Homework 4 for AMAT840: Multiparameter Persistent Homology

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1 Stability of Persistent Homology

1.1 Stability of Sublevel Persistence

Theorem 1.1 ([CCSG$^+$09b]). *For any finite metric spaces P, Q and $i \geq 0$, we have*

$$d_B(\mathcal{B}_{H_i Rips(P)}, \mathcal{B}_{H_i Rips(Q)}) \leq d_{GH}(P, Q)$$

Proof. It is a standard fact that for P and Q compact, the infimum appearing in the definition of $d_{GH}(P, Q)$ is in fact a minimum(?). Moreover, it is clear that for the tripet (Z, γ, κ) realizing this minimum, we may take $Z = \gamma(P) \cup \kappa(Q)$. In particular, if P and Q are finite, Z is also finite because isometric embeddings are injective. (If $\gamma(x) = \gamma(y)$, then $d(x, y) = d(\gamma(x), \gamma(y)) = 0$, so $x = y$.)

We now fix such a triplet with Z finite, and ordering the elements of Z arbitrarily, write $Z = \{z_1, \cdots, z_{|Z|}\}$. We define a map $f : Z \to \mathbb{R}^{|Z|}$ by

$$f(y) = (|y - z_1|, |y - z_2|, \cdots, |y - z_{|Z|}|).$$

Let us consider $(\mathbb{R}^{|Z|}, \ell_\infty)$ as a metric space. Then for $\forall x, y \in Z$,

$$d(f(x), f(y)) = \max_{i=1,\cdots,|Z|} (||x - z_i| - |y - z_i||) = |x - y|.$$

So f is an isometric embedding. Since the Hausdorff distance is preserved by insometric embedding, we have

$$d_H(f \circ \gamma(P), f \circ \kappa(Q)) = d_H(\gamma(P), \kappa(Q)) = d_{GH}(P, Q).$$

Hausdorff Stability gives that:

$$d_B(\mathcal{B}_{H_i O(f\circ\gamma(P))}, \mathcal{B}_{H_i O(f\circ\kappa(Q))}) \leq d_H(f \circ \gamma(P), f \circ \kappa(Q)) = d_{GH}(P, Q).$$

*This is a course taught by Michael Lesnick from SUNY Albany.

From Colollary 5.26, we have $O(f \circ \gamma(P)) \simeq \check{C}ech(f \circ \gamma(P))$.

In metric space $(\mathbb{R}^{|Z|}, \ell_\infty)$, we have $\check{C}ech(f \circ \gamma(P)) = Rips(f \circ \gamma(P))$. (The proof is in the following exercise)

$f \circ \gamma$ is isometric, so $Rips(f \circ \gamma(P)) \cong Rips(P)$. Thus

$$O(f \circ \gamma(P)) \simeq \check{C}ech(f \circ \gamma(P)) = Rips(f \circ \gamma(P)) \cong Rips(P),$$

$$\mathcal{B}_{H_i O(f \circ \gamma(P))} = \mathcal{B}_{H_i Rips(P)}$$

and similarly for Q. So we have

$$d_B(\mathcal{B}_{H_i Rips(P)}, \mathcal{B}_{H_i Rips(Q)}) \leq d_{GH}(P, Q)$$

□

Exercise 1.2 (9.10). *Check that Čech and Rips complexes are equal in* $(\mathbb{R}^n, \ell_\infty)$*, as required by the above proof.*

Proof. Let finite $P \subseteq \mathbb{R}^n$. We want to prove $\check{C}ech(P)_r = Rips(P)_r$ for $\forall r \geq 0$.

If $[x_1, \cdots, x_k] \in \check{C}ech(P)_r$ where $k \geq 0$, then $\bigcap_{i=1}^{k} B(x_i, r) \neq \varnothing$. So for $\forall x_i, x_j$ where $1 \leq i \leq j \leq k$, we have $B(x_i, r) \cap B(x_j, r) \neq \varnothing$, so $\|x_i - x_j\|_\infty \leq 2r$ and $[x_1, \cdots, x_k] \in Rips(P)$.

