PROBABILITY, RANDOM VARIABLES &NORMAL DISTRIBUTION

Syllabus

Axioms of probability - Conditional probability - Total probability – Baye's theorem - Random variable – Distribution function – properties - Probability mass function - Probability density function – moments and moment generating function .

Basic Concepts:

Random experiment: It is an experiment which can be repeated any number of times under the same conditions, but does not give unique results. The result will be any one of several possible outcomes, but for each trial, the result will not be known in advance. A random experiment is also called a trial and the outcomes are called events.

Eg: Tossing a coin is a trial, getting head is an event.

Sample space: The totality of all possible outcomes of a random experiment is called a sample space S and a possible outcome, or element in a sample space is called a sample point.

Eg: in throwing a die, S = {1,2,3,4,5,6}

Equally likely events: Two or more events are equally likely if each of them has an equal chance of happening.

Mutually exclusive events: Two events are said to be mutually exclusive if the occurrence of any one of them excludes the occurrence of the other in a single experiment.

Eg. If a coin is tossed, the events of getting H and T are mutually exclusive.

Eg: Consider the following two events:

A — a randomly chosen person has blood type A, and

B — a randomly chosen person has blood type B.

Events A and B are mutually exclusive or disjoint.

Eg: Consider the following two events:

A — a randomly chosen person has blood type A

B — a randomly chosen person is a woman.

In this case, it **is** possible for events A and B to occur together.

Events A and B are not disjoint

Independent events. Two or more events are independent if the occurrence of one does not affect the occurrence of the other.

Eg. If a coin is thrown twice, the result of the second throw is not affected by the result of the first throw.

Exhaustive events: When a list of possible events that can result from an experiment includes every possible outcome, the list is said to be exhaustive.

Complementary events: If A and B are mutually exclusive and exhaustive events, then A is the complementary event of B and vice versa

Eg: When a die is thrown, getting an even number and getting an odd number are complementary events

Classical or mathematical or apriori definition of probability

If there are m equally likely, mutually exclusive and exhaustive outcomes and m of them are favourable to an event A, then

$$P(A) = \frac{m}{n} = \frac{\text{Number of favourable cases}}{\text{Total number of exhaustive cases}}$$

Note: 1. $P(A) \leq 1$

2. Probability that A does not happen = $P(\bar{A}) = \frac{n-m}{n}$

3. $\therefore P(A) + P(\bar{A}) = 1$

Statistical or posterior definition

If an experiment is repeated a large number of times under identical and homogeneous conditions, then the limiting value of the ratio of the number of times the event A happens to the total number of trials, as the number of trials increases indefinitely , is called the probability of A

If the event A happens m times out of n repititions of a random experiment, then $P(A) = \lim_{n \to \infty} \frac{m}{n}$

Axioms of Probability

Let S be a sample space and A be an event. Let P be a real valued function defined on $\wp(S)$. $P(A)$ is called the probability of A if P satisfies the following axioms:

(i) For every event A, $0 \leq P(A) \leq 1$

(ii) $P(S) = 1$

(iii) If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$

Note: Axiom (i) – Axiom of non-negativity states that the probability of an event is a non – negative real number

Axiom (ii) – Axiom of certainty states that the probability of a sure event is 1.

Axiom (iii) – Axiom of union

Theorem 1: Probability of an impossible event is zero. i.e., $P(\varphi) = 0$.

Theorem 2: $P(\bar{A}) = 1 - P(A)$ where $\bar{A}$ is the complementary event of A.

Theorem 3: Addition law of probability

If A and B are any two events, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Corollary: If A, B, C are any three events, then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

Theorem 4: If A and B are two events such that $B \subseteq A$, then $P(B) \leq P(A)$

Independent events: Two events A and B are independent if

$$P(A \cap B) = P(A).P(B)$$

Problems

1. One card is selected at random from a pack of 52 cards. What is the probability that it is either a king or an ace?

Answer:

Let A be the event that the card is a king

B be the event that the card is an ace.

2. A and B are events with $P(A) = \frac{3}{8}, P(B) = \frac{1}{2}, P(A \cap B) = \frac{1}{4}$. Find $P(A^c \cap B^c)$

Answer:

3. Four persons are chosen at random from a group containing 3 men, 2 women and 4 children. Find the probability that exactly two of them will be children.

