Geophys. J. Int. (2003) **154,** 253–267

On post-glacial sea level: I. General theory

Jerry X. Mitrovica[1] and Glenn A. Milne[2]

[1] *Department of Physics, University of Toronto,* 60 *St George Street, Toronto, Ontario, Canada* M5S 1A7
[2] *Department of Geological Sciences, University of Durham, Science Labs, South Road, Durham* DH1 3LE

Accepted 2002 November 23. Received 2002 October 30; in original form 2001 December 4

SUMMARY

Modern analyses of sea level changes due to glacial isostatic adjustment (GIA) are based on the classic sea level equation derived by Farrell & Clark (1976, *Geophys. J.R. astr. Soc.*, **46**, 647–667). The connection between global sea level variations and changes to ocean height that is assumed within this equation breaks down in the presence of a time-varying shoreline geometry. We present a generalized sea level equation that overcomes this difficulty. We also derive analytic expressions for, and present schematic illustrations of, the error in the ocean height change over finite time intervals introduced in published efforts to incorporate shoreline evolution into the theory of GIA-induced sea level change. This comparison includes studies of shoreline migration due to either local sea level changes or the growth and ablation of marine-based ice. We conclude that the theories applied by Johnston (1993, *Geophys. J. Int.*, **114**, 615–634), Milne (1998, PhD thesis, University of Toronto, Toronto) and co-workers are more accurate than the procedure advocated by Peltier (1994, *Science*, **265**, 195–201; 1998a, *Geophys. Res. Lett.*, **25**, 3955–3958; 1998b, *Rev. Geophys.*, **114**, 615–634), although an improvement in the latter has recently been reported (Peltier & Drummond (2002, *Geophys. Res. Lett.*, **29**, 10.1029/2001GL014273). Our generalized theory is valid for any Earth model. In a companion paper we derive the equations necessary to treat the special case of a spherically symmetric, linear viscoelastic and rotating Earth, and we quantify errors associated with previous work.

Key words: glacial isostasy, sea-level.

1 INTRODUCTION

The calculation of gravitationally self-consistent sea level changes driven by the melting of ice sheets is a classic problem in geophysics. Woodward (1988), following earlier work, showed that melting of ice sheets would be accompanied by highly non-uniform (or non-eustatic) changes in sea level as a consequence of the direct gravitational effects of the surface load. His rigid Earth calculations were extended by a variety of authors (e.g. Daly 1925) to include the effects of solid-Earth deformation, culminating in the seminal treatment of Farrell & Clark (1976) in which an 'exact method (was) presented for calculating the changes in sea level that occur when ice and water masses are rearranged on the surface of elastic and viscoelastic non-rotating Earth models' (Farrell & Clark 1976, p. 647). The ultimate aim of the Farrell & Clark (1976) paper was to provide a formalism for predicting sea level changes driven by the growth and ablation of the Late Pleistocene ice complexes. In this regard, their formalism was intended to replace earlier, more ad hoc, methods of analysing geological markers of post-glacial sea level change in applications involving inferences of Late Pleistocene ice geometries and/or Earth rheology.

The Farrell & Clark (1976) formalism remains a standard pillar of modern research in glacial isostatic adjustment (henceforth GIA) and interest in post-glacial sea level change has broadened to reflect an ever-increasing set of geophysical applications. Since the redistribution of ocean mass constitutes, together with the ice load, the total surface mass load, a robust prediction of any GIA-related observable, whether it involves sea level markers or not, requires an accurate prediction of the sea level change.

Farrell & Clark (1976) derived the following 'sea level equation' governing GIA-induced perturbations in global sea level, $\Delta SL(\theta, \psi, t)$:

$$\Delta SL(\theta, \psi, t) = \frac{\rho_{\rm I}}{g}\Phi^*_{\rm I} I + \frac{\rho_{\rm W}}{g}\Phi^*_{\rm O}\Delta SL + C_{SL}(t), \quad \text{(I1)}$$

where $\rho_{\rm I}$ and $\rho_{\rm W}$ are the densities of ice and water, respectively, g is the gravitational acceleration, θ is the co-latitude and ψ is the longitude of the sea level change and t is the time. The space–time evolution of the ice cover is given by I. The symbol Φ represents a Green's function for the potential perturbation that is constructed by suitably combining viscoelastic surface load Love numbers (Peltier 1974) and the superscript $*$ denotes a convolution over both geographic coordinates and time. In this case, the subscripts I and O refer to convolutions limited to the geometry of the ice cover and oceans, respectively. Finally, C_{SL} is a time-varying and geographically uniform shift in the geoid that is constrained by invoking conservation of mass. Integrating the above equation over the ocean

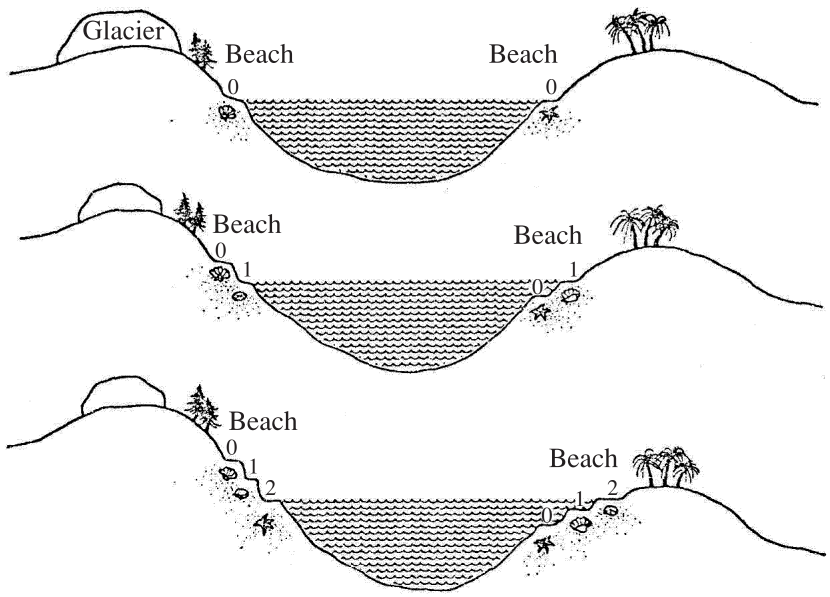

Figure 1. Schematic illustration of sea-level changes on a viscoelastic earth model, taken from Farrell & Clark (1976, Fig. 6).

geometry yields

$$C_{SL}(t) = -\frac{M_{\mathrm{I}}(t)}{A_{\mathrm{O}}\rho_{\mathrm{W}}} - \frac{1}{A_{\mathrm{O}}}\left\langle \frac{\rho_{\mathrm{I}}}{g}\Phi_{\mathrm{I}}^{*}I + \frac{\rho_{\mathrm{W}}}{g}\Phi_{\mathrm{O}}^{*}\Delta SL\right\rangle_{\mathrm{O}}, \tag{I2}$$

where M_{I} is the change in ice mass relative to some reference state, A_{O} is the area of the ocean and the symbol $\langle\ \rangle_{\mathrm{O}}$ refers to a spatial integration limited to the location of ocean. The first term on the right-hand side of eq. (I2) is the so-called eustatic sea level change.

A primary assumption made in deriving the sea level eq. (I1) is that the Earth's 'response to loads can be accurately represented by computing the impulsive response of a spherically symmetric model with a linear viscosity and perfect Hookean elasticity' (Farrell & Clark 1976, p. 666). The sea level equation is clearly an integral equation because the global sea level change being sought also appears in two terms on the right-hand side. In practice, the equation is usually solved iteratively by successively refining a first guess to the global sea level change (e.g. Farrell & Clark 1976). The space–time convolution of the Green's function with global sea level change is limited to the oceans since it is clear that only ocean height changes constitute a surface mass loading of the Earth model.

Fig. 1 is a schematic illustration taken from Farrell & Clark (1976) showing incremental ocean level changes driven by the melting of a land-based glacier on a viscoelastic planet. The figure involves a shoreline that migrates landward as local sea level rises and oceanward as it falls. Nevertheless, their subsequent calculations, and indeed predictions of post-glacial sea level change over the next 15 years (e.g. Peltier & Andrews 1976; Wu & Peltier 1983; Nakada & Lambeck 1989; Tushingham & Peltier 1991), all assumed that the shoreline remained fixed as sea level rose and fell through a glacial cycle. That is, these calculations assumed that the edge of the ocean basin was characterized by steep cliffs that prevented any onlap or offlap of water.

The initial numerical procedure developed to solve the Farrell & Clark (1976) sea level equation was based on a spatial discretization of the Earth's surface, including all of the ocean basins and any regions covered by Late Pleistocene ice, into a set of 'finite elements' (e.g. Clark *et al.* 1978; Peltier *et al.* 1978; Wu & Peltier 1983). In this case, the spatial convolution appearing in the sea level equation could be replaced by a sum over a set of 'active elements'. A table providing the response of the Earth as a function of distance from a set of elements of varying size was pre-computed by assuming that the element could be replaced by a circular cap of equal area. The convolution in time was performed analytically by discretizing the surface mass load into a series of Heaviside step increments.

This early approach to solving the sea level eq. (I1) was cumbersome and not widely adopted. As an example, changes in the spatial resolution of the sea level calculation would require time-consuming global rediscretizations. Furthermore, the level of error introduced by using spherical caps as proxies for the actual elements in computing the response matrices was not established. To remedy this, Mitrovica & Peltier (1991) derived both fully spectral and pseudospectral approaches to solving the sea level equation. The latter scheme has now become the standard tool within the GIA community. Their approach was formulated for the case of a fixed ocean–continent geometry and it was based on a slightly altered (but equivalent) form of the Farrell & Clark (1976) sea level eqs (I1) and (I2):

$$\Delta S(\theta, \psi, t) = C(\theta, \psi)\Delta SL(\theta, \psi, t), \tag{I3}$$

where

$$\Delta SL(\theta, \psi, t) = \frac{\rho_{\mathrm{I}}}{g}\Phi^{*}I + \frac{\rho_{\mathrm{W}}}{g}\Phi^{*}\Delta S + C_{SL}(t), \tag{I4}$$

and

$$C_{SL}(t) = -\frac{M_{\mathrm{I}}(t)}{A_{\mathrm{O}}\rho_{\mathrm{W}}} - \frac{1}{A_{\mathrm{O}}}\left\langle \frac{\rho_{\mathrm{I}}}{g}\Phi^{*}I + \frac{\rho_{\mathrm{W}}}{g}\Phi^{*}\Delta S\right\rangle_{\mathrm{O}}. \tag{I5}$$

In these equations the *ocean height change*, $\Delta S(\theta, \psi, t)$, is computed from the *global sea level variation*, $\Delta SL(\theta, \psi, t)$, by projecting the latter on to the so-called ocean function $C(\theta, \psi)$ (Munk & MacDonald 1960); the ocean function has a value of unity over the ocean and zero elsewhere. By explicitly defining such a projection in eq. (I3), the spatial convolution need not be *a priori* 'limited' to the ocean as in eqs (I1) and (I2); rather, the convolution is performed globally on the field representing ocean height changes.

Using the form (I3)–(I5) of the sea level equation, the pseudospectral algorithm proceeds as follows.

