Mechatronics for Rehabilitation Engineering: Course Notes

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Update on June 5, 2025

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Chapter 2

Review of Basic Circuit Theory

This chapter is a quick review of basic circuit theory. Prior knowledge of these topics is assumed, along with basic understanding of linear time invariant systems - Fourier/Laplace transforms, linear constant coefficient differential equations, transfer functions, and frequency response. The reader is encouraged to review the material in this chapter before proceeding to the next chapter.

The interaction of electromagnetic fields with matters is the basis of all electrical and electronic devices. These interactions are often described, analyzed and synthesized through the abstractions of electrical circuit theory. The following are the most basic circuit two-terminal elements we will need for now. We will introduce new ones as and when they are required. Each circuit element has a unique voltage-current relationship, and it is *important* that you know these by heart.

2.1 Basic

Independent Voltage source. An Ideal voltage source provides a fixed voltage V between its two terminals, and can provide any amount of current. Notice that the voltage V can be fixed or time varying. For example, for a DC voltage source with $V = 5V$, the voltage across the two terminals will be $5V$ for all time. But for a time varying AC source, $V = 5\sin(100\pi t)$, the voltage across its terminal will vary with time. We will often drop the adjective "independent" when we are sure that the context is

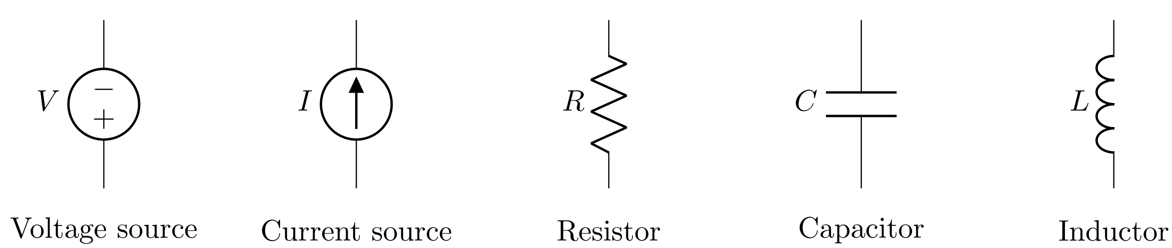


Fig. 2.1: Basic circuit elements: voltage source, current source, resistor, capacitor, and inductor.

clear. We will look at dependent sources later, and we will always use the adjective "dependent" to refer to them.

Independent Current source. An ideal current source provides a fixed amount of current to flow throgh its terminals (out through one and in through the other), irrespective of the voltage across its terminals. Current sources can also be time-varying.

Resistor. A passive element where the current i_R flowing through the element is proportional to the voltage v_R across its terminals.

$$v_R \propto i_R$$

In the case of linear resistors, the proportionality factor is constant, resulting in Ohm's law,

$$v_R = R\, i_R \tag{2.1}$$

The units of R are $V.A^{-1}$ or $Omhs\,(\Omega)$. R in general is positive. The power absorbed by a resistor is given by the product of the voltage across it and the current flowing through it,

$$P = v_R\, i_R = i_R^2\, R = \frac{v_R^2}{R} \tag{2.2}$$

This power is dissipated as heat by the resistor. Note that the power absorbed by a resistor is always positive, since R is positive.

We will later see non-linear resistors, where the resistance varies as a function of the applied votlage, temperature and other factors.

Capacitor. A capacitor is another passive element with the following voltage current relationship.

$$i_C = C\frac{dv_C}{dt} \tag{2.3}$$

The current i_C through the capacitor is proportional to the rate of change of voltage across its terminals v_C. The proportionality factor is called the capacitance C, and has units of F (Farads) or $C \cdot V^{-1}$. The voltage across the capacitor at any given time is proportional to the integral of the current flowing through it or the charge stored in the capacitor. The voltage across the capacitor is given by

$$v_C = \frac{q}{C} = \frac{1}{C}\int i_C\, dt \tag{2.4}$$

The instantaneous power absorbed by the capacitor is given by,

$$P = v_C\, i_C = C\, v_C\frac{dv_C}{dt} \tag{2.5}$$

The power absorbed by the capacitor can be positive or negative, depending on the direction of current flow. If the current is flowing into the capacitor, then the voltage across it is increasing, and the power absorbed is positive. If the current is flowing out of the capacitor, then the voltage across it is decreasing, and the power absorbed is negative. A capacitor stores energy in the form of electric field between its plates. The energy stored in a capacitor at any given time depends on the charge stored in it, and is given by,

$$E = \frac{1}{2} C \, v_C^2 \tag{2.6}$$

Inductor. An inductor is another passive element with the following voltage current relationship.

$$v_L = L \frac{di_L}{dt} \tag{2.7}$$

The voltage v_L across the inductor is proportional to the rate of change of current i_L flowing through it. The proportionality factor is called the inductance L, and has units of H (Henries) or $V \cdot s \cdot A^{-1}$. The current through the inductor at any given time is proportional to the integral of the voltage across it. The current through the inductor is given by

$$i_L = \frac{1}{L} \int v_L \, dt \tag{2.8}$$

The instantaneous power absorbed by the inductor is given by,

$$P = v_L \, i_L = L \, i_L \frac{di_L}{dt} \tag{2.9}$$

The power absorbed by the inductor can be positive or negative, depending on the direction of current flow. If the current is flowing into the inductor, then the voltage across it is increasing, and the power absorbed is positive. If the current is flowing out of the inductor, then the voltage across it is decreasing, and the power absorbed is negative. An inductor stores energy in the form of magnetic field around it. The energy stored in an inductor at any given time depends on the current flowing through it, and is given by,

$$E = \frac{1}{2} L \, i_L^2 \tag{2.10}$$

2.2 Kirchoff's Laws

The five elements alone are not that interesting. But interesting things can be done by connecting these elements together in different ways to form an electrical circuit. The elements are connected together by wires, which are assumed to be perfect conductors, i.e. zero resistance. Consider the following circuit (Figure 2.2),

How do we find out the voltages and currents in the circuit? Kirchoff's laws can be used for analysing such circuits, which are based on the conservation of charge and energy. The circuit in Figure 2.2 is a simple electrical circuit with a voltage source, a current source, and a bunch of resistors. The voltage source provides a fixed voltage V_s between its two terminals, and the current source provides a fixed amount of current I_s to flow through its terminals. The voltages and currents in the rest of the elements

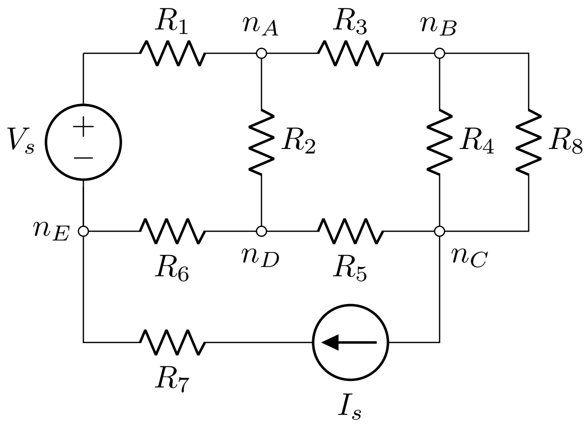


Fig. 2.2: A simple electrical circuit with a voltage sourse, a current source, and a bunch of resistors.

will be determined by Kirchoff's laws with the constraints imposed by the voltage and current sources. The two laws are:

1. **Kirchoff's current law (KCL):** The sum of the currents entering a node is equal to the sum of the currents leaving the node. A *node* is a point at which two or more circuit elements are connected together. In Figure 2.2, n_A, n_B, n_C, n_D and n_E are examples of nodes where three elements are connected together. There are two other nodes in the circuit, can you identify them?

   The sum of the currents at a node is equal to zero. This is based on the conservation of charge, and can be expressed mathematically as:

$$\sum_{i=1}^{n} i_i = 0 \tag{2.11}$$

   where i_i is the current flowing into or out of the node, and n is the number of elements connected to the node. The current flowing into a node is positive, and the current flowing out of a node is negative.

2. **Kirchoff's voltage law (KVL):** The sum of the voltages around a closed loop in a circuit is equal to zero. A *closed loop* is a path in the circuit that starts and ends at the same node, and does not cross itself. In Figure 2.2, the path starting from n_A, to n_D, to n_B, and back to n_A is a closed path. This path includes the resistors R_2, R_5, R_4, and R_3.

   This is based on the conservation of energy, and can be expressed mathematically as:

$$\sum_{i=1}^{n} v_i = 0 \tag{2.12}$$

where v_i is the voltage across each element in the loop, and n is the number of elements in the loop.

