Uva Wellassa University

SIMPLIFICATION OF BOOLEAN EXPRESSIONS

Dr. Suneth Pathirana

B.Sc.(UWU-CST)(Hons), M.Sc.(AI)(Moratuwa), MIEEE, Ph.D.(MSU)

Senior Lecturer

Department of Computer Science & Informatics

Uva Wellassa University of Sri Lanka

suneth@uwu.ac.lk

Basic Rules for Boolean Algebra

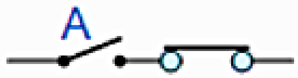
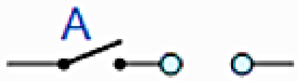
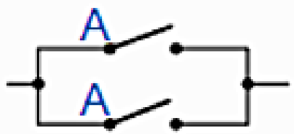
Basic Rules of Boolean Algebra

1. $A + 0 = A$	7. $A \cdot A = A$
2. $A + 1 = 1$	8. $A \cdot \overline{A} = 0$
3. $A \cdot 0 = 0$	9. $\overline{\overline{A}} = A$
4. $A \cdot 1 = A$	10. $A + AB = A$
5. $A + A = A$	11. $A + \overline{A}B = A + B$
6. $A + \overline{A} = 1$	12. $(A + B)(A + C) = A + BC$

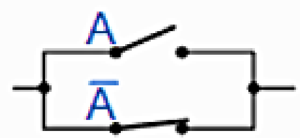
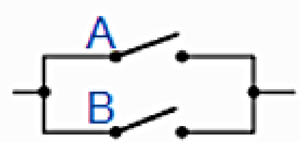
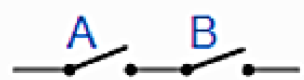
DeMorgan's Theorem

$$\overline{(AB)} = (\overline{A} + \overline{B}) \qquad \overline{(A + B)} = (\overline{A}\ \overline{B})$$

Explained: in Detail

Boolean Expression	Description	Equivalent Switching Circuit	Boolean Algebra Law or Rule
A + 1 = 1	A in parallel with closed = "CLOSED"		Annulment
A + 0 = A	A in parallel with open = "A"		Identity
A . 1 = A	A in series with closed = "A"		Identity
A . 0 = 0	A in series with open = "OPEN"		Annulment
A + A = A	A in parallel with A = "A"		Idempotent
A . A = A	A in series with A = "A"		Idempotent

Explained: in Detail

NOT $\overline{A} = A$	NOT NOT A (double negative) = "A"		Double Negation
$A + \overline{A} = 1$	A in parallel with NOT A = "CLOSED"		Complement
$A . \overline{A} = 0$	A in series with NOT A = "OPEN"		Complement
A+B = B+A	A in parallel with B = B in parallel with A		Commutative
A.B = B.A	A in series with B = B in series with A		Commutative
$\overline{A+B} = \overline{A}.\overline{B}$	invert and replace OR with AND		de Morgan's Theorem
$\overline{A.B} = \overline{A}+\overline{B}$	invert and replace AND with OR		de Morgan's Theorem

Comprehensive Version Ahead!

1.	Law of Identity	$A = A$ $\overline{A} = \overline{A}$
2.	Commutative Law	$A \cdot B = B \cdot A$ $A + B = B + A$
3.	Associative Law	$A \cdot (B \cdot C) = A \cdot B \cdot C$ $A + (B + C) = A + B + C$
4.	Idempotent Law	$A \cdot A = A$ $A + A = A$
5.	Double Negative Law	$\overline{\overline{A}} = A$
6.	Complementary Law	$A \cdot \overline{A} = 0$ $A + \overline{A} = 1$
7.	Law of Intersection	$A \cdot 1 = A$ $A \cdot 0 = 0$
8.	Law of Union	$A + 1 = 1$ $A + 0 = A$
9.	DeMorgan's Theorem	$\overline{AB} = \overline{A} + \overline{B}$ $\overline{A + B} = \overline{A}\,\overline{B}$
10.	Distributive Law	$A \cdot (B + C) = (A \cdot B) + (A \cdot C)$ $A + (BC) = (A + B) \cdot (A + C)$
11.	Law of Absorption	$A \cdot (A + B) = A$ $A + (AB) = A$
12.	Law of Common Identities	$A \cdot (\overline{A} + B) = AB$ $A + (\overline{A}B) = A + B$

Simplifications

- Using the above laws, simplify the following expression:
- $(A + B)(A + C)$

Simplifications

$Q =$	$(A + B).(A + C)$	
	$A.A + A.C + A.B + B.C$	– Distributive law
	$A + A.C + A.B + B.C$	– Idempotent AND law ($A.A = A$)
	$A(1 + C) + A.B + B.C$	– Distributive law
	$A.1 + A.B + B.C$	– Identity OR law ($1 + C = 1$)
	$A(1 + B) + B.C$	– Distributive law
	$A.1 + B.C$	– Identity OR law ($1 + B = 1$)
$Q =$	$A + (B.C)$	– Identity AND law ($A.1 = A$)

Simplifications

- Simplify the following:
  - $C + \overline{BC}$
  - $\overline{AB}(\overline{A} + B)(\overline{B} + B)$
  - $(A + C)(AD + A\overline{D}) + AC + C$
  - $\overline{A}(A + B) + (B + AA)(A + \overline{B})$

- $C + \overline{BC}$

$$\overline{AB}(\overline{A} + B)(\overline{B} + B)$$

- $(A + C)(AD + A\overline{D}) + AC + C$

$$\overline{A}(A + B) + (B + AA)(A + \overline{B})$$

Answers

Simplify: $C + \overline{BC}$:

Expression	Rule(s) Used
$C + \overline{BC}$	Original Expression
$C + (\overline{B} + \overline{C})$	DeMorgan's Law.
$(C + \overline{C}) + \overline{B}$	Commutative, Associative Laws.
$T + \overline{B}$	Complement Law.
T	Identity Law.

Answers

Simplify: $\overline{AB}(\overline{A} + B)(\overline{B} + B)$:

Expression	Rule(s) Used
$\overline{AB}(\overline{A} + B)(\overline{B} + B)$	Original Expression
$\overline{AB}(\overline{A} + B)$	Complement law, Identity law.
$(\overline{A} + \overline{B})(\overline{A} + B)$	DeMorgan's Law
$\overline{A} + \overline{B}B$	Distributive law. This step uses the fact that or distributes over and. It can look a bit strange since addition does not distribute over multiplication.
$\overline{A}$	Complement, Identity.

Answers

Simplify: $(A + C)(AD + A\overline{D}) + AC + C$:

Expression	Rule(s) Used
$(A + C)(AD + A\overline{D}) + AC + C$	Original Expression
$(A + C)A(D + \overline{D}) + AC + C$	Distributive.
$(A + C)A + AC + C$	Complement, Identity.
$A((A + C) + C) + C$	Commutative, Distributive.
$A(A + C) + C$	Associative, Idempotent.
$AA + AC + C$	Distributive.
$A + (A + T)C$	Idempotent, Identity, Distributive.
$A + C$	Identity, twice.

Answers

Simplify: $\overline{A}(A + B) + (B + AA)(A + \overline{B})$:

Expression	Rule(s) Used
$\overline{A}(A + B) + (B + AA)(A + \overline{B})$	Original Expression
$\overline{A}A + \overline{A}B + (B + A)A + (B + A)\overline{B}$	Idempotent (AA to A), then Distributive, used twice.
$\overline{A}B + (B + A)A + (B + A)\overline{B}$	Complement, then Identity. (Strictly speaking, we also used the Commutative Law for each of these applications.)
$\overline{A}B + BA + AA + B\overline{B} + A\overline{B}$	Distributive, two places.
$\overline{A}B + BA + A + A\overline{B}$	Idempotent (for the A's), then Complement and Identity to remove $B\overline{B}$.
$\overline{A}B + AB + AT + A\overline{B}$	Commutative, Identity; setting up for the next step.
$\overline{A}B + A(B + T + \overline{B})$	Distributive.
$\overline{A}B + A$	Identity, twice (depending how you count it).
$A + \overline{A}B$	Commutative.
$(A + \overline{A})(A + B)$	Distributive.
$A + B$	Complement, Identity.