Answer:

Total number of persons = $3 + 2 + 4 = 9$.

4. Three coins are tossed together. Find the probability that there are exactly 2 heads.

Answer:

Sample space =

5. The probability that A solves a problem is $\frac{2}{3}$ and the probability of A and B solving the problem is $\frac{14}{25}$. The probability of A or B solving the problem solving the problem is $\frac{4}{5}$. Calculate the probability of B solving the problem.

Answer:

6. From a well shuffled deck of 52 playing cards, 4 cards are selected at random. Find the probability that the selected cards are

i) 3 spades and 1 heart

ii) 2 kings, 1 ace and 1 queen

iii) All are diamonds

iv) There is one card of each suit.

v) All the four are hearts and one of them is a jack.

vi) A spade, a heart, a diamond and a club in that order

Answer:

7. What is the chance that a leap year selected at random will contain 53 Sundays?

Answer:

8. Three horses A, B, C are in a race. A is twice as likely to win as B and B is twice as likely to win as C. What are their respective probabilities of winning?

Answer:

Let A, B, C be respectively the events that A, B, C win the race.

9. If $P(A) = 0.6$, $P(\bar{B}) = 0.5, P(\bar{A} \cup \bar{B}) = 0.8$, find $P(A \cup B)$.

Answer:

10. Let A and B be two events, such that $P(A) = \frac{3}{4}$, $P(B) = \frac{5}{8}$

Show that (i) $P(A \cup B) \geq \frac{3}{4}$ (ii) $\frac{3}{8} \leq P(A \cup B) \leq \frac{5}{8}$

Answer:

11. Three groups of children contain 3 girls, 1 boy; 2 girls, 2 boys; 1 girl and 3 boys respectively. One child is selected at random from each group. Show that the chance that the three selected consists of 1 girl and 2 boys is 13/32.

Answer:

12. A problem is given to 3 students A, B, C whose chances of solving it are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ respectively. What is the probability that

(i) The problem is solved

(ii) Exactly one of them solves the problem.

Answer:

13. A problem in Statistics is given to five students A, B, C, D and E. Their chances of solving it are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$. What is the probability that the problem will be solved?

Answer:

14. A is known to hit the target in 2 out of 5 shots whereas B is known to hit the target in 3 out of 4 shots. Find the probability of the target being hit when both try independently?

Answer:

15. The odds that person X speaks the truth are 3:2 and the odds that person Y speaks the truth are 5:3. In what percentage of cases are they likely to contradict each other on an identical point?

Answer:

16. The odds in favour of A solving a mathematical problem are 3 to 4 and the odds against B solving the problem are 5 to 7. Find the probability that the problem will be solved by at least one of them.

Answer:

Multiplication law of probability

If A and B are dependent events in a sample space S, then,

$P(A \cap B) = P(A)P(B/A)$ if $P(B) \neq 0$

$= P(B)P(A/B)$ if $P(A) \neq 0$

Conditional probability

If A and B are dependent events, the conditional probability of A given B means the probability of occurrence of A when the event B has already happened. It is denoted by P(A/B) and is defined by

$P(A/B) = \frac{P(A \cap B)}{P(B)}$ if $P(B) \neq 0$

$P(B/A) = \frac{P(A \cap B)}{P(A)}$ if $P(A) \neq 0$

Note: If two events A and B are independent,

$P(A/B) = P(A), P(B/A) = P(B)$

$\therefore P(A \cap B) = P(A).P(B)$

17. A box contains 4 bad and 6 good tubes. Two are drawn from the box at a time. One of them is tested and found to be good. What is the probability that the other one is also good?

Answer:

18. If the probability that a communication system will have high fidelity is 0.81 and the probability that it will have high fidelity and high selectivity is 0.18, what is the probability that a system with high fidelity will also have high selectivity?

Answer:

19. In a certain group of computer personnel, 65% have insufficient knowledge of hardware, 45% have inadequate idea of software and 70% are in either one or both of the two categories. What is the percentage of people who know software among those who have a sufficient knowledge of hardware?

