}{n!} v_2 = e^{i\theta\lambda_2} v_2$$

→ same eigenvectors.

$$\Rightarrow \lambda_1' = e^{i\theta\lambda_1} ; \lambda_2' = e^{i\theta\lambda_2}$$

b) Using this result, prove that U is unitary.

$$UU^\dagger = I.$$

$$U^\dagger = (e^{igM})^\dagger = e^{-igM^\dagger}$$

Since M is hermitian, $M^\dagger = M$

$$UU^\dagger = e^{-igM^\dagger} e^{igM}$$

$$= e^{-igM^\dagger} e^{igM^\dagger} = e^{0}\hat{I} = \hat{I}$$

By analogy to $U_z(\chi) = e^{-i\frac{\chi}{2}\sigma^z}$,

we have

$$U_x(\chi) = e^{-i\frac{\chi}{2}\sigma^x}$$

$$U_y(\chi) = e^{-i\frac{\chi}{2}\sigma^y}$$

and around an arbitrary $\hat{\omega}$ axis we

$$U_{\hat{\omega}}(\chi) = e^{-i\frac{\chi}{2}\hat{\omega}\cdot\vec{\sigma}}$$

where $\vec{\omega}\cdot\vec{\sigma} = \omega_x\sigma^x + \omega_y\sigma^y + \omega_z\sigma^z$

the 3 components of the qubit spin polarization

$\vec{S} = \frac{1}{2}(\sigma^x, \sigma^y, \sigma^z)$ are the generators (the Lie group of) rotations in the spin space (rotations on the Bloch sphere)

$-i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}|1\rangle$

Exercise 3.12 Prove that

$$U_x\left(\tfrac{\pi}{2}\right)|+z\rangle = |-y\rangle$$

$$U_x\left(\tfrac{\pi}{2}\right)|-z\rangle = -i|+y\rangle,$$

$$-i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = -i\begin{pmatrix}0\\1\end{pmatrix} = -i|1\rangle$$

$$U_x\left(\tfrac{\pi}{2}\right) = \cos\tfrac{\pi}{4}\hat{I} - i\sin\tfrac{\pi}{4}\sigma^x$$

$$= \begin{pmatrix}\cos\frac{\pi}{4} & 0\\ 0 & \cos\frac{\pi}{4}\end{pmatrix} - i\begin{pmatrix}0 & \sin\frac{\pi}{4}\\ \sin\frac{\pi}{4} & 0\end{pmatrix}$$

$$= \begin{pmatrix}\cos\frac{\pi}{4} & -i\sin\frac{\pi}{4}\\ -i\sin\frac{\pi}{4} & \cos\frac{\pi}{4}\end{pmatrix}$$

$$U_x\left(\tfrac{\pi}{2}\right)|+z\rangle = \cos\tfrac{\pi}{4}|0\rangle - i\sin\tfrac{\pi}{4}|1\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right) = |-y\rangle$$

$$U_x\left(\frac{\pi}{2}\right)|-z\rangle = \cos\frac{\pi}{4}|1\rangle - \sin\frac{\pi}{4}\, i|0\rangle$$

$$= -i\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle\right)$$

$$= -i\frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right) = -i|+y\rangle$$

How to actually do rotations?

↳ In classical mech, we need 3 Euler angles to rotate an initial config → final config.

↳ For single qubit quantum states, we only need to specify 2 angles to move $|0\rangle$ to an arbitrary final state $|+\hat{n}\rangle$.

Example: θ about y & ϕ about z.

$$U|0\rangle = e^{-i\frac{\varphi}{2}Z}\, e^{-i\frac{\theta}{2}Y}\left[|0\rangle\right] \qquad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= e^{-i\frac{\varphi}{2}Z}\left[\cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} Y\right]|0\rangle$$

$$= e^{-i\frac{\varphi}{2}Z}\left[\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle\right]$$

$$= \left[e^{-i\frac{\varphi}{2}}\cos\frac{\theta}{2}|0\rangle + e^{-i\frac{\varphi}{2}}\sin\frac{\theta}{2}|1\rangle\right]$$

$$= e^{-i\frac{\varphi}{2}}\left[\cos\frac{\theta}{2}|0\rangle + e^{-i\varphi}\sin\frac{\theta}{2}|1\rangle\right]$$

$$= \underbrace{e^{-i\frac{\varphi}{2}}}_{\text{irrelevant global phase}}|+\hat{n}\rangle$$

↳ we can get rid of the phase by a 3rd & final rotation around the $\hat{n}$ axis. (which would correspond to the 3rd Euler angle)

↳ this phase is not important if we are only looking at $|+\hat{n}\rangle$ or $|-\hat{n}\rangle$.