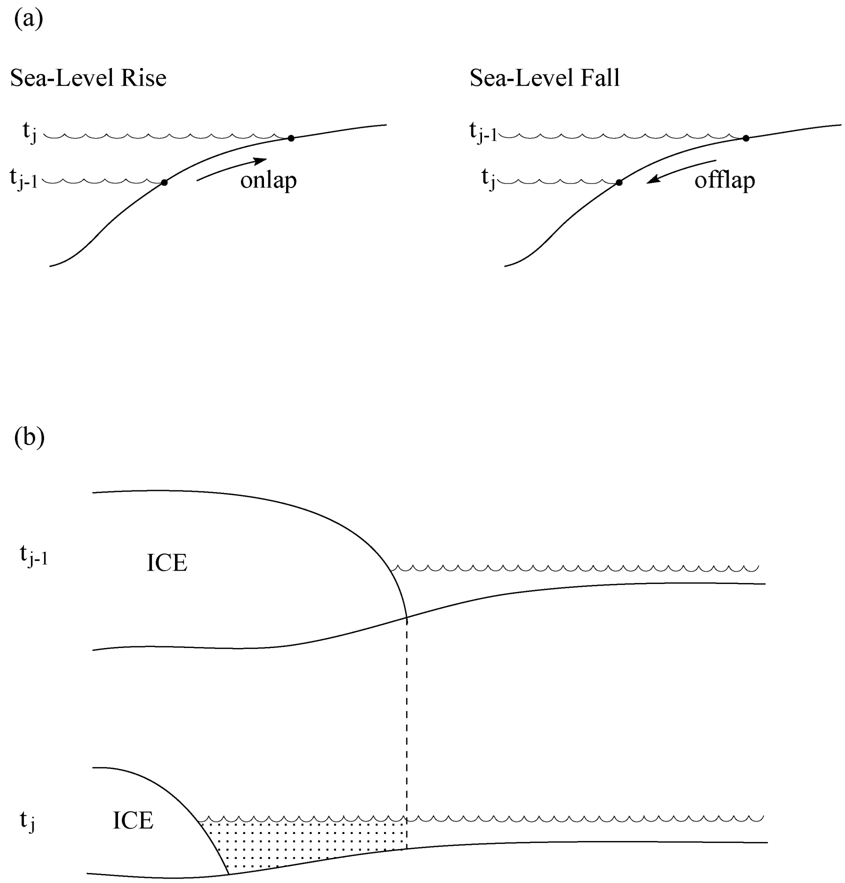

Figure 2. (a) A schematic illustration of shoreline evolution in the vicinity of a sea level rise or fall (after Milne *et al.* 1999, Fig. 1). (b) An illustration of the water load influx into regions vacated by marine-based ice between $t = t_{j-1}$ and $t = t_j$ (Milne *et al.* 1999, Fig. 2).

(1) Beginning with an initial guess to ΔS, the global sea level change ΔSL is computed from eq. (I4) using a purely spectral formulation in which the ocean and ice height changes are expressed in terms of spherical harmonic expansions and the spatial convolutions are performed analytically.

(2) To perform the projection on to the ocean function in eq. (I3), and for this purpose alone, the global sea level change ΔSL computed from step (1) is transformed into the space domain. The projection is thus reduced to a simple multiplication of grid elements.

(3) This product is transformed back into the spectral domain to yield the next estimate of the ocean height change, and the process is repeated until convergence.

The pseudospectral approach is highly efficient numerically because it applies fast-transform methods in the move from the spectral to space domain and back. Furthermore, an increase in spatial resolution is achieved trivially by increasing the truncation level (in powers of two) of the various spherical harmonic expansions. Indeed, truncation levels of 512 or more are certainly now feasible using the approach.

Although the sea level eqs (I3)–(I5) solved by Mitrovica & Peltier (1991) are equivalent to the Farrell & Clark (1976) eqs (I1) and (I2), the newer treatment involved two somewhat subtle differences beyond the obvious change in numerical algorithm. First, Mitrovica & Peltier (1991) described the global sea level change as a perturbation in the distance between two surfaces, namely the geoid (ocean surface) and the solid surface, and they provided distinct Green's functions for each. More importantly, the connection between ocean height variations and global sea level changes, at least for the case of a fixed shoreline, was made explicit in the new treatment (eq. I3), rather than being implicitly invoked by limiting the bounds of the spatial convolution.

Over the last decade, numerous efforts to extend previous sea level predictions based on the Farrell & Clark (1976) theory and/or the Mitrovica & Peltier (1991) pseudospectral numerical algorithm to incorporate a time-varying shoreline geometry have appeared (e.g. Lambeck & Nakada 1990; Johnston 1993; Peltier 1994, 1998a,b; Milne 1998; Milne *et al.* 1999; Peltier & Drummond 2002). Fig. 2 illustrates two processes that can lead to a migration of the shoreline. Fig. 2(a), as in Fig. 1, demonstrates how a local rise or fall in sea level produces an onlap or offlap, respectively, at the ocean–continent interface and thus a migration of the shoreline. In Fig. 2(b), the melting of marine-based ice is followed by an influx of water and a rather dramatic shift in the edge of the ocean; similarly, the growth of marine-based ice would lead to a local removal (or replacement) of water.

The question arises as to whether the Farrell & Clark (1976) theory embodied in eqs (I1) and (I2) remains valid in either of the situations in Fig. 2? Unfortunately, recent efforts to deal with these situations have led to significant confusion and ongoing debate. As an example, Johnston (1993), Peltier (1994) and Milne (1998) have provided distinct sea level equations for the case of shoreline migration in the absence of marine-based ice. Peltier (1994), for example, incorporated shoreline migration by making the ocean function time-dependent in eq. (I3); that is by replacing $C(\theta, \psi)$ by $C(\theta, \psi, t)$ in the projection between global sea level variations and ocean height changes. The sea level equations provided by both Johnston (1993) and Milne (1998) involved a more significant, albeit ill-defined, level of revision to the Farrell & Clark (1976) theory. Most recently, Peltier & Drummond (2002) have introduced a 'broad shelf effect' that improves the theory derived by Peltier (1994). The connections between all of these various algorithms, and the underlying approximations made in each of them, are not clear.

The literature is no less confusing when the process shown in Fig. 2(b) is considered. Milne (1998) and Milne *et al.* (1999), who coined the term 'water dumping' to describe the scenario, described a further revision to their sea level equation to model this process. Peltier (1998a) independently argued that the process is inherently 'non-perturbative' and he performed what appears to be an *a posteriori* calculation of the amount of ocean water that would have replaced the ablating ice mass; he then computed the equivalent ice thickness of this inundation and defined this to be a (dominant) contributor to the 'implicit' component of the ice load. Once again, the connection and differences between the two algorithms is unclear.

To what extent the Farrell & Clark (1976) theory remains valid in the presence of the processes shown in Fig. 2 is actually a combination of two, somewhat unrelated issues. First, Farrell & Clark (1976) showed that, in the case of spherically symmetric, linear viscoelastic non-rotating Earth models, the GIA-induced perturbation in global sea level could be written in the form of a convolution of the potential perturbation Green's function with the surface mass load (including both ice and ocean height changes) and conservation of mass terms (eqs I1 or I4). This is certainly still correct. Indeed, the only recent extension in this regard is to complement eqs (I3)–(I5) to include changes in global sea level driven by contemporaneous GIA-induced variations in the Earth's rotation vector (e.g. Han & Wahr 1989; Bills & James 1996; Milne & Mitrovica 1996, 1998a; Peltier 1998b; Mitrovica *et al.* 2001). The second issue is whether the ocean height *change* can be determined by spatially limiting the convolution of the global sea level *change*, as in eqs (I1) and (I2), when the processes in Fig. 2 are active? Or, in the context of the Mitrovica & Peltier (1991) formulation of the sea level theory, whether the ocean height *change* is always defined as a simple projection of the global sea level *change* on to (in the case of Fig. 2) a time-dependent ocean function, as in eq. (I3)?

This paper deals in detail with the second of these issues by deriving an exact, generalized relationship between GIA-induced global sea level variations and ocean height changes across finite time intervals. We conclude, on this basis, that the traditional theory embodied in eqs (I1) and (I2) cannot accurately treat the case envisaged by Farrell & Clark (1976) in their Fig. 6 (our Fig. 1), or the processes illustrated in Fig. 2. We demonstrate the validity of our generalized sea level theory using a series of schematic illustrations. We also provide both analytic expressions and schematic illustrations of the error in the ocean height change introduced in previous efforts to incorporate the processes shown in Fig. 2 in the sea level modelling.

Our theory holds for any type of Earth model because it makes no assumption as to how the global sea level change is computed. That is, ΔSL may be computed from analytic expressions, as in eq. (I4), for the case of spherically symmetric Earth models, or be output from a new generation of software being developed to treat laterally varying Earth models. In a companion paper (henceforth Paper II) we provide complete analytic expressions that are required to solve for sea level change on rotating, spherically symmetric Earth models with realistic shoreline geometries and marine-based ice. This companion paper will also compare quantitative results generated from previous forms of the governing sea level theory.

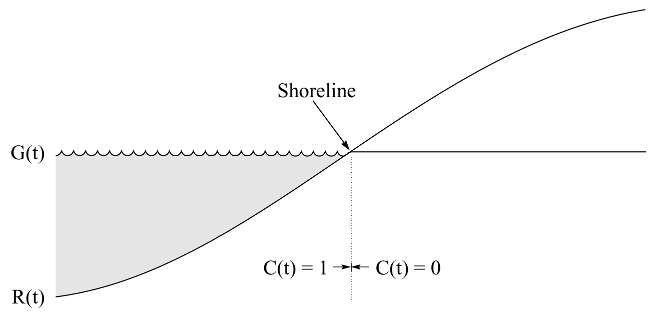

Figure 3. Schematic illustration showing 'sea level' as the difference between the geoid and solid surfaces, and 'ocean height' (the vertical height of the shaded region) as the projection of sea level on to the ocean function. In this and all subsequent figures the dependence of the various fields on the geographic coordinates θ and ψ is suppressed.

2 THEORY

Let us consider the simple geometry shown in Fig. 3. In the figure R is the position of the solid surface and G denotes the geoid. We define 'sea level' as the region bounded by these two surfaces, specifically

$$SL(\theta, \psi, t) \equiv G(\theta, \psi, t) - R(\theta, \psi, t). \tag{1}$$

The surfaces $G(\theta, \psi, t)$ and $R(\theta, \psi, t)$ are absolute heights measured (for example) relative to the centre of the Earth and they are not to be confused with perturbations associated with a geophysical process such as GIA. Our definition (1) follows standard practice in the GIA literature, where sea level is referenced to the local height of the solid surface. This practice is motivated by the geological markers of sea level change, such as uplifted strandlines, that reflect variations in $SL(\theta, \psi, t)$ as opposed to absolute sea level change. A definition of topography, T, follows trivially from eq. (1) and Fig. 3:

$$T(\theta, \psi, t) \equiv R(\theta, \psi, t) - G(\theta, \psi, t), \tag{2}$$

$$= -SL(\theta, \psi, t). \tag{3}$$

Within oceanic regions the globally defined field, $SL(\theta, \psi, t)$, is equivalent to the ocean height. While this connection might be deemed trivial, as we illustrate below (and as discussed in the introduction) the relationship between *changes* in *SL* and *changes* in the ocean height lies at the heart of much of the complexity associated with recent predictions of GIA-induced sea level change.

To begin, we will derive a sea level theory for the somewhat special case where ice complexes include no marine-based components. That is, we will assume that the ocean geometry evolves due to the onlap and offlap of water at shorelines (as local sea level rises and falls) and not due to the growth or ablation of marine-based ice. The extension required to deal with the latter case is straightforward and will be described in a subsequent section. There are several reasons for our approach. First, it will permit us to isolate, for the purposes of illustration, the rather distinct physical processes occurring at shorelines with and without marine-based ice. Secondly, it will permit a clearer comparison of results from the GIA literature.

2.1 Special case: no marine-based ice

In the absence of marine-based ice, the 'ocean' will be defined to exist at any location where sea level, SL, is positive. In this case, the ocean height, $S(\theta, \psi, t)$, can easily be derived from eqs (1)–(3) as

$$S(\theta, \psi, t) = SL(\theta, \psi, t)C(\theta, \psi, t), \tag{4a}$$

$$= [G(\theta, \psi, t) - R(\theta, \psi, t)]C(\theta, \psi, t), \tag{4b}$$

$$= -T(\theta, \psi, t)C(\theta, \psi, t), \tag{4c}$$

where, as described in the introduction, C is the ocean function (Munk & MacDonald 1960) defined such that

$$\begin{aligned} C(\theta, \psi, t) &= 1 \quad \text{where } SL(\theta, \psi, t) > 0 \text{ (or } T(\theta, \psi, t) < 0), \\ &= 0 \quad \text{where } SL(\theta, \psi, t) < 0 \text{ (or } T(\theta, \psi, t) > 0). \end{aligned} \tag{5}$$

Eqs (4) represent a projection of the globally defined sea level, SL, on to the ocean function. To avoid a common source of confusion, in what follows we will *always* refer to SL as sea level and to S as the ocean height (see Fig. 3 and its caption).

