Lecture 7

- Goals: General rotations + measurements.
  Multi-Qubit States.

  ↳ N qubits: Dimension 2^N,
  Tensor product of N 2d vector spaces

- every ($N \times N$) Hermitian operator H has a complete set of eigenvectors $H|\psi_j\rangle = e_j|\psi\rangle$.
  (e_j ← real.)

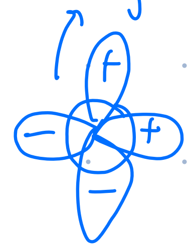
↳ You can construct a complete basis with these eigenvectors.

$$\sum_j |\psi_j\rangle\langle\psi_j| = I.$$

think more on that

Degenerate eigenvectors ⇓ orthogonalize those.

these 4 orbitals are degenerate



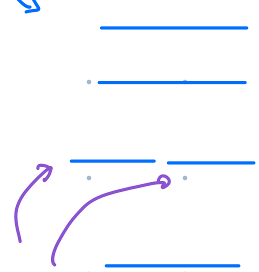
↳ spectral representation:

$$H = \sum_{j=1}^{N} \epsilon_j |\psi_j\rangle\langle\psi_j|$$

$$H|\psi_k\rangle = \sum_{j=1}^{N} \epsilon_j |\psi_j\rangle \underbrace{\langle\psi_j|\psi_k\rangle}_{\delta_{jk}}$$

$$= \epsilon_k |\psi_j\rangle$$

atom energy levels



2D-case:

$$H = \epsilon_+ |+\hat{n}\rangle\langle+\hat{n}| + \epsilon_- |-\hat{n}\rangle\langle-\hat{n}|$$

for example, these 2 states have the same energy, but they are distinguishable quantum states. (different spin/angular momentum)

- Every physical observable is represented by a Hermitian operator.
- Every measurement of physical quantity yields a result equal to one of the eigenvalues of H.
  - ↳ ($\hat{x}$ would be infinite dimensional)
  - ↳ if measurement is eigenvalue ϵ_k.
    state collapses to $|\psi_k\rangle$.
    * caution on degenerate eigenvalues.

if measurement result is ϵ_k,

then $$|\psi\rangle \rightarrow \frac{\hat{P}_k|\psi\rangle}{\langle\psi|\hat{P}_k^\dagger\hat{P}_k|\psi\rangle^{\frac{1}{2}}}$$, where $P_k \equiv |\psi_k\rangle\langle\psi_k|$

this collapse is non-unitary transformation.

$$\frac{\alpha_k}{(\alpha_k^*\alpha_k)^{\frac{1}{2}}}|\psi_k\rangle$$

the coefficients are normalized to 1, so you only get the state left.

$$|+x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$Z|b\rangle = \underbrace{(1-2b)}_{\text{you are measuring this coefficient}}|b\rangle \quad , b \in \{0,1\}.$$

What is the average measurement result:

$$\overline{\epsilon} = \sum_{j=1}^{N} |\alpha_j|^2 \epsilon_j \qquad |\psi\rangle = \sum_{j=1}^{N} \alpha_j |\psi_j\rangle.$$

compare this to $\parallel$

$$\underbrace{\langle\psi|H|\psi\rangle}_{} = \langle\psi| \sum_{j=1}^{N} \epsilon_j \alpha_j |\psi_j\rangle = \alpha^*_k \langle\psi_k| \sum_{j=1}^{N} \epsilon_j \alpha_j |\psi_j\rangle$$

$$= \sum_{j=1}^{N} \epsilon_j |\alpha_j|^2 = \overline{\epsilon} \qquad \boxtimes$$

expectation of H in the state $|\psi\rangle$.

* Difference between Gates & Measurements ?

Every unitary rotation is generated by a Hermitian matrix:

Lie algebra stuff.

$$U(\theta) = e^{-i\theta h}$$

real. Hermitian.

