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Sessional Test

Model Solution.

Course: Btech

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Sub: Maths-4

code BAS 403

Section: CSE, CSE AIML, CSE DS, ME EN CS

sem: IV

year II

Section A

Q1 solve $(D-D'+3)(D'+1)^2 z = 0$

Sol $Z = e^{-3x} f_1(y+x) + e^{-y} f_2(x) + y e^{-y} f_3(x)$ — 1 — 1

Q2 Find PDE whose solution given by $f(2x-y)$

Sol $\frac{\partial z}{\partial x} = f'(2x-y) \cdot 2$ — (1)

$\frac{\partial z}{\partial y} = f'(2x-y)(-1)$ — (2) — 1

Divide (1) by (2)

$\frac{\partial z}{\partial x} \Big/ \frac{\partial z}{\partial y} = \frac{2}{(-1)} \Rightarrow \frac{\partial z}{\partial x} + 2\frac{\partial z}{\partial y} = 0$ — 1

Q3 classify the PDE

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial t} + t\frac{\partial^2 u}{\partial t^2} = 0$$

Sol $A = x,\ B = 2,\ C = t$

$\Rightarrow B^2 - 4AC = 4 - 4xt$ — 1

If $4-4xt > 0 \Rightarrow$ Hyperbolic PDE

$4-4xt < 0 \Rightarrow$ elliptic PDE

$4-4xt = 0 \Rightarrow$ parabolic PDE — 1

Q4 Write wave eq^n with boundary condi... and initial condition.

Sol $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ subject to

$$\left.\begin{array}{ll} x = 0, & y = 0 \\ x = l & y = 0 \end{array}\right\} \text{ Boundary condition}$$ 1

$$\left.\begin{array}{ll} t = 0, & y = f(x) \\ t = 0 & \frac{\partial y}{\partial t} = 0 \end{array}\right\}$$ if string is made to vibrate by displacing in shape $y = f(x)$ from rest. 1

OR

$$\left.\begin{array}{ll} t = 0, & y = 0 \\ t = 0 & \frac{\partial y}{\partial t} = f(x) \end{array}\right\}$$ if string is made to vibrate by giving initial velocity.

Q5 Solve PDE $r - 4s + 4t = e^{2x+y}$

Sol $(D^2 - 4DD' + 4D'^2)y = e^{2x+y}$

CF $m^2 - 4m + 4 = 0 \Rightarrow m = 2, 2$

$\Rightarrow$ CF $= f_1(y+2x) + x f_2(y+2x)$ 1

PI $\frac{1}{D^2 - 4DD' + 4D'^2} e^{2x+y}$

$= x \frac{1}{2D - 4D'} e^{2x+y}$

$= x^2 \frac{1}{2} e^{2x+y}$ 1

$\Rightarrow y = f_1(y+2x) + x f_2(y+2x) + \frac{x^2}{2} e^{2x+y}$

Section - B

Solve PDE $y^2(x+y)p + x^2(x+y)q = (x^2+y^2)z$

Sol Aux Eq: $\frac{dx}{y^2(x+y)} = \frac{dy}{x^2(x+y)} = \frac{dz}{(x^2+y^2)z}$ — ①

From I & II fraction

$\frac{dx}{y^2} = \frac{dy}{x^2}$ Int both side

$\Rightarrow \int x^2 dx = \int y^2 dy$

$\frac{x^3}{3} = \frac{y^3}{3} + c_1$

$\boxed{u = x^3 + y^3 = a}$ — ②

Taking multiplier 1, 1, 0

Each fraction $= \frac{dx + dy}{(x+y)(x^2+y^2)}$

Taking this with III fraction

$\frac{dx+dy}{(x+y)(x^2+y^2)} = \frac{dz}{(x^2+y^2)z}$

$\int \frac{dx+dy}{x+y} = \int \frac{dz}{z}$

$\log(x+y) = \log z + \log b$

$(x+y) = bz \Rightarrow \boxed{v = \frac{x+y}{z} = b}$ — ②

$\Rightarrow$ complete sol:

$\phi(u, v) = 0$

Ans

Solve $(D^2 - DD' - 2D'^2 + 2D + 2D')z = \sin(2x+y)$

Sol^n. Given $(D^2 - DD' - 2D'^2 + 2D + 2D')z = \sin(2x+y)$

or $\{D^2 - D'^2 + 2D - D + 2D' - 2 + DD' - DD'\}z = \sin(2x+y)$

$\{D^2 - DD' + 2D + DD' - D'^2 + 2D' - D + D' - 2\}z = \sin(2x+y)$

$$(D - D' + 2)(D + D' - 1)z = \sin(2x+y)$$

$CF = e^{-2x} f_1(y+x) + e^x f_2(y-x)$ —— (2)

$$PI = \frac{1}{(D^2 - DD' - 2D'^2 + 2D + 2D')} \sin(2x+y)$$

$$= \frac{1}{-2^2 - (-2\cdot1) - 2(-1^2) + 2D + 2D'} \sin(2x+y)$$

$$= \frac{1}{2(D+D')} \times \frac{(D-D')}{(D-D')} \sin(2x+y)$$

$$= \frac{(D-D')}{2(D^2 - D'^2)} \sin(2x+y)$$

$$= \frac{(D-D')}{2\{-2^2 - (-1^2)\}} \sin(2x+y) = \frac{2\cos(2x+y) - \cos(2x+y)}{2\{-3\}}$$

$PI = -\frac{1}{6}\cos(2x+y)$ —— (3)

Hence Gen. Sol^n given by

$$z = CF + PI = e^{-2x} f_1(y+x) + e^x f_2(y-x) - \frac{1}{6}\cos x$$

Ans

Solve the PDE by Method of separation of Variable

$$3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0 \text{ with } u(x,0) = 4e^{-x} \longrightarrow (1)$$

Sol^n: Let $u = XY \rightarrow (2)$ be Sol^n of given PDE.

$$(1) \Rightarrow 3X'Y + 2XY' = 0 \Rightarrow \frac{X'}{X} = -\frac{2Y'}{3Y} = -p^2 \longrightarrow (3)$$ 1

(i) $\frac{X'}{X} = -p^2 \Rightarrow X' + p^2X = 0 \Rightarrow \boxed{X = C_1 e^{-p^2 x}}$ 1

(ii) $-\frac{2}{3}\frac{Y'}{Y} = -p^2 \Rightarrow 2Y' - 3p^2Y = 0 \Rightarrow Y' - \frac{3}{2}p^2Y = 0$

$$\boxed{Y = C_2 e^{\frac{3}{2}p^2 y}}$$ 1

$$\therefore (2) \Rightarrow u(x,y) = C_1C_2 e^{(x+\frac{3}{2}y)p^2} \longrightarrow (4)$$

When $u(x,0) = 4e^{-x}$

then (4) at $y=0 \Rightarrow 4e^{-x} = C_1C_2 e^{(-x+0)p^2}$

On Comparing $C_1C_2 = 4$, $p^2 = 1$

$$\therefore (4) \Rightarrow \boxed{u(x,y) = 4\,e^{-x+\frac{3}{2}y}}$$ 2

Hence the Solution of given PDE.

Solve one dimensional heat equation.