The field of GIA is concerned with the time evolution of sea level and the ocean height. We assume that the time dependence embodied in eqs (4) arises from the GIA process. The influence of GIA enters our theory through its impact on what we might call the 'primary' surfaces $G(\theta, \psi, t)$ and $R(\theta, \psi, t)$, or alternatively their combination $SL(\theta, \psi, t)$ or $T(\theta, \psi, t)$. The perturbation to these primary surfaces leads to variations in the ocean function according to eq. (5) and the ocean height following any of eqs (4).

Let us denote $t = t_0$ as any time prior to the onset of glaciation, then the perturbation to the ocean height S from $t = t_0$ to $t = t_j$ represents the change due to GIA during this interval. As in the introduction, we will denote the change arising from GIA by the symbol Δ. We can therefore write

$$G(\theta, \psi, t_j) = G(\theta, \psi, t_0) + \Delta G(\theta, \psi, t_j), \tag{6}$$

$$R(\theta, \psi, t_j) = R(\theta, \psi, t_0) + \Delta R(\theta, \psi, t_j), \tag{7}$$

$$SL(\theta, \psi, t_j) = SL(\theta, \psi, t_0) + \Delta SL(\theta, \psi, t_j), \tag{8}$$

$$T(\theta, \psi, t_j) = T(\theta, \psi, t_0) + \Delta T(\theta, \psi, t_j), \tag{9}$$

where, following eqs (1) and (2),

$$\Delta SL(\theta, \psi, t_j) = \Delta G(\theta, \psi, t_j) - \Delta R(\theta, \psi, t_j), \tag{10}$$

$$\begin{aligned} \Delta T(\theta, \psi, t_j) &= \Delta R(\theta, \psi, t_j) - \Delta G(\theta, \psi, t_j) \\ &= -\Delta SL(\theta, \psi, t_j). \end{aligned} \tag{11}$$

Expressions for the perturbations to the geoid and solid surfaces, ΔG and ΔR, respectively, can be derived, for example, from 'standard' theoretical approaches for the case of linear viscoelastic and spherically symmetric Earth models (see the introduction and Paper II), or from the output of a new generation of numerical codes being developed to treat more complex Earth models.

We next turn our attention to the ocean height, S. Using eqs (6)–(11), eqs (4) for the total ocean height can be rewritten as (for time t_j):

$$S(\theta, \psi, t_j) = [SL(\theta, \psi, t_0) + \Delta SL(\theta, \psi, t_j)]C(\theta, \psi, t_j), \tag{12a}$$

$$\begin{aligned} = [&G(\theta, \psi, t_0) - R(\theta, \psi, t_0) + \Delta G(\theta, \psi, t_j) \\ &- \Delta R(\theta, \psi, t_j)]C(\theta, \psi, t_j), \end{aligned} \tag{12b}$$

$$= -[T(\theta, \psi, t_0) + \Delta T(\theta, \psi, t_j)]C(\theta, \psi, t_j). \tag{12c}$$

Although GIA forward predictions will provide ΔG, ΔR, ΔSL and ΔT, they do not directly provide the total value of these fields at some reference time t_0. However, while we do not know, *a priori*, the topography at $t = t_0$, we do know the topography at the present time, $t = t_p$. This knowledge permits one to iteratively refine a first guess to $T(\theta, \psi, t_0)$ (Peltier 1994). This suggests, for the purposes of computation, a useful hybrid version of eqs (12):

$$S(\theta, \psi, t_j) = [-T(\theta, \psi, t_0) + \Delta SL(\theta, \psi, t_j)]C(\theta, \psi, t_j). \tag{13}$$

Given a time-varying global sea level, $SL(\theta, \psi, t)$, an exact expression for the GIA-induced change in the ocean height since the onset of loading, ΔS, can be determined using any one of eqs (12) or (13). However, we begin with a simpler expression derived from eq. (4a):

$$\Delta S(\theta, \psi, t_j) = SL(\theta, \psi, t_j)C(\theta, \psi, t_j) - SL(\theta, \psi, t_0)C(\theta, \psi, t_0). \tag{14}$$

A schematic illustration of this expression is provided in Fig. 4. Specifically, the original ocean is the region between dashed lines to the left of the 'original shoreline', while the new ocean is the region between solid lines to the left of the new, migrated, shoreline. Consequently, the ocean height *change* is given by the total vertical thickness of the shaded region on the figure.

For the purposes of computation, a more useful expression for ΔS can be generated using our so-called 'hybrid' eq. (13):

$$\begin{aligned} \Delta S(\theta, \psi, t_j) = &\Delta SL(\theta, \psi, t_j)C(\theta, \psi, t_j) \\ &- T(\theta, \psi, t_0)[C(\theta, \psi, t_j) - C(\theta, \psi, t_0)], \end{aligned} \tag{15}$$

where we have also used $\Delta SL(\theta, \psi, t_0) \equiv 0$.

Eq. (15) demonstrates that the change in ocean height over the interval from t_0 to t_j is not simply the change in sea level over this time period projected on to the ocean function at t_j. Instead, it is this projection minus the value of the *original* topography (or, plus the value of the *original* sea level) in a region bounded by the shoreline evolution from t_0 to t_j.

In the introduction we indicated that the sea level perturbation due to ocean loading effects is determined, in the Farrell & Clark (1976) sea level eqs (I1) and (I2), by limiting the spatial convolution between the potential perturbation Green's function and the global sea level change to the location of the ocean. This is equivalent to assuming that the change in ocean height is a simple projection of the change in global sea level on to the ocean function. In the case of a fixed shoreline, eq. (15) simplifies to eq. (I3) (see also Section 2.1.1 below) and thus the assumption holds. In the presence of a time-varying shoreline, eq. (15) indicates that the assumption made by Farrell & Clark (1976) breaks down.

We examine the validity of eq. (15) in more detail in Appendix A1, with reference to the individual regions identified in Fig. 4. Each of the terms in eq. (15) is associated with the height of a set of regions in the figure and the net result of applying the equation is shown in the appendix to be equivalent to the ocean height change indicated by the shaded region in Fig. 4 (and the simple eq. 14).

GIA predictions of post-glacial sea level change are generally solved at discrete times that define the ice model input into the calculation. In this case, the sea level algorithms are often formulated to solve for the sea level or ocean height change over two successive times, say t_{j-1} and t_j. Using eqs (4a) and (13) we can write

$$\begin{aligned} \delta S(\theta, \psi, t_j) &\equiv S(\theta, \psi, t_j) - S(\theta, \psi, t_{j-1}) \\ &= SL(\theta, \psi, t_j)C(\theta, \psi, t_j) \\ &\quad - SL(\theta, \psi, t_{j-1})C(\theta, \psi, t_{j-1}) \end{aligned} \tag{16}$$

$$\begin{aligned} = &[-T(\theta, \psi, t_0) + \Delta SL(\theta, \psi, t_j)]C(\theta, \psi, t_j) \\ &- [-T(\theta, \psi, t_0) + \Delta SL(\theta, \psi, t_{j-1})]C(\theta, \psi, t_{j-1}). \end{aligned} \tag{17}$$

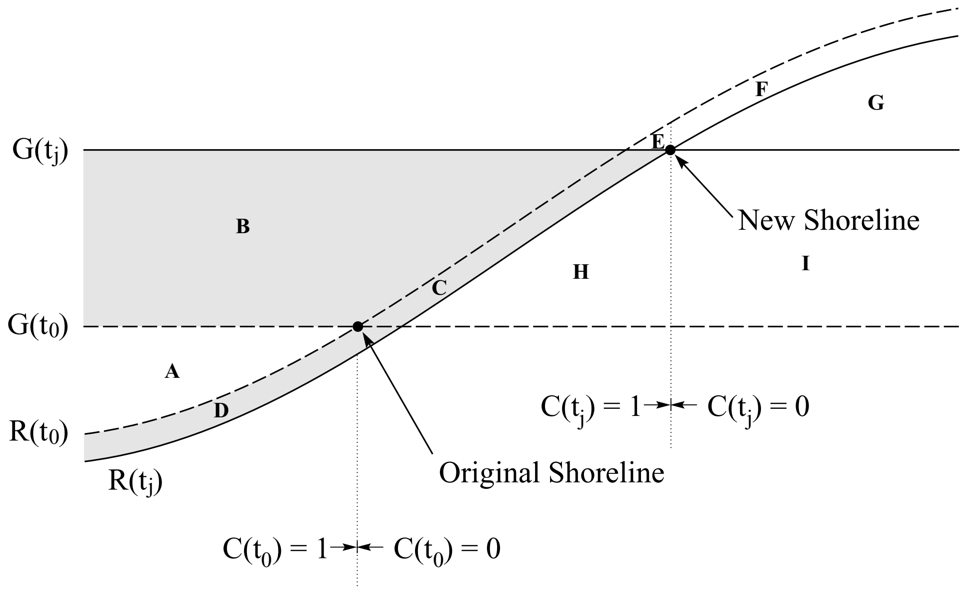

Figure 4. Schematic showing the change in ocean height over the period from the onset of loading, $t = t_0$, to $t = t_j$, in the case of a time-varying shoreline (the vertical height of the shaded region). The figure is an illustration of eq. (14) in the text. With reference to the various labelled regions on this figure, Appendix A1 demonstrates the validity of the sea level eq. (15) governing the change in ocean height over the period from $t = t_0$ to $t = t_j$. Note that labels A–I are not to be interpreted as areas; rather, they represent heights of the associated region as a function of position.

If we use $SL\,(\theta, \psi, t_j) = SL\,(\theta, \psi, t_{j-1}) + \delta SL\,(\theta, \psi, t_j)$ in eq. (16), we can derive a second useful form of eq. (17):

$$\delta S(\theta, \psi, t_j) = \delta SL(\theta, \psi, t_j) C(\theta, \psi, t_j) + SL(\theta, \psi, t_{j-1})[C(\theta, \psi, t_j) - C(\theta, \psi, t_{j-1})]. \tag{17b}$$

Eq. (16) is analogous to eq. (14), with the exception that the former represents the exact change in ocean height across a single time increment (given the global sea level *SL* at the beginning and end of this finite time interval). Similarly, eq. (17b) is analogous to eq. (15). The validity of eq. (17) is demonstrated in the main frame of the schematic Fig. 5 and Appendix A2.

Eqs (15) and (17) represent the so-called 'sea level equations' we sought at the outset of this section. As we discussed above, the derivation of these equations assumes that ice sheets at any time t_j did not extend into regions defined by a non-zero ocean function. That is, these equations hold for the case in which the time-dependent ice sheet perimeters played no role in defining the evolution of the ocean perimeter.

In general, the solution of these equations at each time step requires two separate iterations. Since the GIA-induced perturbations ΔSL or δSL arise from surface mass loading, these perturbations implicitly involve a dependence on the time-varying ocean height through the ocean load component of the mass load (see eq. I4 for the case of spherically symmetric Earth models). The ocean height variation also appears explicitly on the left-hand side of the sea level eqs (15) and (17), and thus these expressions are ultimately integral equations. As discussed in the introduction, the solution of this integral form has commonly involved an iterative scheme in which an initial guess to the sea level perturbation is successively refined (e.g. Peltier *et al.* 1978; Wu & Peltier 1983; Mitrovica & Peltier 1991). As also discussed above, a second iteration is required to deal with the appearance of the reference topography, $T\,(\theta, \psi, t_0)$ (or, alternatively, the reference sea level, $SL(\theta, \psi, t_0)$), which is not known *a priori*. To consider this issue further, we can combine eqs (3) and (9) to yield

$$T(\theta, \psi, t_p) = T(\theta, \psi, t_0) - \Delta SL(\theta, \psi, t_p). \tag{18}$$

Eq. (18) provides a method for iteratively refining our knowledge of $T(\theta, \psi, t_0)$ (see Peltier 1994). Our initial guess for $T(\theta, \psi, t_0)$ would be the known field $T(\theta, \psi, t_p)$. Adopting this guess within the sea level equation ultimately yields a prediction of the present-day change in sea level $\Delta SL(\theta, \psi, t_p)$. This prediction can then be used, via eq. (18), to provide a second iterate for $T(\theta, \psi, t_0)$, and so on.