↳ However, if we are looking at a superposition of the 2 states, the final spin would produce an important <u>relative phase</u> between the two states in superposition.

↳ How?

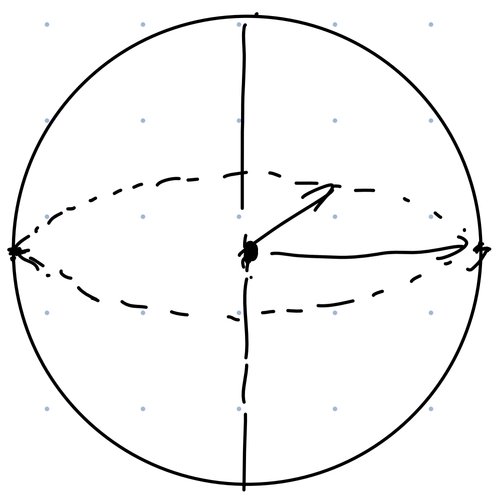
let's say we have

$$e^{-i\frac{\varphi}{2}}\left[|+\hat{n}\rangle + |-\hat{n}\rangle\right]$$

times

$$\left[e^{-i\frac{\varphi}{2}\hat{n}\cdot\vec{\sigma}}\right] = e^{-i\frac{\varphi}{2}\left[\underbrace{\hat{n}_x\sigma^x + \hat{n}_y\sigma^y + \hat{n}_z\sigma^z}_{\text{these act on}}\right]}$$

$|+\hat{n}\rangle$ & $|-\hat{n}\rangle$ differently, and might cause a relative phase to emerge.



Some authors define the Z-rotation unitary to be:

$$U_1(\varphi) = e^{-i\frac{\varphi}{2}[Z-I]} = e^{i\frac{\varphi}{2}}U_Z(\varphi)$$

$$U_1(\varphi)|0\rangle = e^{i\frac{\varphi}{2}}U_Z(\varphi)|0\rangle = e^{i\frac{\varphi}{2}}\left[e^{-i\frac{\varphi}{2}}|0\rangle\right]$$

$$U_Z(\varphi)|0\rangle = e^{-i\frac{\varphi}{2}}|0\rangle$$

$$= \left[|0\rangle + e^{i\varphi}|1\rangle\right]$$

$$U_Z(\varphi)|1\rangle = e^{i\frac{\varphi}{2}}|1\rangle$$

$$U_1(\varphi) = e^{i\frac{\varphi}{2}}U_Z(\varphi)|1\rangle$$

$$= e^{i\frac{\varphi}{2}}e^{i\frac{\varphi}{2}}|1\rangle$$

$$= e^{i\varphi}|1\rangle$$

To be consistent with the phase choice we made, we have

$$U_1(\varphi)e^{-i\frac{\chi}{2}Y}|0\rangle = |+\hat{n}\rangle.$$

3.4.1 Hadamard Gate

↳ Invented to swap the x & z component more conveniently.

$$H = |+x\rangle\langle -z| + |-x\rangle\langle -z|$$

$$= \frac{1}{\sqrt{2}}(\sigma^x + \sigma^z) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

It has the following nice properties:

$H|0\rangle = |+\rangle$

$H|1\rangle = |-\rangle$

$H|+\rangle = |0\rangle$

$H|0\rangle = |1\rangle$

$H|b\rangle = \frac{1}{\sqrt{2}}[|0\rangle + (-1)^b|1\rangle], \quad b \in [0,1]$

3.5 Hilbert Space and Operators for Multiple Qubits

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\begin{pmatrix}1\\0\end{pmatrix}\\ 0\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$|01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\begin{pmatrix}0\\1\end{pmatrix}\\ 0\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

...

