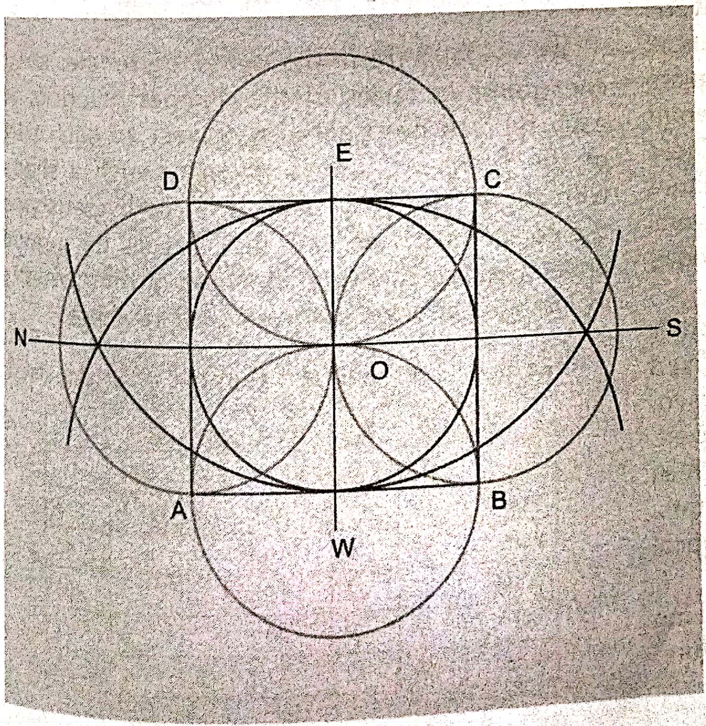
CHAPTER

8 Mathematics

LEARNING OUTCOMES

After finishing this chapter, you will be able to:

- Understand the nature of contributions made by ancient Indian Mathematicians
- Appreciate the unique nature of the Indian Mathematical contributions
- Understand specific contributions in Arithmetic, Geometry and Trigonometry
- Know the key role played by Āryabhaṭa and others in the field of Mathematics
- Appreciate the role of Piṅgala in the field of Binary Mathematics



Geometry is an ancient Science in India. Just with a pole anchored on the ground and a thread attached to it, Indians were able to generate complex geometrical shapes. What we see here is a procedure for construction of a square mentioned in Baudhāyana-śulba-sūtra, an ancient mathematical text taught in the Department of Mathematics in some Universities in the West as 'Rope Geometry'.

IKS IN ACTION 8.1

Ancient Indians: Tryst with Mathematics

Mathematics on the Indian subcontinent has a rich and long history going back over 5,000 years and thrived for centuries before advances were made in Europe. Its influence spread to China, Southeast Asia, the Middle East, and Europe. Apart from introducing the concept of zero, Indian mathematicians made seminal contributions to the study of geometry, arithmetic, binary mathematics, the notion of negative numbers, algebra, trigonometry, and calculus among other areas. The decimal place value system that is employed worldwide today was first developed in India.

For a large part, European mathematicians were reluctant to accept negative numbers as meaningful. Many took the view that negative numbers were absurd. This reluctance to adopt negative numbers, and indeed zero, held European mathematics back for many years. Although the reputation of Indian mathematics continues to suffer from the Eurocentric bias. Notwithstanding this, the Indian subcontinent has a strong and continuing mathematical heritage, the pinnacle of its expression being the Mathematical genius in Ramanujan who lived in the 20th Century CE.

Let us look at the genius of the mathematicians in the Vedic age by taking an example. It is a regular feature for the Vedic people to make several offerings to the Gods in a sacrifice. The sacrificial altars were not a standard shape such as a square or a rectangle. There were more than 70 different shapes of altars used in various sacrifices. These include shapes such as tortoise, falcon, and chariot wheels. The construction of these involved several complex shapes including, the isosceles triangle, rhombus, and circle.

The construction complexity had certain other dimensions. For example, there were a fixed number of bricks of certain shapes to be used. The number of layers in the altar was also fixed. Furthermore, there were other constraints such as the area of certain altars to be equal to other altars. Among other things, this is a tough mathematical problem to solve.

The problem is best demonstrated by taking the example Śyena-citi, a falcon-shaped structure. Figure 8.1 is the schematic representation of a Śyena-citi (Falcon bird-shaped altar).

As seen from Figure 8.1, the falcon has five components: the head, the body, the tail, and the two wings. Five differently shaped bricks have been employed to construct this structure. These include a square, a right-angled triangle, an equilateral triangle, and two compound shapes made from a rectangle, triangle, and a trapezium. Table 8.1 has the details on the number of bricks used in the construction of the Śyena-citi.

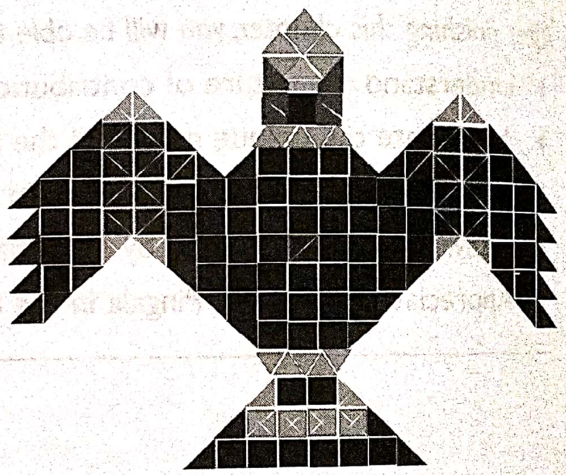


FIGURE 8.1 The Śyena-citi

As evident from the table there are strict constraints in terms of the number of bricks of each type to be used with respect to each part of the Śyena-citi. Construction of such a shape with these differently shaped bricks by satisfying the constraints is no easy task. It calls for a good understanding of several branches of mathematics including geometry, arithmetic, and algebra.

TABLE 8.1 Type and the Number of Bricks Used

	B1	B2	B3	B4	B5	
Head	1		6	6	1	14
Body	30	6	10			46
Wings	30	62	16			108
Tail	8	4	20			32
Total	69	72	52	6	1	200

Mathematics, referred to as Gaṇita is an integral part of Indians from very ancient times[1]. Ancient Indians developed several concepts of mathematics primarily in the course of solving several real-life problems that they were facing. For example, one of the Vedāṅgas is Jyotiṣa, which confines itself with required mathematical concepts, measurements, and approximation techniques to make certain predictions of the movement of celestial bodies in the sky. Similarly, another Vedāṅga, Kalpa required principles of geometry to construct Vedic altars of many different shapes. Buddhists and Jains were also deeply interested in mathematical concepts and produced canonical works discussing some useful mathematical concepts.

What started as a requirement at an early stage continued through the post-Vedic period as an uninterrupted tradition and many useful contributions to the field of mathematics were made throughout Indian history. Over the years, the contributions spanned almost all modern areas of mathematics. These include algebra, geometry, trigonometry, indeterminate equations, square, cube, and their roots, permutation and combinations, numerical methods and approximations, infinite series, to name a few.

8.1 UNIQUE ASPECTS OF INDIAN MATHEMATICS

Indian mathematics is unique, and it vastly differs from the modern approaches. A few of them are worthy of mention:

(a) There is a popular thinking among many that the world is divided into those who know and love mathematics and those who don't. This separation was primarily because we are educated that mathematics works with left-brain functionalities and literature works with right-brain functionalities. Therefore, the design of the pedagogy and delivery have kept these away in two separate compartments. In contrast, Indian mathematics is a seamless blend of poetry, literature, logic, and mathematical thinking weaved into a single work. All great works of mathematics are invariably great literary works too and they appeal to everyone on account of this natural blend. Therefore, there is no fear or stress in learning Mathematics.

> - Indian Mathematics is a seamless blend of poetry, literature, logic, and mathematical thinking weaved into a single work.
> - The use of sūtras and pithy verses is characteristic in the Indian Mathematical tradition to convey complex ideas and concepts.

(b) Mathematics was considered as a part of life. This is why Mathematics could be found in temple inscriptions, literary work addressing issues of life, and in a discussion on religion or spirituality. Bhāskarācārya in his Līlāvatī, for example, brings interesting mathematical concepts by posing interesting riddles to a student and solving them. Similarly, in Brahmasūtrabhāṣya of Śaṅkarācārya and the Vyāsabhāṣya on Patañjali's Yogasūtras refers to the decimal place value system as an example while discussing a philosophical issue.

(c) The use of sūtras is characteristic in the Indian tradition to convey ideas and concepts. These improve the retention of complex ideas and details very easily. Indian mathematics uses these mechanisms as much as possible. For example, there is a simple sūtra for remembering the pattern that constitutes a binary cycle of length three based on eight groups defined by Piṅgala in his Chandaḥ-śāstra. Āryabhaṭa's uniqueness lies in his ability to use sūtras to bring the utmost simplicity to describe complex information. A shining example is his description of the entire sine table using a verse of just two lines (see the last Endnote in Chapter 1). One who is familiar with Āryabhaṭa's system of numeration will be able to remember the sine table with very little effort.

(d) There has been an uninterrupted tradition of mathematical thinking and it has widely spread across the length and breadth of India. Mathematical concepts were developed by those from Gāndhāra (modern-day Afghanistan) to those in Bengal, as well as by those from Kashmir to Kerala.

(e) One of the important characteristics of Indian Mathematics is its algorithmic approach and the fact that it allows for approximate solutions based on the needs of real-life situations. Indian mathematicians also adopted what is today referred to as the constructive approach where the emphasis is on finding a procedure to solve a problem rather than merely seeking proofs of existence of a solution.

8.2 GREAT MATHEMATICIANS AND THEIR CONTRIBUTIONS

Before we discuss some key mathematical concepts developed by Indians, it will be interesting to have a bird's eye view of some of the great mathematicians and their contributions. The number of mathematicians India has produced, and their contributions are significant. Table 8.2 illustrates the major contributions. A closer look at the table reveals a continuous stream of mathematical theory building. Several areas of modern mathematics have been addressed. These include for example Fibonacci series, Pell's equation, and Pascal's triangle. In this chapter, we shall take a few examples from different areas of mathematics and briefly sketch the contributions made by Indians in the past and illustrate a few methods. This is to provide a curtain riser view of the Indian mathematical tradition and its contribution to the development of science, engineering, and technology.

- Indian Mathematics dealt with almost all areas of modern mathematics for more than 1000 years.
- Mathematical concepts found in Vedic texts, Buddhist and Jain works suggest that this culture is several thousand years old.

