Taylor Table method and Matlab code

Objective:

1. Derive the Fourth order approximation of the second order derivatives using
   a) Central difference scheme
   b) Skewed right sided difference scheme
   c) Skewed left sided difference scheme
2. Prove that they are fourth order accurate
3. A program to evaluate the second-order derivative of the analytical function exp(x)*cos(x) and compare it with the 3 numerical approximations.
4. Plot the absolute errors of 3 approximations.
5. Explain the use of skewed schemes and their benefit over Central difference schemes.

Solution:

1. **Central Difference Scheme**

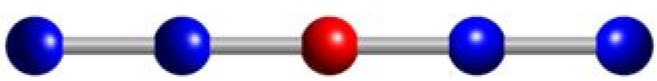
Number of nodes = p+q-1

Where p=order of approximations

q= order of derivative

Here number of nodes = 4+2-1=5

Here the central node is the main node and 2 nodes on either side are considered.



Therefore the central difference scheme equation is

$$\Delta x^2 (\frac{\delta^2 f}{\delta x^2}) = af(i-2) + bf(i-1) + cf(i) + df(i+1) + ef(i+2)$$

where,

$$af(i-2) = af(i) - (2af'(i)\Delta x)/(1!) + (4af''(i)\Delta x^2)/(2!) - (8af'''(i)\Delta x^3)/(3!) + (16af''''(i)\Delta x^4)/(4!) - (32af'''''(i)\Delta x^5)/(5!) + (64af''''''(i)\Delta x^6)/(6!)$$

…..(1)

$$bf(i-1) = bf(i) - (bf'(i)\Delta x)/(1!) + (bf''(i)\Delta x^2)/(2!) - (bf'''(i)\Delta x^3)/(3!) + (bf''''(i)\Delta x^4)/(4!) - (bf'''''(i)\Delta x^5)/(5!) + (bf''''''(i)\Delta x^6)/(6!)$$

…..(2)

$$cf(i) = cf(i)$$

…..(3)

$$df(i+1) = df(i) + (df'(i)\Delta x)/(1!) + (df''(i)\Delta x^2)/(2!) + (df'''(i)\Delta x^3)/(3!) + (df''''(i)\Delta x^4)/(4!) + (df'''''(i)\Delta x^5)/(5!) + (df''''''(i)\Delta x^6)/(6!)$$

…..(4)

$$ef(i+2) = ef(i) + (2ef'(i)\Delta x)/(1!) + (4ef''(i)\Delta x^2)/(2!) + (8ef'''(i)\Delta x^3)/(3!) + (16ef''''(i)\Delta x^4)/(4!) + (32ef'''''(i)\Delta x^5)/(5!) + (64ef''''''(i)\Delta x^6)/(6!)$$

…..(5)

Now let's form a Taylor's table,

	$f(i)$	$\Delta x\, f'(i)$	$\Delta x^2\, f''(i)$	$\Delta x^3\, f'''(i)$	$\Delta x^4\, f^{iv}(i)$	$\Delta x^5\, f^{v}(i)$	$\Delta x^6\, f^{vi}(i)$
$af(i-2)$	*a*	*-2a*	*4a/2*	*-8a/6*	*16a/24*	*-32a/120*	*64a/720*
$bf(i-1)$	*b*	*-b*	*b/2*	*-b/6*	*b/24*	*-b/120*	*b/720*
$cf(i)$	*c*	*0*	*0*	*0*	*0*	*0*	*0*
$df(i+1)$	*d*	*d*	*d/2*	*d/6*	*d/24*	*d/120*	*d/720*
$ef(i+2)$	*e*	*2e*	*4e/2*	*8e/6*	*16e/24*	*32e/120*	*64e/720*
$\Delta x^2\ \frac{(\delta^2 f)}{\delta x^2}$	*0*	*0*	*1*	*0*	*0*	*?*	*?*

$$\Delta x^2(\frac{\delta^2 f}{\delta x^2}) = [(2a + (b/2) + 0 + (d/2) + 2e)f''(x)\Delta x^2] + [(64a/720) + (b/720) + 0 + (d/720) + (64e/720)f''''''(x)\Delta x^6]$$

$$(\frac{\delta^2 f}{\delta x^2}) = [(2a + (b/2) + 0 + (d/2) + 2e)f''(x)] + [(64a/720) + (b/720) + 0 + (d/720) + (64e/720)f''''''(x)\Delta x^4]$$

The above equation proves that the central difference scheme is fourth order accurate delx^4.

