Homework 2 for AMAT840: Multiparameter Persistent Homology

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1 Review of Abstract Algebra and Homology Coefficients

Exercise 1.1 (2.39). *(i) Consider R-modules $M' \subset M$ and $N' \subset N$.Given a homomorphism $f : M \to N$ such that $f(M') \subset N'$, show that f induces a homomorphism $f : M/M' \to N/N'$, given by $f[m] = [f(m)]$. (In particular, check that this definition does not depend on the choice of representative of the equivalence class $[m]$, hence is well defined.)*
(ii) Show that for any chain map $f : C \to D$ and $i \in \mathbb{N}$, the map f_i induces a homomorphism $H(f_i) : H_i(C) \to H_i(D)$, given by $H(f_i)[x] = [f_i[x]]$.

Proof. (i) Assume $m, m' \in M'$ and $m \sim m'$. Then we have $m - m' \in M'$, so $f(m - m') = f(m) - f(m') \in N'$ according to the definition of modules homomorphism and the known conditions. So $f(m) \sim f(m')$ and the induced homomorphism is well defined.

(ii)

$$\begin{tikzcd} \cdots \arrow[r, "\alpha_3"] & C_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & C_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & C_0 \arrow[d, "f_0"] \\ \cdots \arrow[r, "\beta_3"] & D_2 \arrow[r, "\beta_2"] & D_1 \arrow[r, "\beta_1"] & D_0 \end{tikzcd}$$

From the definition of homology group, $H_i(C) = ker(\alpha_i)/im(\alpha_{i+1})$ and $H_i(D) = ker(\beta_i)/im(\beta_{i+1})$. It's obvious that $im(\alpha_{i+1}) \subset ker(\alpha_i)$ and $im(\beta_{i+1}) \subset ker(\beta_i)$ since $\alpha_i \circ \alpha_{i+1} = 0$ and $\beta_i \circ \beta_{i+1} = 0$.
For $\forall x \in im(\alpha_{i+1})$, $\exists y \in C_{i+1}$ such that $\alpha_{i+1}(y) = x$, so $f_i \circ \alpha_i(y) = f_i(x)$. Since the commutativity of the diagram, we have $\beta_{i+1} \circ f_{i+1} = f_i \circ \alpha_i$. Then $\beta_{i+1} \circ f_{i+1}(y) = f_i \circ \alpha_i(y) = f_i(x)$, so $f_i(x) \in im(\beta_{i+1})$, which means $f_i(im(\alpha_{i+1})) \subset im(\beta_{i+1})$. For the same reason (the commutativity of the diagram), we know that $f_i(ker(\alpha_i)) \subset ker(\beta_i)$. Now the homomorphism f_i satisfies all the conditions of (i), so $H(f_i)$ is well defined. □

*This is a course taught by Michael Lesnick from SUNY Albany.

2 1-Parameter Persistent Homology

Exercise 2.1 (4.10). *Show that the intervals in the barcode of an essentially discrete persistence module have one for the following two forms:*
1. $[a,b)$ for $a < b \in \mathbb{R} \cup \{\infty\}$.
2. $(-\infty, b)$, for $b \in \mathbb{R} \cup \{\infty\}$.

Proof. Given an essentially discrete persistence module $M : \mathbb{R} \to \mathbf{Vec}$, $M \circ j$ is a $\mathbb{Z}$-indexed module. We apply the structure theorem to $M \circ j$ and get $M \circ j \cong \bigoplus_{I \in \mathcal{B}_{M \circ j}} K^I$ where each $I \subset \mathbb{Z}$, so each I has one of the following forms:
$[a,b]$, $(a,b]$, $[a,b)$, (a,b), $(-\infty,b]$, $(-\infty,b)$, $(-\infty,+\infty)$, $[a,+\infty)$, $(a,+\infty)$

Since I is the subset of $\mathbb{Z}$, we know that
1. $[a,b] \cong [a,b+1)$
2. $(a,b] \cong [a+1,b+1)$
3. $[a,b) \cong [a,b)$
4. $(a,b) \cong [a+1,b)$
5. $(-\infty,b] \cong (-\infty,b+1)$
6. $(-\infty,b) \cong (-\infty,b)$
7. $(-\infty,+\infty) \cong (-\infty,+\infty)$
8. $[a,+\infty) \cong [a,+\infty)$
9. $(a,+\infty) \cong [a+1,+\infty)$

All the right parts have one for the two forms given in the exercise. If the left part is I, we denote the right part is $\overline{I}$, and $K^I \cong K^{\overline{I}}$.
From Prop 4.5, we can deduce that $M \circ j \cong \bigoplus_{I \in \mathcal{B}} K^{\overline{I}}$.

