

Contents

1	Introduction to Vectors	3
1.1	Vectors and Linear Combinations	3
1.2	Lengths and Dot Products	3
1.3	Matrices	4
1.3.1	Linear Equations	4
1.3.2	The Inverse Matrix	4
1.3.3	Cyclic Differences	4
2	Solving Linear Equations	5
2.1	Vectors and Linear Equations	5
2.2	The Idea of Elimination	5
2.3	Elimination Using Matrices	5
2.4	Rules for Matrix Operations	6
2.5	Inverse Matrices	6
2.6	Elimination = Factorization: $A = LU$	6
2.7	Transposes and Permutations	6
3	Vector Spaces and Subspaces	7
3.1	Spaces of Vectors	7
3.2	The Nullspace of A : Solving $Ax = 0$	7
3.3	The Rank and the Row Reduced Form	7
3.4	The Complete Solution to $Ax = b$	7
3.5	Independence, Basis and Dimension	7
3.6	Dimensions of the Four Subspaces	7
4	Orthogonality	9
4.1	Orthogonality of the Four Subspaces	9
4.2	Projections	9
4.3	Least Squares Approximations	9
4.4	Orthogonal Bases and Gram-Schmidt	9
5	Determinants	11
5.1	The Properties of Determinants	11
5.2	Permutations and Cofactors	11

5.3	Cramer's Rule, Inverse, and Volumes	11
6	Eigenvalues and Elgenvectors	13
6.1	Introduction to Eigenvalues	13
6.2	Diagonalizing a Matrix	13
6.3	Applications to Differential Equations	13
6.4	Symmetric Matrices	13
6.5	Positive Definite Matrices	13
6.6	Similar Matrices	13
6.7	Singular Value Decomposition	13
7	Linear Transformations	15
7.1	The idea of a Linear Transformation	15
7.2	The Matrix of a Linear Transformation	15
7.3	Diagonalization and the Pseudoinverse	15
8	Applications	17
8.1	Matrices in Engineering	17
8.2	Graphs and Networks	17
8.3	Markov Matrices, Population, and Economics	17
8.4	Linear Programming	17
8.5	Fourier Series: Linear Algebra for Functions	17
8.6	Linear Algebra for Statistics and Probability	17
8.7	Computer Graphics	17
9	Numerical Linear Algebra	19
9.1	Gaussian Elimination in Practice	19
9.2	Norms and Condition Numbers	19
9.3	Iterative Methods and Preconditioners	19
10	Complex Vectors and Matrices	21
10.1	Complex Numbers	21
10.2	Hermitian and Unitary Matrices	21
10.3	The Fast Fourier Transform	21

Chapter 1

Introduction to Vectors

1.1 Vectors and Linear Combinations

REVIEW OF THE KEY IDEAS

1. A vector \mathbf{v} in two-dimensional space has two components v_1 and v_2 .
2. $\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2)$ and $c\mathbf{v} = (cv_1, cv_2)$ are found a component at a time.
3. A linear combination of three vectors \mathbf{u} and \mathbf{v} and \mathbf{w} is $c\mathbf{u} + d\mathbf{v} + e\mathbf{w}$.
4. Take all linear combination of \mathbf{u} , or \mathbf{u} and \mathbf{v} , or \mathbf{u} , \mathbf{v} , \mathbf{w} . In three dimensions, those combinations typically fill a line, then a plane, and the whole space \mathbf{R}^3 .

1.2 Lengths and Dot Products

REVIEW OF THE KEY IDEAS

1. The dot product $\mathbf{v} \cdot \mathbf{w}$ multiplies each component v_i by w_i and adds all $v_i w_i$.
2. The length $\|\mathbf{v}\|$ of a vector is the square root of $\mathbf{v} \cdot \mathbf{v}$.
3. $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a **unit vector**, its length is 1.
4. The cosine product is $\mathbf{v} \cdot \mathbf{w} = 0$ when vector \mathbf{v} and \mathbf{w} are perpendicular.
5. The cosine of θ (the angle between any nonzero \mathbf{v} and \mathbf{w}) never exceeds 1:

$$\cos \theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} \quad (\text{Schwarz inequality}) \quad |\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \|\mathbf{w}\|$$

Problem 21 will produce the **triangle inequality** $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$.