The voltage across an element is positive if the current is flowing into the positive terminal of the element, and negative if the current is flowing out of the positive terminal of the element.

Note that the two laws apply for any type of circuit element used in the circuits, indepenndnet or dependent voltage/current sources, resistors, capacitors, inductors, either? two, three or four terminal elements.

2.3 Series and Parallel Connections

Two elements that share the same voltage across them between a given pair of nodes are said to be **parallel** to each other. In Figure 2.2, R_4 and R_8 are parallel to each other. In a single loop, two elements that share the same current are said to be in **series** with each other. In Figure 2.2, V_s and R_1 are in series, I_s and R_7 are in series.

2.3.1 Resistors in series and parallel

When n resistors $R_1, R_2, \cdots R_n \geq 0$ are in series, these can be combined to an equivalent resistor with resistance R_{eq} given by the following,

$$R_{eq} = \sum_{i=1}^{n} R_i \implies R_{eq} \geq \max_{1 \leq i \leq n} R_i. \tag{2.13}$$

Note that in a series connection the equivalent resistance is at least as large as the largest value of R_1 to R_n.

When n resistors $R_1, R_2, \cdots R_n \geq 0$ are in parallel, these can be combined to an equivalent resistor with resistance R_{eq} given by the following,

$$\frac{1}{R_{eq}} = \sum_{i=1}^{n} \frac{1}{R_i} \implies R_{eq} = \frac{R_1 R_2 \cdots R_n}{R_1 + R_2 + \cdots + R_n} \implies R_{eq} \leq \min_{1 \leq i \leq n} R_i \tag{2.14}$$

Note that in a parallel connection, the equivalent resistance cannot be larger than the smallest value of R_1 to R_n.

2.3.2 Capacitors in series and parallel

Series connection of n capacitors $C_1, C_2, \cdots C_n \geq 0$

$$C_{eq} = \frac{C_1 C_2 \cdots C_n}{C_1 + C_2 + \cdots + C_n} \tag{2.15}$$

Parallel connection of n capacitors $C_1, C_2, \cdots C_n \geq 0$

$$C_{eq} = C_1 + C_2 + \cdots + C_n \tag{2.16}$$

2.3.3 Inductors in series and parallel

Series connection of n inductors $L_1, L_2, \cdots L_n \geq 0$

$$L_{eq} = L_1 + L_2 + \cdots + L_n \tag{2.17}$$

Parallel connection of n inductors $L_1, L_2, \cdots L_n \geq 0$

$$L_{eq} = \frac{L_1 L_2 \cdots L_n}{L_1 + L_2 + \cdots + L_n} \tag{2.18}$$

Its left as an exercise for you to verify these expressions.

What does the equivalent resistance actually mean? The equivalent resistor with resistance R_{eq} has the same voltage-current relationship as the individual elements in series or parallel connection. We can replace the series or parallel connection of the individual resistors R_1 to R_n by a single resistor with value R_{eq} without changing the volatage current relationships in the circuit. The same argument applies for equivalent capacitors and inductors.

2.3.4 Voltage sources in series and parallel

Series connection of n voltage sources $V_1, V_2, \cdots V_n$ will result in an equivalent voltage source V_{eq} given by

$$V_{eq} = V_1 + V_2 + \cdots + V_n \tag{2.19}$$

Voltage soruces should not be connected in parallel, as this will result in a short circuit. Ideally, an infinite current will flow through the connection, because the volatage sources force a potential difference between the two ends of the wire to be zero and it has zero resistance. Parallel connections are allowed only when the two sources have the same voltage and polarity.

2.3.5 Current sources in series and parallel

Parallel connection of n current sources $I_1, I_2, \cdots I_n$ will result in an equivalent current source I_{eq} given by

$$I_{eq} = I_1 + I_2 + \cdots + I_n \tag{2.20}$$

Current sources should not be connected in series; series connections are allowed only when the two sources have the same current and polarity.

2.4 Superposition Principle

Linear approximations of circuits are often employed as first order approximations when analysing circuits. A linear ciruit is one that consists of linear passive elements, independent sources and linear dependent sources. Linear circuits follow the superposition

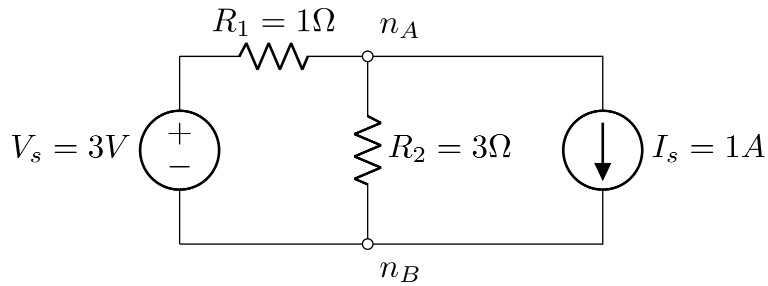


Fig. 2.3: A simple circuit with two sources

principle, which states that the response of a linear circuit to a linear combination of inputs is equal to the corresponding linear comibnation of the responses to each input applied separately. Solving the following circuit (Figure 2.3) should make this concept clear. For the circuit in Figure 2.3, perform the following calculation and compare your results.

Step 1. Solve for the voltage across and the current through the resistors R_1 and R_2; we will refer to these are v_{R1}, i_{R1} and v_{R2}, i_{R2}, respectively. You should use both Kirchoff's current and voltages laws to compute these variables.

Let's now consider of the sources in the circuit seperately. This would mean making the source values "zero". This corresponds to two different operations on the circuit. Zeroing a voltage source corresponds to replacing it with a wire (a short circuit), while zeroing a current source corresponds to simple removing the current source (an open circuit).

Step 2. Zero the voltage source V_s and compute the voltages and currents associated with the two resistors. We will refer to these as $v_{R1,V_s=0}$, $v_{R2,V_s=0}$, $i_{R1,V_s=0}$, and $i_{R2,V_s=0}$.

Step 3. Zero the current source I_s and compute the voltages and currents associated with the two resistors. We will refer to these as $v_{R1,I_s=0}$, $v_{R2,I_s=0}$, $i_{R1,I_s=0}$, and $i_{R2,I_s=0}$.

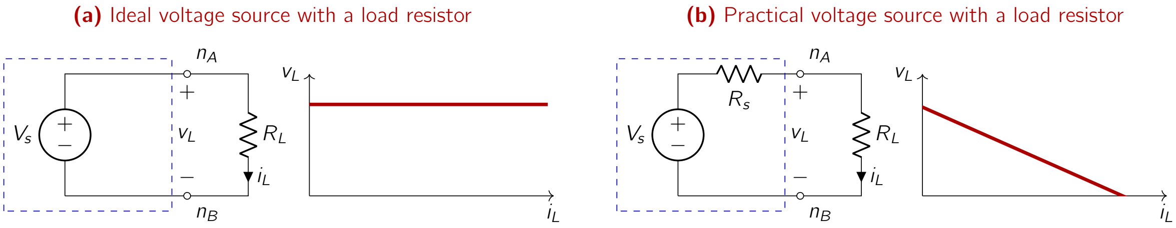
Step 4. For a linear circuit, shown in Figure 2.3, the following will always be true.

$$\begin{aligned} v_{R1} &= v_{R1,V_s=0} + v_{R1,I_s=0} \\ v_{R2} &= v_{R2,V_s=0} + v_{R2,I_s=0} \\ i_{R1} &= i_{R1,V_s=0} + i_{R1,I_s=0} \\ i_{R2} &= i_{R2,V_s=0} + i_{R2,I_s=0} \end{aligned} \tag{2.21}$$

Let's assume that I am only interested in i_{R2}. Can I use $i_{R2,V_s=0}$ and $i_{R2,I_s=0}$ to compute the current i_{R2} if $V_s = 1V$ and $I_s = -2A$?

2.5 Practical Voltage and Current Sources

The independent voltage and current sources we have discussed so far are "ideal" sources. Practical or real sources do not behave like them - a battery cannot pro-



(a) Ideal voltage source **(b)** Practical voltage source with a load

Fig. 2.4: Comparison of the voltage-current relationship of an ideal and a practical voltage source.

vide any amount of current for a load without any changes to the voltage across its terminals.