□

Exercise 2.2 (4.14). *Explicitly describe the equivalence classes of $\sim$ for the compatible set of bases given in Examples 4.12 and 4.13.*

Solution. In Exercise 4.12, the first equivalence class is $\{1 \in B_0, (1,0) \in B_1, 1 \in B_2\}$ and the second quivalence class is $\{(0,1)\}$.
In Exercise 4.13, the first equivalence class is $\{1, (1,1)\}$. The second equivalence class is $\{(-1,1), (2,2), 2\}$. The third equivalence class is $\{(1,0)\}$. □

Exercise 2.3 (4.16). *Consider the filtration $F : \{0,1\} \to$ **Simp** where*

$$F_0 = \{[a],[b],[c],[d],[a,b],[b,c],[c,d],[a,d],[a,c]\},$$

$$F1 = F_0 \cup \{[a,b,c],[e],[a,e],[b,e]\}.$$

(i) Sketch F.
(ii) Explicitly compute a set of compatible bases for $H_1 F$.
(iii) What is $\mathcal{B}(H_1 F)$?

Solution. (i)

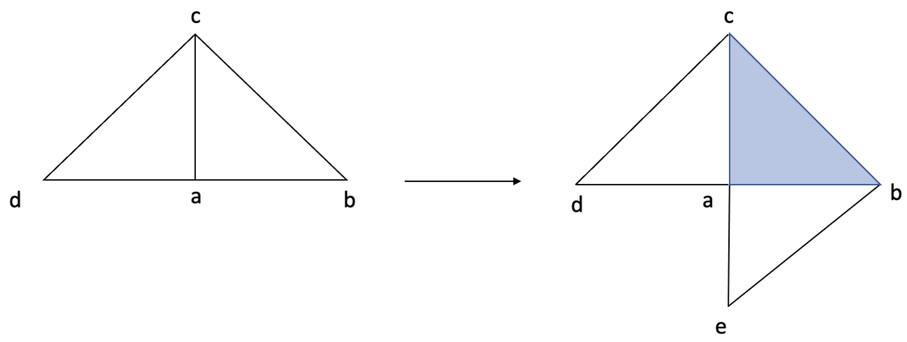


Figure 2.1: Geometry realization of F

(ii) H_1F is the functor: $\{0,1\} \to$ **Vec**. Let $K = \mathbb{R}$, the functor is:

$$\mathbb{R}^2 \xrightarrow{\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}} \mathbb{R}^2$$

and the set of compatible bases for H_1F is $B_0 = \{(1,1), (-1,1)\}$ and $B_1 = \{(1,0), (2,2)\}$.

(iii) $\mathcal{B}(H_1F) = \{[0,1], \{0\}, \{1\}\}$. □

3 Filtrations in Topological Data Analysis

Exercise 3.1 (5.14). *Prove that if F and G are weakly equivalent, then $H_iF \cong H_iG$ for all $i \geq 0$.*

Proof. For the objectwise homotopy equivalence $\mathcal{N} : W_1 \to F$, each $x \in Ob(\mathcal{C})$ we have $\mathcal{N}_x : W_1(x) \to F(x)$ is a homotopy equivalence. Then $\mathcal{N}_x$ can induce a homomorphism $\eta_x : Hi(W_1(x)) \to Hi(F(x))$ because homology is invariant under homotopy equivalence. So $Hi(W_1(x)) \cong Hi(F(x))$ for each $x \in Ob((C))$, which means the natural transformation η is a natural isomorphism. So $HiW_1 \cong HiF$, and with the same reason we have $HiF \cong HiW_1 \cong \cdots \cong HiW_n \cong H_iG$. Finally we have $HiF \cong HiG$ because of the transitivity. □

Exercise 3.2 (5.16). *Consider the following diagrams $F, G : \{0,1\} \to$ **Top**:*

$$F : \{0,1\} \hookrightarrow [0,1] \qquad G : \{0,1\} \xrightarrow{0} \{0\}$$

(i) Give an objectwise homotopy equivalence $F \to G$.
(ii) Show that there is no objectwise homotopy equivalence $G \to F$, and hence no homotopy equivalence between F and G.

Proof. (i) Let N be the natural transformation: $F \to G$. Define $N_0 = Id_{\{0,1\}}$ and $N_1 = 0$. Now N_x for each $x \in \{0,1\}$is a homotopy equivalence, so N is an objectwise homotopy equivalence.

