Mechatronics for Rehabilitation Engineering: Course Notes

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Chapter 1

Review of Basic Circuit Theory

This chapter is a quick review of basic circuit theory. Prior knowledge of these topics is assumed, along with basic understanding of linear time invariant systems - Fourier/Laplace transforms, linear constant coefficient differential equations, transfer functions, and frequency response. The reader is encouraged to review the material in this chapter before proceeding to the next chapter.

The interaction of electromagnetic fields with matters is the basis of all electrical and electronic devices. These interactions are often described, analyzed and synthesized through the abstractions of electrical circuit theory. The following are the most basic circuit two-terminal elements we will need for now. We will introduce new ones as and when they are required. Each circuit element has a unique voltage-current relationship, and it is *important* that you know these by heart.

1.1 Basic

Independent Voltage source. An Ideal voltage source provides a fixed voltage V between its two terminals, and can provide any amount of current. Notice that the voltage V can be fixed or time varying. For example, for a DC voltage source with $V = 5V$, the voltage across the two terminals will be $5V$ for all time. But for a time varying AC source, $V = 5\sin(100\pi t)$, the voltage across its terminal will vary with time. We will often drop the adjective "independent" when we are sure that the context is

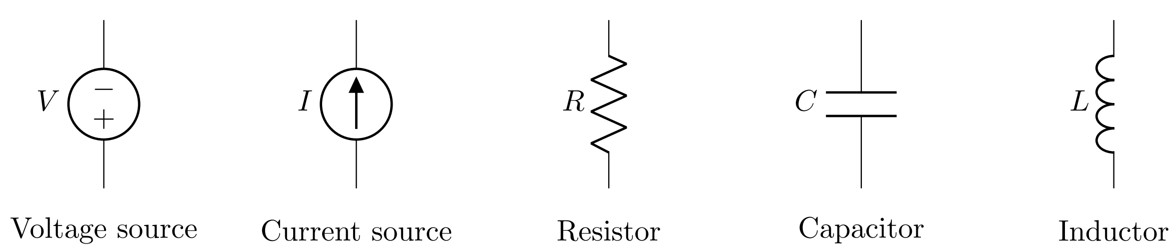


Fig. 1.1: Basic circuit elements: voltage source, current source, resistor, capacitor, and inductor.

clear. We will look at dependent sources later, and we will always use the adjective "dependent" to refer to them.

Independent Current source. An ideal current source provides a fixed amount of current to flow throgh its terminals (out through one and in through the other), irrespective of the voltage across its terminals. Current sources can also be time-varying.

Resistor. A passive element where the current i_R flowing through the element is proportional to the voltage v_R across its terminals.

$$v_R \propto i_R$$

In the case of linear resistors, the proportionality factor is constant, resulting in Ohm's law,

$$v_R = R\, i_R \tag{1.1}$$

The units of R are $V.A^{-1}$ or $Omhs\,(\Omega)$. R in general is positive. The power absorbed by a resistor is given by the product of the voltage across it and the current flowing through it,

$$P = v_R\, i_R = i_R^2\, R = \frac{v_R^2}{R} \tag{1.2}$$

This power is dissipated as heat by the resistor. Note that the power absorbed by a resistor is always positive, since R is positive.

We will later see non-linear resistors, where the resistance varies as a function of the applied votlage, temperature and other factors.

Capacitor. A capacitor is another passive element with the following voltage current relationship.

$$i_C = C\frac{dv_C}{dt} \tag{1.3}$$

The current i_C through the capacitor is proportional to the rate of change of voltage across its terminals v_C. The proportionality factor is called the capacitance C, and has units of F (Farads) or $C \cdot V^{-1}$. The voltage across the capacitor at any given time is proportional to the integral of the current flowing through it or the charge stored in the capacitor. The voltage across the capacitor is given by

$$v_C = \frac{q}{C} = \frac{1}{C}\int i_C\, dt \tag{1.4}$$

The instantaneous power absorbed by the capacitor is given by,

$$P = v_C\, i_C = C\, v_C\frac{dv_C}{dt} \tag{1.5}$$

The power absorbed by the capacitor can be positive or negative, depending on the direction of current flow. If the current is flowing into the capacitor, then the voltage across it is increasing, and the power absorbed is positive. If the current is flowing out of the capacitor, then the voltage across it is decreasing, and the power absorbed is negative. A capacitor stores energy in the form of electric field between its plates. The energy stored in a capacitor at any given time depends on the charge stored in it, and is given by,

$$E = \frac{1}{2} C\, v_C^2 \tag{1.6}$$

Inductor. An inductor is another passive element with the following voltage current relationship.

$$v_L = L \frac{di_L}{dt} \tag{1.7}$$

The voltage v_L across the inductor is proportional to the rate of change of current i_L flowing through it. The proportionality factor is called the inductance L, and has units of H (Henries) or $V \cdot s \cdot A^{-1}$. The current through the inductor at any given time is proportional to the integral of the voltage across it. The current through the inductor is given by

$$i_L = \frac{1}{L} \int v_L\, dt \tag{1.8}$$

The instantaneous power absorbed by the inductor is given by,

$$P = v_L\, i_L = L\, i_L \frac{di_L}{dt} \tag{1.9}$$

The power absorbed by the inductor can be positive or negative, depending on the direction of current flow. If the current is flowing into the inductor, then the voltage across it is increasing, and the power absorbed is positive. If the current is flowing out of the inductor, then the voltage across it is decreasing, and the power absorbed is negative. An inductor stores energy in the form of magnetic field around it. The energy stored in an inductor at any given time depends on the current flowing through it, and is given by,

$$E = \frac{1}{2} L\, i_L^2 \tag{1.10}$$

1.2 Kirchoff's Laws

The five elements alone are not that interesting. But interesting things can be done by connecting these elements together in different ways to form an electrical circuit. The elements are connected together by wires, which are assumed to be perfect conductors, i.e. zero resistance. Consider the following circuit (Figure 1.2),

How do we find out the voltages and currents in the circuit? Kirchoff's laws can be used for analysing such circuits, which are based on the conservation of charge and energy. The circuit in Figure 1.2 is a simple electrical circuit with a voltage source, a current source, and a bunch of resistors. The voltage source provides a fixed voltage V_s between its two terminals, and the current source provides a fixed amount of current I_s to flow through its terminals. The voltages and currents in the rest of the elements

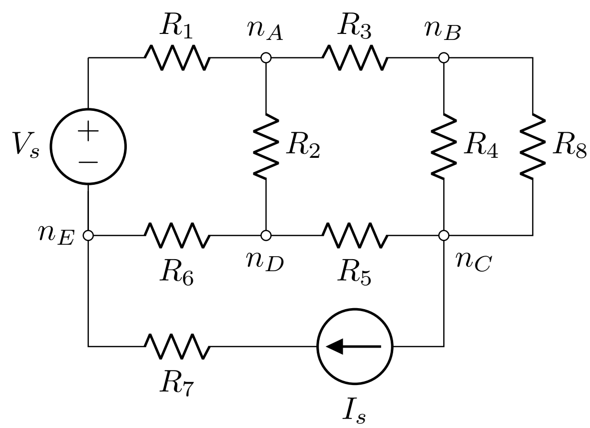


Fig. 1.2: A simple electrical circuit with a voltage sourse, a current source, and a bunch of resistors.

will be determined by Kirchoff's laws with the constraints imposed by the voltage and current sources. The two laws are:

1. **Kirchoff's current law (KCL):** The sum of the currents entering a node is equal to the sum of the currents leaving the node. A *node* is a point at which two or more circuit elements are connected together. In Figure 1.2, n_A, n_B, n_C, n_D and n_E are examples of nodes where three elements are connected together. There are two other nodes in the circuit, can you identify them?

   The sum of the currents at a node is equal to zero. This is based on the conservation of charge, and can be expressed mathematically as:

   $$\sum_{i=1}^{n} i_i = 0 \tag{1.11}$$

   where i_i is the current flowing into or out of the node, and n is the number of elements connected to the node. The current flowing into a node is positive, and the current flowing out of a node is negative.

2. **Kirchoff's voltage law (KVL):** The sum of the voltages around a closed loop in a circuit is equal to zero. A *closed loop* is a path in the circuit that starts and ends at the same node, and does not cross itself. In Figure 1.2, the path starting from n_A, to n_D, to n_B, and back to n_A is a closed path. This path includes the resistors R_2, R_5, R_4, and R_3.

   This is based on the conservation of energy, and can be expressed mathematically as:

   $$\sum_{i=1}^{n} v_i = 0 \tag{1.12}$$

where v_i is the voltage across each element in the loop, and n is the number of elements in the loop.

The voltage across an element is positive if the current is flowing into the positive terminal of the element, and negative if the current is flowing out of the positive terminal of the element.

Note that the two laws apply for any type of circuit element used in the circuits, independnet or dependent voltage/current sources, resistors, capacitors, inductors, either? two, three or four terminal elements.

1.3 Series and Parallel Connections

Two elements that share the same voltage across them between a given pair of nodes are said to be **parallel** to each other. In Figure 1.2, R_4 and R_8 are parallel to each other. In a single loop, two elements that share the same current are said to be in **series** with each other. In Figure 1.2, V_s and R_1 are in series, I_s and R_7 are in series.

1.3.1 Resistors in series and parallel

When n resistors $R_1, R_2, \cdots R_n \geq 0$ are in series, these can be combined to an equivalent resistor with resistance R_{eq} given by the following,

$$R_{eq} = \sum_{i=1}^{n} R_i \implies R_{eq} \geq \max_{1 \leq i \leq n} R_i. \tag{1.13}$$

Note that in a series connection the equivalent resistance is at least as large as the largest value of R_1 to R_n.

When n resistors $R_1, R_2, \cdots R_n \geq 0$ are in parallel, these can be combined to an equivalent resistor with resistance R_{eq} given by the following,

$$\frac{1}{R_{eq}} = \sum_{i=1}^{n} \frac{1}{R_i} \implies R_{eq} = \frac{R_1 R_2 \cdots R_n}{R_1 + R_2 + \cdots + R_n} \implies R_{eq} \leq \min_{1 \leq i \leq n} R_i \tag{1.14}$$

Note that in a parallel connection, the equivalent resistance cannot be larger than the smallest value of R_1 to R_n.

1.3.2 Capacitors in series and parallel

Series connection of n capacitors $C_1, C_2, \cdots C_n \geq 0$

$$C_{eq} = \frac{C_1 C_2 \cdots C_n}{C_1 + C_2 + \cdots + C_n} \tag{1.15}$$

Parallel connection of n capacitors $C_1, C_2, \cdots C_n \geq 0$

$$C_{eq} = C_1 + C_2 + \cdots + C_n \tag{1.16}$$

1.3.3 Inductors in series and parallel

Series connection of n inductors $L_1, L_2, \cdots L_n \geq 0$

$$L_{eq} = L_1 + L_2 + \cdots + L_n \tag{1.17}$$

Parallel connection of n inductors $L_1, L_2, \cdots L_n \geq 0$

$$L_{eq} = \frac{L_1 L_2 \cdots L_n}{L_1 + L_2 + \cdots + L_n} \tag{1.18}$$

Its left as an exercise for you to verify these expressions.

What does the equivalent resistance actually mean? The equivalent resistor with resistance R_{eq} has the same voltage-current relationship as the individual elements in series or parallel connection. We can replace the series or parallel connection of the individual resistors R_1 to R_n by a single resistor with value R_{eq} without changing the volatage current relationships in the circuit. The same argument applies for equivalent capacitors and inductors.

1.3.4 Voltage sources in series and parallel

Series connection of n voltage sources $V_1, V_2, \cdots V_n$ will result in an equivalent voltage source V_{eq} given by

$$V_{eq} = V_1 + V_2 + \cdots + V_n \tag{1.19}$$

Voltage soruces should not be connected in parallel, as this will result in a short circuit. Ideally, an infinite current will flow through the connection, because the volatage sources force a potential difference between the two ends of the wire to be zero and it has zero resistance. Parallel connections are allowed only when the two sources have the same voltage and polarity.

1.3.5 Current sources in series and parallel

Parallel connection of n current sources $I_1, I_2, \cdots I_n$ will result in an equivalent current source I_{eq} given by

$$I_{eq} = I_1 + I_2 + \cdots + I_n \tag{1.20}$$

Current sources should not be connected in series; series connections are allowed only when the two sources have the same current and polarity.

1.4 Superposition Principle

Linear approximations of circuits are often employed as first order approximations when analysing circuits. A linear ciruit is one that consists of linear passive elements, independent sources and linear dependent sources. Linear circuits follow the superposition

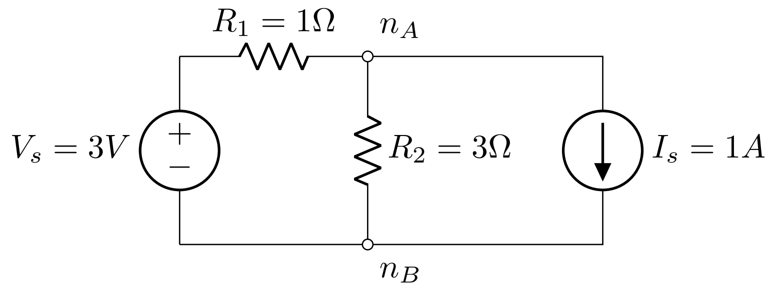


Fig. 1.3: A simple circuit with two sources

principle, which states that the response of a linear circuit to a linear combination of inputs is equal to the corresponding linear comibnation of the responses to each input applied separately. Solving the following circuit (Figure 1.3) should make this concept clear. For the circuit in Figure 1.3, perform the following calculation and compare your results.

Step 1. Solve for the voltage across and the current through the resistors R_1 and R_2; we will refer to these are v_{R1}, i_{R1} and v_{R2}, i_{R2}, respectively. You should use both Kirchoff's current and voltages laws to compute these variables.

Let's now consider of the sources in the circuit seperately. This would mean making the source values "zero". This corresponds to two different operations on the circuit. Zeroing a voltage source corresponds to replacing it with a wire (a short circuit), while zeroing a current source corresponds to simple removing the current source (an open circuit).

Step 2. Zero the voltage source V_s and compute the voltages and currents associated with the two resistors. We will refer to these as $v_{R1,V_s=0}$, $v_{R2,V_s=0}$, $i_{R1,V_s=0}$, and $i_{R2,V_s=0}$.

Step 3. Zero the current source I_s and compute the voltages and currents associated with the two resistors. We will refer to these as $v_{R1,I_s=0}$, $v_{R2,I_s=0}$, $i_{R1,I_s=0}$, and $i_{R2,I_s=0}$.

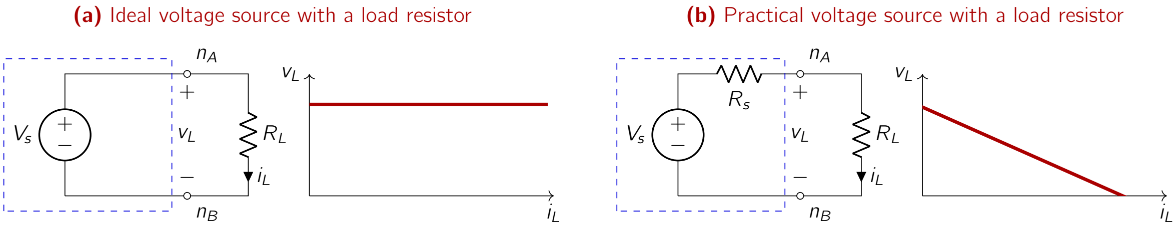
Step 4. For a linear circuit, shown in Figure 1.3, the following will always be true.

$$\begin{aligned} v_{R1} &= v_{R1,V_s=0} + v_{R1,I_s=0} \\ v_{R2} &= v_{R2,V_s=0} + v_{R2,I_s=0} \\ i_{R1} &= i_{R1,V_s=0} + i_{R1,I_s=0} \\ i_{R2} &= i_{R2,V_s=0} + i_{R2,I_s=0} \end{aligned} \tag{1.21}$$

Let's assume that I am only interested in i_{R2}. Can I use $i_{R2,V_s=0}$ and $i_{R2,I_s=0}$ to compute the current i_{R2} if $V_s = 1V$ and $I_s = -2A$?

