RIEMANNIAN GEOMETRY CONFERENCE COURSE

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These are notes and exercises we compiled during a reading course with Professor Neitzke in the spring of 2018.

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Week 1: Lie Groups and Vector Fields

This week's talk will be given by Jeffrey.

Definition 1.1. A ***Lie group*** is a group $(G, \cdot)$ such that G is a smooth manifold and the mappings

$$(g, h) \mapsto gh$$
$$g \mapsto g^{-1}$$

are smooth.

If we have a Lie group G, each element g determines a map $L_g : G \to G$ where $L_g(h) = gh$. This map is smooth, and is in fact a diffeomorphism since it has a smooth inverse given by $L_{g^{-1}}$

Examples of Lie groups include $GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$, and $\mathbb{S}^1$

Definition 1.2. A ***Lie group homomorphism*** $F : G \to H$ is a smooth map that is also a group homomorphism.

A simple example would be the map $t \mapsto e^{2\pi i t}$, which gives a Lie group homomorphism from $\mathbb{R} \to \mathbb{S}^1$.

Theorem 1.3. *Every Lie group homomorphism is constant rank*

Proof. If we show that dF_g has the same rank as dF_e for arbitrary $g \in G$, we are done. To see this, we note that

$$F(gh) = F(g)F(h)$$
$$\implies (F \circ L_g)(h) = (L_{F(g)} \circ F)(h)$$

By the chain rule, this gives us $dF_g \circ d(L_g)_e = d(L_{F(g)})_e \circ dF_e$. Then since left multiplication is a diffeomorphism, we are composing with a isomorphism so we conclude dF_g has the same rank as dF_e. ∎

Corollary 1.4. *Let $F : G \to H$ be a Lie group homomorphism. Then* $\ker F$ *is a Lie subgroup of G with codimension equal to the rank of F.*

Theorem 1.5. *For any Lie group G, the* ***identity component****, denoted G_0 is the path component containing the identity element. We claim that G_0 is a normal Lie subgroup of G, and that every connected component of G is diffeomorphic to G_0*

Proof. We first prove that G_0 is a Lie subgroup. Let $g, h \in G_0$. Then let $\gamma_g, \gamma_h : I \to G$ be smooth paths from e to g and h respectively. Then the mapping

$$t \mapsto \gamma_g(t)\gamma_h(t)$$

gives us a smooth path from e to gh, so $gh \in G_0$. To prove it is normal, let $g \in G$ be an arbitrary group element, and $h \in G_0$. Let γ_h again denote a smooth path from e to h. Then we note that the mapping

$$t \mapsto g\gamma_h(t)g^{-1}$$

gives us a smooth path from e to ghg^{-1}, so $ghg^{-1} \in G_0$. Then since G_0 is a connected component, it is closed, so G_0 indeed forms a closed submanifold of G, so G_0 is a Lie subgroup.

For the second part, let $C \subset G$ be another connected component of G not containing the identity. Then let $g \in C$. If we then consider the diffeomorphsim given by left translation map L_g, the restriction of L_g to G_0 determines a diffeomorphism onto its image, which must be open and closed (i.e. a path component of G). Since the image contains g, $L_g\big|_{G_0}$ then determines a diffeomorphism from $G_0 \to C$. ∎

An object closely related to a Lie group G is its Lie algebra.

Definition 1.6. A ***Lie algebra*** $(\mathfrak{g}, [-,-])$ is a real vector space equipped with a bilinear map

$$[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

such that

(1) $[X, Y] = -[Y, X]$

(2) $[X,[Y,Z]]+[Z,[X,Y]]+[Y,[X,Z]]=0$

Examples of Lie algebras include $\mathfrak{X}(M)$, $\mathfrak{gl}(n,\mathbb{R})$, and $\mathfrak{o}(n)$. As you expect with an algebraic object, there is a concept of a homomorphism.

Definition 1.7. A ***Lie algebra homomorphism*** is a linear map $F:\mathfrak{g}\to\mathfrak{h}$ such that

$$F([X,Y])=[F(X),F(Y)]$$

Given a Lie group, we can associate a Lie algebra to it as follows

Definition 1.8. Let G be a Lie group, then define $\mathrm{Lie}(G)\subset\mathfrak{X}(M)$ to be the set of all ***left invariant vector fields***. A vector field X is left invariant if for all g, we have

$$(L_g)_*X=X$$

or equivalently,

$$X_{gh}=d(L_g)_h(X_h)$$

Proposition 1.9. *The set* $\mathrm{Lie}(G)$ *forms a Lie algebra under the Lie bracket of vector fields.*

Proof. We first prove that $\mathrm{Lie}(G)$ is a vector space. The 0 vector field is clearly left invariant, and will be the identity element. Then given $X,Y\in\mathrm{Lie}(G)$, $\lambda\in\mathbb{R}$, we have that $(L_g)_*(X+Y)$ and $(L_g)_*(\lambda X)$ are in $\mathrm{Lie}(g)$ due to linearity of the differential. In addition, $\mathrm{Lie}(G)$ is closed under brackets, since

$$(L_g)_*[X,Y]=[(L_g)_*X,(L_g)_*Y]=[X,Y]$$

Therefore, $\mathrm{Lie}(G)$ is a Lie subalgebra of $\mathfrak{X}(M)$. ∎

The Lie algebra might look a bit strange, but there is a natural identification with the tangent space at the identity

Proposition 1.10. *There exists a basis-independent isomorphism* $\mathrm{Lie}(G)\to T_eG$

