Tutorial_6_Question

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COMP2610/COMP6261 – Information Theory

Tutorial 6

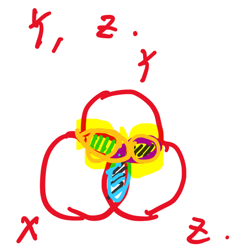
Zhifeng Tang (zhifeng.tang@anu.edu.au)

Question 1. Entropy

Let $p = (p_1, p_2, \cdots, p_m)$ be a probability distribution on m elements, i.e, $p_i \geq 0$, and $\sum_{i=1}^m p_i = 1$. Define a new distribution q on $m-1$ elements as $q_1 = p_1, q_2 = p_2, \cdots, q_{m-2} = p_{m-2}$, and $q_{m-1} = p_{m-1} + p_m$, i.e., the distribution q is the same as p on any $i \in \{1, 2, \cdots, m-2\}$, and the probability of the last element in q is the sum of the last two probabilities of p. Show that

$$H(p) = H(q) + (p_{m-1} + p_m) H\left(\frac{p_{m-1}}{p_{m-1} + p_m}, \frac{p_m}{p_{m-1} + p_m}\right).$$

X, Y, Z.



Question 2. Mutual Information and Relative Entropy

Let X, Y, Z be three random variables with a joint probability mass function $p(X, Y, Z)$.

(a). Show that

$$\boxed{\boxed{I(X,Y;Z)} - \boxed{I(Y,Z;X)} = I(Y;Z) - I(X;Y).}$$

(b). The relative entropy between the joint distribution and the product of the marginals is $D(p(x,y,z)||p(x)p(y)p(z))$. Show that

$$\boxed{D(p(x,y,z)||p(x)p(y)p(z)) = I(X;Y) + I(X,Y;Z).}$$

$$LHS = H(Z) - H(Z|X,Y) - (H(X) - H(X|Y,Z)) = \boxed{H(Z) - H(X)} + \boxed{H(X|Y,Z) - H(Z|X,Y)} \quad ①$$

$$\|$$

$$RHS = H(Z) - H(Z|Y) - (H(X) - H(X|Y)) = \boxed{H(Z) - H(X)} + \boxed{H(X|Y) - H(Z|Y)} \quad ②$$

$$① - ② = H(X|Y,Z) + H(Z|Y) - H(Z|X,Y) - H(X|Y)$$

$$H(X|Y,Z) + H(Z|Y) = H(X,Z|Y)$$

$$H(Z|X,Y) + H(X|Y) = H(X,Z|Y)$$

$$= 0 \Rightarrow ① = ② \Rightarrow LHS = RHS.$$

b.

$$D(p(x,y,z)||p(x)p(y)p(z))$$

$$= \sum_{x,y,z} p(x,y,z) \cdot \log \frac{p(x,y,z)}{p(x)p(y)p(z)}$$

$$= \boxed{\sum_{x,y,z} p(x,y,z) \log p(x,y,z)} - \sum_{x,y,z} p(x,y,z) \log p(x)p(y)p(z)$$

$$= -H(X,Y,Z) - \boxed{\sum_{x,y,z} p(x,y,z) \log p(x)} = \sum_x p(x) \log p(x) = -H(X)$$

$$- \boxed{\sum_{x,y,z} p(x,y,z) \log p(y)} = -H(Y)$$

$$- \boxed{\sum_{x,y,z} p(x,y,z) \log p(z)} = -H(Z).$$

$$= -H(X,Y,Z) + H(X) + H(Y) + H(Z).$$

$$= -(\cancel{H(X)} + H(Y|X) + H(Z|X,Y)) + \cancel{H(X)} + H(Y) + H(Z)$$

$$= I(X;Y) + I(Z;X,Y).$$

Question 3. Markov Chain

Suppose a Markov chain $X_1 \to X_2 \to X_3$, starts in one of n states, i.e., $X_1 \in \{1, 2, \cdots, n\}$. Suppose X_2 will go down to $k < n$ states, i.e., $X_2 \in \{1, 2, \cdots, k\}$. Then X_3 go back to $m > k$ states, i.e., $X_3 \in \{1, 2, \cdots, m\}$.

(a). What is the upper bound of $I(X_1; X_3)$?

(b). Evaluate $I(X1; X3)$ for $k = 1$, and conclude that no dependence can survive.

Question 4. Inequalities

A coin is known to land heads with probability $\frac{1}{5}$. The coin is flipped N times for some even integer N.

(a). Using Markov's inequality, provide a bound on the probability of observing $\frac{N}{2}$ or more heads.

(b). Using Chebyshev's inequality, provide a bound on the probability of observing $\frac{N}{2}$ or more heads. Express your answer in terms of N.

4. head 1/5, tail 4/5

$$X = \begin{cases} 1 & \frac{1}{5} \quad \text{head} \\ 0 & \frac{4}{5} \quad \text{tail.} \end{cases} \qquad E[X] = \frac{1}{5} \cdot 1 + \frac{4}{5} \cdot 0$$

X^N $X_1, \cdots, X_N$

(a). $\lambda = \frac{N}{2}$

$$\boxed{P\left(X^N \geq \frac{N}{2}\right)} \leq \frac{E[X^N]}{N/2} = \frac{\boxed{NE[X]}}{N/2} = \frac{N/5}{N/2} = \boxed{\frac{2}{5}}$$

(b).

$$\boxed{P(|X^N - E[X^N]| \geq \lambda)} \leq \frac{V[X^N]}{\lambda^2}$$

$$\boxed{P(|X^N - E[X^N]| \geq \lambda)} \quad ① \ldots$$

$$= P(X^N - E[X^N] \geq \lambda) + P(X^N - E[X^N] \leq -\lambda)$$

$$= P(X^N \geq \lambda + E[X^N]) + P(X^N \leq E[X^N] - \lambda)$$

$$E[X^N] \to \frac{N}{5}$$

$$= P\left(X^N \geq \lambda + \frac{N}{5}\right) + P\left(X^N \leq \frac{N}{5} - \lambda\right).$$

$X^N \geq \frac{N}{2}$ $\quad \lambda = \frac{3N}{10}$. $\quad X^N$

$$= P\left(X^N \geq \frac{N}{2}\right) + P\left(X^N \leq -\frac{N}{10}\right)^{=0}.$$

$$= P\left(X^N \geq \frac{N}{2}\right). \quad \frac{N}{2} \text{ more heads.}$$

$N \cdot V[X] = \frac{1}{5} \cdot \frac{4}{5}$

$$① \quad P\left(\left|X^N - \frac{N}{5}\right| \geq \frac{3N}{10}\right) \leq \frac{V[X^N]}{\left(\frac{3N}{10}\right)^2} = \frac{\frac{4N}{25}}{\frac{9N^2}{100}} = \frac{16}{9N}.$$