Sol: The heat eqn in one dimension is $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \rightarrow (1)$

Let the Soln. of (1) be $u(x,t) = X(x).T(t) \longrightarrow (2)$ — 1

where $X \equiv X(x)$, $T \equiv T(t)$ — 1

then (1) $\Rightarrow XT' = c^2 X''T \Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = k$ (say) $\longrightarrow (3)$

(i) if $k = -p^2$, (3) $\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = -p^2$ — 1

$\Rightarrow X'' - kX = 0 \Rightarrow X'' + p^2 X = 0 \Rightarrow X(x) = (C_1 \cos px + C_2 \sin px)$

and $T' + p^2c^2T = 0 \Rightarrow T(t) = C_3 e^{-c^2p^2t}$

$\Rightarrow$ (2) $\Rightarrow u(x,t) = X.T = (C_1 \cos px + C_2 \sin px).C_3 e^{-c^2p^2t} \longrightarrow (4)$

(ii) if $k = p^2$, (3) $\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = p^2$ — 1

$\Rightarrow X'' - p^2X = 0 \Rightarrow X(x) = C_4 e^{px} + C_5 e^{-px}$

and $T' - p^2c^2T = 0 \Rightarrow T(t) = C_6 e^{p^2c^2t}$

$\therefore$ (2) $\Rightarrow u(x,t) = (C_4 e^{px} + C_5 e^{-px}) C_6 e^{p^2c^2t} \longrightarrow (5)$

(iii) if $k = 0$, (3) $\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = 0$

$\Rightarrow X'' = 0 \quad (C_7x + C_8)$
and $T' = 0 \Rightarrow T(t) = C_9$
$\Big\} \; u(x,t) = (C_7x + C_8).C_9 \longrightarrow (6)$ — 1

Due to physical nature of problem i.e. if $t \to \infty$, $u \to 0$
Soln in (5) & (6) are rejected
Hence the Suitable Soln. given by (4) Since in (4)
u decreases as time t increases.

Q.10. Solve the PDE $x^2r+2xys+y^2t=x^3y^4$

Sol.^{n} Let $x=e^X$, $y=e^Y \Rightarrow X=\log x$, $Y=\log y$ and $\frac{\partial}{\partial X}\equiv D\equiv x\frac{\partial}{\partial x}$

$\frac{\partial}{\partial Y}\equiv D'\equiv y\frac{\partial}{\partial y}$

$\frac{\partial^2}{\partial X\partial Y}=DD'=xy\frac{\partial^2}{\partial x\partial y}$

then given PDE reduced into

$$[D(D-1)+2DD'+D'(D'-1)]z=e^{3X+4Y}$$

$$(D+D')(D+D'-1)z=e^{3X+4Y}$$

$CF=f_1(Y-X)+e^X f_2(Y-X)$ —— (2)

$$PI=\frac{1}{(D+D')(D+D'-1)}e^{3X+4Y}=\frac{1}{(3+4)(3+4-1)}e^{3X+4Y}$$

$$=\frac{1}{42}e^{3X+4Y}$$

$$\therefore\ z=CF+PI=f_1(Y-X)+e^X f_2(Y-X)+\frac{1}{42}e^{3X+4Y}$$

$$=f_1\{\log y-\log x\}+x f_2\{\log y-\log x\}+\frac{1}{42}x^3y^4$$

$z=g_1(y/x)+x g_2(y/x)+\frac{1}{42}x^3y^4$ —— (3)

Ans

Section C

Q11 Solve PDE $q + xp = p^2$

Sol $f = q + xp - p^2 = 0$

Auxiliary eqn

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\frac{dp}{p} = \frac{dq}{0} = \frac{dz}{-p(x-2p)-q} = \frac{dx}{-(x-2p)} = \frac{dy}{-1}$$

3

Taking I & V fraction, Int both side

$$\int \frac{dp}{p} = \int -dy \Rightarrow \log p = -y + \log A$$

$$\Rightarrow \boxed{p = Ae^{-y}}$$

From given eqn we have

$q + xp = p^2$, put $\boxed{p = Ae^{-y}}$

$1\frac{1}{2}$

$$q + x \cdot Ae^{-y} = A^2 e^{-2y}$$

$$\Rightarrow \boxed{q = A^2 e^{-2y} - Axe^{-y}}$$

1

Put p and q in

$$dz = p\,dx + q\,dy$$

$$dz = Ae^{-y}\,dx + (A^2e^{-2y} - Axe^{-y})\,dy$$

$$dz = Ae^{-y}\,dx - Axe^{-y}\,dy + A^2e^{-2y}\,dy$$

$$dz = A\,d(xe^{-y}) + A^2 e^{-2y}\,dy$$

Integrate both side

2

$$z = Axe^{-y} + A^2(-2)e^{-2y} + B$$

Ans

Q12 Find deflection of tightly stretched vibrating string of unit length that is initially at rest whose initial position is given by $\sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x$