In the following we first treat the special case of a time-independent ocean shoreline. We then turn to a discussion of recent efforts to incorporate time-dependent shorelines in GIA sea level theory.

2.1.1 *A time-independent shoreline*

Prior to the 1990s all numerical predictions of post-glacial sea level change assumed that the location of shorelines remained fixed in time. Physically, these calculations assumed that the entire ocean–continent interface was characterized by steep cliffs, which would preclude onlap and offlap of water as sea level varied in the vicinity of the shoreline. In this case $C(\theta, \psi, t_{j-1}) = C(\theta, \psi, t_j) = C(\theta, \psi)$, and thus eqs (15) and (17) become

$$\Delta S(\theta, \psi, t_j) = \Delta SL(\theta, \psi, t_j) C(\theta, \psi), \tag{19}$$

$$\delta S(\theta, \psi, t_j) = [\Delta SL(\theta, \psi, t_j) - \Delta SL(\theta, \psi, t_{j-1})] C(\theta, \psi). \tag{20}$$

In this special case, topography obviously does not enter into the predictions. The ocean height change is computed from perturbations in sea level derived, following eq. (10), from perturbations to the geoid and solid surface, and knowledge of the present-day ocean function. Eqs (19) and (20) represent the sea level equation that has been adopted, in one form or the other, by the vast majority of GIA analyses (with expressions for ΔSL derived under the assumption

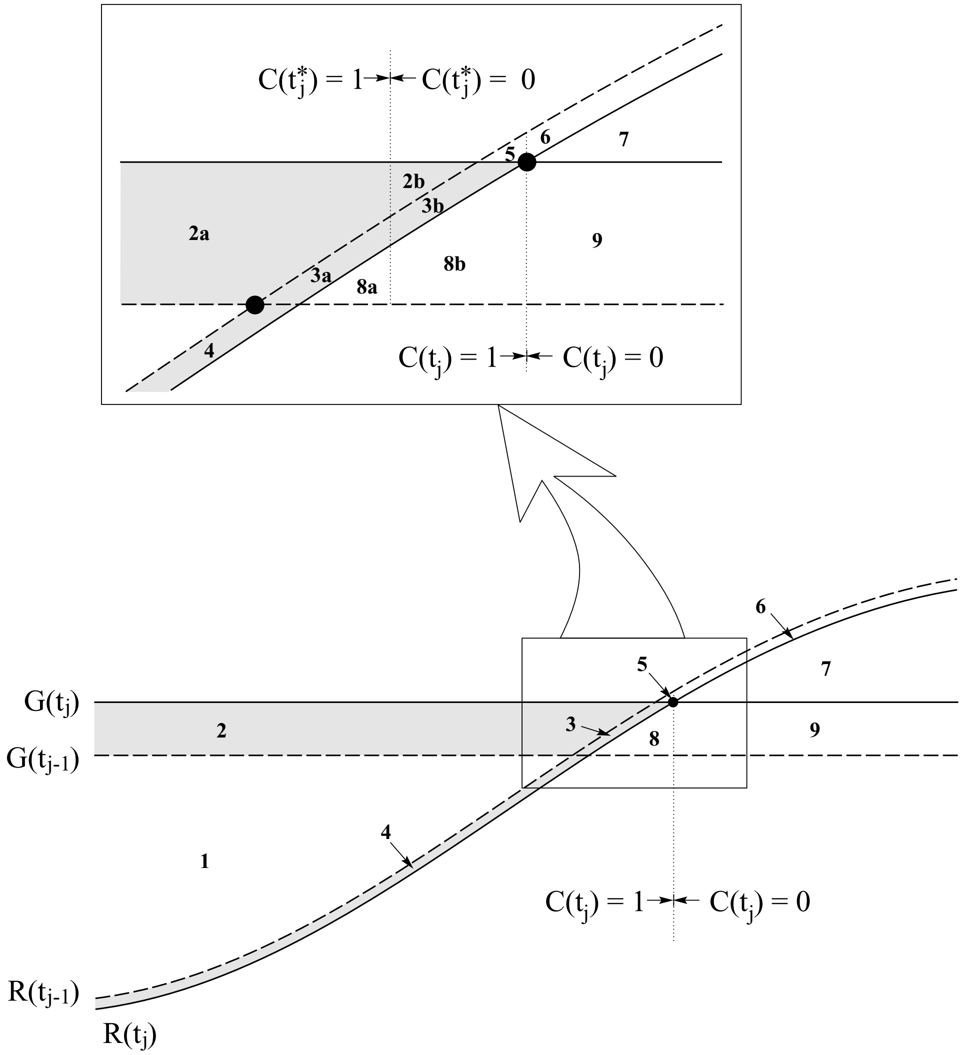

Figure 5. Schematic showing the change in ocean height over the period from $t = t_{j-1}$ to $t = t_j$ in the case of a time-varying shoreline (the vertical height of the shaded region). The figure, combined with the results in Appendix A2, illustrates the validity of eq. (17) in the text. Inset—a blow-up of the region close to the evolving shoreline. In this case, $C(\theta, \psi, t_j^*)$ represents the ocean function at some time between t_{j-1} and t_j; the regions 2, 3 and 8 on the main figure are split into areas oceanward of this shoreline (2a, 3a and 8a) and landward of this shoreline (2b, 3b and 8b) within the inset. The inset is described below eq. (23) of the text and in Appendix A3. Note that labels 1–9 (and subregions) are not to be interpreted as areas; rather, they represent heights of the associated region as a function of position.

of a spherically symmetric linear viscoelastic Earth model; as in eqs I3–I5).

2.1.2 *Johnston (1993)*

Researchers at the Australian National University were the first to implement a time-varying coastline geometry into predictions of post-glacial sea level change (Lambeck & Nakada 1990; Johnston 1993). Johnston (1993) provided a detailed description of the algorithm developed for this purpose, and his eq. (10) provides a useful starting point for comparison. Using symbols introduced above, his equation reads

$$S_{\mathrm{J}}(\theta, \psi, t_j) = \sum_{i=1}^{j} [\Delta SL(\theta, \psi, t_i) - \Delta SL(\theta, \psi, t_{i-1})] C(\theta, \psi, t_i^*), \tag{21}$$

where the time t_i^* is chosen to be 'representative of the time interval $t_{i-1} < t < t_i$' (Johnston 1993, p. 618), and the subscript J denotes the Johnston (1993) approximation. The *j*th increment in ocean height is, from eq. (21)

$$\delta S_{\mathrm{J}}(\theta, \psi, t_j) = [\Delta SL(\theta, \psi, t_j) - \Delta SL(\theta, \psi, t_{j-1})] C(\theta, \psi, t_j^*), \tag{22}$$

which we can rewrite as

$$\delta S_{\mathrm{J}}(\theta, \psi, t_j) = [-T(\theta, \psi, t_0) + \Delta SL(\theta, \psi, t_j) - (-T(\theta, \psi, t_0) + \Delta SL(\theta, \psi, t_{j-1}))] C(\theta, \psi, t_j^*). \tag{23}$$

Eq. (23) can be compared with our correct expression for the ocean height change between successive time increments, eq. (17). Instead of taking the difference between total sea level change projected on to the associated ocean function, Johnston (1993) approximates this difference by adopting a 'representative' ocean function projection for the total sea level change at both t_{j-1} and t_j.

It is difficult to quantify the error introduced by this approximation given that the choice of t_j^* was not defined. However, in the inset of

Fig. 5 and Appendix A3 we explore the error in the ocean height change introduced, for the simple case illustrated in the main frame of the figure, when t_j^* is chosen such that the shoreline at t_j^* is half way between the shorelines at t_{j-1} and t_j. According to eq. (A.8), the correct expression for the change in ocean height is the addition of the heights of regions 2–4 (see the shaded region in Fig. 5). The approximation (23) correctly captures region 4: however, a section of region 2 (2b in Fig. 5, inset) is omitted; region 3 is replaced by a double count of subregion 3a; and a portion of region 8 (8a in Fig. 5, inset) is included in the approximation.

The error in the ocean height change introduced by the Johnston (1993) approximation is a function of the choice of t_j^*, the size of the time step and the geometry of the shoreline in the vicinity of the ocean migration. It is clear from Fig. 5 and Appendix A3 that the error in the ocean load within each time increment is confined to a relatively small region close to the location of the shoreline.

2.1.3 *Peltier (1994)*

Peltier (1994) was next to describe a method for incorporating a time-dependent shoreline in predictions of post-glacial sea level change. His governing equation (2) was derived through an extension of the equation for the time-independent shoreline case (our eq. 19) to include a time-varying ocean function:

$$\Delta S_{\mathrm{P}}(\theta, \psi, t_j) = \Delta SL(\theta, \psi, t_j) C(\theta, \psi, t_j), \tag{24}$$

where the subscript P denotes the Peltier (1994) approximation. A comparison of this equation with the correct expression, eq. (15), for the total ocean height change due to GIA since the onset of loading indicates that the former omits the second term on the right-hand side of the latter. Thus, the amplitude of the error in the ocean height change introduced by eq. (24) is equal to the initial topography within the region of shoreline migration from $t = t_0$ to $t = t_j$.

Consider again the illustration in Fig. 4. The analysis in Appendix A1 (see eq. A4) indicates that the error incurred in eq. (24) is equivalent to the total height of regions C + H + E. This region is highlighted in Fig. 6. In comparison with the Johnston (1993) approximation, the error introduced by eq. (24) is relatively large and it accumulates rapidly as time progresses from $t = t_0$. Over a single deglaciation (or glaciation) phase, the global shift in ocean height is ∼120 m, and this value serves as a bound on the amplitude of the error in ΔS_{P}. The horizontal scale of the error, defined by the shoreline migration from $t = t_0$ to $t = t_j$, depends on the geometry of the local shoreline.

2.1.4 *Milne (1998) and Milne et al. (1999)*

Milne and colleagues also extended the traditional sea level equation to consider the case of a time-dependent shoreline migration (e.g. Milne & Mitrovica 1998b; Milne 1998; Milne *et al.* 1999). Their algorithm was first described in Milne (1998) and it was applied to compute successive increments of ocean height change. Specifically, they used the equation

$$\delta S_{\mathrm{M}}(\theta, \psi, t_j) = [\Delta SL(\theta, \psi, t_j) - \Delta SL(\theta, \psi, t_{j-1})] C(\theta, \psi, t_j). \tag{25}$$

This equation is similar to the approximation adopted by Johnston (1993), with the exception that the projection on to the ocean function at $t = t_j^*$ is replaced by a projection on to the ocean function as it existed at the end of the time increment (i.e. at $t = t_j$).

To investigate the error in the ocean height change introduced by eq. (25), we can rewrite it as

$$\begin{aligned}\delta S_{\mathrm{M}}(\theta, \psi, t_j) = [&-T(\theta, \psi, t_0) + SL(\theta, \psi, t_j) \\ &- (-T(\theta, \psi, t_0) + SL(\theta, \psi, t_{j-1}))] C(\theta, \psi, t_j),\end{aligned} \tag{26}$$

which can be compared with our correct expression for the ocean height change between successive time increments (17). Appendix A4 demonstrates that the error incurred by eq. (26) is equal to the height of regions 3 + 5 + 8 in Fig. 5. This region is the topography at time $t = t_{j-1}$ within the region of shoreline migration from $t = t_{j-1}$ to $t = t_j$(see Fig. 5).

