CLIFFORD ALGEBRAS AND SPIN GROUPS

JEFFREY JIANG

One motivating reason for studying Clifford algebras and Spin groups is to better understand the orthogonal groups O_n and special orthogonal groups SO_n, which consist of the transformations that preserve the standard inner product (and orientation in the case of SO_n) on $\mathbb{R}^n$. These groups are of physical importance, since they are exactly the linear maps preserving our usual notions of length and angle, which are derived from the standard inner product.

If you care about the representations of these groups, you might have noticed some problems – there exist representations of the Lie algebra $\mathfrak{so}_n$ that fail the exponentiate to representations of SO_n (though they do exponentiate to projective representations). The source of the problem lies in the topology of the SO_n and O_n.

Theorem 0.1. *For $n > 2$, we have*

$$\pi_1(SO_n) \cong \mathbb{Z}/2\mathbb{Z}$$

In the case $n = 3$, this can be visualized using the plate/belt loop trick. The failure of the orthogonal groups to be simply connected tells us that there is information that we cannot recover from the Lie algebra. This motivates the question :

Question. *What is the universal cover of* SO_n*? What is the representation theory of this group?*

You might already know the answer to this question in some simple cases.

Example 0.2. Consider the case of SO_3. If you are familiar with computer graphics, you might know that we can represent rotations in $\mathbb{R}^3$ (i.e. elements of SO_3) by quaternions. Let $\mathbb{H}$ denote the algebra of quaternions, which are elements of the form

$$q = a + bi + cj + dk$$

with $a, b, c, d \in \mathbb{R}$ and i, j, k formal symbols satisfying the relations

$$i^2 = j^2 = k^2 = ijk = -1$$

embedding $\mathbb{R}^3$ into $\mathbb{H}$ via the mapping $(x, y, z) \mapsto xi + yj + zk$, we define an action of the unit quaternions $\mathrm{Sp}(1) = S^3 \subset \mathbb{H}^\times$ on $\mathbb{R}^3$ (here $\mathrm{Sp}(1)$ denotes the compact symplectic group) by

$$q \cdot v = qv\overline{q}$$

where $\overline{q}$ is the quaternionic conjugate, i.e.

$$\overline{a + bi + cj + dk} = a - bi - cj - dk$$

Any orthogonal transformation $A \in SO_n$ can be represented by the action of some $q \in S^3$. In addition q and $-q$ define the same orthogonal transformation, which tells us that S^3 is the universal cover of SO_3, and the covering map is the quotient by antipodal points, which tells us that $SO_3 \cong \mathbb{RP}^3$.

The central ingredient to answering both of these questions stems from the classical theorem

Theorem 0.3 (***Cartan-Dieudonné***). *Any orthogonal transformation $A \in \mathrm{O}_n$ can be expressed as the compositions of at most n reflections about hyperplanes.*

By a hyperplane, we mean a $n-1$ dimensional subspace $H \subset \mathbb{R}^n$, which can be identified with a choice of a unit normal vector $v \in H^\perp$. Reflection about H is then given by the map $R_H : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$R_H(w) = w - 2\langle w, v\rangle v$$

The Cartan-Dieudonné theorem is the key piece in understanding the Clifford algebra.

Definition 0.4. The Clifford algebra Cliff_n is the freest unital associative $\mathbb{R}$-algebra generated by elements of $\mathbb{R}^n$ subject to the relations

(1) $v^2 = -1$ for any unit vector $v \in \mathbb{R}^n \subset \mathrm{Cliff}_n$
(2) $vw = -wv$ for $v, w \in \mathbb{R}^n \subset \mathrm{Cliff}_n$ where $\langle v, w\rangle = 0$.

By "generated by $\mathbb{R}^n$, we mean that every element in Cliff_n can be written as a formal sum of formal products of vectors in $\mathbb{R}^n$. In particular, if we fix a basis for $\mathbb{R}^n$, all elements in Cliff_n can be written as sums of formal products of these basis elements. Let $\{e_i\}$ denote the standard orthonormal basis for $\mathbb{R}^n$ with the standard inner product. Then the relations we specified tells us that the set

$$\left\{e_{i_1} \cdots e_{i_k} \;:\; 0 \le k \le n, 1 \le i_1 < i_2 < \ldots < i_k \le n\right\}$$

forms a basis for Cliff_n, much like how they form a basis for the exterior algebra of $\mathbb{R}^n$. Note that this implies that $\dim \mathrm{Cliff}_n = 2^n$.