Answer:

Total probability theorem

$$P(A_i) \neq 0, i = 1, 2, \dots . n$$

If the events $A_1, A_2, \dots \dots A_n$ constitute a partition of the sample space S and $P(A_i) \neq 0, i = 1, 2, \dots . n$, then for any event B in S we have

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(A_i)P(B/A_i) = P(A_1)P(B/A_1) + P(A_2)P(B/A_2) + P(A_3)P(B/A_3)$$

Baye's theorem

Let $A_1, A_2, \dots \dots A_n$ be n mutually exclusive and exhaustive events. Let B be an independent event such that $B \subset \bigcup_{i=1}^{n} A_i$.

Then the conditional probability of Ai given that B has already occurred, is $P(A_i/B) = \frac{P(A_i)P(B/A_i)}{\sum_{i=1}^{n} P(A_i)P(B/A_i)}$

20. In 1989 there were three candidates for the position of principal – Mr. Chatterji, Mr. Ayangar and Dr. Singh – whose chances of getting the appointment are in the proportion 4 : 2 : 3 respectively. The probability that Mr. Chatterji if selected would introduce co-education in the college is 0.3. The probabilities of Mr. Ayangar and Dr. Singh doing the same are respectively 0.5 and 0.8. What is the probability that there was co-education in the college in 1990?

Ans:

21. In a bolt factory, machines A, B and C manufacture respectively 25%, 35% and 40% of the total. Of their output 5, 4, 2 percent are defective bolts. A bolt is drawn at random from the product and is found to be defective. What are the probabilities that it was manufactured by machines A, B and C?

Ans:

22. The contents of urns I, II, III are as follows: 1 white, 2 black and 3 red balls; 2 white, 1 black and 1 red balls; 4 white, 5 black and 3 red balls respectively. One urn is chosen at random and two balls are drawn from it. They happen to be white and red. What is the probability that they come from urns I, II, or III ?

Answer:

23. An urn contains 5 balls. Two balls are drawn and are found to be white. What is the probability of all the balls being white?

24. For a certain binary, communication channel, the probability that a transmitted ‘0’ is received is 0.95 and the probability that a transmitted ‘1’ is received as 1 is 0.90. If the probability that a ‘0’ is transmitted s 0.4, find the probability that (1) a ‘1’ is received and (2) a ‘1’ was transmitted given that a ‘1’ was received.

Answer:

25. The chances of three candidates A, B and C becoming the manager of a company are in the ratio 3: 5: 4. The probability that a special bonus scheme will be introduced by them if selected are 0.6, 0.4 and 0.5 respectively. If the bonus scheme is introduced, what is the probability that B has become the manager?

Answer:

26. A lab blood test is 95% effective in detecting a certain disease when it is present. However, the test also gives a false positive result for 1% of the healthy persons tested. If 0.5% of the population actually has the disease, what is the probability that a person has the disease given that his test is positive?

Answer:

Random variables

A real variable X whose value is determined by the outcome of a random experiment is called a random variable.

Eg. A random experiment consists of two tosses of a coin. Consider the random variable representing the number of heads (0, 1, or 2)

Outcome:	HH	HT	TH	TT
Value of X:	2	1	1	0

Two types of random variable:

1. Discrete random variable 2. Continuous random variable

Discrete random variable

A random variable which can assume only a countable number of real values is called a discrete random variable.

Eg: Number of telephone calls per minute, marks obtained in a test etc.

Probability mass function (p.m.f.)

If X is a discrete random variable assuming values $x_1, x_2, \ldots \ldots ..$, then

$P(x_i) = P(X = x_i), i = 1,2, \ldots.$

is called the probability mass function of X.

$P(x_i) = P(X = x_i), i = 1,2, \ldots.$ should satisfy the following conditions

(i) $P(x_i) \geq 0, \forall i$

(ii) $\sum_{i=1}^{\infty} P(x_i) = 1$

Continuous random variable

A random variable X is said to be continuous if it can take all possible values between certain limits

Probability density function (p.d.f.)