1.5 Practical Voltage and Current Sources

The independent voltage and current sources we have discussed so far are "ideal" sources. Practical or real sources do not behave like them - a battery cannot pro-



(a) Ideal voltage source **(b)** Practical voltage source with a load

Fig. 1.4: Comparison of the voltage-current relationship of an ideal and a practical voltage source.

vide any amount of current for a load without any changes to the voltage across its terminals.

A good model of practical voltage source is an ideal voltage source V_s in series with a *internal*, *source* or *output* resistor R_s. And for a practical current source, it is an ideal current source I_s in parallel with a resistor R_s. The voltage across the terminal of a voltage source as a function of the current drawn from it is depicted for an ideal and practical voltage source in Figure 1.4. For the ideal source (Figure 1.4a), the voltage v_L is independent of the current i_L drawn from it. For the practical voltage source (Figure 1.4b), the voltage across the terminals is a function of the current drawn from it,

$$v_L = V_s - R_s\, i_L \tag{1.22}$$

When $R_L = \infty$ ($i_L = 0$), the voltage across the terminals is equal to the voltage of the source V_s. This is maximum voltage the practical source can provide. This is also know as the *open circuit voltage* v_{oc} of the source. When $R_L = 0$ ($v_L = 0$), the current drawn from the source $i_L = \frac{V_s}{R_s}$. This is the maximum current the voltage source can provide. This is also known as the *short circuit current* i_{sc} of the source.

Problem 1.1. Plot the voltage-current relationship of a practical current source with $I_s = 2A$ and $R_s = 10\Omega$. What are v_{oc} and i_{sc}?

1.6 Thevenin's and Norton's Theorems

Thevenin's and Norton's theorems are two important theorems in circuit theory that allow us to simplify complex circuits into simpler equivalent circuits. These theorems are based on the superposition principle, and can be used to analyze linear circuits with independent and dependent sources.

Thevenin's Theorem. Thevenin's theorem states that any linear circuit with independent and dependent sources can be replaced by an equivalent circuit with a single voltage source V_{th} in series with a resistor R_{th}, connected to the load resistor R_L.

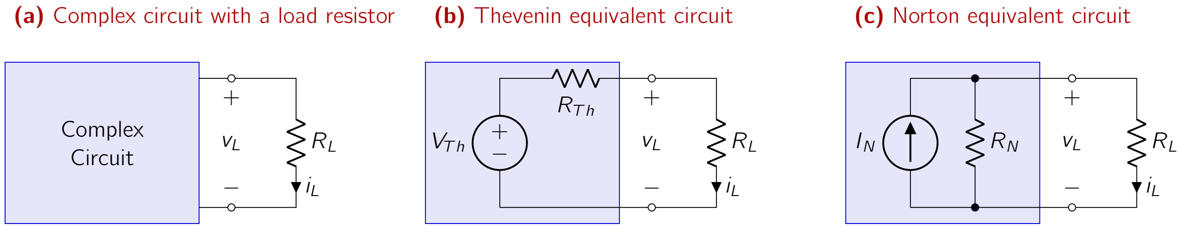


Fig. 1.5: Thevenin's and Norton's circuits for a complex linear circuit. The Thevenin and Norton equivalent circuits have the same voltage-current relationship for any load.

Norton's Theorem. Norton's theorem states that any linear circuit with independent and dependent sources can be replaced by an equivalent circuit with a single current source I_N in parallel with a resistor R_N, connected to the load resistor R_L.

Computing the Thevenin and Norton equivalent circuits. The Thevenin and Norton equivalent circuits can be computed using the following steps:

1. Remove the load resistor R_L from the circuit.

2. Compute the open circuit voltage v_{oc} across the terminals of the load resistor. This is the Thevenin voltage V_{th}. This can be done using the superposition principle, where we zero all the independent sources except one and compute the open circuit voltage. The overall open circuit voltage is the sum of the open circuit voltages when all other sources are zeroed.

3. Compute the short circuit current i_{sc} through the terminals of the load resistor. This is the Norton current I_N. This too can be calculated using the superposition principle.

4. Compute the Thevenin resistance R_{th} by zeroing all independent sources in the circuit and computing the equivalent resistance seen from the terminals of the load resistor (without the load resistor). This is also equal to the Norton resistance R_N.

5. The Thevenin and Norton equivalent circuits are then given by:

$$\begin{aligned} V_{th} &= v_{oc} \\ I_N &= i_{sc} \\ R_{th} &= R_N \end{aligned} \tag{1.23}$$

6. The load resistor R_L can be connected to either the Thevenin or Norton equivalent circuit, and the voltage and current across it can be computed using the voltage-current relationship of the equivalent circuit.

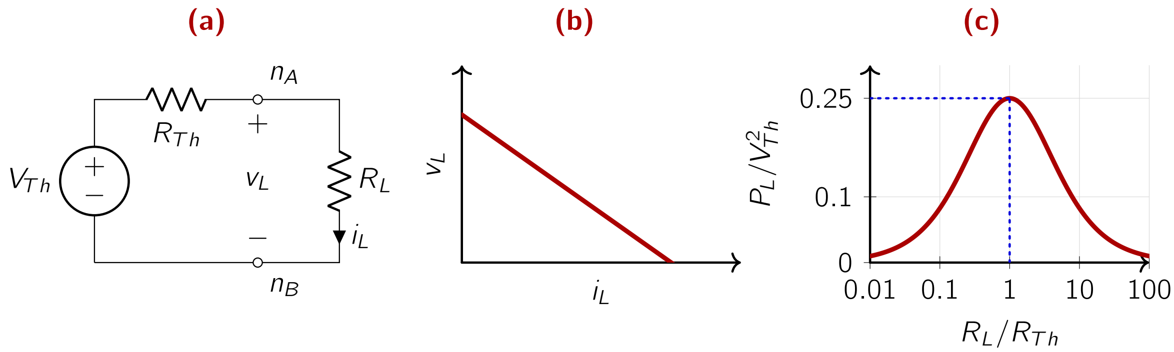


Fig. 1.6: Maximum power transfer theorem.

Problem 1.2. Compute the Thevenin and Norton equivalent circuits for the circuit shown in Figur 1.2 assuming the following as the load resistor: (a) R_8; (b) R_2; and (c) R_7.

1.7 Maximum Power Transfer Theorem

The maximum power transfer theorem states that the maximum power is transferred to the load resistor R_L when the load resistance is equal to the Thevenin resistance R_{th} of the circuit. Consider the following circuit (Figure 1.6). The power absorbed by the load resistor R_L is given by,

$$P_{R_L} = \frac{R_L}{(R_{th} + R_L)^2} V_{th}^2 \tag{1.24}$$

Its easy to check that the optimal value of R_L that maximizes the power absorbed by the load resistor is given by,

$$\begin{aligned} R_L^* &= \arg\max_{R_L} \frac{R_L}{(R_{th} + R_L)^2} V_{th}^2 = R_{Th} \\ P_L^* &= \max_{R_L} \frac{R_L}{(R_{th} + R_L)^2} V_{th}^2 = \frac{1}{4}\left(\frac{V_{Th}^2}{R_{Th}}\right) \end{aligned} \tag{1.25}$$

Problem 1.3. Prove the statements in Eq. 1.25. (*Hint:* Use the first order condition for maximization of a continuous function.)

1.8 RC, RL, and RLC Circuits

Unlike resistors, that neither remember the past values of its votlage or current, capacitors and indutors retain memory of the past current and past voltage, respectively. This can be easily seen from their respective voltage-current relationships given by Eq. 1.3 and Eq. 1.7, respectively.

Capacitor:

$$i_C = C\frac{dv_C}{dt} \implies v_C(t) = \frac{1}{C}\int_0^t i_C(\tau)\, d\tau + v_C(0) \tag{1.26}$$

The instantaneous voltage across a capacitor contains some information about the past history the current that has flown through the capacitor. This voltage $v_C(t)$ is determined by the charge on the capacitor at time t. Note, that the voltage across the capacitor cannot change instantaneously, as this would require an infinite current to flow through the capacitor. Theoretically, however, an impulse (Dirac delta function) current applied to the capacitor can produce instantaneous change in the capacitor's voltage.

Inductor:

$$v_L = L\frac{di_L}{dt} \implies i_L(t) = \frac{1}{L}\int_0^t v_L(\tau)\, d\tau + i_L(0) \tag{1.27}$$

Similarly, the instantaneous current through the inductor contains some information about the history of voltage applied across the inductor. Current through an inductor cannot change instantaneously, as this would require an infinite voltage to be applied across the inductor. An impulse voltage applied to the inductor can produce instantaneous change in the inductor's current.

RC Circuit. Consider a simple RC circuit shown in Figure 1.7a. We can write Kirchoff's voltage law for the circuit as follows,

$$RC\frac{dv_C}{dt} + v_C = V_s \implies v_C(t) = e^{-t/RC}\left[v_C(0) + \int_0^t \frac{1}{RC}e^{\tau/RC}V_s(\tau)\, d\tau\right] \tag{1.28}$$

The above euqation gives the general solution for the voltage across the capactior. We can derive all other variables of interest from v_C. The response to an impulse input, step input, sinusoidal input, or any arbitrary V_s can be computed using the above equation. The response for a step input obtained using a fixed sources and a switch is given in Figure 1.7a. RC is the time constant of the circuit, and is a measure of how fast the capacitor charges or discharges and it has the units of time.

Using Laplace transform to analyze the circuit, we can write the following expression for the voltage across the capacitor,

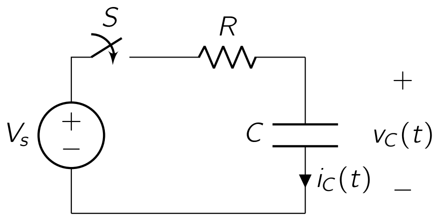
$$V_C(s) = \frac{1}{sRC+1}V_s(s) + \frac{v_C(0)}{sRC+1} \tag{1.29}$$

$V_C(s)$ and $V_s(s)$ are the Laplace transforms of $v_C(t)$ and $V_s(t)$. Replacing $s = j\omega$ will give us the frequency response of the system with $V_C(s)$ as the output and $V_s(s)$ as the input.

RL Circuit. Consider a simple RL circuit shown in Figure 1.7b. Kirchoff's voltage law for the circuit is as follows,

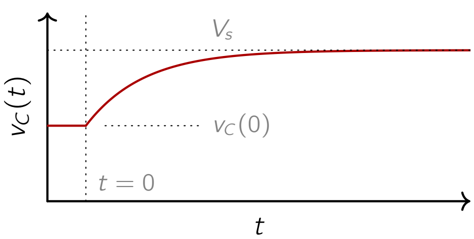
$$\frac{L}{R}\frac{di_L}{dt} + i_L = \frac{1}{R}V_s \implies i_L(t) = e^{-tR/L}\left[i_L(0) + \int_0^t \frac{1}{L}e^{\tau R/L}V_s(\tau)\, d\tau\right] \tag{1.30}$$

(a) A Simple RC Circuit



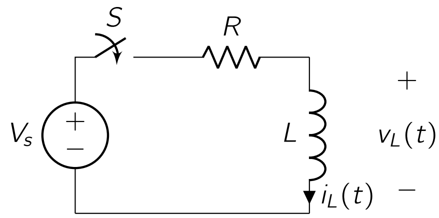
Assuming the switch S is closed at $t = 0$, the voltage across the capacitor $v_C(t)$ is given by,

$$v_C(t) = V_s \cdot \left(1 - e^{-\frac{t}{RC}}\right) + v_C(0)e^{-\frac{t}{RC}}, \;\; t \geq 0$$



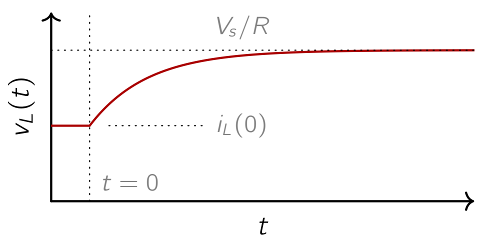
(a) Ideal voltage source

(b) A Simple RL Circuit



Assuming the switch S is closed at $t = 0$, the current through the inductor $i_L(t)$ is given by,

$$i_L(t) = \frac{V_s}{R} \cdot \left(1 - e^{-\frac{t}{L/R}}\right) + i_L(0)e^{-\frac{t}{L/R}}, \;\; t \geq 0$$



(b) Practical voltage source with a load

Fig. 1.7: Simple RC and RL circuits and their transient responses.

The above euqation gives the general solution for the current through the inductor. The response to an impulse input, step input, sinusoidal input, or any arbitrary V_s can be computed using the above equation. The response for a step input obtained using a fixed sources and a switch is given in Figure 1.7a. $\frac{L}{R}$ is the time constant of the circuit, and is a measure of how fast the current through the inductor can chamge; it has the units of time.

Similarly, applying the Laplace transform, we have,

$$I_L(s) = \frac{1}{sL + R}V_s(s) + \frac{i_L(0)}{sL + R} \tag{1.31}$$

RLC Circuit. Consider a simple series RLC circuit shown in Figure 1.8. Kirchoff's voltage law for the circuit is as follows,

$$LC\frac{d^2 v_C}{dt^2} + RC\frac{dv_C}{dt} + v_C = V_s \tag{1.32}$$

The response of the circuit when the switch is closed at $t = 0$ is qualitatively different depending on the values of R, L and C. This is better understood in the Laplace domain. The Laplace transform of the response $v_c(t)$ is given by,

$$V_C(s) = \frac{V_s(s)}{LCs^2 + RCs + 1} + v_C(0)\frac{C\,(Ls + R)}{LCs^2 + RCs + 1} + i_L(0)\frac{RC}{LCs^2 + RCs + 1} \tag{1.33}$$

(a) A Simple RLC Circuit **(b)** Four different step responses

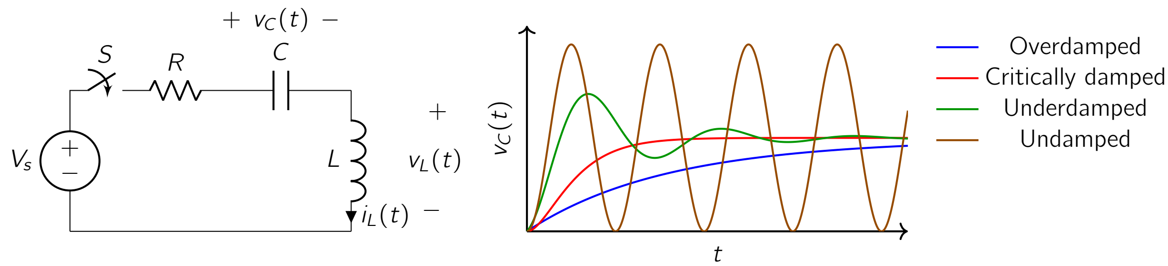


Fig. 1.8: A simple RLC circuit and the different respones of the circuit when the switch S is closed at time $t = 0$, with zero initial capacitor voltage and inductor current whent .

Note that $i_L(0) = \frac{dv_C}{dt}(0)$. The denominator of the above equation is a second order polynomial in s, and thus the response of the circuit will depend on the roots of the polynomial. The four different responses are shown in Figure 1.8.

