Lecture 6 IBM Circuit Composer, Qiskit

- Move on to Dirac notation, projectors, and unitary gates, Pauli matrix
- Measurements & Non-unitary state collapse.

$$\begin{cases} Z = |0\rangle\langle 0| - |1\rangle\langle 1| \\ I = |0\rangle\langle 0| + |1\rangle\langle 1| \end{cases}$$

Linear Operators take the form $|\psi\rangle\langle\Phi|$ or $\sum_j |\psi_j\rangle\langle\Phi_j|$

$$X = |0\rangle\langle 1| + |1\rangle\langle 0|$$

apply to $|1\rangle$, get $|0\rangle$ apply to $|0\rangle$, get $|1\rangle$.

$$X\left[\frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)\right] = \frac{1}{\sqrt{2}}(|1\rangle \pm |0\rangle)$$

$$= \pm|x\rangle.$$

$$Y\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\i\end{bmatrix}$$

$$Y\begin{bmatrix}0\\i\end{bmatrix} = \begin{bmatrix}-i\\0\end{bmatrix}.$$

$$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$Y = |+i\rangle\langle +i| - |-i\rangle\langle -i|$$

$$\cancel{\begin{pmatrix} 0 & +i \\ -i & 0 \end{pmatrix}} \qquad (?)$$

should be

$$\boxed{\begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}} \checkmark$$

$$|+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix} + i\begin{pmatrix}0\\1\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$$

$$|-i\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$

$\Rightarrow$ ① − ②

$$\frac{1}{2}\begin{pmatrix} 1-1 & i-(-i) \\ i+i & -1+1 \end{pmatrix}$$

$$|+i\rangle\langle +i| = \frac{1}{2}\begin{pmatrix}1\\i\end{pmatrix}\begin{pmatrix}1 & -i\end{pmatrix} = \begin{pmatrix} 1 & -i \\ i & +1 \end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix} 0 & -2i \\ +2i & 0 \end{pmatrix}$$

$$|-i\rangle\langle -i| = \frac{1}{2}\begin{pmatrix}1\\-i\end{pmatrix}\begin{pmatrix}1 & +i\end{pmatrix} = \begin{pmatrix} 1 & +i \\ -i & +1 \end{pmatrix}$$

Pauli Matrices are both unitary

$$UU^\dagger = U^\dagger U = I.$$

preserves the inner product of states.

$|\psi\rangle$

$|\Phi\rangle$, $\langle\Phi|\psi\rangle = ?$

$|\psi'\rangle = U|\psi\rangle$

$|\Phi'\rangle = U|\Phi\rangle.$

$$\langle\Phi'|\psi'\rangle = (\langle\Phi|U^\dagger U|\psi\rangle)$$

$$= \langle\Phi|\psi\rangle.$$

& Hermitian $Z = Z^\dagger$

Rotations

$$U_Z(\varphi) = e^{-i\frac{\varphi}{2}Z}$$

what is the derivation?

the generator

Define exponential via power series.

$$= \sum_{n=0}^{\infty} \frac{(-i\varphi/n)^n}{n!} Z^n$$

$($

y

Very conveniently, $Z^2 = I$.

that means:

$$\begin{cases} Z^{2n} = I \\ Z^{2n+1} = Z. \end{cases}$$

$$U_Z(\varphi) = \left\{ \sum_{n \text{ even}} \frac{(-i\varphi/2)^n}{n!} \right\} I + \left\{ \sum_{n \text{ odd}} \frac{(-i\varphi/2)^n}{n!} \right\} Z$$

$$= \cos(\varphi/2)\, I - i \sin(\varphi/2)\, Z.$$

↑ Euler-Pauli identity.