The probability density function $f_X(x)$of a continuous random variable X is defined as

$P\{x \leq X \leq x + dx\} = f_X(x)$, where $(x, x + dx)$

is an infinitesimally small interval and satisfies the following conditions:

(i) $f_X(x)$ is integrable over the range $(-\infty, \infty)$

(ii) $f_X(x) \geq 0$ for all $x, -\infty < x < \infty$

(iii) $\int_{-\infty}^{\infty} f_X(x)dx = 1$

Properties of probability density function

(i) $f(x) \geq 0$

(ii) $P(x_1 < X < x_2) = \int_{x_1}^{x_2} f_X(x)dx$

(iii) $P(x_1 < X \leq x_2) = P(x_1 < X < x_2) = P(x_1 \leq X \leq x_2) = P(x_1 \leq X < x_2)$

(iv) $P(X = a) = \int_a^a f_X(x)dx = 0$

Distribution function

Let X be a random variable. The function

$F(x) = P(X \leq x), -\infty < x < \infty$

is called the distribution function or cumulative distribution function (c.d.f) of X.

(iv) For a continuous random variable, $F(x) = \int_{-\infty}^{x} f_X(x)dx$

Properties of distribution function

Let F be the distribution function of the random variable X

(i) If a < b, then $P(a < X \leq b) = F(b) - F(a)$

(ii) $0 \leq F(x) \leq 1$

(iii) If $x < y$, then $F(x) \leq F(y)$, ie, $F(x)$ is a monotonically non-decreasing function

(iv) $F(-\infty) = 0,\ F(\infty) = 1$

Problems

1. A discrete random variable X has the probability function given below:

X:	0	1	2	3	4	5	6	7
$P(X = x)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

Find (i) the value of k

(ii) $P(X < 6)$, $P(X \geq 6$, $P(0 < X < 4)$

(iii) the distribution function of X.

(iv) the minimum value of 'a' such that $P(X \leq a) > \frac{1}{2}$

Answer:

2. A discrete random variable has the following probability distribution

x:	0	1	2	3	4	5	6	7	8
$P(x)$:	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

Find (i) the value of a (ii) $P(2 \leq X < 6)$

(iii) $P(X > 3)$ (iv) distribution function of X.

3. If $P(X = x) = \frac{x}{15}, x = 1, 2, 3, 4, 5,$ find (i)$P(X = 1 \text{ or } X = 2)$

(ii) $P\left(\frac{1}{2} < X < \frac{5}{2} / X > 1\right)$

Answer:

x:	1	2	3	4	5
$P(x)$:	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

4. If a random variable x takes the value 1, 2, 3, 4 such that

$2P(X = 1) = 3P(X = 2) = P(X = 3) = 5P(X = 4)$

find the probability distribution of X.

Answer:

5. A continuous random variable X has a p.d.f $f(x) = 3x^2, 0 \leq x \leq 1$. Find a such that $P(X \leq a) = P(X > a)$

Answer:

6. A continuous random variable X that can assume any value between $x = 2$ and $x = 5$ has the density function given by $f(x) = k(1 + x)$

Find $P(X < 4)$ and $P(3 < X < 4)$

Answer:

7. The mileage in thousands of miles which car owners get with a certain kind of tyre is a random variable having p.d.f

$f(x) = \frac{1}{20} e^{-x/20}$, for $x > 0$

and $\boldsymbol{x \leq 0.}$ Find the probabilities that one of these tyres will last

(i) atmost 10,000 miles (ii) anywhere from 16,000 to 24,000 miles

(iii) atleast 30,000 miles

Answer:

8. If X is a continuous random variable with p.d.f.

$$F(x) = \begin{Bmatrix} 0, \quad x \leq 1 \\ k(x-1)^4, \quad 1 \leq x \leq 3 \\ 1, \quad x > 3 \end{Bmatrix}$$

Find k, the density function of X and $P(X < 2)$

Answer:

Mathematical expectation and moments

The mean value of a random variable X is called the expectation of X and is denoted by E(X).

$$E(X) = \begin{cases} \sum_i x_i p(x_i) & \text{if X is a discrete random variable} \\ \int_{-\infty}^{\infty} x f_X(x) & \text{if X is a continuous random variable} \end{cases}$$

Note: If X is a random variable and g(X) is a real function of X, then

$$E[g(X)] = \begin{cases} \sum_i g(x_i) p(x_i) & \text{if X is a discrete random variable} \\ \int_{-\infty}^{\infty} g(x) f_X(x) & \text{if X is a continuous random variable} \end{cases}$$

Properties of Mathematical expectation

If X and Y are random variables and a, b are constants, then

(i) E(a) = a

(ii) E(aX) = aE(X)

(iii) E(aX+b) = aE(X) + b

(iv) E(X+Y) = E(X) + E(Y)

(v) E(XY) = E(X).E(Y) if X and Y are independent random variables.