If $[x_1, \cdots, x_k] \in Rips(P)_r$ where $k \geq 0$, then for $\forall i$ such that $1 \leq i \leq j \leq k$, we have $\|x_i - x_j\|_\infty \leq 2r$. Let $x_i = (x_i^1, x_i^2, \cdots, x_i^n) \in \mathbb{R}^n$ and $x_j = (x_j^1, x_j^2, \cdots, x_j^n) \in \mathbb{R}^n$, we have $\max\limits_{d=1,\cdots,n} |x_i^d - x_j^d| \leq 2r$. Let $x^d = (\max\limits_{i=1,\cdots,k} x_i^d + \min\limits_{i=1,\cdots,k} x_i^d)/2$ for all $d \in \{1, \cdots, n\}$ and $x = (x^1, \cdots, x^n)$. Now we have $|x^d - x_i^d| \leq r$ for all $i \in \{1, \cdots, k\}$ and all $d \in \{1, \cdots, n\}$, so $\|x - x_i\|_\infty \leq r$ for all $i \in \{1, \cdots, d\}$, which means $x \in \bigcap_{i=1}^{k} B(x_i, r) \neq \varnothing$, so $[x_1, \cdots, x_k] \in \check{C}ech(P)_r$.

Therefore, we prove that $\check{C}ech(P) = Rips(P)$. □

1.2 Algebraic Stability

Theorem 1.3 (Isometry Theorem). *For any persistence modules M, N, we have*

$$d_B(\mathcal{B}_M, \mathcal{B}_N) = d_I(M, N).$$

To prove this theorem, one first proves the ***(forward) algebraic stability theorem*** *[CCSG+09a, CDSGO16]:*

$$d_B(\mathcal{B}_M, \mathcal{B}_N) \leq d_I(M, N),$$

and then the ***converse algebraic stability*** *[Les15]*

$$d_B(\mathcal{B}_M, \mathcal{B}_N) \geq d_I(M, N).$$

Exercise 1.4 (Proof of converse algebraic stability).

(i) Show that converse algebraic stability holds in the special case that M and N are both interval modules or trivial modules.

(ii) Check that if M and N are δ-interleaved and M' and N' are also δ-interleaved, then $M \oplus M'$ and $N \oplus N'$ are δ-interleaved.

(iii) Show that converse algebraic stability follows from these two facts.

Proof. (i) Let $M = K^{I_1}$ and $N = K^{I_2}$ where intervals $I_1, I_2 \subseteq \mathbb{R}$ and $I_1 = [a_1, b_1), I_2 = [a_2, b_2)$, $a_1 \leq b_1, a_2 \leq b_2$. Let $d_B(\mathcal{B}_M, \mathcal{B}_N) = \delta = \max\{|a_1 - a_2|, |b_1 - b_2|, |b_1 - a_1|/2, |b_2 - a_2|/2\}$.

Let $\gamma : M \to N^\delta$ be the morphism such that $\gamma_r = Id_{\mathbb{R}}$ if $r \in I_1$ and $r + \delta \in I_2$, otherwise $\gamma_r = 0$. Let $\kappa : N \to M^\delta$ be the morphism such that $\kappa_r = Id_{\mathbb{R}}$ if $r \in I_2$ and $r + \delta \in I_1$, otherwise $\kappa_r = 0$.

First, we prove that γ and κ are truly morphisms. γ and κ are symmetric, so we only prove the case for γ. For $\forall x, y \in \mathbb{R}$, consider the following diagram:

$$\begin{tikzcd} M_x \arrow[r] \arrow[d, "\gamma_x"] & M_y \arrow[d, "\gamma_y"] \\ N_{x+\delta} \arrow[r] & N_{y+\delta} \end{tikzcd}$$

If $\delta = |a_2 - a_1|$ or $\delta = |b_2 - b_1|$:
1) If $x, y \in I_1$, then $x + \delta, y + \delta \in I_2$, so $M_{x,y} = N_{x+\delta,y+\delta} = \gamma_x = \gamma_y = Id_{\mathbb{R}}$ and the above diagram commutes.
2) If $x \notin I_1$, then $M_x = 0$ and $\gamma_x = 0$, so the above diagram commutes.
3) If $y \notin I_1$, then $M_y = 0$ and $\gamma_y = 0$, so the above diagram commutes.