The values of a,b,c,d,e are as derived using the following script:

```
%Matrix solving for cds
A1=[1 -2 4/2 -8/6 16/24 ;1 -1 1/2 -1/6 1/24 ;1 0 0 0 0 ;1 1 1/2 1/6 1/24 ;1 2 4/2 8/6 16/24];
B1=A1';
C1=[0;0;1;0;0];
X1=B1\C1;
```

Output:

X1	double	5x1	[-0.083333; 1.3333; -2.5000; 1.3333; -0.083333]

Substituting the coefficients of the Taylor's series terms after solving the matrix,

$$(\frac{\delta^2 f}{\delta x^2}) = [-0.083333f(i-2) + 1.333333f(i-1) - 2.500000f(i) + 1.333333f(i+1) - 0.083333f(i-2)]/\Delta x^2$$

2. **Skewed right sided difference scheme**
   Number of nodes = p+q
   Where p=order of approximations
   q= order of derivative
   Here number of nodes = 4+2=6

$$\Delta x^2(\frac{\delta^2 f}{\delta x^2}) = af(i) + bf(i+1) + cf(i+2) + df(i+3) + ef(i+4) + gf(i+5)$$

Here,

$$af(i) = af(i)$$

$$bf(i+1) = bf(i) + (bf'(i)\Delta x)/(1!) + (bf''(i)\Delta x^2)/(2!) + (bf'''(i)\Delta x^3)/(3!) + (bf''''(i)\Delta x^4)/(4!) + (bf'''''(i)\Delta x^5)/(5!) + (bf''''''(i)\Delta x^6)/(6!)$$

$$cf(i+2) = cf(i) + (2cf'(i)\Delta x)/(1!) + (4cf''(i)\Delta x^2)/(2!) + (8cf'''(i)\Delta x^3)/(3!) + (16cf''''(i)\Delta x^4)/(4!) + (32cf'''''(i)\Delta x^5)/(5!) + (64cf''''''(i)\Delta x^6)/(6!)$$

$$df(i+3) = df(i) + (3df'(i)\Delta x)/(1!) + (9df''(i)\Delta x^2)/(2!) + (27df'''(i)\Delta x^3)/(3!) + (81df''''(i)\Delta x^4)/(4!) + (243df'''''(i)\Delta x^5)/(5!) + (729df''''''(i)\Delta x^6)/(6!)$$

$$ef(i+4) = ef(i) + (4ef'(i)\Delta x)/(1!) + (16ef''(i)\Delta x^2)/(2!) + (64ef'''(i)\Delta x^3)/(3!) + (256ef''''(i)\Delta x^4)/(4!) + (1024ef'''''(i)\Delta x^5)/(5!) + (4096ef''''''(i)\Delta x^6)/(6!)$$

$$gf(i+5) = ef(i) + (5gf'(i)\Delta x)/(1!) + (25gf''(i)\Delta x^2)/(2!) + (125gf'''(i)\Delta x^3)/(3!) + (625gf''''(i)\Delta x^4)/(4!) + (3125gf'''''(i)\Delta x^5)/(5!) + (15625gef''''''(i)\Delta x^6)/(6!)$$

Now let's form a Taylor's table.

	$f(i)$	$\Delta x\, f'(i)$	$\Delta x^2\, f''(i)$	$\Delta x^3\, f'''(i)$	$\Delta x^4\, f^{iv}(i)$	$\Delta x^5\, f^{v}(i)$	$\Delta x^6\, f^{vi}(i)$
$af(i)$	a	0	0	0	0	0	0
$bf(i+1)$	b	b	b/2	b/6	b/24	b/120	b/720
$cf(i+2)$	c	2c	2c	8c/6	16c/24	32c/120	64c/720
$df(i+3)$	d	3d	9d/2	27d/6	81d/24	243d/120	729d/720
$ef(i+4)$	e	4e	8e	64e/6	256e/24	1024e/120	4096e/720
$gf(i+5)$	g	5g	25g/2	125g/6	625g/24	3125g/120	15625g/720
$\Delta x \frac{(\delta^2 f)}{\delta x^2}$	0	0	1	0	0	?	?