8.3 ARITHMETIC

Arithmetic is the branch of mathematics dealing with computation using numbers. Arithmetic is part of the daily life of any society. Commercial operations and trade require handling numbers and basic mathematical operators such as addition, subtraction, multiplication, and division. Indian arithmetic was quite sophisticated by the time of Āryabhaṭa in the 5th century CE. This is primarily because of a fully developed decimal place value system employing 0 to 9. In the 7th century, Brahmagupta established the use of 0 as a number and gave rules for multiplication, division, addition, and subtraction of zero. Indian decimal place value system was well known by the middle of 7th century as evident from the following observation of the Syriac bishop Severus Sebokht, "Indians possess a method of calculation that no word can praise enough. Their rational system of mathematics, or their method of calculation. I mean the system using nine symbols.[2]"

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TABLE 8.2 Ancient Indian Mathematicians and Their Salient Contributions

Sl. No.	*Details of the Work/Mathematician*	*Period, Location*	*Salient Contributions*
1	**Vedic Texts**	3000 BCE or earlier	The earliest recorded mathematical knowledge, number system, Pythagorean type triplets; Decimal system of naming numbers, the concept of infinity;
2	**Lagadha** – Vedāṅga-jyotiṣa	~ 1300 BCE	Astronomical concepts; a mathematical model for sun-moon movement in time; equinoxes and solstices;
3	**Śulba-sūtras** (Baudhāyana, Āpasthamba, Kātyāyana and Mānava Śulba-sūtras)	800–600 BCE	Earliest Texts of Geometry; Approximate value of the square root of 2, and π. Exact procedures for the construction and transformations of squares, rectangles, trapezia, etc.
4	**Pāṇini** – *Aṣṭādhyāyī*	500 BCE; Śalātura (in Khyber province in Pakistan)	Algorithmic approaches; Originator of the Backus–Naur Form (BNF), used in programming languages today, Context-sensitive rules, Arrays, inheritance, polymorphism;
5	**Piṅgala** – *Chandaḥ-śāstra*	300 BCE	Binary sequences; Conversion of Binary to Decimal system and vice versa; 'Meru Prastara' (Pascal's triangle); Optimal Algorithms to calculate powers; Zero as a Symbol;
6	**Bauddha Mathematical Works**	about 500 BCE to 500 CE	Multi-valued logic, Discussion of indeterminate and infinite numbers;
7	**Jaina Mathematical Works** – *Sūrya-Prajñapti, Jambūdvīpa-prajñapti, Bhagavatī-sūtra, Sthānāṅga-sūtra, Uttarādhyayana-sūtra, Tiloyapannati, Anuyoga-dvāra-sūtra*	200 BCE to 300 CE	Concept of logarithms, large numbers; algorithms for raising a number by a power; the arc of a circle; combinatorics; mensuration; Decimal system; Approximation of π;
8	**Āryabhaṭa** – *Āryabhaṭīyam*	476–550 CE; Kusumapura, near Pataliputra, Bihar	Concise verses; Algorithm for square root, cube root, Place value system; Sine table; geometry; quadratic equations; Linear indeterminate equations; Sums of squares and cubes of numbers; Planetary astronomy; Plane and spherical trigonometry;
9	**Varāha Mihira** – *Bṛhat Saṃhitā, Bṛhat-jātaka, Pañca-siddhāntikā*	482–565 CE, Ujjain, Madhya Pradesh	Summary of five ancient siddhāntas; Sine table, trigonometric identities; ($\sin^2 + \cos^2$); combinatorics; Magic squares;
10	**Bhāskara I** – *Commentary on Āryabhaṭīya, Laghu-bhāskarīyam and Mahā-bhāskarīyam*	600–680 CE; Vallabhi region, Saurashtra, Gujarat	Expanded Āryabhaṭa's work on Integer solution for indeterminate equations; Approximate formula for the sine function, Planetary Astronomy;

(*Contd.*)

TABLE 8.2 Ancient Indian Mathematicians and Their Salient Contributions (*Contd.*)

Sl. No.	*Details of the Work/Mathematician*	*Period, Location*	*Salient Contributions*
11	**Brahmagupta** – *Brāhmasphuṭa-siddhānta; Khaṇḍakhādyaka*	598–668 CE; Bhillamala in Rajasthan	Rules of arithmetic operations with zero and negative numbers, Algebra (Bījagaṇita); linear and quadratic indeterminate equations; Pythagorean triplets, Formula for the diagonals and area of a cyclic quadrilateral; notion of arithmetic mean.
12	**Virahāṅka** – *Vṛttajātisamuccaya (in Prākṛt)*	~600 CE	Fibonacci numbers; Moric metres.
13	**Śrīdharācārya** – *Triśatikā and Pāṭīgaṇita*	870–930 CE; Bhūrisṛṣṭi or Bhurshut village, Hugli, West Bengal	Arithmetic, Algebra, and Commercial Mathematics; Approximation of square root of a non-square number; Quadratic equations; Practical applications of algebra.
14	**Mahāvīrācārya** – *Gaṇita-sāra-saṅgraha*	800–870 CE; Gulbarga Karnataka;	A comprehensive, exclusive textbook on mathematics covering arithmetic-geometry- algebra. Continuing the ancient Jaina mathematics tradition; permutations and combinations; arithmetic and geometric series; the sum of squares and cubes of numbers in arithmetic progression.
15	**Jayadeva**	10th Century CE or earlier	Cakravāla (cyclic) method for solution of the second-order indeterminate equation.
16	**Shrīpati** – *Gaṇita-tilaka, Siddhānta-śekhara, Dhikoṭidākaraṇa, etc.*	1019–1066 CE; Rohiṇīkhaṇḍa, Maharashtra	Planetary Astronomy
17	**Bhāskarācārya (Bhaskara-II)** – *Līlāvatī on arithmetic and geometry; Bījagaṇita on algebra; Siddhānta-śiromaṇi on astronomy; Vāsanābhāṣya on Siddhānta-śiromaṇi.*	1114–1185 CE; Hailed from Bijjadavīda	Canonical textbooks used all over India, Detailed explanations including Upapatti (demonstration or proof); addition formula for sine function. Surds; permutations, and combinations; Solution of indeterminate equations, Ideas of calculus, including mean value theorem, planetary astronomy; construction of several instruments.
18	**Nārāyaṇa Paṇḍita** – *Gaṇita-kaumudī – a treatise on arithmetic and Bījagaṇita Vataṁśa – a treatise on algebra.*	1325–1400 CE;	Advanced textbooks taking forward the works of Bhāskarācārya, further properties of cyclical quadrilaterals, summation and repeated summations of arithmetic series, theory and construction of Magic squares, further developments in combinatorics.

Sl. No.	Details of the Work/Mathematician	Period, Location	Salient Contributions
19	**Mādhava of Saṅgamagrāma**	1340–1425 CE; Sangama Grama, in Kerala	Founder of Kerala School of Mathematics – a pioneer in the development of calculus; Infinite series and approximations for π, Infinite series and approximations for cosine and sine functions.
20	**Parameśvara,** – *Dṛggaṇita, Siddhāntadīpikā;* Commentaries on Āryabhaṭīyam, Mahā-bhāskarīya; Laghu-bhāskarīya, Līlāvatī, and Sūryasiddhānta	1360–1460 CE; Alathiyur, (near Tirur), Kerala	Properties of Cyclic quadrilateral; iterative techniques.
21	**Nīlakaṇṭha Somayājī,** *Tantra-saṅgraha; Āryabhaṭīya-bhāṣya, Siddhānta-darpaṇa*	1444–1544 CE; Near Tirur, Kerala	Irrationality of π, basic ideas of calculus; revised planetary theory, which is a close approximation to Kepler's model; Exact results in spherical astronomy.
22	**Jyeṣṭhadeva** – *Yukti-bhāṣā*	1500–1575 CE; Kerala	Hailed as the first textbook of Calculus; detailed explanations and proofs of the infinite series given by Mādhava.
23	**Śaṅkaravāriyar** – *Kriyākramakarī commentary on Līlāvātī and commentary on Tantra-saṅgraha*	1500–1569 CE Kerala	Explanations and Proofs of the results and procedures given in Līlāvatī.
24	**Gaṇeśa Daivajña** – *Buddhi-vilāsinī (commentary on Līlāvatī);*	1504 CE; Nandi Grama, Nadod, Gujarat	Explanations and Proofs of the results and procedures given in Līlāvatī.
25	**Kṛṣṇa Daivajña** – *Bījapallva-Commentary on Bījagaṇita of Bhāskarācārya*	1600 CE Delhi	Explanations and Proofs of results and procedures given in Bījagaṇita.
26	**Munīśvara** – *Siddhānta-sārvabhauma, commentary on Līlāvatī; Pātisāra;*	17th Century CE; Varanasi	Explanations and Proofs of the results and procedures given in Līlāvatī; trigonometric identities.
27	**Kamalākara** – *Siddhānta-tattva-viveka*	1616–1700 CE; Varanasi, Uttara Pradesh	Addition and subtraction theorems for the sine and the cosine; Sines and cosines of double, triple, etc., angles.

8.3.1 Square of a Number

Āryabhaṭa identifies the geometric aspect and the mathematical operation of the square in his definition. As per his definition[3] varga is a square and is also a geometric object whose sides are equal. Bhāskara–I (629 CE) in his commentary on Āryabhaṭīya provides an algorithm for finding the square of any number. The verse that explains the algorithm is given below:

- An algorithm for computation of cube root was given for the first time by Āryabhaṭa.
- Brāhma-sphuṭa-Siddhānta, gives a good description of calculations with positive and negative numbers, Zero, fractions and surds.

अन्त्यपदस्य वर्गं कृत्वा द्विगुणं तदेव चान्त्यपदम् ।
शेषपदैराहन्यात् उत्सार्योत्सार्य वर्गविधौ ॥

antyapadasya vargaṃ kṛtvā dviguṇaṃ tadeva cāntyapadam |
śeṣapadairāhanyāt utsāryotsārya vargavidhau ||

The steps of the algorithm as provided by the above verse can be enumerated as follows:

Step 1: अन्त्यपदस्य वर्गं कृत्वा (*antyapadasya vargaṃ kṛtvā*).
Square the last digit (most significant digit) first. Place it in a new row (two places to the right of digits in the previous row).

Step 2: द्विगुणं तदेव चान्त्यपदम् (*dviguṇaṃ tadeva cāntyapadam*).
Multiply the last digit with two and each of the remaining digits to the right and place them to the right in the same row.

Step 3: शेषपदैराहन्यात् उत्सार्योत्सार्य (*śeṣapadairāhanyāt utsāryotsārya*).
Remove the current most significant digit.
If there are no more digits to operate go to step 5.

Step 4: The next digit becomes the last digit now. Go to step 1.

Step 5: Perform the final addition to get the square of the number.

It may be noted that this ancient rule for squaring, uses $\frac{n(n-1)}{2}$ multiplications for squaring an n-digit number.