A good model of practical voltage source is an ideal voltage source V_s in series with a *internal*, *source* or *output* resistor R_s. And for a practical current source, it is an ideal current source I_s in parallel with a resistor R_s. The voltage across the terminal of a voltage source as a function of the current drawn from it is depicted for an ideal and practical voltage source in Figure 2.4. For the ideal source (Figure 2.4a), the voltage v_L is independent of the current i_L drawn from it. For the practical voltage source (Figure 2.4b), the voltage across the terminals is a function of the current drawn from it,

$$v_L = V_s - R_s\, i_L \tag{2.22}$$

When $R_L = \infty$ ($i_L = 0$), the voltage across the terminals is equal to the voltage of the source V_s. This is maximum voltage the practical source can provide. This is also know as the *open circuit voltage* v_{oc} of the source. When $R_L = 0$ ($v_L = 0$), the current drawn from the source $i_L = \frac{V_s}{R_s}$. This is the maximum current the voltage source can provide. This is also known as the *short circuit current* i_{sc} of the source.

Problem 2.1. Plot the voltage-current relationship of a practical current source with $I_s = 2A$ and $R_s = 10\Omega$. What are v_{oc} and i_{sc}?

2.6 Thevenin's and Norton's Theorems

Thevenin's and Norton's theorems are two important theorems in circuit theory that allow us to simplify complex circuits into simpler equivalent circuits. These theorems are based on the superposition principle, and can be used to analyze linear circuits with independent and dependent sources.

Thevenin's Theorem. Thevenin's theorem states that any linear circuit with independent and dependent sources can be replaced by an equivalent circuit with a single voltage source V_{th} in series with a resistor R_{th}, connected to the load resistor R_L.

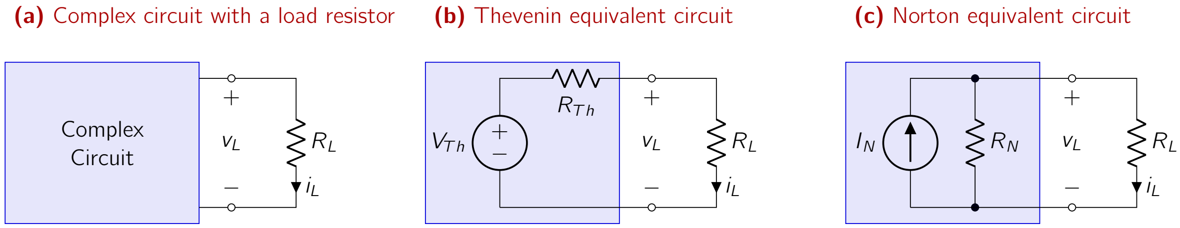


Fig. 2.5: Thevenin's and Norton's circuits for a complex linear circuit. The Thevenin and Norton equivalent circuits have the same voltage-current relationship for any load.

Norton's Theorem. Norton's theorem states that any linear circuit with independent and dependent sources can be replaced by an equivalent circuit with a single current source I_N in parallel with a resistor R_N, connected to the load resistor R_L.
Computing the Thevenin and Norton equivalent circuits. The Thevenin and Norton equivalent circuits can be computed using the following steps:

1. Remove the load resistor R_L from the circuit.

2. Compute the open circuit voltage v_{oc} across the terminals of the load resistor. This is the Thevenin voltage V_{th}. This can be done using the superposition principle, where we zero all the independent sources except one and compute the open circuit voltage. The overall open circuit voltage is the sum of the open circuit voltages when all other sources are zeroed.

3. Compute the short circuit current i_{sc} through the terminals of the load resistor. This is the Norton current I_N. This too can be calculated using the superposition principle.

4. Compute the Thevenin resistance R_{th} by zeroing all independent sources in the circuit and computing the equivalent resistance seen from the terminals of the load resistor (without the load resistor). This is also equal to the Norton resistance R_N.

5. The Thevenin and Norton equivalent circuits are then given by:

$$\begin{aligned} V_{th} &= v_{oc} \\ I_N &= i_{sc} \\ R_{th} &= R_N \end{aligned} \tag{2.23}$$

6. The load resistor R_L can be connected to either the Thevenin or Norton equivalent circuit, and the voltage and current across it can be computed using the voltage-current relationship of the equivalent circuit.

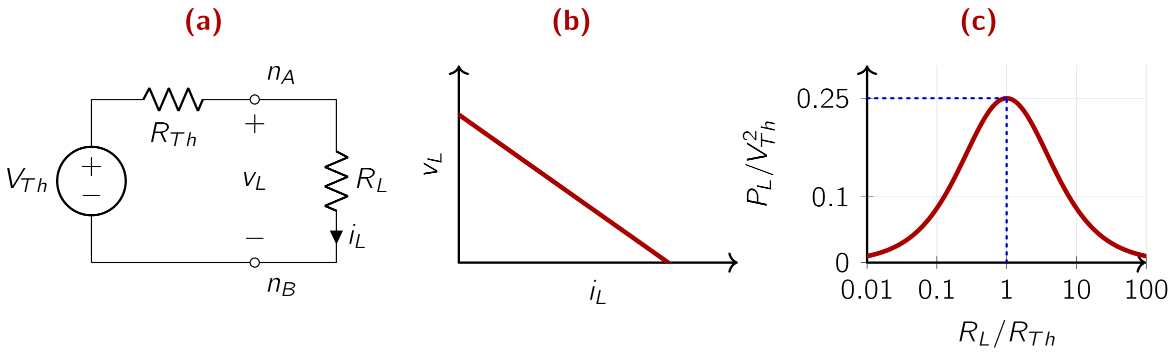


Fig. 2.6: Maximum power transfer theorem.

Problem 2.2. Compute the Thevenin and Norton equivalent circuits for the circuit shown in Figur 2.2 assuming the following as the load resistor: (a) R_8; (b) R_2; and (c) R_7.

2.7 Maximum Power Transfer Theorem

The maximum power transfer theorem states that the maximum power is transferred to the load resistor R_L when the load resistance is equal to the Thevenin resistance R_{th} of the circuit. Consider the following circuit (Figure 2.6). The power absorbed by the load resistor R_L is given by,

$$P_{R_L} = \frac{R_L}{(R_{th} + R_L)^2} V_{th}^2 \tag{2.24}$$

Its easy to check that the optimal value of R_L that maximizes the power absorbed by the load resistor is given by,

$$\begin{aligned} R_L^* &= \arg\max_{R_L} \frac{R_L}{(R_{th} + R_L)^2} V_{th}^2 = R_{Th} \\ P_L^* &= \max_{R_L} \frac{R_L}{(R_{th} + R_L)^2} V_{th}^2 = \frac{1}{4}\left(\frac{V_{Th}^2}{R_{Th}}\right) \end{aligned} \tag{2.25}$$

Problem 2.3. Prove the statements in Eq. 2.25. (*Hint:* Use the first order condition for maximization of a continuous function.)

2.8 RC, RL, and RLC Circuits

Unlike resistors, that neither remember the past values of its votlage or current, capacitors and indutors retain memory of the past current and past voltage, respectively. This can be easily seen from their respective voltage-current relationships given by Eq. 2.3 and Eq. 2.7, respectively.

Capacitor:

$$i_C = C\frac{dv_C}{dt} \implies v_C(t) = \frac{1}{C}\int_0^t i_C(\tau)\,d\tau + v_C(0) \tag{2.26}$$

The instantaneous voltage across a capacitor contains some information about the past history the current that has flown through the capacitor. This voltage $v_C(t)$ is determined by the charge on the capacitor at time t. Note, that the voltage across the capacitor cannot change instantaneously, as this would require an infinite current to flow through the capacitor. Theoretically, however, an impulse (Dirac delta function) current applied to the capacitor can produce instantaneous change in the capacitor's voltage.

Inductor:

$$v_L = L\frac{di_L}{dt} \implies i_L(t) = \frac{1}{L}\int_0^t v_L(\tau)\,d\tau + i_L(0) \tag{2.27}$$

Similarly, the instantaneous current through the inductor contains some information about the history of voltage applied across the inductor. Current through an inductor cannot change instantaneously, as this would require an infinite voltage to be applied across the inductor. An impulse voltage applied to the inductor can produce instantaneous change in the inductor's current.

RC Circuit. Consider a simple RC circuit shown in Figure 2.7a. We can write Kirchoff's voltage law for the circuit as follows,

$$RC\frac{dv_C}{dt} + v_C = V_s \implies v_C(t) = e^{-t/RC}\left[v_C(0) + \int_0^t \frac{1}{RC}e^{\tau/RC}V_s(\tau)\,d\tau\right] \tag{2.28}$$

The above euqation gives the general solution for the voltage across the capactior. We can derive all other variables of interest from v_C. The response to an impulse input, step input, sinusoidal input, or any arbitrary V_s can be computed using the above equation. The response for a step input obtained using a fixed sources and a switch is given in Figure 2.7a. RC is the time constant of the circuit, and is a measure of how fast the capacitor charges or discharges and it has the units of time.

