Article

parameterize

Version August 9, 2020 submitted to Journal Not Specified

1. Unified function

$$\min_{\substack{\mathbf{U}^{(i)},\mathbf{V}^{(i)},\\ \alpha_i,\mathbf{S}^{(i)},\mathbf{S}^*}} \sum_{i=1}^{m} \left\|\mathbf{X}^{(i)} - \mathbf{U}^{(i)}\mathbf{V}^{(i)}\right\|_F^2 + \lambda_1 \left\|\mathbf{V}^{(i)} - \mathbf{V}^{(i)}\mathbf{S}^{(i)}\right\|_F^2 - \lambda_2 \mathrm{Tr}(\mathbf{F}\mathbf{F}^\top \sum_{i=1}^{m} \mathbf{W}^{(i)\top}\mathbf{S}^{(i)}\mathbf{W}^{(i)})$$
$$\text{s.t. } \mathbf{V}^{(i)} \geq 0,\ 0 \leq S_{j,k}^{(i)} \leq 1,\ \mathbf{S}^{(i)\top}\mathbf{1} = \mathbf{1},\ \mathbf{F}\mathbf{F}^\top = \mathbf{I} \tag{1}$$

2. Optimization

2.1. Update $\mathbf{U}^{(i)}$

With $\mathbf{V}^{(i)}$, $\mathbf{Z}^{(i)}$, α_i and $\mathbf{Z}^*$ fixed, for each $\mathbf{U}^{(i)}$, we need to solve the following problem in Eq.(2)

$$\min_{\mathbf{U}^{(i)}} \left\|\mathbf{X}^{(i)} - \mathbf{U}^{(i)}\mathbf{V}^{(i)}\right\|_F^2 \tag{2}$$

Each of $\mathbf{U}^{(i)}$ can be solved separately since views are independent from each other. Therefore the objective function that we minimize can be rewritten as

$$\mathcal{L}(\mathbf{U}^{(i)}) = \min_{\mathbf{U}^{(i)}} \mathrm{Tr}(\mathbf{X}^{(i)\top}\mathbf{X}^{(i)} - 2\mathbf{V}^{(i)\top}\mathbf{U}^{(i)\top}\mathbf{X}^{(i)} + \mathbf{V}^{(i)\top}\mathbf{U}^{(i)\top}\mathbf{U}^{(i)}\mathbf{V}^{(i)}). \tag{3}$$

The closed-form solution of Eq.(3) can be easily obtained by setting $\partial\mathcal{L}(\mathbf{U}^{(i)})/\partial\mathbf{U}^{(i)} = -2\mathbf{X}^{(i)}\mathbf{V}^{(i)\top} + 2\mathbf{U}^{(i)}\mathbf{V}^{(i)}\mathbf{V}^{(i)\top}$ to zero. Then we can get the optimal solution

$$\mathbf{U}^{(i)} = \mathbf{X}^{(i)}\mathbf{V}^{(i)\top}(\mathbf{V}^{(i)}\mathbf{V}^{(i)\top})^{-1} \tag{4}$$

2.2. Update $\mathbf{V}^{(i)}$

Fixing $\mathbf{U}^{(i)}$, $\mathbf{Z}^{(i)}$, α_i and $\mathbf{Z}^*$, the minimum problem for optimizing $\mathbf{V}^{(i)}$ is reduced into the following problem,

$$\mathcal{L}(\mathbf{V}^{(i)}) = \min_{\mathbf{V}^{(i)}} \sum_{i=1}^{m} \left\|\mathbf{X}^{(i)} - \mathbf{U}^{(i)}\mathbf{V}^{(i)}\right\|_F^2 + \lambda_1 \left\|\mathbf{V}^{(i)} - \mathbf{V}^{(i)}\mathbf{S}^{(i)}\right\|_F^2 \quad \text{s.t. } \mathbf{V}^{(i)} \geq 0. \tag{5}$$

The partial derivation of $\mathcal{L}(\mathbf{V}^{(i)})$ with respect to $\mathbf{V}^{(i)}$ is

$$\frac{\partial\mathcal{L}(\mathbf{V}^{(i)})}{\partial\mathbf{V}^{(i)}} = \mathbf{V}^{(i)}\mathbf{P}^{(i)} + \mathbf{U}^{(i)\top}\mathbf{U}^{(i)}\mathbf{V}^{(i)} - \mathbf{U}^{(i)\top}\mathbf{X}^{(i)} + \Gamma, \tag{6}$$

where Γ is the Lagrange Multiplier and $\mathbf{P}^{(i)} = \lambda_1(\mathbf{I} - \mathbf{S}^{(i)} - \mathbf{S}^{(i)^\top} + \mathbf{S}^{(i)}\mathbf{S}^{(i)^\top})$. According to the optimization of semi-NMF [?] and the KKT condition, we can get

$$(\mathbf{V}^{(i)}\mathbf{P}^{(i)} + \mathbf{U}^{(i)^\top}\mathbf{U}^{(i)}\mathbf{V}^{(i)} - \mathbf{U}^{(i)^\top}\mathbf{X}^{(i)})_{jk}\mathbf{V}^{(i)}_{jk} = 0. \tag{7}$$