EXAMPLE 8.1: Find the square of the number 1638.

Final Digit Position	7	6	5	4	3	2	1
Row 1: Last digit = 1 Multiplication with other digits Digit '1' is removed	$(1^2) = 1$	(2*1*6) = 12	(2*1*3) = 6	(2*1*8) = 16			
Row 2: Last digit = 6 Multiplication with other digits Digit '6' is removed			$(6^2) = 36$	(2*6*3) = 36	(2*6*8) = 96		
Row 3: Last digit = 3 Multiplication with other digits Digit '3' is removed					$(3^2) = 9$	(2*3*8) = 48	
Row 4: Last digit = 8 Digit '8' is removed No more digits available							$(8^2) = 64$

Final Addition of the numbers:

Digit Position						
7	**6**	**5**	**4**	**3**	**2**	**1**
1						
1	2					
		6				
		1	6			
	3	6				
		3	6			
			9	6		
				9		
				4	8	
					6	4
2	6	8	3	0	4	4

The answer is 26,83,044.

8.3.2 Square Root

Āryabhaṭa presents an algorithm for determining the square root of a number. The methodology of finding the square root revolves around the concept of splitting the digits into pairs, starting from the least significant digit, the odd place being designated as varga (V), and the even place as avarga (A). For example, if we take two numbers 55225 and 205209 the split will look as follows:

V	A	V	A	V		A	V	A	V	A	V
5	5	2	2	5		2	0	5	2	0	9

Āryabhaṭa then provides a simple recursive algorithm that starts from the most significant varga place and progressively introduces one digit at a time until all digits are exhausted. The algorithm for square root is provided in the following verse:

भागं हरेत् अवर्गात् नित्यं द्विगुणेन वर्गमूलेन ।
वर्गाद्वर्गे शुद्धे लब्धं स्थानान्तरे मूलम् ॥
bhāgaṃ haret avargāt nityaṃ dviguṇena vargamūlena |
vargādvarge śuddhe labdhaṃ sthānāntare mūlam ||

The above verse has indicated the following steps to the algorithm:

1. Designate the varga and avarga digits starting from right to left (least significant to most significant).
2. Take the first varga sthāna at the leftmost (most significant) along with an avarga digit at its left (if any).
3. Remove (subtract) the maximum possible square from this number, and the square root of the square that we can remove will be added to the square root line (this is a place where we are accumulating the answer as we perform the operation).
4. Along with the remainder of the previous operation, bring the next digit down. The next digit is avarga digit (as we have just completed the varga digit related operation). भागं हरेत् अवर्गात् नित्यं द्विगुणेन वर्गमूलेन (*bhāgaṃ haret avargāt nityaṃ dviguṇena vargamūlena*)

   Whenever we operate at avarga digit, we need to divide the number by two times the current value of the square root that we have stored in the square root line. The quotient obtained in this division will be added to the square root line.
5. Along with the remainder of the previous operation, we will bring the next digit down. The next digit is varga digit (as we have just completed the avarga digit related operation).
   वर्गाद्वर्गे शुद्धे (*vargādvarge śuddhe*)

   Whenever we operate at a varga digit, we need to remove the square of the quotient obtained in the previous step.
6. If some more digits are remaining go to step 4, else go to step 7.
7. लब्धं स्थानान्तरे मूलम् (*labdhaṃ sthānāntare mūlam*).
   The final result in the square root line is the answer.

As evident from the above algorithm, this procedure runs in a recursive fashion bringing down one digit at a time. When the procedure stops after exhausting all the digits the final value accumulated in the square root line is the answer. If the number is not a perfect square this procedure aborts mid-way or leaves a remainder at the end of the operation.

EXAMPLE 8.2: Find the square root of the number 19,881.

		V	A	V	A	V	1 4 1
		1	9	8	8	1	(Square root line)
Maximum Square = 1		1					
Twice the value at sq. root line (2*1)	2)	0	9 (4				
		0	8				
			1	8			
Square of the previous quotient (4^2)			1	6			
Twice the value at sq. root line (2*14)	28)			2	8 (1		
				2	8		
					0	1	
Square of the previous quotient (1^2)						1	
						0	

The final answer = 141.

Cube Root

Āryabhaṭa was the first to establish a procedure to calculate the cube root (*Ghana-mūla*) of a number. Āryabhaṭa provided an algorithm for determining the cube root of a number that is strikingly similar to that of the algorithm for square root. If a number has '*n*' digits, the number of digits of the cube of that number will be $\geq 3n - 2$ and $\leq 3n$. With this in mind, Āryabhaṭa designates the digits in a number using three different nomenclatures. Āryabhaṭa then provides a simple recursive algorithm that starts from the most significant place and progressively introduces one digit at a time until all digits are exhausted. An illustration of this is available in Shukla and Sarma's critical edition of Āryabhaṭīya[4].

8.3.3 Series and Progressions

A sense of the knowledge of progressions of ancient Indians is evident in the Vedic text. For example, in the *Camaka-praśna* (Taittirīya-saṃhitā 4.5.11) there is a mention of an arithmetic progression of odd numbers starting from one to 33 followed by another arithmetic progression of even numbers starting from four to 48 in steps of four[5]. In the *Vājasaneyī-saṃhitā* we have the *yugma* (even) and *ayugma* (odd) series: 4, 8, 12, 16, 48 and 1, 3, 5, 7, ... Similarly, there is a geometric series (12, 24, 48, 96, 196608, ..., 393216) mentioned in the *Pañcaviṃśa-brāhmaṇa*. This indicates an appreciation of odd and even numbers as distinct entities in the Vedic period and also the formation of different types of series. Āryabhaṭa I (499 CE), and Brahmagupta (628 CE) considered the cases of the sums of the sums, the squares and the cubes of the natural numbers. Mahāvīra (850 CE) gave a rule for the summation of an interesting geometric series. Nārāyaṇa (1356 CE) provided a more generalised method for repeated summation of partial series. We shall see some of them in this section[6].

IKS IN ACTION 8.2

Square Root of Imperfect Squares

Several sources in the ancient Indian texts point to multiple attempts and methods to obtain the square root of an imperfect square. Let us look at some of them here.

Śulba-sūtra Formula for $\sqrt{2}$

The square root of two has been explored as early as the Vedic times (Śulba-sūtra). According to Bodhāyana-śulba-sūtra (BSS 2.12), the value of the square root of two can be obtained using the following sūtra:

प्रमाणं तृतीयेन वर्धयेत् तच्चतुर्थेन आत्मचतुस्त्रिंशेनोनेन सविशेषः ।

pramāṇaṃ tṛtīyena vardhayet taccaturthena ātmacatustriṃśenonena saviśeṣaḥ |

प्रमाणं तृतीयेन वर्धयेत् *(pramāṇaṃ tṛtīyena vardhayet) essentially meaning, add 1/3 to 1;* तच्चतुर्थेन *(taccaturthena), meaning add one fourth of this* $\frac{1}{4}*\frac{1}{3}$, *however,* आत्मचतुस्त्रिंशेनोनेन *(ātma-catustriṃśenonena) take out* $\frac{1}{34}$ *from this.* सविशेषः *(saviśeṣaḥ)* essentially denotes it is a special case (meaning an approximate number). Using this sūtra,

The value of $\sqrt{2} \approx 1+\frac{1}{3}+\frac{1}{3\times 4}\left(1-\frac{1}{34}\right)$

$$=\frac{577}{408} \approx 1.4142156863$$

Bakshali Method

Bakshali manuscripts (estimated to have been written sometime during 300–600 CE) discusses several unique mathematical issues, including that of finding the square root of an imperfect square. Any imperfect square N may be expressed as $\sqrt{A^2+b}$. According to Bakshali Manuscript, the square root of the number N may be expressed as:

$$\sqrt{N}=\sqrt{A^2+b} \approx A+\frac{b}{2A}-\frac{\left(\frac{b}{2A}\right)^2}{2\left(A+\frac{b}{2A}\right)}$$

Approximations to the Square Root of a Non-square Number

Śrīdhara (850 CE) in his Triśatikā, has explained how Āryabhaṭa method can be used to get better approximations to the square root of a non-square number. For instance, if D is a non-square number, we can use the expression below to calculate $\sqrt{D}$ to any desired accuracy:

$$\sqrt{D}=\frac{\sqrt{D*10^{2n}}}{10^n}$$

Source: Based on the NPTEL Video on "Mathematics in India – From Vedic period to modern times. https://nptel.ac.in/courses/111/101/111101080/. Last accessed on Oct. 1, 2021.

The term *upaciti* or *citi* is used to indicate a series in general. For example, the series 1 + 2 + 3 + ⋯ + *n*, which starts with '1' and has a common difference of '1' is called एकोत्तराद्युपचिति (*ekottarādi-upaciti*). Let us consider the case of the following the series of natural numbers up to *n* terms: 1, (1 + 2), (1 + 2 + 3), …, (1 + 2 + 3 + ⋯ + *n*). The term चितिघन:(*citighana*) is used to denote the sum of this series. Āryabhaṭa gives the formula for computing the sum in the following verse:

एकोत्तराद्युपचितेर्गच्छाद्यकोत्तरत्रिसंवर्गः ।
षड्भक्तः स चितिघनः सैकपदघनो विमूलो वा ॥ २१ ॥

ekottarādy-upaciter-gacchādyakottara-trisaṃvargaḥ |
ṣaḍbhaktaḥ sa citighanaḥ saikapada-ghano vimūlo vā || *Gaṇita-pāda 21* ||

Of the series (*upaciti*) which begins with the term '1' and has a common difference '1', (*ekottarādi*), take three terms in continuation, of which the first is equal to the given number of terms, and find their continued product (*gacchādyakottara-trisaṃvargaḥ*). That (product) divided by 6 gives the citighana (*ṣaḍbhaktaḥ sa citighanaḥ*). Alternatively (*vā*), it can be obtained by the number of terms plus one subtracted from the cube of that (*saikapada-ghano vimūlo*), divided by 6 (*ṣaḍbhaktaḥ*). This can be expressed in the following notation:

$$1+(1+2)+(1+2+3)+\cdots+(1+2+3+\cdots+n)=\frac{n(n+1)(n+2)}{6} \text{ or } \frac{(n+1)^3-(n+1)}{6}$$

The use of the term citighana is indeed interesting. It literally means the contents of a pile (of say balls) in the shape of a pyramid on a triangular base. The equivalent physical interpretation of the above series is that the pyramid is so constructed that there is 1 ball in the topmost layer, $1+2$ balls in the next lower layer, $1+2+3$ balls in the further next lower layer, and so on.