Karnaugh Maps

Karnaugh maps

- The Karnaugh map provides a simple and straight-forward method of minimising boolean expressions
- With the Karnaugh map Boolean expressions having up to four and even six variables can be simplified

Karnaugh maps

- A Karnaugh map provides a pictorial method of grouping together expressions with common factors and therefore eliminating unwanted variables
- The Karnaugh map can also be described as a special arrangement of a truth table

Karnaugh maps

- The correspondence between the Karnaugh map and the truth table for the general case of a two variable problem

A	B	F
0	0	a
0	1	b
1	0	c
1	1	d

Truth Table.

B \ A	0	1
0	a	b
1	c	d

F.

Karnaugh maps

- Example

A	B	F
0	0	0
0	1	1
1	0	1
1	1	1

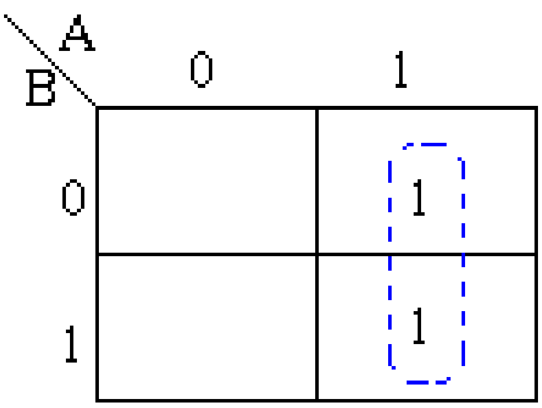
Truth Table.

B \ A	0	1
0	0	1
1	1	1

F.

Using Karnaugh maps for simplifications

- Example:
  - $Z = A\overline{B} + AB$
- Using algebraic simplification,
  - $Z = A(\overline{B} + B)$
  - $Z = A$
- Consider the following map:
- Referring to the map above, the two adjacent 1's are grouped together



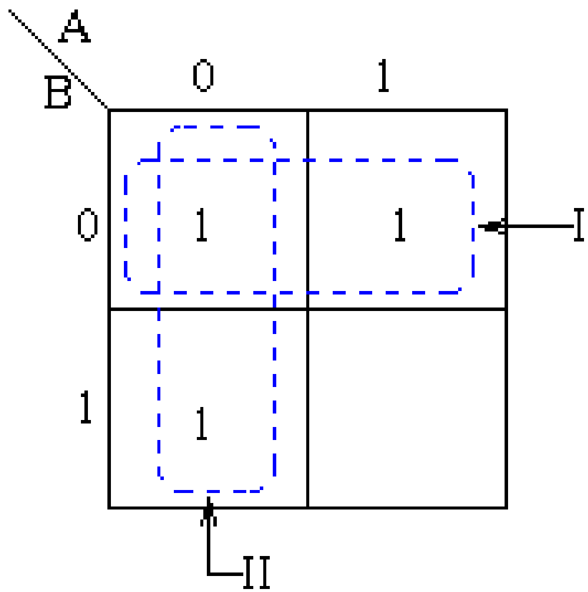
Using Karnaugh maps for simplifications

B \ A	0	1
0		1
1		1

- Through inspection it can be seen that variable B has its true and false form within the group
- This eliminates variable B leaving only variable A which only has its true form
- The minimised answer therefore is Z = A

Example 2

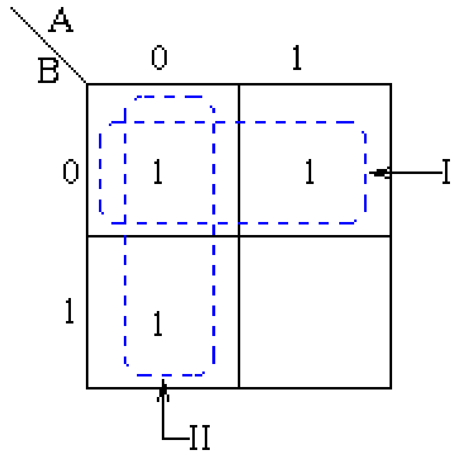
- Consider the expression $Z = \overline{A}.\overline{B} + A\overline{B} + \overline{A}B$



- Pairs of 1's are *grouped* as shown above, and the simplified answer is obtained by using the following steps:
- Notice that a 1 can belong to more than one group

Example 2

- $Z = \overline{A}.\overline{B} + A\overline{B} + \overline{A}B$

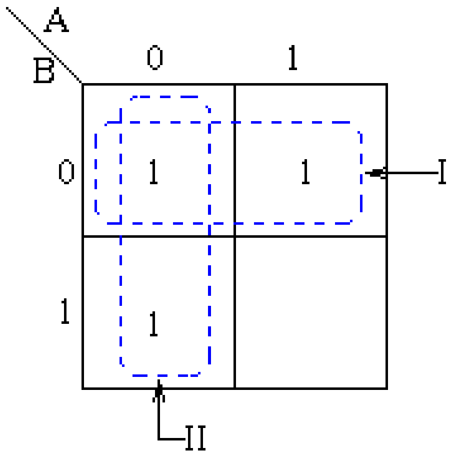


- The first group labelled I, consists of two 1s which correspond to

  A = 0, B = 0 and A = 1, B = 0
- Put in another way, all squares that correspond to the area of the map where B = 0 contains 1s, independent of the value of A
- So **when B = 0 the output is 1**
- The expression of the **output will contain** the term $\overline{B}$

Example 2

- $Z = \overline{A}.\overline{B} + A\overline{B} + \overline{A}B$



- For group labelled II corresponds to the area of the map where A = 0
- The group can therefore be defined as $\overline{A}$
  - This implies that when A = 0 the output is 1
- The **output is** therefore **1 whenever B = 0 and A = 0**
- Hence the simplified answer is Z = $\overline{\mathbf{A}} + \overline{\mathbf{B}}$

Examples with more than 2 variables

Let's start with simple…

A	B	C	Z
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Examples with more than 2 variables

Let's start with simple…

A	B	C	Z
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

A \ BC	00	01	11	10
0	0	0	1	0
1	0	1	1	1

Examples with more than 2 variables

Let’s start with simple…

A	B	C	Z
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

A \ BC	00	01	11	10
0	0	0	1	0
1	0	1	1	1

$A.\overline{B}.C$

Examples with more than 2 variables

Let’s start with simple…

A	B	C	Z
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

A \ BC	00	01	11	10
0	0	0	1	0
1	0	1	1	1

$A.\overline{B}.C$ $A.B.C$

Examples with more than 2 variables

Let's start with simple…

A	B	C	Z
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

A \ BC	00	01	11	10
0	0	0	1	0
1	0	1	1	1

$$\underbrace{A.\overline{B}.C \qquad A.B.C}_{A.C}$$

Examples with more than 2 variables

Let's start with simple…

A	B	C	Z
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

A \ BC	00	01	11	10
0	0	0	1	0
1	0	1	1	1

Accordingly…. A.B

Examples with more than 2 variables

Let's start with simple…

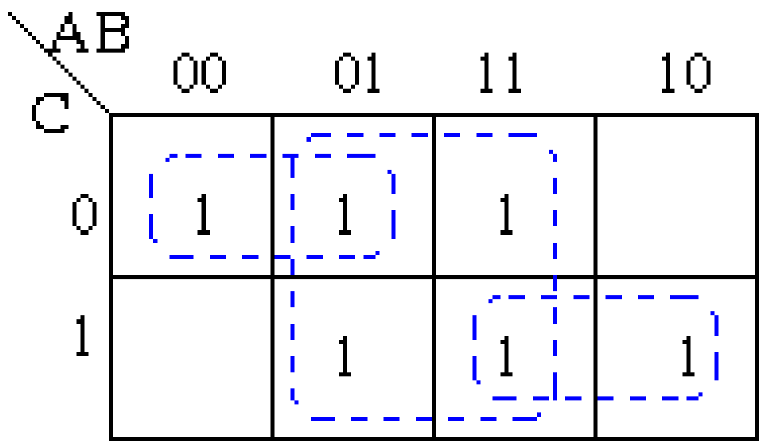
A	B	C	Z
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

