Chapter 3: Intro to Hilbert Space.

What is Hilbert Space?

↳ Space for allowed states for a qubit

↳ 2D vector space over a field of complex numbers.

Qubit State

↳ Can be written as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

α, β : quantum amplitudes.

$|0\rangle, |1\rangle$: basis states

Inner product / Overlap

↳ $\langle\psi'|\psi\rangle = \alpha'^{*}\alpha + \beta'^{*}\beta$

↳ Rules: $\langle\psi'|\psi\rangle = \alpha'^{*}\alpha + \beta'^{*}\beta$ → implies

$= (\alpha'\alpha^{*} + \beta'\beta^{*})^{*}$ $\quad \underbrace{\langle\psi|\psi\rangle}_{\text{must be real}} = |\alpha|^2 + |\beta|^2 = 1$

$= (\langle\psi|\psi'\rangle)^{*}$

↑ Born's law.

↳ $\langle 0|1\rangle = \langle\uparrow|\downarrow\rangle = 0 \Rightarrow$ these basis states are orthogonal.

even though they are colinear on the Bloch sphere.

why?

$$\left.\begin{array}{l} |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \langle 0| = (1, 0). \end{array}\right\} \Rightarrow \langle 0|1\rangle = (1\ 0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \qquad D.$$

3.1 linear Operators on Hilbert Space

What is a linear operator?

↳ Operator: maps from Hilbert space back to the same Hilbert space.

↳ Represented by matrices.

↳ Hermitian matrices specifically.

↳ they can express all observable outcomes.

↳ Example: $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow$ "The z-comp. of the polarization vec. on Bloch sph. could turn out to be +1 or −1"

$$\sigma^z |0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (+1)\,|0\rangle$$

$$\sigma^z |1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = (-1)\,|1\rangle$$

$\Rightarrow$ $|0\rangle$ is an eigenstate of σ^z with eigenvalue +1

$|1\rangle$ is an eigenstate of σ^z with eigenvalue −1

↓

all possible measurement result.

* $\sigma^z|\psi\rangle$ does not represent the state of the system after the measurement of σ^z.

Instead, the state always collapses to one of the eigenstates of the operation being measured.

The measurement result is always one of the eigenvalues.

example: $\alpha|0\rangle + \beta|1\rangle \xrightarrow{\text{measurement}} |0\rangle$.

The system has to collapse into one of its eigenstates such that when you apply measurement again, you always get the same resuls.

Another justification is written in the following pages (the variance part).

The average measurement result of an observable is the expectation value of that observable (operator) in the quantum state being measured.

$$\langle\psi|\sigma^z|\psi\rangle = \underbrace{(\alpha^*\langle 0| + \beta^*\langle 1|)}_{\langle\psi}\underbrace{(\alpha|0\rangle - \beta|1\rangle)}_{\sigma^z|\psi\rangle}$$

$$= |\alpha|^2 - |\beta|^2 .$$

We determine how random the measurement will be by calculating the variance.

$$Var(\xi) = \overline{\xi^2} - \overline{\xi}^2$$

$$Var(Q) = \langle\psi|Q^2|\psi\rangle - (\langle\psi|Q|\psi\rangle)^2$$

We can only be certain about the outcome of measurement when we know that $|\psi\rangle$ is an eigenstate of Q.

if $|\psi\rangle$ is an eigenstate of Q

$$\Rightarrow Q|\psi\rangle = v_m|\psi\rangle .$$

$$Var(Q) = \langle\psi|v_m^2|\psi\rangle - (\langle\psi|v_m|\psi\rangle)^2$$

$$= v_m^2 \langle\psi|\psi\rangle - (v_m\langle\psi|\psi\rangle)^2$$

$$= v_m^2 - v_m^2 = 0$$

$\Rightarrow$ no variance $\Rightarrow$ result is certain.

↳ Hence measurement must collapse $|\psi\rangle$ into an eigenstate of Q.

