Basics

$1 \in G$ is the unit.

- Groups G
- Subgroups $H \leq G$ $H < G \Rightarrow$ proper.
- Normal subgroups $N \trianglelefteq G$
- Quotient groups G/N
- Group homomorphisms $\varphi: G_1 \to G_2$
  (isomorphisms)
  - $\operatorname{Ker} \varphi \trianglelefteq G_1$
  - $\operatorname{Im} \varphi \leq G_2$
  - Injective
  - Surjective

Isomorphism Laws

Prop (First Isomorphism): Let $N \trianglelefteq G$ and let $\pi: G \to G/N$ be the canonical projection $g \mapsto gN = \bar{g}$

π is a surjective homomorphism with Kernel N

Let $\varphi: G \to Q$ with kernel N.

Then $\exists!$ $\hat{\varphi}: G/N \to Q$ $\hat{\varphi}(gN) = \varphi(g)$

is well-defined and an isomorphism.

This gives the commutative diagram.

$$\begin{tikzcd} G \arrow[r, two heads, "\pi"] \arrow[dr, two heads, "\varphi"'] & G/N \arrow[d, dashed, "\hat{\varphi}"] \\ & Q \end{tikzcd}$$

We say φ factors through π.

Informally $\{\text{Normal subgroups}\} \longleftrightarrow \left\{\begin{matrix}\text{surjective} \\ \text{homomorphisms}\end{matrix}\right\}$

Prop: Let $N \trianglelefteq G$ and $\varphi: G \to H$ be a homomorphism, and $N < \operatorname{Ker}\varphi$.

Then φ factors uniquely through $\pi: G \to G/N$, i.e.

$$\begin{tikzcd} G \arrow[r, two heads, "\pi"] \arrow[dr, "\varphi"'] & G/N \arrow[d, dashed, "\hat{\varphi}"] \\ & H \end{tikzcd}$$ commutes

Moreover, $\operatorname{Ker}\hat{\varphi} = \pi(\operatorname{Ker}\varphi) = \operatorname{Ker}\varphi/N$

$\operatorname{Im}\hat{\varphi} = \operatorname{Im}\varphi$

Def: Given subsets $X, Y \subset G$, let

$$XY = \{xy \mid x \in X, y \in Y\}$$

Remark: Even if X, Y are subgroups, XY may not be a subgroup.

Defn: The *normalizer* of X in G, denoted $N_G(X)$ is defined to be

$$N_G(X) = \{g \in X \mid gXg^{-1} = X\}$$

We say Y *normalizes* X if $Y \subset N_G(X)$

Remark: Suppose $Y \leq G$.

Y normalizes $X \iff yXy^{-1} \subseteq X$

for any $y \in Y$.

Remark: For any $X \subset G$, $N_G(X)$ is a subgroup.

Prop (Second Isomorphism): Let $N, H \leq G$ such that H normalizes N. Then

1. $NH = HN$ and is a subgroup.
2. $N \trianglelefteq NH$ and $N \cap H \trianglelefteq H$

③ We have the diagram

$$\begin{tikzcd}[column sep=small]
 & G \arrow[d, dash] & \\
 & NH \arrow[dl, dash] \arrow[dr, dash] & \\
N \arrow[dr, dash] & & H \arrow[dl, dash] \\
 & N \cap H &
\end{tikzcd}$$

and $NH/_N \cong H/_{N \cap H}$

<u>Proof</u>: ① $hNh^{-1} = N \Rightarrow hN = Nh$

② H normalizes N
N normalizes N
$\Rightarrow$ NH normalizes N

③

$$\begin{tikzcd}
N \cap H \arrow[r, hook] & H \arrow[r, two heads] \arrow[d, hook] \arrow[dr, "\varphi"] & H/_{N \cap H} \\
N \arrow[r, hook] & NH \arrow[r, two heads, "\pi"'] & NH/_N
\end{tikzcd}$$

Le φ be the restriction of $\pi : NH \to NH/N$ to H.

