

# Computing the rank invariant for persistence bimodules

Jingyi Li

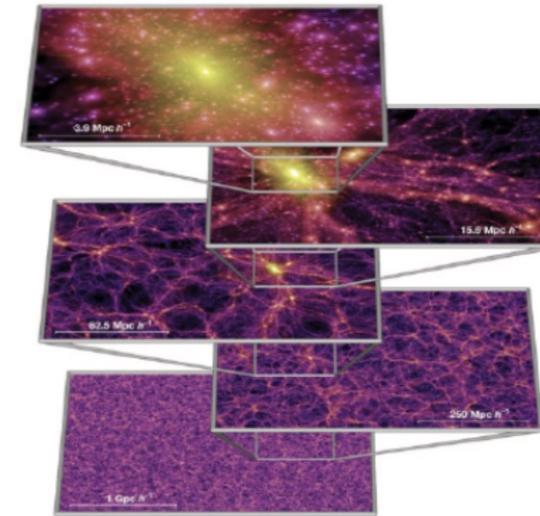
# Outline

- Introduction
- Background
  - Single-parameter persistence
  - Bipersistence
- Algorithm
  - Description
  - Implementation
  - Complexity
- Experimental results
  - Correctness validation
  - Running time

# Context: the data deluge

Data are generated at an unprecedented rate by:

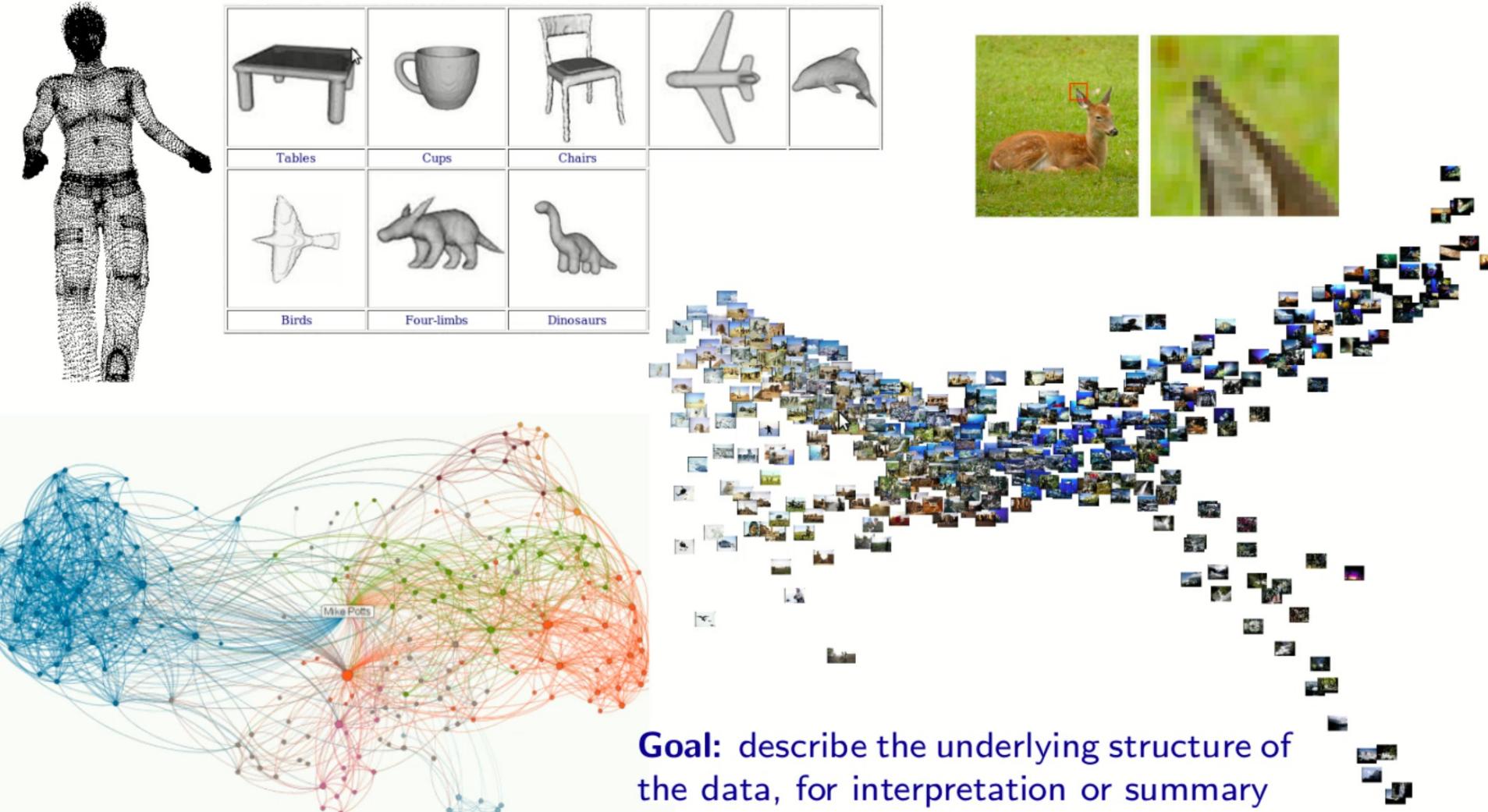
- academia
- industry
- general public



# Exploratory analysis of geometric data

**Input:** set of data points with metric or (dis-)similarity measure

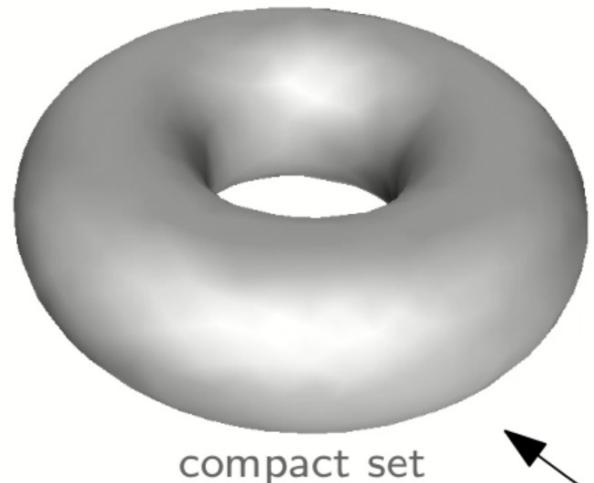
**data point**  $\equiv$  3d point, image patch, image or 3d shape in collection, Facebook user, etc.



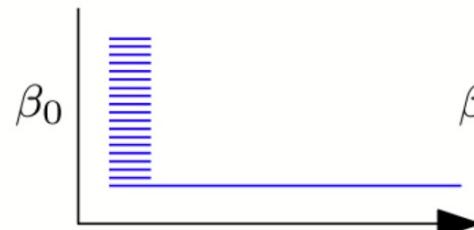
# Topological data analysis (TDA)