Mean and Variance

Mean of X = $\overline{X} = E(X)$

Variance of X = $\sigma_X{}^2 = E\left(\left(X-\overline{X}\right)^2\right) = E\left(\left(X-E(X)\right)^2\right) = E\left(X^2\right)-\left(E(X)\right)^2$

Properties of variance

(i) $Var(X) \geq 0$

(ii) $Var(a) = 0$, where a is a constant

(iii) $Var(aX) = a^2 Var(X)$

(iv) $Var(aX \pm b) = a^2 Var(X)$

(v) $Var(aX \pm bY) = a^2 Var(X) + b^2 Var(Y)$ if X and Y are independent

Moments

Raw moment: (Moments about the origin)

The r^th raw moment of a random variable X about the origin is defined as he expected value of the r^th power of X.

$$\mu_r' = E\left(X^r\right) = \begin{cases} \sum_i x_i^r p_i & \text{if X is a discrete r.v, } r \geq 1 \\ \int_{-\infty}^{\infty} x^r f_X(x)dx & \text{if X is a continuous r.v., } r \geq 1 \end{cases}$$

Central moments: (Moments about the mean)

$$\mu_r = E\left(\left(X-\overline{X}\right)^r\right) = \begin{cases} \sum_i \left(x_i-\overline{X}\right)^r p_i & \text{if X is a discrete r.v, } r \geq 1 \\ \int_{-\infty}^{\infty} \left(x-\overline{X}\right)^r f_X(x)dx & \text{if X is a continuous r.v., } r \geq 1 \end{cases}$$

Relationship between raw moments and central moments

$\mu_1 = 0$

$\mu_2 = \mu_2' - \left(\mu_1'\right)^2$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\left(\mu_1'\right)^3$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\left(\mu_1'\right)^2 - 3\left(\mu_1'\right)^4$

9. When a die is thrown, X denotes the number that throws up. Find E(X), E(X^2), Var(X) and standard deviation.

Ans: X is a discrete random variable.

p $=\frac{1}{6}$, for x = 1, 2, 3, 4, 5, 6

The p.m.f. of X is

10. If a random variable X has density function

$$f(x) = \begin{cases} \dfrac{x}{6} \text{ for } x = 1,2,3 \\ 0 \text{ otherwise} \end{cases}$$

find mean, variance, S.D. of X and E($4X^3$+3X+11)

Answer:

11. The random variable X can take only the values 2 and 5. Given that the value 5 is twice as likely as the value 2, determine the expectation of X.

Answer:

12. A coin is tossed till a head appears. What is the expectation of the number of tosses required?

Answer:

13. By throwing a fair die, a player gains Rs. 20 if 2 turns up, gains Rs. 40 if 4 turns up, and loses Rs.30 if 6 turns up. and loses Rs.30 if 6 turns up. He never loses or gains if any other number turns up. Find the expected value of money he gains.

Ans: Let X denote the money won on any trial.

14. A man draws 3 balls from an urn containing 5 white and 7 black balls. He gets Rs. 10 for each white ball and Rs. 5 for each black ball. Find his expectation.

Ans:

15. If $Var(X) = 4$, find $Var\ (4X + 5)$ where X is a random variable.

Ans:

16. If X and Y are independent random variables with variances 2 and 3 respectively, find the variance of $3X + 4Y$.

Ans: $Var\ (aX + bY)\ = a^2 Var(X) + b^2 Var(Y)$

17. Let X be a random variable with $E(X) = 1, E(X(X-1)) = 4.$

Find $Var(X), Var(2-3X), Var\left(\frac{X}{2}\right).$

Answer:

18. Find the mean, variance of the random variable X which has the following density function.

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

Answer:

19. Let X be a random variable with distribution function F given by

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & 0 \le x < \infty \\ 0, & \text{otherwise} \end{cases}$$

Find the pdf of X. Determine r^{th} raw moment, mean and variance of the distribution.