1.9 Steady State Sinusoidal Analysis

The previous subsection looked at the transient response of the some simple circuits involving R, L, and C. However, we are often interested in the steady state response of these circuits to sinusoidal excitations. This is because sinusoidal signals are eigenfunctions of linear systems, such as the linear circuits we have discussed so far. With sinusoidal excitations, all currents and voltages in the circuit will also be sinusoidal with the same frequency as the input excitations, althought with different amplitudes and phases. Complex numbers are used in this analysis for compact representation of ampltiude and phase of signals.

For steady state frequency analysis for circuits involving R, L, and C, we extend the idea of resistance to impedance. Fourier transforms provide a natural way to analyze the steady state response of linear circuits to sinusoidal excitations. We first recast the voltage-current relationships of the circuit elements in the Fourier domain.

$$\begin{aligned}
\textbf{Resistor:} \quad & v_R = R\, i_R \implies V_R\,(j\omega) = R \cdot I_R\,(j\omega) \\
\textbf{Capacitor:} \quad & i_C = C\frac{dv_C}{dt} \implies V_C\,(j\omega) = \frac{1}{j\omega C} \cdot I\,(j\omega) \\
\textbf{Inductor:} \quad & v_L = L\frac{di_L}{dt} \implies V_L\,(j\omega) = j\omega L \cdot I_L\,(j\omega)
\end{aligned} \tag{1.34}$$

The Fourier transformed voltage-current relationships of the circuit elements look like Ohms law, with the concept of resistance extended to impedance. The impedance of a capacitor and inductor defined as the ratio of the Fourier transformed voltage to the Fourier transformed current. They are functions of the frequency of the voltage/current.

The impedance of a resistor, capacitor, and inductor are given by,

$$\begin{aligned} \textbf{Resistor:} \quad & Z_R = R \\ \textbf{Capacitor:} \quad & Z_C = \frac{1}{j\omega C} \\ \textbf{Inductor:} \quad & Z_L = j\omega L \end{aligned} \tag{1.35}$$

Equivalent Impedance. The equivalent impedance of a circuit with impedances $Z_1, Z_2, \cdots Z_n$ in series is given by $Z_{eq} = Z_1 + Z_2 + \cdots + Z_n$. When the impedances $Z_1, Z_2, \cdots Z_n$ are in parallel, the equivalent impedances is given by $\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_n}$.

Amplitude and phase modification by an impedance. If a sinusoidal current excitation $i(t) = I_o \sin(\omega t)$ is passed through an impednace Z. Then the voltage across the impedance is given by,

$$v(t) = |Z| I_o \sin(\omega t + \arg Z) \tag{1.36}$$

The amplitude of the sinusoidal voltage is $|Z| I_o$, and the phase of the sinusoidal voltage is $\arg Z$, with respect to the current $i(t)$.

1.10 Exercise

1. Plot the current through a resistor $R = 10\Omega$, capacitor $C = 5\mu F$, and inductor $L = 2mH$ for the following voltage inputs.

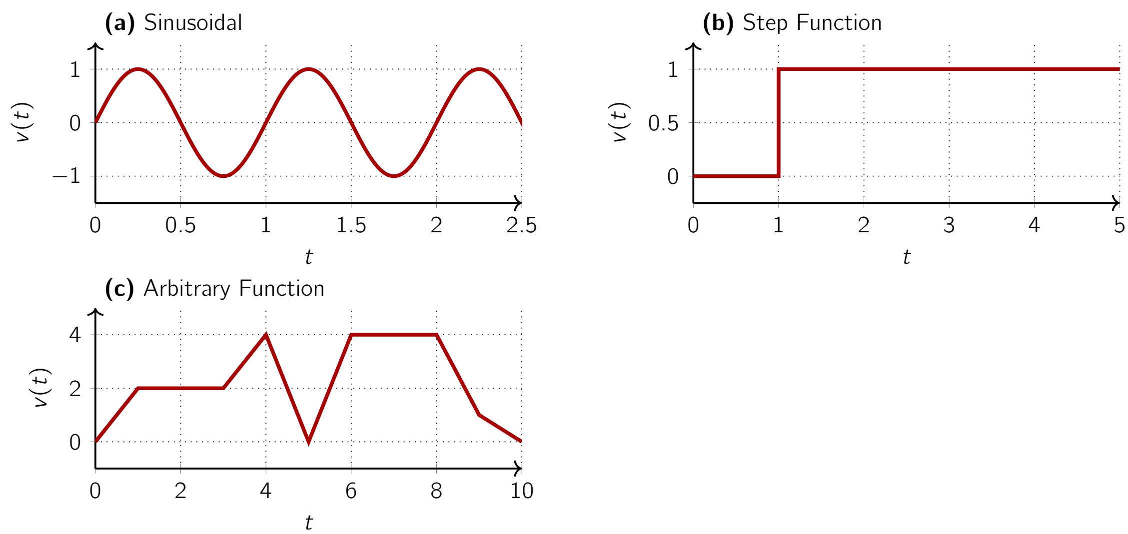


Fig. 1.9: [Exercise 1] Voltage inputs applied to a resistor, capacitor, or an inductor. The units of voltage are in volts, and the time is in seconds.

2. Find the Thevenin and Norton equivalent circuits of the following circuits (Figure 1.10). Note that one of the problems has a dependent source. Dependent sourcrs are voltage or current sources (diamond shaped elements), whose voltage or current, respectively, depend on the voltage or current across another element in the circuit.

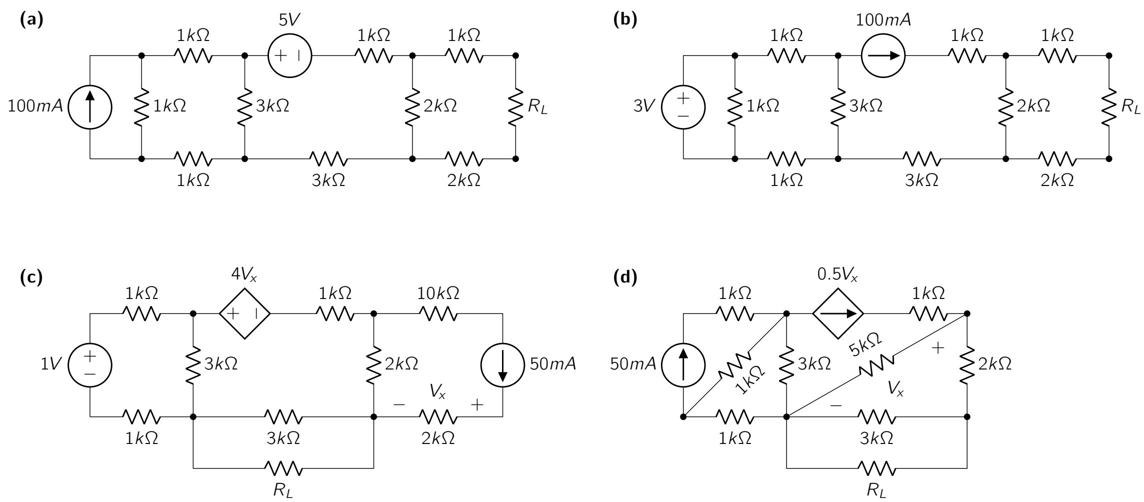


Fig. 1.10: [Exercise 2] Find the equivalent impedance of the circuit.

3. A certain red LED has a maximum current rating of $35mA$, and if this value is exceeded, overheating and catastrophic failure will result. The resistance of the LED is a nonlinear function of its current, but the manufacturer warrants a minimum resistance of 47Ω and a maximum resistance of 117Ω. Only $9V$ batteries are available to power the LED. Design a suitable circuit to deliver the maximum power possible to the LED without damaging it. Use only combinations of the standard resistor values. (*This problem is from Engineering Circuit Analysis by Hayt Jr. et al.*)

4. The load resistor in Figure 1.11 can safely dissipate up to $1W$ before overheating and bursting into flame. The lamp can be treated as a 10.6Ω resistor if less than $1A$ flows through it and a 15Ω resistor if more than $1A$ flows through it. What is the maximum permissible value of I_s? Verify your answer with an appropriate computer simulation. (*This problem is from Engineering Circuit Analysis by Hayt Jr. et al.*)

5. In the following circuit (Figure 1.12), find the voltage across the capacitors C_1 and C_2. The single pole double throw swtich S is connected to the top terminal before time $t = 0$. At time $t = 0$, swtich is flipped to the bottom terminal. Assume that before time $t = 0$, the voltage across the capacitor C_2, $v_{C2} = 1V$. Draw the plot of the voltage across the capcitors C_1 and C_2 as function of time.

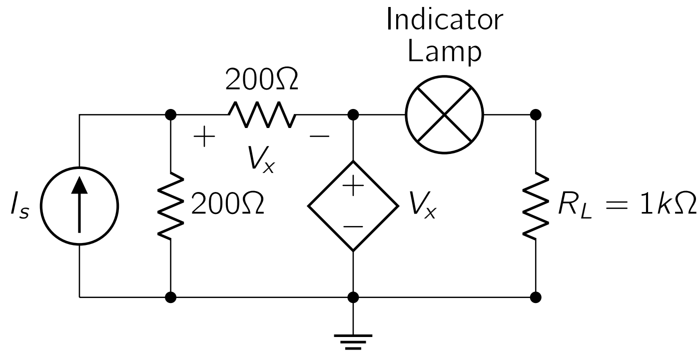


Fig. 1.11: [Exercise 4]Find the equivalent impedance of the circuit.

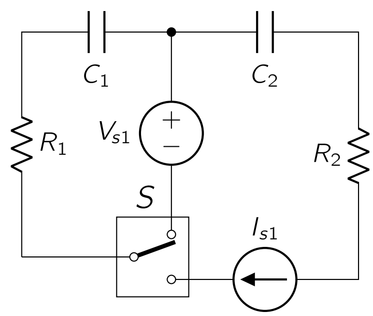


Fig. 1.12: [Exercise 5]Find the equivalent impedance of the circuit.

6. Figure 1.13 shows a voltage divider circuit with a sinusoidal voltage source $V_s(t) = 10\sin(1000\pi t)$. The impedane $Z_1 = 3 + j4\Omega$. What should the impedance Z_2 be if the the voltage across Z_2 has one-fourth the amplitude of V_s and with a phase difference of 45 deg?

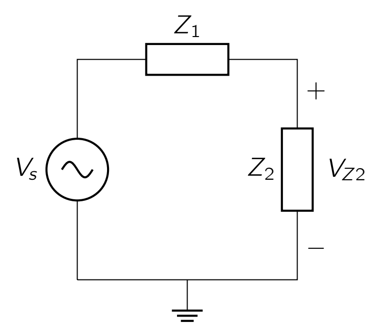


Fig. 1.13: [Exercise 5]Find the equivalent impedance of the circuit.

Chapter 2

Electronics Review

This chapter is a quick review of basic electronics. The previous chapter involved simple, linear passive two terminal elements, which are essential for building most electrical circuits. This chapter will focus on some essential non-linear, two/three terminal electronics components which form the basis for more useful electronics cricuits such as amplifiers, oscillators,filters, power supplies, switches, etc. We will review three important electronic components and their circuits in this chapter: *diode*, *bipolar junction transistor (BJT)*, *metal oxide field effect transistor*, and *operational amplifiers*.

2.1 Basic semiconductor concepts

Conductivity of a material is proportional the concentration of free electrons.

$$
\begin{aligned}
\textbf{Conductor:} \quad & n_{cond} \approx 10^{28} electrons/m^3 \\
\textbf{Insulator:} \quad & n_{ins} \approx 10^{7} electrons/m^3 \\
\textbf{Semiconductor:} \quad & n_{ins} < n_{sem} < n_{cond}
\end{aligned}
\tag{2.1}
$$

2.1.1 Intrinsic semiconductors

Silicon or Germanium have crystalline strcuture with four covalent bonds between neighbouring atoms. At $0°K$ semiconductors all covalent bonds are in place tightly holding on

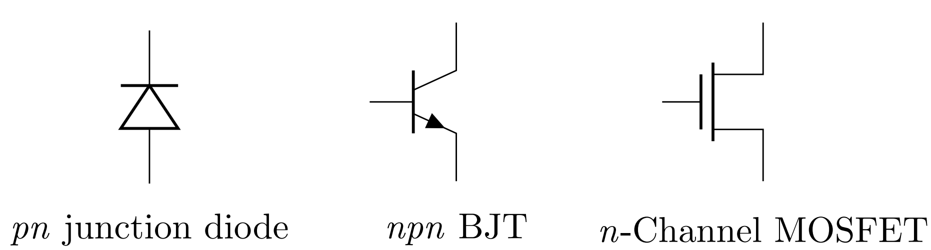


Fig. 2.1: Three most essential electronic components reviewed in this chapter.

to electrons in the bonds. Thus these behave as perfect insulators at low temperatures. With increase in temperature, covalent bonds are ruptured, releasing *free electrons* to roam around in the crystal and become available for conduction when there is an externally applied electric field. The missing electron in the ruptured covalent bond is called a hole, which too act as carrier of electricity. When an electric field is applied, electrons from neighbouring covalent bonds jump into a hole creating a hole in their previous bond. This results in the hole effectively moving the direction opposite to the jumping electrons from the covalent bonds. Thus, unlike conductors, semiconductors can have two types of charge carriers: *electrons* and *holes*. The concentration of free electrons and holes in a semiconductor determines it conductivity, which is controlled by temperature. In intrinsic semiconductors, the concentration of free electrons and holes is equal, i.e. $n_{e^-} = n_{h^+} = n_i$. This concentration is given by,

$$n_i = B\,T^{3/2}\,e^{-\frac{E_g}{k_B T}} \tag{2.2}$$

where B is a material constant ($7.3 \times 10^15 \text{cm}^{-3}\text{K}^{-3/2}$ for Si), E_g is the band gap energy, k_B is the Boltzmann constant, and T is the absolute temperature in Kelvin. The band gap energy is the energy required to break a covalent bond and create a free electron-hole pair. It should be noted that the conductivity of semiconductors increases with temperature, making them suitable for thermometry.

2.1.2 Doped Semiconductors

With intrinsic semiconductors, the conductivity is still quite small at room temperature. Another precise and controlled way to change a semiconductor's conductivity is by adding impurities, called *doping*. The two types of doping are: *n*-**type** and *p*-**type** doping.

In a *n*-**type** semiconductor, a small amount of pentavalent atoms (e.g. Phosphorus, Arsenic) are added to the semiconductor. These atoms have five valence electrons, and when added to the silicon crystal, four of these electrons form covalent bonds with the neighbouring silicon atoms, while the fifth electron is free to roam around in the crystal. This results in an increase in the concentration of free electrons, making the semiconductor more conductive. The concentration of free electrons in an *n*-type semiconductor is given by,

$$n_{e^-} = n_i + N_D \quad n_{h^+} = n_i \tag{2.3}$$

where, N_D is concentration of the doping element.

In a *p*-**type** semiconductor, a small amount of trivalent atoms (e.g. Boron) is added to the silicon crystal structure. This leaves the dopped elements with four covalent bonds with one of the bonds lacking an electron or a hole. This holes acts as a charge carrier. The concentration of the holes in a *p*-type semiconductor is give by,

$$n_{h^+} = n_i + N_A \quad n_{e^-} = n_i \tag{2.4}$$

where, N_A is concentration of the doping element in the p-type semiconductor.

The doped semiconductors will have higher conductivity than the intrinsic semiconductors depending on the concentration of the impurities added to the silicon crystal.

2.1.3 Flow of current in semiconductors

There are two types of current flow mechanisms in seminconductors unlike pure conudctors – (a) **drift current** and (b) **diffusion current**. Both are imporant for understanding the operation of the basic electronic components.