topological invariants for classification

$$\begin{aligned}\beta_0 &= \beta_2 = 1 \\ \beta_1 &= 2\end{aligned}$$

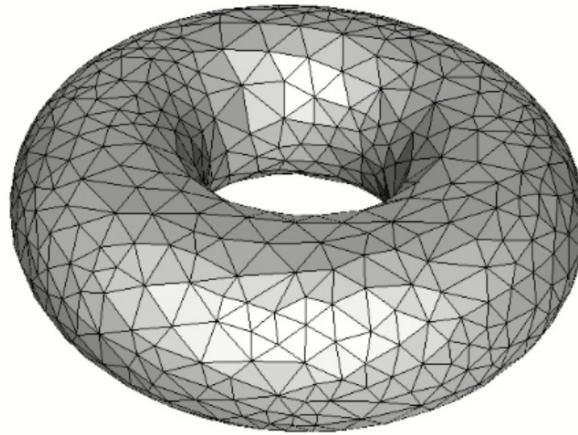


topological descriptors for inference and comparison

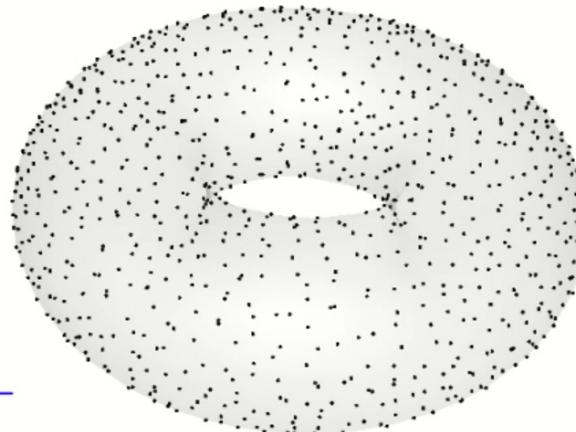


Algebraic topology

Applied algebraic topology



triangulation



point cloud

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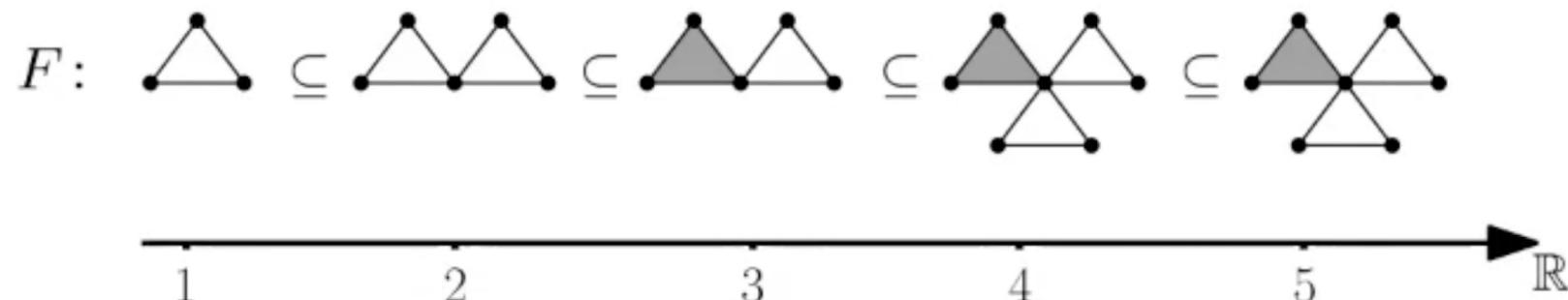
# Filtration

**Definition 2.2** (Filtration). *A filtration over  $U$  is a family  $\mathcal{F} = (F_t)_{t \in U}$  of increasing (for inclusion) topological spaces:*

$$\forall s, t \in U, s \leq t \Rightarrow F_s \subset F_t$$

Example:

$$U = \{1, 2, 3, 4, 5\}$$

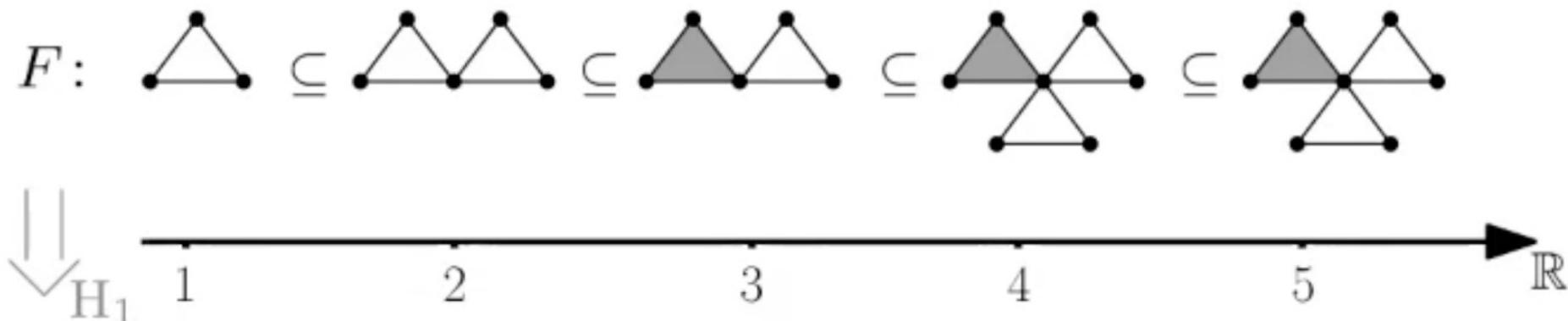


# A persistence module from the filtration

**Definition 2.3.** Let  $\mathcal{F} = (F_t)_{t \in U}$  be a filtration. We apply the homology functor  $H_*$ :

- $\forall t \in U$ , we define  $V_t := H_*(F_t, k)$
- $\forall s \leq t$ , let  $v_s^t$  be the linear map induced by the canonical inclusion  $F_s \xrightarrow{i} F_t$

$$U = \{1, 2, 3, 4, 5\}$$



$$M: \quad k \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} k^2 \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} k \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} k^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} k^2$$

# Single-parameter persistence

Let  $U \subseteq \mathbb{R}$ . Let  $\mathbf{k}$  be a field.

**Definition 2.1** (Single-parameter persistence module). A persistence module  $\mathbb{V}$  over  $U$  is a collection of vector spaces  $\{V_t\}_{t \in U}$  and linear maps  $v_s^t := V(s \leq t) : V_s \rightarrow V_t$  with the property that  $v_s^s$  is the identity map and  $v_t^u \circ v_s^t = v_s^u$  for all  $s \leq t \leq u \in U$ .

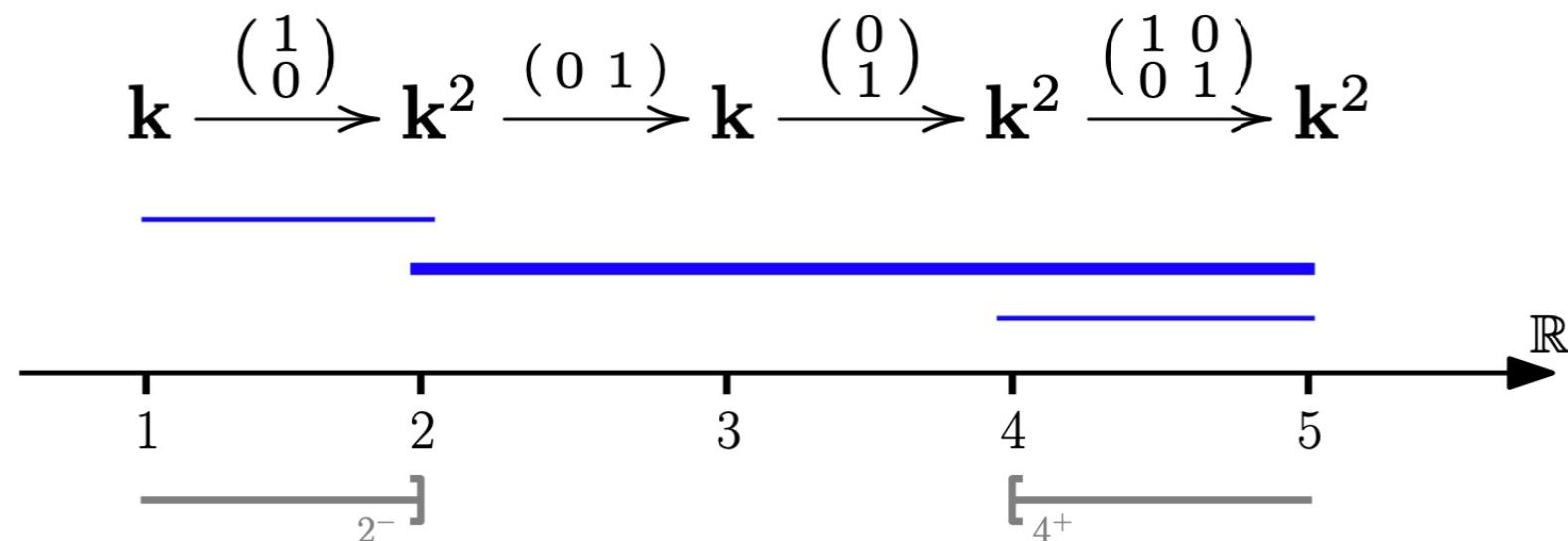
- Vector spaces:  $\{V_t\}_{t \in U}$
- Linear maps:  $v_s^t = V(s \leq t) : V_s \rightarrow V_t$

$$v_s^s : V_s \rightarrow V_s = id_{V_s}$$

$$v_t^u \circ v_s^t = v_s^u : V_s \rightarrow V_t \rightarrow V_u = V_s \rightarrow V_u$$

# Decomposition theorem

**Theorem 2.1.** A single-parameter persistence module  $\mathbb{V}$  can be uniquely decomposed as a direct sum of interval modules, written as:  $\mathbb{V} \simeq \bigoplus \mathbb{I}_{[b_j, d_j]}$ , when all the vector spaces in  $\mathbb{V}$  are finite-dimensional. The multi-set of intervals in the decomposition is called the (persistence) barcode of  $\mathbb{V}$ , denoted as  $Dgm \mathbb{V}$ .