Physical observables are always represented by Hermitian matrices because

① Their eigenvalues are always real (aligns with experiment: measured values of physical observables are always real)

② Their eigenvalues form a basis set that spans the Hilbert space (also aligns with experiment, we can observe $|0\rangle$ & $|1\rangle$, they form a basis for the 2D Hilbert space).

$H^\dagger$ means transpose + complex conjugation.

Demonstration that Hermitian matrices have <u>real eigenvalues</u> & their <u>eigenvectors are ortho</u>.

Consider j^{th} & k^{th} eigvec.

$$H|\psi_j\rangle = \lambda_j|\psi_j\rangle$$

$$H|\psi_k\rangle = \lambda_k|\psi_k\rangle$$

take adjoint of 2nd equation

$$\langle\psi_k|H^\dagger = \lambda_k^*\langle\psi_k|$$

Assuming $H^\dagger = H$ (Hermitian).

$$\langle\psi_k|\underbrace{H|\psi_j\rangle} = \underbrace{\lambda_j}\langle\psi_k|\underbrace{\psi_j\rangle}$$

$$\langle\psi_k|H|\psi_j\rangle = \lambda_k^*\langle\psi_k|\psi_j\rangle$$

Subtracting the two yields:

$$0 = (\lambda_j - \lambda_k^*)\langle\psi_k|\psi_j\rangle$$

Case 1: $j = k$: $\langle\psi_k|\psi_j\rangle = 1$

$$\Rightarrow (\lambda_j - \lambda_j^*)\underbrace{\langle\psi_j|\psi_j\rangle}_{=1} = 0$$

$\lambda_j = \alpha + bi$, $\lambda_j^* = \alpha - bi$

$\lambda_j - \lambda_j^* = \alpha + bi - \alpha + bi = 2bi = 0 \Rightarrow$

Hence $b = 0$, $\lambda_j = \lambda_j^*$ are REAL!

Case 2: $j \neq k$

if the eigvals are not degenerate $(\lambda_j \neq \lambda_k)$, then $0 = (\lambda_j - \lambda_k^*) \langle \psi_k | \psi_j \rangle$ requires

$\langle \psi_k | \psi_j \rangle = 0 \Rightarrow$ eig vec are orthogonal.

Thus if the full spectrum is non-degenerate, the set of eigenvectors is orthonormal. → every unit vector has length = 1 $\langle \psi_j | \psi_j \rangle = 1$.

$$\langle \psi_k | \psi_j \rangle = \delta_{kj} \quad \begin{pmatrix} =0 & k \neq j \\ =1 & k = j \end{pmatrix}$$

3.2 Dirac Notation For Operators

- Inner product: $\langle \Phi | \psi \rangle$: a number
- Outerproduct: $G = |\psi\rangle\langle\Phi|$: an operator

↳ why?

$$G|\chi\rangle = (|\psi\rangle\langle\Phi|)|\chi\rangle = (|\psi\rangle)(\langle\Phi|\chi\rangle)$$

$$= \underbrace{g}_{\text{inner product, a complex \#}} \underbrace{|\psi\rangle}_{\text{a vector}}.$$

↳ It's linear because

$$G(\alpha|\chi\rangle + \beta|\phi\rangle) = \alpha G|\chi\rangle + \beta G|\phi\rangle.$$

- In Dirac's notation,
  Hermitian operators take the form:

$$V = \sum_{j=1}^{M} v_j |\psi_j\rangle\langle\psi_j|$$

eigenvalue v_j. the j^{th} eigenvector of V.

example: $G = |0\rangle\langle 0| - |1\rangle\langle 1|$

↳ $G|0\rangle = (+1)|0\rangle$

↳ $G|1\rangle = (-1)|1\rangle$

verification (exercise 3.3).

$$V|\psi_m\rangle = (v_m|\psi_m\rangle\langle\psi_m|)|\psi_m\rangle$$

$$= v_m|\psi_m\rangle \cdot 1$$

$$= v_m|\psi_m\rangle \quad W$$

exercise 3.4.

show that

$$\sigma^x = \begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

in z-basis.