Sum of the Series of Squares and Cubes

Let us now consider the summation of the series of squares and the summation of the series of cubes, of the first n natural numbers indicated as ΣN^2 and ΣN^3 respectively.

$$\Sigma N^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$$

and

$$\Sigma N^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3$$

Āryabhaṭa designated ΣN^2 as वर्गचितिघनः (*varga-citighanaḥ*) and ΣN^3 as घनचितिघनः (*ghana-citighanaḥ*). The formulae for the summation of the series of squares and cubes is provided in the following verse:

सैक-सगच्छ-पदानां क्रमात् त्रिसंवर्गितस्य षष्ठोंऽशः ।
वर्गचितिघनः स भवेत् चितिवर्गो घनचितिघनश्च ॥ २२ ॥
saika-sagaccha-padānāṃ kramāt trisaṃvargitasya ṣaṣṭhoṃ'śaḥ |
varga-citighanaḥ sa bhavet, citivargo ghana-citighanaśca || Gaṇita-pāda 22 ||

The product of the three quantities (*trisaṃvargitasya*), viz., the number of terms (n), the number of terms plus one ($n+1$), and the same increased by the number of terms ($n+1+n$), (*saika-sagaccha-padānāṃ kramāt*), when divided by 6 (*ṣaṣṭhoṃ'śaḥ*) gives the sum of the series of squares of natural numbers (*varga-citighanaḥ sa bhavet*). The square of the sum of the series of natural numbers (*citivargo ghana*) gives the sum of the series of cubes of natural numbers (*ghana-citighanaśca*).

According to the above verse,

$$\Sigma N^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

and

$$\Sigma N^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Repeated Summation of Series (Vārasaṅkalita)

In the previous case we saw the *saṅkalita* (sum of a series) for the first n natural numbers. A more general case of the repeated summation of such series was provided by Nārāyaṇa. Vāra means repeated (or again). Therefore, this method of repeated summation of a series can be designated as *vārasaṅkalita*.

Let the symbol $^nV_1$ denote the arithmetic series of the first 'n' natural numbers. This can be expressed as: $^nV_1 = 1 + 2 + 3 + \dots + n$. Similarly, let $^nV_2$ denote the series of the partial sums of the series $^nV_1$. If we use 'r' to denote the number of terms up to which we want to sum up, then we can create partial sums of $^nV_1$. For example,

For $n = 1$, $^1V_1 = 1$; $n = 2$, $^2V_1 = 1 + 2$; $n = 3$, $^3V_1 = 1 + 2 + 3$; and so on. Using this we can write the second order sum of partial series as:

$$^nV_2 = \sum_{r=1}^{r=n} {}^rV_1$$

$$^nV_2 = 1 + (1 + 2) + (1 + 2 + 3) + \dots + (1 + 2 + 3 + \dots + n)$$

Similarly one can write the other sums of the partial series for the higher order in the following fashion:

$$^nV_3 = \sum_{r=1}^{r=n} {}^rV_2;$$

In general, the mth order sum of partial series of number can be represented as:

$$^nV_m = \sum_{r=1}^{r=n} {}^rV_{m-1}$$

Nārāyaṇa provided a formula for the computation of the mth order *vāra-saṅkalita*. It denotes in this case the operation of forming a new series by taking the sums of the previous series. He provided an expression to calculate the sum using the following verse:

एकाधिकवारमिताः पदादिरूपोत्तरा पृथक् तेंऽशाः ।
एकाद्येकचयहरास्तद्घातो वारसङ्कलितम् ॥
ekādhika-vāramitāḥ padādi-rūpottarā pṛthak teṃ'śā: |
ekādy-ekacayaharās-tadghāto vārasaṅkalitam ||

- Āryabhaṭīya gives a good indication of the knowledge of algorithmic approach and use of recursive algorithms for problem-solving.
- The birth of Indian geometry could be traced to the Vedic time.

The terms of the sequence beginning with the pada (number of terms, i.e. n) and increasing by 1 (पदादिरूपोत्तराः) taken up to $(m+1)$ times (*ekādhika-vāramitāḥ*) are successively the numerators (*pṛthak teṃ'śā:*) and the terms of the sequence beginning with unity and increasing by 1 (*ekādy-ekacayaharāḥ*) are respectively the denominators. The continued products of these (fractions) (*tadghātaḥ*) gives the vāra-saṅkalita (*vāra-saṅkalitam*). According to the above, since n is the number of terms and m the order, we get the following sequence of fractions:

$$\frac{n}{1}, \frac{n+1}{2}, \frac{n+2}{3}, \dots, \frac{n+m}{m+1}$$

and the sum of the series is the product of this sequence of fractions, given by:

$$^nV_m = \frac{n\,(n+1)(n+2)\dots(n+m)}{1.2.3\dots(m+1)}$$

This expression is the most generalised form for summation of series. Substituting $m = 1$ above will fetch us the sum of the first 'n' natural numbers. Substituting $m = 2$ in the above equation will give us one of the two formulae of Āryabhaṭa that we derived in the previous section.

8.4 GEOMETRY

As we saw in Chapter 2, Yajñas formed a very important part of Vedic life. The performance of Vedic rituals involved the construction of a variety of Vedic altars (*yajña-vedīs*) as per certain specifications. The Brāhmaṇa portion of the Yajurveda contains details about the arrangement of the sacrificial ground and the construction of the altars. For this purpose, one first fixes cardinal directions and then goes on to construct altars of different shapes and dimensions using prefabricated bricks. Śulba-sūtras give exact methods for the construction of such altars. On account of these, the birth of Indian geometry can clearly be traced to the Vedic times.

Śulba-sūtras are a part of the Vedāṅga. These sūtras give information on the methods of the layout of the vedī, citi, and the maṇḍapa. Śulba means a thread and with the help of a thread and a pin (or a pole), construction methods for various shapes have been described in the śulba-sūtras. The four śulba-sūtras attributed to Bodhāyana, Manu, Āpastambha, and Kātyāyana are the predominant ones. We also have śulba-sūtras due to Maitrāyaṇa, Varāha, and Vādhūla. The oldest of the śulba-sūtras attributed to Bodhāyana is estimated to have been written prior to 800 BCE. Śulba-sūtras are manuals for the construction of Vedic altars. Therefore, Śulba-sūtras also discuss the manufacturing of bricks, thereby indicating a good knowledge of the materials and manufacturing processes which seems to have prevailed in those times.

8.4.1 Property of Right-angled Triangle in Śulba-sūtras

In current mathematical texts, we are taught an important theorem being attributed to Pythagoras (570–495 BCE). However, the first general statements of this so-called Pythagoras theorem is actually found in the Śulba-sūtras. In the Baudhāyana-śulba-sūtra (prior to 800 BCE), the theorem is stated in the following form (this result is known by name *Bhuja-koṭi-karṇa-nyāya)*: "The sum of the areas of the squares formed by the length and breadth of a rectangle equals the area produced by the diagonal of the rectangle"[7]. Let us consider a rectangle of length a and breadth b. Let the diagonal of the rectangle be c. According to the Baudhāyana formula, $a^2 + b^2 = c^2$. This is graphically illustrated in Figure 8.2.

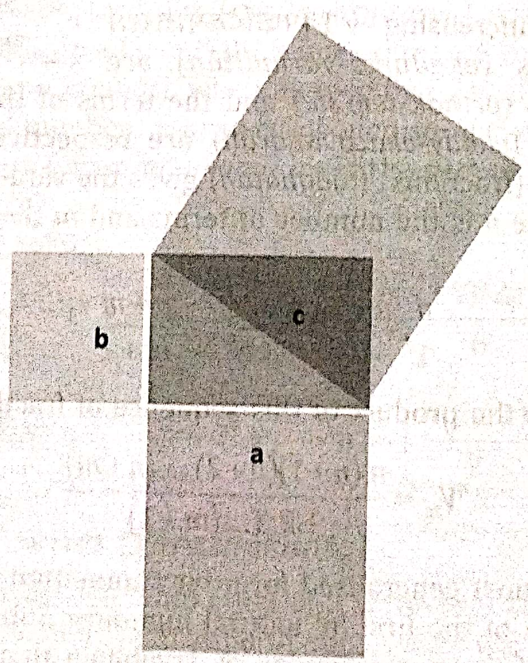

FIGURE 8.2 Baudhāyana Formula for Right-angled Triangle

IKS IN ACTION 8.3

Fun and Practicality in Indian Mathematics

Indian mathematical works have several interesting problems of day-to-day importance solved using mathematical principles. Further, it is described as a game in poetic verses. Āryabhaṭīya, Līlāvatī and other Indian mathematical works describe several such entertaining and interesting problems. Let us see an example.

Ancient Indians used *śaṅku*, a gnomon, as a device for measurement to fix the direction and coordinates, to measure the length scale and host of such applications. One verse poses a problem related to gnomonic shadow.

शङ्कुगुणं शङ्कुभुजाविवरं शङ्कुभुजयोर्विशेषहृतम् ।
यल्लब्धं सा छाया ज्ञेया शङ्कोः स्वमूलाद्धि ॥

śaṅku-guṇaṃ śaṅku-bhujāvivaraṃ śaṅku-bhujayor-viśeṣahṛtam |
yal-labdhaṃ sā chāyā jñeyā śaṅkoḥ sva-mūlāddhi ||

According to this verse, the height of the gnomon multiplied by the distance between gnomon and the lamp post is to be divided by the difference between the lamp post and the gnomon. The quotient thus obtained, should be known as the length of the shadow measured from the foot of the gnomon. This can be better understood graphically using in Figure 8.3.

In Figure 8.4, there is a lamp post (denoted by AC) and a śaṅku (denoted by EF). The objective is to find the length of the shadow cast by the śaṅku on account of the lamp post.

Triangles FEB and CDF are similar.

Therefore, $EB = \dfrac{EF * DF}{DC} = \dfrac{EF * AE}{AC - EF}$

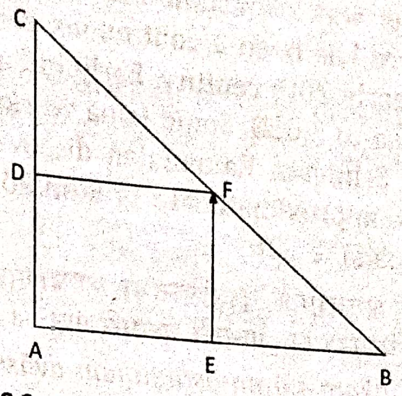

FIGURE 8.3 The Shadow Problem – An Illustration

Such formulations are used in the design and construction of temples when the architect has to design for the sunlight to fall on the shrine. Bhāskara defines an inverse problem and provides the solution, i.e., how to calculate the height of the lamp post knowing the height of śaṅku and the length of the shadow. The same principle is applied to several astronomical calculations including calculation of eclipses as shown in Figure 8.5.