A \ BC	00	01	11	10
0	0	0	1	0
1	0	1	1	1

B.C

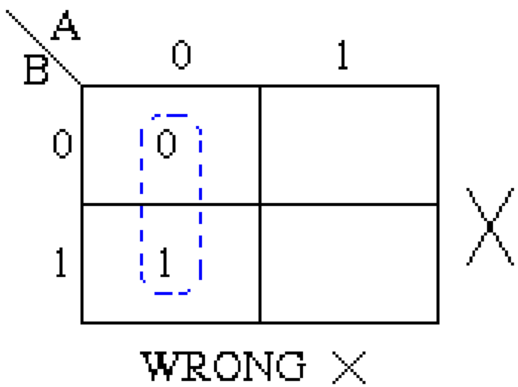
Examples with more than 2 variables

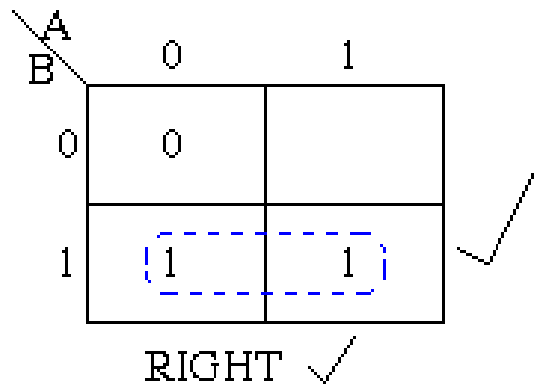
- $Z = \overline{A}.\overline{B}.\overline{C} + \overline{A}.B + AB\overline{C} + AC$



Karnaugh Maps - Rules of Simplification

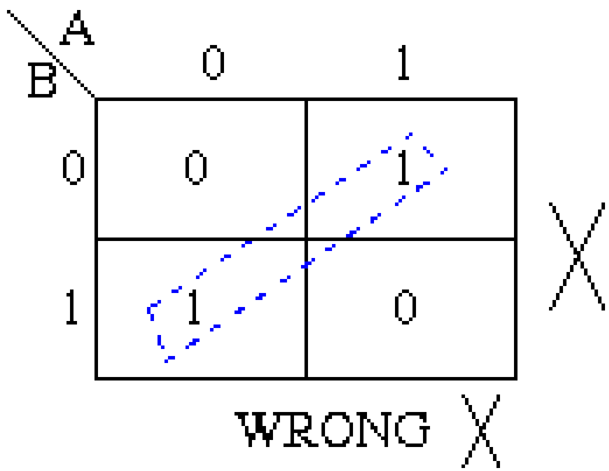
- The Karnaugh map uses the following rules for the simplification of expressions by grouping together adjacent cells containing ones
  - Groups may not include any cell containing a zero

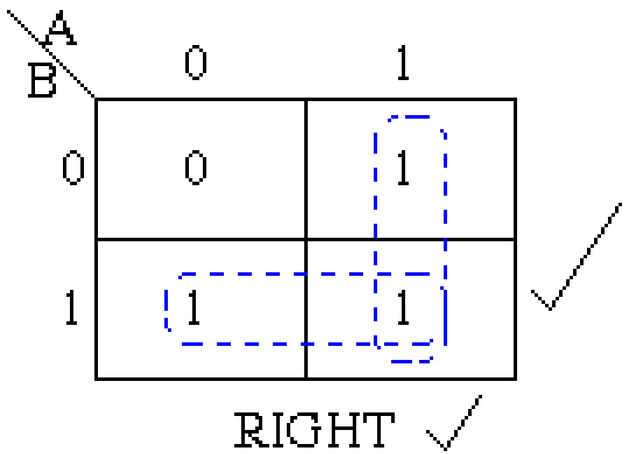




Karnaugh Maps - Rules of Simplification

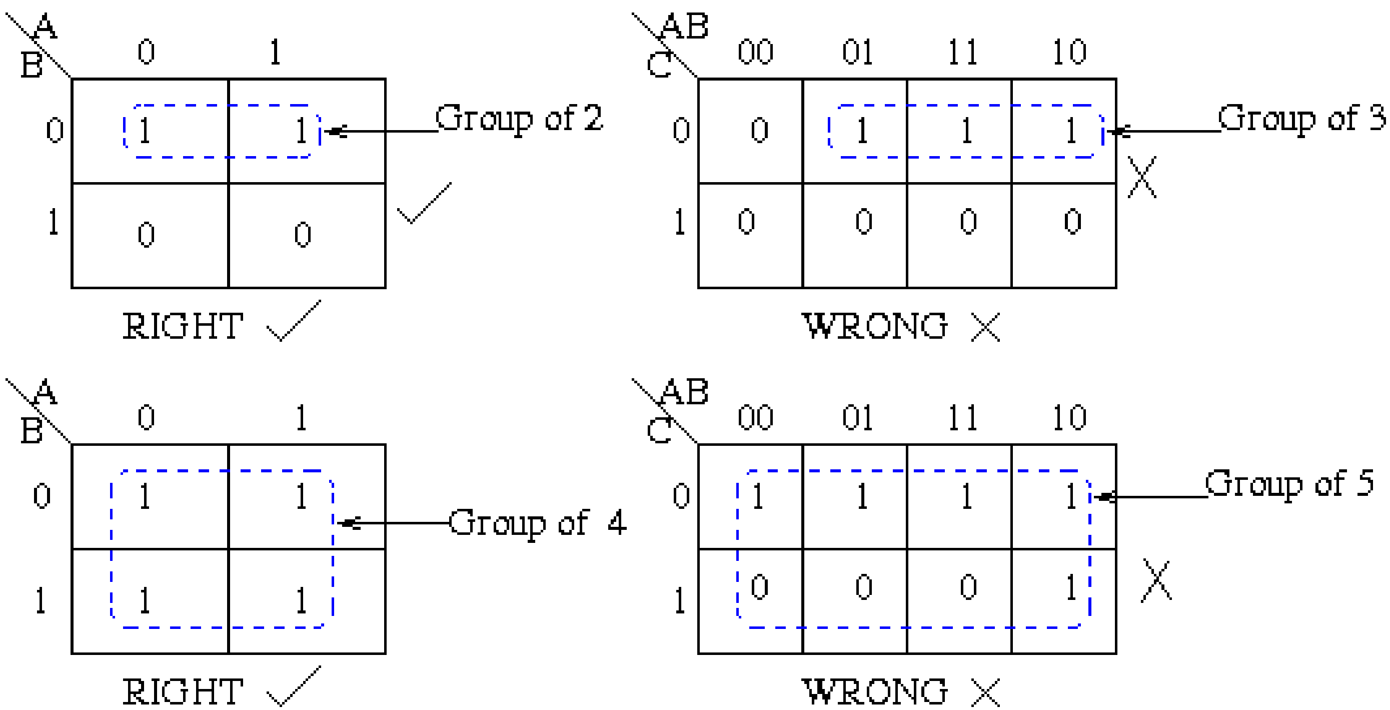
- Groups may be horizontal or vertical, but not diagonal