Drift current. Drift current is established by an external electric fields, for examples when a semiconductor is connected to a battery. Holes will accelerate in the direction of the field, while the electrons accelerate in the opposite direction. Bumping into the atoms of the cystral structure, the holes and electrons acquire an average drift velocity given by,

$$\nu_{p-drift} = \mu_p E \quad \& \quad \nu_{n-drift} = -\mu_n E \tag{2.5}$$

This constitute a drift current through the semiconductor given by the following,

$$I_{drift} \propto q \left(n_{h^+} \mu_p + n_{e^-} \mu_n \right) E \tag{2.6}$$

where, q is the magnitude of electron charge.

Diffusion current. Diffusion currents result whena concentration gradient exists across a semiconductor.

$$I_{diff} \propto -qD_q \frac{dp\,(x)}{dx} \tag{2.7}$$

where, $p\,(x)$ is the concentration of the charge carrier along the x direction. Diffusion currents play an important role in the functioning of the BJT.

2.2 Diode

Whena p-type and n-type semiconductors are brought in close contact to each other, there is a big concentration difference between the charge carriers across the interface. A pn-junction diode made by bringing a n-type and p-type semiconductors together as shown in Figure 2.2. Due to the concentration difference, electrons from the n-type semiconductor diffuse into the p-type semiconductor, while holes from the p-type semiconductor will diffuse into the n-type semiconductor. Note that this is the diffusion current I_D, which flows even when the diode is open circuited. This diffusion current sets up the depletion region at the interface, where there are no free charge carriers. This region consists of fixed positive and negative charges, which creates an electric field across the junction. This electric field opposes further diffusion of charge carriers across the junction, and the potential or *barrier voltage* V_0 due to the electric field must

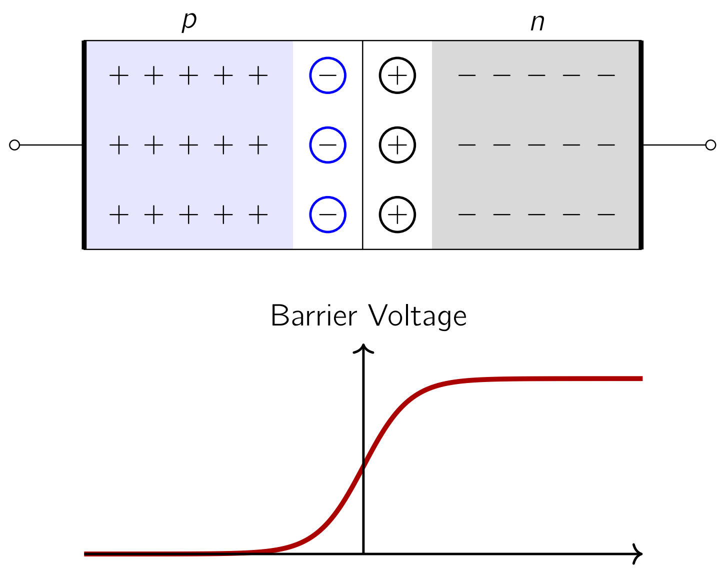


Fig. 2.2: *pn* junction diode

be overcome for th diffusion current to flow across the junction. Note that the diffusion current I_D is due to the majority charge carriers – holes from the p side and *electrons* from the n-side.

There is a drift current I_S that flows across the junction due to the minority charge carriers - the *electrons* from the p-side and the *holes* from the n-side. These thermally generated minority charge carriers, will be swept across and transported across the junction due to the electric field, which tends to reduce the barrier voltage V_0. Thus, when the diode is open circuited, we have an equilibrium $I_D = I_S$.
Barrier voltage of a diode depends on several factors, the amount fo doping of the n and p semiconductors, the temperature, etc. The exact relationship is give by,

$$V_0 = V_T \ln\left(\frac{N_A N_D}{n_i^2}\right) \tag{2.8}$$

where, V_T is the thermal voltage given by $V_T = \frac{k_B T}{q}$, where k_B is the Boltzmann constant, T is the absolute temperature in Kelvin. For silicon diodes, this is approximately $V_T \approx 26mV$ at room temperature. The barrier voltage V_0 is typically around $0.7V$ for silicon diodes, and around $0.3V$ for germanium diodes.

2.2.1 Applying a voltage across a diode

When we connect a voltage source across the diode (Figure 2.3), we can either apply a forward bias or a reverse bias. When the positive terminal of the voltage source is connected to the p-side and the negative terminal is connected to the n-side, we have a **forward bias**. This reduces the barrier voltage V_0, and allows the diffusion current I_D to flow across the junction. The diode is said to be *on* in this case, and the current I

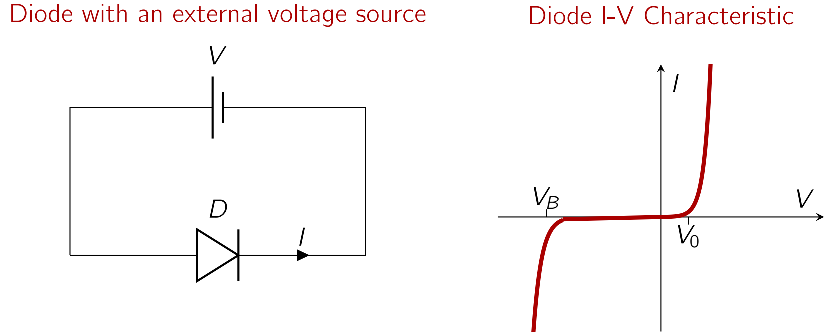


Fig. 2.3: V-I characteristics of a diode

flowing through the diode is called the *forward current*. The voltage-current realtionship between the forward current I and the voltage V across the diode is given by,

$$I = I_S \left(e^{\frac{V}{V_T}} - 1 \right) \tag{2.9}$$

where, I_S is the saturation current, which is the reverse current that flows when the diode is reverse biased, and , and q is the magnitude of electron charge.

When the negative terminal of the voltage source is connected to the p-side and the positive terminal is connected to the n-side, we have a **reverse bias**. This increases the barrier voltage V_0, and prevents the diffusion current I_D from flowing across the junction. The diode is said to be *off* in this case, and only a small reverse saturation current I_S flows through the diode. When the reverse biased voltage is increased beyond a certain threshold, called the *breakdown voltage* V_B, the diode will start conducting in the reverse direction. This is called *avalanche breakdown*, and the diode can be damaged if the current is not limited by a resistor or some other means. The avalanche breakdown is the reason for the sharp rise in the current at the V_B.

2.2.2 A simple diode circuit

In the circuit shown in Figure 2.3, the moment the forward bias-voltage gets close tot 0.6, the current starts to rise drammatically. Slight changes in the voltage will lead to

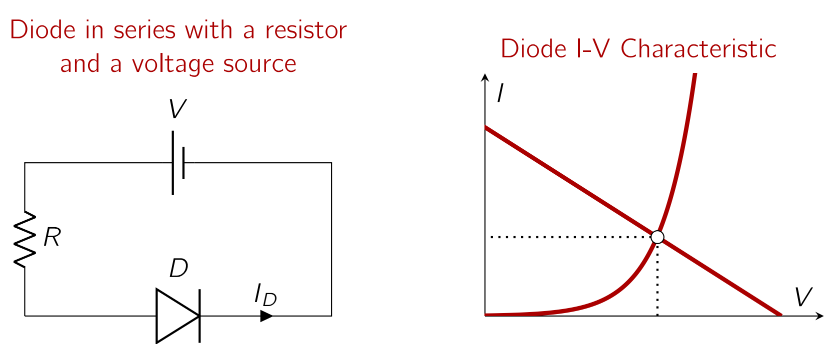


Fig. 2.4: Diode voltage and current in a simple diode circuit with a series resistor.

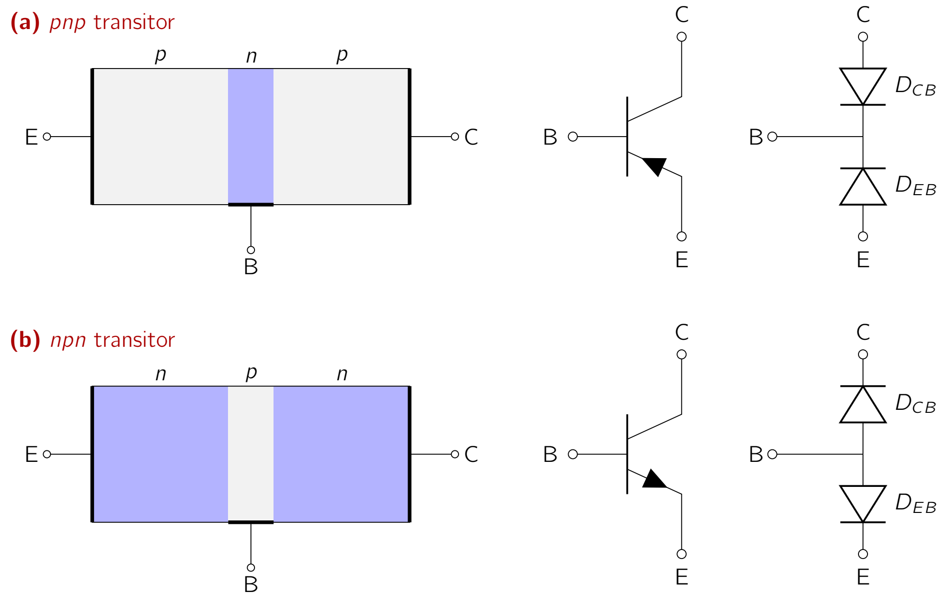


Fig. 2.5: Bipolar junction transistor structure and symbol

large changes in the current if it is not limited by a series resistor. Without a resistor, we can easily burn the diode due to large power dissipation across the diode. Consider a more practical cicruit with a voltage source, resistor, and a diode, as shown in Figure 2.4. The voltage across the diode V_D and the current through the diode I_D by writing the Kirchoff voltage law (KVL) around the loop,

$$V - I_D R - V_D = 0 \tag{2.10}$$

The current I_D and V_D from Eq. 2.9 can be substituted into Eq. 2.10 to solve for V_D and I_D,

$$V - RI_S\left(e^{\frac{V_D}{V_T}} - 1\right) - V_D = 0 \tag{2.11}$$

Unfortunately, this is a non-linear solution will need to be solved to find the voltage across the diode V_D and the current through the diode I_D.

Let's assume that $V = 5V$ and $R = 1k\Omega$ in Figure 2.4. A first order approximation can be made by assuming that the diode is *switched off*, i.e. no current is flow through it and see if it leads to a contradiction. If we assume the diode is off and no current is flowing $I_D = 0mA$ through it, then $V_D = 5V$ The diode cannot be off if $V_D \geq 0.7V$. So, the diode must be swtiched on. We will assume that the diode is on and then find out the current flowing through it and check that it is not close to 0. In the current circuit, have

$$I_D \approx \frac{V - 0.7}{R} = 4.3mA \tag{2.12}$$

I_D is large enough for to flow through the didode when its switched on. Thus, $V_D = 0.7V$ and $I_D = 4.3mA$ is a reasonable first approximation of the diode's voltage and current in the current circuit.

2.3 Bipolar junction transistor (BJT)

The bipolar junction transistor (BJT) is the first of some of the three terminal devices we will come across int his course. There are two types of BJTs – *npn* and *pnp*. BJTs are formed by sandwiching a thin layer of *p*-type semiconductor between two *n*-type semiconductors or vice versa. This is depicted in Figure 2.5. Metal contacts are made with these three semiconductor regions to form the three terminals of the BJT. In an *npn* BJT, one of the *n* regions is the *emitter* which is heavily dopped, and the other *n* region is the *collector*. The *p* region is the *base*, which is lightly doped and very thin compared to the widths of the emitter and collector regions. The collector is moderately doped. Three terminal devices like the BJT and MOSFET can be used as an amplifier or a switch, depending on how it is biased. We will only discuss *npn* transistors for the rest of the section; the *pnp* transistor is similar, but with the polarities of the voltages and currents reversed.A BJT is essentially two diodes connected back to back as shown in Figure 2.5

A BJT can be operated in three different modes, depending on the biasing of the terminals. The three modes are: (a) **Cut-off mode**, (b) **Active mode**, and (c) **Saturation mode**. The difference between the three modes are summarized in Table 2.1.

2.3.1 Cut-off mode

In this mode, the emitter-base (EB) and the collector-base (CB) junctions are reverse biased, and not current flows through the transistor.

$$I_E = 0 \quad I_B = 0 \quad I_C = 0 \tag{2.13}$$

2.3.2 Active mode

In this mode, the EB junction is forward biased, while the CB junction is reverse biased. Consider the circuit in Fig. 2.6. When the *npn* BJT's EB junction is forward biased ($V_{BE} \approx 0.7V$), electrons (the majority carrier) from the highly doped emitter region are injected just across the EB junction. The electrons diffuse into the base region, which is lighly doped and very thin. Because $V_C > V_B$, the injected electrons are swept across the base region by the electric field created by the reverse biased CB junction. A small percentage of these injected electrons recombibe with the holes in the base region, which contributes to the small base current. The collector current will be proportional the base current; the current gain is given by the parameter β.

$$I_C = \beta I_B \quad \text{where, } \beta \approx 100 - 1000 \tag{2.14}$$

An equivalent circuit of the BJT in the active model is shown in the following figure (Fig. 2.7). In this circuit, assume that the supply voaltges V_{BB}, V_{CC}, and the resistors

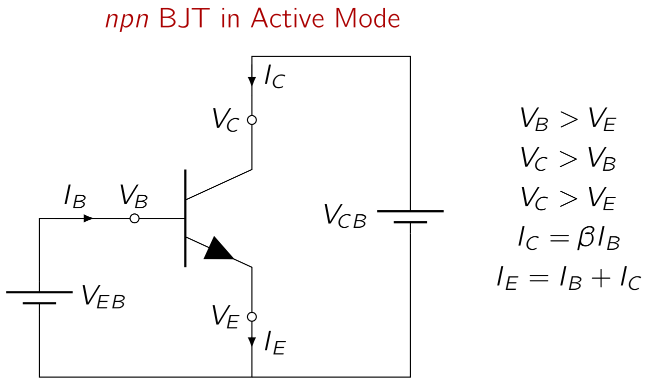


Fig. 2.6: *npn* BJT in active mode.

R_B and R_C are chosen appropriately to place the *npn* transistor in the active mode. The AC source v_s of sufficiently small in amplitude (in mV) will result in small changes in I_B, which lead to corresponding change in I_C. The voltage at the collector V_C have a larger amplitude variations riding on a DC voltage, given by the following,

$$V_C = V_{CC} - \beta \frac{R_C}{R_B} (V_{BB} - 0.7) - \beta \frac{R_C}{R_B} v_s \tag{2.15}$$

This is easy to derive and is left as an exercise for the reader. The input voltage v_s get amplified by a factor of $\beta \frac{R_C}{R_B}$, which is the voltage gain of this the common emitter amplifier circuit. We will no disucss such circuits any further in the course. This was just to illustrate the utility of the BJT in amplifying small AC signals when operated in the active mode.

Table 2.1: Modes of operation of a bipolar junction transistor (BJT)

Mode	Emitter-Base Junction	Collector-Base Junction	Operation
Cut-off	Reverse biased	Reverse biased	Transistor is off; no current flows.
Active	Forward biased	Reverse biased	I_C is proportional to I_B
Saturation	Foward biased	Forward biased	I_C and V_{CE} saturate.