Using Laplace transform to analyze the circuit, we can write the following expression for the voltage across the capacitor,

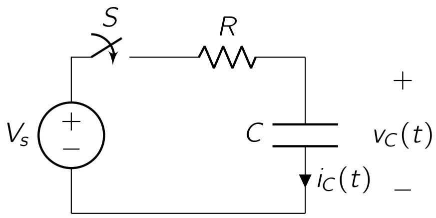
$$V_C(s) = \frac{1}{sRC+1}V_s(s) + \frac{v_C(0)}{sRC+1} \tag{2.29}$$

$V_C(s)$ and $V_s(s)$ are the Laplace transforms of $v_C(t)$ and $V_s(t)$. Replacing $s = j\omega$ will give us the frequency response of the system with $V_C(s)$ as the output and $V_s(s)$ as the input.

RL Circuit. Consider a simple RL circuit shown in Figure 2.7b. Kirchoff's voltage law for the circuit is as follows,

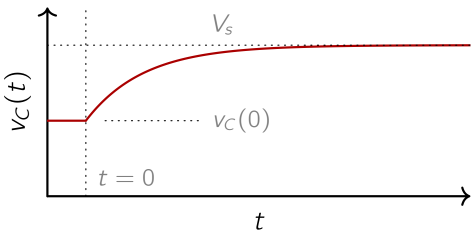
$$\frac{L}{R}\frac{di_L}{dt} + i_L = \frac{1}{R}V_s \implies i_L(t) = e^{-tR/L}\left[i_L(0) + \int_0^t \frac{1}{L}e^{\tau R/L}V_s(\tau)\,d\tau\right] \tag{2.30}$$

(a) A Simple RC Circuit



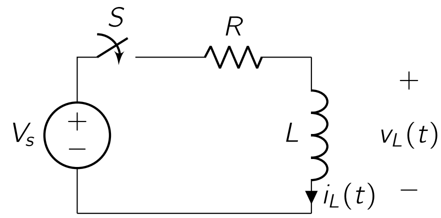
Assuming the switch S is closed at $t = 0$, the voltage across the capacitor $v_C(t)$ is given by,

$$v_C(t) = V_s \cdot \left(1 - e^{-\frac{t}{RC}}\right) + v_C(0)e^{-\frac{t}{RC}}, \;\; t \geq 0$$



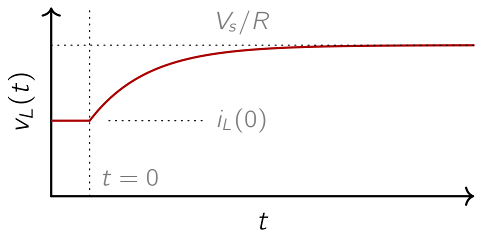
(a) Ideal voltage source

(b) A Simple RL Circuit



Assuming the switch S is closed at $t = 0$, the current through the inductor $i_L(t)$ is given by,

$$i_L(t) = \frac{V_s}{R} \cdot \left(1 - e^{-\frac{t}{L/R}}\right) + i_L(0)e^{-\frac{t}{L/R}}, \;\; t \geq 0$$



(b) Practical voltage source with a load

Fig. 2.7: Simple RC and RL circuits and their transient responses.

The above euqation gives the general solution for the current through the inductor. The response to an impulse input, step input, sinusoidal input, or any arbitrary V_s can be computed using the above equation. The response for a step input obtained using a fixed sources and a switch is given in Figure 2.7a. $\frac{L}{R}$ is the time constant of the circuit, and is a measure of how fast the current through the inductor can chamge; it has the units of time.

Similarly, applying the Laplace transform, we have,

$$I_L(s) = \frac{1}{sL + R} V_s(s) + \frac{i_L(0)}{sL + R} \tag{2.31}$$

RLC Circuit. Consider a simple series RLC circuit shown in Figure 2.8. Kirchoff's voltage law for the circuit is as follows,

$$LC\frac{d^2 v_C}{dt^2} + RC\frac{dv_C}{dt} + v_C = V_s \tag{2.32}$$

The response of the circuit when the switch is closed at $t = 0$ is qualitatively different depending on the values of R, L and C. This is better understood in the Laplace domain. The Laplace transform of the response $v_c(t)$ is given by,

$$V_C(s) = \frac{V_s(s)}{LCs^2 + RCs + 1} + v_C(0)\frac{C\,(Ls + R)}{LCs^2 + RCs + 1} + i_L(0)\frac{RC}{LCs^2 + RCs + 1} \tag{2.33}$$

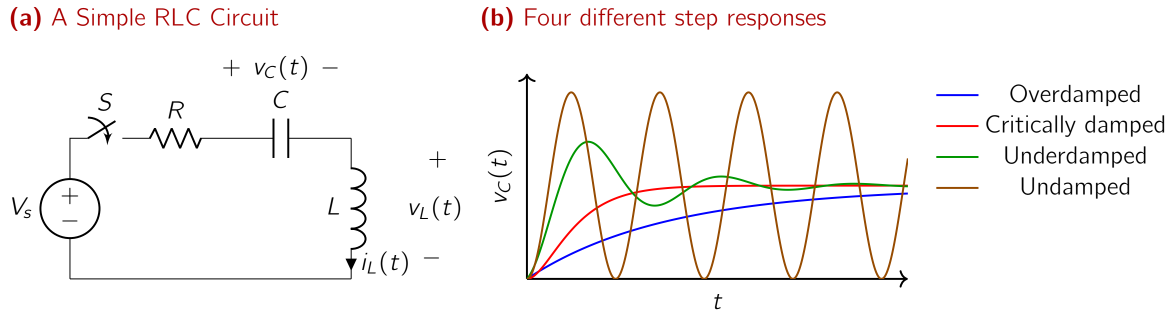


Fig. 2.8: A simple RLC circuit and the different respones of the circuit when the switch S is closed at time $t = 0$, with zero initial capacitor voltage and inductor current whent .

Note that $i_L(0) = \frac{dv_C}{dt}(0)$. The denominator of the above equation is a second order polynomial in s, and thus the response of the circuit will depend on the roots of the polynomial. The four different responses are shown in Figure 2.8.

2.9 Steady State Sinusoidal Analysis

The previous subsection looked at the transient response of the some simple circuits involving R, L, and C. However, we are often interested in the steady state response of these circuits to sinusoidal excitations. This is because sinusoidal signals are eigenfunctions of linear systems, such as the linear circuits we have discussed so far. With sinusoidal excitations, all currents and voltages in the circuit will also be sinusoidal with the same frequency as the input excitations, althought with different amplitudes and phases. Complex numbers are used in this analysis for compact representation of ampltiude and phase of signals.

For steady state frequency analysis for circuits involving R, L, and C, we extend the idea of resistance to impedance. Fourier transforms provide a natural way to analyze the steady state response of linear circuits to sinusoidal excitations. We first recast the voltage-current relationships of the circuit elements in the Fourier domain.

$$\begin{aligned}
\textbf{Resistor:} \quad & v_R = R\, i_R \implies V_R\,(j\omega) = R \cdot I_R\,(j\omega) \\
\textbf{Capacitor:} \quad & i_C = C\frac{dv_C}{dt} \implies V_C\,(j\omega) = \frac{1}{j\omega C} \cdot I\,(j\omega) \\
\textbf{Inductor:} \quad & v_L = L\frac{di_L}{dt} \implies V_L\,(j\omega) = j\omega L \cdot I_L\,(j\omega)
\end{aligned} \tag{2.34}$$

The Fourier transformed voltage-current relationships of the circuit elements look like Ohms law, with the concept of resistance extended to impedance. The impedance of a capacitor and inductor defined as the ratio of the Fourier transformed voltage to the Fourier transformed current. They are functions of the frequency of the voltage/current.

The impedance of a resistor, capacitor, and inductor are given by,

$$
\begin{aligned}
\textbf{Resistor:} \quad & Z_R = R \\
\textbf{Capacitor:} \quad & Z_C = \frac{1}{j\omega C} \\
\textbf{Inductor:} \quad & Z_L = j\omega L
\end{aligned}
\tag{2.35}
$$

Equivalent Impedance. The equivalent impedance of a circuit with impedances $Z_1, Z_2, \cdots Z_n$ in series is given by $Z_{eq} = Z_1 + Z_2 + \cdots + Z_n$. When the impedances $Z_1, Z_2, \cdots Z_n$ are in parallel, the equivalent impedances is given by $\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_n}$.