Answer:

20. The cumulative distribution function of a random variable X is $F(x) = 1 - (1+x)e^{-x},\ x > 0$.

Find the probability density function of X and the mean and variance of X.

Answer:

Moment Generating function (M.G.F.)

M.g.f of a r.v X is:

$$M_X(t) = E(e^{tx}) = \begin{cases} \sum e^{tx}\, p(x), & \text{if } X \text{ is discrete} \\ \int\limits_{-\infty}^{\infty} e^{tx}\, f_X(x)\, dx, & \text{if } X \text{ is continuous} \end{cases}$$

$$\begin{aligned} M_X(t) &= E(e^{tX}) \\ &= E\left[1 + \frac{tX}{1!} + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + \ldots\right] \\ &= 1 + t\,E(X) + \frac{t^2}{2!}\,E(X^2) + \ldots + \frac{t^r}{r!}\,E(X^r) + \ldots \end{aligned}$$

$$\boxed{M_X(t) = 1 + t\,\mu_1' + \frac{t^2}{2!}\,\mu_2' + \ldots + \frac{t^r}{r!}\,\mu_r' + \ldots}$$

where $\mu_r' =$ coefficient of $\frac{t^r}{r!}$ in $M_X(t)$

$$\begin{aligned} &= E[X^r] \\ &= \begin{cases} \sum\limits_x x^r p(x) & \text{(Discrete case)} \\ \int x^r\, f(x) dx & \text{(Continuous case)} \end{cases} \end{aligned}$$

Since $M_X(t)$ generates moments, it is known as moment generating function.

Note:

1. Differentiating (1) $w.r.$ to t and then putting $t = 0$, we get

$$\mu_r' = \left[\frac{d^r}{dt^r}\, M_X(t)\right]_{t=0}$$

2. $M_{CX}(t) = M_X(Ct), C$ being a constant

Proof:

By definition $L.H.S = M_{CX}(t)$

$$= E[e^{tCX}]$$

$$R.H.S = M_X(Ct) = E[e^{tCX}] = L.H.S$$

3. The moment generating function of the sum of a given number of independent random variables is equal to the product of their respective moment generating functions.

   i.e $M_{X_1+X_2+\cdots+X_n(t)} = M_{X_1}(t).M_{X_2}(t)\cdots M_{X_n}(t)$

Proof:

By definition $M_{X_1+X_2+\cdots+X_n(t)}$

$$= E[e^{t(X_1+X_2+\cdots+X_n)}]$$

$$= E\left[e^{tX_1}\cdot e^{tX_2}\ldots e^{tX_n}\right]$$

$$= E[e^{tX_1}].E[e^{tX_2}]\cdots .E[e^{tX_n}]$$

$$(\because\ X_1, X_2\cdots X_n \text{ are independent})$$

$$= M_{X_1}(t).M_{X_2}(t)\cdots M_{X_n}(t)$$

1. A random variable X has the probability function

   $f(x) = \frac{1}{2^x}$, x=1, 2, 3,.....

   Find its (i) M.G.F, (ii) Mean.

Answer:

To find MGF:

2. Find the m.g.f , mean and variance of the distribution whose p.m.f is

$$p(x) = \begin{cases} q^x p, & x = 0,1,2...... \\ 0, & otherwise \end{cases}$$

Answer:

<u>To find MGF</u>

$$M_X(t) = E(e^{tX}) = \sum e^{tX} f(x) = \sum_{x=0}^{\infty} e^{tX} q^x p$$

5. Find the m.g.f of the random variable X which has the following density function.

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

Answer:

Normal Distribution or Gaussian distribution

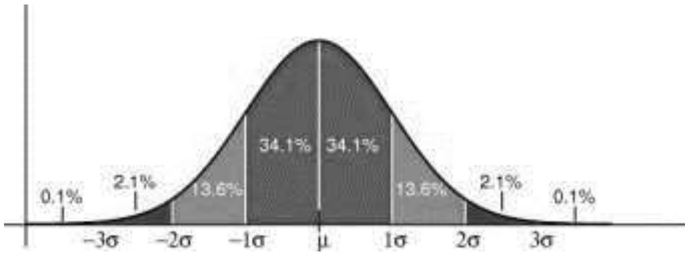
A normal distribution is a distribution that occurs naturally in many situations. For example, the bell curve is seen in tests like the SAT and GRE. The bulk of students will scorethe average (C), while smaller numbers of students will score a B or D. An even smaller percentage of students score an F or an A. This creates a distribution that resembles a bell . The bell curve is symmetrical. Half of the data will fall to the left of the mean; half will fall to the right.

Many groups follow this type of pattern. That is why it’s widely used in business, statistics and in government organizations

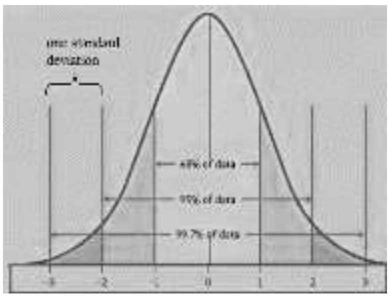
- Heights of people.
- Measurement errors.
- Blood pressure.
- Points on a test.
- IQ scores.
- Salaries.

The empirical rule gives us the percentage of data that falls within a certain number of standard deviations from the mean:

- 68.2% of the data falls within one standard deviation of the mean.
- 95.4% of the data falls within two standard deviations of the mean.
- 99.7% of the data falls within three standard deviations of the mean.



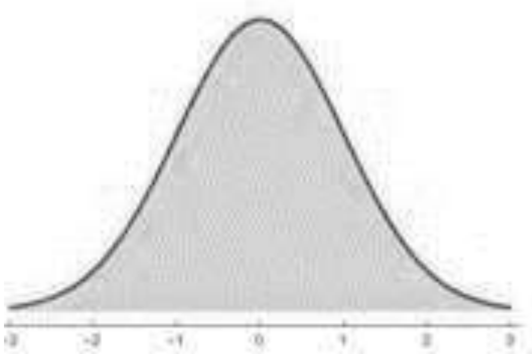
The standard deviation controls the spread of the distribution. A smaller standard deviation indicates that the data is tightly clustered around the mean; the normal distribution will be taller. A larger standard deviation indicates that the data is spread out around the mean; the normal distribution will be flatter and wider.

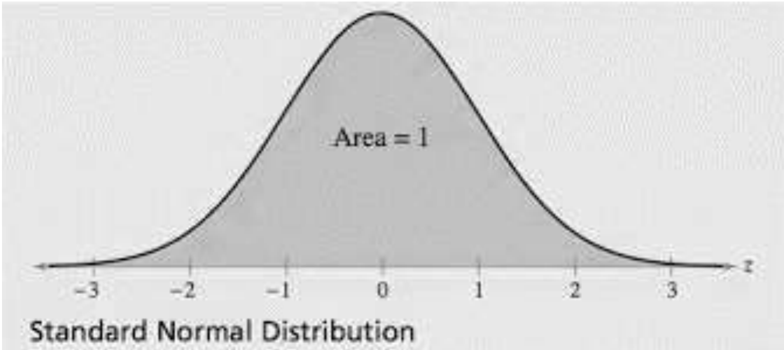


Probability density function

A continuous random variable X follows normal distribution (or Gaussian distribution) if its p.d.f is $\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $-\infty < x < \infty$, $-\infty < \mu < \infty$, $\sigma > 0$

