THE AUSTRALIAN NATIONAL UNIVERSITY

Final-Semester Examination 2022

COMP2610/6621

Information Theory

Tuesday 15 November 2022

Reading Period: 15 minutes
Writing Period: 150 minutes

Permitted materials:
Any material.

Instructions:

The mark for each question is indicated next to the question. Wherever applicable, you must explain and show all steps taken to arrive at your answer. The clarity and precision of your explanations and answers will be taken into account when marking. This examination is worth 50% of the course mark. It will be marked out of 100 and then scaled.

For TRUE/FALSE questions, providing ONLY the correct TRUE/FALSE answer constitutes 1/3 of the mark for that question. You need to fully and correctly justify each TRUE/FALSE answer to get the full mark.

Question 1 [18 marks total]

Determine whether the following statements are TRUE or FALSE. For each answer briefly, but clearly provide your reason.

(a) The entropy for discrete random variables $X \in \mathcal{X}$ satisfies

$$H(X) \leq 3$$

bits. We can conclude that the number of elements in $\mathcal{X}$ is smaller or equal than 8, i.e., $|\mathcal{X}| \leq 8$. **[3 marks]**

(b) $Y \in \mathcal{Y}$ is a function of the random variable $X \in \mathcal{X}$. If $H(Y) = 3$ bits, then we can conclude that $H(X) \geq 3$ bits. **[3 marks]**

(c) A source has 3 symbols. The average codeword length of the binary Shannon code for this source can be 2.6 bits. **[3 marks]**

(d) For three discrete random variables X, Y, Z, we have $I(X;Z|Y) = 0$. Then $Z \to Y \to X$ forms a Markov chain. **[3 marks]**

(e) If a discrete memoryless channel has capacity C bits/channel use, the size of the input alphabet is at least 2^C. **[3 marks]**

(f) Consider a $(15, 11)$ block code C. There should be $2^{11} = 2048$ different codewords with the length of each codeword is 15 bits. **[3 marks]**

Question 2 [16 marks total]

Answer the following two questions.

(a) Suppose X and Y are random variables with the following joint distribution $p(x, y)$:

X \ Y	0	1	2	3
0	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
1	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{6}$

Answer the following questions.

(i) Are X and Y independent? Explain your answer. **[2 marks]**

(ii) Compute the expected value of X and Y, i.e., $\mathbb{E}[X]$ and $\mathbb{E}[Y]$. **[3 marks]**

(iii) Compute the expected value of XY, i.e., $\mathbb{E}[XY]$. **[3 marks]**

(b) A 6-sided dice with each side as $\{1, 2, 3, 4, 5, 6\}$ is rolled twice independently of each other and in secret. Denote the corresponding random variables to these two dice rolls as X_1 and X_2, respectively. We denote $Z = |X_1 - X_2|$.

(i) Find $H(X_1, Z)$. **[3 marks]**

(ii) Find $I(X_1; Z)$. **[3 marks]**

(iii) Find $I(X_1, X_2; Z)$. **[2 marks]**

Question 3 [16 marks total]

Consider the random variable Answer the following questions.

X	x_1	x_2	x_3	x_4	x_5
$p(x_i)$	0.4	0.25	0.15	0.1	0.1

We observe $Y = X + Z$ where $Z \in \{0, 1\}$ is a Bernoulli$\left(\frac{1}{2}\right)$ random variable, which is independent of X and $+$ denotes real addition.

Answer the following questions.

(a) What is the entropy $H(X)$? **[2 marks]**

(b) Find the binary Huffman code for X and the expected codelength for Huffman code. **[4 marks]**

(c) What is the entropy $H(Y)$? **[3 marks]**

(d) What is the mutual information $I(X;Y)$ and $I(X,Y;Z)$? **[3 marks]**

(e) Find the binary Huffman code for Y and the expected codelength for Huffman code. **[4 marks]**

Question 4 [17 marks total]

Suppose a single fair ten-sided die with the number $\{0,0,0,0,0,0,0,1,1,1\}$ on each face of the die (7 faces with the number 0 and 3 faces with the number 1). The die is rolled N times, and let X_i, $i = \{1,2,\cdots,N\}$, be the outcome of the number of these N times, respectively. Let denote $Y_n = \sum_{i=1}^{n} X_i$, $n = \{1,2,\cdots,N\}$, which is the summation of the first n times outcome. Answer the following questions.