If $\delta = |b_1 - a_1|/2$ or $\delta = |b_2 - a_2|/2$, we have $\delta \geq |a_2 - a_1|$ and $\delta \geq |b_2 - b_1|$:
1) If $x, y \in I_1$, which means $M_x = M_y = \mathbb{R}$, then the only case that the diagram does not commute is when $N_{x+\delta} = 0$ but $N_{y+\delta} = \mathbb{R}$. But if $N_{x+\delta} = 0$, we have $a_1 \leq x \leq x + \delta < a_2$ which implies that $|a_2 - a_1| > \delta$. It's a contradiction for the range of δ.
2) If $x \notin I_1$, then $M_x = 0$ and $\gamma_x = 0$, so the above diagram commutes.
3) If $y \notin I_1$, then $M_y = 0$ and $\gamma_y = 0$. The only case that the diagram does not commute is when $M_x = N_{x+\delta} = \mathbb{R}$, then we have $a_1 \leq x < b_1 \leq y \leq y + \delta < b_2$ and $a_2 \leq x + \delta < y + \delta < b_2$, which imply that $b_1 \leq y \leq y + \delta < b_2$ and $|b_2 - b_1| > \delta$. It's a contradiction for the range of δ, so the diagram commutes.

Second, we prove that M and N are δ-interleaved, which means $\kappa^\delta \circ \gamma = \phi^{M,2\delta}$ and $\gamma^\delta \circ \kappa = \phi^{N,2\delta}$.
$\forall r \geq 0$, $\phi_r^{M,2\delta} : M_r \to M_{r+2\delta}$ and $(\kappa^\delta \circ \gamma)_r : M_r \to N_{r+\delta} \to M_{r+2\delta}$. We need to prove that $r, r + 2\delta \in I_1 \Rightarrow r + \delta \in I_2$. Since $r, r + 2\delta \in I_1$, we have $a_1 \leq r \leq r + 2\delta < b_1$. Then $|b_1 - a_1| > 2\delta$, so $|a_2 - a_1| \leq \delta$ and $|b_2 - b_1| \leq \delta$. Then we can deduce that $a_2 < a_1 + \delta \leq r + \delta \leq b_1 - \delta < b_2$, so $r + \delta \in I_2$.
It's similar for $\gamma^\delta \circ \kappa = \phi^{N,2\delta}$.

Therefore, F and G are δ-interleaved and $d_I(M, N) \leq \delta = d_B(\mathcal{B}_M, \mathcal{B}_N)$.

(ii) If M and N are δ-interleaved, there exists a pair of morphism $\gamma : M \to N^\delta$ and $\kappa : N \to M^\delta$ such that $\kappa^\delta \circ \gamma = \phi^{M,2\delta}$ and $\gamma^\delta \circ \kappa = \phi^{N,2\delta}$. If M' and N' are δ-interleaved, there also exists a pair of morphism $\gamma' : M' \to N'^\delta$ and $\kappa' : N' \to M'^\delta$ such that $\kappa'^\delta \circ \gamma' = \phi^{M',2\delta}$ and $\gamma'^\delta \circ \kappa' = \phi^{N',2\delta}$.

Let $\gamma \oplus \gamma' : M \oplus M' \to N \oplus N'$ such that $(\gamma \oplus \gamma')_r : M_r \oplus M'_r \to \gamma_r(M_r) \oplus \gamma'_r(M'_r) \subseteq N_r \oplus N'_r$. Similarly, let $\kappa \oplus \kappa' : N \oplus N' \to M \oplus M'$ such that $(\kappa \oplus \kappa')_r : N_r \oplus N'_r \to \kappa_r(M_r) \oplus \kappa'_r(M'_r) \subseteq N_r \oplus N'_r$. It's obvious that $\gamma \oplus \gamma'$ and $\kappa \oplus \kappa'$ are still morphisms.