# Direct sum of modules

**Definition 2.6** (Direct sum of modules). *Given  $\mathbb{V} = (V_t)_t$ ,  $\mathbb{W} = (W_t)_t$  two persistence modules with corresponding linear maps  $(v_s^t)$ ,  $(w_s^t)$ , we define  $\mathbb{V} \oplus \mathbb{W}$  by*

- $\mathbb{V} \oplus \mathbb{W} = (V_t \oplus W_t)_t$
- *The corresponding linear maps are denoted by:*

$$(v \oplus w)_s^t : V_s \oplus W_s \rightarrow V_t \oplus W_t$$

$$(x, y) \mapsto (v_s^t(x), w_s^t(y))$$

This definition extends naturally to any family of persistence modules  $(\mathbb{V}_j)_{j \in J}$ , denoted by  $\mathbb{V} := \bigoplus_{j \in J} \mathbb{V}_j$ .

# Interval Module

**Definition 2.4** (Interval). An interval of  $U$  is a convex subset  $I \subset U$ , i.e.  $\forall s \leq t \leq u \in U, (s, u) \in I^2 \Rightarrow t \in I$ .

**Definition 2.5** (Interval module). An interval module over  $I \subset U$  is a persistence module  $\mathbb{V}$  defined by:

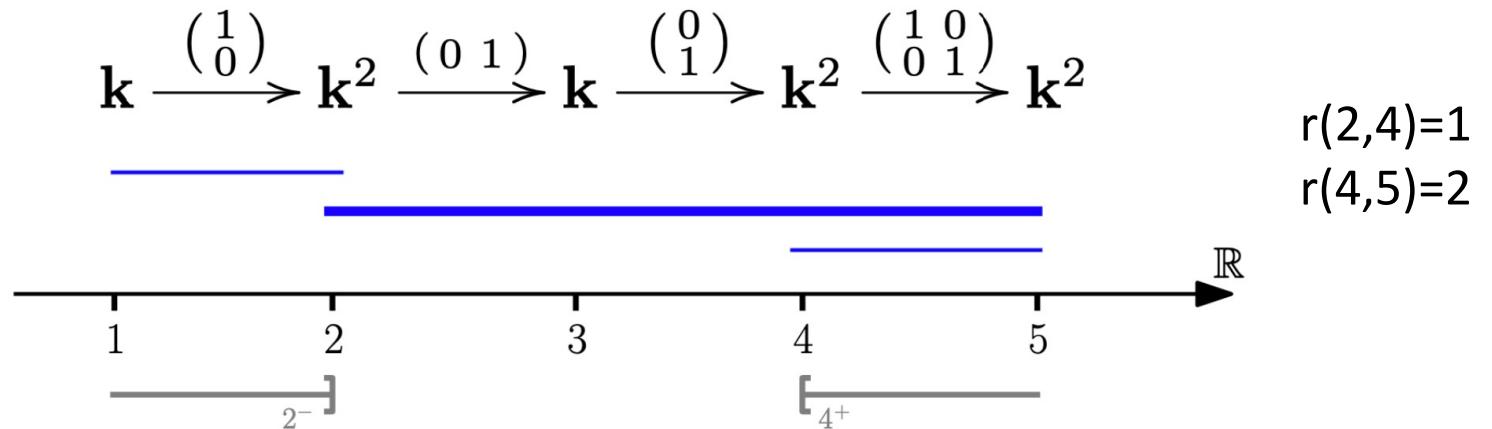
- $V_t = k$  if  $t \in I$ .  $V_t = \{0\}$  otherwise.
- $\forall s, t \in U, v_s^t = id_k$  if  $(s, t) \in I^2$ .  $v_s^t = 0$  otherwise.

For example, if  $I=[b,d]$ , the interval module is

$$\mathbb{I}_{[b,d]} := \underbrace{\{0\} \xrightarrow{0} \cdots \xrightarrow{0} \{0\}}_{t < b} \xrightarrow{0} \underbrace{k \xrightarrow{id} \cdots \xrightarrow{id} k}_{b \leq t \leq d} \xrightarrow{0} \underbrace{\{0\} \xrightarrow{0} \cdots \xrightarrow{0} \{0\}}_{d < t}$$

By convention, we set  $\mathbb{I}_\emptyset = 0$ .

# Multiplicity

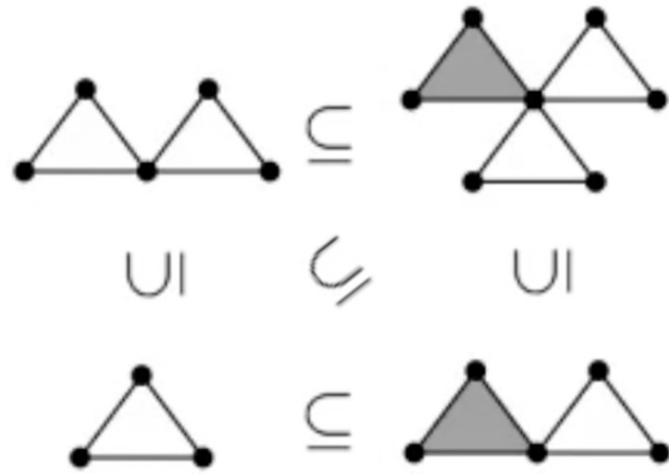


**Theorem 2.2** (Multiplicity). When  $U = \mathbb{Z}$ , the multiplicity of interval  $[s, t] \in Dgm\mathbb{V}$  is given by the following inclusion-exclusion formula:

$$m(s, t) = r(s, t) - r(s - 1, t) - r(s, t + 1) + r(s - 1, t + 1). \quad (2)$$

**Definition 2.7** (Rank invariant). Given a persistence module  $\mathbb{V}$  over  $U$ , the rank invariant of  $\mathbb{V}$  is the collection of ranks  $r(s, t) = \text{rank}(V(s \leq t))$  for all  $s, t \in U$  such that  $s \leq t$ .

# Bifiltration



**Definition 2.9** (Bifiltration). *A bifiltration over  $U$  is a family  $\mathcal{F} = (F_t)_{t \in U}$  of increasing (for inclusion) topological spaces:*

$$\forall s, t \in U, s \leq t \Rightarrow F_s \subset F_t$$

**Definition 2.10.** *Let  $\mathcal{F} = (F_t)_{t \in U}$  be a bifiltration. We apply the homology functor  $H_*$ :*

- $\forall t \in U$ , we define  $V_t := H_*(F_t, \mathbf{k})$
- $\forall s \leq t$ , let  $v_s^t$  be the linear application induced by the canonical inclusion  $F_s \xrightarrow{i} F_t$

# Bipersistence

Let  $U$  be a subset of  $\mathbb{R}^2$  equipped with the product order:  $s \leq t$  iff  $s.x \leq t.x$  and  $s.y \leq t.y$ .