Another equivalent definition of the Clifford algebra Cliff_n is via a universal property.

Definition 0.5. The Clifford algebra Cliff_n is a unital associative equipped with a linear map $i : \mathbb{R}^n \to \mathrm{Cliff}_n$ such that for any linear map $\varphi : V \to A$ into another unital associative $\mathbb{R}$-algebra A satisfying $(\varphi(v))^2 = -\langle v, v\rangle$, there exists a unique map $\widetilde{\varphi} : \mathrm{Cliff}_n \to A$ such that the diagram

$$\begin{tikzcd} V \arrow[d, "i"'] \arrow[dr, "\varphi"] & \\ \mathrm{Cliff}_n \arrow[r, "\widetilde{\varphi}"'] & A \end{tikzcd}$$

commutes.

Remark. Some choose the convention that unit vectors square to 1 in the Clifford algebra, which is one of the several places where people will have differing sign conventions.

What is the motivation for this construction? Inside Cliff_n, we want vector $v \in \mathbb{R}^n$ to represent a hyperplane reflection about $v^\perp$. The fact that hyperplane reflections defined by orthogonal vectors commute explains the second relation. Before going further, we prove a helpful formula.

Lemma 0.6. *Let $v, w \in \mathbb{R}^n \subset \mathrm{Cliff}_n$. Then $vw + wv = -2\langle v, w\rangle$*

Proof. Let $v = \sum_i v^i e_i$ and $w = \sum_j w^j e_j$. We compute

$$\begin{aligned} vw + wv &= \sum_{i,j} (v^i w^j e_i e_j + v^i w^j e_j e_i) \\ &= \sum_{i,j} v^i w^j (e_i e_j + e_j e_i) \end{aligned}$$

We note that since $\langle e_i, e_j \rangle = 0$, we have that $e_i e_j = -e_j e_i$. This, along with the fact that $e_i^2 = -1$ gives us that this sum becomes

$$\sum_i v^i w^i (e_i^2 + e_i^2) = -2 \sum_i v^i w^i = -2\langle v, w \rangle$$

■

Let $\alpha : \mathrm{Cliff}_n \to \mathrm{Cliff}_n$ be the automorphism extending the map $v \mapsto -v$ on $\mathbb{R}^n \subset \mathrm{Cliff}_n$ (i.e. for $v_1, \ldots v_k \in \mathbb{R}^n$, $\alpha(v_1 \cdot v_k) = (-1)^k v_1 \cdots v_k$.) and let $G \subset \mathrm{Cliff}_n^\times$ denote the subgroup of the group of units generated by taking products of unit vectors in $\mathbb{R}^n$. We then define a group action of G on $\mathbb{R}^n \subset \mathrm{Cliff}_n$ by the formula

$$g \cdot v = \alpha(g) v g^{-1}$$

(Note the similarity of this action to the one of the quaternions on R^3.) We first need to verify that this defines a group action, i.e. that it maps $\mathbb{R}^n$ back to itself. To show this, it suffices to check on the generating set of unit vectors. Let $v \in \mathbb{R}^n$ be a unit vector. We note that $v^{-1} = -v$, and $\alpha(v) = -v$. We then compute for $w \in \mathbb{R}^n$

$$\begin{aligned} v \cdot w &= \alpha(v) w v^{-1} \\ &= vwv \\ &= (-2\langle v, w \rangle - wv)v \\ &= -2\langle v, w \rangle v + w \end{aligned}$$

Note that this just the definition of hyperplane reflection about $v^\perp$! Therefore, not only do we have a group action, but we also have that G acts on $\mathbb{R}^n$ by orthogonal transformations, giving us a map $G \to O_n$. This map is surjective by the Cartan-Dieudonné theorem, and some additional work shows that the kernel of this map is $\{\pm 1\}$, giving us a $2-1$ map. This tells us that G is the double cover of O_n that we are looking for. The group G called the ***Pin group*** Pin_n. Other thing to note is that a single hyperplane reflection is orientation reversing, so the composition of an even number of reflections is orientation preserving. Restricting to even products of unit vectors gives us the ***Spin group*** Spin_n, which is the simply connected double cover of SO_n.