2.3.3 Saturation mode

Consider the circuit in Fig. 2.8. If for fixed V_{BB}, we reduce R_B, this will increase I_B, which would result in progressively increasing I_C. However, I_C cannot continue to increase forever. As I_C increases, $V_C = V_{CC} - I_C R_C$ decreases. Once its value falls sufficiently below the base voltage V_B, the collector-base (CB) junction becomes forward biased. This is called the **saturation mode** of operation of the BJT. In this mode, the BJT behaves like a closed switch, and the collector current I_C saturates at a value

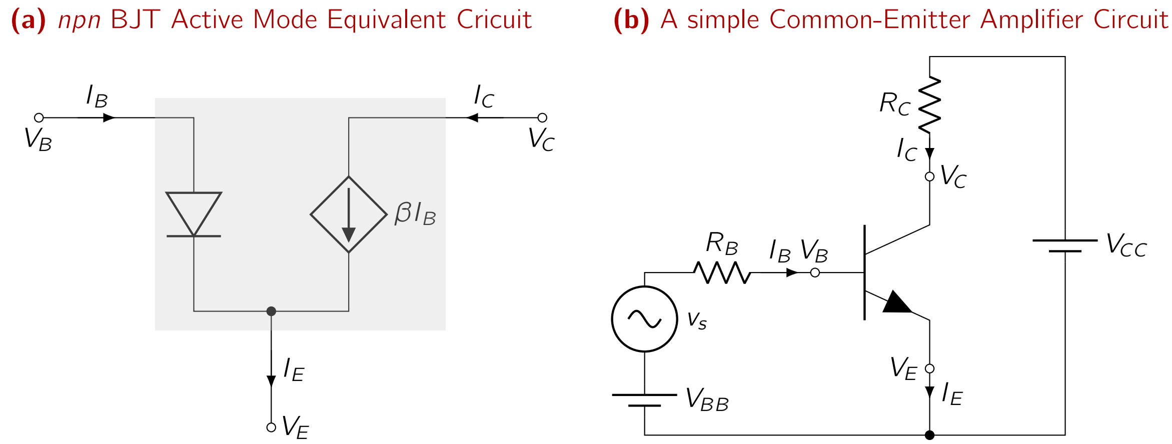


Fig. 2.7: *npn* BJT active mode equivalent circuit, and a simple common emitter voltage amplifier circuit.

that is independent of the base current I_B. The collector-emitter voltage saturates at $V_{CE(sat)} \approx 0.2V$ for silicon BJTs, making the collector current independent of the base current.

$$I_C \approx \frac{V_{CC} - V_{CE(sat)}}{R_C} \quad \text{and } V_{CE(sat)} \approx 0.2V \tag{2.16}$$

The saturation mode is used in switching applications, where the BJT is used as a switch to turn on or off a load connected to the collector; think of the collector resistor as the load. We will discuss some applications later in the chapter.

2.4 Metal oxide field effect transistor (MOSFET)

While the BJT might be a popular choice for many applications where circuits are made on a breadboard or a PCB. However, when it comes to integrated circuits, the MOSFET is the most popular choice. MOSFETS can be made quite small, are easier

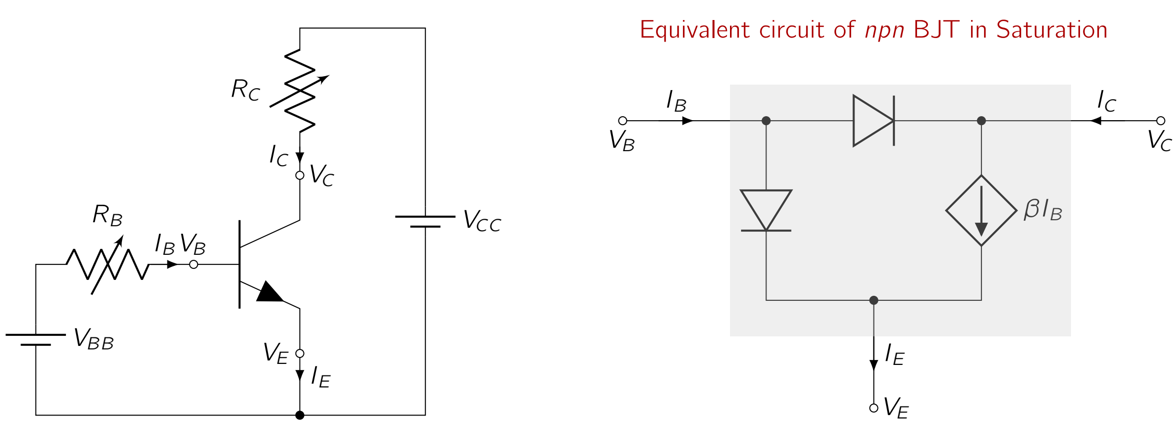


Fig. 2.8: *npn* BJT active mode equivalent circuit, and a simple common emitter voltage amplifier circuit.

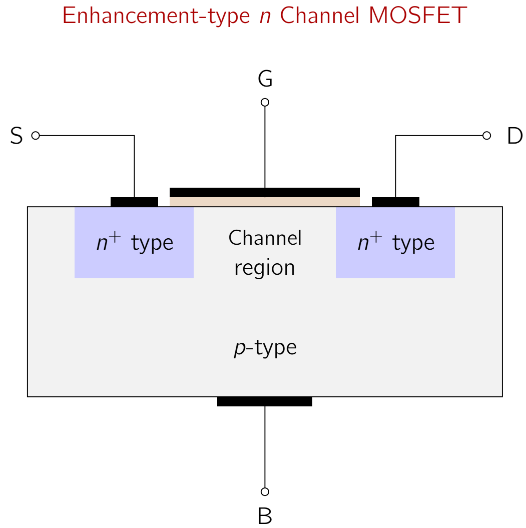


Fig. 2.9: *n* Channel Enhancement MOSFET.

to fabricate, and consume less power than BJTs. They canb be use for both digitial and analog integrated circuits. There are four types of MOSFETs, *n*-channel or *p*-channel textitenhancement or *depeltion* type MOSFETs. In this section, we will only focus on the enhancement-type *n*-channel MOSFET to understand the fundamental principle of operation of the MOSFETs.

The physical structure of an enhancement-type *n*-channel MOSFET is shown in Figure 2.9. The MOSFET in this figure has four terminals - source (S), drain (D), gate (G), adn body (B). The source and drain are heavily doped *n*-type regions, while the channel is lightly doped *p*-type region. The region between drain and the source is called the *channel region*. The gate is insulated from the channel by a thin layer of oxide, which acts as a dielectric. The gate terminal is used to control the formation of a conductive channel between the source and drain terminals. The body terminal is connected to the source terminal, and is usually shorted to the source in most applications. The MOSFET is a four terminal device, but it can be treated as a three terminal device by shorting the body and source terminals.

The drain is always kept at then higher potential than then source. The connection between the drain to the source is essentially two diodes connected back-to-back, implying that applying a voltage between the drain and source does not result in any current because one of the diodes if reverse biased. However, when a positive voltage is applied to the gate with respect to the body/source, things get interesting as shown in Figure 2.10. Applying a positive gate voltage V_{GS} with respect to the source terminal creates an electric field across the oxide layer, which penetrates into the channel region. This electric field attracts electrons in to the channel region right below the gate oxide layer. As the voltage in increased beyond a threshold V_t, a conductive channel is formed between the source and drain terminals, which can now conduct current between the

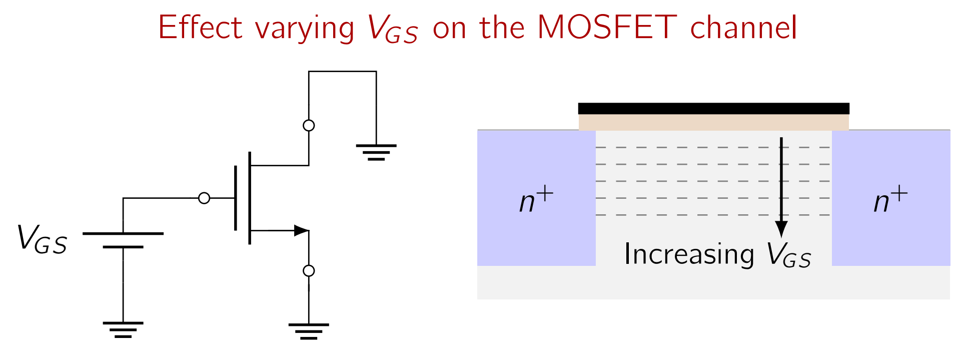


Fig. 2.10: n-Channel MOSFET V_{GS}-V_{DS} Characteristics.

source and the drain. The threshold voltage is typically between $0.3V - 1.0V$. Any v_{GS} applied beyond V_t is called the overdrive votlage or *effective* voltage v_{OV}.

In Figure 2.10 no current flows because both the source and drain are grouded. When a drain source voltage is applied, as shown in Figure **??**, the MOSFET can be turned on by applying a positive gate voltage V_{GS} with respect to the source. The current flowing through the MOSFET is called the *drain current* I_D. The relationship between the drain current i_D, the gate-source voltage V_{GS}, and the drain-source voltage V_{DS} is given by,

$$i_D = k_n \left((v_{GS} - V_t)\, v_{DS} - \frac{1}{2} v_{DS}^2 \right) \tag{2.17}$$

where, k_n is the process transconductance parameter, and V_t is the threshold voltage. This expression hold only when $V_{GS} > V_t$ and $0 \leq V_{DS} < V_{GS} - V_t$. Something very interesting happens to the channel when a drain-source voltage is applied and its magnitude is increased, as shown in Figure 2.11. The channel becomes trapizoidal in shape as v_{DS} is increased due to different is g_{GS} and v_{GD}. This causes the resistance of the channel to increase, tapering off the current flowing through it. When the v_{DS} is increased further, the channels gets pinched off and the current saturates.

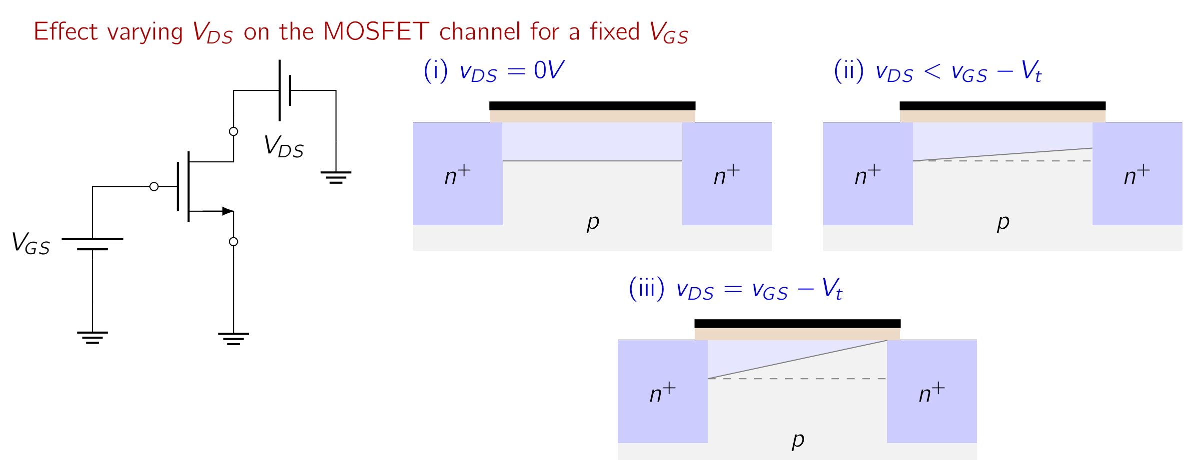
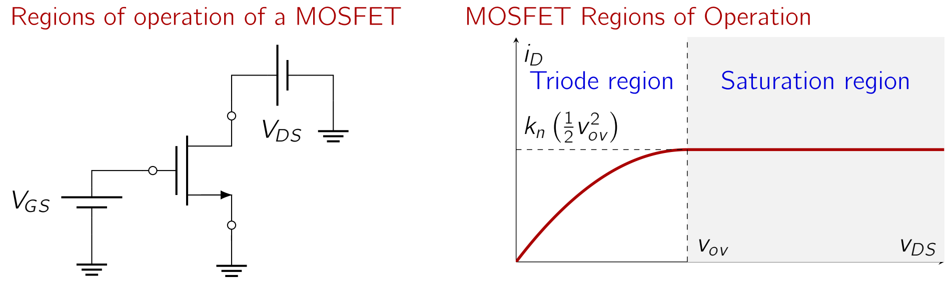


Fig. 2.11: n-Channel MOSFET V_{DS} Characteristics.



MOSFET Triode Region

$v_{GS} > V_t$, $v_{DS} < v_{ov}$

$i_D = k_n\left(v_{ov} - \frac{1}{2}V_{DS}\right)V_{DS}$

MOSFET Saturation Region

$v_{DS} \geq v_{ov}$

$i_D = \frac{1}{2}k_n v_{ov}^2$

Fig. 2.12: Regions of operation of a MOSFET.

MOSFET regions of operation

The MOSFET can be operated in three different regions, depending on the values of V_{GS} and V_{DS}. The three regions are: (a) **Cut-off region**, (b) **Triode region**, and (c) **Saturation region**. The difference between the three regions are summarized in Table 2.2.

Table 2.2: Modes of operation of a MOSFET

Mode	Gate-Source Voltage	Drain-Source Voltage	Operation
Cut-off	$V_{GS} < V_t$	$V_{DS} = 0$	Transistor is off; no current flows.
Triode	$V_{GS} > V_t$	$V_{DS} < V_{GS} - V_t$	I_D is proportional to V_{DS}.
Saturation	$V_{GS} > V_t$	$V_{DS} \geq V_{GS} - V_t$	I_D is constant.

Figure 2.12 shows i_D versus v_{DS} for a fixed value of $v_{GS} > V_t$. When v_{DS} is increased, the relationship is approximately linear for small values of v_{DS}, while increasing v_{DS} slowly constricts the channel, until its pinched off, while i_D saturates. When v_{DS} ¡ $V_{GS} - V_t$, the MOSFET is in the *triode region*, where the drain current I_D is proportional to the drain-source voltage V_{DS}. In this region, the MOSFET behaves like a variable resistor. When v_{DS} is increased beyond $V_{GS} - V_t$, the MOSFET enters the *saturation region*, where the drain current I_D becomes constant and independent of the drain-source voltage V_{DS}. In the saturation mode, the MOSFET behaves like a voltage controlled current source, where the drain current I_D is controlled by the gate-source voltage V_{GS}. When $V_{GS} < V_t$, the MOSFET is in the *cut-off region*, where no current flows through the MOSFET. The current versus voltage characteristics of a MOSFET is shown in Figure 2.13. When v_{DS} is large enough to keep the MOSFET in the saturation region, the drain current I_D

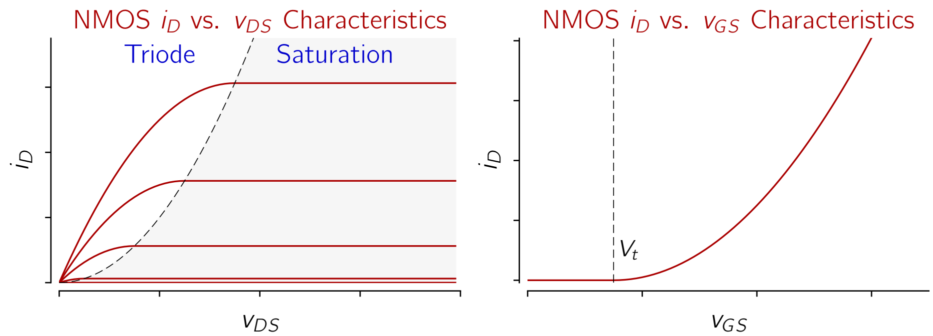


Fig. 2.13: Current vs. voltage characteristics of a MOSFET.

is given by,

$$i_D = \frac{1}{2} k_n \left(v_{GS} - V_t\right)^2 \tag{2.18}$$

2.5 Some Diode Circuits

The diode is used in many circuits, such as rectifiers, votlage regulators, clipper, clampers, etc. In this section, we will briefly presents some common and basic diode circuits. The analysis of these circuits are left as an exercise for the reader. The reader is encouraged to simulate these circuits using a circuit simulation software such as LTSpice to gain a better understanding of the operation of these circuits.