Then we need to prove that $(\kappa^\delta \oplus \kappa'^\delta) \circ (\gamma \oplus \gamma') = \phi^{M \oplus M',2\delta}$ and $(\gamma^\delta \oplus \gamma'^\delta) \circ (\kappa \oplus \kappa') = \phi^{N \oplus N',2\delta}$.

For $\forall r \geq 0$, $((\kappa^\delta \oplus \kappa'^\delta) \circ (\gamma \oplus \gamma'))_r(M_r \oplus M'_r) = (\kappa^\delta \oplus \kappa'^\delta)(\gamma_r(M_r) \oplus \gamma'_r(M'_r)) = (\kappa^\delta \circ \gamma)_r(M_r) \oplus (\kappa'^\delta \circ \gamma')_r(M'_r) = \phi_r^{M,2\delta}(M_r) \oplus \phi_r^{M',2\delta}(M'_r) = \phi_r^{M \oplus M',2\delta}(M_r \oplus M'_r)$. It's similar for $(\gamma^\delta \oplus \gamma'^\delta) \circ (\kappa \oplus \kappa') = \phi^{N \oplus N',2\delta}$.

Therefore, $M \oplus M'$ and $N \oplus N'$ are δ-interleaved.

(iii) From structure theorem, there exist unique multiset of intervals $\mathcal{B}_M$ and $\mathcal{B}_N$ such that

$$M \cong \bigoplus_{I \in \mathcal{B}_M} K^I,$$

$$N \cong \bigoplus_{I \in \mathcal{B}_N} K^I.$$

If $d_B(\mathcal{B}_M, \mathcal{B}_N) = \delta$, then for $\forall I_1 \in \mathcal{B}_M$ and $\forall I_2 \in \mathcal{B}_N$, we have $d_B(I_1, I_2) \leq \delta$, so $d_I(K^{I_1}, K^{I_2}) \leq \delta$ from (i). Then $d_I(M, N) \leq \delta$ from (ii). Finally, we have $d_B(\mathcal{B}_M, \mathcal{B}_N) \geq d_I(M, N)$.

□

1.3 The Single-Morphism Approach to Algebraic Stability

This part is to prove forward algebraic stability.

Theorem 1.5 (Induced Matchings [BL13]). *Suppose $\mathcal{X}_f[b, d) = [b', d')$.*

(i) $b' \leq b < d' \leq d$.

(ii) If each interval in $\mathcal{B}_{\ker f}$ has length at most δ, then

$$|d - d'| \leq \delta$$

and $\mathcal{X}_f$ matches each interval in $\mathcal{B}_M$ of length greater than δ.

(iii) *Dually, if each interval in $\mathcal{B}_{coker f}$ has length at most δ, then*

$$|b - b'| \leq \delta$$

and $\mathcal{X}_f$ matches each interval in $\mathcal{B}_N$ of length greater than δ.

Partical Functoriality of Induced Matchings: Let **Barc** denote the category whose objects are barcodes and whose morphisms are matchings. The matchings $\mathcal{X}_f$ are not functorial (i.e. they do not define a functor $\mathbf{Vec}^{\mathbb{R}} \to \mathbf{Barc}$). However, they are functorial on the subcategory of $\mathbf{Vec}^{\mathbb{R}}$ consisting of only monomorphisms (i.e., morphisms that are injective at each index.) That is, if $j : M \to N$ and $j' : N \to O$ are monomorphisms of persistence modules, then $\mathcal{X}_{j' \circ j} = \mathcal{X}'_j \circ \mathcal{X}_j$. The same is true for epimorphisms.

Exercise 1.6. *Check that the induced matchings are indeed functorial on monomorphisms as claimed above.*

Proof. For $d \in \mathbb{R}$, consider all intervals of the form $[\cdot, d)$ in $\mathcal{B}_M$, $\mathcal{B}_N$ and $\mathcal{B}_O$. In each barcode, sort intervals of the form $[\cdot, d)$ by length in descending order. Then we have $b_1 \leq \cdots \leq b_m \leq d$ and for each $i \in \{1, \cdots, m\}$, $[b_i, d) \in \mathcal{B}_M$. Similarly, let $b'_1 \leq \cdots \leq b'_n \leq d$ and for each $i \in \{1, \cdots, n\}$, $[b'_i, d) \in \mathcal{B}_N$. Let $b''_1 \leq \cdots \leq b''_o \leq d$ and for each $i \in \{1, \cdots, o\}$, $[b''_i, d) \in \mathcal{B}_O$.