To investigate these groups further, we study the structure of the Clifford algebras they came from. The first goal is to identify the Clifford algebras in terms of algebras we understand well (e.g. matrix algebras). It's not too hard to identify the first few

Example 0.7.

(1) $\mathrm{Cliff}_0 \cong \mathbb{R}$
(2) $\mathrm{Cliff}_1 \cong \mathbb{C}$ via the map $e_1 \mapsto i$.
(3) $\mathrm{Cliff}_2 \cong \mathbb{H}$ via the map $e_1 \mapsto i$, $e_2 \mapsto j$.

The name of the game here is to find algebras with an anti-commuting basis of elements that square to -1. For the low dimensional cases, this is easy, but things will quickly become difficult, since $\dim \mathrm{Cliff}_n = 2^n$. To find some more (in fact, to find them all), we develop a little more technology.

Definition 0.8. Let A be an associative $\mathbb{R}$-algebra. A $\mathbb{Z}/2\mathbb{Z}$-grading on A is a direct sum decomposition

$$A = A^0 \oplus A^1$$

such that multiplication respects the grading, i.e. for $a \in A^i$ and $b \in A^j$, $ab \in A^{i+j \mod 2}$. The subspaces A^0 and A^1 are called ***even*** and ***odd*** respectively. Note in particular that A^0 forms a subalgebra of A, since the product of even elements is even. The elements of A^0 and A^1 are called ***homogeneous elements***. For a homogeneous element a, we let $|a| \in \{0, 1\}$ denote the index of subspace it lies in.

Example 0.9.

(1) For a vector space V, $\Lambda^\bullet V$ has a $\mathbb{Z}/2\mathbb{Z}$-grading

$$\Lambda^\bullet V = \Lambda^{\text{even}} V \oplus \Lambda^{\text{odd}} V$$

(2) The Clifford algebra has a $\mathbb{Z}/2\mathbb{Z}$-grading in much the same way, where Cliff_n^0 is the algebra of even products of basis vectors, and Cliff_n^1 is the subspace of odd products of basis vectors.

Something that will come in handy later :

Theorem 0.10. *For $n \geq 1$, the even subalgebra $\mathrm{Cliff}_n^0 \subset \mathrm{Cliff}_n$ is isomorphic to Cliff_{n-1} as ungraded algebras.*

Proof. We note that for the standard basis $\{e_i\}$ for $\mathbb{R}^n$, we have that

$$(e_i e_j)^2 = e_i e_j e_i e_j = -e_i^2 e_j^2 = -1$$

and that a generating set for Cliff_n^0 is

$$\{e_1 e_i \ : \ 2 \leq i \leq n\}$$

The mapping $e_1 e_i \mapsto e_{i-2}$ defines the desired isomorphism. ∎

Another helpful operation is taking tensor products. Since things are graded, we need a small modification of the definition.

Definition 0.11. Let A and B be $\mathbb{Z}/2\mathbb{Z}$-graded algebras. Define the ***graded tensor product*** of A and B, denoted $A \otimes B$ to be the algebra with the underlying vector space $A \otimes B$ with the multiplication defined on homogeneous elements by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|b_1||a_2|} a_1 a_2 \otimes b_1 b_2$$

This algebra is once again graded, where

$$(A \otimes B)^0 = (A^0 \otimes B^0) \oplus (A^1 \otimes B^1)$$
$$(A \otimes B)^1 = (A^0 \otimes B^1) \oplus (A^1 \otimes B^0)$$

Remark. A heuristic reason for this definition is that when we are multiplying, we are formally "commuting b_1 with a_2, so we should pick up an extra sign depending on their parities.

The graded tensor product allows us to express higher dimensional Clifford algebras in terms of their smaller brethren.

Theorem 0.12.
$$\mathrm{Cliff}_p \otimes \mathrm{Cliff}_q = \mathrm{Cliff}_{p+q}$$
where $\otimes$ denotes the graded tensor product.

