Self Study

Based on my Understanding of the Subject

Functional Analysis

Compilation of my study materials

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Hahn-Banach Theorems

2.1 Helly, Hahn-Banach Analytic Form

The theorem is named for the mathematicians Hans Hahn and Stefan Banach, who proved it independently in the late 1920s. The special case of the theorem for the space $C[a,b]$ of continuous functions on an interval was proved earlier (in 1912) by Eduard Helly, and a more general extension theorem, the M. Riesz extension theorem, from which the Hahn–Banach theorem can be derived, was proved in 1923 by Marcel Riesz.

Definition 1: Sublinear Function

Let X be a vector space over a field $\mathbb{K}$, where $\mathbb{K}$ is either the real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. A real-valued function $p : X \to \mathbb{R}$ on X is called a sublinear function (or a sublinear functional if $\mathbb{K} = \mathbb{R}$), and also sometimes called a quasi-seminorm or a Banach functional, if it has these two properties:

1. **Positive homogeneity/Nonnegative homogeneity**: $p(rx) = rp(x)$ for all real $r \geq 0$ and all $x \in X$. This condition holds if and only if $p(rx) \leq rp(x)$ for all positive real $r > 0$ and all $x \in X$.
2. **Subadditivity/Triangle inequality**: $p(x+y) \leq p(x)+p(y)$ for all $x, y \in X$. This subadditivity condition requires p to be real-valued.

A real-valued function $f : M \to \mathbb{R}$ defined on $M \subseteq X$ is said to be dominated (above) by a function $p : X \to \mathbb{R}$ if $f(m) \leq p(m)$ for every $m \in M$. Hence the reason why the following version of the Hahn-Banach theorem is called the dominated extension theorem [1].

Theorem 2.1.1: Helly, Hahn-Banach Analytic Form

If $p : X \to \mathbb{R}$ is a sublinear function (such as a norm or seminorm for example) defined on a real vector space X then any linear functional defined on a vector subspace of X that is dominated above by p has at least one linear extension to all of X that is also dominated above by p. In other words, any linear functional defined on a vector subspace of X that is dominated above by a sublinear function $p : X \to \mathbb{R}$ has at least one linear extension to all of X that is also dominated above by p.

Explicitly, if $p : X \to \mathbb{R}$ is a sublinear function, which by definition means that it satisfies

$$p(x + y) \leq p(x) + p(y) \quad \text{and} \quad p(tx) = tp(x) \quad \forall\, x, y \in X \quad \text{and} \quad \forall t \in \mathbb{R}_{\geq 0},$$

and if $f : M \to \mathbb{R}$ is a linear functional defined on a vector subspace M of X such that

$$f(m) \leq p(m) \quad \text{for all } m \in M$$

then there exists a linear functional $F : X \to \mathbb{R}$ such that

$$F(m) = f(m) \quad \text{for all } m \in M,$$

$$F(x) \leq p(x) \quad \text{for all } x \in X.$$

Moreover, if p is a seminorm then $|F(x)| \leq p(x)$ necessarily holds for all $x \in X$.

Proof of theorem 2.1.1. The proof of depends on Zorn's lemma. ■

Bibliography

[1] Wikipedia contributors. *Hahn–Banach theorem — Wikipedia, The Free Encyclopedia.* [Online; accessed 24-May-2022]. 2022. URL: https://en.wikipedia.org/w/index.php?title=Hahn%E2%80%93Banach_theorem&oldid=1089256573 (page - 2).