Considering half symmetry, of sun, earth, moon along the horizontal axis in the orbital system, and getting the equivalence of the triangle ABC above, we have:

EF is the half diameter of the earth (śaṅku)

AC is the half diameter of the sun (lamp post)

AE is the distance between sun and earth

EB is the earth's shadow (to be determined)

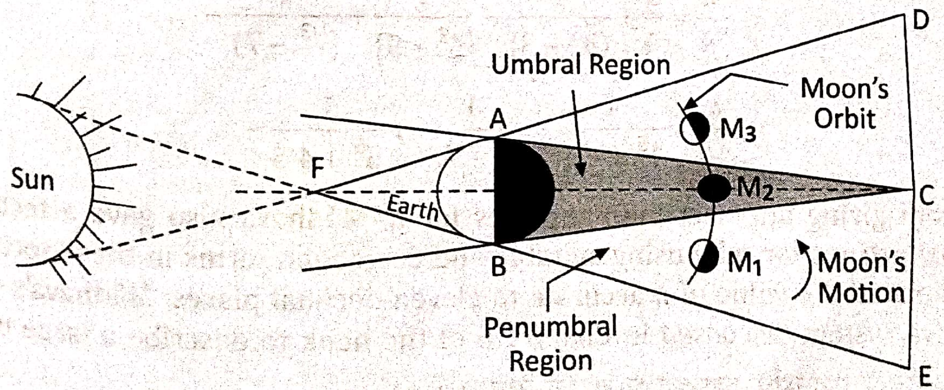

FIGURE 8.4 Shadow Problem – An Astronomy Example

8.4.2 The Value of π

Right from the Vedic period, Indians understood that the ratio of the area of a circle to its perimeter was a constant and that this value can only be approximated. What is interesting is that there has been a continuous evolution of the approximation of π from the Vedic period till the early 20th century. Early Indian works took the value of π as 3. Śulba-sūtras estimated π as close to 3.088. Some Jaina works approximated π as the square root of 10. In early 20th century Srinivasa Ramanujan discovered an infinite series which was later used during the 1980s to approximate π to 17.5 million places. Āryabhaṭīyam discusses π in the following verse (Gaṇitapāda, 2.10):

चतुरधिकं शतमष्टगुणं द्वाषष्टिस्तथा सहस्राणाम् ।
अयुतद्वयविष्कम्भस्य आसन्नो वृत्तपरिणाहः ॥
caturadhikaṃ śatam-aṣṭaguṇaṃ dvāṣaṣṭis-tathā sahasrāṇām |
ayuta-dvaya-viṣkambhasya āsanno vṛttapariṇāhaḥ ||

According to the verse, the circumference of a circle is eight times (100 + 4) added to 62,000 (*caturadhikaṃ śatam-aṣṭaguṇaṃ-dvāṣaṣṭis-tathā-sahasrāṇām*). The diameter of the circle is 20,000 (*ayuta-dvaya*). Since the ratio of the circumference to the diameter yields the value of π, we can compute π to be:

$$\pi = \frac{(100+4)\times 8+62000}{20000} = \frac{62832}{20000} = 3.1416$$

Bhāskarācārya in his work Līlāvatī, (verse 199), derives the value of π using a different approach and arrives at the same value.[8] According to this,

$$\pi = \frac{3927}{1250} = 3.1416$$

In the 14th century CE, the celebrated astronomer mathematician Mādhava discovered several infinite series for π which were later re-discovered in Europe by James Gregory (1671), Gottfried Leibniz (1674), and Abraham Sharp (1699). Some of Mādhava's series are given below:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

$$\frac{\pi}{\sqrt{12}} = 1 - \frac{1}{3.3} + \frac{1}{3^2\times 5} - \frac{1}{3^3\times 7} + \cdots$$

$$\frac{\pi}{4} = \frac{3}{4} + \frac{1}{(3^3-3)} - \frac{1}{(5^3-5)} + \frac{1}{(7^3-7)} - \cdots$$

$$\frac{\pi}{16} = \frac{1}{1^5+4.1} - \frac{1}{3^5+4.3} + \frac{1}{5^5+4.5} - \cdots$$

Apart from giving an exact infinite series for π, Mādhava also gave a technique finding better approximations for π by using suitable end correction terms in these series. In this way Mādhava estimated the value of π accurate to eleven decimal places. Mādhava's verse uses the bhūta-saṃkhyā system discussed in Chapter 6 of the book to describe a large number:

विबुध-नेत्र-गज-अहि-हुताशन-त्रि-गुण-वेद-भ-वारण-बाहवः ।
नव-निखर्व-मिते वृत्तिविस्तरे परिधिमानमिदं जगदुर्बुधाः ॥
vibudha-netra-gaja-ahi-hutāśana-tri-guṇa-veda-bha-vāraṇa-bāhavaḥ |
nava-nikharva-mite vṛttivistare paridhimānam-idaṃ jagadurbudhāḥ ||

In this verse, vibudha means devas, who are 33 in number. Using the bhūta-saṃkhyā system discussed in Chapter 6 of the book, we can compute π as follows:

$$\pi = \frac{2827433388233}{9*10^{11}} = 3.141592653592...$$

The following Table 8.3 summarises the different approximations to value of π that were discovered in the Indian tradition right from the Vedic times to the time of Ramanujan.

TABLE 8.3 History of Approximations to π by Indian Mathematicians

	Value of π	*Accuracy (Decimal places)*	*Method*
Śulba-sūtras (around 800 BCE)	3.08888	1	Geometrical
Jaina texts (500 BCE)	$\sqrt{10} = 3.1623$	1	Geometrical
Āryabhaṭa (499 CE)	$\frac{62832}{20000} = 3.1416$	4	Polygon doubling ($4.2^8 = 1024$ sides)
Bhāskarācārya (Līlāvatī)	3927/1250 = 3.1416	4	Polygon doubling
Mādhava (1375 CE)	$\frac{2827433388233}{9*10^{11}} = 3.141592653592...$	11	Infinite series with end corrections
Ramanujan (1914 CE)		17 million	Modular equation

8.5 TRIGONOMETRY

Trigonometry is called *jyotpatti,* the science of computation of chords in Indian mathematics. Consider a circle of radius R as shown in Figure 8.5. DA = $R\theta$, is an arc. AB = $R \sin \theta$, is called the jyā corresponding to the arc, $R\theta$. Earlier it was called jyārdha or 'half a bow-string', but later it was just called 'jyā'. R is taken to be $\frac{21600}{2\pi} \approx 3438$, where the circumference of the circle is 21,600 units (number of 'minutes' in a radian). OB = $R \cos \theta$ is called the koṭijyā or kojyā or cojyā. Jīvā was another name for jyā. When this was transmitted to Arab countries,

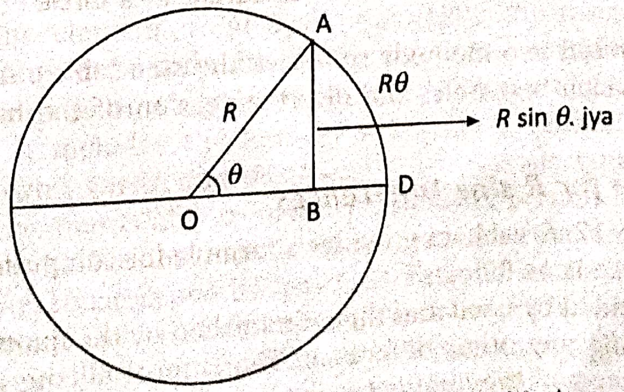

FIGURE 8.5 The Jyā and Cojyā in Indian Mathematics

jīvā became jība, which was read as 'jayb' in Arabic. In common parlance, 'jayb' was 'pocket' or 'fold' in Arabic. When many mathematical texts in Arabic were translated into Latin, 'jayb' got translated as 'Sinus', which means 'pocket' or 'fold' in Latin. This eventually became 'sine'. Since the complement of jyā is known as kojyā, the complement of sine became 'cosine' in Europe.

Computation of the 'R sines'

R sine values are required if we need to compute any quantity which involves the sine and cosine functions. Let us consider the first quadrant in a circle, since the values of the R sines and R cosines in other quadrants can be related in a simple manner to the values in the first quadrant. One way to obtain these values is to divide the quadrant into certain parts and determine R sine values for these. Once these values are available, one can compute the other values of the angles by interpolation. Let us consider one quadrant of a circle as shown in Figure 8.6. Let this quadrant be divided into 24 parts (P_0, P_1, P_2, ..., P_{24}). Each arc measures $\frac{90}{24} = 3°45'$. In the figure, we are interested in the chord lengths P_iN_i, which is $R \sin i\theta$. If we see the triangle P_2N_2O, the chord length P_2N_2 is the sine of the right-angle triangle of angle θ_2. For any chord in between these 24 chords, 1st order or 2nd order interpolation is used to get the intermediate values.

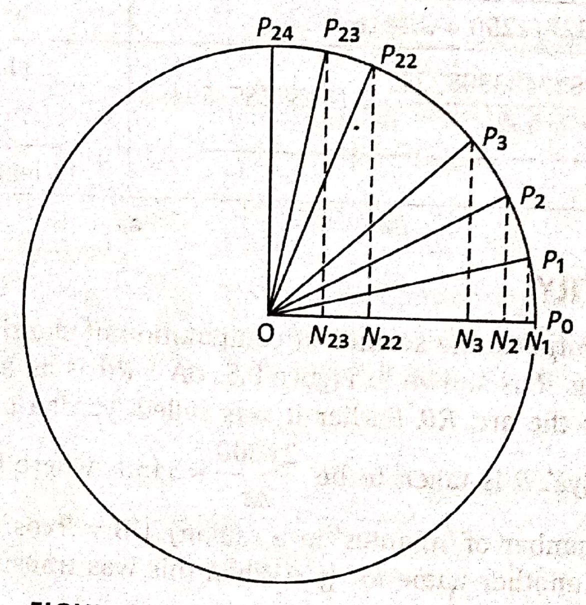

FIGURE 8.6 Arcs of Quadrant of a Circle

Āryabhaṭa has provided two methods to derive the sine tables. A geometric method, and an analytical method which resembles the discrete version of the harmonic equation as we know today.

Āryabhaṭa's Formula for R sine Differences

In the Gaṇita-pāda verse 12, Āryabhaṭa provides a formula for computing the r sine differences. The meaning of the verse is as follows[9]:

"The first R sine divided by itself and then diminished by the quotient will give the second difference. For computing any other difference. The sum of all the preceding differences is divided by the first R sine and the quotient is subtracted from the preceding difference. Thus, all the remaining differences (can be calculated)."