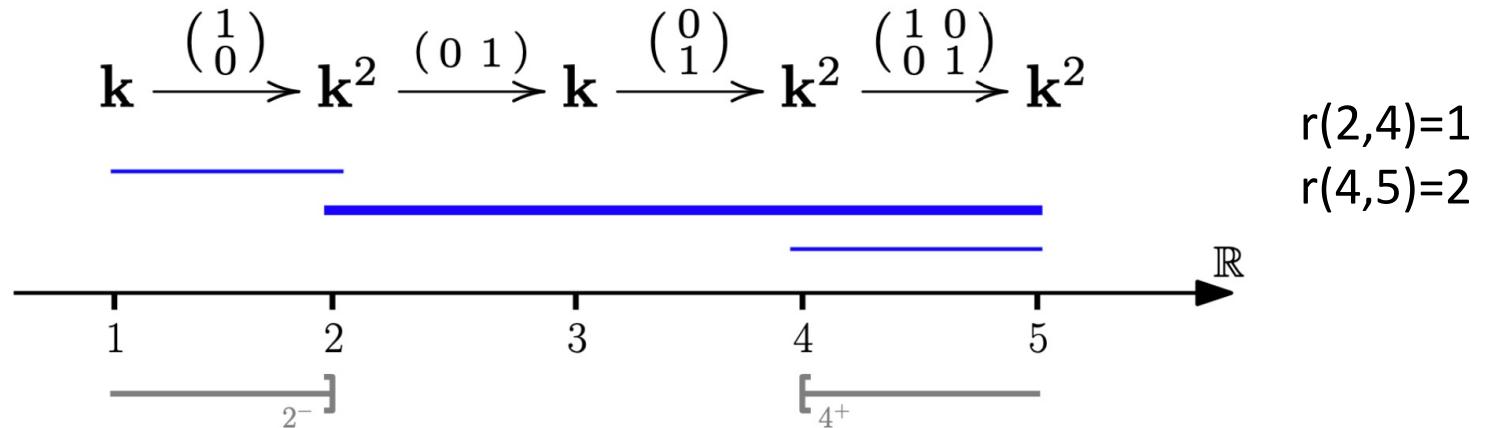
**Definition 2.8** (Bimodule). *A bimodule  $\mathbb{V}$  over  $U$  is a collection of vector spaces  $\{V_t\}_{t \in U}$  and linear maps  $v_s^t := V(s \leq t) : V_s \rightarrow V_t$  with the property that  $v_s^s$  is the identity map and  $v_t^u \circ v_s^t = v_s^u$  for all  $s \leq t \leq u \in U$ , where  $s \leq t$  if and only if  $s.x \leq t.x$  and  $s.y \leq t.y$ .*

**Definition 2.1** (Single-parameter persistence module). *A persistence module  $\mathbb{V}$  over  $U$  is a collection of vector spaces  $\{V_t\}_{t \in U}$  and linear maps  $v_s^t := V(s \leq t) : V_s \rightarrow V_t$  with the property that  $v_s^s$  is the identity map and  $v_t^u \circ v_s^t = v_s^u$  for all  $s \leq t \leq u \in U$ .*

# Bipersistence

$$\begin{array}{ccccccc} \mathbf{k} & \xrightarrow{id} & \mathbf{k} & \longrightarrow & 0 \\ id \uparrow & \left[ \begin{matrix} 1 \\ 0 \end{matrix} \right] & \uparrow & \left[ \begin{matrix} 1 & 0 \end{matrix} \right] & \uparrow & & \\ \mathbf{k} & \xrightarrow{\quad} & \mathbf{k}^2 & \xrightarrow{\quad} & \mathbf{k} & \cong & \oplus \quad ? \\ \uparrow & \left[ \begin{matrix} 0 \\ 1 \end{matrix} \right] \uparrow & \left[ \begin{matrix} 1 & 1 \end{matrix} \right] \uparrow & & \uparrow id & & \\ 0 & \xrightarrow{\quad} & \mathbf{k} & \xrightarrow{id} & \mathbf{k} & & \end{array}$$

# Multiplicity



**Theorem 2.2** (Multiplicity). When  $U = \mathbb{Z}$ , the multiplicity of interval  $[s, t] \in Dgm\mathbb{V}$  is given by the following inclusion-exclusion formula:

$$m(s, t) = r(s, t) - r(s - 1, t) - r(s, t + 1) + r(s - 1, t + 1). \quad (2)$$

**Definition 2.7** (Rank invariant). Given a persistence module  $\mathbb{V}$  over  $U$ , the rank invariant of  $\mathbb{V}$  is the collection of ranks  $r(s, t) = \text{rank}(V(s \leq t))$  for all  $s, t \in U$  such that  $s \leq t$ .

$$Rk\mathbb{V} = Rk\left(\bigoplus_{I \in Dgm\mathbb{V}} \mathbf{k}_I\right) = \sum_{I \in Dgm\mathbb{V}} Rk\mathbf{k}_I.$$

# Rank decomposition

$$Rk\mathbb{V} = Rkk_{\mathcal{R}} - Rkk_{\mathcal{S}} = \bigoplus_{R \subset \mathcal{R}} k_R - \bigoplus_{S \subset \mathcal{S}} k_S$$

where  $\mathcal{R}$  and  $\mathcal{S}$  are multi-sets of rectangles

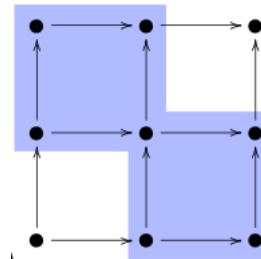
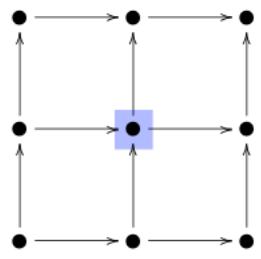
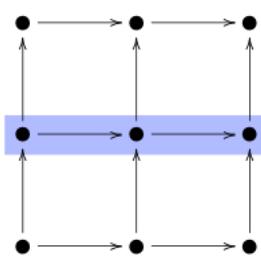
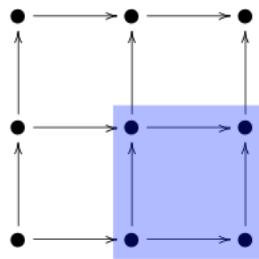
It is a minimal rank decomposition if  $\mathcal{R}$  and  $\mathcal{S}$  are disjoint as multi-sets.

$$\begin{aligned} Rk \left( \begin{array}{ccc} k & \xrightarrow{\text{id}} & k \\ & \uparrow & \uparrow \\ & id & [1 \ 0] \\ k & \xrightarrow{\left[ \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right]} & k^2 \xrightarrow{\left[ \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right]} k \\ & \uparrow & \uparrow \\ 0 & \xrightarrow{\quad} & k \xrightarrow{\text{id}} k \end{array} \right) &= Rk \left( \begin{array}{c} \text{Diagram 1} \oplus \text{Diagram 2} \oplus \text{Diagram 3} \oplus \text{Diagram 4} \\ \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right) \\ - Rk \left( \begin{array}{c} \text{Diagram 5} \oplus \text{Diagram 6} \\ \text{Diagram 5} \\ \text{Diagram 6} \end{array} \right) \end{aligned}$$

The diagrams consist of nodes arranged in a grid-like structure with arrows indicating connections between them. The nodes are represented by small black dots. The connections are shown as horizontal and vertical arrows. The diagrams are labeled with numbers 1 through 6, corresponding to the components in the rank decomposition. Diagrams 1 through 4 are highlighted with blue boxes, while Diagrams 5 and 6 are highlighted with red boxes.

# Rectangles

**Definition 2.16** (Rectangle). A set  $R \subset \mathbb{R}^2$  is a rectangle if  $R = I \times J$  where  $I$  and  $J$  are intervals in  $\mathbb{R}$ .