Amplitude and phase modification by an impedance. If a sinusoidal current excitation $i(t) = I_o \sin(\omega t)$ is passed through an impednace Z. Then the voltage across the impedance is given by,

$$
v(t) = |Z| I_o \sin(\omega t + \arg Z) \tag{2.36}
$$

The amplitude of the sinusoidal voltage is $|Z| I_o$, and the phase of the sinusoidal voltage is $\arg Z$, with respect to the current $i(t)$.

2.10 Exercise

1. Plot the current through a resistor $R = 10\Omega$, capacitor $C = 5\mu F$, and inductor $L = 2mH$ for the following voltage inputs.

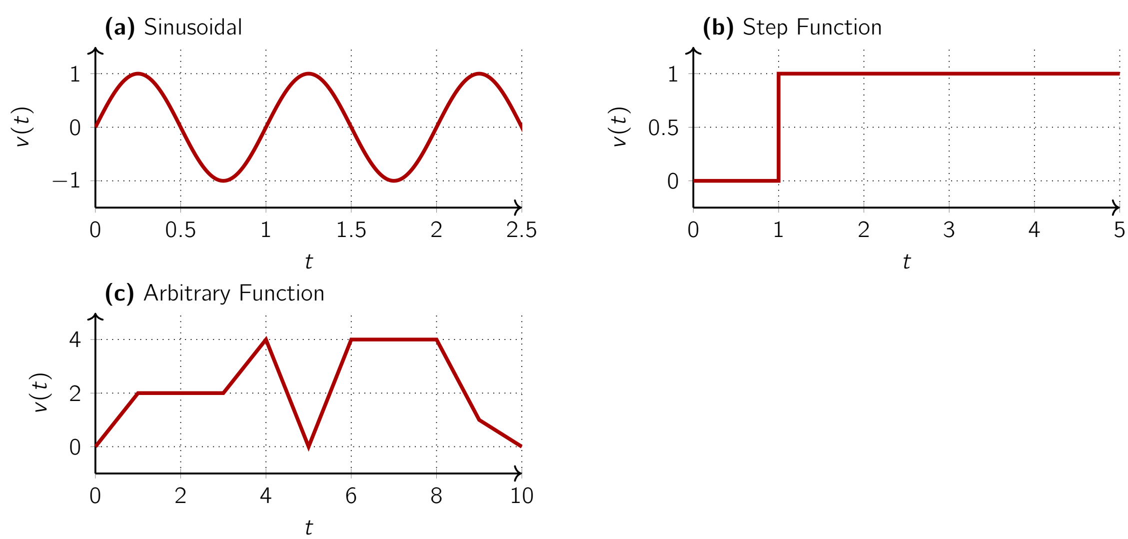


Fig. 2.9: [Exercise 1] Voltage inputs applied to a resistor, capacitor, or an inductor. The units of voltage are in volts, and the time is in seconds.

2. Find the Thevenin and Norton equivalent circuits of the following circuits (Figure 2.10). Note that one of the problems has a dependent source. Dependent sourcrs are voltage or current sources (diamond shaped elements), whose voltage or current, respectively, depend on the voltage or current across another element in the circuit.

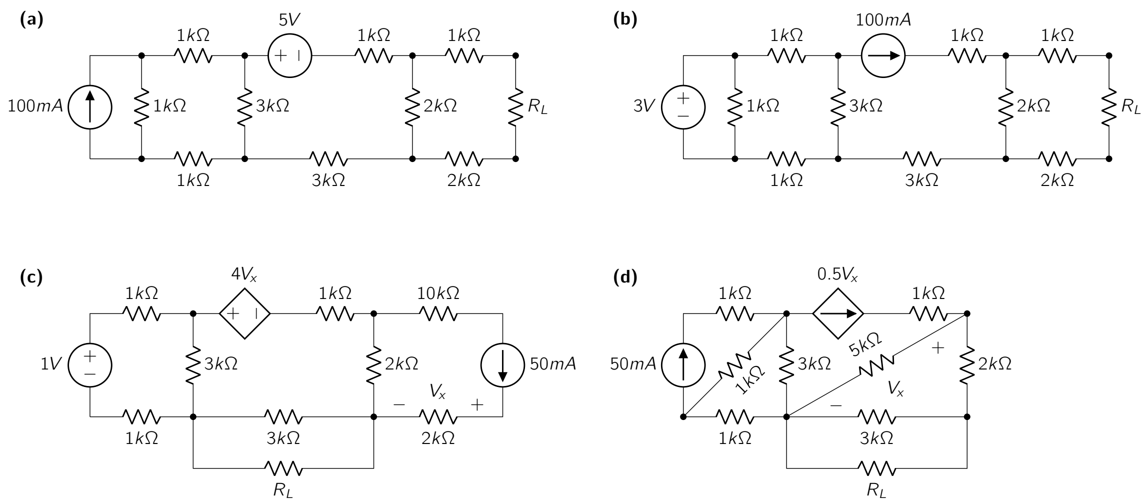


Fig. 2.10: [Exercise 2] Find the equivalent impedance of the circuit.

3. A certain red LED has a maximum current rating of 35mA, and if this value is exceeded, overheating and catastrophic failure will result. The resistance of the LED is a nonlinear function of its current, but the manufacturer warrants a minimum resistance of 47Ω and a maximum resistance of 117Ω. Only 9V batteries are available to power the LED. Design a suitable circuit to deliver the maximum power possible to the LED without damaging it. Use only combinations of the standard resistor values. (*This problem is from Engineering Circuit Analysis by Hayt Jr. et al.*)

4. The load resistor in Figure 2.11 can safely dissipate up to 1W before overheating and bursting into flame. The lamp can be treated as a 10.6Ω resistor if less than 1A flows through it and a 15Ω resistor if more than 1A flows through it. What is the maximum permissible value of I_s? Verify your answer with an appropriate computer simulation. (*This problem is from Engineering Circuit Analysis by Hayt Jr. et al.*)

5. In the following circuit (Figure 2.12), find the voltage across the capacitors C_1 and C_2. The single pole double throw swtich S is connected to the top terminal before time $t = 0$. At time $t = 0$, swtich is flipped to the bottom terminal. Assume that before time $t = 0$, the voltage across the capacitor C_2, $v_{C2} = 1V$. Draw the plot of the voltage across the capcitors C_1 and C_2 as function of time.

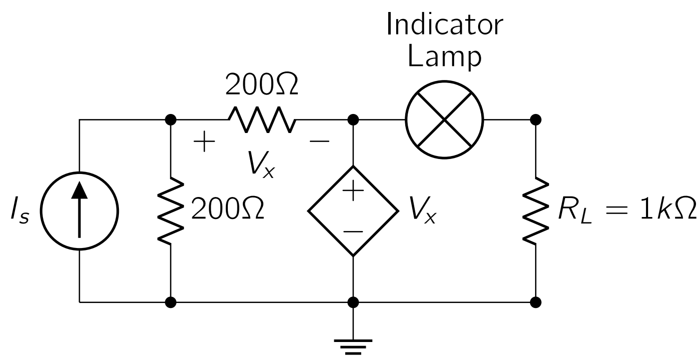


Fig. 2.11: [Exercise 4]Find the equivalent impedance of the circuit.

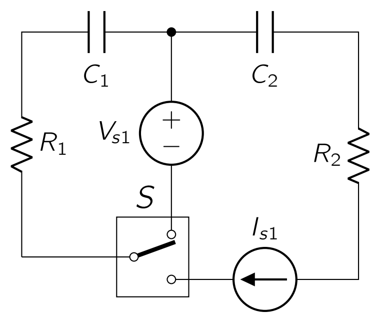


Fig. 2.12: [Exercise 5]Find the equivalent impedance of the circuit.

6. Figure 2.13 shows a voltage divider circuit with a sinusoidal voltage source $V_s(t) = 10\sin(1000\pi t)$. The impedane $Z_1 = 3 + j4\Omega$. What should the impedance Z_2 be if the the voltage across Z_2 has one-fourth the amplitude of V_s and with a phase difference of 45 deg?

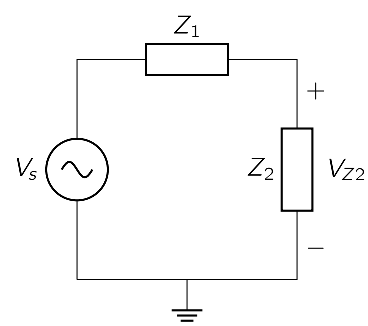


Fig. 2.13: [Exercise 5]Find the equivalent impedance of the circuit.