Denoting the positive element M^+_{jk} in matrix $\mathbf{M}$ as $M^+_{jk} = (|M_{jk}| + M_{jk})/2$ and the absolute value of negative element M^-_{jk} as $M^-_{jk} = (|M_{jk}| - M_{jk})/2$, we can get $M_{jk} = M^+_{jk} - M^-_{jk}$. Similarly, $\left(\mathbf{U}^{(i)^\top}\mathbf{U}^{(i)}\right)_{jk} = \left(\mathbf{U}^{(i)^\top}\mathbf{U}^{(i)}\right)^+_{jk} - \left(\mathbf{U}^{(i)^\top}\mathbf{U}^{(i)}\right)^-_{jk}$. Based on this, we can write the updating rule for $\mathbf{V}^{(i)}$:

$$V^{(i)}_{jk} \leftarrow V^{(i)}_{jk}\sqrt{\frac{[\mathbf{V}^{(i)}(\mathbf{P}^{(i)})^-]_{jk} + [(\mathbf{U}^{(i)^\top}\mathbf{U}^{(i)})^-\mathbf{V}^{(i)}]_{jk} + (\mathbf{U}^{(i)^\top}\mathbf{X}^{(i)})^+_{jk}}{[\mathbf{V}^{(i)}(\mathbf{P}^{(i)})^+]_{jk} + [(\mathbf{U}^{(i)^\top}\mathbf{U}^{(i)})^+\mathbf{V}^{(i)}]_{jk} + (\mathbf{U}^{(i)^\top}\mathbf{X}^{(i)})^-_{jk}}} \tag{8}$$

5 *2.3. Update* $\mathbf{S}^{(i)}$

When $\mathbf{U}^{(i)}$, $\mathbf{V}^{(i)}$, α_i and $\mathbf{S}^*$ are fixed, the optimization for $\mathbf{S}^{(i)}$ can be simplified as

$$\begin{aligned}\min_{\mathbf{S}^{(i)}} \sum_{i=1}^{m} \lambda_1 \left\|\mathbf{V}^{(i)} - \mathbf{V}^{(i)}\mathbf{S}^{(i)}\right\|_F^2 - \lambda_2 \mathrm{Tr}(\mathbf{W}^{(i)^\top}\mathbf{S}^{(i)^\top}\mathbf{W}^{(i)}\mathbf{F}\mathbf{F}^\top) \\ \text{s.t. } 0 \le S^{(i)}_{j,k} \le 1,\ \mathbf{S}^{(i)^\top}\mathbf{1} = \mathbf{1}.\end{aligned} \tag{9}$$

The closed-form solution of optimizing Eq.(9) can be attained by setting its derivation w.r.t. $S^{(i)}_{:,j}$ to zero

$$\mathbf{S}^{(i)}{}_{j,:} = (2\lambda_1\mathbf{V}^{(i)^\top}\mathbf{V}^{(i)})^{-1}(2\lambda_1\mathbf{V}^{(i)^\top}\mathbf{V}^{(i)}_{:,j} + \lambda_2\mathbf{W}^{(i)}\mathbf{F}\mathbf{F}^\top\mathbf{W}^{(i)^\top}_{j,:}) \tag{10}$$

6 *2.4. Update* $\mathbf{F}$

With $\mathbf{U}^{(i)}$, $\mathbf{V}^{(i)}$ and $\mathbf{S}^{(i)}$ fixed, we need to minimize the following objective for $\mathbf{F}$.

$$\begin{aligned}\min_{\mathbf{F}} -\mathrm{Tr}(\mathbf{F}\mathbf{F}^\top \sum_{i=1}^{m} \mathbf{W}^{(i)^\top}\mathbf{S}^{(i)}\mathbf{W}^{(i)}) \\ s.t.\ \mathbf{F}^\top\mathbf{F} = \mathbf{I}\end{aligned} \tag{11}$$

Using $\mathbf{M}$ to represent $\sum_{i=1}^{m} \mathbf{W}^{(i)^\top}\mathbf{S}^{(i)}\mathbf{W}^{(i)}$ in Eq.(11), we can transform the objective into the following form

$$\begin{aligned}\max_{F} \mathrm{Tr}(\mathbf{F}^\top\mathbf{M}\mathbf{F}) \\ s.t.\ \mathbf{F}^\top\mathbf{F} = \mathbf{I}.\end{aligned} \tag{12}$$

7 The optimal solution for $\mathbf{F}$ can be attached by taking the eigenvectors corresponding to the largest
8 k eigenvalues of $\mathbf{M}$.