## Homework 3 for AMAT840: Multiparameter Persistent Homology

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## 1 Algebraic Aspects of Persistence Modules

**Exercise 1.1** (6.8). For which  $a, b \in \mathbb{N}^n$  is there a non-zero morphism (i.e., natural transformation)  $Q^a \to Q^b$ ?

*Solution.* Let  $x, y \in \mathbb{N}^n$  and consider the following diagram:

$$Q_{x}^{a} \xrightarrow{Q_{x,y}^{a}} Q_{y}^{a}$$

$$\downarrow^{N_{x}} \qquad \downarrow^{N_{y}}$$

$$Q_{x}^{b} \xrightarrow{Q_{x,y}^{b}} Q_{y}^{b}$$

If the diagram looks like this:

$$\begin{array}{ccc}
\mathcal{K} & \xrightarrow{Id} & \mathcal{K} \\
\downarrow 0 & & \downarrow Id \\
0 & \xrightarrow{0} & \mathcal{K}
\end{array}$$

it's impossible to commute. So, if  $Q_x^a = \mathcal{K}$ ,  $Q_x^b$  must be  $\mathcal{K}$ , which means  $a \geq b$ .

**Exercise 1.2** (6.13). For which  $a, b \in \mathbb{N}^d$  do we have  $Q^a \subset Q^b$ .

*Solution.* For  $b \le a$ , i.e.  $b_i \le a_i$  where  $i \in \{1, 2, ..., d\}$ , we have  $Q^a \subset Q^b$ . Because in this case, for  $x \ge a$ ,  $Q_x^a = K = Q_x^b$  and for x < a,  $Q_x^a = 0 \subset Q_x^b$ .

**Exercise 1.3** (6.15).  $f: M \to N$  is a morphism of persistence modules. Check that ker f and im f are in fact well-defined submodules.

<sup>\*</sup>This is a course taught by Michael Lesnick from SUNY Albany.

*Proof.* Consider the following commutative diagram:

$$M_{x} \xrightarrow{M_{\gamma}} M_{y}$$

$$\downarrow f_{x} \qquad \downarrow f_{y}$$

$$N_{x} \xrightarrow{N_{\gamma}} N_{y}$$

First we prove that  $M_{\gamma}((kerf)_x) \subset (kerf)_y$ :

For  $\forall m \in (kerf)_x$ ,  $f_x(m) = 0$ . According to the commutativity of the diagram, we have  $N_\gamma \circ f_x(m) = f_y \circ M_\gamma(m) = 0$ , so  $M_\gamma(m) \in (kerf)_y$ .

Then we prove that  $N_{\gamma}((imf)_x) \subset (imf)_{\eta}$ :

For  $\forall n \in (imf)_x$ ,  $\exists m \in M_x$  such that  $f_x(m) = n$ . According to the commutativity of the diagram, we have  $N_\gamma \circ f_x(m) = f_y \circ M_\gamma(m)$ , so  $f_y \circ M_\gamma(m) = N_\gamma(n)$  and  $N_\gamma(n) \in (imf)_y$ .

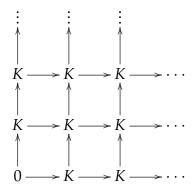
**Exercise 1.4** (6.18). Given submodules W and W' of a persistence module M, let  $W + W' \subset M$  be given by

$$(W + W')_z = \{w + w' \mid w \in W_z, w \in W'_z\}.$$

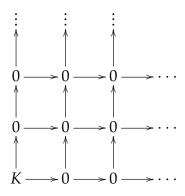
This is clearly also a submodule of M.

- 1. Draw the diagram of vector spaces  $Q^{0,1} + Q^{1,0}$ .
- 2. Up to isomorphism, what is the module  $Q^{0,0}/(Q^{0,1}+Q^{1,0})$ ?

Solution. 1.



2.



**Exercise 1.5** (6.20). Give an example of an  $\mathbb{N}$ -indexed persistence module whose vector spaces have dimension at most one, and for which any minimal generating set has two elements.

Solution.

$$K \xrightarrow{0} 0 \xrightarrow{0} K \xrightarrow{Id} K \xrightarrow{Id} \cdots$$

**Exercise 1.6** (6.21). Give an example of an N-indexed persistence module which is not finitely generated, but whose vector spaces are all of finite dimension.