(ii) Let $\gamma \in hom(0,1)$ and consider the following diagram

$$\begin{tikzcd} G_0 \arrow[r, "G_\gamma"] \arrow[d, "N_0"] & G_1 \arrow[d, "N_1"] \\ F_0 \arrow[r, "F_\gamma"] & F_1 \end{tikzcd}$$

which is equal to

$$\begin{tikzcd} \{0,1\} \arrow[r, "0"] \arrow[d] & \{0\} \arrow[d] \\ \{0,1\} \arrow[r, "i"] & [0,1] \end{tikzcd}$$

If we want N_0 to be homotopy equivalence, N_0 should be at least a bijection (otherwise two separate 2 points is impossible to deform continuously into one point). Without loss of generality, let $N_0(0) = 0$ and $N_0(1) = 1$. Then we find that $F_\gamma \circ N_0(0) = 0 \neq F_\gamma \circ N_0(1) = 1$ but $N_1 \circ G_\gamma(0) = N_1 \circ G_\gamma(1)$, so the diagram is not commutative and hence there is no objectwise homotopy equivalence $G \to F$.

□

Exercise 3.3 (5.23). *Check that $N(U)_\gamma$ is indeed a simplicial map.*

Proof. Let $x, y \in Ob\mathcal{C}$ and $\gamma \in hom(x,y)$.
$U_x = \{G_x \mid G \in U\}$ and $N(U)_x = N(U_x) = \{\sigma \subset U_x \; finite \mid \cap_{S\in\sigma} \neq \varnothing\}$.
We define $N(U)_\gamma$ as $N(U)_\gamma(G_x) = G_y$. We have to prove that G_y is not empty if G_x is not empty.
Let $z \in G_x$. Since $\gamma \in hom(x,y)$, we have $G(\gamma) \in hom(G_x, G_y)$, so for $z \in G_x$, $G(\gamma)(z) \in G_y$, which means $G_y \neq \varnothing$. Then let $\sigma = [G_x^1, G_x^2, ..., G_x^n]$, we define $N(U)\gamma(\sigma) = [G_y^1, G_y^2, ..., G_y^n]$ where each $G^i \in U$ for $i \in \{1, 2, ..., n\}$.
Then we need to prove that $\cap_{i=1}^n G_x^i \neq \varnothing$ implies $\cap_{i=1}^n G_y^i \neq \varnothing$. Let $z \in \cap_{i=1}^n G_x^i$, for each $i \in \{1, 2, ..., n\}$ we have $G_\gamma^i(z) = F_\gamma|_{G_x}(z) = F_\gamma(z) \in G_y^i$ because G^i is the subfunctor of F. $F_\gamma(z) \in G_y^i$ for all i, so $F_\gamma(z) \in \cap_{i=1}^n G_y^i \neq \varnothing$.
(The injectivity is because of the definition of the subfunctor.) □

Exercise 3.4 (5.27). *Show that for $X = \{0,1\} \subset \mathbb{R}$ there is not objectwise homotopy equivalence $O(X) \to \check{C}ech(X)$.*

Proof. Let $\gamma \in hom(\frac{1}{3}, \frac{1}{2})$ and consider the following diagram:

$$\begin{tikzcd} O(X)_{1/3} \arrow[r, "O(X)_\gamma"] \arrow[d, "N_{1/3}"] & O(X)_{1/2} \arrow[d, "N_{1/2}"] \\ \check{C}ech(X)_{1/3} \arrow[r, "\check{C}ech(X)_\gamma"] & \check{C}ech(X)_{1/2} \end{tikzcd}$$