Proof. Define the map

$$\begin{aligned}\varphi:\mathrm{Lie}(G)&\to T_eG\\ X&\mapsto X_e\end{aligned}$$

This map is visibly linear. To show that it is injective, let $X\in\ker\varphi$. Then $X_e=0$. Since X is left invariant, we have that

$$X_g=d(L_g)_e(0)=0$$

So $X=0$. For surjectivity, let $v\in T_eG$. Then define the vector field Y such that $Y_g=d(L_g)_e(v)$, which is almost tautologically left invariant, and $Y_e=v$. ∎

In essence, a left invariant vector field is entirely determined by its value at the identity, since left invariance requires us to push the vector at T_eG around to find its values at all the other points of G. This identification allows us to identity the Lie algebra of the matrix groups we know and love

Example 1.11. The Lie algebra of the unitary group $U(n)$ is

$$\mathfrak{u}(n)=\left\{X\in\mathcal{M}_{nn}(\mathbb{C})\ :\ X=-X^\dagger\right\}$$

Proof. We know that $U(n)$ is the kernel of the map $F:GL(n,\mathbb{C})\to GL(n\mathbb{C})$ where $F(X)=XX^\dagger$. Therefore, we have that $T_eU(n)=\ker dF_e\subset\mathcal{M}_{nn}(\mathbb{C})$. To compute this, we probe neighborhoods of the identity with smooth curves. Fix some $A\in T_eGL(n\mathbb{C})=\mathcal{M}_{nn}(\mathbb{C})$, and define the curve $\gamma:J\to GL(n,\mathbb{C})$ (where $0\in J$) where $\gamma(t)=\mathrm{id}_{\mathbb{C}^n}+tA$. We note that $\gamma'(0)=A$, so

$$dF_e(A)=\frac{d}{dt}\bigg|_{t=0}F\circ\gamma$$

We then compute

$$\begin{aligned}F\circ\gamma(t)&=(\mathrm{id}_{\mathbb{C}^n}+tA)(\mathrm{id}_{\mathbb{C}^n}+tA)^\dagger\\&=(\mathrm{id}_{\mathbb{C}^n}+tA)(\mathrm{id}_{\mathbb{C}^n}+tA^\dagger)\\&=\mathrm{id}_{\mathbb{C}^n}+tA^\dagger+tA+t^2AA^\dagger\end{aligned}$$

which has linear term $t(A + A^\dagger)$, so $\ker dF_e = T_e U(n)$ consists of all skew Hermitian matrices. ∎

Example 1.12. The Lie algebra of $SL(n, \mathbb{R})$ is

$$\mathfrak{sl}(n\mathbb{R}) = \{X \in \mathcal{M}_{nn}(\mathbb{R}) \ : \ \text{trace}\, X = 0\}$$

Proof. We note that $SL(n, \mathbb{R})$ is the kernel of the determinant map $GL(n\mathbb{R}) \to \mathbb{R}$, so it suffices to find the kernel of $d(\det)_{\mathrm{id}_{\mathbb{R}^n}}$. As we proved earlier,

$$\left.\frac{d}{dt}\right|_{t=0} \det(\mathrm{id}_{\mathbb{R}^n} + tA = \text{trace}\, A$$

For any matrix A. The mapping $t \mapsto \mathrm{id}_{\mathbb{R}^n} + tA$ defines a smooth curve with the derivative at 0 being A, so we have that $d(\det)_{\mathrm{id}_{\mathbb{R}^n}}(A) = \text{trace}\, A$. So $\mathfrak{sl}(n, \mathbb{R}) = T_e SL(n\mathbb{R})$. ∎

There's more to Lie algebras than just their identification as tangent spaces though. Every Lie group homomorphism induces a homomorphism between their Lie algebras.

Proposition 1.13. *Given a Lie group homomorphism $F : G \to H$, given a left invariant vector field $X \in \mathfrak{g}$, there exists a unique vector field $Y \in \mathfrak{h}$ that is F related to X. We denote this vector field as $F_* X$.*

Proof. Noting that any vector field in $\mathfrak{h}$ is uniquely determined by its value at the identity, let $Y_e = dF_e(X_e)$, which determines the vector field Y by the rule

$$Y_g = d(L_g)_e(Y_e)$$

if we show that Y is F related to X, it is clear that this will be the unique vector field in $\mathfrak{h}$ that satisfies this property. To show this, we have the the condition that F is a Lie group homomorphism implies

$$\begin{aligned} &F(g_1 g_2) = F(g_1)F(g_2) \\ \implies &(F \circ L_{g_1})(g_2) = (L_{F(g_1)} \circ F)(g_2) \\ \implies &(dF_{g_1} \circ d(L_{g_1})_e) = (d(L_{F(g_1)})_e \circ dF_e) \end{aligned}$$

Therefore, if we consider X_g, we have

$$dF_g(X_g) = (dF_g \circ d(L_g)_e)(X_e) = (d(L_{F(g)})_e \circ dF_e)(X_e) = d(L_{F(g)})_e(Y_e) = Y_{F(g)}$$

So X and Y are F related. ∎

Proposition 1.14. *Given a Lie group homomorphism $F : G \to H$, the mapping $F_* : \mathfrak{g} \to \mathfrak{h}$ given by $X \mapsto F_* X$ is a Lie algebra homomorphism.*

Proof. Linearity follows from the fact that the differential is linear. We also know that for each pair of F related vector fields X_1, Y_1 and X_2, Y_2 with $X_i \in \mathfrak{g}$ and $Y_i \in \mathfrak{h}$, we have that $[X_1, X_2]$ is F related to $[Y_1, Y_2]$. Therefore, since there exists a unique F related vector field in $\mathfrak{h}$ for each vector field $X \in \mathfrak{g}$, this map is well defined, and is a Lie algebra homomorphism. ∎