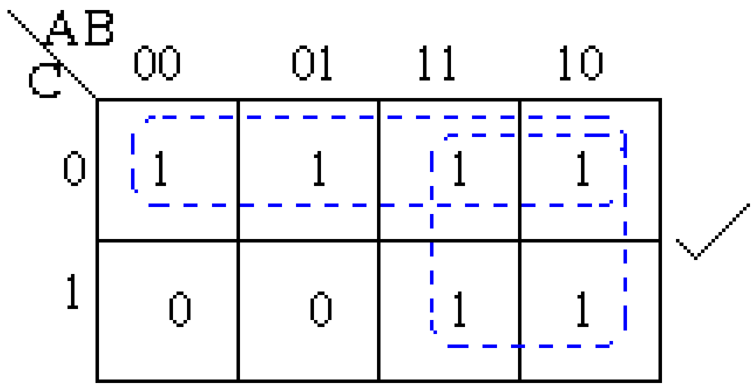
Karnaugh Maps - Rules of Simplification

- Groups must contain 1, 2, 4, 8, or in general 2^n cells
  - That is if n = 1, a group will contain two 1's since $2^1 = 2$
  - If n = 2, a group will contain four 1's since $2^2 = 4$

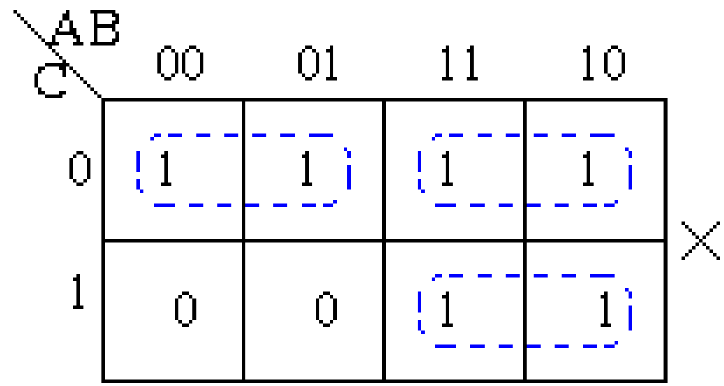


Karnaugh Maps - Rules of Simplification

- Each group should be as large as possible



RIGHT ✓

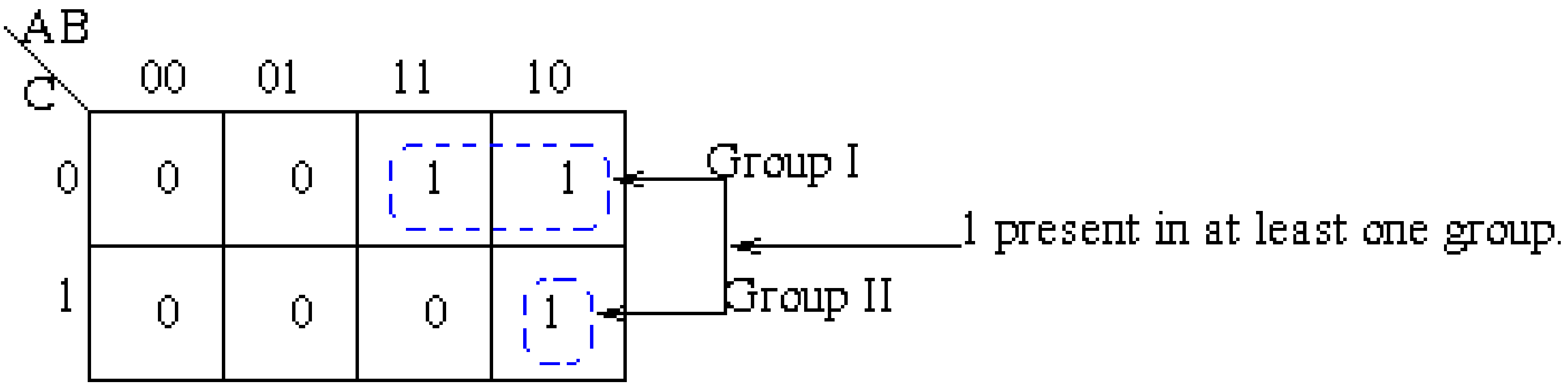


WRONG ×

(Note that no Boolean laws broken, but not sufficiently minimal)

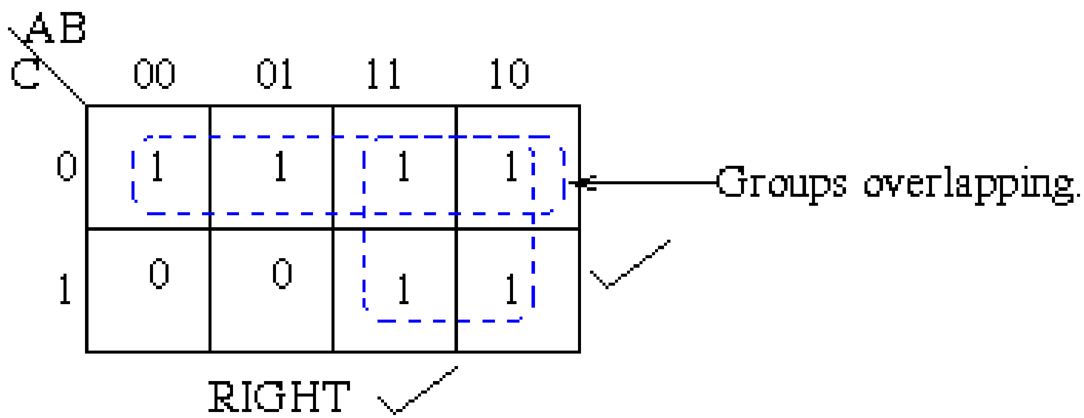
Karnaugh Maps - Rules of Simplification

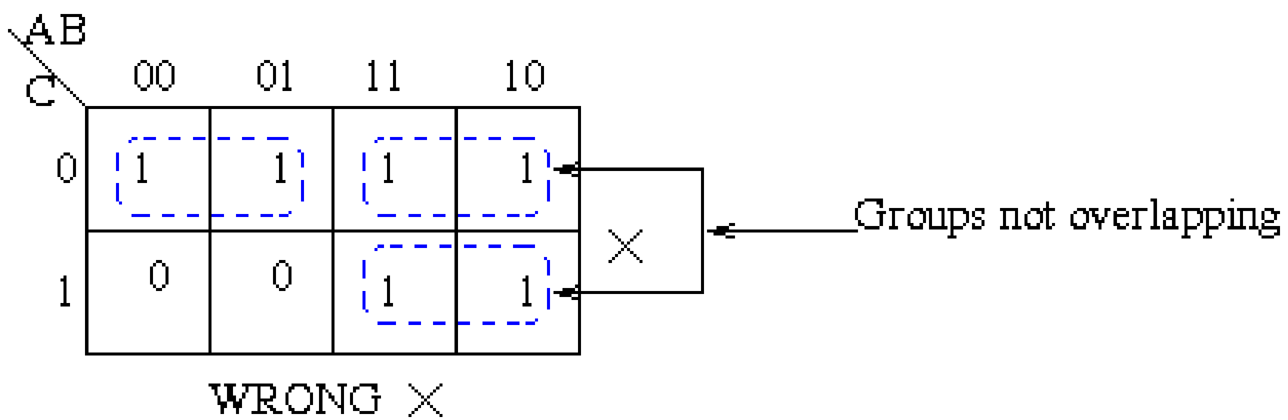
- Each cell containing a one must be in at least one group