which is equal to

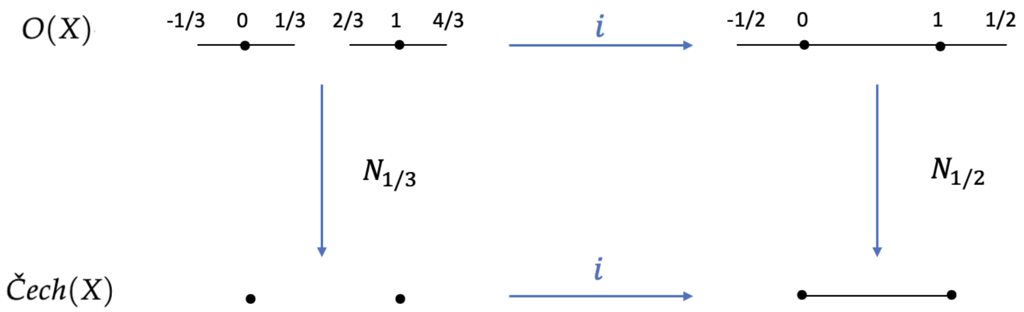


Figure 3.1: Geometry realization of F

If $N_{1/3}$ is homotopy equivalence, $[-1/3, 1/3]$ and $[2/3, 4/3]$ should be mapped to different points (otherwise these two separate intervals is impossible to deform continuously into two points). Without the loss of generality, let $N_{1/3}([-1/3, 1/3]) = 0$ and $N_{1/3}([2/3, 4/3]) = 1$. Then we have $\check{C}ech(X)_\gamma \circ N_{1/3}([-1/3, 1/3]) = 0$ and $\check{C}ech(X)_\gamma \circ N_{1/3}([2/3, 4/3]) = 1$. If this diagram is commutative, it must satisfy that $N_{1/2} \circ O(X)_\gamma([-1/3, 1/3]) = N_{1/2}([-1/3, 1/3]) = 0$ and $N_{1/2} \circ O(X)_\gamma([1/3, 4/3]) = N_{1/2}([2/3, 4/3]) = 1$. Actually it should satisfy that $N_{1/2}((-1/2, 1/2)) = 0$ and $N_{1/2}((1/2, 3/2)) = 1$ because for $\forall r < 1/2$ and $\gamma \in hom(r, 1/2)$, the diagram should be commutative. But in this way $N_{1/2}$ is impossible to be a homotopy equivalence because an interval is impossible to deform continuously into two points. Hence there is not objectwise homotopy equivalence $O(X) \to \check{C}ech(X)$.

$\square$

Exercise 3.5 (5.28). *By adapting Exercise 5.27, give an example of a pair of filtrations X, Y which are weakly equivalent, but for which there exists no objectwise homotopy equivalence from one to the other.*

Solution. For $X = \{(-1, 0), (1, 0), (0, \sqrt{3})\} \subset \mathbb{R}^2$. $O(X) \simeq \check{C}ech(X)$ because of the Nerve theorem, but there exists no objectwise homotopy equivalence from one to the other. $\square$

Exercise 3.6 (5.29). *Let $X = \{(0, 0), (2, 0), (0, 1)\}$. Give an explicit expression for $\check{C}ech(X)$, i.e. specify $\check{C}ech(X)_r$ for each $r \geq 0$.*

Solution. $0 \leq r < 1/2$

$1/2 \leq r < 1$

$1 \leq r < \sqrt{5}/2$

$r \geq \sqrt{5}/2$

□

Exercise 3.7 (5.30)**.** *Using Corollary 5.26, prove that $O(X)$ is essentially discrete for any finite $X \subset \mathbb{R}^n$.*

Proof. Well...I don't know how to prove it using Corollary 5.26, but it seems that we can indeed construct a monotonic injection $j : \mathbb{Z} \to \mathbb{R}$ such that $\lim_{z \to \pm\infty} j(z) = \pm\infty$ and for all $z \in \mathbb{Z}$ and $r \leq s \in [j(z), j(z+1))$, $M_{r,s}$ is an isomorphism.

Here is the map j:
When $z \leq 0$, $j(z) = z$;
When $z > 0$, let $z = 1$ and we increase r from 0 to $+\infty$. Each time when one or some new simplices are created, let $j(z) = r$ and $z = z + 1$. Then continue to increase r until no more new simplices will be created. Now $r = r'$, and let $j(z) = z$ when $z > r'$.
It's obvious that $\lim_{z \to \pm\infty} j(z) = \pm\infty$.
For all $z \in \mathbb{Z}$, and $r \leq s \in [j(z), j(z+1))$, $M_{r,s}$ is the identity map, so it's also an isomorphism.

□

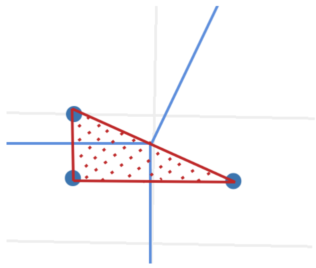
Exercise 3.8 (5.39)**.** *Show that for $P \subset \mathbb{R}^n$, not necessarily generic, $D(P)$ can have as many as $2^{|P|} - 1$ simplices.(HINT: assume that $P \subset S^{n-1}$.)*

Proof. Assume that $P \subset S^{n-1}$ and $p \in P$, then $V(p) = \{y \in \mathcal{R}^n \mid \|y - p\| \leq \|y - x\|, \forall x \in P\}$. For $\forall p \in P$, we have $0 \in V(p)$ because $1 = \|0 - p\| \leq \|0 - x\| = 1$ for $\forall x \in P$. Then we $\forall \sigma \subset V(P)$ and $\sigma \neq \varnothing$, $0 \in \cap_{V(p) \in \sigma} V(p) \neq \varnothing$, so $\sigma \in D(P)$. The number of nonempty subsets of $V(P)$ is $2^{|P|} - 1$, so $D(P)$ has $2^{|P|} - 1$ simplies.

