Homework 3 for AMAT840: Multiparameter Persistent Homology

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1 Algebraic Aspects of Persistence Modules

Exercise 1.1 (6.8). For which $a, b \in \mathbb{N}^n$ is there a non-zero morphism (i.e., natural transformation) $Q^a \to Q^b$?

Solution. Let $x, y \in \mathbb{N}^n$ and consider the following diagram:

$$Q_{x}^{a} \xrightarrow{Q_{x,y}^{a}} Q_{y}^{a}$$

$$\downarrow^{N_{x}} \qquad \downarrow^{N_{y}}$$

$$Q_{x}^{b} \xrightarrow{Q_{x,y}^{b}} Q_{y}^{b}$$

If the diagram looks like this:

$$\begin{array}{c}
\mathcal{K} \xrightarrow{Id} \mathcal{K} \\
\downarrow 0 & \downarrow Id \\
0 \xrightarrow{0} \mathcal{K}
\end{array}$$

it's impossible to commute. So, if $Q_x^a = \mathcal{K}$, Q_x^b must be \mathcal{K} , which means $a \geq b$.

Exercise 1.2 (6.13). For which $a, b \in \mathbb{N}^d$ do we have $Q^a \subset Q^b$.

Solution. For $b \le a$, i.e. $b_i \le a_i$ where $i \in \{1, 2, ..., d\}$, we have $Q^a \subset Q^b$. Because in this case, for $x \ge a$, $Q_x^a = K = Q_x^b$ and for x < a, $Q_x^a = 0 \subset Q_x^b$.

Exercise 1.3 (6.15). $f: M \to N$ is a morphism of persistence modules. Check that ker f and im f are in fact well-defined submodules.

^{*}This is a course taught by Michael Lesnick from SUNY Albany.

Proof. Consider the following commutative diagram:

$$M_{x} \xrightarrow{M_{\gamma}} M_{y}$$

$$\downarrow f_{x} \qquad \downarrow f_{y}$$

$$N_{x} \xrightarrow{N_{\gamma}} N_{y}$$

First we prove that $M_{\gamma}((kerf)_x) \subset (kerf)_y$:

For $\forall m \in (kerf)_x$, $f_x(m) = 0$. According to the commutativity of the diagram, we have $N_\gamma \circ f_x(m) = f_y \circ M_\gamma(m) = 0$, so $M_\gamma(m) \in (kerf)_y$.

Then we prove that $N_{\gamma}((imf)_x) \subset (imf)_{\eta}$:

For $\forall n \in (imf)_x$, $\exists m \in M_x$ such that $f_x(m) = n$. According to the commutativity of the diagram, we have $N_\gamma \circ f_x(m) = f_y \circ M_\gamma(m)$, so $f_y \circ M_\gamma(m) = N_\gamma(n)$ and $N_\gamma(n) \in (imf)_y$.

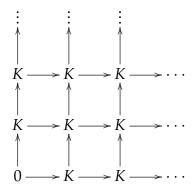
Exercise 1.4 (6.18). Given submodules W and W' of a persistence module M, let $W + W' \subset M$ be given by

$$(W + W')_z = \{w + w' \mid w \in W_z, w \in W'_z\}.$$

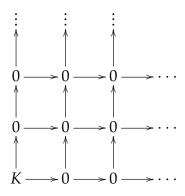
This is clearly also a submodule of M.

- 1. Draw the diagram of vector spaces $Q^{0,1} + Q^{1,0}$.
- 2. Up to isomorphism, what is the module $Q^{0,0}/(Q^{0,1}+Q^{1,0})$?

Solution. 1.



2.



Exercise 1.5 (6.20). Give an example of an \mathbb{N} -indexed persistence module whose vector spaces have dimension at most one, and for which any minimal generating set has two elements.

Solution.

$$K \xrightarrow{0} 0 \xrightarrow{0} K \xrightarrow{Id} K \xrightarrow{Id} \cdots$$

Exercise 1.6 (6.21). Give an example of an N-indexed persistence module which is not finitely generated, but whose vector spaces are all of finite dimension.