Karnaugh Maps - Rules of Simplification

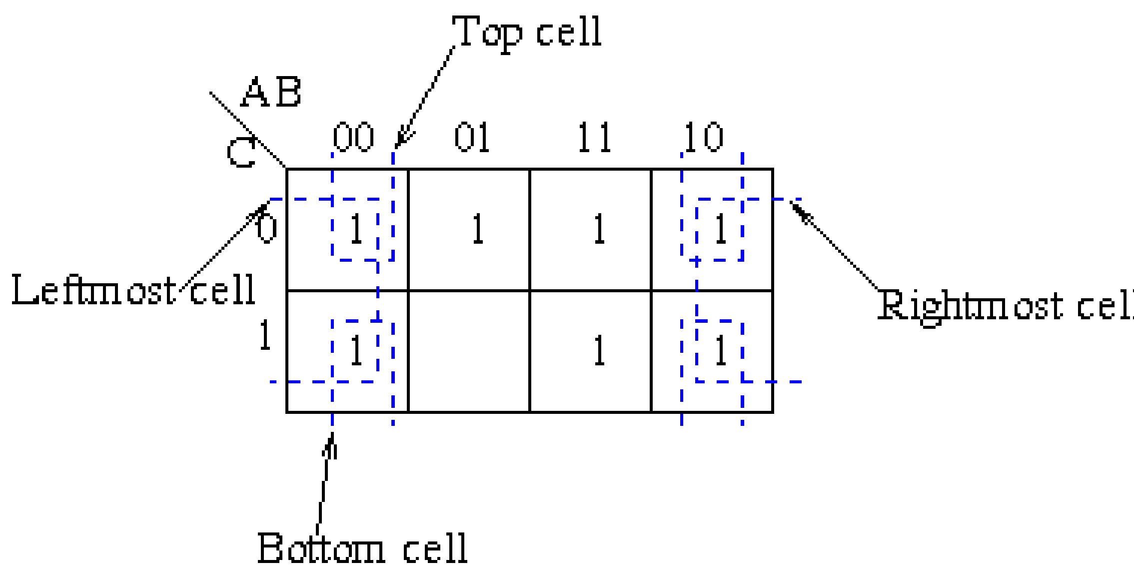
- Groups may overlap





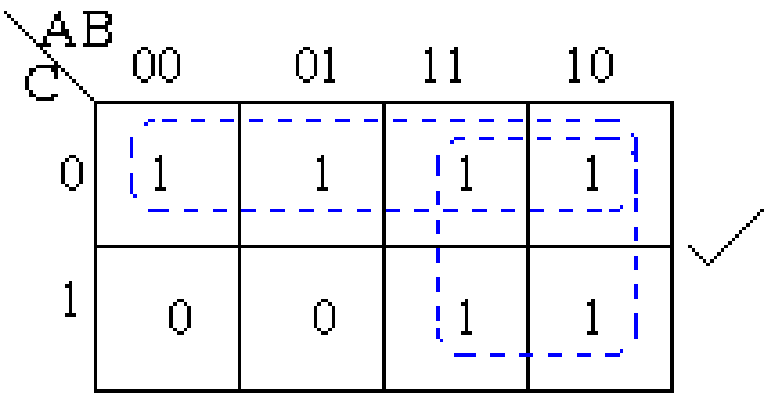
Karnaugh Maps - Rules of Simplification

- Groups may wrap around the table
- The leftmost cell in a row may be grouped with the rightmost cell and the top cell in a column may be grouped with the bottom cell

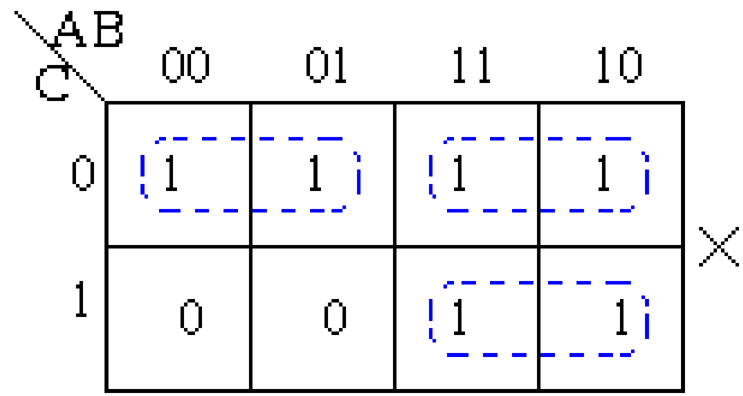


Karnaugh Maps - Rules of Simplification

- There should be as few groups as possible, as long as this does not contradict any of the previous rules



RIGHT ✓



WRONG ✗

Summary

- No zeros allowed
- No diagonals
- Only power of 2 number of cells in each group
- Groups should be as large as possible
- Every one (1) must be in at least one group
- Overlapping allowed
- Wrap around allowed
- Fewest number of groups possible

Examples

A \ BC	00	01	11	10
0	1	0	0	0
1	1	1	0	1

$A.\overline{B}$

Examples

A \ BC	00	01	11	10
0	1	0	0	0
1	1	1	0	1

$$\overline{B}.\overline{C}$$

Examples

A \ BC	00	01	11	10
0	1	0	0	0
1	1	1	0	1

$A.\overline{C}$

Examples with more than 2 variables

- $Z = \overline{A}.B + B.\overline{C} + B.C + A.\overline{B}.\overline{C}$

C \ AB	00	01	11	10
0		1	1	1
1		1	1	

Examples with more than 2 variables

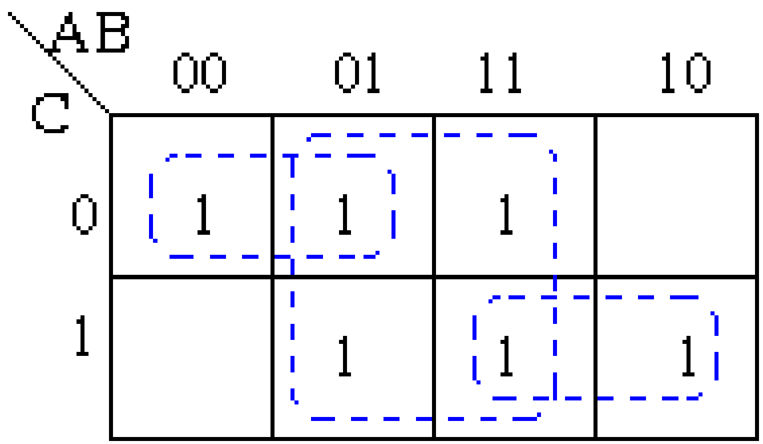
- $Z = \overline{A}.B + B.\overline{C} + B.C + A.\overline{B}.\overline{C}$

C \ AB	00	01	11	10
0		1	1	1
1		1	1	

- The minimized result obtained is $B + A\overline{C}$

Examples with more than 2 variables

- $Z = \overline{A}.\,\overline{B}.\,\overline{C} + \overline{A}.B + AB\overline{C} + AC$



