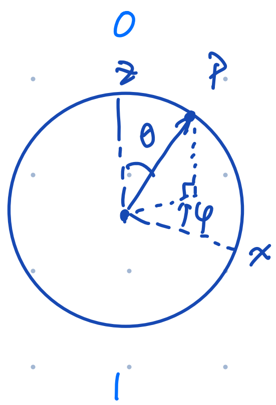
Lecture 3: Mathematical Representation of Quantum States.

- Goal: turn the Bloch sphere picture into an abstract vector space picture.

why is $\frac{\theta}{2}$ involved?

Bloch Sphere



$\boxed{0}$ $\uparrow$ $P_0 = \cos^2\frac{\theta}{2}$

$\boxed{1}$ $\downarrow$ $P_1 = \sin^2\frac{\theta}{2}$

Mathematical States of a Quantum Bit:

↳ represented by an element of a complex vector space (Hilbert Space)

↳ Computational basis states.

$$|0\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

complex number

$$|1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Born's Rule:

I ↳ In a superposition state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$,

the probability of $\begin{cases} \text{measuring } 0, & P_0 = |\alpha|^2 \\ \text{measuring } 1, & P_1 = |\beta|^2 \end{cases}$

$$|\alpha|^2 + |\beta|^2 = 1$$

$\hookrightarrow \vec{v}\cdot\vec{v} = (x,y)\cdot(x,y) = x^2+y^2 = 1$

↗
unit vector

What is the analog of inner product for a complex vector space?

$$\text{ket } |\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$\text{bra } \langle\psi| = \alpha^*\langle 0| + \beta^*\langle 1| = \begin{pmatrix}\alpha^*\\ \beta^*\end{pmatrix}^\dagger = \begin{pmatrix}\alpha^*\\ \beta^*\end{pmatrix}^T$$

"dual vector"

$$= (\alpha^* \ \ \beta^*)$$

$$\vec{v}_1\cdot\vec{v}_2 = x_1x_2+y_1y_2 = |v_1||v_2|\cos\theta_{12}$$

$$\langle\psi_2|\psi_1\rangle = (\alpha_2^* \ \ \beta_2^*)\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}$$

$$= \alpha_2^*\alpha_1 + \beta_2^*\beta_1$$

$$= \left(\langle\psi_1|\psi_2\rangle\right)^*$$

The inner product btwn 2 states is a measure of how different the 2 states are.

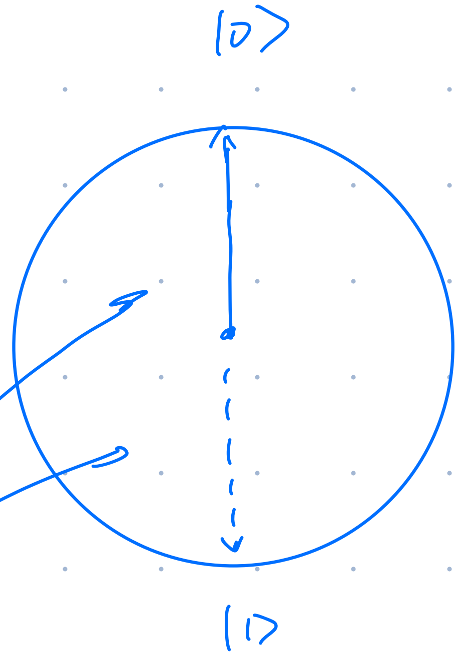
if you set $\psi_2=\psi_1$, there would be no ambiguity.

$\langle 0|0\rangle = (1\ 0)^* \begin{pmatrix}1\\0\end{pmatrix} = 1$

$\langle 1|1\rangle = (0\ 1)^* \begin{pmatrix}0\\1\end{pmatrix} = 1$

$\langle 0|1\rangle = (1\ 0)^* \begin{pmatrix}0\\1\end{pmatrix} = 0$

orthogonal.



but the vectors are not orthogonal, they are anti-parallel!

II. Born Rule's 2nd Implication:

"Global phases do not matter"

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix}\alpha\\\beta\end{pmatrix},\quad P_0 = |\alpha|^2,\ P_1 = |\beta|^2$$

$$\theta \in \mathbb{R},\ e^{i\theta}|\psi\rangle = \begin{pmatrix}\alpha e^{i\theta}\\ \beta e^{i\theta}\end{pmatrix},\quad P_0 = |\alpha e^{i\theta}|^2 = \alpha^2$$

$$P_1 = |\beta e^{i\theta}|^2 = \beta^2$$

"ray" = equivalence class of vector $|\psi\rangle$

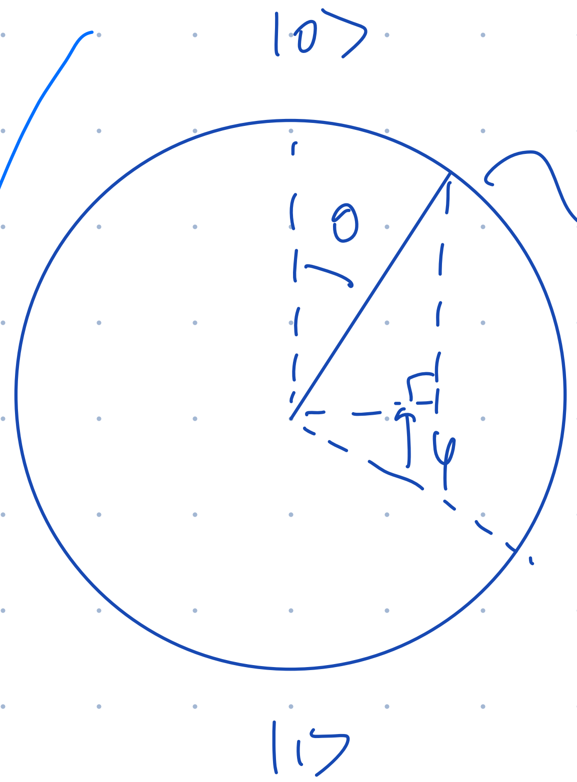
$$e^{i\theta}|\psi\rangle \sim |\psi\rangle$$

equivalent means experimentally indistinguishable

Relative Phase:

$$|\psi\rangle = \alpha|0\rangle + e^{i\varphi}\beta|1\rangle.$$

What does the global phase & relative phase represent physically?



$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

could put a $e^{i\varphi'}$ here, but that would be a global phase.

So we choose φ such that the coefficient before $\cos\frac{\theta}{2}$ is 1.

there are states around the equator that we can also measure, & φ takes us to these states.

$|0\rangle \leftrightarrow \theta = 0, \varphi = 0 \quad |\uparrow\rangle$

$|1\rangle \leftrightarrow \theta = \pi, \varphi = 0 \quad |\downarrow\rangle$

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle.$$

when $\theta = \pi$, you want $P_0 = 0$