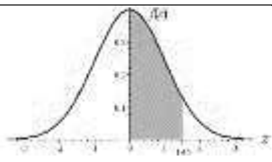
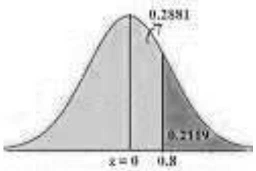
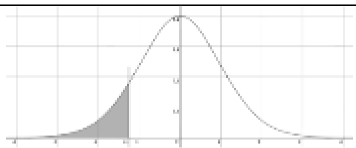
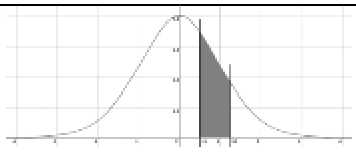
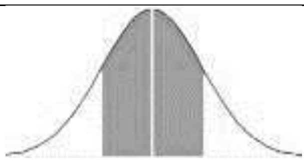
1. The parameters are μ and σ.
2. μ is the mean and σ is the standard deviation of the distribution.
3. The graph of the distribution is bell shaped and is called the normal probability curve.
4. The curve is symmetrical about the ordinate at x = μ.
5. x – axis is an asymptote to the curve.
6. For the normal distribution, mean = median = mode.
7. The normal distribution with mean = 0 and variance = 1, is called the standard normal distribution and is denoted as N(0,1).
8. The following is the plot of the standard normal probability density function.

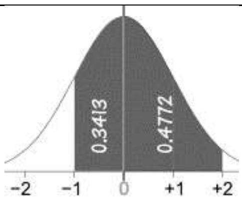
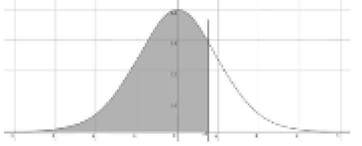
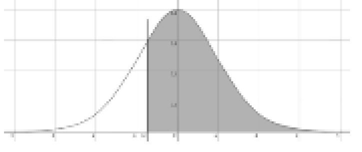




9. If a random variable X follows the normal distribution, then we write: $X \sim N(\mu, \sigma^2)$

Calculation of various probabilities

Required probability	Shaded region	Calculation
$P(0 < z < 1.5)$		$P(0 < z < 1.5)$
$P(z > 0.8)$ **(Green area)**		$P(z > 0.8) = 0.5 - P(0 < z < 0.8)$
$P(z < -1.2)$		$P(z < -1.2) = P(z > 1.2) = 0.5 - P(0 < z < 1.2)$
$P(0.5 \le z < 1.2)$		$P(0.5 \le z < 1.2) = P(0 < z < 1.2) - P(0 < z < 0.5)$
$P(-1.2 \le z < 0.8)$		$P(-1.2 \le z < 0.8) = P(0 < z < 1.2) + P(0 < z < 0.8)$
$P(-1 < z < 1)$		$P(-1 < z < 1) = 2 \times P(0 < z < 1)$

$P(-1 \le z < 2)$		$P(-1 \le z < 2) = P(0 < z < 1) + P(0 < z < 2)$
$P(z < 0.8)$		$P(z < 0.8) = 0.5 + P(0 < z < 0.8)$
$P(z > -0.8)$		$P(z > -0.8) = 0.5 + P(0 < z < 0.8)$

Problems

1.X is a Normal variate with μ=30,*and* σ=5.

$(i)P(26 \leq X \leq 40)$ (ii) $P(X \geq 45)$ (iii) $P(|X - 30|) \geq 5$.

Given:

$\mu = 30, \sigma = 5$

2. X is normally distributed, and the mean of **X** is 12 and S.D = 4. Find

(i) $P(X \geq 20)$(ii) $P(X \leq 20)$ (iii) $P(0 \leq X \leq 12)$. (iv) Find a when $P(X > a) = 0.24$. (v) Find b and c when $P(b < X < c) = 0.5$ and $P(X > c) = 0.25$

Solution:

Given:

3. In a test of 2000 electric bulbs it was found that the life of a particular make was normally distributed with an average life of 2040hrs and SD of 60 hrs. Estimate (i) The number of bulbs likely to burn for (i) more than 2150hrs(ii) less than 1950hrs and (iii) more than 1920 hrs but less than 2160hrs.

Solution:

Given: $\mu = 2040, \sigma = 60, N = 2000$

4. Suppose the heights of men of a certain country are normally distributed with average68 inches and standard deviation 2.5, find the percentage of men who are

(i) between 66 inches and 71 inches inheight

(ii) approximately 6 feet tall (ie, between 71.5 inches and 72.5inches)

Solution:

Given: $\mu = 68, \sigma = 2.5$

(i) $P(66 < X < 71) = P\left(\frac{66-68}{2.5} < \frac{X-68}{2.5} < \frac{71-68}{2.5}\right)$

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