For monomorphism $j : M \to N$, we have $\mathcal{X}_j([b_i, d)) = [b'_i, d)$ for $i \leq \min(m, n)$. And for monomorphism $j' : N \to O$, we have $\mathcal{X}_{j'}([b'_i, d)) = [b''_i, d)$ for $i \leq \min(n, o)$. Thus $\mathcal{X}_{j'} \circ \mathcal{X}_j[b_i, d_i) = \mathcal{X}_{j'}([b'_i, d)) = [b''_i, d)$, for $i \leq \min(m, n, o)$. Obviously, $j' \circ j : M \to O$ is also monomorphism, so $\mathcal{X}_{j' \circ j}([b_i, d)) = [b''_i, d) = \mathcal{X}_{j'} \circ \mathcal{X}_j[b_i, d_i)$, for $i \leq \min(m, n, o)$. When $\min(m, n, o) < i \leq m$, $\mathcal{X}_{j' \circ j}([b_i, d)) = \mathcal{X}_{j'} \circ \mathcal{X}_j[b_i, d_i) = \emptyset$. □

Proposition 1.7 (Structure theorem for submodules and quotients)**.**

(i) *Given a monomorphism $j : M \to N$, $\mathcal{X}_j : \mathcal{B}_M \to \mathcal{B}_N$ matches each intervals in $\mathcal{B}_M$, and for each $[b, d) \in \mathcal{B}_M$, we have $\mathcal{X}_j[b, d) = [b', d)$ for some $b' \leq b$.*

(ii) *Given an epimorphism $q : M \to N$, $\mathcal{X}_q : \mathcal{B}_M \to \mathcal{B}_N$ matches each intervals in $\mathcal{B}_N$, and for each $[b, d) \in \mathcal{B}_N$, we have $\mathcal{X}_q[b, d') = [b, d)$ for some $d' \geq d$.*

Exercise 1.8. *Prove Proposition 1.7(i), as follows. Fix $b < d \in \mathbb{R} \cup \{\infty\}$.*

(i) *Using the fact that M and N are p.f.d., show that there exists $b < d' < d$ such that neither $\mathcal{B}_M$ or $\mathcal{B}_N$ have interval $[y, z)$ with $y \leq b$ and $d' < z < d$.*

(ii) *Consider the functor $F : \mathbf{Vec}^{\mathbb{N}} \to \mathbf{Vec}$ which sends a persistence module Z to $\ker Z_{b,d} / \ker Z_{b,d'}$, with the action of F on morphism defined in the expected way. Show that $dim(F_Z)$ is the number of intervals in $\mathcal{B}_Z$ of the form $[b', d)$ with $b' \leq b$.*

(iii) *Show that F preserves monomorphisms.*

(iv) *Considering the monomorphism F_j, it follows that the number of intervals in $\mathcal{B}_M$ of the form $[b', d)$ with $b' \leq b$ is at most the number of such intervals in $\mathcal{B}_N$. Show that this implies the claimed result.*

Proof. (i) Assume that for $\forall d' \in (b, d)$, $\mathcal{B}_M \cup \mathcal{B}_N$ have an interval $[y, z)$ with $y \leq b$ and $d' < z < d$. Let z_0 be any value in (b, d), then there exists $[y_1, z_1) \in \mathcal{B}_M \cup \mathcal{B}_N$ with $y_1 \leq b$ and $z_0 < z_1 < d$. The rest can be done in the same manner and we can find a sequence $(z_n)_{z \in \mathbb{N}}$ such that $z_0 < z_1 < \cdots < z_i < \cdots < d$. All those intervals $[y_i, z_i) \in \mathcal{B}_M \cup \mathcal{B}_N$, so $dim(M_{z_0}) = +\infty$ or $dim(N_{z_0}) = +\infty$, which contradicts the fact that M and N are p.f.d.