Let $R_1, R_2, R_3, \ldots, R_{24}$, denote the 24 R sines and δ_1 $(=R_1)$, $\delta_2, \delta_3, \delta_4, \ldots, \delta_{24}$ denote the 24 R sine-differences. Then, according to the above verse,

$$\delta_2 = R_1 - \frac{R_1}{R_1}$$

$$\delta_{n+1} = \delta_n - \frac{\delta_1 + \delta_2 + \delta_3 + \cdots + \delta_n}{R_1} = \delta_n - \frac{R_n}{R_1}$$

Nīlakaṇṭha's Formula for R sine Differences

Nīlakaṇṭha (1500 CE) provided a much accurate estimate of R sine differences by providing a correction to the above formula. According to him, "the first R sine divided by itself and then diminished by the quotient gives the second R sine difference. To obtain the other R sine differences, divide the preceding R sine by the first R sine and multiply the quotient by the difference between the first and second R sine differences and subtract the resulting product form the preceding R sine difference". It is stated mathematically below:

$$\delta_{n+1} = \delta_n - \left(\frac{R_n}{R_1}\right)(\delta_1 - \delta_2)$$

This is the same as the relation

$$2 \sin nx - \sin\{(n+1)x\} - \sin\{(n-1)x\} = (\sin nx/\sin x)\ (2 \sin x - \sin 2x)$$

This approach to computing R sine difference was extraordinary to be conceived at the time of Āryabhaṭa. French mathematician and astronomer Jean Baptiste Joseph Delambre, (1749–1822) paid rich tributes to Āryabhaṭa's analytical method, "the method is curious: it indicates a method of calculating the table of sines using their second differences ..., the differential process has not up to now been employed except for Briggs, who himself did not know that the constant factor was the square of the chord. Here then is a method, the Indians possessed, and which is found neither among Greeks not amongst the Arabs[10]".

8.6 ALGEBRA

Algebra became the core of mathematics in the 20th century, which many historians termed as 'algebraization of mathematics'. Algebra provided simplicity, clarity, and precision, to the mathematicians. Algebra is one of the main areas of contribution by ancient Indians as they considered it as a subject of great utility. The ancient Indian name for the science of algebra is bīja-gaṇita. Bīja means 'element' or 'analysis' and gaṇita 'the science of calculation.'

The science of algebra is broadly divided by the Indian mathematicians into two major parts. Of these, one deals with analysis (bīja). The other part discusses concepts that are essential for analysis. It includes the laws of signs, the arithmetic of zero (and infinity), operations with unknowns, surds, and the linear indeterminate equation (known as kuṭṭaka or pulveriser). Therefore, one is able to identify several important features of algebra in the Indian mathematical works. For instance, symbols were used for unknowns by Brahmagupta (it is also indicated in Āryabhaṭīya and its commentary by Bhāskara). Operations with negative numbers were also introduced by Brahmagupta. Furthermore, we can notice that linear and quadratic equations were solved by Āryabhaṭa and Brahmagupta. The Indian mathematicians recognised few fundamental operations in algebra, viz, addition, subtraction, multiplication, division, squaring and the extraction of the square-root.

Recognition of negative numbers and their treatment in mathematical operations were evident very early. Brahmagupta (628 CE) says: "The sum of two positive numbers is positive, of two negative numbers is negative; of a positive and a negative number is their difference." The use of symbols to denote unknowns, working on equations to solve unknown were the key contributions of ancient Indians. Indians discovered the usage of fundamental arithmetic operators *'yu'* for *yuta* – addition; *'kṣa'* for *kṣaya* – subtraction; *'gu'* for *guṇa* – multiplication; *'bhā'* for *bhāga* – division. They used *'mū'* or *'ka'* for *mūla* or *karaṇi* denoting root. They used the first letter of the names of different colors to denote different unknown variables. With these, Indians analysed and classified equations – called *samīkaraṇa*.

Āryabhaṭa in his work *Āryabhaṭīya (Gaṇitapāda,* Verse 24), discusses an interesting case of how to determine the two numbers whose product and difference are known[11]. The procedure explained in the verse can be algebraically stated as follows. Let us consider two numbers x and y. Let $x - y = a$ and $xy = b$. Then as per the verse, x and y can be determined as follows:

$$x = \frac{\sqrt{4b + a^2} + a}{2}; \; y = \frac{\sqrt{4b + a^2} - a}{2}$$

8.7 BINARY MATHEMATICS AND COMBINATORIAL PROBLEMS IN CHANDAḤ-ŚĀSTRA OF PIṄGALA (300 BCE)

We saw the work of Piṅgala known as Chandaḥ-śāstra in Chapter 2 and Chapter 6. In this section, we shall study his contributions in the field of binary mathematics and combinatorial problems. The metres in Sanskrit (chandas) were analysed by means of two kinds of syllables, laghu (L), and guru (G) from a prosody perspective. Any syllabic metre (varṇa-vṛtta) is therefore characterized as a sequence of L and G. If we replace laghu (L) with '1' and guru (G) with '0', then it transforms the metrical pattern into a binary sequence. A binary sequence is a sequence composed of '1's or '0's. For example, 1,0,0,1 and 0,1,0,0 or examples of binary sequences of length 4. In Chapter 8 of Chandaḥ-śāstra, Piṅgala has analysed a host of combinatorial problems relating to metrical patterns or, equivalently, binary sequences. These are as follows:

- *Prastāra:* A procedure by which all possible metrical patterns or, equivalently, binary sequences of a given length are generated sequentially as an array (prastāra).
- *Saṃkhyā:* The process of finding the total number of binary sequences (rows) in the prastāra (array).
- *Naṣṭa:* For a given row number in an array, the process of identifying the corresponding binary sequence directly.
- *Uddiṣṭa:* Given a binary sequence, the process of identifying the corresponding row number in the array (prastāra), directly.
- *Lagakriyā:* The process of finding the number of binary sequences in the array with a given number of '1's or '0's.
- *Adhvayoga:* The process of finding the space occupied by the array (prastāra) (to determine the floor area needed).

These concepts introduced by Piṅgala around 300 BCE are perhaps the earliest instance of computations involving binary numbers. The basic building block of this binary analysis is the generation of the prastāras. Sūtras 20–23 in Chapter 8 of Chandaḥ-śāstra provide the details for the generation of the binary arrays[12]. The method indicated in the sūtras is as follows:

To begin with, the array of binary sequences of length one is a table of two rows (see Figure 8.7), row 1 = 0, row 2 = 1. Every new array of increasing length is generated using a simple procedure. This is demonstrated in Figure 8.7. At every iteration, the existing array is replicated, followed by adding one more column to the replicated array. In the first half of the new column '0' is inserted and in the second half, '1'. The procedure can be repeated in this manner to generate array of binary sequences of length. As seen in Figure 8.7, in the second array, column 1 in rows 1 to 2 and rows 3 to 4 are merely the repeat of the one-syllable array of length 1. In the second column '0' and '1'. are repeated. Similarly, in the third array, the first two columns in rows 1 to 4 and rows 5 to 8 are the repeat of the array of sequences of length two. In the last array, the first three columns in rows 1 to 8 and rows 9 to 16 are the repeat of the array of sequences of length three. One can proceed in this manner to generate arrays of binary sequences of any length in a recursive fashion.

1	0
2	1

Sequence of length = 1

1	0	0
2	1	0
3	0	1
4	1	1

Sequence of length = 2

1	0	0	0
2	1	0	0
3	0	1	0
4	1	1	0
5	0	0	1
6	1	0	1
7	0	1	1
8	1	1	1

Sequence of length = 3

1	0	0	0	0
2	1	0	0	0
3	0	1	0	0
4	1	1	0	0
5	0	0	1	0
6	1	0	1	0
7	0	1	1	0
8	1	1	1	0
9	0	0	0	1
10	1	0	0	1
11	0	1	0	1
12	1	1	0	1
13	0	0	1	1
14	1	0	1	1
15	0	1	1	1
16	1	1	1	1

Sequence of length = 4

FIGURE 8.7 Progressive Generation of Binary Tables of Increasing Length (Prastāras of Piṅgala)

As we can see, the number of binary sequences of length n is 2^n. This is called the Saṃkhyā or the total number of rows in the array (prastāra). We can also see that there is a relation between a binary sequence and the row in which it occurs in the array. This relation is the following:

Row-number of a binary sequence = Mirror image of the binary sequence (viewed as a binary number) +1

For example, the fifth row in the array of sequences of length 3 is 001. The mirror image of this is 100. This, when viewed as binary number, is: $2^2.1 + 2.0 + 1.0 = 4$; and $4 + 1 = 5$.

Similarly, the 14th row of array of sequences of length 4 is 1011. The mirror image of this is 1101. This, when viewed as a binary number, is: $2^3.1 + 2^2.1 + 2.0 + 1.1 = 13$; and $13 + 1 = 14$.

We shall now describe the methods given by Piṅgala to identify the binary sequence that is associated with a particular row number in a given array and vice-versa. Piṅgala called these processes *Naṣṭa and Uddiṣṭa.*

Let us consider the array of length 4 (the last array in Figure 8.7). There are some interesting mathematical problems that one encounters. For example, given a row-number, how can we identify the binary sequence associated with that row in the array. Piṅgala provides a simple algorithm to identify the binary sequence, known as Naṣṭa[13]. The steps are as follows:

- Start with the desired row number.
- Divide it by 2. If it is perfectly divisible, place '1' in the sequence.
- If the number is not divisible by two, place '0' in the sequence, add one to the number and divide.
- Repeat the process until the binary sequence of required length is obtained.

For the array of length 4, let us identify the binary sequence which is associated with row 13 using the above algorithm:

13 is not divisible by 2, we place '0' in the sequence, add one to it and divide it by two. The new number is '7'.
7 is not divisible by 2, we place '0' in the sequence, add one to it and divide it by two. The new number is '4'.
4 is divisible by 2, we place '1' in the sequence. The new number is 2.
2 is divisible by 2, we place '1' in the sequence.
The required sequence of length 4 is obtained. We stop the operation. The answer is '0011'.

The converse of the above problem is that given a binary sequence, how to identify the row position in the array? This was referred to as Uddiṣṭa by Piṅgala, for which he has given an algorithm[14]. The steps of the algorithm are as follows:

1. Start with number 1. Current number = 1.
2. Scanning the binary sequence from the right, locate the first '1' in the sequence. Multiply the current number by 2.
3. Move the next number to the left. When a '1' is encountered, double the current number. If '0' is encountered double it and subtract 1 from it.
4. Continue this till the end of the binary sequence to arrive at the row number of the sequence.

Let us consider the following sequence from the array of length 4–'0111'. Applying the above algorithm, we get the following:

Start with the current number = 1.
The first number is '1'. Therefore, the current number is $1 \times 2 = 2$.
The second and third numbers are also '1'. So, we double the earlier result twice. $2 \times 2 \times 2 = 8$.
The last number is '0'. So, we double the last result and then subtract 1 from it. $8 \times 2 - 1 = 15$.
Therefore, the row position is 15.