1.3 Matrices

1.3.1 Linear Equations

1.3.2 The Inverse Matrix

this part is so hard

1.3.3 Cyclic Differences

REVIEW OF THE KEY IDEAS

1. **Matrix times vector:** $\mathbf{Ax} = \text{combination of the columns of } A$.
2. The solution to $Ax = b$ is $x = A^{-1}b$, when A is an invertible matrix.
3. The difference matrix A is inverted by the sum matrix $\mathbf{S} = \mathbf{A}^{-1}$.
4. The cyclic matrix \mathbf{C} has no inverse. Its three columns lie in the same plane
Those dependent columns add to the zero vector. $\mathbf{Cx} = \mathbf{0}$ has many solutions.
5. This section is looking ahead to key ideas. not fully explained yet.

Chapter 2

Solving Linear Equations

2.1 Vectors and Linear Equations

REVIEW OF THE KEY IDEAS

- 1.

2.2 The Idea of Elimination

Second lecture

Elimination

Back substitution

Here, we need to pay attention to the particularity of the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix} \quad (2.1)$$

REVIEW OF THE KEY IDEAS

- 1.

2.3 Elimination Using Matrices

REVIEW OF THE KEY IDEAS

- 1.

2.4 Rules for Matrix Operations

REVIEW OF THE KEY IDEAS

- 1.

2.5 Inverse Matrices

REVIEW OF THE KEY IDEAS

- 1.

2.6 Elimination = Factorization: $A = LU$

REVIEW OF THE KEY IDEAS

- 1.

2.7 Transposes and Permutations

REVIEW OF THE KEY IDEAS

- 1.

Chapter 3

Vector Spaces and Subspaces

3.1 Spaces of Vectors

3.2 The Nullspace of A : Solving $Ax = 0$

3.3 The Rank and the Row Reduced Form

3.4 The Complete Solution to $Ax = b$

3.5 Independence, Basis and Dimension

3.6 Dimensions of the Four Subspaces

Chapter 4

Orthogonality

4.1 Orthogonality of the Four Subspaces

4.2 Projections

4.3 Least Squares Approximations

4.4 Orthogonal Bases and Gram-Schmidt

Chapter 5

Determinants

5.1 The Properties of Determinants

5.2 Permutations and Cofactors

5.3 Cramer's Rule, Inverse, and Volumes

Chapter 6

Eigenvalues and Eigenvectors

6.1 Introduction to Eigenvalues

6.2 Diagonalizing a Matrix

6.3 Applications to Differential Equations

6.4 Symmetric Matrices

6.5 Positive Definite Matrices

6.6 Similar Matrices

6.7 Singular Value Decomposition

Chapter 7

Linear Transformations

7.1 The idea of a Linear Transformation

7.2 The Matrix of a Linear Transformation

7.3 Diagonalization and the Pseudoinverse

Chapter 8

Applications

8.1 Matrices in Engineering

8.2 Graphs and Networks

8.3 Markov Matrices, Population, and Economics

8.4 Linear Programming

8.5 Fourier Series: Linear Algebra for Functions

8.6 Linear Algebra for Statistics and Probability

8.7 Computer Graphics

Chapter 9

Numerical Linear Algebra

9.1 Gaussian Elimination in Practice

9.2 Norms and Condition Numbers

9.3 Iterative Methods and Preconditioners

Chapter 10

Complex Vectors and Matrices

10.1 Complex Numbers

10.2 Hermitian and Unitary Matrices

10.3 The Fast Fourier Transform