Chapter 3

Electronics Review

This chapter is a quick review of basic electronics. The previous chapter involved simple, linear passive two terminal elements, which are essential for building most electrical circuits. This chapter will focus on some essential non-linear, two/three terminal electronics components which form the basis for more useful electronics cricuits such as amplifiers, oscillators,filters, power supplies, switches, etc. We will review three important electronic components and their circuits in this chapter: *diode*, *bipolar junction transistor (BJT)*, and *metal oxide field effect transistor*.

3.1 Basic semiconductor concepts

Conductivity of a material is proportional the concentration of free electrons.

$$\begin{aligned} \textbf{Conductor:} \quad & n_{cond} \approx 10^{28} electrons/m^3 \\ \textbf{Insulator:} \quad & n_{ins} \approx 10^{7} electrons/m^3 \\ \textbf{Semiconductor:} \quad & n_{ins} < n_{sem} < n_{cond} \end{aligned} \tag{3.1}$$

3.1.1 Intrinsic semiconductors

Silicon or Germanium have crystalline strcuture with four covalent bonds between neighbouring atoms. At $0°K$ semiconductors all covalent bonds are in place tightly holding on

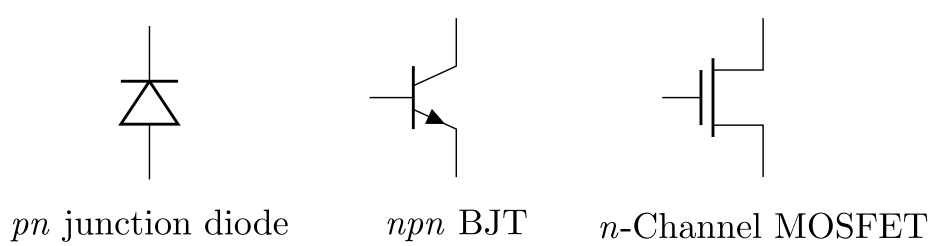


Fig. 3.1: Three most essential electronic components reviewed in this chapter.

to electrons in the bonds. Thus these behave as perfect insulators at low temperatures. With increase in temperature, covalent bonds are ruptured, releasing *free electrons* to roam around in the crystal and become available for conduction when there is an externally applied electric field. The missing electron in the ruptured covalent bond is called a hole, which too act as carrier of electricity. When an electric field is applied, electrons from neighbouring covalent bonds jump into a hole creating a hole in their previous bond. This results in the hole effectively moving the direction opposite to the jumping electrons from the covalent bonds. Thus, unlike conductors, semiconductors can have two types of charge carriers: *electrons* and *holes*. The concentration of free electrons and holes in a semiconductor determines it conductivity, which is controlled by temperature. In intrinsic semiconductors, the concentration of free electrons and holes is equal, i.e. $n_{e^-} = n_{h^+} = n_i$. This concentration is given by,

$$n_i = B\,T^{3/2}\,e^{-\frac{E_g}{k_B T}} \tag{3.2}$$

where B is a material constant ($7.3 \times 10^15 \text{cm}^{-3}\text{K}^{-3/2}$ for Si), E_g is the band gap energy, k_B is the Boltzmann constant, and T is the absolute temperature in Kelvin. The band gap energy is the energy required to break a covalent bond and create a free electron-hole pair. It should be noted that the conductivity of semiconductors increases with temperature, making them suitable for thermometry.

3.1.2 Doped Semiconductors

With intrinsic semiconductors, the conductivity is still quite small at room temperature. Another precise and controlled way to change a semiconductor's conductivity is by adding impurities, called *doping*. The two types of doping are: *n*-**type** and *p*-**type** doping.

In a *n*-**type** semiconductor, a small amount of pentavalent atoms (e.g. Phosphorus, Arsenic) are added to the semiconductor. These atoms have five valence electrons, and when added to the silicon crystal, four of these electrons form covalent bonds with the neighbouring silicon atoms, while the fifth electron is free to roam around in the crystal. This results in an increase in the concentration of free electrons, making the semiconductor more conductive. The concentration of free electrons in an *n*-type semiconductor is given by,

$$n_{e^-} = n_i + N_D \quad n_{h^+} = n_i \tag{3.3}$$

where, N_D is concentration of the doping element.

In a *p*-**type** semiconductor, a small amount of trivalent atoms (e.g. Boron) is added to the silicon crystal structure. This leaves the dopped elements with four covalent bonds with one of the bonds lacking an electron or a hole. This holes acts as a charge carrier. The concentration of the holes in a *p*-type semiconductor is give by,

$$n_{h^+} = n_i + N_A \quad n_{e^-} = n_i \tag{3.4}$$

where, N_A is concentration of the doping element in the *p*-type semiconductor.

The doped semiconductors will have higher conductivity than the intrinsic semiconductors depending on the concentration of the impurities added to the silicon crystal.

3.1.3 Flow of current in semiconductors

There are two types of current flow mechanisms in seminconductors unlike pure conudctors – (a) **drift current** and (b) **diffusion current**. Both are imporant for understanding the operation of the basic electronic components.

Drift current. Drift current is established by an external electric fields, for examples when a semiconductor is connected to a battery. Holes will accelerate in the direction of the field, while the electrons accelerate in the opposite direction. Bumping into the atoms of the cystral structure, the holes and electrons acquire an average drift velocity given by,

$$\nu_{p-drift} = \mu_p E \quad \& \quad \nu_{n-drift} = -\mu_n E \tag{3.5}$$

This constitute a drift current through the semiconductor given by the following,

$$I_{drift} \propto q \left(n_{h^+}\mu_p + n_{e^-}\mu_n \right) E \tag{3.6}$$

where, q is the magnitude of electron charge.

Diffusion current. Diffusion currents result whena concentration gradient exists across a semiconductor.

$$I_{diff} \propto -qD_q \frac{dp\left(x\right)}{dx} \tag{3.7}$$

where, $p\left(x\right)$ is the concentration of the charge carrier along the x direction. Diffusion currents play an important role in the functioning of the BJT.

3.2 Diode

Whena *p*-type and *n*-type semiconductors are brought in close contact to each other, there is a big concentration difference between the charge carriers across the interface. A *pn*-junction diode made by bringing a *n*-type and *p*-type semiconductors together as shown in Figure 3.2. Due to the concentration difference, electrons from the *n*-type semiconductor diffuse into the *p*-type semiconductor, while holes from the *p*-type semiconductor will diffuse into the *n*-type semiconductor. Note that this is the diffusion current I_D, which flows even when the diode is open circuited. This diffusion current sets up the depletion region at the interface, where there are no free charge carriers. This region consists of fixed positive and negative charges, which creates an electric field across the junction. This electric field opposes further diffusion of charge carriers across the junction, and the potential or *barrier voltage* V_0 due to the electric field must

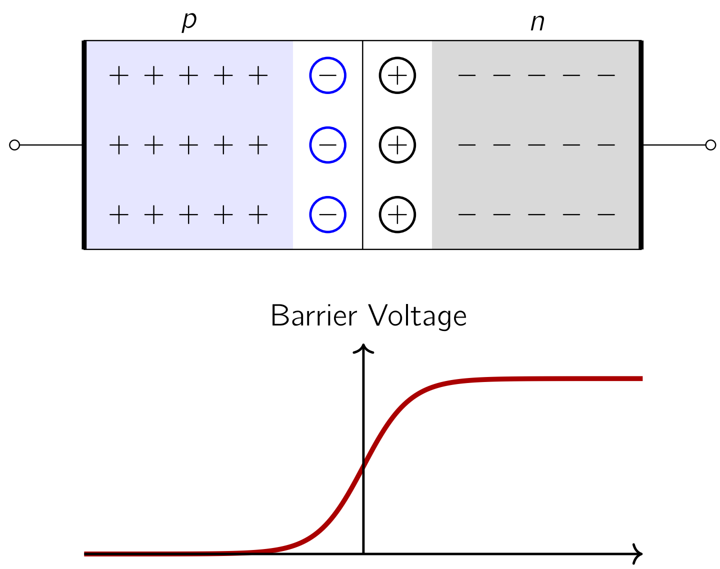


Fig. 3.2: *pn* junction diode

be overcome for th diffusion current to flow across the junction. Note that the diffusion current I_D is due to the majority charge carriers – holes from the p side and *electrons* from the n-side.