EXAMPLE 8.4: Find the binary sequence associated with 37th array of length of 6.

Using the Naṣṭa algorithm of Piṅgala, we obtain the sequence as follows:

37 is not divisible by 2, we place '0' in the sequence, add one to it and divide it by two. The new number is '19'.

19 is not divisible by 2, we place '0' in the sequence, add one to it and divide it by two. The new number is '10'.

10 is divisible by 2, we place '1' in the sequence. The new number is '5'.

5 is not divisible by 2, we place "0" in the sequence, add one to it and divide it by two. The new number is '3'.

3 is not divisible by 2, we place '0' in the sequence, add one to it and divide it by two. The new number is '2'.

2 is divisible by 2, we place '1' in the sequence.

The required binary sequence of length 6 is obtained. We stop the operation. The answer is '001001'.

EXAMPLE 8.5: Identify the row number of the binary sequence '1010100' array of length 7.

We use the Uddiṣṭa algorithm to arrive at the position in the array.

We start with number 1.

The first letter '1' is found in the third position from the left. Therefore, the number is $1 \times 2 = 2$.

The next letter is '0'. So, we double the last result and then subtract 1 from it. $2 \times 2 - 1 = 3$.

The next letter is '1'. We double the last result. The number is $3 \times 2 = 6$.

The next letter is '0'. We double the last result and then subtract 1 from it. $2 \times 6 - 1 = 11$.

The last number is '1'. We double the last result. $11 \times 2 = 22$.

Therefore, the row position is 22.

Another interesting problem is to find out how many binary sequences of length '*n*' there are, that contain '*r*' number of '1's. Essentially, this boils down to the combinatorial problem of choosing '*r* out of *n*', which leads to the binomial co-efficient ^nC_r. Piṅgala refers to this problem as lagakriyā. The procedure is best explained by the "Varṇa-Meru" of Piṅgala. Figure 8.8 has a schematic representation of Varṇa-Meru. The construction follows a simple algorithm. Start with a single square. Place '1' in the square. Below this are successive rows with one more number of square compared to the previous row in the top. In each square, numbers have to be placed as per the following rule:

The number to be placed inside any square is the sum of the numbers contiguous to this square in the row above. Continue this procedure to construct the triangle shown in Figure 8.8. For the two squares in the second row, there is only one square above. Therefore, we place the number 1 in each of the squares. In the third row, the second square is contiguous to both the squares above. Therefore, we place the number $1 + 1 = 2$. In the same way we get 1,3,3,1 in the fourth row. In this manner we may proceed to construct the Varṇa-Meru shown in Figure 8.8. The same triangle of binomial coefficients was re-discovered by the French mathematician Blaise Pascal (1655 CE). It is therefore generally referred to as Pascal's triangle in modern books.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

FIGURE 8.8 Varṇa Meru (due to Piṅgala)

Thus, we can see that Piṅgala laid a solid foundation for combinatorial theory and binary mathematics, which was followed by later-day Mathematicians in the country. *Varāhamihira's Bṛhatsaṃhitā* (550 CE), mentions 1820 different combinations that can be obtained by choosing 4 perfumes from a set of 16 basic perfumes. ($^{16}C_4 = 1820$). Varāhamihira also discusses the construction of a *Meru*–tabular form for arriving at this binomial coefficient.

8.8 MAGIC SQUARES IN INDIA

Mathematics can be utilitarian and at the same time a tool for human curiosity, fun, and a matter for mind-expanding exercise. One such example is the magic square problems in arithmetic. An array of numbers containing an equal number of rows and columns is called a magic square when the summation of the numbers in every column and those in every row and each diagonal happens to be the same. On the other hand, if the summation of the 'other' diagonals also is the same, then such a square is called a pan-diagonal magic square. Figure 8.9 is a simple illustration of the magic squares. In the first square [Figure 8.9(a)] the row sums, column sums, and the sum of the main diagonals are 34. However, the other diagonals do not add up to 34. For example, the shaded diagonal (6, 5, 7, and 8) does not add up to 34. Please note that the matrix structure is to be imagined like a torus (a rolled version like a cylinder). Therefore, after 7, the next number in the diagonal will be 8. On the other hand, in Figure 8.9(b), all the diagonals add up to 34. This is a pan-diagonal magic square.

12	3	6	13
14	5	4	11
7	16	9	2
1	10	15	8

(a) Magic square

10	3	13	8
5	16	2	11
4	9	7	14
15	6	12	1

(b) Pan-diagonal magic square

FIGURE 8.9 Magic and Pan-diagonal Magic Square; Magic Sum = 34

The construction of such magic squares has been known in India from very early times. Indian mathematicians specialized in the construction of a special class of magic squares called *sarvatobhadra* or *pan-diagonal magic squares*. Bhadra-gaṇita is the name for the study of magic squares in the Indian mathematical tradition. Various 3 × 3 magic squares are attributed to the ancient astronomer Garga (100 BCE). The work Kakṣapuṭa of Nāgārjuna (100 CE) describes a method for constructing 4 × 4 magic squares, which gives pan-diagonal squares when the magic sum is even. Magic squares have been used to address certain combinatorial issues involving arithmetic. For example, Varāramihira (587 CE) made use of a 4 × 4 magic square to specify varying proportions of four ingredients that can be mixed to prepare different perfumes. 4 × 4 magic squares are found on the gates of buildings, on the walls where shopkeepers transact their business, and on the covers of calendars used by astrologers even to this day. A 4 × 4 square occurs in a Jaina inscription of the 11th century CE, found in the ancient town of Khajuraho. A similar square dated to 1480 CE was found in the Gwalior fort. The first chapter of the notebooks of Srinivasa Ramanujan is on magic squares.

Construction of a 4 × 4 Pan-diagonal Magic Square

Perhaps the easiest magic square to understand is the one presented by Nāgārjuna in his work Kakṣapuṭa, where he provided a mnemonic and a formula to construct a generic magic square. The mnemonic is as follows: "अर्क इन्दुनिधा नारी तेन लग्न विनासनम्" (*arka indunidhā nārī tena lagna vināsanam*). There are 16 syllables in this mnemonic. Using the katapayādi formula (see Chapter 6 for details), the above mnemonic can be converted into 16 numbers. All vowels are zero in kaṭapyādi. Therefore, in the mnemonic, the first number is zero. Populating the 4 × 4 magic square with this number gives us the first template. See Figure 8.10(a) for details. Let the desired sum of the magic square be $2n$. Then using the formula given in Figure 8.10(b) the individual cells of the magic square can be arrived at. For example, if the desired sum is 100, then n = 50. The magic square for this is available in Figure 8.10(c).

0	1	0	8
0	9	0	2
6	0	3	0
4	0	7	0

(a)

n-3	1	n-6	8
n-7	9	n-4	2
6	n-8	3	n-1
4	n-2	7	n-9

(b)

47	1	44	8
43	9	46	2
6	42	3	49
4	48	7	41

(c)

FIGURE 8.10 Nāgārjuna's Scheme for Generating a 4 × 4 Pan-diagonal Magic Square with an Illustration for n = 50

The mathematics involved in the construction of magic squares and other magic figures was first systematically and elaborately discussed by the mathematician Nārāyaṇa (1356 CE) in his Gaṇitakaumudī. Some of his methods were unknown in the west and were recently discovered by the efforts of several scholars. The last chapter of the seminal work Gaṇitakaumudī is devoted to Bhadragaṇita, which contains 55 verses giving rules and 17 verses giving examples. Consider an nth order magic square. Nārāyaṇa classifies the magic squares into three categories:

(a) if $n = 4m$, then it is called *Samagarbha* (double even squares)

(b) if $n = 4m + 2$, then it is *Viṣamagarbha*, and (semi-double even squares)

(c) if $n = 2m + 1$, then it is *Viṣama* (odd squares), where $m = 1, 2, 3, \ldots$

After classifying the squares into samagarbha, viṣamagarbha, and viṣama, Nārāyaṇa discusses a generalized method to construct the magic square for the desired sum, S and the size of the square, n. The n^2 numbers to be filled in an $n \times n$ magic square follow an arithmetic progression with an initial number a, and the common difference d. Nārāyaṇa presented a general method for filling up a 4 × 4 pan-diagonal magic square by an arithmetic sequence using a succession of *turaga-gati* (horse movements)[15]. Positioning 1 at the top left corner he generated 24 such squares and concluded that in all there are 384 such pan-diagonal squares. In the 20th century a few mathematicians studied the 4 × 4 pan-diagonal magic squares and also concluded that only 384 such pan-diagonal squares are possible. Further, using some more analysis they came up with certain properties of the 4 × 4 pan-diagonal magic squares. These properties are useful for constructing a 4 × 4 pan-diagonal magic square.

	3		
5	16	2	
	9		
			1

(a)

10	3	13	
5	16	2	
4	9	7	
			1

(b)

10	3	13	8
5	16	2	
4	9	7	14
15		12	1

(c)

10	3	13	8
5	16	2	11
4	9	7	14
15	6	12	1

(d)

FIGURE 8.11 Construction of a 4 x 4 Pan-diagonal Magic Square

Property 1: In a pan-diagonal 4 × 4 magic, the entries of any 2 × 2 sub-square formed by consecutive rows and columns add up to the magic.

Property 2: In a pan-diagonal 4 × 4 magic square the sum of an entry with another which is two squares away from it along a diagonal is always half the magic sum.

Property 3: In a 4 × 4 magic square with entries, 1, 2, ..., 16, each element has the same set of neighbours in each of the 384 pan-diagonal versions. In particular, the element 16 has as neighbours 2, 3, 5, 9.

Using these properties, we can construct a 4 × 4 pan-diagonal matrix using numbers 1, 2, 3, ..., 16 as follows:

Step 1: First we place 1 in any of the cells and place 16 two cells diagonally away from 1.

Step 2: Using property 3, we generate the first few elements of the 4 × 4 pan-diagonal magic square by placing 2, 3, 5, and 9 in any order as neighbours of 16 (Figure 8.11(a)).

Step 3: The magic sum for a 4 × 4 pan-diagonal magic square for numbers 1 to 16 is 34. We use the above properties to fill all the remaining cells in the magic square. For example,

(a) Using property 1 we fill in some of the adjacent cells as shown in Figure 8.11(b).

(b) Using property 2 to fill in some more cells (highlighted cells in Figure 8.11(c)).

(c) Finally using property 1, we fill the balance cells (highlighted cells in Figure 8.11(d)).

