Homework 1 for AMAT840: Multiparameter Persistent Homology

Jingyi Li [*]

Institut Polytechnique de Paris , Palaiseau, France

jingyi.li@ip-paris.fr

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1 Posets and Basic Category Theory

Exercise 1.1 (3.12). *Draw the Hasse diagrams of all the posets (up to isomorphism) with 4 elements.*

Solution. There are 17 Hasse diagrams of posets with 4 elements:

(1)
$$\bullet \quad \bullet \quad \bullet \quad \bullet$$

(2)
$$\bullet \to \bullet \quad \bullet \quad \bullet$$

(3)
$$\bullet \to \bullet \to \bullet \quad \bullet$$

(4)
$$\bullet \to \bullet \quad \bullet \to \bullet$$

(5)
$$\bullet \leftarrow \bullet \to \bullet \quad \bullet$$

(6)
$$\bullet \to \bullet \leftarrow \bullet \quad \bullet$$

(7)
$$\bullet \to \bullet \to \bullet \to \bullet$$

(8)
$$\bullet \leftarrow \bullet \leftarrow \bullet \to \bullet$$

(9)
$$\bullet \to \bullet \to \bullet \leftarrow \bullet$$

[*]This is a course taught by Michael Lesnick from SUNY Albany.

(10)

$$\bullet \leftarrow \bullet \rightarrow \bullet \leftarrow \bullet$$

(11)

$$\begin{tikzcd}
 & \bullet & \\
\bullet & \bullet \arrow[l] \arrow[r] \arrow[u] & \bullet
\end{tikzcd}$$

(12)

$$\begin{tikzcd}
 & \bullet & \\
\bullet \arrow[r] & \bullet \arrow[r] \arrow[u] & \bullet
\end{tikzcd}$$

(13)

$$\begin{tikzcd}
 & \bullet & \\
\bullet \arrow[r] & \bullet \arrow[u] & \bullet \arrow[l]
\end{tikzcd}$$

(14)

$$\begin{tikzcd}
 & \bullet \arrow[d] & \\
\bullet & \bullet \arrow[l] \arrow[r] & \bullet
\end{tikzcd}$$

(15)

$$\begin{tikzcd}
 & \bullet \arrow[d] & \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

(16)

$$\begin{tikzcd}
 & \bullet \arrow[d] & \\
\bullet \arrow[r] & \bullet & \bullet \arrow[l]
\end{tikzcd}$$

(17)

$$\begin{tikzcd}
\bullet \arrow[r] & \bullet \\
\bullet \arrow[r] \arrow[u] & \bullet \arrow[u]
\end{tikzcd}$$

□

Exercise 1.2 (3.26)**.** *Show that a fully faithful functor F is **essentially injective**, i.e., $F_x \cong F_y$ implies $x \cong y$.*

Proof. If $F_x \cong F_y$, there exists a morphism $\gamma : F_x \to F_y$ such that γ is an isomorphism. The morphism $\gamma^{-1} : F_y \to F_x$ is also an isomorphism. The functor F is fully faithful, so there exist a unique morphism η such that $F(\eta) = \gamma$ and a unique morphism η' such that $F(\eta') = \gamma^{-1}$. Then use the composition operation:

$$F(\eta \circ \eta') = F(\eta) \circ F(\eta') = \gamma \circ \gamma^{-1} = Id_{F_y}$$

$$F(\eta' \circ \eta) = F(\eta') \circ F(\eta) = \gamma^{-1} \circ \gamma = Id_{F_x}$$

For $\forall x \in ObC$, the function $F : hom(x, x) \to hom(F_x, F_x)$ is bijective. So for $Id_{F_x} \in hom(F_x, F_x)$, there exists a unique $\delta \in \mathrm{hom}(x, x)$ such that $F(\delta) = Id_{F_x}$, and we know that $F(Id_x) = Id_{F_x}$ according to the definition of functor, so $\delta = Id_x$, which means that the preimage of an identity map must be an identity map for a fully faithful functor. Thus, $\eta \circ \eta' = Id_y$ and $\eta' \circ \eta = Id_x$, so η is bijective and $x \cong y$.

□

Exercise 1.3 (3.27)**.** *Recall the construction of a category from a poset given in Example 3.16. Show that this extends to a fully faithful functor* ***Pst*** $\to$ ***Cat****.*

Proof. Consider the functor F : **Pst** $\to$ **Cat**.