Define #Z and $U_Z(\varphi)$ on eigenvectors of Z

$$e^{-i\frac{\varphi}{2}Z}|0\rangle = |0\rangle = e^{-i\varphi/2}|0\rangle$$

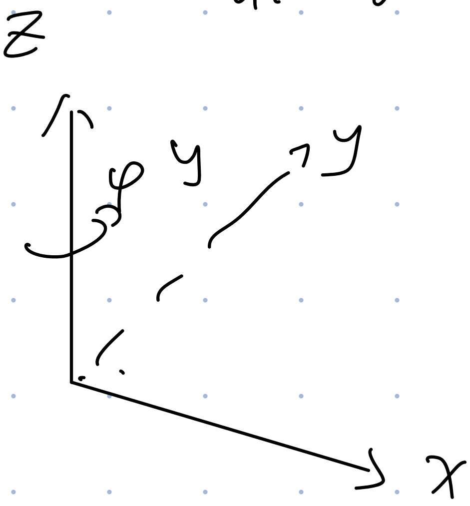
$$e^{-i\frac{\varphi}{2}Z}|1\rangle = -|0\rangle = e^{+i\frac{\varphi}{2}}|1\rangle.$$

?

because $Z = (-1)$?

why does (-1) correspond to $(+i\frac{\varphi}{2})$

Claim: $U_z(\varphi)$ rotates the state by any φ around the z-axis.



$$U_z(\varphi)\left[\alpha|0\rangle + \beta|1\rangle\right]$$

$$= \alpha \cdot e^{-i\varphi/2}|0\rangle + \beta e^{+i\varphi/2}|1\rangle$$

$$= e^{-i\varphi/2}\left(\alpha|0\rangle + \beta e^{i\varphi}|1\rangle\right)$$

$$= e^{-i\varphi/2}|+\hat{n}\rangle$$

the relative phase btwn $|0\rangle$ & $|1\rangle$ changed by φ.

Suppose:

$$U_z(\varphi)\left[\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\varphi_0}|1\rangle\right]$$

$$= e^{-i\varphi/2}\left[\cos\frac{\theta}{2}|0\rangle + \cos\frac{\theta}{2}e^{i(\varphi_0+\varphi)}|1\rangle\right]$$

→ is this saying that applying the rotation to a superposition of states give you a relative phase?

forgot the question asked in class

$$i\sin\frac{\theta}{2}Y = i\sin\frac{\theta}{2}\begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & 0 \end{pmatrix}$$

$$U_y(\theta) = \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} Y$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ +\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$U_y(\theta)|0\rangle = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

Suppose $\theta = \pi$ $= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$=$

$$U_y(\theta)|0\rangle = \cos\frac{\theta}{2}|0\rangle - i\sin\frac{\theta}{2}(+i)|1\rangle$$

$$= \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}(e^{i\theta})|1\rangle$$

if $\theta = \frac{\pi}{2}$ $\quad = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$ ✓

Rotations to & back gives you a global phase → berry phase.

General rotation axis $\hat{n} = (\hat{n}_x, \hat{n}_y, \hat{n}_z)$

$$\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z) = (X, Y, Z)$$

↳ a vector whose elements are pauli matrices

$$U_{\hat{n}}(\chi) = e^{-i\frac{\chi}{2}\underbrace{(\hat{n}\cdot\vec{\sigma})}}$$

→ a single rotation about the $\hat{n}$ axis

could expand. this relationship exists because space is curved. → Lie Algebra

$\hat{n}_x \sigma^x + \hat{n}_y \sigma^y + \hat{n}_z \sigma^z$

1 by 1   2x2

$$\neq e^{-i\frac{\chi}{2}\hat{n}_x\sigma^x} \underbrace{e^{-i\frac{\chi}{2}\hat{n}_y\sigma^y} e^{-i\frac{\chi}{2}\hat{n}_z\sigma^z}}$$

→ 3 sequential rotations.

$e^{A+B} = e^A e^B$ in standard scalar algebra; but for operators, this holds true only when $[A,B] = AB - BA = 0$

$\boxed{\text{why?}}$

However, $[\sigma_x, \sigma_y] = 2i\sigma_z$

Hence the rotation about $\hat{n}$ cannot be represented by 3 consecutive rotations about the z, y, and x axes.