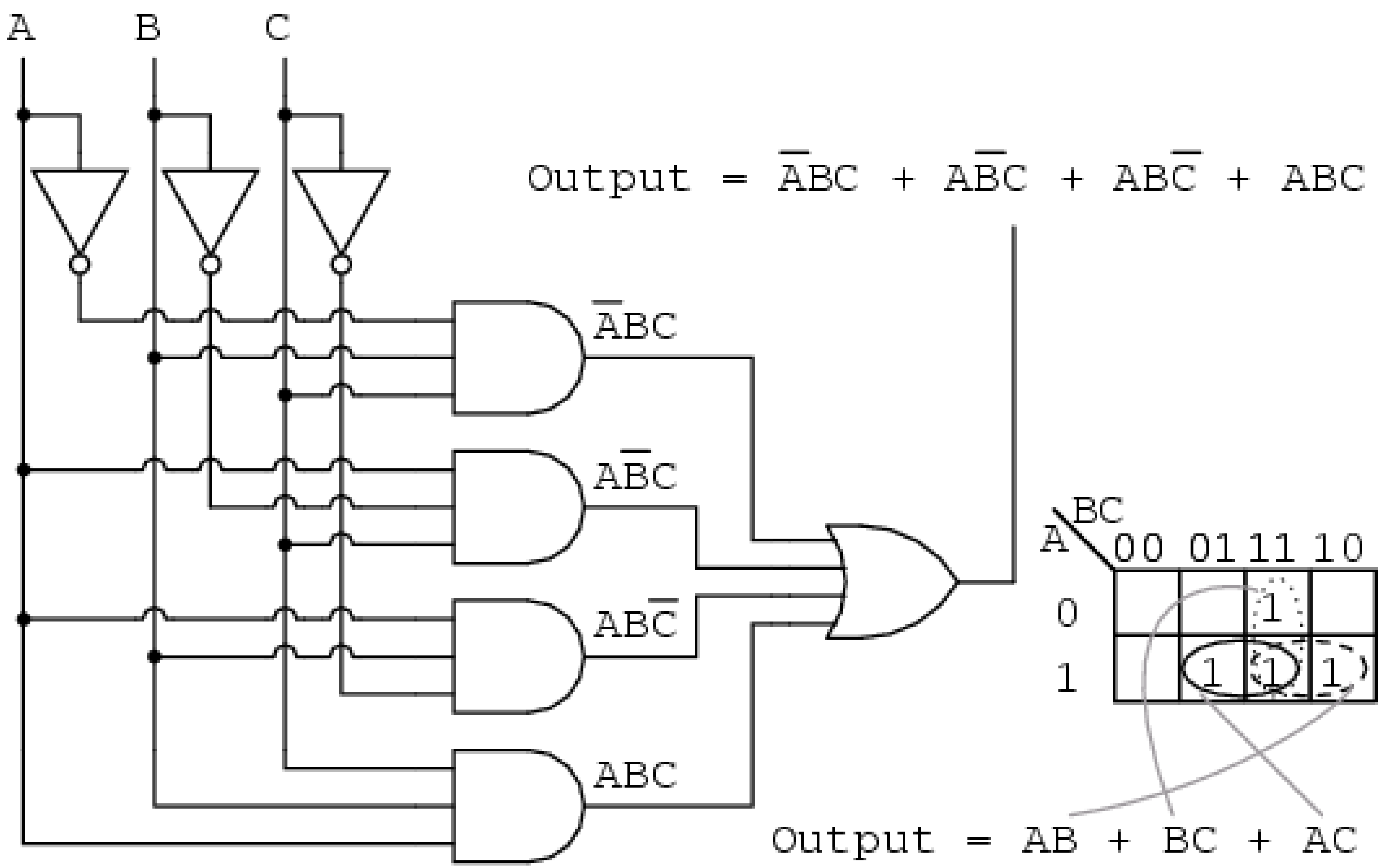
Examples with more than 2 variables

- $Z = \overline{A}.\,\overline{B}.\,\overline{C} + \overline{A}.B + AB\overline{C} + AC$

C \ AB	00	01	11	10
0	1	1	1	
1		1	1	1

- The minimized result obtained is $B + AC + \overline{A}.\,\overline{C}$

Examples with 3 variables



Examples with 4 variables

AB \ CD	00	01	11	10
00	1	0	0	1
01	1	1	0	0
11	1	0	0	0
10	1	0	0	1

Examples with 4 variables

AB \ CD	00	01	11	10
00	0	0	0	0
01	1	1	0	0
11	0	0	0	0
10	0	0	0	0

- The variable D changes from a 0 to a 1 as you move from the left cell to the right cell
- Consequently, these two 1s are not dependent upon the value of D, and therefore:
- **D will not appear in the product term that results**
- The Result is **$Z = \overline{A}.B.\overline{C}$**

Examples with 4 variables

AB \ CD	00	01	11	10
00	0	0	0	0
01	0	0	0	0
11	0	1	0	0
10	0	1	0	0

- Accordingly:
- The Result is **$Z = A.\overline{C}.D$**

More Examples

AB \ CD	00	01	11	10
00	0	0	0	0
01	0	0	0	0
11	0	1	1	0
10	0	1	1	0

Answer: Z = **A.D**

More Examples

AB \ CD	00	01	11	10
00	0	0	0	0
01	0	1	1	0
11	0	1	1	0
10	0	0	0	0

- Answer: Z = **B.D**

More Examples

AB \ CD	00	01	11	10
00	0	0	0	0
01	1	1	1	1
11	0	0	0	0
10	0	0	0	0

- Answer: $Z = \overline{A}.B$

More Examples

AB \ CD	00	01	11	10
00	0	0	0	0
01	1	1	1	1
11	1	1	1	1
10	0	0	0	0

- Answer: Z = **B**

Examples with 4 variables

AB \ CD	00	01	11	10
00	1	0	0	1
01	0	0	0	0
11	0	0	0	0
10	1	0	0	1

- In the upper left: $Z = \overline{A}.\overline{B}.\overline{C}.\overline{D}.$
- In the upper right: $Z = \overline{A}.\overline{B}.C.\overline{D}.$
- In the lower left: $Z = A.\overline{B}.\overline{C}.\overline{D}.$
- In the lower right: $Z = A.\overline{B}.C.\overline{D}.$

Examples with 4 variables

AB \ CD	00	01	11	10
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01	0	0	0	0
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- In the upper right: $Z = \overline{A}.\overline{B}.C.\overline{D}.$
- In the lower left: $Z = A.\overline{B}.\overline{C}.\overline{D}.$
- In the lower right: $Z = A.\overline{B}.C.\overline{D}.$
- By combining the first two terms above: $Z = \overline{A}.\overline{B}.\overline{D}.$
- By combining the last two terms above: $Z = A.\overline{B}.\overline{D}.$
- Then, these two germs can be combined to give: $Z = \overline{B}.\overline{D}.$

Further Observations

AB \ CD	00	01	11	10
00	0	0	0	0
01	0	1	1	1
11	0	1	0	0
10	0	1	0	0

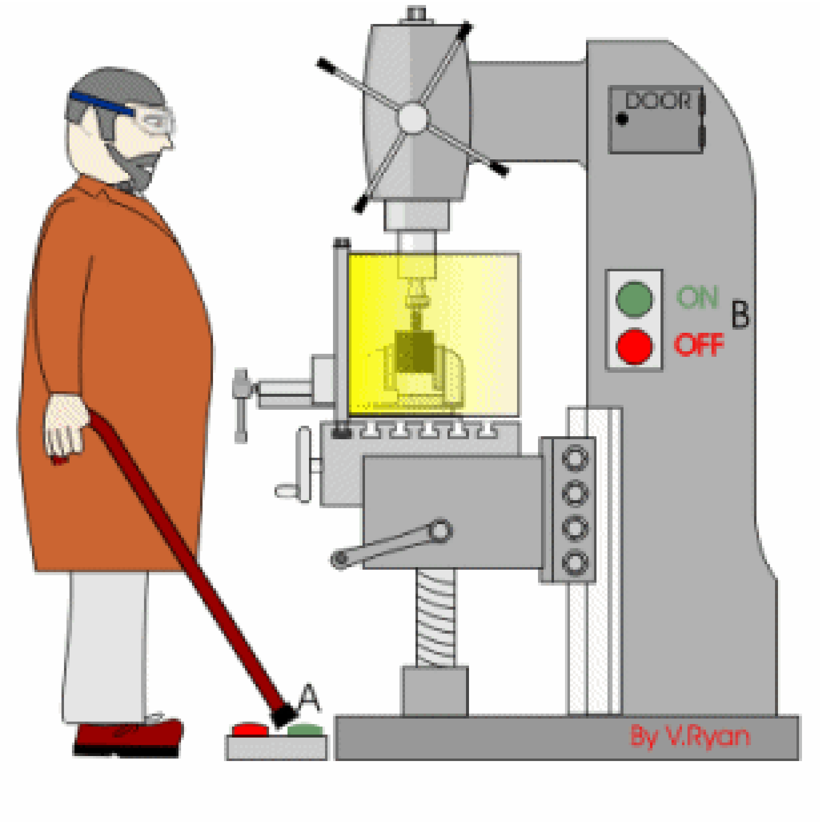
- There may well be more than one solution of equal complexity
- $A.\overline{C}.D. + \overline{A}.B.C.$?