First, let's define the functor.
The category **Pst** has objects all posets and morphisms all poset maps.
For $\forall P \in Ob$**Pst**, $F(P)$ is the catogory whose objects are all elements in P and whose morphisms are the partially ordered relation between two elements. So we can say that the elements in P and the objects in $F(P)$ are one-to-one, and we denote the corresponding object in F(P) as $\tilde{x}$ if $x \in P$.
For $\forall P, P' \in Ob$**Pst** and $\gamma \in hom(P, P')$, $F(\gamma) \in hom(F(P), F(P'))$ is a functor from $F(P)$ to $F(P')$. For $x \in P$ and $y \in P'$, if $y = \gamma(x)$, then let $\tilde{y} = F(\gamma)(\tilde{x})$. It's obvious that F respects the composition operation and maps identity morphisms to identity morphisms, so F is indeed a functor.

Then, prove F is faithful.
For $\forall P, P' \in ob$**Pst** and $\forall \gamma, \gamma' \in hom(P, P')$, if $F(\gamma) = F(\gamma')$, we have $F(\gamma)(\tilde{x}) = F(\gamma')(\tilde{x})$ for $\forall \tilde{x} \in F(P)$. Thus $\gamma(x) = \gamma'(x)$ for $\forall x \in P$ according to the above definition of the functor. So $F(\gamma) = F(\gamma')$ implies $\gamma = \gamma'$, i.e., F is faithful.

Finally, prove F is full.
For $\forall \eta \in hom(F(P), F(P'))$, we define a morphism $\gamma \in hom(P, P')$ such that if $\tilde{y} = \eta(\tilde{x})$, then $y = \gamma(x)$. Thus $F(\gamma) = \eta$ according to the definition of the functor F. So F is full. □

Exercise 1.4 (3.30)**.** *As in the previous example, a natural transformation between functors* $F, G : \{0,1\} \times \{0,1\} \to$ ***Top*** *can be thought of as a commutative diagram. Sketch this diagram.*

Solution.

$$\begin{tikzcd}
G_{0,1} \arrow[rrr] & & & G_{1,1} \\
& F_{0,1} \arrow[r] \arrow[ul, dashed, "N_{0,1}"'] & F_{1,1} \arrow[ur, dashed, "N_{1,1}"] & \\
& F_{0,0} \arrow[r] \arrow[u] \arrow[dl, dashed, "N_{0,0}"'] & F_{1,0} \arrow[u] \arrow[dr, dashed, "N_{1,0}"] & \\
G_{0,0} \arrow[rrr] \arrow[uuu] & & & G_{1,0} \arrow[uuu]
\end{tikzcd}$$

□

Exercise 1.5 (3.34)**.** *Show that a morphism in* $\mathcal{D}^{\mathcal{C}}$ *is an isomorphism if and only if it is a natural isomorphism in the sense defined above.*

Proof. Let F, G be two functors $\mathcal{C} \to \mathcal{D}$ and N be a natural transformation from F to G.

If N is an isomorphism, then there exists a natural transformation M from G to F such that $M \circ N = Id_F$ and $N \circ M = Id_G$, i.e., for $\forall x \in Ob\mathcal{C}$, $M_x \circ N_x = Id_{F_x}$ and $N_x \circ M_x = Id_{G_x}$, so each N_x is an isomorphism and N is a natural isomorphism.

If N is a natural isomorphism, then for $\forall x \in Ob\mathcal{C}$, N_x is an isomorphism. Thus $\exists M_x \in hom(G_x, F_x)$ such that $M_x \circ N_x = Id_{F_x}$ and $N_x \circ M_x = Id_{G_x}$. So $M = (M_x : G_x \to F_x)_{x \in Ob\mathcal{C}}$ is a natural transformation such that $M \circ N = Id_F$ and $N \circ M = Id_G$, then N is an isomorphism. □

Exercise 1.6 (3.39)**.**
(a) Describe all functors $\mathbb{Z} \to \mathbb{Z}$.
(b) Describe all equivalences $\mathbb{Z} \to \mathbb{Z}$.
(c) Describe all equivalences $\mathbb{Z} \to \mathbb{Z}^{op}$.