2.5.1 Rectifiers

Figure 2.14 shows a half-wave and full-wave rectifier circuit. Assume that the input signal $v_{in}(t)$ is sinusoidal, i.e., $v_{in}(t) = v_o \sin(\omega t)$. The output signal, i.e., the voltage across the load resistance R_L can be computed with different levels of accuracy based on the model we assume for the diode. The output of the half-wave rectifier circuit is

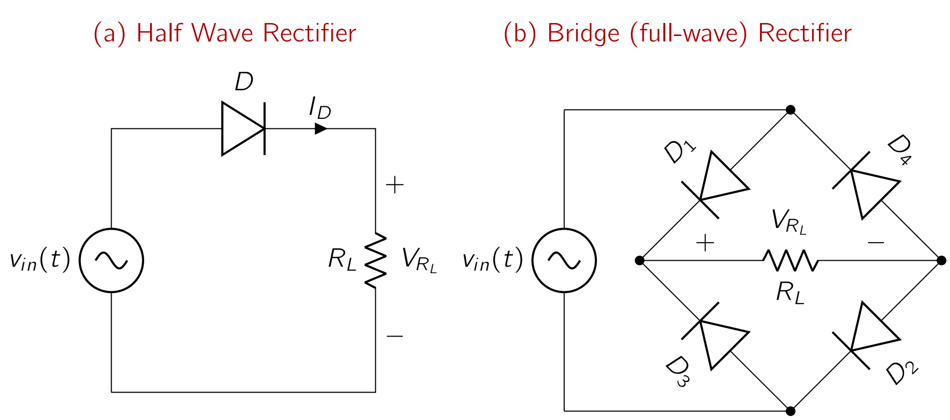


Fig. 2.14: Half-wave and full-wave rectifier circuits.

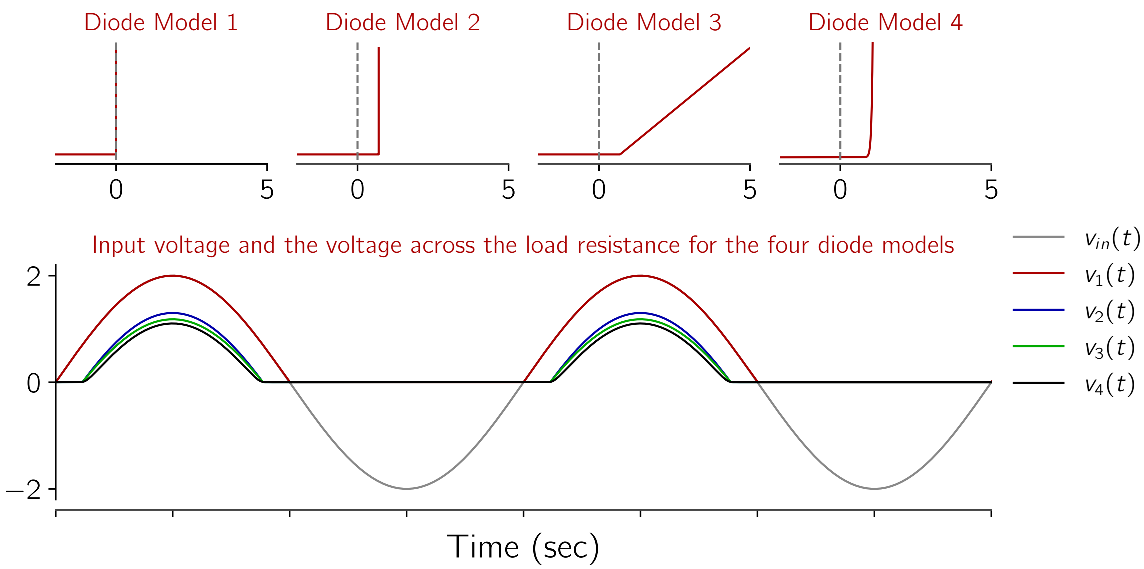


Fig. 2.15: The top row plots show the current-voltage relationship for the assumed diode model. Model 1 is an ideal diode model, while model 4 is a one described in Eq. 2.9. The plot in the bottom row shows the input voltage $v_{in}(t) = 2\sin(2\pi t)$ in gray, and the voltage across the load resistance $R_L = 100\Omega$ for the different assumed model.

shown in Figure 2.15 for four different models, starting with the simplest idea model (model 1) to a more realistic model (model 4).

Problem 2.1. Can you explain the four output graphs in Figure 2.15 using the four models?

The problem of generating a similar output for the full-wave bridge rectifier is left as an exercise at the end of the chapter.

2.5.2 Voltage regulator

A voltage regulator is a circuit that maintains a constant voltage across its output terminals, even when the input voltage or load current changes. Voltage regulators are widely used in power supplies to provide a stable output voltage. Diodes can be used for building simple voltage regulators. Consider the simple circuit in Figure 2.16. Here, we want a stable 2.8V supply for a circuit and we have a voltage supply V_s that is higher than 2.8V. We can build a stable 2.8V using the arrangement shown in this figure ($R_1 = 1k\Omega$). The variation of V_{out} with variations in $V_{in} \in [5, 15]V$ are shown in the adjacent plots with ($R_L = \infty\Omega$) and without a load resistance (top plot; $R_L = 1k\Omega$). The bottom plot shows the change in the V_{out} as a function of the load resistance $R_L \in [0.1k, 10K]\Omega$. Both of these plots show that that V_{out} is fairly robust to large variations in V_{in} and R_L; for very low values of V_{in} and R_L, the circuit fails as expected.

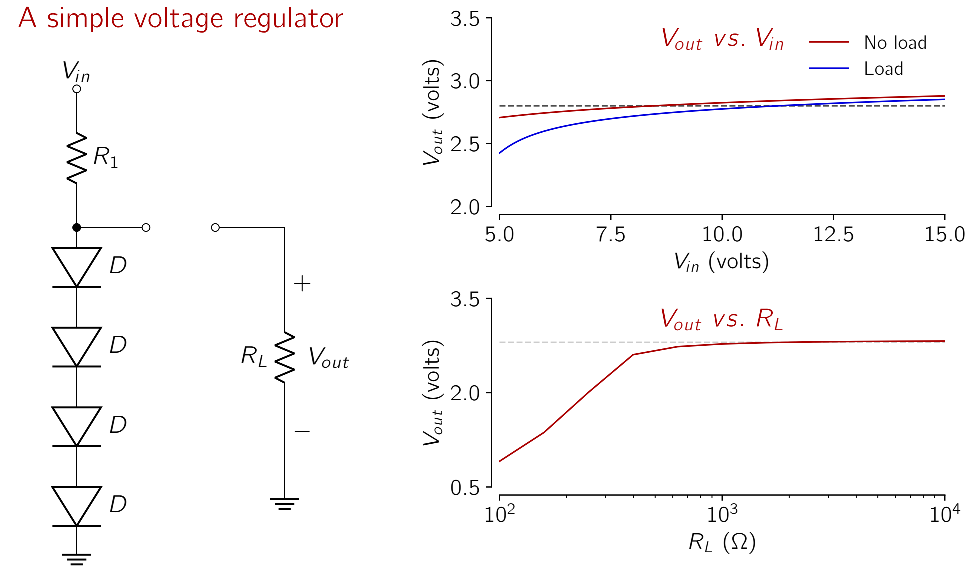


Fig. 2.16: A simple voltage regulator circuit using 4 diodes to obtain a stable voltage of 2.8V. The plots shown in the figure were obtained by simulating this circuit in LTSpice.

Zener diode voltage regulator. A Zener diode is a special type of diode that is designed to operate in the reverse breakdown region, with sufficient power dissipation capability. When a Zener diode is reverse biased, it allows current to flow in the reverse direction when the voltage across it exceeds a certain value, called the *Zener voltage* V_Z. This makes it useful for voltage regulation applications. The Zener diode can be used to build a simple voltage regulator circuit as shown in Figure 2.17. The Zener diode is connected in reverse bias in series with a resistor R_1 and the input voltage V_{in} is slowly increased. The current-voltage relationship of the Zener diode (V_Z vs. I_Z) is shown in the figure. For now assume that the load resistance R_L is not connected to the the circuit. In this case, the Zener breakdown voltage is $5.1V$, when $V_Z < 5.1V$, hardly any current flows through the Zener diode and $V_Z = V_{in}$. But once V_{in} get closed to $5.1V$, the Zener diode starts to conduct and V_Z clamps to $5.1V$.

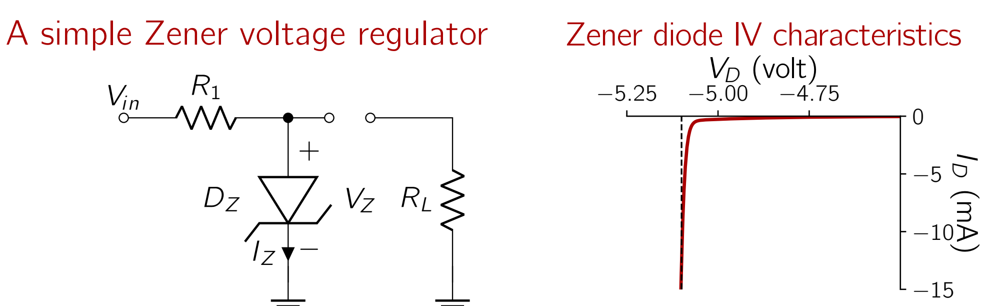


Fig. 2.17: A simple Zener voltage regulator circuit using a Zener diode.

The plot of V_{in} versus V_Z with and without the load resistance is shown in Figure 2.18.

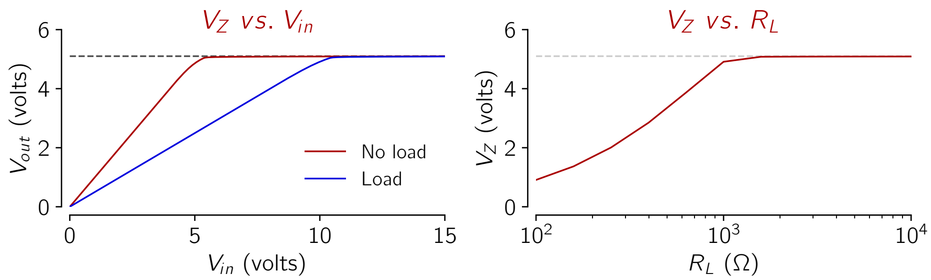


Fig. 2.18: The plot on the left shows the performance of the Zener diode voltage regulator with ($R_L = 1k\Omega$) and without ($R_L = \infty\Omega$) the load. The second plot shows the Zener diode's performance for varying load resistance.

2.6 Some BJT Circuits

These days the most common practical use of the BJT is as a switch. This is what we will focus on; we will also briefly talk about a simple voltage amplifier circuit using the BJT for the sake of demonstration.

2.6.1 BJT as a switch

The BJT can be used as a voltage controlled switch to control the flow of current through different types of loads such as resistors, inductors, LEDs, etc. To act as a switch the BJT is operated in the cut-off or the saturation modes, to turn it off or on, respectively, as shown in Figure 2.19. In the ideal case, it behave alike an open or closed switch between the collector and emitter terminals.

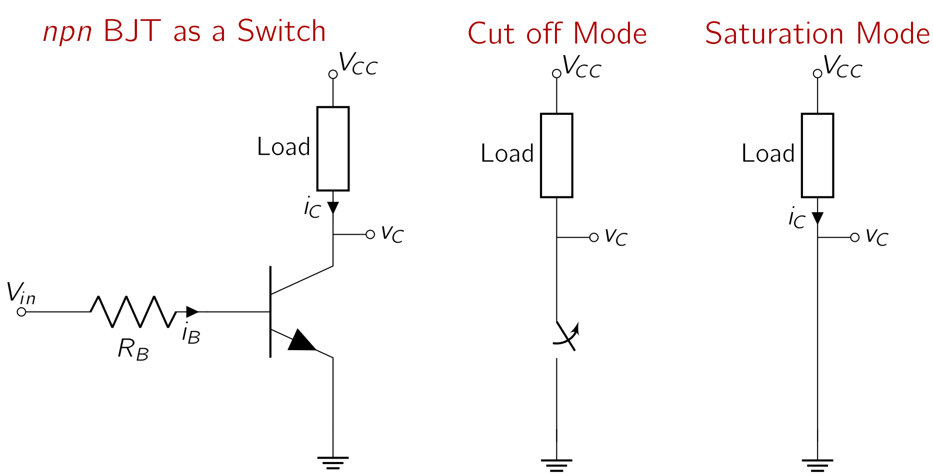


Fig. 2.19: Circuit of a typical *npn* BJT based switch.

Consider a resistive load in Figure 2.19 with load resistance R_L. In the cut-off mode, when the base-emitter junction is reverse biased, i.e. $V_{BE} < 0.7V$, the BJT is off. One way to achieve this is to set $V_{in} = 0V$ for a *npn* BJT. In the cut-off mode, $i_B = 0$, which means that $i_C = 0$ and $i_E = 0$. The voltage at the collector of the BJT will be equal to the supply voltage V_{CC}, i.e. $v_C = V_{CC} - i_C R_L = V_{CC}$. In saturation mode, the base-emitter and collector-base junctions are both forward biased. In a real *npn* BJT circuit, the collector-emitter voltage is $v_{CE} =\approx 0.2V$. Thus, we have, $i_C = \frac{V_{CC}-v_{CE}}{R_L} = \frac{V_{CC}-0.2}{R_L}$. For this collector current, the base current must be at least $i_B = \frac{i_C}{\beta}$. Thus, we have $i_B = \frac{V_{in}-0.7}{R_B}$. This can be achieve by choosing a sufficiently large V_{in} or sufficiently small R_B.

For non-resistive loads, the analysis is similar, but we must consider the load's voltage-current characteristics and appropriately design the circuit. Let's look at a couple of examples.

npn BJT to switch on/off an LED

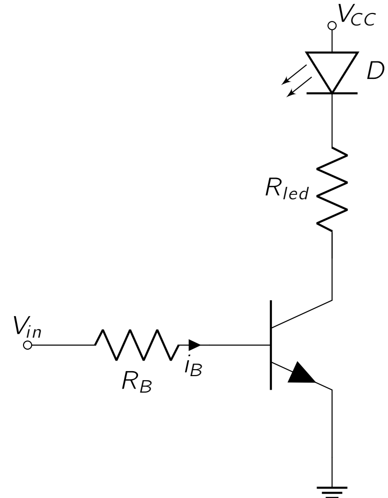


Fig. 2.20: BJT LED control circuit

Example 2.1 (**Switch on an LED**). We have an LED with a forward voltage drop of $2.0V$ and a forward current of $20mA$. We want to control the LED through a transistor, such that when $3.3V$ is applied to the base of the transistor, the LED switches ON, else it is OFF (Figure 2.20). Assume that the *npn* transistor has a current gain in the range $\beta = 100 - 200$. We need to choose R_{led} and R_B for this circuit.

We need i_C must be $20mA$ when the LED is turned on, and it has a drop of $2V$. When the transistor is in saturation, we have,

$$V_{CC} - 2V - i_C R_L = 0 \implies R_L = \frac{V_{CC} - 2V}{i_C} = \frac{3.3V - 2V}{20mA} = 65\Omega$$

To get an collector current of $20mA$, we need a base current of $i_B = \frac{i_C}{\beta} = \frac{20mA}{\beta}$. Since we do not have the exact value of β, we will assume the worst case, i.e. lowest possible value for β, which in this case is 100. Thus, we have $i_B = \frac{20mA}{100} = 0.2mA$.