Solution.

$$K \xrightarrow{0} 0 \xrightarrow{0} K \xrightarrow{0} 0 \xrightarrow{0} K \xrightarrow{0} 0 \longrightarrow \cdots$$

Exercise 1.7 (6.34). Prove Proposition 6.33:

Let  $\gamma: V \to W$  be a linear map of finite-dimensional vector spaces, and let  $A = \{a_1, \dots, a_n\}$ and  $B = \{b_1, \dots, b_m\}$  be bases for V and W, respectively. For any  $c \in K$ ,

- If A' is obtained from A by replacing  $a_i$  with  $a_i + ca_j$ , then  $[\gamma]^{B,A'}$  is obtained from  $[\gamma]^{B,A}$ by adding  $c[\gamma]_{*,i}^{B,A}$  to  $c[\gamma]_{*,i}^{B,A}$ .
- Similarly, if B' is obtained from B by replacing  $b_i$  with  $b_i + cb_j$ , then  $[\gamma]^{B',A}$  is obtained from  $[\gamma]^{B,A}$  by subtracting  $c[\gamma]_{i,*}^{B,A}$  from  $[\gamma]_{i,*}^{B,A}$ .

*Proof.* • If A' is obtained from A by replacing  $a_i$  with  $a_i + ca_j$ 

For 
$$k \neq i$$
,  $[\gamma]_{*k}^{B,A'} = [\gamma(a_k)]^B = [\gamma]_{*k}^{B,A}$ .

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For  $k = i$ ,  $[\gamma]_{*,k}^{B,A'} = [\gamma(a_i + ca_j)]^B = [\gamma(a_i)]^B + c[\gamma(a_j)]^B = [\gamma]_{*,i}^{B,A} + c[\gamma]_{*,i}^{B,A}$ .

• if B' is obtained from B by replacing  $b_i$  with  $b_i + cb_i$ For  $\forall k \in \{1, 2, \dots, n\},\$ 

$$\gamma(a_k) = \sum_{t=1}^{m} [\gamma]_{t,k}^{B,A} b_t = \sum_{t=1}^{m} [\gamma]_{t,k}^{B',A} b_t'$$
(1)

where  $b'_t = b_t$  if  $t \neq i$  and  $b'_t = b_i + cb_j$  if t = i, so

$$\gamma(a_k) = \sum_{t \neq i} [\gamma]_{t,k}^{B',A} b_t + [\gamma]_{i,k}^{B',A} (b_i + cb_j) = \sum_{t \neq i} [\gamma]_{t,k}^{B',A} b_t + ([\gamma]_{j,k}^{B',A} + c[\gamma]_{i,k}^{B',A}) b_j$$
 (2)

Compare (1) and (2), we have  $[\gamma]_{t,k}^{B,A} = [\gamma]_{t,k}^{B',A}$  if  $t \neq j$ , and

$$[\gamma]_{j,k}^{B,A} = [\gamma]_{j,k}^{B',A} + c[\gamma]_{i,k}^{B',A} = [\gamma]_{j,k}^{B',A} + c[\gamma]_{i,k}^{B,A}$$
(3)

Therefore,  $[\gamma]_{j,k}^{B',A} = [\gamma]_{j,k}^{B,A} - c[\gamma]_{i,k}^{B,A}$  for any k, which means  $[\gamma]^{B',A}$  is obtained from  $[\gamma]^{B,A}$  by subtracting  $c[\gamma]_{i,*}^{B,A}$  from  $[\gamma]_{j,*}^{B,A}$ .

## **Computing Persistent Homology** 2

**Exercise 2.1** (8.3). Supposing  $M: \mathbb{N} \to Vec$  has the following presentation matrix (over  $\mathbb{Z}/2\mathbb{Z}$ ), compute  $\mathcal{B}_M$ .

Solution.

So 
$$\mathcal{B}_M = \{[2,4), [0,6), [1,+\infty)\}.$$

**Exercise 2.2** (8.13). Consider the simplicial filtration  $F : \mathbb{N} \to Vec$  specified by the table below. Compute all barcodes of  $H_iF$  using Theorem 8.7. Also compute the reduced presentation matrix Q' for  $\bigoplus_i H_iF$  appearing in the proof of Theorem 8.7.

SIMPLEX	[1]	[2]	[3]	[4]	[2 3]	[1 2]	[1 3]	[2 4]	[1 2 3]	[3 4]	[2 3 4]
BIRTH INDEX	1	2	3	4	5	6	7	8	9	10	11

Solution.

$$Q' = \begin{array}{c} 5 & 6 & 8 & 9 & 11 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 & 0 \\ 7 & 0 & 0 & 0 & 1 & 0 \\ 10 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$$\mathcal{B}_0(F) = \{[3,5), [2,6), [4,8), [1,+\infty)\}$$
  
 $\mathcal{B}_1(F) = \{[7,9), [10,11)\}$