Milne (1998) recognized that the error in the ocean height change incurred by applying the approximation (24) would be large (see

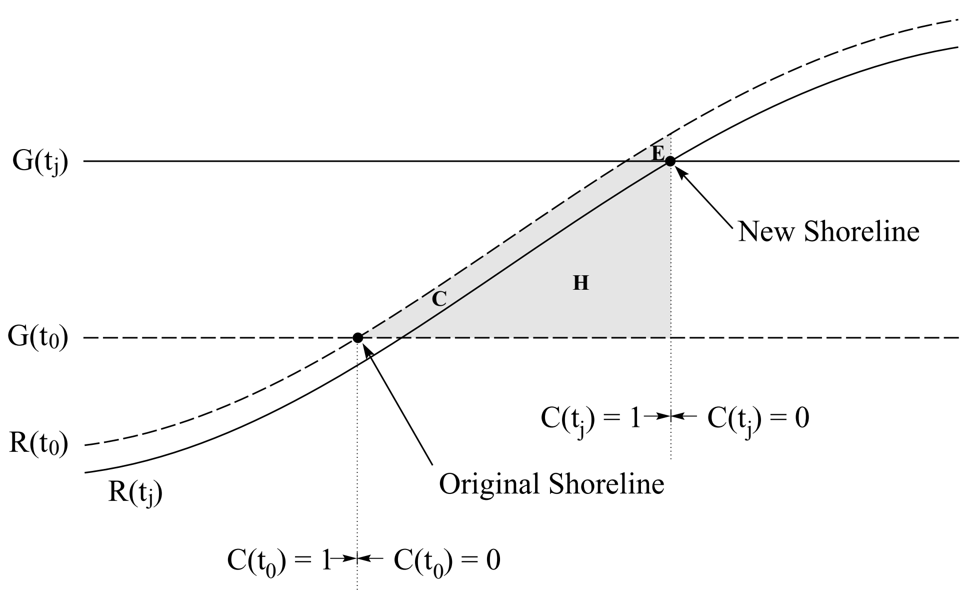

Figure 6. Re-illustration of Fig. 4 showing the accumulated error in the ocean height change (given by the vertical height of the shaded region) incurred by adopting the sea level theory outlined in Peltier (1994) from $t = t_0$ to time $t = t_j$ (see Section 2.1.3). Note that the labels C, E and H are not to be interpreted as areas; rather, they represent heights of the associated shaded region as a function of position.

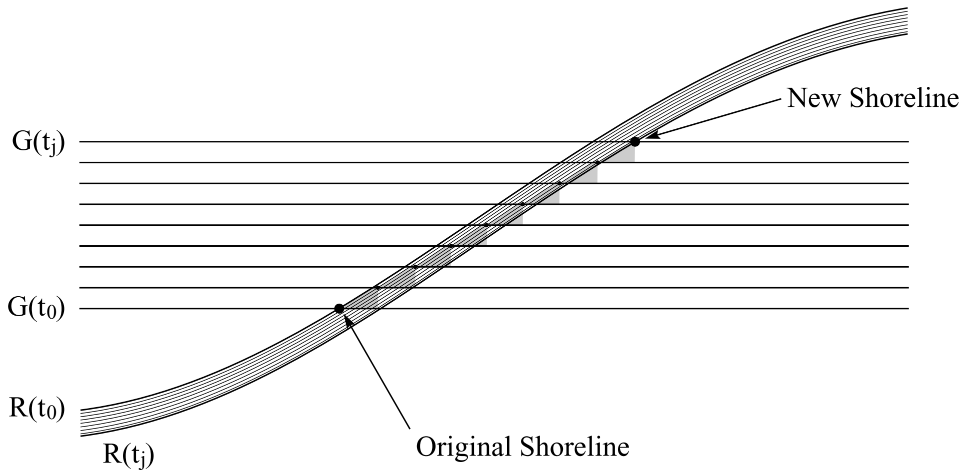

Figure 7. The accumulated error in the ocean height (given by the vertical height of the small shaded triangular regions) incurred by adopting the sea level theory outlined in Milne (1998) from the onset of deglaciation (at $t = t_0$) to some time $t = t_j$ in eight discrete steps (see Section 2.1.4).

eq. 22 of Milne *et al.* 1999, and related discussion) and the algorithm defined by his eq. (25) was intended to improve the accuracy of predictions in the vicinity of a time-varying shoreline. To understand the nature of this improvement, consider the schematic Fig. 7, which shows a sequence of geoid and solid surfaces extending from $t = t_0$ to $t = t_j$ in a series of (in this example) eight discrete time steps. At each time step, the error in the ocean height increment incurred by eq. (25) is the topography at the last time value within the region defined by shoreline migration over the time step. The shaded triangular regions in the figure illustrate the total error after the eight steps. This total error in the ocean height (or ocean load) change can be compared with the error introduced by eq. (24), given by the shaded region in Fig. 6. The error in approximating $\delta S(\theta, \psi, t_j)$ by $\delta S_{\mathrm{M}}(\theta, \psi, t_j)$ will diminish as the time increments become smaller, and will be a function of the geometry of the shoreline.

There is an intriguing special case in which the error introduced by eq. (25) vanishes. To derive this case we need to re-arrange eq. (17) into a form that involves the approximation (25) plus an 'error' term. We begin by rewriting eq. (17) as

$$\begin{aligned}\delta S(\theta, \psi, t_j) = & -T(\theta, \psi, t_0)[C(\theta, \psi, t_j) - C(\theta, \psi, t_{j-1})] \\ & + \Delta SL(\theta, \psi, t_j) C(\theta, \psi, t_j) \\ & - \Delta SL(\theta, \psi, t_{j-1}) C(\theta, \psi, t_{j-1}). \end{aligned} \tag{27}$$

Following eqs (9) and (11), we can replace the original topography $T(\theta, \psi, t_0)$ in eq. (27) by $T(\theta, \psi, t_{j-1}) + \Delta SL(\theta, \psi, t_{j-1})$. Applying this substitution, and re-arranging terms gives

$$\begin{aligned}\delta S(\theta, \psi, t_j) &= C(\theta, \psi, t_j)[\Delta SL(\theta, \psi, t_j) - \Delta SL(\theta, \psi, t_{j-1})] \\ &\quad - T(\theta, \psi, t_{j-1})[C(\theta, \psi, t_j) - C(\theta, \psi, t_{j-1})] \\ &= \delta S_{\mathrm{M}}(\theta, \psi, t_j) - T(\theta, \psi, t_{j-1})[C(\theta, \psi, t_j) \\ &\quad - C(\theta, \psi, t_{j-1})]. \end{aligned} \tag{28}$$

This equation verifies our earlier conclusion based on the schematic Fig. 5 (and Appendix A4); namely, that the error introduced by the Milne (1998) approximation across the time step $t = t_{j-1}$ to $t = t_j$ is equal to the topography at $t = t_{j-1}$ in the region of shoreline migration across the time step.

When does the error in the Milne (1998) approximation vanish? Clearly, it does so at any location in which there is no shoreline migration (i.e. when $C(\theta, \psi, t_{j-1}) = C(\theta, \psi, t_j)$). However, it also vanishes wherever the topography $T(\theta, \psi, t_{j-1})$ is zero within the region of shoreline migration between $t = t_{j-1}$ and $t = t_j$. This latter case is illustrated in the schematic Fig. 8. To require that $T(\theta, \psi, t_{j-1}) = 0$ in the region of shoreline migration between $t = t_{j-1}$ and $t = t_j$ is equivalent to requiring that the geoid and solid surface at $t = t_{j-1}$ coincide within the zone of migration. As illustrated in Fig. 8, this requirement is physically satisfied when the shoreline is defined by a series of small cliffs and plateaux, and where the ocean height change between time steps is such that the geoid (ocean surface) moves from the level of one plateau to the next in successive time increments.

This special case is notable because it is precisely the situation described by Farrell & Clark (1976) in the figure reproduced as our Fig. 1 in their classic discussion of post-glacial sea level change.

2.1.5 Peltier (1998b)

Peltier (1998b) was aware that the sea level equation adopted by Peltier (1994) (eq. 24) neglected a term involving shoreline migration. In generalizing the theory he described in Peltier (1994), Peltier (1998b) took a perturbation through the mapping between ocean height and sea level (eq. 4b), and obtained (Peltier 1998b, eq. 2):

$$\partial S = C[\partial G - \partial R] + \partial C[G - R]. \tag{29}$$

The second term in this expression for the ocean height change, which involves the shoreline migration ∂C, was subsequently dropped because it 'is always very much smaller than the first term' (Peltier 1998b, p. 624). Furthermore, in implementing eq. (29), the perturbation ∂ was interpreted as a change since the onset of loading. Thus, using our symbolism, the final equation governing ocean height changes became (Peltier 1998b, eq. 3):

$$\begin{aligned}\Delta S_{\mathrm{P}}(\theta, \psi, t) &= C(\theta, \psi, t)[\{G(\theta, \psi, t) - G(\theta, \psi, t_0)\} \\ &\quad - \{R(\theta, \psi, t) - R(\theta, \psi, t_0)\}] \\ &= C(\theta, \psi, t)[\Delta G(\theta, \psi, t) - \Delta R(\theta, \psi, t)] \\ &= C(\theta, \psi, t) \Delta SL(\theta, \psi, t), \end{aligned} \tag{30}$$

which is identical to eq. (24) in Section 2.1.3.

Peltier's (1998b) expression (29) provides insight into the problem of shoreline evolution because it makes clear that the change in the ocean load will, in general, involve terms in both the evolution of the shoreline location and the perturbation in global sea level.

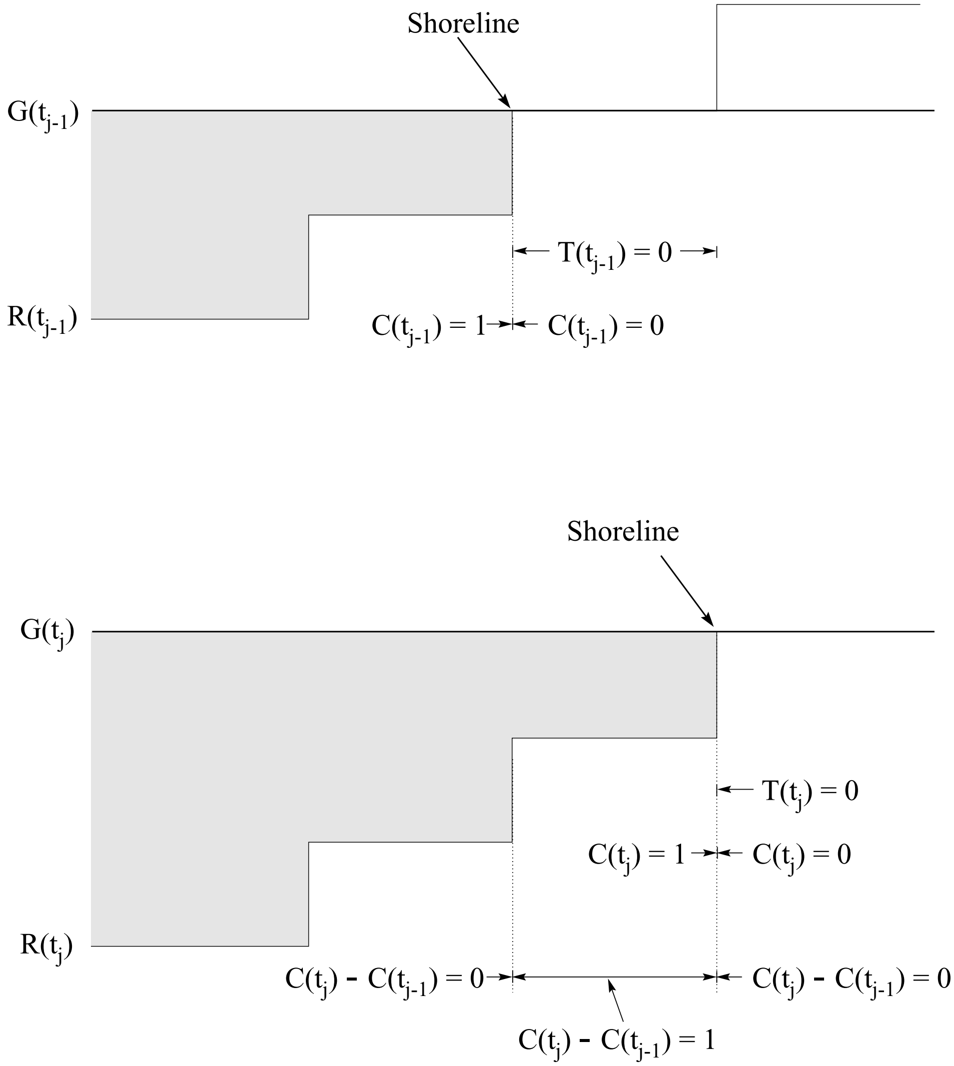

Figure 8. Change in ocean height from $t = t_{j-1}$ to $t = t_j$ on a shoreline defined by a set of discrete cliffs. The change in ocean height (which arises from a combination of changes in the geoid height and solid surface position) is such that at each time increment the ocean surface reaches the foot of the next cliff on the shoreline. A detailed discussion of this special case can be found in Section 2.1.4 of the text.