$$\Delta x^2(\frac{\partial^2 f}{\partial x^2}) = (0 + b/2 + 2c + 9d/2 + 8e + 25g/2)f''(x)\Delta x^2 + (0 + b/720 + 64c/720 + 729d/720 + 4096e/720 + 15625g/720)f''''''(x)\Delta x^6$$

$$(\frac{\partial^2 f}{\partial x^2}) = (0 + b/2 + 2c + 9d/2 + 8e + 25g/2)f''(x) + (O)f''''''(x)\Delta x^4$$

The above equation is fourth order accurate for skewed right sided difference schemes.
The values of a,b,c,d,e are as derived using the following script:

%Matrix solving for srds
A2=[1 0 0 0 0 0; 1 1 1/2 1/6 1/24 1/120; 1 2 2 8/6 16/24 32/120; 1 3 9/2 27/6 81/24 243/120; 1 4 8 64/6 256/24 1024/120; 1 5 25/2 125/6 625/24 3125/120];

B2=A2';
C2=[0;0;1;0;0;0];
X2=B2\C2;

Output:

X2	double	6x1	[3.7500; -12.833; 17.833; -13.000; 5.0833; -0.83333]

3. **Skewed left sided difference scheme**
   Number of nodes = p+q
   Where p=order of approximations
   q= order of derivative
   Here number of nodes = 4+2=6

$$\Delta x^2\left(\frac{\delta^2 f}{\delta x^2}\right) = af(i) + bf(i-1) + cf(i-2) + df(i-3) + ef(i-4) + gf(i-5)$$

	$f(i)$	$\Delta x\, f'(i)$	$\Delta x^2\, f''(i)$	$\Delta x^3\, f'''(i)$	$\Delta x^4\, f^{iv}(i)$	$\Delta x^5\, f^{v}(i)$	$\Delta x^6\, f^{vi}(i)$
$gf(i-5)$	g	-5g	25g/2	-125g/6	625g/24	-3125g/120	15625g/720
$ef(i-4)$	e	-4e	8e	-64e/6	256e/24	-1024e/120	4096e/720
$cf(i-2)$	c	-2c	2c	-8c/6	16c/24	-32c/120	64c/720
$df(i-3)$	d	-3d	9d/2	-27d/6	81d/24	-243d/120	729d/720
$bf(i-1)$	b	-b	b/2	-b/6	b/24	-b/120	b/720
$af(i)$	a	0	0	0	0	0	0
$\Delta x^2\ \frac{(\delta^2 f)}{\delta x^2}$	0	0	1	0	0	?	?

$$gf(i-5) = ef(i) - (5gf'(i)\Delta x)/(1!) + (25gf''(i)\Delta x^2)/(2!) - (125gf'''(i)\Delta x^3)/(3!) + (625gf''''(i)\Delta x^4)/(4!) - (3125gf'''''(i)\Delta x^5)/(5!) + (15625gef''''''(i)\Delta x^6)/(6!)$$

$$ef(i-4) = ef(i) - (4ef'(i)\Delta x)/(1!) + (16ef''(i)\Delta x^2)/(2!) - (64ef'''(i)\Delta x^3)/(3!) + (256ef''''(i)\Delta x^4)/(4!) - (1024ef'''''(i)\Delta x^5)/(5!) + (4096ef''''''(i)\Delta x^6)/(6!)$$

$$df(i-3) = df(i) - (3df'(i)\Delta x)/(1!) + (9df''(i)\Delta x^2)/(2!) - (27df'''(i)\Delta x^3)/(3!) + (81df''''(i)\Delta x^4)/(4!) - (243df'''''(i)\Delta x^5)/(5!) + (729df''''''(i)\Delta x^6)/(6!)$$

$$cf(i-2) = cf(i) - (2cf'(i)\Delta x)/(1!) + (4cf''(i)\Delta x^2)/(2!) - (8cf'''(i)\Delta x^3)/(3!) + (16cf''''(i)\Delta x^4)/(4!) - (32cf'''''(i)\Delta x^5)/(5!) + (64cf''''''(i)\Delta x^6)/(6!)$$

$$bf(i-1) = bf(i) - (bf'(i)\Delta x)/(1!) + (bf''(i)\Delta x^2)/(2!) - (bf'''(i)\Delta x^3)/(3!) + (bf''''(i)\Delta x^4)/(4!) - (bf'''''(i)\Delta x^5)/(5!) + (bf''''''(i)\Delta x^6)/(6!)$$