Computation wise

$$= \cos\frac{\chi}{2} I - i\sin\frac{\chi}{2}(\hat{n}\cdot\vec{\sigma})$$

because $(\hat{n}\cdot\vec{\sigma})(\hat{n}\cdot\vec{\sigma}) = I$

$$(\hat{n}\cdot\vec{\sigma}) = |+\hat{n}\rangle\langle+\hat{n}| - |-\hat{n}\rangle\langle-\hat{n}|$$

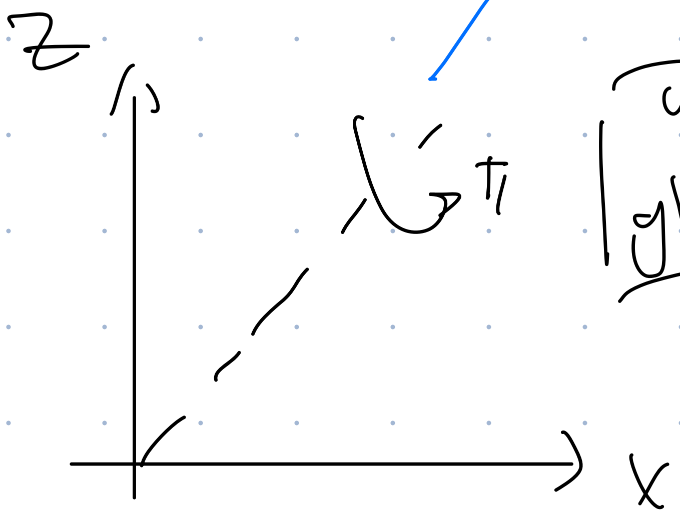
$$X(XY) = iXZ$$

$$Y = iXZ$$

$\sigma_y = i\sigma_x\sigma_z$

H-gate is a rotation about the funny axis.

The Hadamard Gate



up to a global phase

$H|0\rangle = |+\rangle$

$H|1\rangle = |-\rangle$

$H^2 = I$

$$e^{-i\frac{\pi}{2}Z} = \cos\frac{\pi}{2} I - i\sin\frac{\pi}{2} Z$$
$$= -iZ.$$

Can be represented by rotation about the z axis.

relationship needs clarification.

$\to$ Z-basis.

$$H = |+\rangle\langle 0| + |-\rangle\langle 1|$$

$$= \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]\langle 0| + \frac{1}{\sqrt{2}}\left[|0\rangle - |1\rangle\right]\langle 1|$$

$\hookrightarrow$ Maps one orthogonal basis to another orthogonal basis.

$\to$ the x basis.

$$= \frac{1}{\sqrt{2}}\left[|0\rangle\langle 0| + |1\rangle\langle 0|\right] + \frac{1}{\sqrt{2}}\left[|0\rangle\langle 1| - |1\rangle\langle 1|\right]$$

$$= \frac{1}{\sqrt{2}}\left[|0\rangle\langle 0| - |1\rangle\langle 1|\right] + \frac{1}{\sqrt{2}}\left[|1\rangle\langle 0| + |0\rangle\langle 1|\right]$$

$$= \frac{1}{\sqrt{2}}\left[Z + X\right]$$ how?

which also corresponds to

$$= \underbrace{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)}_{\text{the axis}} \cdot \underset{(X, Y, Z)}{\vec{\sigma}}$$

rotation of x about axis $\hat{n}$ is:

$$\boxed{\cos\frac{\alpha}{2} I - i\sin\frac{\alpha}{2}(\hat{n}\cdot\vec{\sigma})}$$

$\to$ a gate operator

$\to$ the Hadamard axis is $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

$$H^2 = H^{-1}H = \frac{1}{2}(Z+X)(Z+X)$$

$$= \frac{1}{2}(Z^2 + X^2 + \underbrace{ZX + XZ}_{\text{anti-commute}})$$

$$= I$$

$ZX = -XZ.$

should try to write it out in matrix multiplication form.

→ Should try to visualize rotations with a few simple examples.