(ii) $dim(F_Z) = dim(\ker Z_{b,d} / \ker Z_{b,d'}) = dim(\ker Z_{b,d}) - dim(\ker Z_{b,d'}) = (dim(Z_b) - rank(Z_{b,d})) - (dim(Z_b) - rank(Z_{b,d'})) = rank(Z_{b,d'}) - rank(Z_{b,d})$. Since $Z_{b,d}$ is equal to the number of intervals passing through $[b,d)$ in $\mathcal{B}_Z$, $rank(Z_{b,d'}) - rank(Z_{b,d})$ is the number of intervals of the form $[b',z)$ with $b' \leq b$ and $d' \leq z < d$. From (i), $z = d'$ and $\#intervals\ of\ the\ form\ [b',d') = \#intervals\ of\ the\ form\ [b',d)$, so $dim(F_Z)$ is the number of intervals in $\mathcal{B}_Z$ of the form $[b',d)$ with $b' \leq b$.

(iii) Let $j : M \to N$ be a monomorphism, then we need to prove that F_j is a monomorphism. Let $\bar{x} = x + \ker M_{b,d'}$ and $\bar{y} = y + \ker M_{b,d'}$. If $F_j(\bar{x}) = F_j(\bar{y})$, then $j_b(x) + \ker N_{b,d'} = j_b(y) + \ker N_{b,d'} \Rightarrow j_b(x-y) \in \ker N_{b,d'} \Rightarrow N_{b,d'}(j_b(x-y)) = 0$. From the commutativity of the diagram, $N_{b,d'} \circ j_b = j_{d'} \circ M_{b,d'}$, so $j_{d'} \circ M_{b,d'}(x-y) = 0$. Since $j_{d'}$ is injective, $M_{b,d'}(x-y) = 0 \Rightarrow x - y \in \ker M_{b,d'} \Rightarrow \bar{x} = \bar{y}$.

(iv) $F_j : \ker M_{b,d} / \ker M_{b,d'} \to \ker N_{b,d} / \ker N_{b,d'}$ is an injective from (iii), so

$$dim(\ker M_{b,d} / \ker M_{b,d'}) \leq dim(\ker N_{b,d} / \ker N_{b,d'}).$$

From (ii), the number of intervals in $\mathcal{B}_M$ of the form $[b',d)$ with $b' \leq b$ is at most the number of such intervals in $\mathcal{B}_N$.

□

Proposition 1.7(ii) follows from Proposition 1.7(i) by a simple duality argument, which we will not give here. Alternatively, one can just dualize the proof of Proposition 1.7(i).

Exercise 1.9. *Use Proposition 1.7 to prove Theorem 1.5(i).*

Proof. $f : M \to N$ can be decomposed as $f = j_f \circ q_f$ where $q_f : M \to imf$ is given by $(q_f)_r(x) = f_r(x)$ and $j_f : imf \to N$ is a given by $(j_f)_r(x) = x$. q_f is epimorphism and j_f is monomorphism. From partial functoriality of induced matchings, we know $(X)_f = (X)_{j_f} \circ (X)_{q_f}$, so $(X)_f[b,d) = (X)_{j_f} \circ (X)_{q_f}[b,d)$. From Proposition 1.7, $\mathcal{X}_{q_f}[b,d) = [b,d')$ for some $d' \leq d$ and $\mathcal{X}_{j_f}[b,d') = [b',d')$ for some $b' \leq b$. So $\mathcal{X}_f[b,d) = [b',d')$ for some $b' \leq b \leq d' \leq d$. If $b = d'$, $\mathcal{X}_f[b,d) = \varnothing$. If we assume that $[b',d') \neq \varnothing$, we have $b' \leq b < d' \leq d$. □

We next establish Theorem 1.5 in the case that f is a monomorphism. In this case, $\ker f$ is trivial, so Theorem 1.5(ii) follows immediately from Proposition 1.7(ii). For f a monomorphism, the proof of Theorem 1.5(iii) proceeds by a sandwiching argument: Let the submodule $N^\delta \subset N$ be defined by

$$N_r^\delta = \{n \in N_r \mid n = N_{r-\delta,r}(n')\ for\ some\ n' \in N_{r-\delta}\}$$