Solution. (i) $\mathbb{Z}$ is a poset, so each object in the category $\mathbb{Z}$ is an integer. $hom(x, y)$ has one element if $x \leq y$, otherwise $hom(x, y)$ is empty.
For a functor $F : \mathbb{Z} \to \mathbb{Z}$, it satisfies that if $x \leq y$, then $F(x) \leq F(y)$, so $F : Ob\mathbb{Z} \to Ob\mathbb{Z}$ is a non-decreasing function.

(ii) If $F : \mathbb{Z} \to \mathbb{Z}$ is an equivalence, then it is fully faithful and essentially surjective, so $F : Ob\mathbb{Z} \to Ob\mathbb{Z}$ should be $F(x) = x + n$ where n is any given interger.

(iii) If $F : \mathbb{Z} \to \mathbb{Z}^{op}$ is an equivalence, it should be $F(x) = n - x$ where n is any given interger.

□

Exercise 1.7. *Show that for any category $\mathcal{C}$, there exists an equivalent category $\mathcal{D}$ in which no two distinct objects are isomorphic.*

Proof. For each isomorphism class in $Ob\mathcal{C}$, we choose one as the representative object. Let $Ob\mathcal{D}$ be the collection of all these representative objects. If $x \in Ob\mathcal{C}$, we denote the corresponding representative object as $\overline{x} \in Ob\mathcal{D}$. Since $x \cong \overline{x}$, there exist isomorphisms $i_x : x \to \overline{x}$ and $i_x^{-1} : \overline{x} \to x$.

Now let's define the functor $F : \mathcal{C} \to \mathcal{D}$.
For $\forall x \in Ob\mathcal{C}$, let $F(x) = \overline{x} \in Ob\mathcal{D}$. For $\forall f \in hom(x, y)$, let $F(f) = i_y \circ f \circ i_x^{-1} \in hom(\overline{x}, \overline{y}) = hom(F(x), F(y))$.

Then let's check that F is a functor.
For $\forall f \in hom(x, y)$ and $\forall g \in hom(y, z)$, $F(g \circ f) = i_z \circ g \circ f \circ i_x^{-1}$. Then $F(g) \circ F(f) = i_z \circ g \circ i_y^{-1} \circ i_y \circ f \circ i_x^{-1} = i_z \circ g \circ f \circ i_x^{-1} = F(g \circ f)$, so F respects composition. For $Id_x \in hom(x, x)$, $F(Id_x) = i_x \circ Id_x \circ i_x^{-1} = Id_{\overline{x}} = Id_{F_x}$, so F is indeed a functor.

Then we define the functor $G : \mathcal{D} \to \mathcal{C}$ as an inclusion.
For $\forall x \in Ob\mathcal{D}, G(x) = x$ and for $\forall g \in hom(x, y), G(g) = g$. It's obvious that G is a functor.

Then, let's consider the natural transformation $N : F \circ G \to Id_{\mathcal{D}}$.
For $\forall \overline{x}, \overline{y} \in Ob(D)$, $(F \circ G)_{\overline{x}} = \overline{x}$ and$(F \circ G)_{\overline{y}} = \overline{y}$, and for $\forall \gamma \in hom(\overline{x}, \overline{y})$, $(F \circ G)_{\gamma} = F_{\gamma} = i_{\overline{y}} \circ \gamma \circ i_{\overline{x}}^{-1} = \gamma$. Thus the following diagram

$$\begin{tikzcd}
(F \circ G)(\overline{x}) \arrow[r, "(F\circ G)(\gamma)"] \arrow[d, "N_{\overline{x}}"] & (F \circ G)(\overline{y}) \arrow[d, "N_{\overline{y}}"] \\
Id_{\mathcal{D}}(\overline{x}) \arrow[r, "Id_{\mathcal{D}}(\gamma)"] & Id_{\mathcal{D}}(\overline{y})
\end{tikzcd}$$

is the same as

$$\begin{tikzcd}
\overline{x} \arrow[r, "\gamma"] \arrow[d, "N_{\overline{x}}"] & \overline{y} \arrow[d, "N_{\overline{y}}"] \\
\overline{x} \arrow[r, "\gamma"] & \overline{y}
\end{tikzcd}$$

Let each $N_{\overline{x}}$ be $Id_{\overline{x}}$ for all $\overline{x} \in Ob\mathcal{D}$, the above diagram commutes. Since identity map is isomorphism, so N is a natural isomorphism and $F \circ G \cong Id_{\mathcal{D}}$.