(a) Compute the expected value of X_1 and the variance of X_1, i.e., $\mathbb{E}[X_1]$ and $V[X_1]$. **[2 marks]**

(b) Compute the the entropy of Y_2 and Y_3, i.e., $H(Y_2)$ and $H(Y_3)$. **[4 marks]**

(c) Compute the expected value of Y_n and the variance of Y_n, i.e., $\mathbb{E}[Y_n]$ and $V[Y_n]$. **[2 marks]**

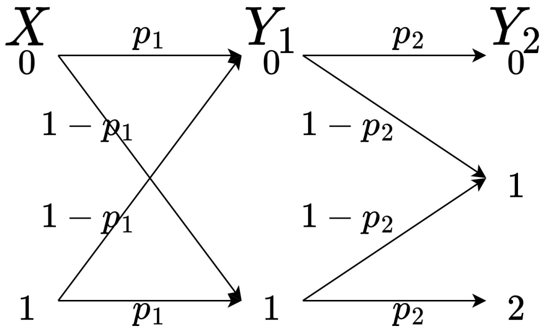
(d) Calculate an upper bound for the quantity $P(Y_n \geq 0.8n)$ using Markov's inequality. **[3 marks]**

(e) Calculate an upper bound for the quantity $P(Y_n \geq 0.8n)$ using Chebyshev's inequality. **[4 marks]**

(f) What is the condition of n that the upper bound obtained by Chebyshev's inequality is closer to the true value of $P(Y_n \geq 0.8n)$ than it obtained by Markov's inequality? **[2 marks]**

Question 5 [15 marks total]

Here we consider a tandem channel as $X \to Y_1 \to Y_2$ with $X \in \mathcal{X} = \{0,1\}$, $Y_1 \in \mathcal{Y}_1 = \{0,1\}$, and $Y_2 \in \mathcal{Y}_2 = \{0,1,2\}$, where the transition diagram is given as



Here we consider a general input probability distribution of X, i.e, $P(X=0) = p_0$ and $P(X=1) = 1-p_0$. Answer the following questions.

(a) Find $P(Y_2|X)$. **[2 marks]**

(b) Find $P(Y_2)$ as a function of the most general input probability distribution. **[2 marks]**

(c) Find $H(Y_1|X)$ and $H(Y_2|X)$. **[3 marks]**

(d) Write the mutual information $I(X;Y_2)$ as a function of the most general input probability distribution. **[3 marks]**

(e) Write the conditional mutual information $I(Y_1;Y_2|X)$ as a function of the most general input probability distribution. **[3 marks]**

(f) Find the channel capacity between X and Y_2. **[2 marks]**

Question 6 [18 marks total]

Answer the following two questions.

(a) There is a discrete memoryless channel (DMC) with the channel input $X \in \mathcal{X} = \{0,1\}$ and channel output $Y \in \mathcal{Y} = \{0,1,2\}$, where the channel is given as

$$Q = \begin{bmatrix} \frac{3}{5}, & 0 \\ \\ \frac{2}{5}, & \frac{2}{5} \\ \\ 0, & \frac{3}{5} \end{bmatrix}$$

Answer the following questions.

(i) Write the mutual information $I(X;Y)$ as a function of the general input probability distribution. **[4 marks]**

(ii) Argue whether or not uniform input distribution achieves the capacity of this channel and find the channel capacity. **[2 marks]**

(iii) An input sequence of length $n = 15$ is drawn according to an i.i.d Bernoulli($\frac{1}{2}$) distribution and the resulting sequence is

$$\mathbf{x}^n = 1,1,0,1,1,1,0,0,1,0,1,1,1,1,0.$$

Is this sequence typical? Explain your answer briefly. **[3 marks]**

(iv) The input given above goes through the channel and the output sequence $\mathbf{y}^n$ is observed. Give at least two possible channel output sequences $\mathbf{y}^n$ that are jointly typical with the given $\mathbf{x}^n$ above. Explain how you chose your sequences. **[3 marks]**

(b) Two 4-bits messages, m_1 and m_2, are encoded by using $(7,4)$ Hamming code, where the generation matrix is given as $G = \begin{bmatrix} I_4 & P^T \end{bmatrix}$, where I_4 is a 4-dimensional identity matrix and P^T is given as

$$P^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

. Answer the following questions.

(i) If $m_1 = [1 \quad 0 \quad 0 \quad 1]$, what is its corresponding Hamming code? **[2 marks]**

(ii) The Hamming code corresponded to m_2 is transmitted over a noise channel, where the channel flips up to 1 bit per code. The received code is $[1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0]$. Find out the 4-bits input message m_2. You may use the syndrome-to-flip-bit table, given by

z	000	001	010	011	100	101	110	111
flip-bit	none	r_7	r_6	r_4	r_5	r_1	r_2	r_3

[4 marks]