Exercise 1.10. *Show that if f is a monomorphism such that each interval in $\mathcal{B}_{cokerf}$ has length at most δ, then $N^\delta \subset imf$.*

Proof. Since each interval in $\mathcal{B}_{cokerf}$ has length at most δ, for $\forall \bar{n}' \in (cokerf)_{r-\delta}$, $(cokerf)_{r-\delta,r}(\bar{n}') = 0$. We know that $cokerf = N/imf$ and $(cokerf)_r = N_r/(imf)_r$. So for $\forall n' \in N_{r-\delta}$, $(cokerf)_{r-\delta,r}(n' + (imf)_{r-\delta}) = N_r(n')/(imf)_r = 0$, which implies $N_r^\delta \subset (imf)_r$. □

Noting that f restricts to an isomprohism $M \to imf$, we thus have a factorization by monomorphisms

$$N_\delta \overset{g}{\hookrightarrow} M \overset{f}{\hookrightarrow} N$$

of the inclusion $j : N^\delta \hookrightarrow N$. We may think of this as a "sandwiching" of M in terms of N.

$\mathcal{B}_{N^\delta}$ is obtained from $\mathcal{B}_N$ by shortening each inverval of $\mathcal{B}_N$ on the left side by δ (with intervals of length less than δ removed altogether). The induced matching $\mathcal{X}_j : \mathcal{B}_{N^\delta} \hookrightarrow \mathcal{B}_N$ is thus especially simple. By the functoriality of matchings induced by monomorphisms, the sequence of induced matchings

$$\mathcal{B}_{N^\delta} \overset{\mathcal{X}_g}{\hookrightarrow} \mathcal{B}_M \overset{\mathcal{X}_f}{\hookrightarrow} \mathcal{B}_N$$

factors $\mathcal{X}_j$. Applying Proposition 1.7 to this factorization yields Theorem 1.5(iii).

Exercise 1.11. *Fill in the details of the part of the argument given in the last sentence.*

Proof. For each $[b,d) \in \mathcal{B}_N$, then $[b+\delta, d) \in \mathcal{B}_{N^\delta}$ (with intervals of length less than δ removed altogether). From the functoriality of matchings induced by monomorphisms, $\mathcal{X}_j[b+\delta, d) = \mathcal{X}_f \circ \mathcal{X}_g[b+\delta, d) = [b,d)$, so $\mathcal{X}_g[b+\delta, d) = [b+\delta', d)$ with $0 \le \delta' \le \delta$ and $\mathcal{X}_f[b+\delta', d) = [b,d)$. Thus, if f is monomorphism and $\mathcal{X}_f[b,d) = [b',d)$, we have $|b-b'| \le \delta$. For each $[b,d) \in \mathcal{B}_N$ such that $|d-b| > \delta$, we can find the corresponding interval $[b+\delta, d) \in \mathcal{B}_{N^\delta}$, so $\mathcal{X}_f$ matches each interval in $\mathcal{B}_N$ of length greater than δ. □

Having established Theorem 1.5(iii) for monomorphisms, a duality argument gives Theorem 1.5(ii) for epimorphisms. Alternatively, this can be shown by dualizing the proof of Theorem 1.5(iii). For epimorphisms, Theorem 1.5(iii) is immediate from Proposition 1.7(ii).