Finally, let's consider the natural transformation $N' : G \circ F \to Id_{\mathcal{C}}$.
For $\forall x, y \in Ob(\mathcal{C})$, $G \circ F(x) = \overline{x}$ and $G \circ F(y) = \overline{y}$. For $\gamma \in hom(x, y)$, $G \circ F(\gamma) = i_x^{-1} \circ \gamma \circ i_y$. Thus, the following diagram

$$\begin{tikzcd}
(G \circ F)(x) \arrow[r, "(G\circ F)(\gamma)"] \arrow[d, "N'_x"] & (G \circ F)(y) \arrow[d, "N'_y"] \\
Id_{\mathcal{C}}(x) \arrow[r, "Id_{\mathcal{C}}(\gamma)"] & Id_{\mathcal{C}}(y)
\end{tikzcd}$$

is the same as

$$\begin{tikzcd}
\overline{x} \arrow[r, "i_x^{-1}\circ\gamma\circ i_y"] \arrow[d, "N'_x"] & \overline{y} \arrow[d, "N'_y"] \\
x \arrow[r, "\gamma"] & y
\end{tikzcd}$$

Let each N'_x be i_x^{-1} for all $x \in Ob\mathcal{C}$. The above diagram commutes and each N'_x is an isomorphism, so N' is a natural isomorphism and $G \circ F \cong Id_{\mathcal{C}}$.

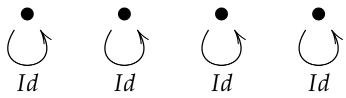
Therefore, category $\mathcal{C}$ and category $\mathcal{D}$ are equivalent and there are no two distinct objects that are isomorphic in $\mathcal{D}$.

$\square$

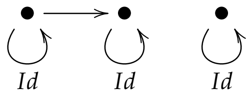
Exercise 1.8 (3.41)**.**
(i) Give an explicit description of all categories (up to isomorphism) with four morphisms.
(ii) Which of these categories are equivalent to each other?
(iii) Which are (isomorphic to) poset categories?

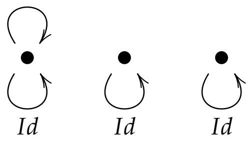
Solution. (i)If a category $\mathcal{C}$ has four morphisms, it has at most four objects because $Id_x \in hom(x, x)$ for all $x \in Ob\mathcal{C}$.
We use bullets to denote objects and arrows to denote morphisms.

If the category has four objects,

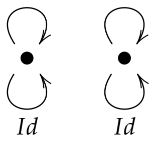
$\mathcal{C}1 =$ 

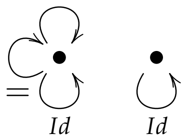
If the category has three objects,

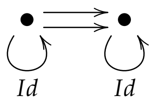
$\mathcal{C}2 =$ 

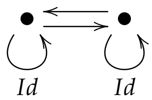
$\mathcal{C}3 =$ 

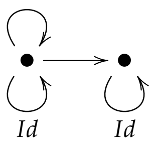
If the category has two objects,

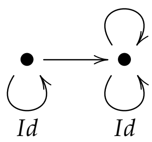
$\mathcal{C}4 =$ 

$\mathcal{C}5 =$ 

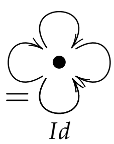
$\mathcal{C}6 =$ 

$\mathcal{C}7 =$ 

$\mathcal{C}8 =$ 

$\mathcal{C}9 =$ 

If the category has one object,

$\mathcal{C}10 =$ 

(ii) $\mathcal{C}7$ and $\mathcal{C}10$ are equivalent because they have the same structure except the size of isomorphism classes differ, i.e. they have the same skeleton. The skeleton is:

$$\mathcal{C} = \begin{tikzcd} \bullet \arrow[loop below, "Id"] \end{tikzcd}$$

(iii) $\mathcal{C}1$, $\mathcal{C}2$, $\mathcal{C}7$ are (isomorphic to) poset categories, because for any objects x, y in the category, $|hom(x, y)| \leq 1$. $\square$