There is a drift current I_S that flows across the junction due to the minority charge carriers - the *electrons* from the p-side and the *holes* from the n-side. These thermally generated minority charge carriers, will be swept across and transported across the junction due to the electric field, which tends to reduce the barrier voltage V_0. Thus, when the diode is open circuited, we have an equilibrium $I_D = I_S$.
Barrier voltage of a diode depends on several factors, the amount fo doping of the n and p semiconductors, the temperature, etc. The exact relationship is give by,

$$V_0 = V_T \ln\left(\frac{N_A N_D}{n_i^2}\right) \tag{3.8}$$

where, V_T is the thermal voltage given by $V_T = \frac{k_B T}{q}$, where k_B is the Boltzmann constant, T is the absolute temperature in Kelvin. For silicon diodes, this is approximately $V_T \approx 26mV$ at room temperature. The barrier voltage V_0 is typically around $0.7V$ for silicon diodes, and around $0.3V$ for germanium diodes.

3.2.1 Applying a voltage across a diode

When we connect a voltage source across the diode (Figure 3.3), we can either apply a forward bias or a reverse bias. When the positive terminal of the voltage source is connected to the p-side and the negative terminal is connected to the n-side, we have a **forward bias**. This reduces the barrier voltage V_0, and allows the diffusion current I_D to flow across the junction. The diode is said to be *on* in this case, and the current I

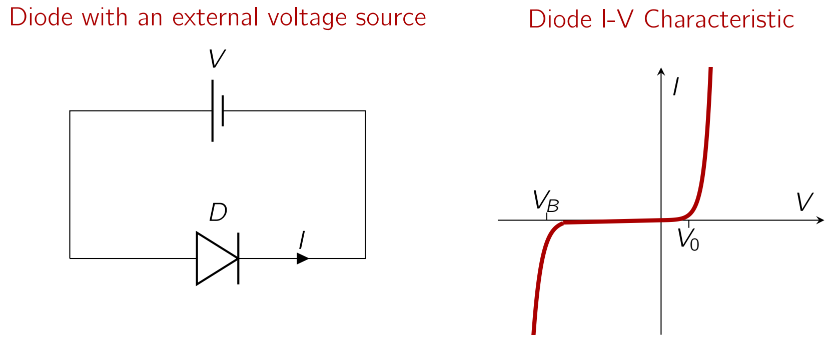


Fig. 3.3: V-I characteristics of a diode

flowing through the diode is called the *forward current*. The voltage-current realtionship between the forward current I and the voltage V across the diode is given by,

$$I = I_S\left(e^{\frac{V}{V_T}} - 1\right) \tag{3.9}$$

where, I_S is the saturation current, which is the reverse current that flows when the diode is reverse biased, and , and q is the magnitude of electron charge.

When the negative terminal of the voltage source is connected to the p-side and the positive terminal is connected to the n-side, we have a **reverse bias**. This increases the barrier voltage V_0, and prevents the diffusion current I_D from flowing across the junction. The diode is said to be *off* in this case, and only a small reverse saturation current I_S flows through the diode. When the reverse biased voltage is increased beyond a certain threshold, called the *breakdown voltage* V_B, the diode will start conducting in the reverse direction. This is called *avalanche breakdown*, and the diode can be damaged if the current is not limited by a resistor or some other means. The avalanche breakdown is the reason for the sharp rise in the current at the V_B.

3.2.2 A simple diode circuit

In the circuit shown in Figure 3.3, the moment the forward bias-voltage gets close tot 0.6, the current starts to rise drammatically. Slight changes in the voltage will lead to

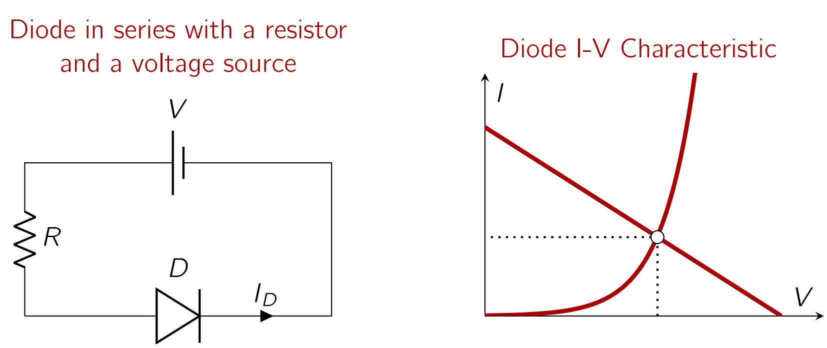


Fig. 3.4: Diode voltage and current in a simple diode circuit with a series resistor.

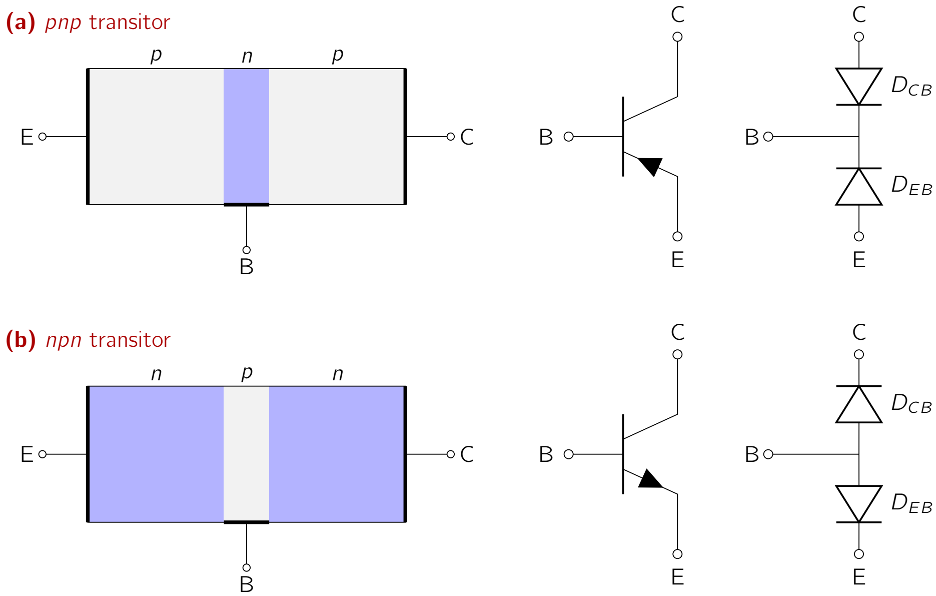


Fig. 3.5: Bipolar junction transistor structure and symbol

large changes in the current if it is not limited by a series resistor. Without a resistor, we can easily burn the diode due to large power dissipation across the diode. Consider a more practical cicruit with a voltage source, resistor, and a diode, as shown in Figure 3.4. The voltage across the diode V_D and the current through the diode I_D by writing the Kirchoff voltage law (KVL) around the loop,

$$V - I_D R - V_D = 0 \tag{3.10}$$

The current I_D and V_D from Eq. 3.9 can be substituted into Eq. 3.10 to solve for V_D and I_D,

$$V - RI_S \left(e^{\frac{V_D}{V_T}} - 1 \right) - V_D = 0 \tag{3.11}$$

Unfortunately, this is a non-linear solution will need to be solved to find the voltage across the diode V_D and the current through the diode I_D.

Let's assume that $V = 5V$ and $R = 1k\Omega$ in Figure 3.4. A first order approximation can be made by assuming that the diode is *switched off*, i.e. no current is flow through it and see if it leads to a contradiction. If we assume the diode is off and no current is flowing $I_D = 0mA$ through it, then $V_D = 5V$ The diode cannot be off if $V_D \geq 0.7V$. So, the diode must be swtiched on. We will assume that the diode is on and then find out the current flowing through it and check that it is not close to 0. In the current circuit, have

$$I_D \approx \frac{V - 0.7}{R} = 4.3mA \tag{3.12}$$

I_D is large enough for to flow through the didode when its switched on. Thus, $V_D = 0.7V$ and $I_D = 4.3mA$ is a reasonable first approximation of the diode's voltage and current in the current circuit.

