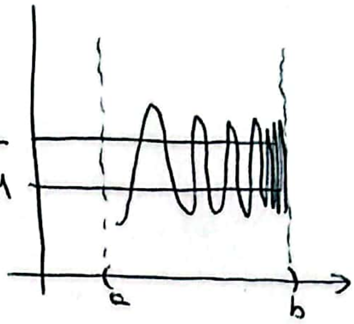
1. $f:(a,b)\to\mathbb{R}$ derivável



$x_n, y_n \to b$

$f(x_n)\to L$, $f(y_n)\to M$ $(L\neq M)$

↳ suponha $L>M$

Como $x_n, y_n \to b$, considere $a_K = x_{n_K}$ de modo que $a_n > y_n, \forall n\in\mathbb{N}$ e $b_K = y_{n_K}$ de modo que $b_n > x_n, \forall n\in\mathbb{N}$.

Pelo Teorema do Valor Médio, temos que para cada $n\in\mathbb{N}$, $\exists\, \tilde{x}_n, \tilde{y}_n$ tais que

$$f'(\tilde{x}_n) = \frac{f(a_n)-f(y_n)}{a_n-y_n} \quad \text{e} \quad f'(\tilde{y}_n) = \frac{f(x_n)-f(b_n)}{x_n-b_n}$$

com $\tilde{x}_n\in(y_n,a_n)$ e $\tilde{y}_n\in(x_n,b_n)$. Assim, temos

$$\lim_{n\to\infty} f'(\tilde{x}_n) = \lim_{n\to\infty} \frac{\overset{\nearrow L}{f(a_n)} - \overset{\nearrow M}{f(y_n)}}{\underbrace{a_n-y_n}_{>0}} = \infty, \text{ já que } a_n-y_n\to 0$$

$$\lim_{n\to\infty} f'(\tilde{y}_n) = \lim_{n\to\infty} \frac{\overset{\nearrow L}{f(x_n)} - \overset{\nearrow M}{f(b_n)}}{\underbrace{x_n-b_n}_{<0}} = -\infty, \text{ já que } x_n-b_n\to 0$$

e $L-M>0$

∎

2. $g_n : [0,1] \to \mathbb{R}$ contínua, $\forall n \in \mathbb{N}$

$|g_n(t)| < \lambda^n$, $\lambda > 0$

Note que $\displaystyle \frac{1}{n!}\int_0^1 (1+t)^n g_n(t)\,dt < \frac{1}{n!}\int_0^1 (1+t)^n \lambda^n = \frac{\lambda^n}{n!}\int_0^1 (1+t)^n\,dt =$

$$= \frac{\lambda^n}{n!}\int_1^2 u^n\,du = \frac{\lambda^n}{n!}\left[\frac{u^{n+1}}{n+1}\right]_1^2 = \frac{\lambda^n}{(n+1)!}(2^{n+1}-1)$$

Sendo $a_n = \dfrac{\lambda^n}{(n+1)!}(2^{n+1}-1)$, temos que

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{\lambda^{n+1}}{(n+2)!}(2^{n+2}-1)}{\dfrac{\lambda^n}{(n+1)!}(2^{n+1}-1)} = \frac{\lambda^{n+1}}{(n+2)!}(2^{n+2}-1)\cdot\frac{(n+1)!}{\lambda^n}\cdot\frac{1}{(2^{n+1}-1)} =$$

$$= \frac{\lambda}{n+2}\cdot\frac{2^{n+2}-1}{2^{n+1}-1}$$

Assim

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left[\frac{\lambda}{n+2}\cdot\frac{2^{n+2}-1}{2^{n+1}-1}\right] = \lim_{n\to\infty}\left[\underbrace{\frac{\lambda}{n+2}}_{\downarrow\, 0}\cdot\underbrace{\frac{2-1/2^{n+1}}{1-1/2^{n+1}}}_{\downarrow\, 2}\right] = 0$$

Logo, $a_n \to 0$ e, portanto, $\displaystyle \lim_{n\to\infty}\frac{1}{n!}\int_0^1 (1+t)^n g_n(t)\,dt = 0$ $\blacksquare$

3. $g: [0,1] \to \mathbb{R}$ derivável, com derivada contínua

$f: \mathbb{R} \to \mathbb{R}$ contínua

$g(0) = g(1)$

$$\int_0^1 f(g(t))\, g'(t)\, dt = \int_{g(0)}^{g(1)} f(u)\, du = 0, \text{ já que } g(0) = g(1) \quad \blacksquare$$

4. $h: [0,1] \to \mathbb{R}$ duas vezes derivável
$h(0)=0$, $h'(0)=0$, $h''(t)>0$

Se existisse $t>0$ tal que $h(t) \leq 0$, então pelo Teorema do Valor Médio, existe $s \in (0,t)$ tal que

$$h'(s) = \frac{h(t)-h(0)}{t-0} = \frac{h(t)}{t} \leq 0$$

Novamente, pelo Teorema do Valor Médio, existe $r \in (0,s)$ tal que

$$h''(r) = \frac{h'(s)-h'(0)}{s-0} = \frac{h'(s)}{s} \leq 0$$

o que é um absurdo, já que $h''(t)>0$, $\forall t$.

Portanto, $h(t)>0$, $\forall t \in (0,1]$ ∎