This physics is also evident in our expressions (15) and (17b) for the ocean height change across a finite time interval. However, these exact expressions are generated by taking the difference in the ocean load across the time intervals (see eqs 14 and 16) and thus they cannot be derived from the perturbation eq. (29) (which is appropriate for infinitesimal time intervals). As an example, consider the second term on the right-hand side of eq. (29). Following eq. (30), this term would be interpreted as the change in the ocean function since the onset of loading multiplied by the sea level at time t: i.e. $\Delta C(\theta, \psi, t)\, SL(\theta, \psi, t)$, or $[C(\theta, \psi, t) - C(\theta, \psi, t_0)]\, SL(\theta, \psi, t)$. However, in the expression (15) for the ocean height change since the onset of loading, the shoreline migration is multiplied by the *original* sea level at the time of load onset, $SL(\theta, \psi, t_0)$ (or $-T(\theta, \psi, t_0)$, as in eq. 15), not the final sea level $SL(\theta, \psi, t)$.

2.1.6 *Peltier & Drummond (2002)*

In two recent articles, Peltier and colleagues have described an improvement to the algorithm of Peltier (1994) for computing ocean load and sea level changes in the vicinity of evolving shorelines (Peltier & Drummond 2002). Specifically, they introduced a 'broad shelf effect', which ensures that 'the magnitude of the water load that can be added to the shelf from the time of last glacial maximum must be constrained so as to conform to the volume bounded by the local paleotopography as defined in Peltier (1994) and the local geoid (mean sea level)' (Peltier & Drummond 2002, pp. 10–12). While a detailed mathematical description of the broad shelf effect is not provided by the authors the correction appears to remedy the error in the ocean height change since the onset of loading introduced by eq. (24) (see Peltier & Drummond 2002, Fig. 2). (Note that this error, shown by the shaded region in Fig. 6, violates the physical constraint on ocean load redistribution described in the above quote.) Peltier & Drummond (2002) described the impact of the improvement on predictions of post-glacial sea level change for a set of sites on the east coast of South America and in northwestern Europe. They found that the error (relative to predictions based on eq. 24) is largest in regions with extensive and shallow continental shelves, as one would also expect from our discussion of Fig. 6.

2.1.7 Ocean loads versus sea level change

In Sections 2.1.2–2.1.4 we discussed, in detail, errors in ocean height changes introduced by previous algorithms for computing post-glacial sea level change in the presence of a time-varying shoreline geometry. Multiplying these errors by the density of water gives the associated error in the ocean load variation. Predictions of sea level change due to GIA are computed by considering the response of a given Earth model (spherically symmetric, or otherwise) to the total surface mass (ice plus ocean) load. Thus, errors in the ocean height change will ultimately map into errors in δSL or ΔSL. These errors will directly impact the prediction of relative sea level curves, which, for a given geographic site, is defined as $\Delta SL(t) - \Delta SL(t_{\rm p})$. The errors introduced by the algorithms also map into errors in the prediction of the time-varying topography (via eq. 11) and therefore the time-varying shoreline geometry (via eq. 5). As we noted above, Paper II quantifies each of these issues in detail.

2.2 Including marine-based ice

In this section we extend the sea level equations derived above to include changes in ocean geometry arising from the growth or ablation of marine-based ice complexes. Consider Fig. 2(b), which is taken from Fig. 2 of Milne *et al.* (1999). From some time t_{j-1} to t_j in Fig. 2(b) a marine-based ice sheet retreats and water floods into the region vacated by the ice complex. Over this time period, the development of the new water load within the ablation zone is clearly governed by the total sea level in this region as it becomes ice-free, rather than by any incremental changes in sea level.

Milne (1998) (see also Milne *et al.* 1999) coined the term 'water dumping' to describe the influx of water illustrated in Fig. 2(b) and extended the sea level algorithm described in Section 2.1.4 to incorporate such effects. The same inundation process was also considered by Peltier (1998a), who adopted the term 'implicit ice' to describe his treatment of the problem. Their two independent approaches to the problem are distinct. Specifically, Milne (1998) revised his sea level equation to model the effect of 'water dumping' (see Section 2.2.1 below) and his approach incorporated, for example, local loading and gravitational effects associated with the water influx as well as the contribution of the effect to ocean mass conservation terms. In contrast, Peltier (1998a) did not revise his basic sea level eq. (see Section 2.1.3 and eq. 1 of Peltier 1998a) to treat the influx process: his predictions of sea level change were generated using eq. (24) together with an ice load (termed 'explicit ice') prescribed by a global ice model. Peltier (1998a) computed, *a posteriori*, the ice-equivalent thickness required to fill the depressions vacated by the ablation of marine-based ice. This thickness formed the 'dominant' contributor to the implicit ice field added to both paleotopography (Peltier 1998a, eq. 10) and to estimates of eustatic sea level rise associated with each ice complex (Peltier 1998a, Fig. 1).

The influx of water into regions vacated by ice leads to a global drop in the sea surface, which is computed in the Milne (1998) approach by invoking conservation of mass in their generalized sea level theory (see Milne *et al.* 2001, for details). This effect provides a natural physical link between the ideas of water dumping and implicit ice. Namely, estimates of ice volumes based on efforts to fit far-field sea level records (e.g. from Barbados) using post-glacial sea level theories that do not include the water dumping mechanism will underestimate ice volumes by an amount roughly equivalent to Peltier's (1998a) estimate of the volumes of implicit ice.

Despite this connection, there appear to be fundamental differences in the two approaches for treating the process in Fig. 2(b). Milne (1998) and Milne *et al.* (1999) argued that the water influx influences predictions of site-specific post-glacial sea level curves through the loading and gravitational effects described above, and they quantified this impact. Peltier's (1998a) site-specific sea level predictions appear to be unaltered by the process, since the ice load incorporated into his governing sea level theory (i.e. eq. 24) only includes the explicit ice component, and since the 'implicit' ice does 'not impact in any way' (Peltier 1998a, p. 3958) viscosity inferences based upon these predictions. We have been unable to reconcile these differences, although we caution that our understanding of the implicit ice algorithm is incomplete.

Peltier (1998a) did not incorporate the water influx process of Fig. 2 into the sea level eq. (24) because that equation 'is a construct of first-order perturbation theory and the redefinition of land to sea is a non-perturbative effect' (Peltier 1998a, p. 3957). The generalized sea level equations we have derived (see eqs 15 and 17) include both perturbative and non-perturbative terms. We begin below by demonstrating that these generalized equations can be extended to incorporate the water influx process and we then compare these extended forms with the sea level algorithm derived by Milne (1998).

Eq. (4a) at the beginning of Section 2.1 holds for the case where marine-based ice is present as long as our definition of the ocean function is slightly revised. Specifically, we can write

$$S(\theta, \psi, t) = SL(\theta, \psi, t)C^*(\theta, \psi, t), \tag{31}$$

where C^* is an ocean function defined such that

$$\begin{aligned} C^*(\theta, \psi, t) = 1 \quad & \text{where } SL(\theta, \psi, t) > 0 \,(\text{or } T(\theta, \psi, t) < 0) \\ & \textit{and} \text{ grounded ice is not present,} \\ = 0 \quad & \text{where } SL(\theta, \psi, t) < 0 \,(\text{or } T(\theta, \psi, t) > 0) \\ & \textit{or} \text{ grounded ice is present.} \end{aligned} \tag{32}$$

That is, we have extended the definition (5) so that the ocean function is zero in the location of marine-based ice. It would also be possible to begin our derivation by extending our definition of topography in eq. (3) to include the height of the ice column and then to assign a value to the ocean function on the basis of the height of this new topography field; however, we prefer to proceed with eq. (31) and to associate changes in topography with perturbations in the geoid and solid surface alone (i.e. as in eq. 11). We can rewrite eq. (31) to explicitly include the field $C(\theta, \psi, t)$, rather than $C^*(\theta, \psi, t)$, as

$$S(\theta, \psi, t) = SL(\theta, \psi, t)C(\theta, \psi, t)\beta(\theta, \psi, t), \tag{33}$$

where C is defined as in eq. (5) and

$$\begin{aligned} \beta(\theta, \psi, t) &= 1 \quad \text{where there is no grounded ice,} \\ &= 0 \quad \text{where there is grounded ice.} \end{aligned} \tag{34}$$

In this case, the change in ocean height between two successive time steps becomes (extending eqs 16 and 17):

$$\begin{aligned} \delta S(\theta, \psi, t_j) &\equiv S(\theta, \psi, t_j) - S(\theta, \psi, t_{j-1}) \\ &= SL(\theta, \psi, t_j)C(\theta, \psi, t_j)\beta(\theta, \psi, t_j) \\ &\quad - SL(\theta, \psi, t_{j-1})C(\theta, \psi, t_{j-1})\beta(\theta, \psi, t_{j-1}) \end{aligned} \tag{35}$$

$$\begin{aligned} &= [-T(\theta, \psi, t_0) + \Delta SL(\theta, \psi, t_j)]C(\theta, \psi, t_j)\beta(\theta, \psi, t_j) \\ &\quad - [-T(\theta, \psi, t_0) + \Delta SL(\theta, \psi, t_{j-1})]C(\theta, \psi, t_{j-1})\beta(\theta, \psi, t_{j-1}). \end{aligned} \tag{36}$$

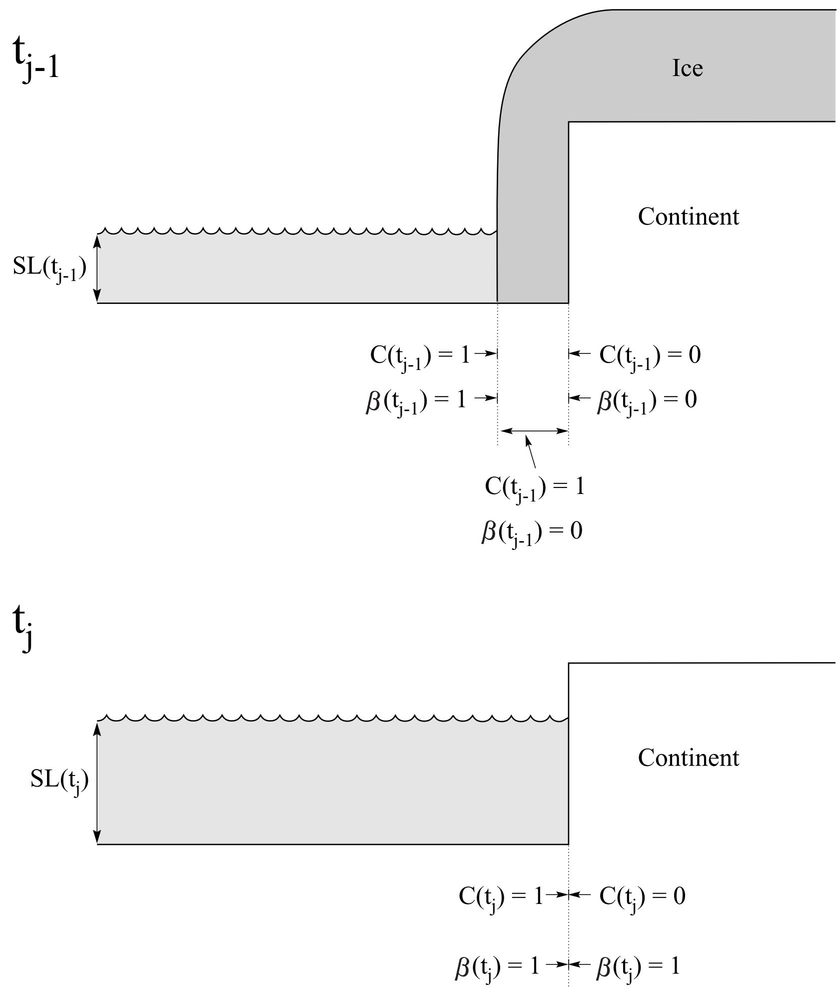

Figure 9. Schematic illustration of a continent margin subject to melting of marine- (and continent-) based ice from $t = t_{j-1}$ to $t = t_j$. The example is used within the text to test the validity of the sea level eq. (35).