# Rectangle module

**Definition 2.13** (Rectangle module). *For a rectangle  $\mathcal{Q} \subset \mathbb{R}^2$ , the rectangle module  $k_{\mathcal{Q}}$  is defined by*

$$k_{Q,t} = \begin{cases} k, & (t \in \mathcal{Q}) \\ 0, & (t \notin \mathcal{Q}) \end{cases} \quad (5)$$

$$k_Q(s \leq t) = \begin{cases} Id_k, & (s \text{ and } t \in \mathcal{Q}) \\ 0, & \text{else} \end{cases} \quad (6)$$

# Decomposition theorem

**Theorem 2.3.** *Given a bimodule  $\mathbb{V}$  over  $\mathcal{X} \times \mathcal{Y} \subset \mathbb{R}^2$  and its rank invariant  $r = \text{Rk } \mathbb{V}$ , the minimal rank decomposition of  $r$  is unique when it exists. Moreover, it exists whenever  $\mathcal{X} \times \mathcal{Y}$  is finite or  $\mathbb{V}$  is finitely presented. The multi-set of rectangles in the decomposition is called signed (persistence) barcode, denoted as  $\text{Dgm } \mathbb{V}$ .*

$$\begin{aligned} \text{Rk} \left( \begin{array}{c} k \xrightarrow{\text{id}} k \longrightarrow 0 \\ \uparrow \text{id} \qquad \uparrow [1 \ 0] \\ k \xrightarrow{[1 \ 0]} k^2 \xrightarrow{[1 \ 1]} k \\ \uparrow \qquad \uparrow [0 \ 1] \\ 0 \longrightarrow k \xrightarrow{\text{id}} k \end{array} \right) &= \text{Rk} \left( \begin{array}{c} \oplus \\ \text{blue rectangle} \\ \oplus \\ \text{blue rectangle} \\ \oplus \\ \text{blue rectangle} \\ \oplus \\ \text{blue rectangle} \end{array} \right) \\ - \text{Rk} \left( \begin{array}{c} \oplus \\ \text{red rectangle} \\ \oplus \\ \text{red rectangle} \end{array} \right) &\quad \begin{aligned} m((0,1), (1,2)) &= 1 \\ m((0,1), (2,1)) &= 1 \\ m((1,0), (2,1)) &= 1 \\ m((1,1), (1,1)) &= 1 \\ m((0,1), (1,1)) &= -1 \\ m((1,1), (2,1)) &= -1 \end{aligned} \end{aligned}$$

# Inclusion-exclusion formula

Similar to single-parameter persistence, when  $\mathcal{X} \times \mathcal{Y} \subset \mathbb{Z}^2$ , the barcode  $\text{Dgm}\mathbb{V}$  is easily obtained from  $Rk\mathbb{V}$  via the following inclusion-exclusion formula [BLO22]:

$$\begin{aligned} m(s, t) = & r(s, t) - r((s_x - 1, s_y), t) \\ & - r((s_x, s_y - 1), t) + r((s_x - 1, s_y - 1), t), \\ & - r(s, (t_x + 1, t_y)) + r((s_x - 1, s_y), (t_x + 1, t_y)) \\ & + r((s_x, s_y - 1), (t_x + 1, t_y)) - r((s_x - 1, s_y - 1), (t_x + 1, t_y)) \\ & - r(s, (t_x, t_y + 1)) + r((s_x - 1, s_y), (t_x, t_y + 1)) \\ & + r((s_x, s_y - 1), (t_x, t_y + 1)) - r((s_x - 1, s_y - 1), (t_x, t_y + 1)) \\ & + r(s, (t_x + 1, t_y + 1)) - r((s_x - 1, s_y), (t_x + 1, t_y + 1)) \\ & - r((s_x, s_y - 1), (t_x + 1, t_y + 1)) + r((s_x - 1, s_y - 1), (t_x + 1, t_y + 1)) \end{aligned} \tag{7}$$

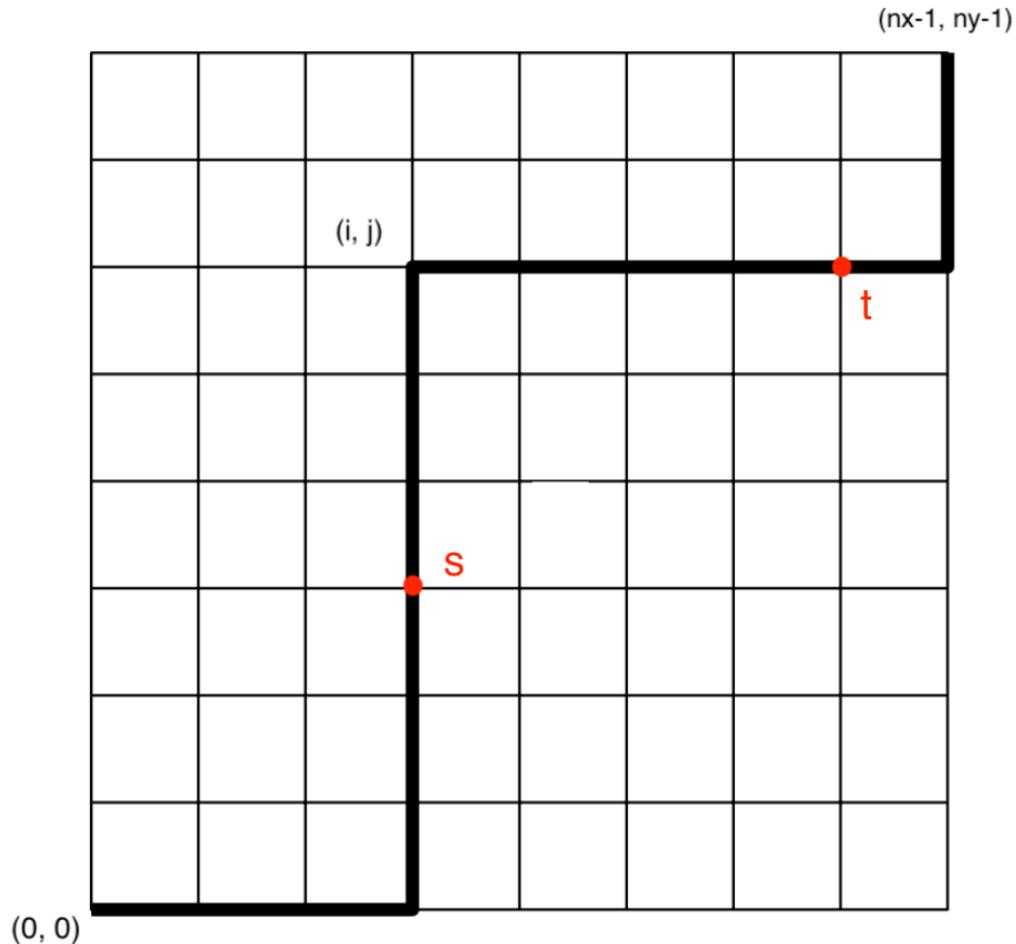
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# Input and Output

- Input
  - A grid  $G = [0, n_x - 1] \times [0, n_y - 1]$
  - A simplicial bifiltration  $\mathcal{F}$  with  $n$  simplices over the grid  $G$
  - $\mathcal{X}$  coordinate is vertical
- Output
  - Rank invariant  $r$  of the bimodule induced by  $\mathcal{F}$
  - $r: \mathbb{Z}^2 \rightarrow \mathbb{Z}$
  - $(s, t) \mapsto r(s, t)$

# Description



For  $\forall(i,j) \in [0, n_x - 1] \times [0, n_y - 1]$

A stair  $S_{i,j}: (0,0) \rightarrow \dots \rightarrow (0,j) \rightarrow \dots \rightarrow (i,j) \rightarrow \dots \rightarrow (i, n_y - 1) \rightarrow \dots \rightarrow (n_x - 1, n_y - 1)$

$r(s,t) = \# \text{intervals in the persistence barcode passing through } s \text{ and } t.$

# Psudocode of the algorithm

- Build a simplex tree for the simplicial complex
- For each elbow/stair:
  - Update the filtration values of the simplex tree
  - Compute the persistence barcode of the single-parameter persistence
  - Calculate the rank invariant
- Calculate the barcode decompsitoin from the rank invariant