The base current is given by $i_B = \frac{V_{in}-0.7}{R_B}$, where $V_{in} = 3.3V$. Thus, we have, $R_B = \frac{3.3V-0.7V}{i_B} = \frac{2.6V}{0.2mA} = 13k\Omega$.

npn BJT to switch on/off a solenoid

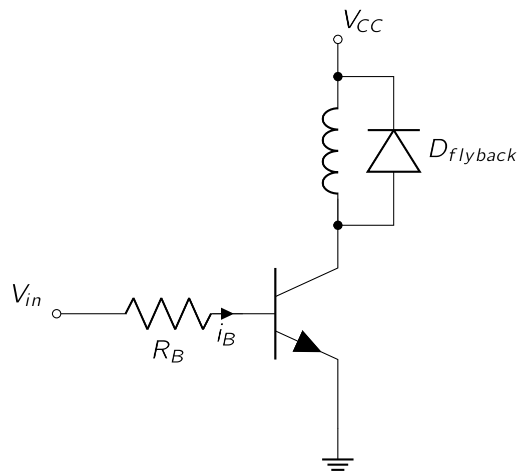


Fig. 2.21: BJT solenoid control circuit

Example 2.2 (Switch on/off a solenoid). Consider a tiny solenoid that switches on at $5V$ across it coil and a current of $0.8A$. We wish to use a *npn* BJT to switch the the solenoid ON and OFF by controlling the BJT's base voltage. The circuit for switching the solenoid ON and OFF is shown in Figure 2.21. We will assume that the BJT has a current gain of $\beta = 500$. We need to choose R_B for this circuit assuming that V_{in} is $0V$ or $3.3V$.

The solenoid has a voltage drop of $5V$ and a current of $0.8A$. When the transistor is in saturation the voltage across the solenoid will be $4.8V$, i.e. close to $5V$ (the solenoid should still work as per the specification). To ensure $0.8A$ flows throught the solenoid, we need to ensure that the base current must be at least $i_B = \frac{i_C}{\beta} = \frac{0.8A}{500} = 1.6mA$. The base current is given by $i_B = \frac{V_{in}-0.7}{R_B}$, where $V_{in} = 3.3V$. Thus, we have, $R_B = \frac{3.3V-0.7V}{1.6mA} = 1.625k\Omega$.

You are probably wondering about the purpose of the $D_{flyback}$ diode. Let's assume that we apply a square pulse as the input V_{in} to the base.

$$V_{in} = \begin{cases} 5V & 0.25s \leq t < 1.25s \\ 0V & \text{otherwise} \end{cases}$$

When the BJT is suddenly turned OFF at time $t = 1.25s$ with $0.8A$ flowing through it, the current is forced to stop suddenly, which causes a large voltage spike across the solenoid, which will result in a large voltage spike across the BJT, which can damage it. The flyback diode $D_{flyback}$ is used to suppress this voltage spike by allowing the current to flow through it, and decay down slowly (relatively), thus avoiding a voltage spike across the inductor and the BJT. You will see this in action in the lab exercise, when you are deliberately asked to remove the flyback diode and see what happens to the BJT.

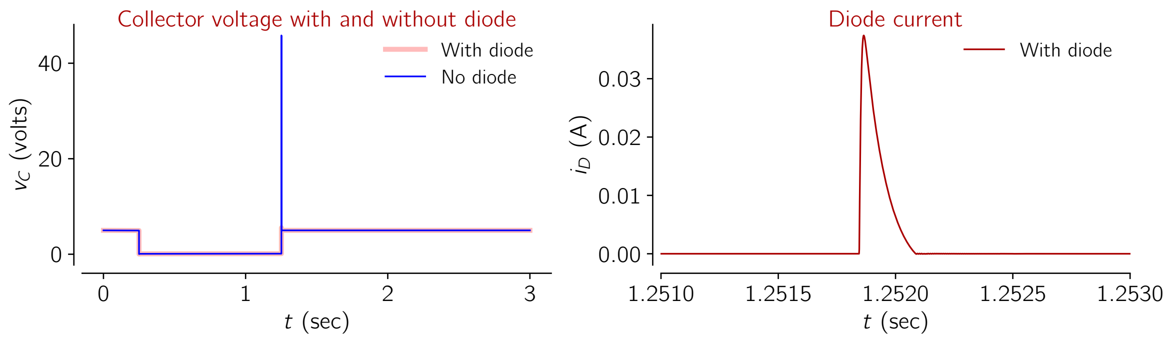


Fig. 2.22: Effect of the flyback diode on the solenoid switching circuit. The left plot shows the collector voltage for a input pulse voltage V_{in} of $5V$ amplitude over $0.25s \leq t \leq 1.25s$. Without the flyback diode we find a large voltage spike when the collector current is suddenly reduced to zero. With the flyback diode suppressed the voltage spike by allowing the current to flow through it when the BJT is turned off. The right plot shows the transient diode current when the BJT is switched off at $t = 1.25sec$.

2.7 Some MOSFET Circuits

We now look at using MOSFETs as switches. A MOSFET operated in the cut-off mode will act like an open switch, while a MOSFET operated in the triode region with sufficient gate overvoltage for it to act like a closed switch (low drain-source resistance). The MOSFET switching circuit design is simpler than the BJT where we needed to ensure that the base current is sufficient to turn on the BJT. In the case of MOSFETs, we only need to ensure that the gate-source voltage V_{GS} is greater than the threshold voltage V_t to turn it on. We demonstrate the MOSFET switch design in the following example.

Example 2.3 (MOSFET switch). Consider a n-channel MOSFET with a threshold voltage of $V_t = 2.5V$. We want to use it to switch on an LED with a forward voltage of $2V$ and current of $50mA$. The drain-source resistance of the MOSFET is $10m\Omega$ at $v_{GS} = 3.3V$. This circuit is shown in Figure 2.23. The supply voltage $V_{DD} = 3.3V$.

Let's assume that $V_{GS} = 3.3V$, which results in the following equation for the drain to source loop, assuming the LED is switched on.

$$V_{DD} - i_D r_{led} - 2 - i_D r_{DS} = 0$$

$$R_{led} = \frac{V_{DD} - 2}{i_D} - r_{DS} = \frac{3.3V - 2V}{50mA} - 10m\Omega \approx 26.0\Omega$$

The votlage divider circuit at the gate in Figure 2.23 is used to control the gate-source voltage V_{GS}. The gate-source voltage is given by,

$$V_{GS} = \frac{R_2}{R_1 + R_2} V_{in}$$

where $V_{in} = 3.3V$. To ensure that the MOSFET is turned on, we need $V_{GS} > V_t = 2.5V$. Thus, we have,

$$\frac{R_2}{R_1 + R_2} V_{in} > 2.5V \implies \frac{R_2}{R_1 + R_2} > \frac{2.5V}{3.3V} \approx 0.7576$$

$$\implies R_2 > 2.5 R_1$$

Typically, R_1 is chosen to be small to prevent current rush when 3.3V is applied to the gate, due to the charging of the gate capacitance. And R_2 is chosen to be a large value to ensure $v_{GS} \approx 3.3V$.

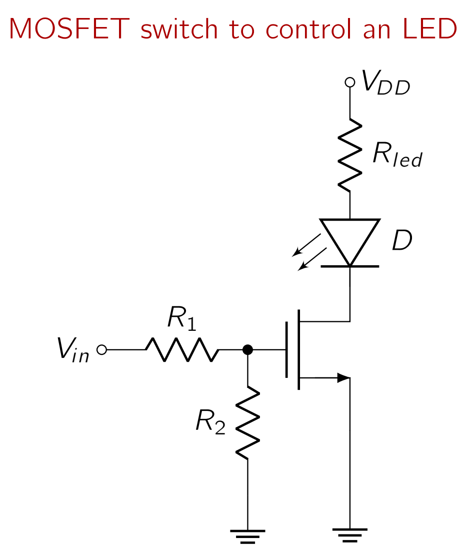


Fig. 2.23: MOSFET switch for controlling an LED. The MOSFET is turned on when $V_{in} = 3.3V$.

2.8 Operational Amplifiers

The electronic components and circuits we have seen so far have been for non-linear applications. The later switching circuits are used most in digital circuits. In this section, we will look at a vital component in analog circuits, the *operational amplifier* or (op-amp). The op-amp is a versatile component that can be used for a wide range of applications such as amplifiers, filters, oscillators, etc. The op-amp is a differential amplifier with two inputs and one output. The two inputs are called the inverting input ($-$) and the non-inverting input ($+$). The output is the amplified difference between the two inputs. The circuit symbol for an op-amp is shown in Figure 2.24. The output

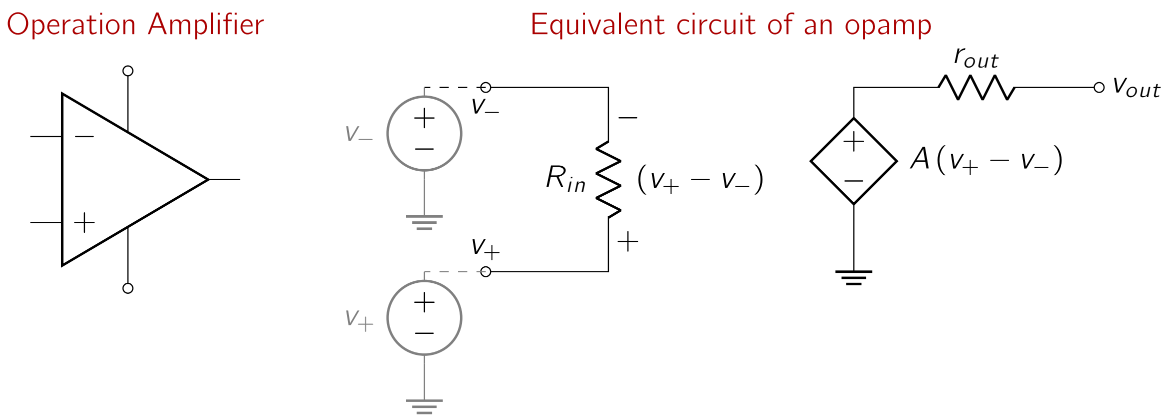


Fig. 2.24: Operational amplifier symbol and equivalent circuit.

v_{out} of the op-amp is the amplified version of the difference in the voltages at the two input terminals, $v_{out} = A \cdot (v_+ - v_-)$, where A is the open-loop gain of the op-amp.

An **ideal op-amp** has the following characteristics, and for most practical applications we can assume that the op-amp behaves like an ideal op-amp.

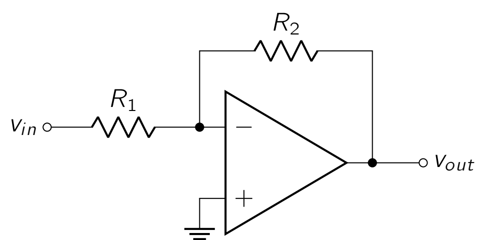
- The input impedance is infinite $R_{in} = \infty\Omega$, i.e., no current flows into the input terminals
- The output impedance is zero $r_{out} = 0\Omega$, i.e., the output can drive any load without affecting the output voltage.
- The open-loop gain is infinite $A = \infty$, i.e., the output voltage is always equal to the amplified difference between the two input voltages.
- The bandwidth of the op-amp is infinite, i.e., it can amplify signals of any frequency without distortion.
- The common mode rejection ratio is infinite, i.e. if the same input is applied to the op-amp, the output will be zero.
- The output voltage is limited to the supply voltage, i.e., the output cannot exceed the positive and negative supply voltages.

For any op-amp, any difference between v_+ and v_- is amplfied and results in the votlage output of the op-amp saturating either to postive or negative supply voltage. Interesting circuits with op-amps are obtained by operating with feedback, i.e. when the output votlage is fed back to the input terminals using a circuit element.

2.9 Some useful Op-Amp circuits

In this section we will present the following useful op-amp circuits: (a) inverting amplifier; (b) non-inverting amplifier; (c) Summing amplifier; (d) Difference amplfier; (e) Instrumentation amplifier; (f) Integrator circuit; (g) Differentiator circuit; and (h) Precision rectifier. We will simply show the circuit diagrams and present the input-output relationships. The derivation of these are left as an exercise for the reader.

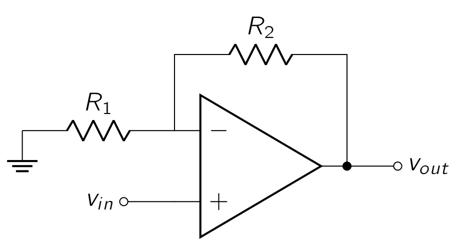
2.9.1 Inverting amplifier



$$v_{out} = -\frac{R_2}{R_1} v_{in}$$
$$Z_{in} = R_1$$
$$Z_{out} = 0$$

Fig. 2.25: Opamp inverting amplifier.

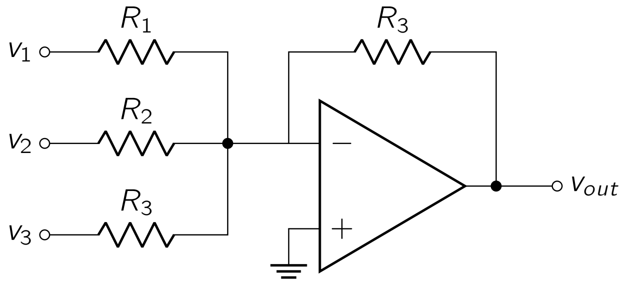
2.9.2 Non-inverting amplifier



$$v_{out} = \left(1 + \frac{R_2}{R_1}\right) v_{in}$$
$$Z_{in} = \infty$$
$$Z_{out} = 0$$

Fig. 2.26: Opamp inverting amplifier.

The op-amp buffer is a special case of the non-inverting amplifier, where $R_1 = 0\Omega$ and $R_2 = \infty\Omega$. The output voltage is equal to the input voltage, i.e., $v_{out} = v_{in}$.



$$v_{out} = -\left(\frac{R_3}{R_1}v_1 + \frac{R_3}{R_2}v_2 + \frac{R_3}{R_3}v_3\right)$$
$$Z_{in}(v_1) = R_1 + R_2\|R_3$$
$$Z_{out} = 0$$

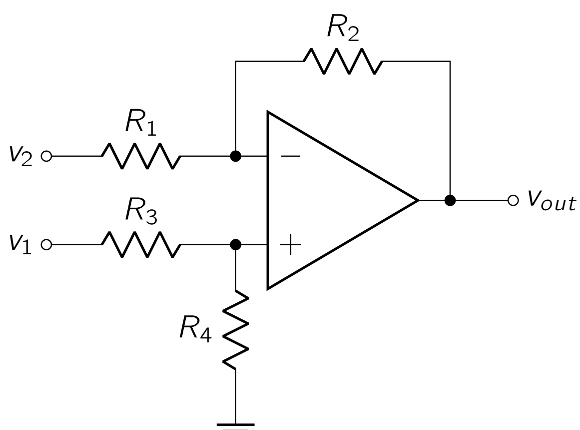
Fig. 2.27: Opamp summing amplifier.

