Assignment 3

CS20BTECH11047

Download all python codes from

https://github.com/JeevanIITH/AI1102/blob/main/assignment3/assignment3.py

and latex codes from

https://github.com/JeevanIITH/AI1102/blob/main/assignment3/assignment3.tex

GATE EC 2012 Q.1

Two independent random variables X and Y are uniformly distributed in the interval [-1,1]. The probability that max $[X, Y]$ is less than $\frac{1}{2}$ is

1) $\frac{3}{4}$
2) $\frac{9}{16}$
3) $\frac{1}{4}$
4) $\frac{2}{3}$

Solution

Since the random variable X is uniformly distributed in the interval [-1,1],let $f_X(x) = k$.Then

$$\int_{-1}^{1} f_X(x)dx = \int_{-1}^{1} kdx = 1 \tag{0.0.1}$$

$$\Rightarrow k = \frac{1}{2} \tag{0.0.2}$$

So

$$f_X(x) = \frac{1}{2} \quad , x \in (-1, 1) \tag{0.0.3}$$

Similarly

$$f_Y(y) = \frac{1}{2} \quad , y \in (-1, 1) \tag{0.0.4}$$

Now

$$max[X, Y] < \frac{1}{2} \Rightarrow \left(X < \frac{1}{2}\right).\left(Y < \frac{1}{2}\right) \tag{0.0.5}$$

So

$$P\left(max[X, Y] < \frac{1}{2}\right) = P\left(\left(X < \frac{1}{2}\right).\left(Y < \frac{1}{2}\right)\right) \tag{0.0.6}$$

Since X and Y are independent

$$P\left(\left(X < \frac{1}{2}\right).\left(Y < \frac{1}{2}\right)\right) = P\left(X < \frac{1}{2}\right).P\left(Y < \frac{1}{2}\right) \tag{0.0.7}$$

Now

$$P\left(X < \frac{1}{2}\right) = \int_{-1}^{\frac{1}{2}} f_X(x)dx \tag{0.0.8}$$

$$= \int_{-1}^{\frac{1}{2}} \frac{1}{2}dx = \frac{1}{2}\left(\frac{1}{2} - (-1)\right) = \frac{3}{4} \tag{0.0.9}$$

similarly

$$P\left(Y < \frac{1}{2}\right) = \frac{3}{4} \tag{0.0.10}$$

Using (1.0.10) and (1.0.9) in (1.0.7), we get

$$P\left(max[X, Y] < \frac{1}{2}\right) = \frac{3}{4}.\frac{3}{4} = \frac{9}{16} \tag{0.0.11}$$

So option 2 is correct answer

Alternative solution

$$P\left(max[X, Y] < \frac{1}{2}\right) = P\left(X > Y, X < \frac{1}{2}\right) + P\left(Y > X, Y < \frac{1}{2}\right) \tag{0.0.12}$$

Now

$$P\left(X > Y, X < \frac{1}{2}\right) = \sum_{x=-1}^{1/2} P\left(Y < x\right).P(X = x) \tag{0.0.13}$$

using eq (0.0.8)

$$\sum_{x=-1}^{1/2} P\left(Y < x\right).P(X = x) = \int_{-1}^{\frac{1}{2}} \left(\int_{-1}^{x} \frac{1}{2} dy\right). \frac{1}{2} dx \tag{0.0.14}$$

$$= \int_{-1}^{\frac{1}{2}} \left(\frac{1}{2}(x+1)\right). \frac{1}{2} dx \tag{0.0.15}$$

$$= \frac{1}{4}\left(\frac{x^2}{2} + x\right)\Big|_{-1}^{1/2} = \frac{9}{32} \tag{0.0.16}$$

So

$$P\left(X > Y, X < \frac{1}{2}\right) = \frac{9}{32} \tag{0.0.17}$$

similarly

$$P\left(Y > X, Y < \frac{1}{2}\right) = \frac{9}{32} \tag{0.0.18}$$

Putting (0.0.17) ,(0.0.18) in (0.0.13) we get

$$P\left(max[X, Y] < \frac{1}{2}\right) = \frac{9}{32} + \frac{9}{32} = \frac{9}{16} \tag{0.0.19}$$