Further Observations

AB \ CD	00	01	11	10
00	1	0	0	0
01	0	0	1	1
11	0	1	0	0
10	0	1	0	0

- There may well be more than one solution of equal complexity
- $\mathbf{A.\overline{C}.D. + \overline{A}.B.C.}$?
- And what about the 1 alone?

Problem Solving with Logic Gates



Problem Solving with Logic Gates

- A metal cutting milling machine has two switches, any one will allow the cutter to run.
- The first switch is on the side of the machine and the second is a foot operated switch.
- However, the machine has two micro-switches (used as safety devices) if any of these is released the cutter will stop.
- The first micro-switch is on a guard, if this is opened the machine will stop.
- The second micro-switch is on a door which allows access to the moving mechanism of the milling machine. If this is opened the machine will stop.
  The micro-switches are normally logic ‘1’ (true, high, on) when pressed.
- Draw the logic diagram for this machine

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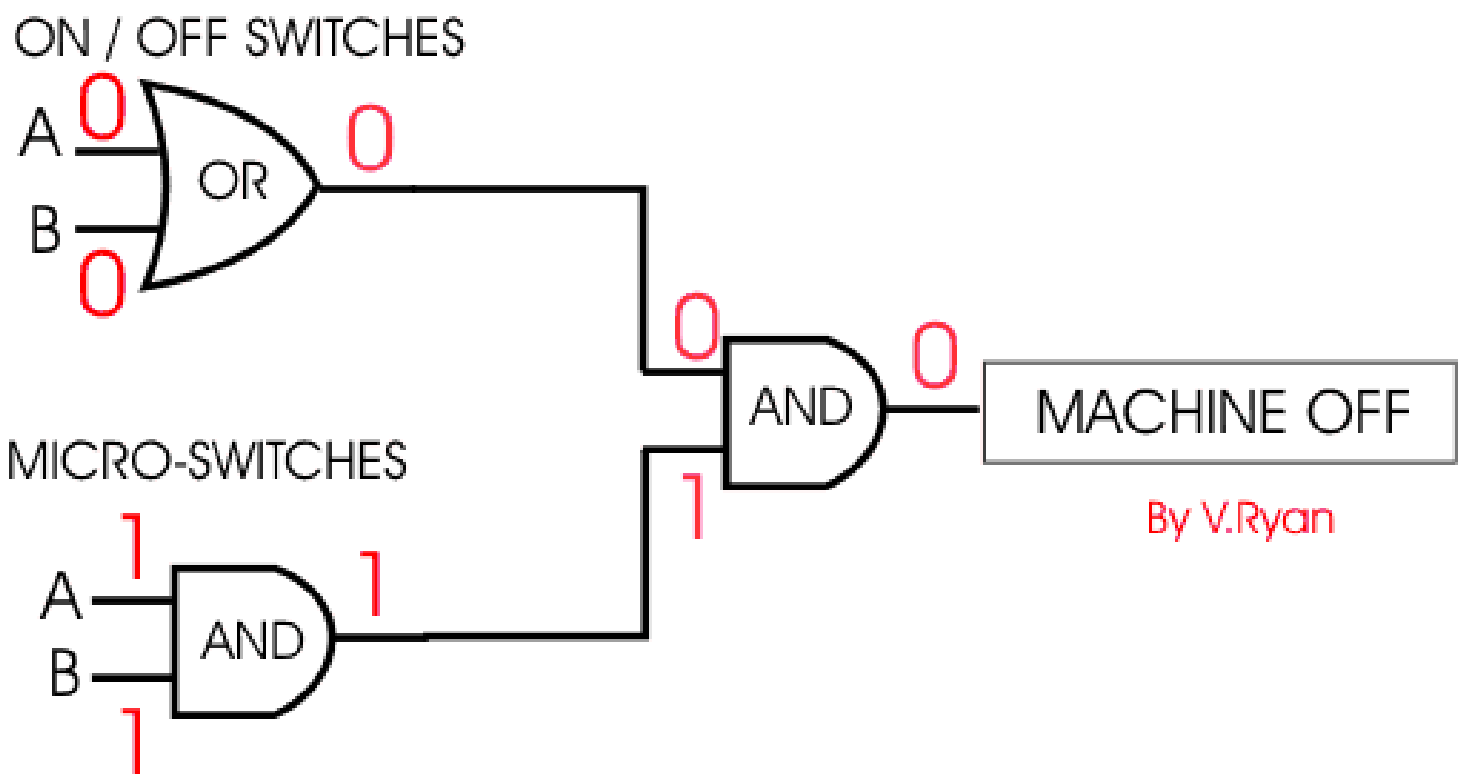
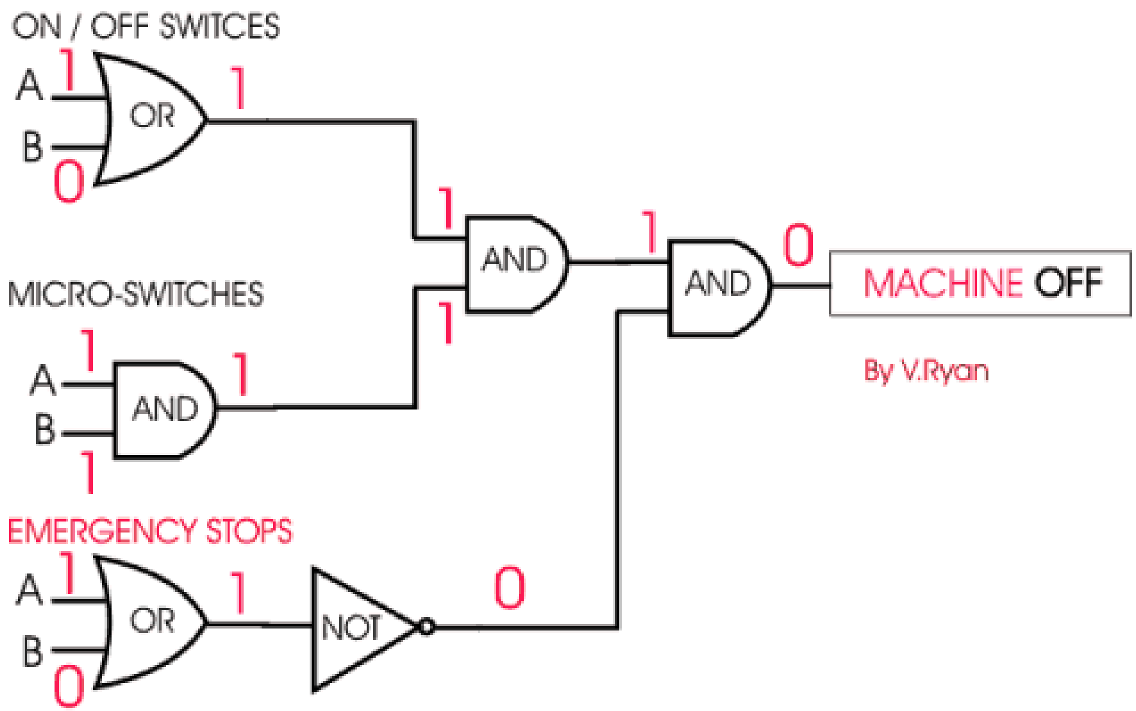
Problem Solving with Logic Gates



Problem Solving with Logic Gates

- In addition to the above (previous) arrangement,
  - The room has two emergency stop buttons at either end of the workshop
  - If either of these are pressed all machinery in the room will stop
- Draw the new logic circuit for this arrangement of buttons and switches

Problem Solving with Logic Gates



Problem Solving with Logic Gates

- An engine has 4 fail-safe sensors. The engine should keep running unless any of the following condition arise:
  - If sensor 2 is activated
  - If sensor 1 and sensor 3 are activated together (meanwhile sensors 2 and 4 are inactive)
  - If sensor 3 and sensor 4 are activated together
- Derive the truth table for this scenario
- Construct a logic gate network to satisfy requirements with basic logic gates
- Re-design the developed logic gate network with NAND and NOR gates only