SUMMARY

- Ancient Indians developed several concepts of mathematics primarily because they needed to solve a lot of real-life problems that they were facing even during the Vedic period.
- Mathematical concepts were developed by those living from Gāndhāra (modern-day Afghanistan) to those in Bengal, as well as by those from Kashmir to Kerala.
- Brahmagupta, in his work Brahma-sphuṭa-siddhānta, gives a good description of working with fractions, calculations with positive, negative numbers, and with zero.
- Several sources in the ancient Indian texts point to multiple attempts and methods to obtain the square root of an imperfect square.
- *Āryabhaṭīya* gives a good indication that ancient Indians had a well-developed algorithmic approach to problem-solving and were able to utilise recursive algorithms.
- Śulba-sūtras, a section of the Kalpa part of the Vedāṅga, have dealt with the subject of geometry in detail.
- Āryabhaṭīya, Līlāvatī and other Indian mathematical works have several interesting problems of day-to-day importance described as a game in poetic verses.
- Right from the Vedic period, there has been a continuous evolution of the approximation of π till the 20th century. Mādhava estimates π using an infinite series with end correction, which is accurate to eleven decimal places.
- Āryabhaṭa has provided two methods to derive the sine tables. A geometric method, and an analytical method which is very unique and not found in works of any other mathematician, till about 15th century.
- In Chapter 8 of Chandaḥ-śāstra, Piṅgala has analysed a host of problems related to handling binary sequences. The concepts developed during 200–300 BCE are relevant for the modern-day computations involving binary numbers.
- The construction of magic squares has been known in India from very early times. Bhadra-gaṇita is the name for the study of magic squares in the Indian mathematical tradition.

REVIEW QUESTIONS

1. Enumerate the unique aspects of Indian Mathematics. Can you illustrate them with some examples?
2. Name three great Indian mathematicians and enumerate their key contributions.
3. Identify three major areas of contribution of Ancient Indian mathematicians.
4. What key inferences can we draw by an examination of the key contributions of Indian mathematicians?

5. Why was knowledge of geometry important for the ancient Indians?
6. Briefly explain about the knowledge of π that Indian Mathematicians possessed. How does it compare with that of other Mathematicians?
7. What was the motivation for Indians to study the right-angled triangle and other geometric shapes?
8. Comment on the statement, "Indian Mathematics is a blend of poetry, literature, and mathematics". Give examples to support your views.
9. What are the key contributions of Piṅgala to modern mathematical thinking?
10. What is the difference between a magic square and a pan-diagonal matrix square?
11. Construct a 4 × 4 pan-diagonal magic square. What is the magic sum for the square that you constructed?

EXERCISE PROBLEMS

1. Calculate the square of the following numbers using the Āryabhaṭa method:
   (a) 149 (b) 2347 (c) 642 (d) 369 (e) 1777
2. Find the varga and avarga sthānas of the following numbers:
   (a) 17,342 (b) 1,23,456 (c) 69,900,342
3. Verify the answers that you obtained in question (1) by deriving the square root of the answers using the Āryabhaṭa method.
4. Find the square root of the following numbers using the Āryabhaṭa method:
   (a) 21,609 (b) 2,85,156 (c) 56,644 (d) 9,27,369 (e) 31,329
5. Verify the answers that you obtained in question (4) by finding the square of the answers using the Āryabhaṭa method.
6. Use the Bhakshali method to find the square root of the following numbers:
   (a) 126 (b) 912 (c) 8,174 (d) 21,924 (e) 83,369
   (*Hint*: Find the nearest perfect square of the number to identify A and b).
7. Construct a table of five-letter binary word (Piṅgala's prastāra).
   (*Hint*: Use the four-letter binary table to construct the five-letter binary table).
8. Consider a five-letter binary table and identify the pattern for the following rows using the naṣṭa algorithm of Piṅgala:
   (a) Row 7 (b) Row 16 (c) Row 25 (d) Row 31
9. For a five-letter binary table identify the row number corresponding to each of the binary word using Piṅgala's uddiṣṭa algorithm:
   (a) 10101 (b) 00101 (c) 10010 (d) 11011 (e) 01010

DISCOVER IKS

1. Indian's contribution to the field of Mathematics is not known as not much effort has been made to bring it to the attention of interested people. Go through the video available at the following site to further reinforce your understanding of the contribution of Ancient Indians in the area of Mathematics: https://www.youtube.com/watch?v=huJDNh0G3kw. After watching the video, develop a three-page note to answer the following questions:
   (a) What are the key contributions of Indians in the area of Mathematics?
   (b) Identify two Mathematicians and their key contributions?
   (c) How do the Indian Mathematicians' contributions compare with that of the Western counterpart?

2. Bhāskarācārya II, one of the greatest Mathematicians of India produced great mathematical works including mathematics, number system, and astronomy. Watch the video on Bhāskarācārya II available in the following site: https://www.youtube.com/watch?v=WoJGJeOyLEc. After watching the video, develop a three-page note to answer the following questions:
   (a) Briefly sketch the biographical sketch of Bhāskarācārya II. When did he compose the Līlāvatī?
   (b) What are the major works of Bhāskarācārya II? Explain what the issues are covered in his works.
   (c) How did Bhāskarācārya II deploy unique real-life examples to introduce interesting mathematical concepts and problems? Give some examples.

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ENDNOTES

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2. For more details see, https://en.wikipedia.org/wiki/Arithmetic. Last accessed on Oct. 1, 2021.
3. वर्गः समचतुरश्रः फलं च सदृशद्वयस्य संवर्गः। vargaḥ samacaturaśraḥ phalaṃ ca sadṛśadvayasya saṃvargaḥ |
4. For more details see, **Shukla, K.S. and Sharma, K.V. (1976).** "Āryabhaṭīya of Āryabhaṭa", Indian National Science Academy, pp. 37–38.
5. एका च मे तिस्रश्च मे पञ्च च मे सप्त च मे नव च म एकादश च मे त्रयोदश च मे पञ्चदश च मे सप्तदश च मे नवदश च म एकविंशतिश्च मे त्रयोविंशतिश्च मे पञ्चविंशतिश्च मे सप्तविंशतिश्च मे नवविंशतिश्च म एकत्रिंशच्च मे त्रयस्त्रिंशच्च मे चतस्रश्च मेऽष्टौ च मे द्वादश च मे षोडश च मे विंशतिश्च मे चतुर्विंशतिश्च मेऽष्टाविंशतिश्च मे द्वात्रिंशच्च मे षट्त्रिंशच्च मे चत्वारिंशच्च मे चतुश्चत्वारिंशच्च मेऽष्टाचत्वारिंशच्च मे ... ekā ca me tisraśca me pañca ca me sapta ca me nava ca ma ekādaśa ca me trayodaśa ca me pañcadaśa ca me saptadaśa ca me navadaśa ca ma ekaviṃśatiśca me trayoviṃśatiśca me pañcaviṃśatiścame saptaviṃśatiśca me navaviṃśatiśca ma ekatriṃśacca me trayastriṃśacca me catasraśca me'ṣṭau ca me dvādaśa ca me ṣoḍaśa ca me viṃśatiśca me caturviṃśatiśca me'ṣṭāviṃśatiśca me dvātriṃśacca me ṣatriṃśacca me catvāriṃśacca me catuścatvāriṃśacca me'ṣṭācatvāriṃśacca me ... **R.L. Kashyap (2005).** "Yajur Veda Taitirīya Samhita" Sri Aurobindo Kapāli Sāstry Institute of Vedic Culture, Banglore, Volume 3, Kāṇḍa 4, pp. 448.
6. For a discussion on this, see, **Dutta, B. and Singh, A.N. (1993).** "Use of Series in India: Revised by Kripa Shankar Shukla" *Indian Journal of History of Science,* **28**(2), pp. 103–129.
7. दीर्घचतुरश्रस्याक्ष्णया रज्जुः पार्श्वमानी तिर्यङ्मानी च यत् पृथग्भूते कुरुतस्तदुभयं करोति dīrghacaturaśrasyākṣṇayā rajjuḥ pārśvamānī tiryaṅmānī ca yat pṛthagbhūte kurutastadubhayaṃ karoti | Baudhāyana Śulba Sūtra (1.12).
8. व्यासे भनन्दाग्निहते विभक्ते खबाणसूर्यैः परिधिः सुसूक्ष्मः । द्वाविंशतिघ्ने विहृतेऽथ शैलैः स्थूलोऽथवा स्याद् व्यवहारयोग्यः ॥ vyāse bhanandāgnihate vibhakte khabāṇasūryaiḥ paridhiḥ susūkṣmaḥ | dvāviṃśatighne vihṛte'tha śailaiḥ sthūlo'thavā syād vyavahārayogyaḥ || **Padmanabha Rao, A.B. (2014).** "Bhāskarācārya's Līlāvatī" Part–II, Chinamaya International Foundation Shodha Sansthan Ernakulam, Kerala, India, Verse 199.
9. प्रथमाच्चापज्यार्धाद्यैरूनं खण्डितं द्वितीयार्धम् । तत्प्रथमज्यार्धांशैस्तैस्तैरूनानि शेषाणि ॥ prathamāccāpajyārdhādyairūnaṃ khaṇḍitaṃ dvitīyārdham | tatprathamajyārdhāṃśaistaistairūnāni śeṣāṇi || (Gaṇita-pāda, Verse 12).
10. **Datta, B. and Singh, A.N. (1983).** "Hindu Trigonometry", *Indian Journal of History of Science,* 18, p. 77.
11. द्विकृतिगुणात् संवर्गाद् द्व्यन्तरवर्गेण संयुतान्मूलम् । अन्तरयुक्तं हीनं तद् गुणकारद्वयं दलितम् ॥ dvikṛtiguṇāt saṃvargād dvyantaravargeṇa saṃyutānmūlam | antarayuktaṃ hīnaṃ tad guṇakāradvayaṃ dalitam || For details see, **Sharma, K.V. (1976).** "Āryabhaṭīya of Āryabhaṭa", Indian National Science Academy, New Delhi, Verse 24.
12. द्विकौ ग्लौ । मिश्रौ च । पृथग्लामिश्राः । वसवस्त्रिकाः । dvikau glau | miśrau ca | pṛthaglāmiśrāḥ | vasavastrikāḥ | Chandaḥ Śāstra 8.20–8.23. See for details, **Paṇḍit Kedārnāth and Vāsudev Laxmaṇ Shāstri Paṇaśikar (1927).** *Chandaḥ Shāstra*, Nirṇaya Sāgar Press, Mumbai.
13. *Ibid.* लर्धे । सैके ग् । lardhe | saike g | 8.24–8.25.
14. *Ibid.* प्रतिलोमगणं द्विर्लाद्यम् । ततोग्येकं जह्यात् । pratilomagaṇaṃ dvirlādyam | tatogyekaṃjahyāt | Chandas Śāstra, 8.26–8.27.
15. For a discussion on the magic squares, see for example, **Dutta, B. and Singh, A.N. (1976).** "Magic Squares in India: revised by Shukla K.S.", *Indian Journal of History of Science,* **27**(1), pp. 51–120.