Sol The given wave eqn is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

let $y = XT$ is sol. put it in PDE

$$\frac{\partial^2 (XT)}{\partial t^2} = c^2 \frac{\partial^2 (XT)}{\partial x^2} \Rightarrow X \frac{\partial^2 T}{\partial t^2} = T c^2 \frac{\partial^2 X}{\partial x^2}$$

$$\frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} = \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -p^2$$

Above equality will hold only for some constant k taking $k = -p^2$

$\therefore \frac{\partial^2 X}{\partial x^2} = -p^2$

$\frac{1}{X}\frac{d^2 X}{dx^2} = -p^2$

$(D^2 + p^2)X = 0$

$\Rightarrow X = A\cos px + B\sin px$

$\frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} = -p^2$

$(D^2 + c^2p^2)T = 0$

$\Rightarrow T = C\cos cpt + D\sin cpt$

3

$$y = XT = (A\cos px + B\sin px)(C\cos cpt + D\sin cpt) \quad (1)$$

Now According to Question we have following conditions

(i) $x = 0 \Rightarrow y = 0$

(ii) $x = 1 \quad y = 0$

(iii) $t = 0, \ \frac{\partial y}{\partial t} = 0$

(iv) $t = 0, \ y = \sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x$

Apply conditions one by one on (1)

Apply (i) condition, $y = 0$ at $x = 0$

$0 = A(.C\cos cpt + D\sin cpt)$

$\Rightarrow \boxed{A = 0}$ — (1) put in (1) we get

$y = B\sin px\,(C\cos cpt + D\sin cpt)$

$y = \sin px\,(C_1\cos cpt + C_2\sin cpt)$ — (2)

Apply (ii) condition $y = 0$ at $x = l$ on (2)

$0 = B\sin p\,(A\cos cpt + B\sin cpt)$

$\Rightarrow \sin p = 0 = \sin n\pi \Rightarrow \boxed{p = n\pi}$

put $\boxed{p = n\pi}$ in (2) we get

$y = \sin n\pi x\,[C_1\cos n\pi ct + C_2\sin n\pi ct]$ — (3)

Apply (iii) condition $\frac{\partial y}{\partial t} = 0$ at $t = 0$ on (3)

$\frac{\partial y}{\partial t} = \sin n\pi x\,(n\pi c)\left[-C_1\sin n\pi ct + C_2\cos n\pi ct\right]$

$0 = C_2 \sin n\pi x\,(n\pi c)$

$\Rightarrow \boxed{C_2 = 0}$ put in (3)

$y = C_1\cos n\pi ct\,\sin n\pi x$

and general solution will be

$$y = \sum_{n=1}^{\infty} C_n \cos n\pi ct\,\sin n\pi x \quad — (4)$$

Apply (iv) condition on (4)

$$\sin \pi x + \frac{1}{3} \sin 3\pi x + \frac{1}{5} \sin 5\pi x$$
$$= \sum_{n=1}^{\infty} C_n \sin n\pi x$$

Comparing like terms of RHS & LHS we get.

$$C_1 = 1, \quad C_3 = \frac{1}{3}, \quad C_5 = \frac{1}{5}$$

all other C_n will be zero. put these constant in (4), we get-

$$y = \sin \pi x \cos \pi ct$$
$$+ \frac{1}{3} \sin 3\pi x \cos 3\pi ct$$
$$+ \frac{1}{5} \sin 5\pi x \cos 5\pi ct$$

$1\frac{1}{2}$

Ans

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