# Simplex tree

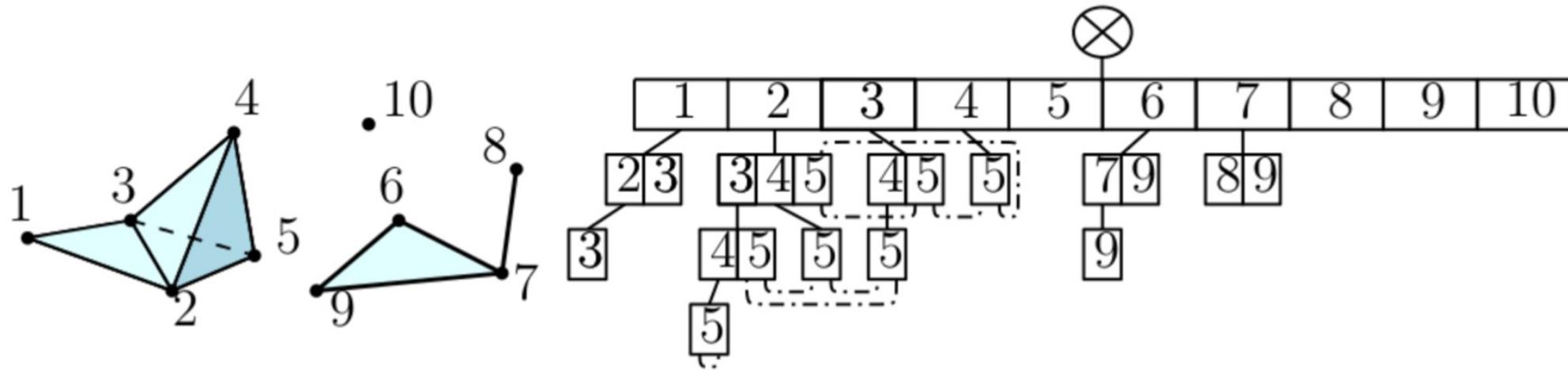
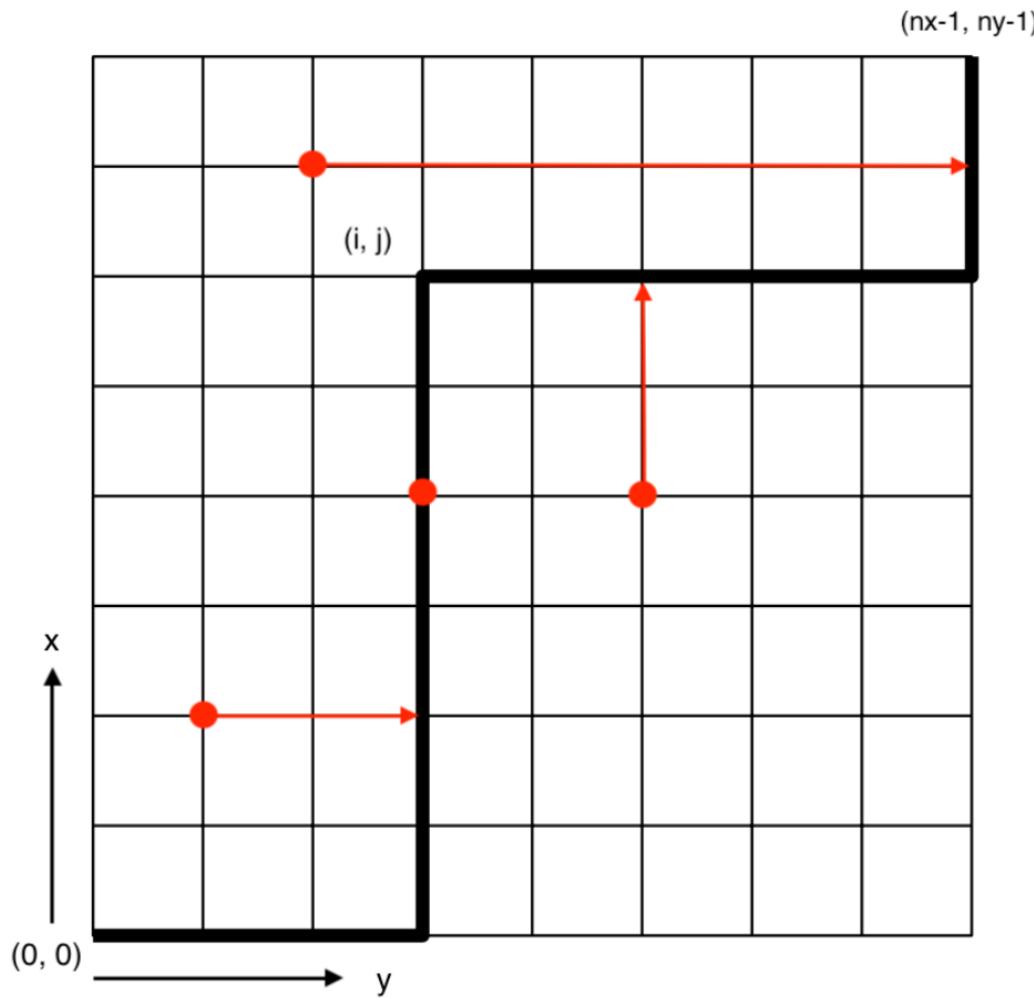


Figure 3.2: Simplex tree representation, from [The15].

# Insertion and projection



The projection  $(x, y)$  of the point  $(x', y')$  onto the path  $S_{(i,j)}$  is given by:

$$(x, y) = \begin{cases} (x', y'), & (x', y') \in S_{(i,j)}) \\ (x', j), & (x', y') \notin S_{(i,j)} \text{ and } x' \leq i \text{ and } y' \leq j \\ (i, y'), & (x', y') \notin S_{(i,j)} \text{ and } x' \leq i \text{ and } y' \geq j \\ (x', n_y - 1), & \text{otherwise} \end{cases}$$

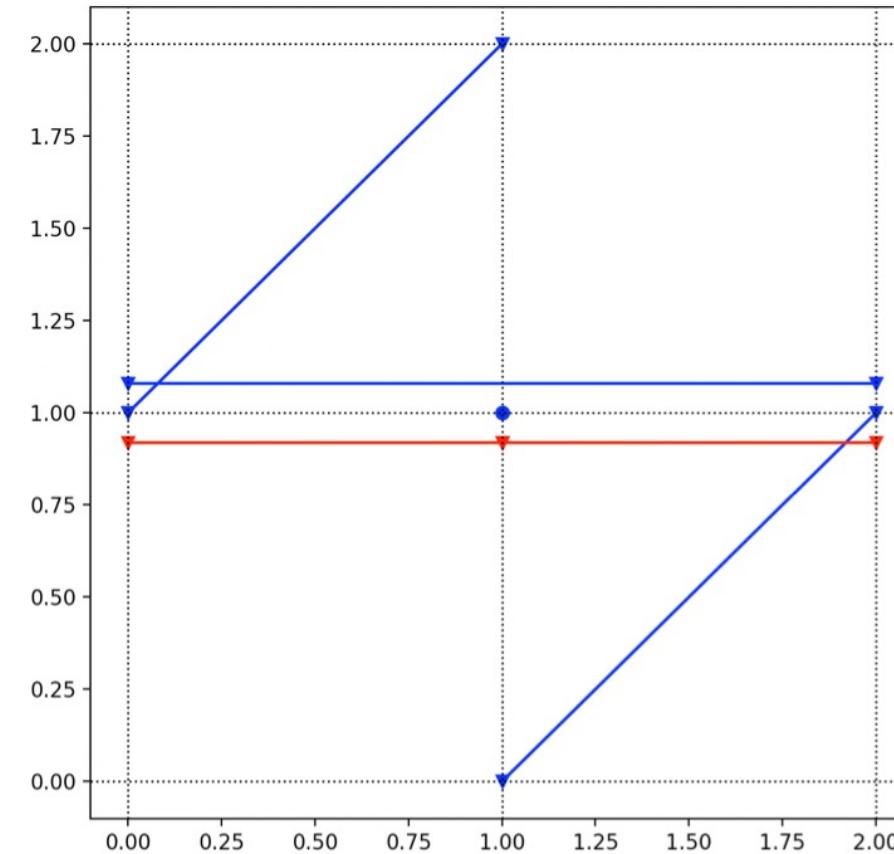
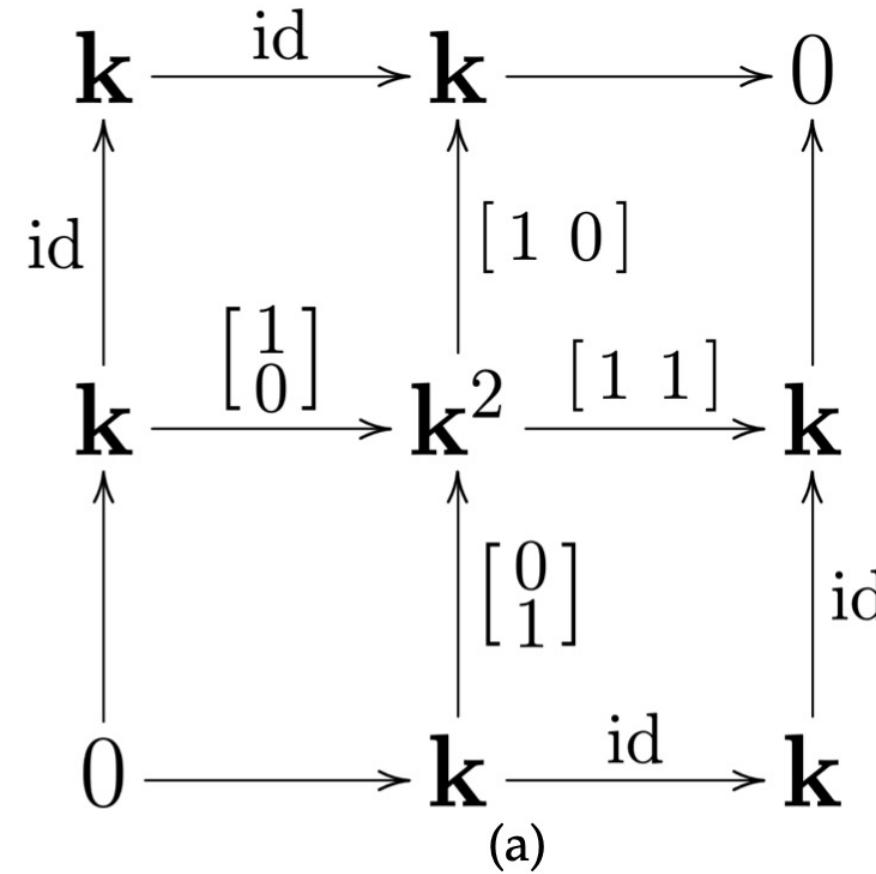
# Complexity

