Supplement: Optimising under Constraint

Lagrange Multipliers

We've talked about optimisation, but life often supplies constraints.

Example: Maximise production but there's a budget

Example: Maximise fun but there's a budget

Example: Live as long as you can on Mars but you have minimum daily calorie requirements to survive

Suppose we have a function $f : \mathbb{R}^n \to \mathbb{R}$ that we want to maximise.

Then look at $\{x \mid \nabla f(x) = 0\}$.

Suppose moreover that we have another function $g : \mathbb{R}^n \to \mathbb{R}$ and that we have some constraint like $g(x) = 0$.

So

$$\text{Maximize } f(x, y) \text{ subject to } g(x, y) = 0$$

Example:

$$\begin{cases} f(x, y) & = xy \\ g(x, y) & = x^2 + y^2 = 1 \end{cases} \tag{1}$$

How do we approach this?

Consider isoclines $f(x, y) = c$.

And consider isoclines $g(x, y) = c'$ (not necessarily the same constant).

We want tangent points.

Recall that gradient is $\perp$ to isoclines.
So we want the points where the ∇ are $\|$.

So we want set of

$$\{x \mid \nabla f(x) = \lambda \nabla g(x)\}$$

Example:

$$
\begin{cases}
\nabla f &= \begin{pmatrix} y \\ x \end{pmatrix} \\
\\
\nabla g &= \begin{pmatrix} 2x \\ 2y \end{pmatrix}
\end{cases}
\tag{2}
$$

So

$$
\begin{cases}
y &= \lambda 2x \\
x &= \lambda 2y \\
1 &= x^2 + y^2
\end{cases}
\tag{3}
$$

From the first of these we get

$$\lambda = \frac{y}{2x} \qquad \text{if } x \neq 0$$

Substituting into the second,

$$x = \frac{y}{2x} \cdot 2y = \frac{y^2}{x} \Rightarrow x^2 = y^2 \Rightarrow x = \pm y$$

And then the third provides

$$2x^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow \boxed{x = \frac{\pm\sqrt{2}}{2} \wedge y = -x}$$

Exercise: Consider $f(x, y) = xy^2$ subject to $2x^2 + 5y^2 = 2$.

Exercise: Consider $f(x, y, z) = x^2y^3z$ subject to $2x^2 + 5y^2 = 2$.

Lagrangian

We can write

$$L = f - \lambda g$$

Then $\nabla L = 0$ is the set of equations we started with, above.

This is generally a nice form for computers.

It also transforms the problem from a constrained optimisation problem into an unconstrained optimisation problem.