$$|+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|+x\rangle = \frac{1}{\sqrt{2}}[|0\rangle + |1\rangle] = \frac{1}{\sqrt{2}}(|+z\rangle + |-z\rangle)$$

$$|-x\rangle = \frac{1}{\sqrt{2}}[|0\rangle - |1\rangle] = \frac{1}{\sqrt{2}}(|+z\rangle - |-z\rangle).$$

$$|+y\rangle = \frac{1}{\sqrt{2}}[|0\rangle + i|1\rangle] = \frac{1}{\sqrt{2}}[|+z\rangle + i|-z\rangle]$$

$$|-y\rangle = \frac{1}{\sqrt{2}}[|0\rangle - i|1\rangle] = \frac{1}{\sqrt{2}}[|+z\rangle - i|-z\rangle].$$

$$|+x\rangle\langle +x| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$|-x\rangle\langle -x| = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}$$

$$\sigma^x = (+1)\,|+x\rangle\langle +x| + (-1)\,|-x\rangle\langle -x|$$

$$= \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}0 & 2\\2 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$$

$$|+y\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + i|1\rangle\right] = \frac{1}{\sqrt{2}}\left[|+z\rangle + i|-z\rangle\right]$$

$$|-y\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle - i|1\rangle\right] = \frac{1}{\sqrt{2}}\left[|+z\rangle - i|-z\rangle\right].$$

$$|+y\rangle\langle +y| = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}$$

this is conjugate transpose

$$= \frac{1}{2}\begin{pmatrix}1 & -i\\i & +1\end{pmatrix}$$

$$|-y\rangle\langle -y| = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & +i\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1 & +i\\-i & +1\end{pmatrix}$$

$$\sigma^Y = (+1)|+Y\rangle\langle +Y| + (-1)|-Y\rangle\langle -Y|$$

$$= \frac{1}{2}\left[\begin{pmatrix} 1 & -i \\ i & +1 \end{pmatrix} - \begin{pmatrix} 1 & +i \\ -i & +1 \end{pmatrix}\right]$$

$$= \frac{1}{2}\left[\begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

3.3 Orthonormal bases for qubit states.

↳ Using standard state parametrization in

$$\begin{cases} |+\hat{s}\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle \\ |-\hat{s}\rangle = \sin\frac{\theta}{2}|0\rangle - e^{i\varphi}\cos\frac{\theta}{2}|1\rangle \end{cases}$$

$$\hat{s} = (x, y, z) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

We can show that any pair of oppositely directed unit vectors $\pm\hat{n}$ on the Bloch sphere are orthogonal.

$$\langle -\hat{n}|+\hat{n}\rangle = \left(\sin\frac{\theta}{2}, -e^{-i\varphi}\cos\frac{\theta}{2}\right)\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}$$

$$= \sin\frac{\theta}{2}\cos\frac{\theta}{2} - \cancel{e^{-i\varphi+i\varphi}}\cos\frac{\theta}{2}\sin\frac{\theta}{2} = 0.$$

$\Rightarrow$ Since Hilbert space are 2D

↳ $|-\hat{n}\rangle$ and $|+\hat{n}\rangle$ are orthogonal.

$\Rightarrow$ they constitute a complete basis for expressing any vector in the Hilbert space.

Hence we can show the completeness relation: $\hat{I} = |+\hat{n}\rangle\langle+\hat{n}| + |-\hat{n}\rangle\langle-\hat{n}|$

$$|+\hat{n}\rangle\langle+\hat{n}| = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & e^{-i\varphi}\sin\frac{\theta}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\frac{\theta}{2} & e^{i\varphi}\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix}$$

$$|-\hat{n}\rangle\langle-\hat{n}| = \begin{pmatrix} \sin\frac{\theta}{2} \\ -e^{i\varphi}\cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \sin\frac{\theta}{2} & -e^{-i\varphi}\cos\frac{\theta}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \sin^2\frac{\theta}{2} & -e^{i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ -e^{i\varphi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \cos^2\frac{\theta}{2} \end{pmatrix}$$

$$|+\hat{n}\rangle\langle+\hat{n}| + |-\hat{n}\rangle\langle-\hat{n}| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The completeness relation is a powerful tool that allows us to express the state $|\psi\rangle$ in any basis.

