**M382D LECTURE NOTES**

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These lecture notes are for Differential Topology (M382D), taught by Professor Gompf in Spring 2017. The main text for this couse is Guillemin and Pollack's *Differential Topology*, with Spivak's *Calculus on Manifolds* as a supplementary text.

Contents

## Preliminary Analysis: January 18, 2017

**Definition 1.1.** For an open subset $U \subset \mathbb{R}^n$, a function $f : U \to \mathbb{R}^m$ is **differentiable** at $a \in U$ iff there exists a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{x \to 0} \frac{\|f(a+x) - f(a) - T(x)\|}{\|x\|} = 0$$

**Proposition 1.2.** *If $T$ exists, $T$ is unique. Consequently, we can denote $T = df_a$ i.e. the unique differential of $f$ at $a$.*

*Proof.* Given $T$, let $x = he_i$ and let $h \to 0$. Then

$$\begin{aligned}
&\lim_{h \to 0} \frac{\|f(a+he_i) - f(a) - hT(e_i)\|}{|h|} = 0 \\
\implies &\lim_{h \to 0} \frac{f(a+he_i) - f(a)}{|h|} - T(e_i) = 0 \\
\implies &\frac{\partial f}{\partial x_i}(a) = T(e_i)
\end{aligned}$$

So in the standard basis, the $i^{th}$ column of $T$ is just all the partial derivative of the component functions $f_j$ with respect to $x_i$. Therefore, differentiability of $f$ implies the existence of partial derivatives, and $T$ is uniquely defined since that patial derivatives are uniquely defined. ■

**Definition 1.3.** A funtion $f : U \to \mathbb{R}^m$ is **smooth** or of class $C^\infty$ if all partial derivatives exist and are continuous, i.e.

$$\frac{\partial f_i^k}{\partial x_{j_1} \dots \partial x_{j_n}} \text{ exists and is continuous}$$

**Proposition 1.4.** *If $f$ is smooth, then $f$ is differentiable*

*Remark.* It is only really necessary that $f$ be of class $C^1$ in order for $f$ to be differentiable.

**Proposition 1.5.** *If $f\mathbb{R}^n \to \mathbb{R}^m$ is linear, then $f$ is differentiable everywhere, and for any $a \in R^n$, $df_a = f$.*

*Proof.*

$$\lim_{x \to 0} \frac{\|f(a+x) - f(a) - f(x)\|}{\|x\|} = \lim_{x \to 0} \frac{\|f(a) + f(x) - f(a) - f(x)\|}{\|x\|} = 0$$

■

**Definition 1.6.** Given $U, V$ open in $\mathbb{R}^n$ and $R^m$ respectively, a function $f : U \to V$ is a **diffeomorphism** if $f$ is smooth and bijective, and $f^{-1}$ is smooth.

## The Chain Rule + Inverse Function Theorem: January 20, 2017

*Remark.* From now on, to avoid repitition, unless specified otherwise, $U$ and $V$ will denote open subsets of $\mathbb{R}^k$ and $\mathbb{R}^\ell$ respectively.

First recall the single variable Chain Rule,

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

in a slightly more general case, given functions

$$\mathbb{R} \xrightarrow{g} \mathbb{R}^n \xrightarrow{f} \mathbb{R}$$

Using the multivariable chain rule this time,

$$\frac{d}{dx}f(g(x)) = \frac{\partial f}{\partial g_1}\frac{\partial g_1}{\partial x} + \ldots + \frac{\partial f}{\partial g_n}\frac{\partial g_n}{\partial x}$$

However, writing it this way obscures what's really going on. If we write this in a more suggestive notation, we get

$$\begin{pmatrix} \frac{\partial f}{\partial g_1} & \cdots & \frac{\partial f}{\partial g_n} \end{pmatrix} \begin{pmatrix} \frac{\partial g_1}{\partial x} \\ \vdots \\ \frac{\partial g_n}{\partial x} \end{pmatrix}$$
$$= df_{g(a)} \cdot df_a$$

This suggests that the chain rule is no more than the composition of the differentials, which is exactly what it is. So in full generality, given functions,

$$\mathbb{R}^k \xrightarrow{g} \mathbb{R}^n \xleftarrow{f} \mathbb{R}^m$$

we get the chain rule generalized to arbitrary dimension

$$d(f \circ g)_a = df_{g(a)} \circ df_a$$

To put the chain rule succinctly, "The differential of the composition is the composition of the differentials."