① Generally, $\begin{pmatrix}v_0\\v_1\end{pmatrix} \otimes \begin{pmatrix}u_0\\u_1\end{pmatrix} = \begin{pmatrix}v_0\begin{pmatrix}u_0\\u_1\end{pmatrix}\\ v_1\begin{pmatrix}u_0\\u_1\end{pmatrix}\end{pmatrix} = \begin{pmatrix}v_0u_0\\v_0u_1\\v_1u_0\\v_1u_1\end{pmatrix}$

② Numbering of qubits

$$|q_{N-1} \dots q_2 q_1 q_0\rangle$$

③ Operations on tensors

$\sigma_0^x \leftarrow$ flip $\qquad \sigma_0^x|00\rangle = |01\rangle$

$\uparrow$ the 0^{th} qubit

$$\sigma_1^z\sigma_1^x|00\rangle = \sigma_1^z|10\rangle = \left[\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right] \otimes \begin{pmatrix}1\\0\end{pmatrix}$$

$$= (-1)\begin{bmatrix}0\\1\end{bmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = -|10\rangle$$

④ Expressing the operations without subscripts

$$\sigma_1^z = \sigma^z \otimes \sigma^0 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix}$$

$$= \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\sigma_0^z = \sigma^0 \otimes \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Kronecker Sum: $A \oplus B = A \otimes I + I \otimes B$

$$S^z \equiv \sigma_1^z + \sigma_0^z = \sigma^z \oplus \sigma^z$$

matrix form $= \sigma^z \otimes \sigma^0 + \sigma^0 \otimes \sigma^z = \begin{pmatrix} +2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$

Exercise 3.14

Prove the following

some kind of distributive law!

$$(\sigma^z \otimes \sigma^x)(|\psi\rangle \otimes |\phi\rangle) = (\sigma^z|\psi\rangle) \otimes (\sigma^x|\phi\rangle)$$

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad |\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad |\phi\rangle = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$$\sigma^z \otimes \sigma^x = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \qquad |\psi\rangle \otimes |\phi\rangle = \begin{pmatrix} \psi_1\phi_1 \\ \psi_1\phi_2 \\ \psi_2\phi_1 \\ \psi_2\phi_2 \end{pmatrix}$$

$$(\sigma^z \otimes \sigma^x)(|\psi\rangle \otimes |\phi\rangle) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \psi_1\phi_1 \\ \psi_1\phi_2 \\ \psi_2\phi_1 \\ \psi_2\phi_2 \end{bmatrix} = \begin{bmatrix} \psi_1\phi_2 \\ \psi_1\phi_1 \\ -\psi_2\phi_2 \\ -\psi_2\phi_1 \end{bmatrix}$$

$$\sigma^z|\psi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ -\psi_2 \end{pmatrix}$$

$$\sigma^x|\phi\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \phi_2 \\ \phi_1 \end{pmatrix}$$

$$[\sigma^z|\psi\rangle) \otimes [\sigma^x|\phi\rangle] = \begin{bmatrix} \psi_1 \\ -\psi_2 \end{bmatrix} \otimes \begin{bmatrix} \phi_2 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} \psi_1\phi_2 \\ \psi_1\phi_2 \\ -\psi_2\phi_2 \\ -\psi_2\phi_1 \end{bmatrix}$$

LHS = RHS.

Exercise 3.5: Write ~~out~~ the matrix representation of the following 2-qubit Pauli matrices

a) $\sigma_1^x \sigma_0^x = \sigma^x \otimes \sigma^x$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

b) $\sigma_1^x + \sigma_0^x = \sigma^x \oplus \sigma^x = (\sigma^x \otimes \hat{I}) + (\hat{I} \otimes \sigma^x)$

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma^x \otimes \hat{I} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\hat{I} \otimes \sigma^x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\sigma_1^x + \sigma_0^x = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

c) $(\sigma_1^0 + \sigma_1^z) + \frac{1}{2}(\sigma_0^0 + \sigma_0^z)$

$$= (\sigma^0 + \sigma^z) \oplus \frac{1}{2}(\sigma^0 + \sigma^z)$$

$$= \left[(\sigma^0 + \sigma^z) \otimes \hat{I}\right] + \frac{1}{2}\left[\hat{I} \otimes (\sigma^0 + \sigma^z)\right]$$