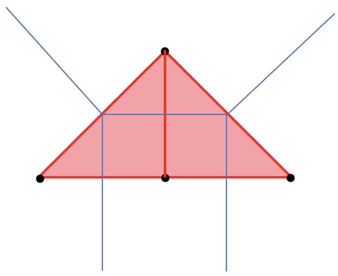
Then we prove that it's impossible for $D(P)$ to have more simplices than $2^{|P|} - 1$.
$|2^{V(P)}| = |2^P| = 2^{|P|}$, so $V(P)$ has at most $2^{|P|} - 1$ non-empty subsets. Since $D(P) = \mathcal{N}(V(P))$ is the subset of power set of $V(P)$ and each element is non-empty, so $D(P)$ can have no more than $2^{|P|} - 1$ elements in it. □

Exercise 3.9 (5.42)**.** *For each of the following sets $X \subset \mathbb{R}^2$, sketch $Vor(X)$ and give explicit expressions for $Del(X)$.*
a. $X = \{(0,0), (2,0), (0,1)\}$,
b. $X = \{(0,0), (1,0), (2,0), (1,1)\}$.

Solution. a. The blue part is $Vol(X)$.The red part is $Del(x)$, which is a filled right triangle.



b. The blue part is $Vol(X)$.The red part is $Del(x)$, which is a filled triangle with a segment in it.



□

Exercise 3.10 (5.44). *For $P \subset \mathbb{R}^n$, clearly $\check{C}ech(P)_r \subset Rips(P)_r$ for all $r \in [0, \infty)$. Perform the easy check that conversely, $Rips(P)_r \subset \check{C}ech(P)_{2r}$.*

Proof. For $[x_1, x_2, ..., x_m] \in Rips(P)_r$, we have $|x_i - x_j| \leq 2r$ for $\forall i, j \in \{1, 2, ..., m\}$. So $|x_1 - x_i| \leq 2r$ for each i. Then $x_1 \in \cap_{i=1}^{m} B(x_i, 2r) \neq \emptyset$, so $[x_1, x_2, ..., x_m] \in \check{C}ech(P)_{2r}$. □

Exercise 3.11 (5.46). *Give explicit expressions for the Vietoris-Rips filtrations of the following sets $X \subset \mathbb{R}^2$:*
a. $X = \{(0,0), (2,0), (0,1)\}$,
b. $X = \{(0,0), (2,0), (0,2), (2,2)\}$.

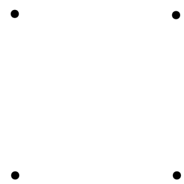
Solution. a. $0 \leq r < 1/2$

$1/2 \leq r < 1$

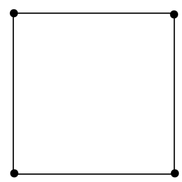
$1 \leq r < \sqrt{5}/2$

$r \geq \sqrt{5}/2$

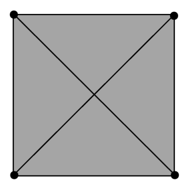
b. $r < 1$



$1 \leq r < \sqrt{2}$



$r \geq \sqrt{2}$.



For $r \geq \sqrt{2}$, let $a = (0,0), b = (2,0), c = (2,2), d = (0,2)$.
$Rips(X)_r = \{[a],[b],[c],[d],[a,b],[a,c],[a,d],[b,c],[b,d],[c,d],[a,b,c],[a,c,d],[a,b,d],[b,c,d],[a,b,c,d]\}$ □

Exercise 3.12 (5.47). *Consider $P = \{(-3,0),(3,0),(0,1)\}$.*
(i)What is the radius at which the edge $[(-3,0),(3,0)]$ appears in Čech(P)?
(ii) What is the radius at which the edge $[(-3,0),(3,0)]$ appears in Del(P)?

Solution. (i) When the radius is 3 the edge $[(-3,0),(3,0)]$ appears in *Čech*(P).
(ii) When the radius is 5 the edge $[(-3,0),(3,0)]$ appears in *Del*(P). □