Problem Solving with Logic Gates

- Conventions:
- A = Sensor1,
  B=Sensor2,
  C=Sensor3,
  D=Sensor4
- Sensor activated = 1,
  F = Shutdown=1

A	B	C	D	F
0	0	0	0	0
0	0	0	1	0
0	0	1	0	0
0	0	1	1	**1**
0	1	0	0	**1**
0	1	0	1	**1**
0	1	1	0	**1**
0	1	1	1	**1**
1	0	0	0	0
1	0	0	1	0
1	0	1	0	**1**
1	0	1	1	**1**
1	1	0	0	**1**
1	1	0	1	**1**
1	1	1	0	**1**
1	1	1	1	**1**

Problem Solving with Logic Gates

AB \ CD	00	01	11	10
00	0	0	1	0
01	1	1	1	1
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Problem Solving with Logic Gates

AB \ CD	00	01	11	10
00	0	0	1	0
01	1	1	1	1
11	1	1	1	1
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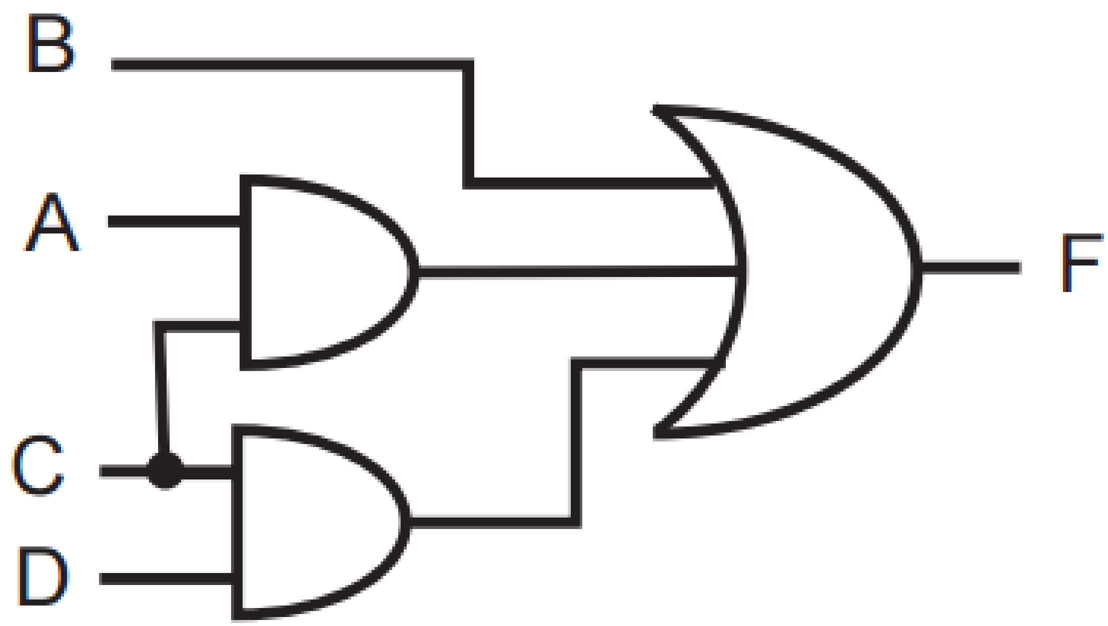
- $F = B + CD + AC$

Problem Solving with Logic Gates

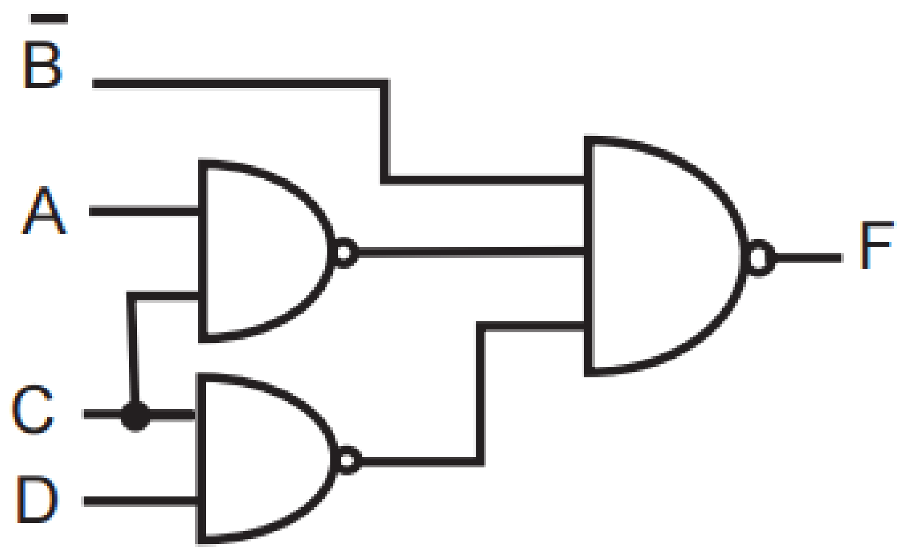


Problem Solving with Logic Gates

- using NAND gates only

$$F = B + CD + AC$$

$$= \overline{\overline{A + CD + AC}} = \overline{\overline{B}.\overline{CD}.\overline{AC}}$$



Problem Solving with Logic Gates

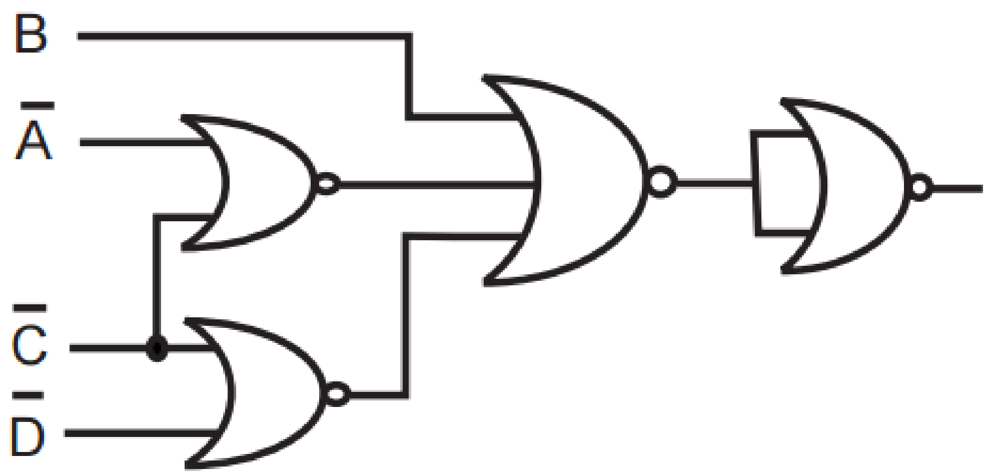
- using NOR gates only

$$F = B + CD + AC$$

$$F = \overline{\overline{B}} + \overline{\overline{CD}} + \overline{\overline{AC}}$$

$$F = \overline{\overline{B}} + (\overline{\overline{C} + \overline{D}}) + (\overline{\overline{A} + \overline{C}})$$

$$F = \overline{\overline{F}} = \overline{\overline{B + (\overline{\overline{C} + \overline{D}}) + (\overline{\overline{A} + \overline{C}})}}$$



Problem Solving with Logic Gates

- An airbag of a vehicle is activated once the following conditions meets:
  - Engine must be running (Logic Level of sensor 'E' is High)
  - The driver / passenger has buckled-up (fasten) the seat belt (Logic level of sensor 'B' is Low)
  - Speed sensor 'S' has exceeded the threshold value. (Logic level High)
  - Vibration sensors both 'V1' and 'V2' together or 'V3' alone is triggered. (Logic level High)

Problem Solving with Logic Gates

- Construct the Karnaugh Map for the scenario
- Construct a logic gate network to satisfy requirements with basic logic gates
- Re-design the developed logic gate network with NAND and NOR gates only