$$|\psi\rangle = I|\psi\rangle = \left(|+\vec{n}\rangle\langle+\vec{n}| + |-\vec{n}\rangle\langle-\vec{n}|\right)|\psi\rangle$$

$$= |+\vec{n}\rangle\Big(\underbrace{\langle+\vec{n}|\psi\rangle}_{\text{coefficient}}\Big) + |-\vec{n}\rangle\Big(\underbrace{\langle-\vec{n}|\psi\rangle}_{\text{coefficient}}\Big).$$

$$= \alpha|+\vec{n}\rangle + \beta|-\vec{n}\rangle.$$

This is similar to projections

$$\vec{r} = (r_x, r_y) = \hat{x}(\hat{x}\cdot\vec{r}) + \hat{y}(\hat{y}\cdot\vec{r})$$

$$= \hat{i}(\hat{i}\cdot\vec{r}) + \hat{j}(\hat{j}\cdot\vec{r})$$

Hence we can think of the identity transformation as:

$$\hat{I} = \hat{x}\hat{x} + \hat{y}\hat{y} = \hat{i}\hat{i} + \hat{j}\hat{j}.$$

↳ We can think of $P_x = \hat{x}\hat{x}$ as a projection operator because:

$$P_x\vec{V} = (\hat{x}\hat{x})\vec{V} = \hat{x}(\hat{x}\cdot\vec{V})$$

↳ Notice that $(P_x)^2 = P_x$ (characteristic of projection operators) (once a vector is projected onto the x-axis, further projections don't do much.)

$$P_{xy} = \hat{x}\hat{x} + \hat{y}\hat{y} = I - \hat{z}\hat{z}.$$

↳ The analogous projector onto $|\hat{n}\rangle$ in the Hilbert space describing the polarization of a qubit is simply

$$P_{\hat{n}} = |\hat{n}\rangle\langle\hat{n}|$$

Example : Measurement not in the standard basis:

if we are given $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ in the standard basis, and are asked to measure a Hermitian operator

$$M = M_+|\hat{n}\rangle\langle\hat{n}| + M_-|-\hat{n}\rangle\langle-\hat{n}|$$

in some nonstandard basis,

we want to express $|\psi\rangle$ in the eigenstates of M.

$$|\psi\rangle = \alpha'|+\hat{n}\rangle + \beta'|-\hat{n}\rangle.$$

where $\begin{cases} M_+ \text{ occurs with } P_+ = |\alpha'|^2 \\ M_- \text{ occurs with } P_- = |\beta'|^2. \end{cases}$

we can perform change of basis:

$$\begin{aligned} |\psi\rangle &= \left[|+\hat{n}\rangle\langle+\hat{n}| + |-\hat{n}\rangle\langle-\hat{n}| \right] |\psi\rangle \\ &= |+\hat{n}\rangle\left(\langle+\hat{n}|\psi\rangle\right) + |-\hat{n}\rangle\left(\langle-\hat{n}|\psi\rangle\right) \\ &= \alpha'|+\hat{n}\rangle + \beta'|-\hat{n}\rangle \end{aligned}$$

$$\begin{cases} \alpha' = \alpha\langle+\hat{n}|0\rangle + \beta\langle+\hat{n}|1\rangle \\ \beta' = \alpha\langle-\hat{n}|0\rangle + \beta\langle-\hat{n}|1\rangle \end{cases}$$

Exercise 3.7. show that $|\pm\hat{n}\rangle$ is an eigenvector of

$$T_{\hat{n}} = \hat{n}\cdot\vec{\sigma} = (\hat{n}\cdot\hat{x})\sigma^x + (\hat{n}\cdot\hat{y})\sigma^y + (\hat{n}\cdot\hat{z})\sigma^z$$

with eigenvalue ± 1.