We can verify the validity of eq. (35) by considering the simple scenario illustrated in Fig. 9. At $t = t_{j-1}$ an ice sheet covering the region has a marine-based component. By $t = t_j$ this region has become ice-free and the ocean height in this region and to the left has increased. The various values of the fields C and β within the region, at the two times, are indicated on the figure. Within the continental region, $C(\theta, \psi, t_{j-1}) = C(\theta, \psi, t_j) = 0$ and thus, from eq. (35), the change in ocean height is $\delta S(\theta, \psi, t_j) = 0$, as required. In the oceanic region to the left of the location of the marine-based ice sheet at $t = t_{j-1}$, both β and C are unity at both time values, and thus the change in ocean height is $\delta S(\theta, \psi, t_j) = SL(\theta, \psi, t_j) - SL(\theta, \psi, t_{j-1})$, also as required. Finally, within the oceanic region vacated by the ice sheet, $\beta(\theta, \psi, t_{j-1}) = 0$ while $\beta(\theta, \psi, t_j) = C(\theta, \psi, t_j) = 1$, and therefore, from eq. (35), $\delta S(\theta, \psi, t_j) = SL(\theta, \psi, t_j)$. Thus, our sea level equation correctly captures the inundation of this region as it becomes ice-free.

Eq. (36) represents a generalized sea level equation. The solution of the equation, as in the case of eq. (17), will require two iterations. One iteration is related to the successive refinement of the unknown topography $T(\theta, \psi, t_0)$ via eq. (18). A second iteration is required to deal with the integral nature of the equation; namely, that the ocean load change appearing on the left-hand side is also required for the computation of the perturbation ΔSL. The evolution of the ocean function C can be determined, using eq. (5), from the time-varying topography.

The field β is prescribed *a priori* from the adopted global ice model. In practical applications this prescription sometimes requires a simple check. Specifically, as the sea level equation is solved through time, any prescribed column of marine-based ice must be more massive than the column of ocean that would exist according to the local sea level. If this is not the case, the ice in this location would be removed from the prescribed ice model (since, in reality, it would simply become buoyant). In general (see Paper II) this requirement yields very minor, if any, modifications to the ice and Earth models we have considered.

It will be instructive, for the analysis of the next section, to write our generalized sea level eq. (36) in a slightly modified form. Using eqs (9) and (11) we have the following identity:

$$T(\theta, \psi, t_j) = T(\theta, \psi, t_{j-1}) + \Delta SL(\theta, \psi, t_{j-1}) - \Delta SL(\theta, \psi, t_j). \tag{37}$$

Substituting this relation into eq. (35), and using the relationship between topography and sea level in eq. (3), yields

$$\begin{aligned}\delta S(\theta, \psi, t_j) = &-[T(\theta, \psi, t_{j-1}) + \Delta SL(\theta, \psi, t_{j-1})\\ &- \Delta SL(\theta, \psi, t_j)]C(\theta, \psi, t_j)\beta(\theta, \psi, t_j)\\ &+ T(\theta, \psi, t_{j-1})C(\theta, \psi, t_{j-1})\beta(\theta, \psi, t_{j-1}).\end{aligned} \tag{38}$$

Re-arranging this expression we obtain

$$\begin{aligned}\delta S(\theta,\psi,t_j) = {} & C(\theta,\psi,t_j)\beta(\theta,\psi,t_j)[\Delta SL(\theta,\psi,t_j) \\ & - \Delta SL(\theta,\psi,t_{j-1})] \\ & + T(\theta,\psi,t_{j-1})[C(\theta,\psi,t_{j-1})\beta(\theta,\psi,t_{j-1}) \\ & - C(\theta,\psi,t_j)\beta(\theta,\psi,t_j)]. \end{aligned} \tag{39}$$

In practice, the form of eq. (39), which is analogous to eq. (17b), provides an efficient algorithm for solving our generalized sea level eq. (36).

2.2.1 *Milne (1998) and Milne et al. (1999)*

Milne (1998) and Milne *et al.* (1999) incorporated water influx associated with deglaciation of marine-based ice (Fig. 2b) into their sea level theory. Of course, during the glaciation phase, the development of marine-based ice would lead to the evacuation of water (rather than an influx); however, the Milne (1998) algorithm was intended to model the deglaciation phase alone. Using our symbolism, this algorithm can be written as (see Milne *et al.* 1999, eq. 24)

$$\begin{aligned}\delta S_{\mathrm{M}}(\theta,\psi,t_j) = {} & C(\theta,\psi,t_j)\beta(\theta,\psi,t_j)[\varepsilon(\theta,\psi,t_j)(\Delta SL(\theta,\psi,t_j) \\ & -\Delta SL(\theta,\psi,t_{j-1})) + (\varepsilon(\theta,\psi,t_j)-1)T(\theta,\psi,t_j)], \end{aligned} \tag{40}$$

where we note that the product $C\beta$ appears in eq. (40) because Milne (1998) and Milne *et al.* (1999) adopted C^* (eq. 32) in their formulation, rather than C (eq. 5). The field ε was defined by Milne (1998) such that

$$\begin{aligned}\varepsilon(\theta,\psi,t_j) &= 0 \quad \text{where ice has retreated from } t = t_{j-1} \text{ to } t_j \\ &= 1 \quad \text{elsewhere.}\end{aligned} \tag{41}$$

Our goal in this section is to assess the error in the ocean load height introduced by eq. (40). To begin, we need to rewrite this equation into a form amenable to comparison with our generalized sea level eqs (38) or (39). In Appendix B (eq. B5) we show that the following is an equivalent form for eq. (40):

$$\begin{aligned}\delta S_{\mathrm{M}}(\theta,\psi,t_j) = {} & C(\theta,\psi,t_j)\beta(\theta,\psi,t_j)[\Delta SL(\theta,\psi,t_j) \\ & - \Delta SL(\theta,\psi,t_{j-1})] - C(\theta,\psi,t_j)[\beta(\theta,\psi,t_j) \\ & - \beta(\theta,\psi,t_{j-1})]T(\theta,\psi,t_{j-1}). \end{aligned} \tag{42}$$

The first terms on the right-hand sides of eqs (42) and (39) are identical. Our comparison between these equations proceeds by both subtracting and adding the second term on the right-hand side of eq. (42) from eq. (39) to generate

$$\begin{aligned}\delta S(\theta,\psi,t_j) = {} & \delta S_{\mathrm{M}}(\theta,\psi,t_j) + C(\theta,\psi,t_j)[\beta(\theta,\psi,t_j) \\ & - \beta(\theta,\psi,t_{j-1})]T(\theta,\psi,t_{j-1}) \\ & + T(\theta,\psi,t_{j-1})[C(\theta,\psi,t_{j-1})\beta(\theta,\psi,t_{j-1}) \\ & - C(\theta,\psi,t_j)\beta(\theta,\psi,t_j)]. \end{aligned} \tag{43}$$

Finally, cancellation in the last two terms on the right-hand side of eq. (43) yields

$$\begin{aligned}\delta S(\theta,\psi,t_j) = {} & \delta S_{\mathrm{M}}(\theta,\psi,t_j) - T(\theta,\psi,t_{j-1})\beta(\theta,\psi,t_{j-1}) \\ & \times [C(\theta,\psi,t_j) - C(\theta,\psi,t_{j-1})]. \end{aligned} \tag{44}$$

The second term on the right-hand side of eq. (44) represents the error in the ocean height change introduced in the sea level equation derived by Milne (1998) and Milne *et al.* (1999) for the case of a time-varying shoreline in the presence of ablating marine-based ice sheets. This equation is an extension of eq. (28), which provided an analogous expression for the error in the Milne (1998) sea level equation in the absence of such ice. In locations where ice existed at the last time value (i.e. $\beta(\theta,\psi,t_{j-1}) = 0$) the error term in eq. (44) would be zero. However, in locations where there was no such ice, the error in eq. (44) would be the same as the error term in eq. (28); namely, it would be equal to the topography at $t = t_{j-1}$ within the region of shoreline migration across the time step $t = t_{j-1}$ to $t = t_j$. Across a single time step this error was illustrated in Fig. 5, with reference to Appendix A4; the error incurred across multiple time steps is shown in Fig. 7. As we described in the context of these figures, the error is zero when: (1) there is no shoreline migration; (2) the time steps are made vanishingly small; or (3) the shoreline is characterized by a series of cliffs, the height of which coincides with ocean height changes across successive time steps (see Fig. 8).

3 FINAL REMARKS

We have presented a generalized sea level equation that governs post-glacial ocean height changes in the case of a time-varying shoreline geometry. The validity of the equation has been demonstrated using a series of schematic illustrations that treat shoreline migration due to either water onlap or offlap associated with local sea level changes or the growth and ablation of marine-based ice sheets.

The basic argument underlying our extension is straightforward. The ocean height (or, if one prefers, depth) can always be computed by projecting global sea level on to the contemporaneous ocean function, as in eq. (4a). However, ocean height *changes* cannot, in general, be defined by a simple projection of global sea level *changes*. In this case, account must also be taken of changes in the ocean function across the time interval of interest. Exact expressions for changes in ocean height over a finite time interval may be derived by considering the difference between the ocean height at the start and end of the time interval. We have provided the required expressions for changes over either the entire post-glacial time window or over two successive time increments in the solution of the sea level equation.

In the last decade, a large number of independent studies have attempted to apply or modify the traditional Farrell & Clark (1976) sea level equation to treat the case of a time-varying shoreline. We have compared these various efforts by deriving, when possible, analytic expressions for the error in the ocean height change incurred relative to our generalized theory, and by considering an additional set of schematic illustrations. We conclude that both the Johnston (1993) and Milne (1998) algorithms for treating sea level change in the vicinity of a migrating shoreline (in the absence of evolving marine-based ice) introduce less error into the prediction than the approach proposed by Peltier (1994). Recent improvements to the latter (Peltier & Drummond 2002) appear to have led to a level of accuracy in modelling ocean load changes in these vicinities that is consistent with the other sea level groups (see also Mitrovica 2002). Our generalized analysis also shows that it is possible to incorporate the influx of water into regions vacated by ablating marine-based ice directly into a sea level equation. Indeed, Milne's (1998) expression for the change in the ocean load in this case collapses to our generalized expression when the time increments defining the ice model are made vanishingly small, in the region vacated by the marine-based ice sheet at the time of ice retreat, or when the shoreline is characterized by a set of small cliffs for which the height coincides with local, incremental geoid heights. The latter example has particular significance in GIA studies, since it was the case discussed qualitatively by Farrell & Clark (1976).

Throughout this paper we have assumed that a method is available for computing global sea level changes, ΔSL, given an input surface mass load. In this case, the equations derived herein provide a practical algorithm for iteratively refining an initial guess to the ocean load component of the surface mass load until it is consistent with the computed global sea level variation and the present-day shoreline geometry. In the case of a spherically symmetric, linear viscoelastic and rotating Earth model, the sea level response due to an arbitrary surface mass load is provided by a version of the classic Farrell & Clark (1976) theory (i.e. eqs I4 and I5) that has been augmented to include rotational effects. In Paper II we provide the full set of analytic results required to combine this augmented theory with the generalized sea level equation described herein. These results will also be applied to quantify the level of error introduced in previous predictions of post-glacial sea level change in the presence of a time-varying shoreline.

ACKNOWLEDGMENTS

We thank Bert Vermeersen and an anonymous reviewer for their constructive comments in regard to this manuscript. This work was support by NSERC.

REFERENCES

Bills, B.G. & James, T.S., 1996. Late Quaternary variations in relative sea level due to glacial cycle polar wander, *Geophys. Res. Lett.,* **23,** 3023–3026.

Clark, J.A., Farrell, W.E. & Peltier, W.R., 1978. Global changes in postglacial sea level: a numerical calculation, *Quat. Res.,* **9,** 265–287.

Daly, R.A., 1925. Pleistocene changes of level, *Am. J. Sci.,* **10,** 281–313.

Farrell, W.E. & Clark, J.A., 1976. On postglacial sea level, *Geophys. J. R. astr. Soc.,* **46,** 647–667.

Johnston, P., 1993. The effect of spatially non-uniform water loads on predictions of sea level change, *Geophys. J. Int.,* **114,** 615–634.

