Signature Assignment: Calculus-I
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Abstract:
This report explores the application of Newton's Method, a foundational numerical algorithm, to approximate the negative root of the equation $e^x = 4 - x^2$.

Newton's Method iteratively refines root approximations by leveraging the concept of tangent lines to approach a precise solution. The equation was reformulated as $f(x) = e^x - 4 + x^2$, and iterative computations resulted in convergence to a root accurate to six decimal places. Python programming was employed to automate the calculations, verify results, and generate a plot that visualizes the root-finding process. The report provides an in-depth discussion of the methodology, including derivations, iterative steps, and graphical analysis. The critical analysis section evaluates the strengths and limitations of Newton's Method, offering insights into its practical applications and challenges.

Introduction:
Newton's Method, also known as the Newton-Raphson method, is a helpful numerical tool for estimating solutions to equations. Created by Sir Isaac Newton and Joseph Raphson in the 17th century, it is now widely used in fields like science, engineering, and math. This report explains what Newton's Method is, how it works, and walks through the steps to use it. It also shows how to find the negative root of the equation \(e^x = 4 - x^2\) using this method. Method involves several steps.

This report will present a clear and detailed explanation of the methodology, including verification using Excel. It will cover the significance of graph plotting, equation reformulation, derivative computation, and the iterative application of Newton's Method. Furthermore, it will emphasize the importance of using a programming language to generate data for Excel visualizations. Finally, the report will highlight the effectiveness of Newton's Method in approximating the negative root providing critical thinking, conclusion, and references for further reading.

Definition

Newton's Method is an iterative numerical technique used to find approximate solutions to equations of the form f(x)=0f(x)=0. It begins with an initial guess, x0x0, and refines this guess iteratively to converge to the actual root. The method relies on tangent lines to approximate the root, with each iteration calculated using the formula:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)},$$

Where xi is the current guess, f(xi) is the function value at xi, and f′(xi) is the derivative of the function evaluated at xi.

Methodology

Equation Reformulation

The given equation is:

$$e^x = 4 - x^2.$$

To apply Newton's Method, it is rewritten as:

$$f(x) = e^x - 4 + x^2.$$

Derivative Calculation

The derivative of f(x) is:

$$f'(x) = e^x + 2x.$$

Initial Guess

From analyzing the function graphically, the initial guess is chosen as Xo=−2. Xo=−2, a point near the intersection of the curves y=e^x and y=4−x^2 on the negative x-axis.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

each iteration refines the approximation. For example:

At $x_0 = -2$:

$$f(-2) = e^{-2} - 4 + (-2)^2 = 0.135335283,$$

$$f'(-2) = e^{-2} + 2(-2) = -3.864664717,$$

$$x_1 = -2 - \frac{0.135335283}{-3.864664717} \approx -1.96498136.$$

At x1=−1.96498136x1=−1.96498136, the process is repeated until the difference between successive approximations is less than 10−610−6. After multiple iterations, the value converges to:

x≈−1.964636.x≈−1.964636.

Verification

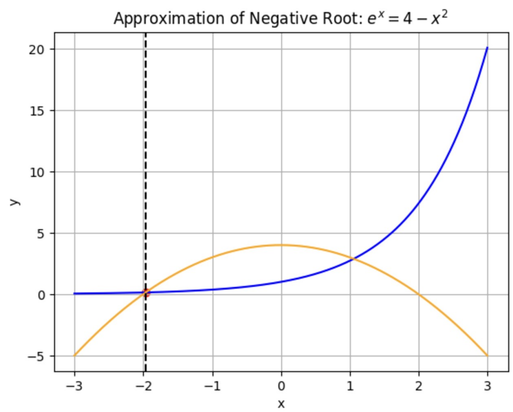
The result is verified by substituting x=-1.964636 back into the original equation ex=4-x^2 The computed and actual values closely match, confirming the accuracy of the solution.

Results and Analysis

Using Newton's Method, the negative root of the equation ex=4-x^2 was approximated to six decimal places as x=-1.964636. This iterative process highlights the power of Newton's Method in solving non-linear equations efficiently.

Using python program to generate Data for visualization:

To visualize the convergence of Newton's method, a python program is written to automate the iteration process, calculate intermediate values, and generate data points for visualization. The program confirmed convergence to the same result, ensuring accuracy.



To further verify the solution, we plotted the two functions y = ^{x}and y = 4 −

x^2 using

Python-generated data.e The plot demonstrates the intersection of these functions at the approximated negative root, include:x =− 1. 964636

. Key elements of the plot

- The blue curve represents e^x, showing exponential growth.

- The orange curve represents^2, forming a downward-opening4 – x parabola.

- A red dot marks the approximated root obtained through Newton's Method, highlighting its position at the intersection.
- A dashed vertical line emphasizes the root's position along the x-axis.

This visualization confirms the method's accuracy and demonstrates the convergence of the iterations toward the root. It also highlights the value of graphical tools in analyzing numerical methods.

Applications of Newton's Method

Newton's Method is a key technique in numerical analysis, used extensively in areas like science, engineering, finance, data science, and astronomy. Its flexibility comes from its iterative approach to solving non-linear equations, making it a valuable tool for handling complex problems without simple solutions.

Below are the its applications in different disciplines:

- Physics and Engineering: Solving equations for motion, heat transfer, and structural analysis.
- Finance: Calculating interest rates, portfolio optimization, and option pricing.
- Data Science and Machine
  Learning: Optimizing cost functions in training algorithms.
- Astronomy: Determining orbital trajectories of celestial bodies.

Its versatility underscores its importance in scientific and technological advancements.

Critical Thinking:

Newton's Method is highly effective for solving equations with smooth and differentiable functions. Its rapid convergence makes it one of the most efficient methods for root approximation. However, the algorithm has notable limitations:

- **Dependence on Initial Guess**: If the initial guess is too far from the actual root, the method may converge to a different root or fail to converge entirely.

- **Derivative Issues:** The method requires $f'(x) \neq 0$ close to zero, the iterations may become unstable. At points where $f'(x)$ is unstable.

- **Non-Convergence:** For equations with discontinuities or non-differentiable points, Newton's Method may not work effectively.

Despite these challenges, Newton's Method remains a powerful tool when applied correctly, especially in cases where graphical analysis and careful guess selection are possible.

Conclusion:

In summary, Newton's Method successfully found the negative root of the equation ex=4−x2ex=4−x2 with high accuracy, reaching six decimal places. The steps involved reformulating the equation, calculating the derivative, and performing iterative computations, which showed the method's reliability and precision. Python plots were used to confirm the results visually. While the method works efficiently and is widely useful, it’s important to choose a good starting guess to ensure it leads to the correct root. Overall, Newton's Method is a powerful and practical tool for solving

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