3.3 Bipolar junction transistor (BJT)

The bipolar junction transistor (BJT) is the first of some of the three terminal devices we will come across int his course. There are two types of BJTs – *npn* and *pnp*. BJTs are formed by sandwiching a thin layer of *p*-type semiconductor between two *n*-type semiconductors or vice versa. This is depicted in Figure 3.5. Metal contacts are made with these three semiconductor regions to form the three terminals of the BJT. In an *npn* BJT, one of the *n* regions is the *emitter* which is heavily dopped, and the other *n* region is the *collector*. The *p* region is the *base*, which is lightly doped and very thin compared to the widths of the emitter and collector regions. The collector is moderately doped. Three terminal devices like the BJT and MOSFET can be used as an amplifier or a switch, depending on how it is biased. We will only discuss *npn* transistors for the rest of the section; the *pnp* transistor is similar, but with the polarities of the voltages and currents reversed.A BJT is essentially two diodes connected back to back as shown in Figure 3.5

A BJT can be operated in three different modes, depending on the biasing of the terminals. The three modes are: (a) **Cut-off mode**, (b) **Active mode**, and (c) **Saturation mode**. The difference between the three modes are summarized in Table 3.1.

3.3.1 Cut-off mode

In this mode, the emitter-base (EB) and the collector-base (CB) junctions are reverse biased, and not current flows through the transistor.

$$I_E = 0 \quad I_B = 0 \quad I_C = 0 \tag{3.13}$$

3.3.2 Active mode

In this mode, the EB junction is forward biased, while the CB junction is reverse biased. Consider the circuit in Fig. 3.6. When the *npn* BJT's EB junction is forward biased ($V_{BE} \approx 0.7V$), electrons (the majority carrier) from the highly doped emitter region are injected just across the EB junction. The electrons diffuse into the base region, which is lighly doped and very thin. Because $V_C > V_B$, the injected electrons are swept across the base region by the electric field created by the reverse biased CB junction. A small percentage of these injected electrons recombibe with the holes in the base region, which contributes to the small base current. The collector current will be proportional the base current; the current gain is given by the parameter β.

$$I_C = \beta I_B \quad \text{where, } \beta \approx 100 - 1000 \tag{3.14}$$

An equivalent circuit of the BJT in the active model is shown in the following figure (Fig. 3.7). In this circuit, assume that the supply voaltges V_{BB}, V_{CC}, and the resistors

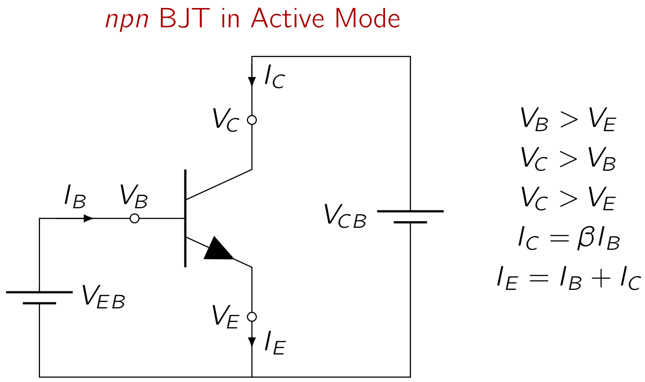


Fig. 3.6: *npn* BJT in active mode.

R_B and R_C are chosen appropriately to place the *npn* transistor in the active mode. The AC source v_s of sufficiently small in amplitude (in mV) will result in small changes in I_B, which lead to corresponding change in I_C. The voltage at the collector V_C have a larger amplitude variations riding on a DC voltage, given by the following,

$$V_C = V_{CC} - \beta \frac{R_C}{R_B} (V_{BB} - 0.7) - \beta \frac{R_C}{R_B} v_s \tag{3.15}$$

This is easy to derive and is left as an exercise for the reader. The input voltage v_s get amplified by a factor of $\beta \frac{R_C}{R_B}$, which is the voltage gain of this the common emitter amplifier circuit. We will no disucss such circuits any further in the course. This was just to illustrate the utility of the BJT in amplifying small AC signals when operated in the active mode.

Table 3.1: Modes of operation of a bipolar junction transistor (BJT)

Mode	**Emitter-Base Junction**	**Collector-Base Junction**	**Operation**
Cut-off	Reverse biased	Reverse biased	Transistor is off and no current flows.
Active	Forward biased	Reverse biased	I_C is proportional to I_B
Saturation	Foward biased	Forward biased	I_C saturates, does not increase with I_B.

3.3.3 Saturation mode

Consider the circuit in Fig. 3.8. If for fixed V_{BB}, we reduce R_B, this will increase I_B, which would result in progressively increasing I_C. However, I_C cannot continue to increase forever. As I_C increases, $V_C = V_{CC} - I_C R_C$ decreases. Once its value falls sufficiently below the base voltage V_B, the collector-base (CB) junction becomes forward biased. This is called the **saturation mode** of operation of the BJT. In this mode, the BJT behaves like a closed switch, and the collector current I_C saturates at a value

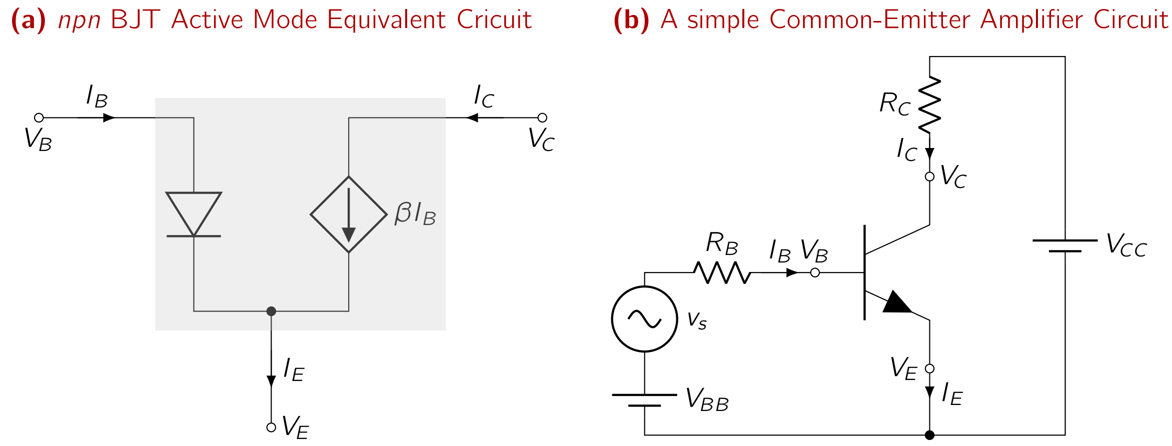


Fig. 3.7: *npn* BJT active mode equivalent circuit, and a simple common emitter voltage amplifier circuit.

that is independent of the base current I_B. The collector-emitter voltage saturates at $V_{CE(sat)} \approx 0.2V$ for silicon BJTs, making the collector current independent of the base current.

$$I_C \approx \frac{V_{CC} - V_{CE(sat)}}{R_C} \quad \text{and } V_{CE(sat)} \approx 0.2V \tag{3.16}$$

The saturation mode is used in switching applications, where the BJT is used as a switch to turn on or off a load connected to the collector; think of the collector resistor as the load. We will discuss some applications later in the chapter.

3.4 Metal oxide field effect transistor (MOSFET)

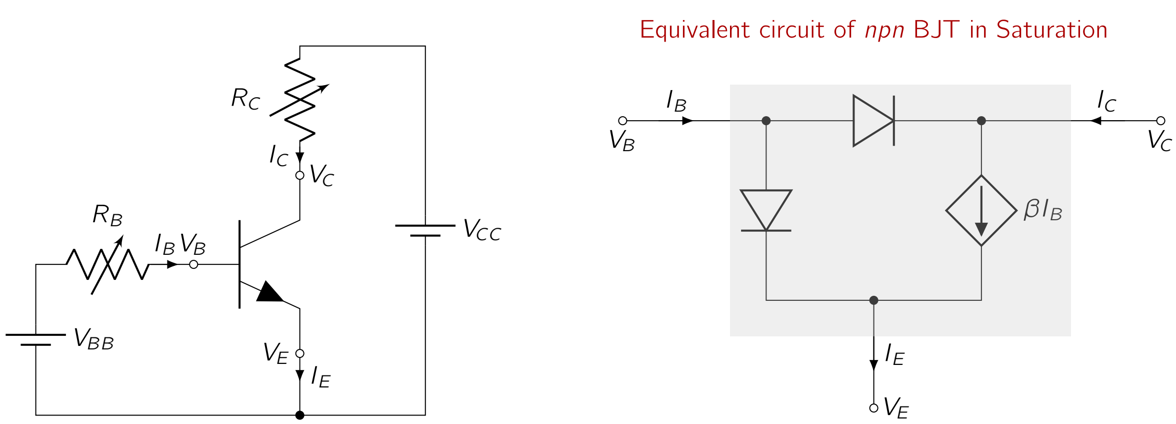


Fig. 3.8: *npn* BJT active mode equivalent circuit, and a simple common emitter voltage amplifier circuit.