Proof. We specify an isomorphism $\varphi : \mathrm{Cliff}_{p+q} \to \mathrm{Cliff}_p \otimes \mathrm{Cliff}_q$ on an orthonormal basis and verifying that it satisfies the Clifford relation. Let $\{e_i\}$ and $\{f_i\}$ be the standard orthonormal bases for $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively, and let $\{b_i\}$ denote the standard orthonormal basis for $\mathbb{R}^{p+q}$
$$\varphi(b_i) = \begin{cases} e_i \otimes 1 & 1 \le i \le p \\ 1 \otimes f_i & p+1 \le i \le p+q \end{cases}$$
to show the Clifford relations are satisfied, we just need to verify that for every i, we have $(\varphi(b_i))^2 = -1$, and for distinct i, j, we have that $\varphi(b_i)$ and $\varphi(b_j)$ anticommute. The first condition is obvious. For the second, there are two cases. If $1 \le i, j \le p$, they anticommute because the e_i anticommute. The same holds for $p \le i, j \le p+q$. In the case that $1 \le i \le p$ and $p+1 \le j \le p+q$, we compute
$$\begin{aligned} \varphi(b_i)\varphi(b_j) + \varphi(b_j)\varphi(b_i) &= (e_i \otimes 1)(1 \otimes f_j) + (1 \otimes f_j)(e_i \otimes 1) \\ &= e_i \otimes f_j - e_i \otimes f_j = 0 \end{aligned}$$
where we use the fact that 1 is an even element and the fact that f_j and e_i are odd. ■

This fact, along with some computations by hand gives us that the first 9 Clifford algebras (starting at Cliff_0) are isomorphic as ungraded algebras to the following list.

$$\mathbb{R} \qquad \mathbb{C} \qquad \mathbb{H} \qquad \mathbb{H} \times \mathbb{H} \qquad M_2\mathbb{H} \qquad M_4\mathbb{C} \qquad M_8\mathbb{R} \qquad M_8\mathbb{R} \times M_8\mathbb{R} \qquad M_{16}\mathbb{R}$$

Beyond this, something curious happens : the graded tensor product with $M_{16}\mathbb{R}$ is the same as the ungraded tensor product on the level of ungraded algebras, so these 9 are the only ones we need to classify them all. In particular, we know that all Clifford algebras are very nice – they are matrix algebras over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. In particular, this implies that they are semisimple algebras, which tells us that they have nice modules. In particular, this implies that all Clifford modules are direct sums of irreducible ones, and from these irreducible modules, we can recover the lost representations of SO_n.

Definition 0.13. The ***Spin representations*** are the representations of Spin_n obtained by restricting the action of Cliff_n on an irredicible module.

The classification of these representations is surprisingly simple, which stems from two observations.

Theorem 0.14. *The even subalgebra Cliff_n^0 is isomorphic as ungraded algebras with Cliff_{n-1}.*

Theorem 0.15. *Let $k = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.*

(1) *The algebra $M_n k$ admits a single irreducible left module, which is $M_n k$ acting on k^n in the standard way.*
(2) *The algebra $M_n k \times M_n k$ admits two irreducible modules, corresponding to the left or right factor acting in the standard way, and the other acting by 0.*

Proof.

(1) We note that M_nk acts transitively on k^n, so k^n is an irreducible module. To prove that it is unique, we note that M_nk admits an increasing chain of left ideals

$$0 = I_0 \subset I_1 \subset \ldots \subset I_n = M_nk$$

where I_k is the left ideal of matrices where all of the entries but the first k columns are 0. These ideals have the property that the quotients I_k/I_{k-1} are all isomorphic to k^n as M_nk modules. Let M be an arbitrary nontrivial irreducible module, and fix $m \in M$. Since M is trivial, the orbit of m under the action of M_nk must necessarily be all of M_nk. We then get a surjective module homomorphism $\varphi : M_nk \to M$ where $A \mapsto A \cdot m$. Therefore, there exists some smallest k such that $\varphi(I_k)$ is nonzero, and by construction, $\varphi|_{I_k}$ factros through I_k/I_{k-1}, which is necessarily an isomorphism by Schur's Lemma.

(2) The proof is much the same, where we use the increasing chain

$$0 = J_0 \subset I_1 \times \{0\} \subset \ldots \subset M_nk \times \{0\} \subset M_nk \times I_1 \subset \ldots \subset M_nk \times M_nk$$

where I_k is defined the same as in the first case. The proof is then the same argument, using the observation that J_k/J_{k-1} is isomorphic to k^n with the standard action of the left factor or the right factor.