Han, D. & Wahr, J., 1989. Post-glacial rebound analysis for a rotating Earth, in *Slow Deformations and Transmission of Stress in the Earth,* pp. 1–6, eds Cohen, S. & Vanicek, P., *AGU Mono. Series 49*. American Geophysical Union, Washington D.C.

Lambeck, K. & Nakada, M., 1990. Late Pleistocene and Holocene sea level change along the Australian coast, *Palaeogeog. Palaeoclimat. Palaeoecol.,* **89,** 143–176.

Milne, G.A., 1998. Refining models of the glacial isostatic adjustment process, *PhD thesis,* University of Toronto, Toronto.

Milne, G.A. & Mitrovica, J.X., 1996. Postglacial sea level change on a rotating Earth: first results from a gravitationally self-consistent sea level equation, *Geophys. J. Int.,* **126,** F13–F20.

Milne, G.A. & Mitrovica, J.X., 1998a. Postglacial sea level change on a rotating Earth, *Geophys. J. Int.,* **133,** 1–10.

Milne, G.A. & Mitrovica, J.X., 1998b. The influence of a time-dependent ocean–continent geometry on predictions of post-glacial sea level change in Australia and New Zealand, *Geophys. Res. Lett.,* **25,** 793–796.

Milne, G.A., Mitrovica, J.X. & Davis, J.L., 1999. Near-field hydro-isostasy: the implementation of a revised sea-level equation, *Geophys. J. Int.,* **139,** 464–482.

Milne, G.A., Mitrovica, J.X. & Schrag, D.P., 2001. Estimating past continental ice volumes from sea-level data, *Quat. Sci. Rev.,* **21,** 361–376.

Mitrovica, J.X., Recent controversies in predicting post-glacial sea-level change: a viewpoint, *Quat. Sci. Rev.,* **22,** 127–133.

Mitrovica, J.X. & Peltier, W.R., 1991. On post-glacial geoid subsidence over the equatorial oceans, *J. geophys. Res.,* **96,** 20 053–20 071.

Mitrovica, J.X., MIlne, G.A. & Davis, J.L., 2001. Glacial isostatic adjustment on a rotating Earth, *Geophys. J. Int.,* **147,** 562–579.

Munk, W.H. & MacDonald, G.J.F., 1960. *The Rotation of the Earth,* Cambridge University Press, New York.

Nakada, M. & Lambeck, K., 1989. Late Pleistocene and Holocene sea-level change in the Australian region and mantle rheology, *Geophys. J. Int.,* **96,** 497–517.

Peltier, W.R., 1974. The impulse response of a Maxwell Earth, *Rev. Geophys.,* **12,** 649–669.

Peltier, W.R., 1994. Ice Age Paleotopography, *Science,* **265,** 195–201.

Peltier, W.R., 1998a. 'Implicit ice' in the global theory of glacial isostatic adjustment, *Geophys. Res. Lett.,* **25,** 3955–3958.

Peltier, W.R., 1998b. Postglacial variations in the level of the sea: implications for climate dynamics and solid-Earth geophysics, *Rev. Geophys.,* **36,** 603–689.

Peltier, W.R., 1998c. The inverse problem for mantle viscosity, *Inverse Problems,* **14,** 441–478.

Peltier, W.R., 1999. Global sea level rise and glacial isostatic adjustment, *Global Planet. Change,* **20,** 93–123.

Peltier, W.R. & Andrews, J.T., 1976. Glacial-isostatic adjustment I—the forward problem, *Geophys. J. R. astr. Soc.,* **46,** 605–646.

Peltier, W.R. & Drummond, R., 2002. A 'broad-shelf effect' upon postglacial relative sea level history, *Geophys. Res. Lett.,* **29,** 10.1029/2001GL014273.

Peltier, W.R., Farrell, W.E. & Clark, J.A., 1978. Glacial isostasy and relative sea level: a global finite element model, *Tectonophysics,* **50,** 81–110.

Tushingham, A.M. & Peltier, W.R., 1991. ICE-3G: a new global model of Late Pleistocene deglaciation based on geophysical predictions of post-glacial relative sea level change, *J. geophys. Res.,* **96,** 4497–4523.

Woodward, R.S., 1888. On the form and position of mean sea level, *US Geol. Surv. Bull.,* **48,** 87–170.

Wu, P. & Peltier, W.R., 1983. Glacial isostatic adjustment and the free air gravity anomaly as a constraint on deep mantle viscosity, *Geophys. J. R. astr. Soc.,* **74,** 377–449.

APPENDIX A

A1 Validation of eq. 15 via Fig. 4

We seek to verify the validity of eq. (15) using the schematic illustration shown in Fig. 4. Using the labels given in that figure we have

$$\Delta G(\theta, \psi, t_j) = B + C + H + I$$
$$\Delta R(\theta, \psi, t_j) = -D - C - E - F, \tag{A1}$$

The first term on the right-hand side of eq. (15) is then

$$[\Delta G(\theta, \psi, t_j) - \Delta R(\theta, \psi, t_j)]C(\theta, \psi, t_j)$$
$$= B + C + H + D + C + E. \tag{A2}$$

Next, the original topography is given by

$$T(\theta, \psi, t_0) = -A + C + H + E + F + G + I, \tag{A3}$$

and the second term on the right-hand side of eq. (15) is thus

$$T(\theta, \psi, t_0)[C(\theta, \psi, t_j) - C(\theta, \psi, t_0)] = C + H + E. \tag{A4}$$

Thus, the difference between eqs (A2) and (A4) gives, following eq. (15),

$$\Delta S(\theta, \psi, t_j) = B + C + D. \tag{A5}$$

Inspection of Fig. 4 verifies that eq. (A5) is correct.

A2 Validation of eq. (17) via Fig. 5

From Fig. 5 (main frame) we have

$$-T(\theta, \psi, t_0) + \Delta SL(\theta, \psi, t_j) = SL(\theta, \psi, t_j) = 1 + 2 + 3 + 4 - 7$$
$$[-T(\theta, \psi, t_0) + \Delta SL(\theta, \psi, t_j)]C(\theta, \psi, t_j) = 1 + 2 + 3 + 4. \tag{A6}$$

Similarly, for $t = t_{j-1}$, we have

$$\begin{aligned} &-T(\theta,\psi,t_0)+\Delta SL(\theta,\psi,t_{j-1}) = SL(\theta,\psi,t_{j-1}) \\ &\quad = 1-3-5-6-7-8-9 \\ &[-T(\theta,\psi,t_0)+\Delta SL(\theta,\psi,t_{j-1})]C(\theta,\psi,t_{j-1}) = 1. \end{aligned} \tag{A7}$$

Accordingly, following eq. (17), we obtain

$$\delta S(\theta,\psi,t_j) = 2+3+4. \tag{A8}$$

Inspection of Fig. 5 verifies that eq. (A8) is correct. Establishing the validity of eq. (17b) is straightforward, and would follow the logic given in Appendix A1 for the regions in Fig. 5 (rather than Fig. 4).

A3 The Johnston (1993) approximation

Eq. (23) in the text provides the approximation adopted by Johnston (1993) for the change in the ocean height between time increments $t = t_{j-1}$ and $t = t_j$. This expression can be explored using the regions defined within the inset of Fig. 5. Specifically, the term within square brackets on the right-hand side of eq. (23) is, using results from Appendix A2,

$$\begin{aligned} &-T(\theta,\psi,t_0)+\Delta SL(\theta,\psi,t_j)-[-T(\theta,\psi,t_0)+\Delta SL(\theta,\psi,t_{j-1})] \\ &\quad = 2+3+4+3+5+6+8+9. \end{aligned} \tag{A9}$$

The projection required in eq. (23) yields, following the regions defined in Fig. 5 (inset):

$$\delta S_{\mathrm{J}}(\theta,\psi,t_j) = 2a+3a+3a+4+8a. \tag{A10}$$

In the text we compare this approximation with the correct expression given by eq. (A8).

A4 The Milne (1998) approximation

Eq. (25) in the text is the approximation adopted by Milne (1998) for the change in the ocean height between time increments $t = t_{j-1}$ and $t = t_j$. We can again explore this expression using the regions defined in Fig. 5. The term within square brackets on the right-hand side of eq. (26) is given by eq. (A9). The projection of this quantity on to the ocean function at time $t = t_j$, as required in eq. (26), yields:

$$\delta S_{\mathrm{M}}(\theta,\psi,t_j) = 2+3+4+3+5+8. \tag{A11}$$

A comparison of this result with the correct expression (eq. A8) indicates that the error in the ocean height change incurred by the Milne (1998) approximation is equivalent to the total height of the regions 3 + 5 + 8 in Fig. 5.

APPENDIX B

We seek to re-arrange eq. (40) into a form that facilitates comparison with our generalized sea level theory. The following relationship holds between the function ε defined by Milne (1998) (eq. 41) and β (eq. 34):

$$\varepsilon(\theta,\psi,t_j) = 1-[\beta(\theta,\psi,t_j)-\beta(\theta,\psi,t_{j-1})], \tag{B1}$$

where, in deriving eq. (B1), we have assumed that if $\beta(\theta,\psi,t_{j-1}) = 1$, then $\beta(\theta,\psi,t_j) = 1$, as is the case for the monotonic deglaciation phase treated by Milne (1998). Using this relation in eq. (40) yields

$$\begin{aligned} \delta S_{\mathrm{M}}(\theta,\psi,t_j) = C(\theta,\psi,t_j)\beta(\theta,\psi,t_j)[&\{1-(\beta(\theta,\psi,t_j) \\ &-\beta(\theta,\psi,t_{j-1}))\}(\Delta SL(\theta,\psi,t_j) \\ &-\Delta SL(\theta,\psi,t_{j-1}))-(\beta(\theta,\psi,t_j) \\ &-\beta(\theta,\psi,t_{j-1}))T(\theta,\psi,t_j)]. \end{aligned} \tag{B2}$$

Substituting the identity (37) into eq. (B2) gives

$$\begin{aligned} \delta S_{\mathrm{M}}(\theta,\psi,t_j) = C(\theta,\psi,t_j)\beta(\theta,\psi,t_j)[&(\Delta SL(\theta,\psi,t_j) \\ &-\Delta SL(\theta,\psi,t_{j-1}))-(\beta(\theta,\psi,t_j) \\ &-\beta(\theta,\psi,t_{j-1}))T(\theta,\psi,t_{j-1})]. \end{aligned} \tag{B3}$$

To continue, we split the two main terms on the right-hand side of equation (B3) to obtain

$$\begin{aligned} \delta S_{\mathrm{M}}(\theta,\psi,t_j) = C(\theta,\psi,t_j)\beta(\theta,\psi,t_j)[&\Delta SL(\theta,\psi,t_j) \\ &-\Delta SL(\theta,\psi,t_{j-1})]-C(\theta,\psi,t_j) \\ &\times[\beta(\theta,\psi,t_j)\beta(\theta,\psi,t_j)-\beta(\theta,\psi,t_j) \\ &\times\beta(\theta,\psi,t_{j-1})]T(\theta,\psi,t_{j-1}). \end{aligned} \tag{B4}$$

Now, $\beta(\theta,\psi,t)\beta(\theta,\psi,t) = \beta(\theta,\psi,t)$. Also, as we discussed below eq. (B1), if $\beta(\theta,\psi,t_{j-1}) = 1$ then it must also be true that $\beta(\theta,\psi,t_j) = 1$ for a monotonic deglaciation phase: hence, the product $\beta(\theta,\psi,t_{j-1})\beta(\theta,\psi,t_j)$ is simply given by $\beta(\theta,\psi,t_{j-1})$. Using these relations in eq. (B4) yields

$$\begin{aligned} \delta S_{\mathrm{M}}(\theta,\psi,t_j) = C(\theta,\psi,t_j)\beta(\theta,\psi,t_j)[&\Delta SL(\theta,\psi,t_j) \\ &-\Delta SL(\theta,\psi,t_{j-1})]-C(\theta,\psi,t_j)[\beta(\theta,\psi,t_j) \\ &-\beta(\theta,\psi,t_{j-1})]T(\theta,\psi,t_{j-1}). \end{aligned} \tag{B5}$$

Eq. (B5) is the form we sought at the outset (see eq. 42).