Homework 3 for AMAT840: Multiparameter Persistent Homology

Jingyi Li *
Institut Polytechnique de Paris , Palaiseau, France
jingyi.li@ip-paris.fr

November 18, 2022

1 Algebraic Aspects of Persistence Modules

Exercise 1.1 (6.8). *For which $a, b \in \mathbb{N}^n$ is there a non-zero morphism (i.e., natural transformation) $Q^a \to Q^b$?*

Solution. Let $x, y \in \mathbb{N}^n$ and consider the following diagram:

$$\begin{tikzcd} Q^a_x \arrow[r, "Q^a_{x,y}"] \arrow[d, "N_x"] & Q^a_y \arrow[d, "N_y"] \\ Q^b_x \arrow[r, "Q^b_{x,y}"] & Q^b_y \end{tikzcd}$$

If the diagram looks like this:

$$\begin{tikzcd} \mathcal{K} \arrow[r, "Id"] \arrow[d, "0"] & \mathcal{K} \arrow[d, "Id"] \\ 0 \arrow[r, "0"] & \mathcal{K} \end{tikzcd}$$

it's impossible to commute. So, if $Q^a_x = \mathcal{K}$, Q^b_x must be $\mathcal{K}$, which means $a \geq b$. □

Exercise 1.2 (6.13). *For which $a, b \in \mathbb{N}^d$ do we have $Q^a \subset Q^b$.*

Solution. For $b \leq a$, i.e. $b_i \leq a_i$ where $i \in \{1, 2, ..., d\}$, we have $Q^a \subset Q^b$. Because in this case, for $x \geq a$, $Q^a_x = K = Q^b_x$ and for $x < a$, $Q^a_x = 0 \subset Q^b_x$. □

Exercise 1.3 (6.15). *$f : M \to N$ is a morphism of persistence modules. Check that $\ker f$ and $\operatorname{im} f$ are in fact well-defined submodules.*

*This is a course taught by Michael Lesnick from SUNY Albany.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} M_x \arrow[r, "M_\gamma"] \arrow[d, "f_x"] & M_y \arrow[d, "f_y"] \\ N_x \arrow[r, "N_\gamma"] & N_y \end{tikzcd}$$

First we prove that $M_\gamma((kerf)_x) \subset (kerf)_y$:
For $\forall m \in (kerf)_x$, $f_x(m) = 0$. According to the commutativity of the diagram, we have $N_\gamma \circ f_x(m) = f_y \circ M_\gamma(m) = 0$, so $M_\gamma(m) \in (kerf)_y$.

Then we prove that $N_\gamma((imf)_x) \subset (imf)_y$:
For $\forall n \in (imf)_x$, $\exists m \in M_x$ such that $f_x(m) = n$. According to the commutativity of the diagram, we have $N_\gamma \circ f_x(m) = f_y \circ M_\gamma(m)$, so $f_y \circ M_\gamma(m) = N_\gamma(n)$ and $N_\gamma(n) \in (imf)_y$. □

Exercise 1.4 (6.18)**.** *Given submodules W and W' of a persistence module M, let $W + W' \subset M$ be given by*

$$(W + W')_z = \{w + w' \mid w \in W_z, w \in W'_z\}.$$

This is clearly also a submodule of M.
1. Draw the diagram of vector spaces $Q^{0,1} + Q^{1,0}$.
2. Up to isomorphism, what is the module $Q^{0,0}/(Q^{0,1} + Q^{1,0})$?

Solution. 1.

$$\begin{tikzcd} \vdots & \vdots & \vdots & \\ K \arrow[r] \arrow[u] & K \arrow[r] \arrow[u] & K \arrow[r] \arrow[u] & \cdots \\ K \arrow[r] \arrow[u] & K \arrow[r] \arrow[u] & K \arrow[r] \arrow[u] & \cdots \\ 0 \arrow[r] \arrow[u] & K \arrow[r] \arrow[u] & K \arrow[r] \arrow[u] & \cdots \end{tikzcd}$$

2.

$$\begin{tikzcd} \vdots & \vdots & \vdots & \\ 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots \\ 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots \\ K \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots \end{tikzcd}$$

□

Exercise 1.5 (6.20)**.** *Give an example of an $\mathbb{N}$-indexed persistence module whose vector spaces have dimension at most one, and for which any minimal generating set has two elements.*

Solution.

$$K \xrightarrow{0} 0 \xrightarrow{0} K \xrightarrow{Id} K \xrightarrow{Id} \cdots$$

□

Exercise 1.6 (6.21). *Give an example of an $\mathbb{N}$-indexed persistence module which is not finitely generated, but whose vector spaces are all of finite dimension.*

Solution.

$$K \xrightarrow{0} 0 \xrightarrow{0} K \xrightarrow{0} 0 \xrightarrow{0} K \xrightarrow{0} 0 \longrightarrow \cdots$$