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Having classified ungraded Clifford modules, this tells us the Spin representations. For example, in the case of $\mathrm{Spin}_3 \cong SU_2 \cong Sp(1)$, we have that the Spin representation is just the action of the Spin group on the irreducible module of the even subalgebra Cliff_3^0, which is isomorphic to $\mathrm{Cliff}_2 \cong \mathbb{H}$. So the Spin representation is just $Sp(1)$ acting on $\mathbb{H}$ in the standard way.

We discuss one final thing about Clifford modules. For a ring A, let ${}_A\mathsf{Mod}$ denote commutative monoid of left A-modules under direct sum. Given a ring homomorphism $\varphi : A \to B$, there is an induced map ${}_B\mathsf{Mod} \to {}_A\mathsf{Mod}$, where a B module M becomes an A-module with multiplication defined by $a \cdot m = \varphi(a) \cdot m$. Studying this induced map on modules proves extremely fruitful for Clifford modules. let $\mathcal{M}_n$ denote the commutative monoid of left Cliff_n modules. The inclusion maps $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ induces an inclusion $\mathrm{Cliff}_n \hookrightarrow \mathrm{Cliff}_{n+1}$, which then gives maps $\mathcal{M}_{n+1} \to \mathcal{M}_n$. Since the maps $\mathrm{Cliff}_n \hookrightarrow \mathrm{Cliff}_{n+1}$ are inclusion, the induce map on $\mathcal{M}_n$ is just the restriction of a module of Cliff_{n+1} to the action of $\mathrm{Cliff}_n \subset \mathrm{Cliff}_{n+1}$. We can then compute the cokernel of this map, which computes to obstruction to extending a Cliff_n-module to a Cliff_{n+1} module. To compute these cokernels, we note that the inclusions $\mathrm{Cliff}_n \hookrightarrow \mathrm{Cliff}_{n+1}$ fall into one of the following cases

(1) $M_nk \hookrightarrow M_nk'$ where $k = \mathbb{R}$ or $\mathbb{C}$, and $k' = \mathbb{C}$ or $\mathbb{H}$ is the division algebra of twice the dimension over $\mathbb{R}$.

(2) $M_nk \times M_nk \hookrightarrow M_{2n}k$, given by the mapping

$$(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

(3) $M_nk' \hookrightarrow M_{2n}k$, where k and k' are defined as in (1)

(4) $M_nk \hookrightarrow M_nk \times M_nk$

Using our classification of Clifford modules, as well as their semisimplicty, we compute the cokernels in all these cases

(1) In this case, the irreuducible module for $M_n k'$ is $(k')^n$, which is twice the dimension of k^n over $\mathbb{R}$. Since both $M_n k$ and $M_n k'$ only admit a single irreducible module, the induced monoid homomorphism is the map $\mathbb{Z}^{\geq 0} \to \mathbb{Z}^{\geq 0}$ where $1 \mapsto 2$. The cokernel is the *group* $\mathbb{Z}/2\mathbb{Z}$.
(2) The irreducible modules k^{2n} decomposes into a direct sum of the two irreducible modules for $M_n k \times M_n k$. The monoid homomorphism is then the map $\mathbb{Z}^{\geq 0} \to \mathbb{Z}^{\geq 0}$ defined by $1 \mapsto (1, 1)$. The cokernel is then the group $\mathbb{Z}$. You might have seen this construction before as the ***Grothendieck completion*** of a commutative monoid.
(3) The irreducible modules $(k')^n$ and k^{2n} are the same dimension so the monoid homomorphism is just $1 \mapsto 1$. so the cokernel is trivial.
(4) Again, both irreducible modules for $M_n k \times M_n k$ are the same dimension as the one for $M_n k$, so the map is given by $(1, 0) \mapsto 1$ and $(0, 1) \mapsto 1$, so the cokernel is again trivial.

Therefore, these cokernels give the following 8-periodic sequence

$$\mathbb{Z}/2\mathbb{Z} \qquad \mathbb{Z}/2\mathbb{Z} \qquad 0 \qquad \mathbb{Z} \qquad 0 \qquad 0 \qquad 0 \qquad \mathbb{Z}$$

which is affectionately called the ***Bott song*** (though perhaps shifted by 1), and is one of the many avatars of ***Bott periodicity***.