The following exercise then completes the proof of the induced matching theorem:

Exercise 1.12. *Assuming that Theorem 1.5 (ii) and (iii) hold when f is either a monomorphism or epimorphism, show that they in fact hold for arbitrary f.*

Proof. As mentioned above, f has a canonical epi-mono factorization $f = j_f \circ q_f$ where q_f is an epimorphism and j_f is a monomorphism. We also define $\mathcal{X}_f := \mathcal{X}_{j_f} \circ \mathcal{X}_{q_f}$. Suppose $\mathcal{X}_f[b,d) = [b',d')$.
For (ii), $\ker f = \ker q_f$, so $\mathcal{B}_{\ker f} = \mathcal{B}_{\ker q_f}$. For each interval $[b,d') \in \mathcal{B}$. If each interval in $\mathcal{B}_f$ has length at most δ, then each interval in $\mathcal{B}_{\ker q_f}$ has length at most δ. $\mathcal{X}_{q_f}[b,d) = [b,d')$ with $d' \le d$, then we have $|d-d'| \le \delta$, and $\mathcal{X}_{q_f}$ matches each interval in $\mathcal{B}_M$ of length greater than δ to $\mathcal{B}_{imf}$. $cokerq_f = 0$, so $\mathcal{X}_{q_f}$ matches each interval in $\mathcal{B}_{imf}$ of length greater than δ.

$cokerf = N/imf = cokerj_f$, so $\mathcal{B}_{cokerf} = \mathcal{B}_{cokerj_f}$. If $\mathcal{B}_{cokerf}$ has length at most δ, $\mathcal{B}_{coker_{j_f}}$ also has length at most δ, then $|b-b'| \le \delta$ and X_{j_f} matches each interval in $\mathcal{B}_N$ of length greater than δ. $kerj_f = 0$, so $\mathcal{X}_{j_f}$ matches each interval in $\mathcal{B}_{imf}$ of length greater than δ.

Therefore, $\mathcal{X}_f$ matches each interval in $\mathcal{B}_M$ and $\mathcal{B}_N$ of length greater than δ. □

Remark 1.1. *The categorical structure on **Barc** can be used to give a slick formulation of the induced matching theorem, which transparently expresses the sense in which passage from a persistence module to its barcode perserves categorical structure[BL20].*

Algebraic Stability from Induced Matchings It remains to explain how the algebraic stability theorem follows from the induced matching theorem. The key observation is the following, which is an easy exercise:

Exercise 1.13. *Let $f : M \to N(\delta)$ be a δ-interleaving morphism. Then each interval in each of the barcode $\mathcal{B}_{kerf}$ and $\mathcal{B}_{cokerf}$ has length at most 2δ.*

Proof. Consider an interval $[b, d) \in \mathcal{B}_{kerf}$. Assume $|d - b| > 2\delta$, then there exists a non-zero vector $v \in (kerf)_b$ such that $(kerf)_{b,b+2\delta}(v) \neq 0$ according to the definition of the barcode associated to the compatible basis. We also have $M_{b,b+2\delta}(v) \neq 0$ from $kerf \subset M$. Since f is a δ-interleaving morphism, $M_{b,b+2\delta}(v) = g \circ f_b(v) = 0$ where g is a morphism from $N_{b+\delta}$ to $M_{b+2\delta}$. We get a contradiction, so each interval in the barcode $\mathcal{B}_{kerf}$ has length at most 2δ.

Similarly, consider an interval $[b, d) \in \mathcal{B}_{cokerf}$. Assume $|d - b| > 2\delta$, then there exists a non-zero $\bar{v} \in (cokerf)_b = N_{b+\delta}/(imf)_b$ such that $(cokerf)_{b,b+2\delta}(\bar{v}) \neq 0$ and $N_{b,b+2\delta}(v) \notin (imf)_{b+\delta}$. Since f is a δ-interleaving morphism, $N_{b,b+2\delta}(v) = f_{b+\delta}(v) \circ g(v) = 0$ where g is a morphism from N_b to $M_{b+\delta}$. So $N_{b,b+2\delta}(v) \in (imf)_{b+\delta}$, which is a contradiction. □

Note that for any $\delta \geq 0$, and p.f.d persistence module N, we have an obvious bijective matching $r^\delta : \mathcal{B}_{N(\delta)} \to \mathcal{B}_N$.

To prove the algebraic stability theorem from the induced matching theorem, we apply the latter to the δ-interleaving morphism $f : M \to N(\delta)$, using Exercise 1.13; This gives us that $r^\delta \circ \mathcal{X}_f : \mathcal{B}_M \to \mathcal{B}_N$ is a δ-matching. Algebraic stability now follows.

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