- Time
  - Time of computing 1-d persistence and updating the rank invariant is  $O(n^\omega)$  per stair
  - Time of updating the filtration values is  $O(n)$  per stair
  - $n_x n_y$  stairs,
  - $O(n_x n_y n^\omega)$  in total. In practice, we expect  $O(n_x n_y n)$
- Space
  - $O(nm)$
  - $n$  is the number of simplices
  - $m$  is the maximum dimension of faces in the simplicial complex

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# A toy example



$$\begin{aligned}m((0,1),(1,2)) &= 1 \\ m((0,1),(2,1)) &= 1 \\ m((1,0),(2,1)) &= 1 \\ m((1,1),(1,1)) &= 1 \\ m((0,1),(1,1)) &= -1 \\ m((1,1),(2,1)) &= -1\end{aligned}$$

# A toy example

$$Rk \left( \begin{array}{ccccccc} & k & \xrightarrow{id} & k & \longrightarrow & 0 \\ id & \uparrow & \left[ \begin{matrix} 1 \\ 0 \end{matrix} \right] & \uparrow & \left[ \begin{matrix} 1 & 0 \end{matrix} \right] & & \\ & k & \xrightarrow{id} & k^2 & \xrightarrow{id} & k & \\ \uparrow & \left[ \begin{matrix} 0 \\ 1 \end{matrix} \right] & \uparrow & \left[ \begin{matrix} 1 & 1 \end{matrix} \right] & & & \\ 0 & \longrightarrow & k & \xrightarrow{id} & k & & \end{array} \right) =$$

$$Rk \left( \begin{array}{c} 0 \longrightarrow 0 \longrightarrow 0 \\ \uparrow \qquad \uparrow \qquad \uparrow \\ 0 \longrightarrow k \xrightarrow{id} k \qquad \qquad \qquad \uparrow id \\ \uparrow \qquad \uparrow \qquad \uparrow \\ 0 \longrightarrow k \xrightarrow{id} k \end{array} \right) \oplus Rk \left( \begin{array}{c} 0 \longrightarrow k \longrightarrow 0 \\ \uparrow id \qquad \uparrow \\ 0 \longrightarrow k \longrightarrow 0 \end{array} \right) \oplus Rk \left( \begin{array}{c} k \xrightarrow{id} k \longrightarrow 0 \\ \uparrow id \qquad \uparrow id \\ k \xrightarrow{id} k \longrightarrow 0 \end{array} \right) \oplus Rk \left( \begin{array}{c} 0 \longrightarrow 0 \longrightarrow 0 \\ \uparrow \qquad \uparrow \\ 0 \longrightarrow k \longrightarrow 0 \\ \uparrow \qquad \uparrow \\ 0 \longrightarrow 0 \longrightarrow 0 \end{array} \right)$$

$$Rk \left( \begin{array}{c} k \xrightarrow{id} k \longrightarrow 0 \\ \uparrow id \qquad \uparrow \\ k \xrightarrow{\left[ \begin{matrix} 1 & 0 \end{matrix} \right]} k^2 \xrightarrow{\left[ \begin{matrix} 1 & 1 \end{matrix} \right]} k \\ \uparrow \qquad \uparrow \\ 0 \longrightarrow k \xrightarrow{id} k \end{array} \right) = Rk \left( \begin{array}{c} \text{blue box} \\ \oplus \\ \text{blue box} \\ \oplus \\ \text{blue box} \\ \oplus \\ \text{blue box} \end{array} \right) - Rk \left( \begin{array}{c} \text{red box} \\ \oplus \\ \text{red box} \end{array} \right)$$