Solution.

$$K \xrightarrow{0} 0 \xrightarrow{0} K \xrightarrow{0} 0 \xrightarrow{0} K \xrightarrow{0} 0 \longrightarrow \cdots$$

Exercise 1.7 (6.34). Prove Proposition 6.33:

Let $\gamma: V \to W$ be a linear map of finite-dimensional vector spaces, and let $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_m\}$ be bases for V and W, respectively. For any $c \in K$,

- If A' is obtained from A by replacing a_i with $a_i + ca_j$, then $[\gamma]^{B,A'}$ is obtained from $[\gamma]^{B,A}$ by adding $c[\gamma]_{*,i}^{B,A}$ to $c[\gamma]_{*,i}^{B,A}$.
- Similarly, if B' is obtained from B by replacing b_i with $b_i + cb_j$, then $[\gamma]^{B',A}$ is obtained from $[\gamma]^{B,A}$ by subtracting $c[\gamma]_{i,*}^{B,A}$ from $[\gamma]_{i,*}^{B,A}$.

Proof. • If A' is obtained from A by replacing a_i with $a_i + ca_j$

For
$$k \neq i$$
, $[\gamma]_{*k}^{B,A'} = [\gamma(a_k)]^B = [\gamma]_{*k}^{B,A}$.

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, $[\gamma]_{*,k}^{B,A'} = [\gamma(a_k)]^B = [\gamma]_{*,k}^{B,A}$.
For $k = i$, $[\gamma]_{*,k}^{B,A'} = [\gamma(a_i + ca_j)]^B = [\gamma(a_i)]^B + c[\gamma(a_j)]^B = [\gamma]_{*,i}^{B,A} + c[\gamma]_{*,i}^{B,A}$.

• if B' is obtained from B by replacing b_i with $b_i + cb_i$ For $\forall k \in \{1, 2, \dots, n\},\$

$$\gamma(a_k) = \sum_{t=1}^{m} [\gamma]_{t,k}^{B,A} b_t = \sum_{t=1}^{m} [\gamma]_{t,k}^{B',A} b_t'$$
(1)

where $b'_t = b_t$ if $t \neq i$ and $b'_t = b_i + cb_j$ if t = i, so

$$\gamma(a_k) = \sum_{t \neq i} [\gamma]_{t,k}^{B',A} b_t + [\gamma]_{i,k}^{B',A} (b_i + cb_j) = \sum_{t \neq i} [\gamma]_{t,k}^{B',A} b_t + ([\gamma]_{j,k}^{B',A} + c[\gamma]_{i,k}^{B',A}) b_j$$
 (2)

Compare (1) and (2), we have $[\gamma]_{t,k}^{B,A} = [\gamma]_{t,k}^{B',A}$ if $t \neq j$, and

$$[\gamma]_{j,k}^{B,A} = [\gamma]_{j,k}^{B',A} + c[\gamma]_{i,k}^{B',A} = [\gamma]_{j,k}^{B',A} + c[\gamma]_{i,k}^{B,A}$$
(3)

Therefore, $[\gamma]_{j,k}^{B',A} = [\gamma]_{j,k}^{B,A} - c[\gamma]_{i,k}^{B,A}$ for any k, which means $[\gamma]^{B',A}$ is obtained from $[\gamma]^{B,A}$ by subtracting $c[\gamma]_{i,*}^{B,A}$ from $[\gamma]_{j,*}^{B,A}$.

Computing Persistent Homology 2

Exercise 2.1 (8.3). Supposing $M: \mathbb{N} \to Vec$ has the following presentation matrix (over $\mathbb{Z}/2\mathbb{Z}$), compute \mathcal{B}_M .

Solution.

So
$$\mathcal{B}_M = \{[2,4), [1,5), [0,+\infty)\}.$$

Exercise 2.2 (8.13). Consider the simplicial filtration $F : \mathbb{N} \to Vec$ specified by the table below. Compute all barcodes of H_iF using Theorem 8.7. Also compute the reduced presentation matrix Q' for $\bigoplus_i H_iF$ appearing in the proof of Theorem 8.7.

SIMPLEX	[1]	[2]	[3]	[4]	[2 3]	[1 2]	[1 3]	[2 4]	[1 2 3]	[3 4]	[2 3 4]
BIRTH INDEX	1	2	3	4	5	6	7	8	9	10	11

Solution.

$$Q' = \begin{array}{c} 5 & 6 & 8 & 9 & 11 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 & 0 \\ 7 & 0 & 0 & 0 & 1 & 0 \\ 10 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$$\mathcal{B}_0(F) = \{[3,5), [2,6), [4,8), [1,+\infty)\}$$

 $\mathcal{B}_1(F) = \{[7,9), [10,11)\}$