I think the idea is that measurements are random, but gates are deterministic?

the eigenvalues live on the unit circle.

$$h|\phi_k\rangle = \omega_k |\phi_k\rangle$$

$$U(\theta)|\phi_k\rangle = e^{-i\theta\omega_k}|\phi_k\rangle$$

they share eigenvectors.

also, go review euler angles.

$$|\psi\rangle = \sum_{j=1}^{N} \alpha_j |\phi_j\rangle$$

$$U(\theta)|\psi\rangle = \sum_{j=1}^{N} e^{-i\theta\omega_j}\alpha_j|\phi_j\rangle$$

so the idea here is that gates can be seen as rotations

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$$

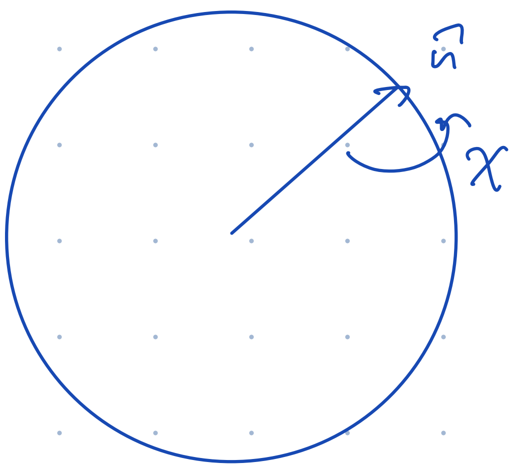
$$Z|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle) = |-\rangle$$

rotated the qubit by π around the Z-axis.

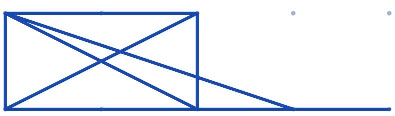
$$e^{-i\frac{\theta}{2}Z} = -iZ$$

still need to internalise

$$e^{-i\frac{\chi}{2}(\hat{n}\cdot\vec{\sigma})} = e^{-i\frac{\chi}{2}(\hat{n}_x X + \hat{n}_y Y + \hat{n}_z Z)}$$



rotation of χ around the $\hat{n}$ axis.



Multi-Qubit Hilbert Space. (2009 Berton Labs!)

- 2 qubits $\{0,1\}, \{0,1\}$.

q_0 —— ← least sig bit (the rightmost number.)

q_1 —— ← most sig bit.

$|b_1 b_0\rangle$

"0" $|00\rangle$
"1" $|01\rangle$
"2" $|10\rangle$
"3" $|11\rangle$

In general, a state can be in a superposition:

$$|\psi\rangle = \psi_{00}|00\rangle + \psi_{01}|01\rangle + \psi_{10}|10\rangle + \psi_{11}|11\rangle.$$

also

$$|\psi\rangle = \underbrace{\begin{pmatrix} \psi_{00} \\ \psi_{01} \\ \psi_{10} \\ \psi_{11} \end{pmatrix}}_{\text{length } = 2^n}$$

$|00\rangle = |0\rangle \otimes |0\rangle$ tensor product $= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$|01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

not commutative.

$|10\rangle = |1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$|11\rangle = |1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 1\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

q_1 q_0

Operator dim: 4×4.

Tensor product of operators:

$$|\psi_{alice}\rangle \otimes |\psi_{bob}\rangle$$

$$= \Big(\underbrace{X}_{2\times2}\,\underbrace{|\psi_{alice}\rangle}_{2\times1}\Big) \otimes \Big(\underbrace{Z}_{2\times2}\,\underbrace{|\psi_{bob}\rangle}_{2\times1}\Big)$$

$$\underset{2\times2}{A} \otimes \underset{2\times2}{B} = C_{4\times4} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \otimes \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$= \begin{bmatrix} A_{11}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} & A_{12}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \\ A_{21}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} & A_{22}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \end{bmatrix}$$

$$C \begin{pmatrix} \psi_{00} \\ \psi_{01} \\ \psi_{10} \\ \psi_{11} \end{pmatrix} = \boxed{}$$

A acting on 1st qubit & B acting on 2nd qubit

$$A \otimes I$$

alice acts

Bob does nothing