$$af(i) = af(i)$$

$$\Delta x^2 (\frac{\partial^2 f}{\partial x^2}) = (0 + b/2 + 2c + 9d/2 + 8e + 25g/2) f''(x) \Delta x^2 + (0 + b/720 + 64c/720 + 729d/720 + 4096e/720 + 15625g/720) f''''''(x) \Delta x^6$$

$$(\frac{\partial^2 f}{\partial x^2}) = (0 + b/2 + 2c + 9d/2 + 8e + 25g/2) f''(x) + (O) f''''''(x) \Delta x^4$$

The above equation is fourth order accurate for skewed right sided difference schemes.
The values of a,b,c,d,e are as derived using the following script:

%Matrix solving for slds
A3=[1 -5 25/2 -125/6 625/24 -3125/120; 1 -4 8 -64/6 256/24 -1024/120;1 -3 9/2 -27/6 81/24 -243/120; 1 -2 2 -8/6 16/24 -32/120; 1 -1 1/2 -1/6 1/24 -1/120; 1 0 0 0 0 0];
B3=A3';
C3=[0;0;1;0;0;0];
*X3=inv(B3)*C3;*

Output:

X3	double	6x1	[-0.83333; 5.0833; -13.000; 17.833; -12.833; 3.7500]

Program:

Main Program to plot dx and error across all the 3 schemes:

```
clear all;
close all;
clc;

x=pi/3;
dx=linspace(pi/4,pi/400,100);


for i=1:length(dx)

error1(i)=cds(x,dx(i));

error2(i)=srds(x,dx(i));

error3(i)=slds(x,dx(i));

end


loglog(dx,error1,'g','Linewidth',3);
hold on
loglog(dx,error2,'b','Linewidth',3);
hold on
loglog(dx,error3,'r','Linewidth',3);
xlabel('dx')
ylabel('Truncation error')
title('Truncation error vs dx')
legend('cds','srds','slds')
```

Function Program for Central Difference Scheme:

```
function error1=cds(x,dx)

%analytical function=exp(x)*cos(x);
%analytical_derivative=-2*exp(x)*sin(x);
f2= -2*exp(x)*sin(x);


%Matrix solving for cds
```

```
A1=[1 -2 4/2 -8/6 16/24 ;1 -1 1/2 -1/6 1/24 ;1 0 0 0 0 ;1 1 1/2 1/6
1/24 ;1 2 4/2 8/6 16/24];
B1=A1';
C1=[0;0;1;0;0];
X1=B1\C1;

cds=
(X1(1)*exp(x-(2*dx))*cos(x-(2*dx))+X1(2)*exp(x-(1*dx))*cos(x-(1*dx))+
X1(3)*exp(x)*cos(x)+X1(4)*exp(x+(1*dx))*cos(x+(1*dx))+X1(5)*exp(x+(2*
dx))*cos(x+(2*dx)))/(dx)^2;
error1=abs(cds-f2);

end
```

Function Program for Skewed Right Difference Scheme:

