Discretization basics

Objective: To compare the first, second and fourth order approximations of the first derivative against the analytical derivative.

Given:

- Function, f(x) = sin(x)/x^3 at x = pi/3

- Formula for the fourth order approximation of the first derivative

$$\frac{(f_{j-2} - 8 * f_{j-1} + 8 * f_{j+1} - f_{j+2})}{12dx}$$

Solution:

Analytical derivative of the given function is

$$\frac{x^3 \cdot (\cos(x)) - \sin(x) \cdot 3 \cdot x^2}{x^6}$$

First order approximation (Numerical Derivative) is

$$\frac{\left(\frac{\sin(x+dx)}{(x+dx)^3}\right)-\left(\frac{\sin(x)}{x^3}\right)}{dx}$$

Second order approximation (Numerical Derivative) is

$$\frac{\left(\frac{\sin(x+dx)}{(x+dx)^3}\right)-\left(\frac{\sin(x-dx)}{(x-dx)^3}\right)}{2\cdot dx}$$

Fourth order approximation (Numerical Derivative) is

$$((sin(x-2*dx)/(x-2*dx)^3)-8*(sin(x-1*dx)/(x-1*dx)^3)+8*(sin(x+1*dx)/(x+1*dx)^3)-(sin(x+2*dx)/(x+2*dx)^3))/(12*dx)$$

Program:

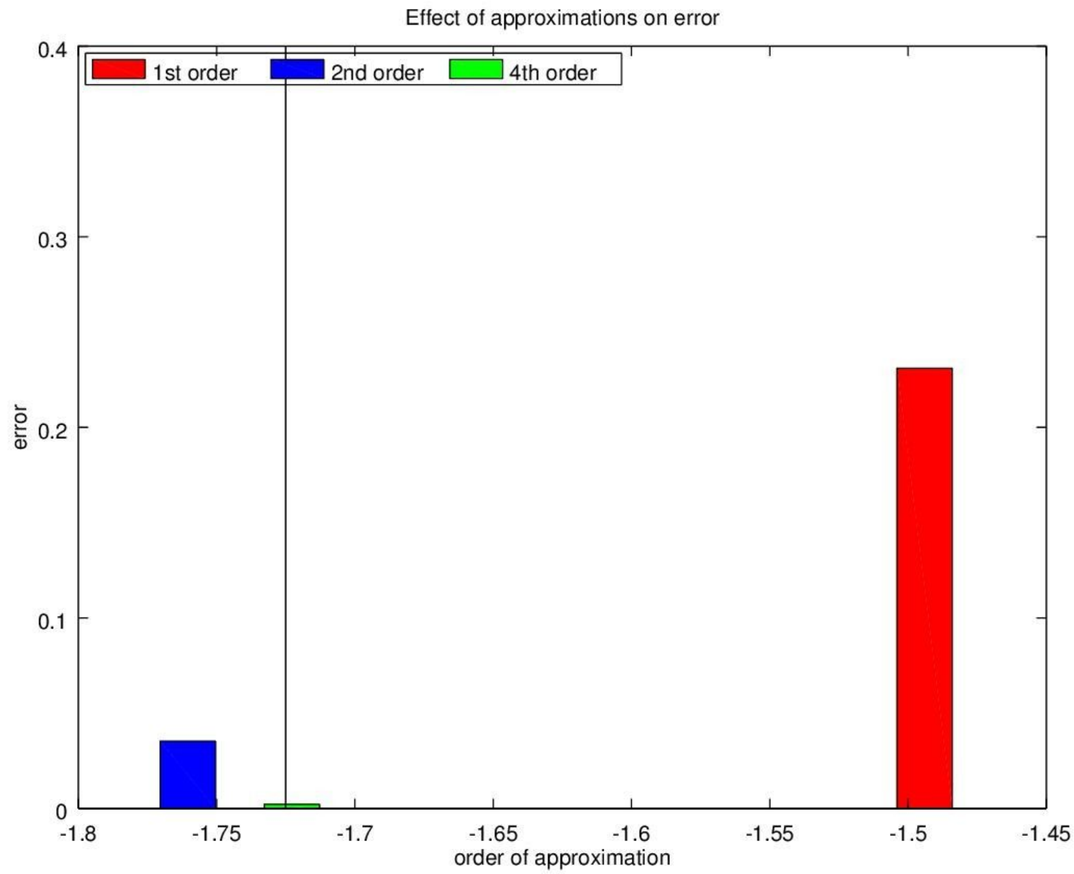
```
%function = sin(x)/x^3;
x=pi/3;
dx=pi/30;

%analytical derivative:
analytical_derivative=(x^3*(cos(x))-sin(x)*3*x^2)/x^6;

%numerical derivative-forward differencing
%first order approx
forward_diff=((sin(x+dx)/(x+dx)^3)-(sin(x)/x^3))/dx;
error_firstorder=abs(forward_diff-analytical_derivative);

%second order approx
central_diff=((sin(x+dx)/(x+dx)^3)-(sin(x-dx)/(x-dx)^3))/(2*dx);
error_secondorder=abs(central_diff-analytical_derivative);

%fourth order approx
central_diff2=((sin(x-2*dx)/(x-2*dx)^3)-8*(sin(x-1*dx)/(x-1*dx)^3)+8*(si
n(x+1*dx)/(x+1*dx)^3)-(sin(x+2*dx)/(x+2*dx)^3))/(12*dx);
error_fourthorder=abs(central_diff2-analytical_derivative);

%plotting;
bar(forward_diff,error_firstorder,0.01,'r');
hold on;
bar(central_diff,error_secondorder,0.01,'b');
hold on;
bar(central_diff2,error_fourthorder,0.01,'g');
```

```
legend({'1st order', '2nd order','4th order'},'Location','northwest','Orientation','horizontal');
hold on;
bar(analytical_derivative,0.4,0.0000001,'y');
xlabel('order of approximation');
ylabel('error');
title('Effect of approximations on error')
```

Result:

Name	Class	Dimension	Value
analytical_derivative	double	1x1	-1.7250
central_diff	double	1x1	-1.7604
central_diff2	double	1x1	-1.7228
dx	double	1x1	0.10472
error_firstorder	double	1x1	0.23102
error_fourthorder	double	1x1	0.0022369
error_secondorder	double	1x1	0.035370
forward_diff	double	1x1	-1.4940
x	double	1x1	1.0472

Above table shows the results of the program in which the analytical derivative corresponds to -1.7250. Same value is plotted in the graph as a single vertical line for better understanding to compare with the numerical approximations.



Inference:

- The plot shows that the error is decreased as the order of approximation is increased.

- The single vertical line at -1.7250 correlates that the fourth order approximation is more accurate to the analytical value.