# Bimodule clustering

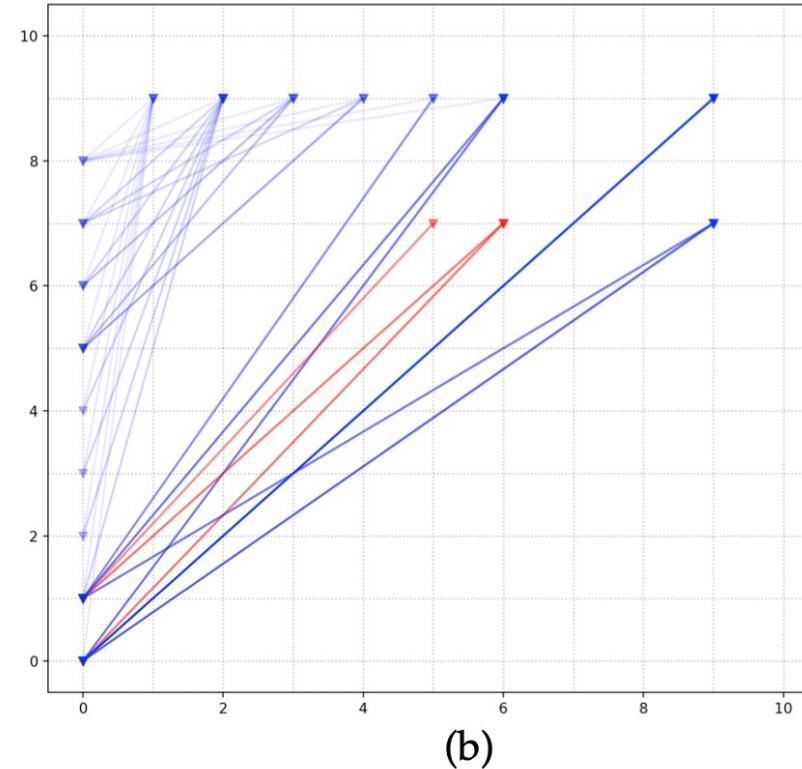
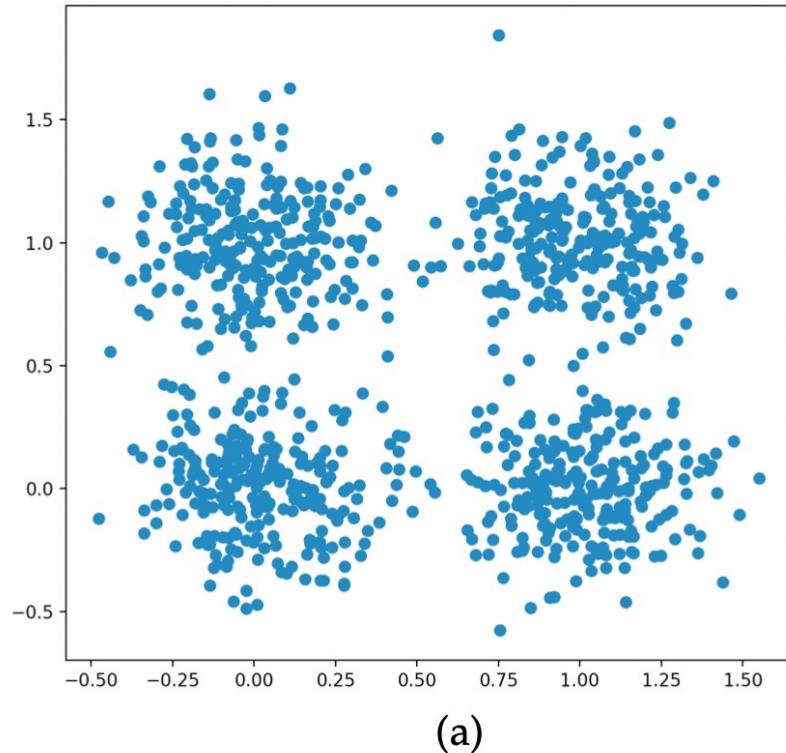
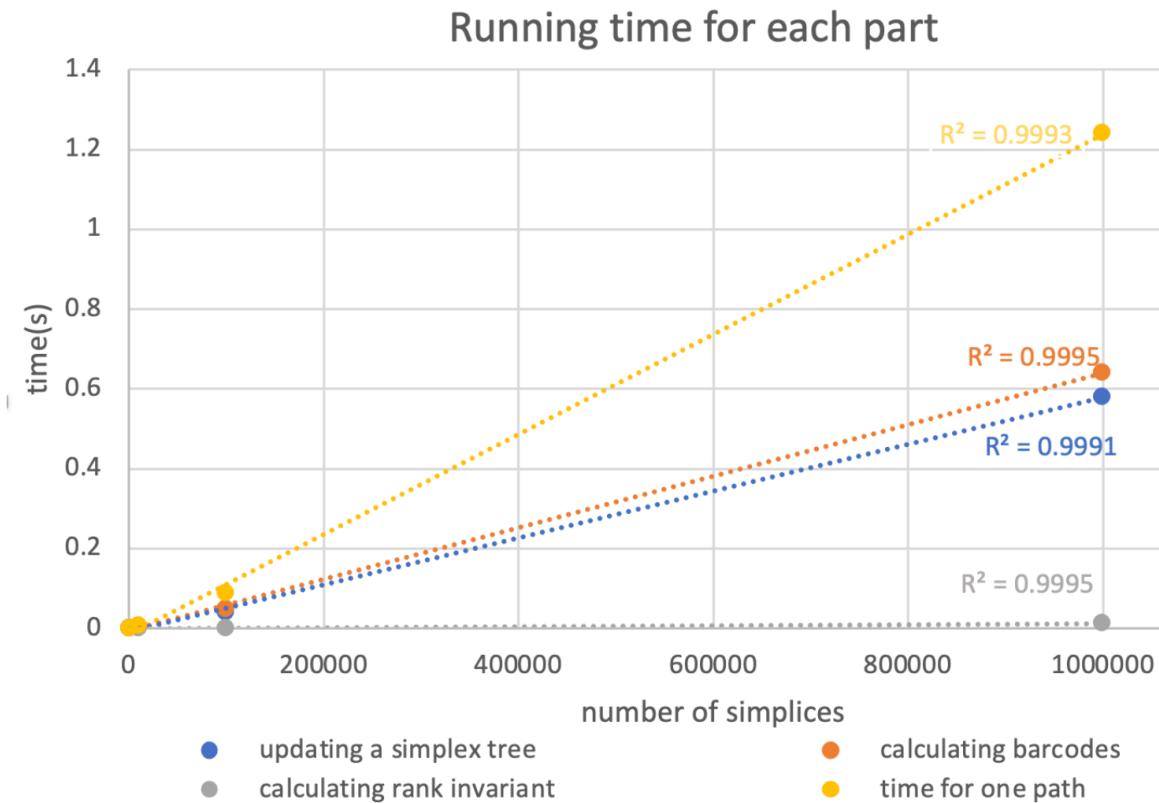


Figure 4.2: (a): Point cloud with 1000 points sampled from 4 Gaussian distributions  
(b): The corresponding signed barcode over a  $10 \times 10$  grid. Each bar with endpoints  $s \leq t$  in the barcode has an intensity proportional to  $\min\{t_x - s_x, t_y - s_y\}$ ; in particular, horizontal and vertical bars are invisible

# Running time



| PART                       | SIMPLICES | GRID SIZE | TIME(s) |
|----------------------------|-----------|-----------|---------|
| UPDATING A SIMPLEX TREE    | 1025      | 20×20     | 0.00022 |
|                            | 10007     | 20×20     | 0.0025  |
|                            | 100000    | 20×20     | 0.039   |
|                            | 1000007   | 20×20     | 0.58    |
| CALCULATING BARCODES       | 1025      | 20×20     | 0.00032 |
|                            | 10007     | 20×20     | 0.0036  |
|                            | 100000    | 20×20     | 0.048   |
|                            | 1000007   | 20×20     | 0.64    |
| CALCULATING RANK INVARIANT | 1025      | 20×20     | 9.6E-05 |
|                            | 10007     | 20×20     | 0.00015 |
|                            | 100000    | 20×20     | 0.001   |
|                            | 1000007   | 20×20     | 0.012   |
| TIME FOR ONE PATH          | 1025      | 20×20     | 0.00076 |
|                            | 10007     | 20×20     | 0.0064  |
|                            | 100000    | 20×20     | 0.088   |
|                            | 1000007   | 20×20     | 1.24    |

# Complexity

- Time
  - Time of computing 1-d persistence and updating the rank invariant is  $O(n^\omega)$  per stair
  - Time of updating the filtration values is  $O(n)$  per stair
  - $n_x n_y$  stairs,
  - $O(n_x n_y n^\omega)$  in total. In practice, we expect  $O(n_x n_y n)$
- Space
  - $O(nm)$
  - $n$  is the number of simplices
  - $m$  is the maximum dimension of faces in the simplicial complex

# Running time

| PART                  | SIMPLICES | GRID SIZE      | TIME(s) |
|-----------------------|-----------|----------------|---------|
| BARCODE DECOMPOSITION | 1025      | $20 \times 20$ | 0.00038 |
|                       | 10007     | $20 \times 20$ | 0.00054 |
|                       | 100000    | $20 \times 20$ | 0.00045 |
|                       | 1000007   | $20 \times 20$ | 0.00054 |
| TOTAL TIME            | 1025      | $20 \times 20$ | 0.31    |
|                       | 10007     | $20 \times 20$ | 2.58    |
|                       | 100000    | $20 \times 20$ | 35.32   |
|                       | 1000007   | $20 \times 20$ | 495.63  |

| SIMPLICES | GRID SIZE      | TIME(s) |
|-----------|----------------|---------|
| 100000    | $10 \times 5$  | 4.19    |
| 100000    | $5 \times 10$  | 4.34    |
| 100000    | $10 \times 10$ | 8.29    |
| 100000    | $15 \times 10$ | 13.12   |
| 100000    | $10 \times 15$ | 12.83   |
| 100000    | $20 \times 20$ | 34.47   |

# Conclusion

- Introduce a new algorithm for rank invariant.
- Decent running time in practice. There exists one algorithm whose theoretical time complexity  $O(n^\omega + n_x n_y n)$  does match with the practical time complexity we observed. With that time complexity, their algorithm is much more complicated than ours, so we already have the best possible running time in practice.
- Future work: higher dimension, degree 0 homology case...

**Thank you!**