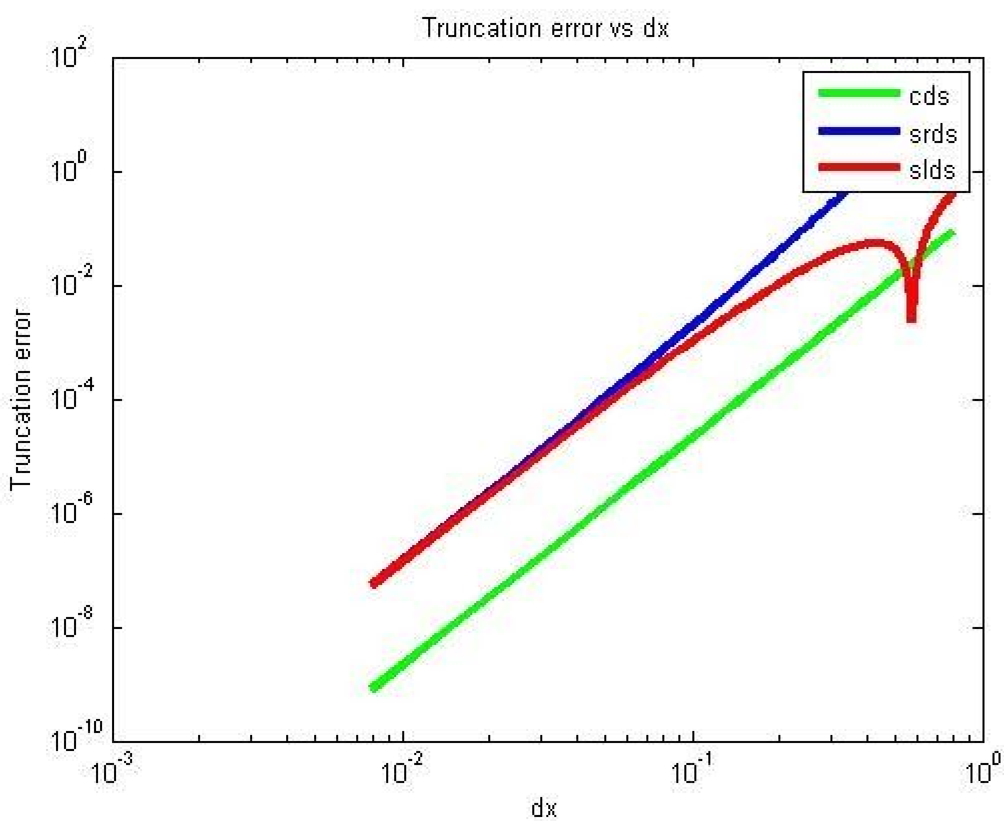
```
function error2=srds(x,dx)

%analytical function=exp(x)*cos(x);
%analytical_derivative=-2*exp(x)*sin(x);
f2= -2*exp(x)*sin(x);


%Matrix solving for srds
A2=[1 0 0 0 0 0; 1 1 1/2 1/6 1/24 1/120; 1 2 2 8/6 16/24 32/120; 1 3
9/2 27/6 81/24 243/120; 1 4 8 64/6 256/24 1024/120; 1 5 25/2 125/6
625/24 3125/120];
B2=A2';
C2=[0;0;1;0;0;0];
X2=inv(B2)*C2;

srds=(X2(1)*exp(x)*cos(x)+X2(2)*exp(x+1*dx)*cos(x+1*dx)+X2(3)*exp(x+2
*dx)*cos(x+2*dx)+X2(4)*exp(x+3*dx)*cos(x+3*dx)+X2(5)*exp(x+4*dx)*cos(
x+4*dx)+X2(6)*exp(x+5*dx)*cos(x+5*dx))/(dx)^2;
error2=abs(srds-f2);

end
```

Function Program for Skewed Left Difference Scheme:

```
function error3=slds(x,dx)

f2= -2*exp(x)*sin(x);
```

```
%Matrix solving for slds
A3=[1 -5 25/2 -125/6 625/24 -3125/120; 1 -4 8 -64/6 256/24
-1024/120;1 -3 9/2 -27/6 81/24 -243/120; 1 -2 2 -8/6 16/24 -32/120; 1
-1 1/2 -1/6 1/24 -1/120; 1 0 0 0 0 0 ];
B3=A3';
C3=[0;0;1;0;0;0];
X3=inv(B3)*C3;

slds=(X3(1)*
exp(x-5*dx)*cos(x-5*dx)+X3(2)*exp(x-4*dx)*cos(x-4*dx)+X3(3)*exp(x-3*d
x)*cos(x-3*dx) +X3(4) *exp(x-2*dx)*cos(x-2*dx)+X3(5)
*exp(x-1*dx)*cos(x-1*dx)+X3(6)*exp(x)*cos(x))/(dx)^2;
error3=abs(slds-f2);

end
```

Result:



Inference:

- All the schemes are fourth order accurate which is proved in the end of all 3 derivations
- From the above plot, it is evident that the central difference scheme has the least truncation error.The reason for that is it takes information from both the sides.
- Skewed Schemes both right and left had similar amounts of error at the initial points whereas as dx increased, the error of the skewed left order scheme dropped below the central scheme and raised again significantly. This may be due to the round off error which arises when the dx is increased.

Why Skewed schemes helpful?

Whenever there is no information about the left and right side nodes, skewed schemes come to the rescue. Skewed right order can be used whenever there is no information about the left nodes and vice versa. This is the advantage skewed schemes have over central difference schemes at the trade off of truncation error.