□

Exercise 1.7 (6.34). *Prove Proposition 6.33:*
Let $\gamma : V \to W$ be a linear map of finite-dimensional vector spaces, and let $A = \{a_1, \cdots, a_n\}$ and $B = \{b_1, \cdots, b_m\}$ be bases for V and W, respectively. For any $c \in K$,
• *If A' is obtained from A by replacing a_i with $a_i + ca_j$, then $[\gamma]^{B,A'}$ is obtained from $[\gamma]^{B,A}$ by adding $c[\gamma]^{B,A}_{*,j}$ to $c[\gamma]^{B,A}_{*,i}$.*
• *Similarly, if B' is obtained from B by replacing b_i with $b_i + cb_j$, then $[\gamma]^{B',A}$ is obtained from $[\gamma]^{B,A}$ by subtracting $c[\gamma]^{B,A}_{i,*}$ from $[\gamma]^{B,A}_{j,*}$.*

Proof. • If A' is obtained from A by replacing a_i with $a_i + ca_j$
For $k \neq i$, $[\gamma]^{B,A'}_{*,k} = [\gamma(a_k)]^B = [\gamma]^{B,A}_{*,k}$.
For $k = i$, $[\gamma]^{B,A'}_{*,k} = [\gamma(a_i + ca_j)]^B = [\gamma(a_i)]^B + c[\gamma(a_j)]^B = [\gamma]^{B,A}_{*,i} + c[\gamma]^{B,A}_{*,j}$.

• if B' is obtained from B by replacing b_i with $b_i + cb_j$
For $\forall k \in \{1, 2, \cdots, n\}$,

$$\gamma(a_k) = \sum_{t=1}^{m} [\gamma]^{B,A}_{t,k} b_t = \sum_{t=1}^{m} [\gamma]^{B',A}_{t,k} b'_t \tag{1}$$

where $b'_t = b_t$ if $t \neq i$ and $b'_t = b_i + cb_j$ if $t = i$, so

$$\gamma(a_k) = \sum_{t \neq i} [\gamma]^{B',A}_{t,k} b_t + [\gamma]^{B',A}_{i,k} (b_i + cb_j) = \sum_{t \neq j} [\gamma]^{B',A}_{t,k} b_t + ([\gamma]^{B',A}_{j,k} + c[\gamma]^{B',A}_{i,k}) b_j \tag{2}$$

Compare (1) and (2), we have $[\gamma]^{B,A}_{t,k} = [\gamma]^{B',A}_{t,k}$ if $t \neq j$, and

$$[\gamma]^{B,A}_{j,k} = [\gamma]^{B',A}_{j,k} + c[\gamma]^{B',A}_{i,k} = [\gamma]^{B',A}_{j,k} + c[\gamma]^{B,A}_{i,k} \tag{3}$$

Therefore, $[\gamma]^{B',A}_{j,k} = [\gamma]^{B,A}_{j,k} - c[\gamma]^{B,A}_{i,k}$ for any k, which means $[\gamma]^{B',A}$ is obtained from $[\gamma]^{B,A}$ by subtracting $c[\gamma]^{B,A}_{i,*}$ from $[\gamma]^{B,A}_{j,*}$.

□

2 Computing Persistent Homology

Exercise 2.1 (8.3). *Supposing $M : \mathbb{N} \to$ **Vec** has the following presentation matrix (over $\mathbb{Z}/2\mathbb{Z}$), compute $\mathcal{B}_M$.*

$$\begin{array}{c|cccc} & 4 & 3 & 6 & 5 \\ \hline 1 & & & & \\ 0 & & & & \\ 3 & & & & \\ 2 & & & & \end{array}$$

$$\begin{array}{cc} & \begin{array}{cccc} 4 & 3 & 6 & 5 \end{array} \\ \begin{array}{c} 1 \\ 0 \\ 3 \\ 2 \end{array} & \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Solution.

$$\begin{array}{cc} & \begin{array}{cccc} 4 & 3 & 6 & 5 \end{array} \\ \begin{array}{c} 1 \\ 0 \\ 3 \\ 2 \end{array} & \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array} \rightarrow \begin{array}{cc} & \begin{array}{cccc} 4 & 3 & 6 & 5 \end{array} \\ \begin{array}{c} 1 \\ 0 \\ 3 \\ 2 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

So $\mathcal{B}_M = \{[2,4), [0,6), [1,+\infty)\}$.

□

Exercise 2.2 (8.13)**.** *Consider the simplicial filtration $F : \mathbb{N} \to$ **Vec** specified by the table below. Compute all barcodes of H_iF using Theorem 8.7. Also compute the reduced presentation matrix Q' for $\bigoplus_i H_iF$ appearing in the proof of Theorem 8.7.*

SIMPLEX	[1]	[2]	[3]	[4]	[2 3]	[1 2]	[1 3]	[2 4]	[1 2 3]	[3 4]	[2 3 4]
BIRTH INDEX	1	2	3	4	5	6	7	8	9	10	11

Solution.

$$[\alpha] = \begin{array}{cc} & \begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 10 & 9 & 11 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 10 \\ 9 \\ 11 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$Q = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 10 \\ 9 \\ 11 \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 10 & 9 & 11 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$Q' = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 7 \\ 10 \end{array} \begin{array}{c} \begin{array}{ccccc} 5 & 6 & 8 & 9 & 11 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\mathcal{B}_0(F) = \{[3,5), [2,6), [4,8), [1,+\infty)\}$$
$$\mathcal{B}_1(F) = \{[7,9), [10,11)\}$$

$\square$