2.9.3 Summing amplifier

The input impedance of this circuit will need to computed with respect to each input. The input impedance of the v_1 is given by,

$$Z_{in}(v_1) = R_1 + \frac{R_2R_3}{R_2 + R_3} \tag{2.19}$$

2.9.4 Difference amplifier

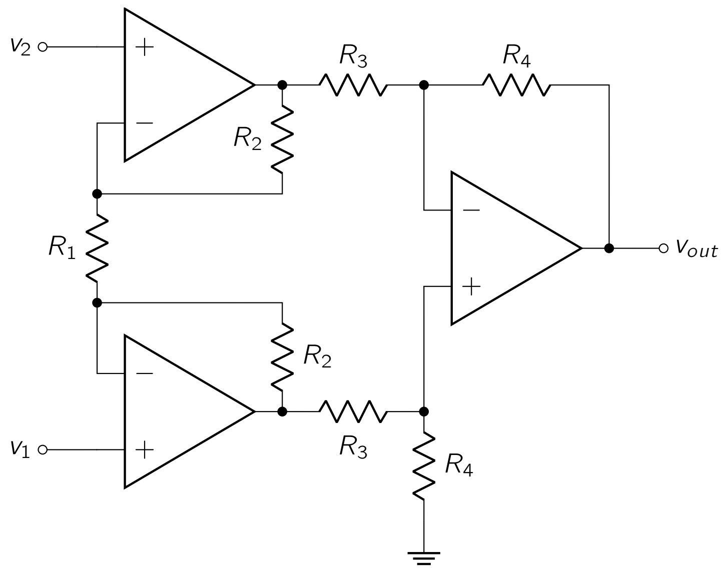


$$v_{out} = \frac{R_4}{R_3 + R_4}v_2\left(1 + \frac{R_2}{R_1}\right) - \frac{R_2}{R_1}v_1$$
$$= \frac{R_2}{R_1}(v_1 - v_2)\,,\ when\,\frac{R_4}{R_3} = \frac{R_2}{R_1}$$
$$Z_{in}(v_1) = R_3 + R_3$$
$$Z_{in}(v_2) = R_2$$
$$Z_{out} = 0$$

Fig. 2.28: Opamp difference amplifier.

2.9.5 Instrumentation amplifier

The three op-amp instrumentation amplifier is a special case of the difference amplifier, where the input impedance with respect to the two inputs is very high. The circuit diagram is shown in Figure 2.29. Note that if $R_2 = 0$ or $R_1 = \infty$, it is equivalent to a difference amplifier circuit where the input are connected to the difference amplifier through an op-amp buffer.

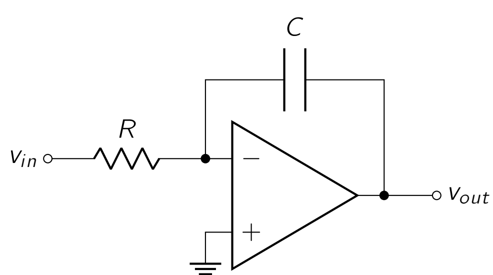


$$v_{out} = \frac{R_4}{R_3}\left(1 + \frac{R_2}{R_1}\right)(v_1 - v_2)$$
$$Z_{in}(v_1) = \infty$$
$$Z_{in}(v_2) = \infty$$
$$Z_{out} = 0$$

Fig. 2.29: Opamp instrumentation amplifier.

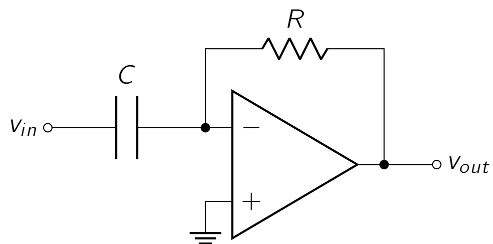
2.9.6 Integrator and differentiator circuits

Op-Amp Integrator Circuit



$$v_{out} = -\frac{1}{RC}\int v_{in}dt$$
$$Z_{in} = R$$
$$Z_{out} = 0$$

Op-Amp Differentiator Circuit



$$v_{out} = -RC\frac{d}{dt}v_{in}$$
$$Z_{in}(s) = \frac{1}{sC}$$
$$Z_{out} = 0$$

Fig. 2.30: Opamp integrator and differentiator.

While RC circuits can approximate the integrator and differentiator behavior, the op-amp integrator and differentiator circuits behave more like the ideal integrator or differentiator. The output, however, is clamped by the op-amp supply voltages.

2.9.7 Precision rectifier

The rectifier circuits we saw earlier have the diode drop in their output voltage. This is due to the diode forward voltage drop. Although this is not a problem when the peak-to-peak voltage of the input signal is large, it can be significant for small signals. For examples, if want to do analog processing of EMG signal to estimate it envelope, the 0.7V diode drop is a significant problem. Op-amps are quite useful in building precision rectifiers where the output voltage accurately follows the input voltage without the diode drop. A simple half-wave precision rectifier is shown in Figure 2.31.

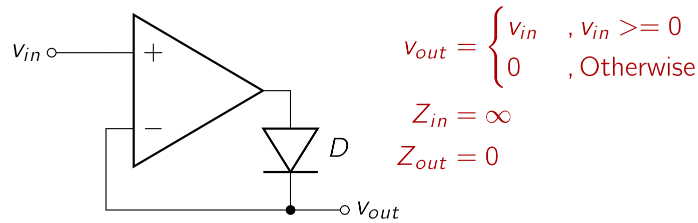


Fig. 2.31: Opamp half wave precision rectifier.

2.10 Exercise

1. Compute the voltage and current across the load resistor R_L in Figure 2.32 for the following: (a) $V_{in} = 1.0V$; (b) $V_{in} = -1.5V$. Assume the following parameters for the diodes: $I_s = 10^{-12}A$, $n = 1$, $V_T = 25.85mV$, and $R_l = 1k\Omega$.

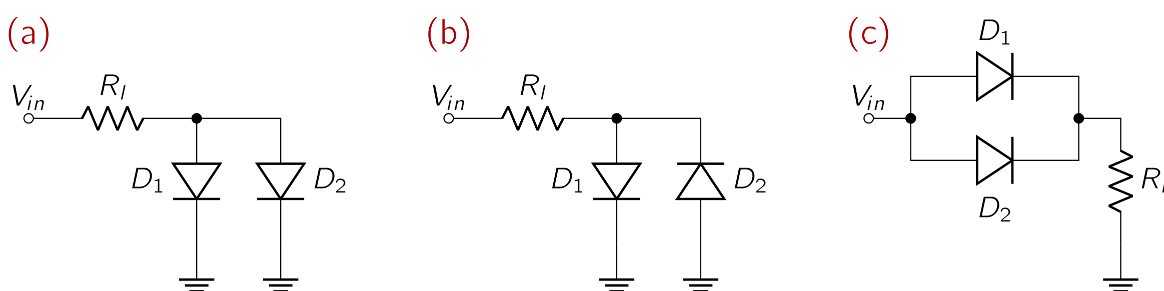


Fig. 2.32: [Exercise 01]Different diode circuits.

2. If the input signal $v_{in} = 2\sin(3\pi t)$, plot the current flowing through the load resistance R_l. You must use Matlab/Python/Julia or any scientific computing software for generating these plots. You are required to generate these plots for the four diode models shown in Figure 2.15. Assume the following parameters for the models:

(a) Model 2: Diode forward voltage $= 0.7V$.

(b) Model 3: Diode forward voltage $= 0.7V$, Diode on resistance $r_d = 1\Omega$.

(c) Model 4: $I_s = 10^{-12}A$, $n = 1$, $V_T = 25.85mV$, and $R_l = 1k\Omega$.

Hand drawn plots will not be accepted.

3. Similar to the plot of the in Figure 2.15 for the half wave rectifier, using a computer program generate the output plot of the full wave bridge rectifier circuit in Figure 2.14.

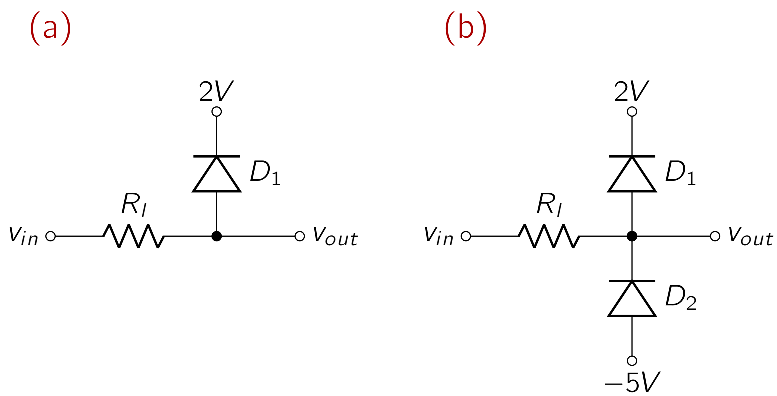


Fig. 2.33: [Exercise 02]More diode circuits.

4. For the following diode circuits, plot the outputs v_{out} for the following input signal,

$$v_{in} = \begin{cases} -10V & , t \leq -10s \\ t & , -10s < t \leq 5s \\ 5 & , t > 5s \end{cases}$$

Generate the plots for the four diode models used in the previous questions. What do think is the purpose of such circuits?

5. Consider the following BJT circuit (Figurer 2.34). If the input is a square wave of amplitude 3V with a time period of 2s, plot the output of the circuit v_C. What does this circuit do? Can you design a similar ciruit with a n-channel MOSFET?

6. We have seen an LED switching circuit using a BJT/MOSFET, were the LED swtiches on when the input is high $v_{in} = 3.3V$ and it turns off when the input is low $v_{in} = 0V$. Can you design a circuit using a BJT and a MOSFET to control an LED with forward voltage of 1.8V and current of 30mA, where we want the LED to turn on when $v_{in} = 0V$ and turn off when $v_{in} = 3.3V$. Draw the circuit diagram, explain how it works,and your design.

7. Draw the circuit diagram for a full-wave precision rectifier and explain its operation.

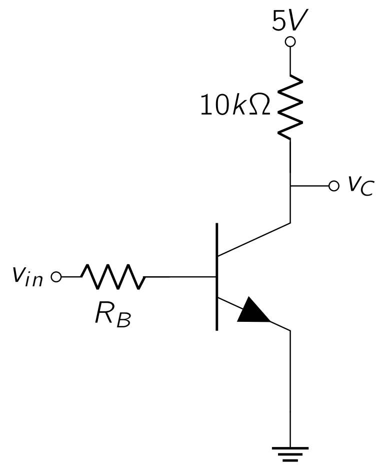


Fig. 2.34: [Exercise 05] BJT switch circuit.

8. Derive the input-output relationship for all the op-amp circuits (except for the precision amplifier) shown in the section on operational amplifiers in the chapter? Assume ideal op-amp characteristics.

Chapter 3

Linear Systems Theory

Linear systems theory is fundamental to understanding, analyzing and designing engineering systems. This chapter introduces some essential concepts and tools employed in linear systems theory. We will only present mathematical formulations and results without proofs or any detailed explanations.

3.1 Linear Systems

In the engineering context, a system is a collection of components that interact to perform a specific function. When we think of a system we usually think of a block that has inputs and outputs; the components of the system, their interconnections and interactions are all abstracted away in the block as shown in Figure 3.1. This system has three inputs, u_1, u_2, and u_3, and two outputs, y_1 and y_2. The inputs are signals that

Fig. 3.1: Block diagram representation of a system with three inputs and two outputs.

can influence the internal behavior and/or output of the system, and the outputs are the readouts of the internal state of the system and the current input. This abstraction allows engineers to analyze and design complex systems without needing to understand every detail of their internal workings. In this type of characterization, we are simply interested in the relationship between the inputs and outputs of the system without worrying about the internal details.

When a system has a single input and a single output, like most systems you learn in an undergraduate signal processing or control theory course, it is referred to as a **single-input single-output** (SISO) system. When a system has multiple inputs and multiple outputs, like the one shown in Figure 3.1, it is referred to as a **multiple-input multiple-output** (MIMO) system. For most of this chapter, will will present ideas for SISO linear system, but extend these to MIMO systems later.

A SISO system $\mathcal{H}$ is said to be *linear* if it satisfies the following two properties for all inputs $u(t)$ and outputs $y(t)$, and all scalars α:

- **Homogeneity (or scaling)**: If $u(t)$ is an input to the system $\mathcal{H}$ and $y(t)$ is the corresponding output, then for any scalar α, the input $\alpha u(t)$ produces the output $\alpha y(t)$.
$$\mathcal{H}(\alpha u(t)) = \alpha \mathcal{H}(u(t)) \tag{3.1}$$

- **Additivity**: If $u(t)$ and $v(t)$ are inputs to the system $\mathcal{H}$ with corresponding outputs $y_u(t)$ and $y_v(t)$, then the input $u(t) + v(t)$ produces the output $y_u(t) + y_v(t)$.
$$\mathcal{H}(u(t) + v(t)) = \mathcal{H}(u(t)) + \mathcal{H}(v(t)) \tag{3.2}$$

Effectively, the scaling factor α (in Eq. 3.1) and the addition operation (in Eq. 3.2) can be factored out of the paranthesis. This is the simplicity offered by the linearity assumption. These two properties together give us the superposition principle,

$$\mathcal{H}(\alpha_1 u_1(t) + \alpha_2 u_2(t)) = \alpha_1 \mathcal{H}(u_1(t)) + \alpha_2 \mathcal{H}(u_2(t)) \tag{3.3}$$

for any inputs $u_1(t)$ and $u_2(t)$ and scalars α_1 and α_2. This means that the response of a linear system to a linear combination of inputs is the same linear combination of the responses to each input. This means that once we know the output of the system for two inputs for a linear system, we automatically know the output for any linear combination of those two inputs.

Problem 3.1. Verify that the system $y(t) = \mathcal{H}(u(t)) = 5 \cdot u(t)$ is linear. Whereas the systems, $y(t) = \mathcal{H}(u(t)) = 3 \cdot u^2(t)$ and $y(t) = \mathcal{H}(u(t)) = u(t) + 1$ are not.

The extension to MIMO systems is straightforward: a MIMO system is linear if it satisfies the homogeneity and additivity properties for all inputs and outputs, where the inputs and outputs are vectors.

Linearity is a strict or highly constraining assumption. For most systems the linearity assumption hold for a restricted range of inputs and output. For instance, a practical resistor from the lab can be very well approximated as a linear system with votlage across it as the input and the current through it as the output for small values of voltage and current, i.e. as long as it is operated well within the power ratings of the resistor. It will not be linear for large voltages and currents.

3.2 Time-invariant systems

A system whose behavior or input-output relationship does not change with time is called a time-invariant system. If my new computer is time-invariant, this essentially means that my computer will continue to behave like a new computer for all time. Like the linearity assumption, for most systems time-invariance would hold over finite durations of time, but not beyond that. My computer might person as well as it did when I bought for a couple of years but may be not beyond that. Mathematically, a system $\mathcal{H}$ is time-invariant if and only if the folloowing hold for all inputs and all time,

$$y(t) = \mathcal{H}(u(t)) \implies y(t-T) = \mathcal{H}(u(t-T)), \;\; \forall\, u(t) \text{ and } T \tag{3.4}$$

In the above equation, it looks like we have simply replaced t by $t - T$, but the issue a bit more subtle. The above equation means that if $y(t)$ is the output of the system to the input $u(t)$ at present, then if I apply the the same input at some point in the future or if I had applied it at some time in the past, the system's output would be exact the same the one at present, except that it appears in the future or the past, respectively.

A system that is both linear and time-invariant is called a linear time-invaraint (LTI) system, which is an important class of systems in engineering. Continuous-time LTI systems can be described by linear constant coefficient ordinary differential equations (ODEs) or linear constant coefficient difference equations in the discrete-time case.

3.3 Impulse response and convolution

We said earlier that if we know the output of a linear system to one or more inputs, then we can easily compute the output of the system for any linear combination of these inputs. It turns out that there is a special input signal that is very useful for characterizing LTI systems. It is the Dirac delta function or the impulse function $\delta(t)$ in the continuous-time case and the Kronecker delta function $\delta[n]$ in the discrete-time case. We will first look at the continuous-time case. The Dirac delta function δ is not a function in the traditional sense, but rather a distribution. It is defined by the following property

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} x(t)\,\delta(t - t_0)\, dt = x(t_0) \tag{3.5}$$