$\overline{nh} = \bar{n}\bar{h} = \bar{h} = \varphi(h) \Rightarrow \operatorname{Ker}\varphi = N \cap H.$

Apply the first isomorphism theorem to get that $H/N\cap H = NH/N$

<u>Prop</u> (Third Isomorphism)

Let $N, K \trianglelefteq G$ with $N \subseteq K$. Then we have the diagram

$$\begin{array}{c} G \\ | \\ K \\ | \\ N. \end{array}$$

Then $K/N \cong G/N$.

$$G/N \Big/ K/N \cong G/K$$

<u>Proof</u>: Consider $\pi : G \to G/K$. By hypothesis $N \subset \operatorname{Ker} \pi$. By the universal property of the quotient, there exists a homomorphism $\hat{\pi} : G/N \to G/K$

$$\operatorname{Ker} \hat{\pi} = K/N$$

$$\operatorname{Im} (\hat{\pi} = G/K)$$

By the first isomorphism thm we are done.

Prop (4th isomorphism): Let $N \trianglelefteq G$

① If $N \subseteq H \subseteq G$
$\Rightarrow H/N \leq G/N$

② If $Q \leq G/N$, $\exists!\ H$ such that $N \subseteq H \subseteq G$ and $Q = H/N$

i.e. There is a bijective inclusion (+normality) preserving correspondence

$$\{\text{Subgroups of } G/N\} \longleftrightarrow \{\text{Subgroups } H \leq G \text{ Containing } N\}$$

Modularity

Given $X, Y, Z \leq G$

Q: Does the "distributive law"

$$X \cap YZ = (X \cap Y)(X \cap Z)$$

hold?

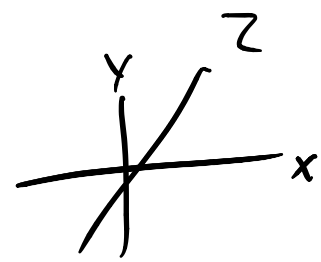
A: No. counterexample:

Take $G = \mathbb{Z}^2$

$Y = \{(z,0) \mid z \in \mathbb{Z}\}$

$Z = \{(0,z) \mid z \in \mathbb{Z}\}$

$X = \{(z,z) \mid z \in \mathbb{Z}\}$



$X \cap (Y+Z) = \mathbb{Z}^2$

$X \cap Y = \{0\}$ $X \cap Z = 0$

$\Rightarrow (X \cap Y) + (X \cap Z) = 0.$

<u>Prop</u> (Dedekind's modular law)

Let $X, Y, Z \leq G$ s.t. $Z \subseteq X$.

Then
$$\begin{array}{ccc} X \cap YZ & = & (X \cap Y)Z \\ \parallel & & \parallel \\ XZ \cap YZ & & (X \cap Y)(X \cap Z) \end{array}$$

<u>Proof</u>: $\supseteq$: $X \cap Y \subset X$, $Z \subset X$

$\Rightarrow (X \cap Y)Z \subset X$

$X \cap Y \subset Y \Rightarrow (X \cap Y)Z \subset YZ$

$\subseteq$ Let $x \in X \cap YZ$, and let

$x = yz \quad x \in X,\ y \in Y,\ z \in Z$

$\Rightarrow y = xz^{-1} \in X$ because $Z \subset X$.

$\Rightarrow x = \underset{\substack{\uparrow \\ X \cap Y}}{y}\underset{\substack{\uparrow \\ Z}}{z} \in (X \cap Y)Z$

<u>Remarks</u>

Let $\mathcal{L}(G) = \{$ Subgroups of $G \}$

$\mathcal{L}(G)$ is a poset with respect to inclusion.

$\langle H_1 \vee H_2 \rangle :=$ join of H_1, H_2

$H_1 \cap H_2 :=$ meet of H_1, H_2

$\Rightarrow \mathcal{L}(G)$ is a lattice