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma^0 + \sigma^z = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$(\sigma^0 + \sigma^z) \otimes \hat{I} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underbrace{\hat{I} \otimes (\sigma^0 + \sigma^z)}_{m} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\frac{1}{2} m = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$c) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Box 3.5. The No-Cloning Theorem

Given an unknown state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and an ancilla qubit initially prepared in a definite state (e.g. $|1\rangle$), there $\times\,\exists$ a unitary operation U that takes the joint initial state

$$|\phi\rangle = [\alpha|0\rangle + \beta|1\rangle] \otimes |1\rangle = \alpha|01\rangle + \beta|11\rangle$$

to the final state

$$U|\phi\rangle = [\alpha|0\rangle + \beta|1\rangle] \otimes [\alpha|0\rangle + \beta|1\rangle]$$

$$= \alpha^2|00\rangle + \alpha\beta\{|01\rangle + |10\rangle\} + \beta^2|11\rangle$$

unless U depend on both $\underbrace{\alpha \,\&\, \beta}$

which we don't know.

Exercise 3.16 Assuming you have knowledge of α & β, construct an explicit $U(\alpha, \beta)$ that ~~out~~ carries out cloning operation.

$$U(\alpha,\beta) = (V^{\dagger} \otimes V^{\dagger})(\bar{I} \otimes X)\,CNOT\,(\bar{I} \otimes X)$$

Exercise 3.4.

Bob attemps to clone an arbitrary state $|\psi\rangle = |\hat{n}\rangle$.

Bob chooses to measure the qubit along an arbitrary measurement axis $\hat{m}$ by measuring $M = \hat{m} \cdot \vec{\sigma}$.

Based on the measurement result Bob guesses that the qubit was in state $|+m\rangle$ or $|-\hat{m}\rangle$.

Knowing this Bob can then make an arbitrary # of approximate clone.

What is the average fidelity of this approximate cloning process?

(you can take $\hat{m} = (0,0,1)$.

$\hookrightarrow$ Measuring along the z-axis.

$$|\psi\rangle = \frac{\cos\frac{\theta}{2}}{\cancel{\sin\frac{\theta}{2}}}|0\rangle + e^{i\varphi}\frac{\sin\frac{\theta}{2}}{\cancel{\cos\frac{\theta}{2}}}|1\rangle$$

$$F = P_+F_+ + P_-F_-$$

$$= \cancel{\sin^2}\cos^2\frac{\theta}{2}\langle 0|\hat{n}\rangle + \sin^2\frac{\theta}{2}\langle 1|\hat{n}\rangle$$

$$= \cancel{3}\cos^4\frac{\theta}{2} + \sin^4\frac{\theta}{2} \qquad \left[2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\right]^2 = [\sin\theta]^2$$

$$= 1 - 2\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2} = 1 - \qquad 2\cancel{4}\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2} = \frac{1}{2}\sin^2\theta$$

$$= 1 - \frac{1}{2}\sin^2\theta$$

$$dP = dA = \frac{\sin\theta\, d\theta\, d\phi}{A} = \frac{\sin\theta\, d\theta\, d\phi}{4\pi}$$

$$P = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} \left(1 - \frac{1}{2}\sin^2\theta\right)\sin\theta\, d\theta$$

$$= \frac{1}{2}\frac{1}{\cancel{4\pi}}(\cancel{2\pi})\int_0^{\pi} \sin\theta - \frac{1}{2}\sin^3\theta\, d\theta$$

$$= \frac{1}{2}\left[\int_0^{\pi}\sin\theta\, d\theta - \frac{1}{2}\int_0^{\pi}\sin^3\theta\, d\theta\right]$$

$$= \frac{1}{2}\left[2 - \frac{1}{2}\frac{4}{3}\right] = \frac{1}{2}\left[2 - \frac{2}{3}\right]$$

$$= \frac{1}{2}\left[\frac{6}{3} - \frac{2}{3}\right] = \frac{2}{3}$$