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Recall that a diffeomorphism is a function between open sets that is smooth, bijective, and with a smooth inverse. Then if $f : U \to V$ is a diffeomorphism,

$$f \circ f^{-1} = \mathrm{id}_V \ , \ f^{-1} \circ f = \mathrm{id}_U$$

Then by applying the chain rule, this implies that

$$df_a \circ df^{-1}_{f(a)} = \mathrm{id}_{\mathbb{R}^n} \ , \ df^{-1}_{f(a)} \circ df_a = id_{\mathbb{R}^m}$$

Therefore, if $f$ is a diffeomorphism $df$ is invertible, and $n = m$. As it's currently defined, a diffeomorphism very much a global concept. However, like differentiability, most of the information we need is local, since there is a very clearly defined notion for what it means to be differentiable *at a point*. For example, while a function like $f(x) = |x|$ isn't differentiable, for any non-zero point $x$, $f$ is differentiable at $x$. It makes sense to talk about diffeomorphisms in a similar vein.

**Definition 1.7.** A function $f : U \to V$ is a **local diffeomorphism** at $a \in U$ iff there exists a neighborhood $W \subset U$ of $a$ such that $f(W)$ is open in $W$ and $f\restriction_W : W \to f(W)$ is a diffeomorphism. If $f$ is a local diffeomorphism for every $a \in U$, we say $f$ is a local diffeomorphism (much like how you would say a function is differentiable if it is differentiable at every point).

**Theorem 1.8.** *If $f$ is a local diffeomorphism at $a$, $df_a$ is invertible and*

$$(df_a)^{-1} = df^{-1}_{f(a)}$$

As an example, consider the polar transformation where

$$f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^2$$
$$(r, \theta) \mapsto (r\cos\theta, r\sin\theta)$$

It's clear that $f$ is not a diffeomorphism, since it's far from bijective. However, $f$ is a local diffeomorphism, which is easily confirmed with following theorem.

**Theorem 1.9** (**Inverse Function Theorem**). *Given a smooth map $f : U \to \mathbb{R}^m$, $a \in U$, if $df_a$ is an isomorphism, then $f$ is a local diffeomorphism at $a$.*

Note that the inverse function theorem is just the converse of the theorem above, but perhaps not as obvious. In essence, it tells us that smooth functions are a particularly nice class of functions that locally behave like linear transformations. With the preliminary calculus established, we are ready to start Guillemin and Pollack proper and begin discussing manifolds, rather than just Euclidean space.

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We'd like to start talking about manifolds. For the time being, we're going to consider manifolds as subsets of $\mathbb{R}^n$, rather than the more abstract approach (though in the end it will end up being that the two are equivalent). But first, we should generalize the notion of a diffeomorphism.

**Definition 2.1.** For an arbitrary subset $X \subset \mathbb{R}^n$ a function $f : X \to \mathbb{R}^m$ is **smooth** iff $\forall x \in X$, there exists a neighborhood $U \subset \mathbb{R}^n$ of $x$ and a smooth function $F : U \to \mathbb{R}^m$ such that

$$F \restriction_{X \cap U} = f$$

Note that in the case that $X$ is open, this just reduces to our previous notion of a smooth function. Now equipped with a more general defintion of what it means to be smooth, we generalize the concept of a diffeomorphism to arbitrary subsets of Euclidean space.

**Definition 2.2.** Given subsets $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$, a function $f : X \to Y$ is a **diffeomorphism** if $f$ is smooth, bijective, and $f^{-1}$ is smooth.

It might seem like we just recited the definition of a diffeomorphism word for word as we did above, but this time, it is with the more general notion of smoothness. One thing to note is that now, $X$ and $Y$ need not be embedded in the same dimensional Euclidean space, since when we consider the smooth extension $F$ in a neighborhood $W_x$ at every point of $X$, $F$ need not be bijective. All that matters is that $F$ agree with $f$ on $X \cap W_x$ and that this restricted function be bijective. For example, the circle minus a point is diffeomorphic to $\mathbb{R}$, and certainly a torus is diffeomorphic to itself, whether it be embedded in $\mathbb{R}^3$ or into $\mathbb{R}^4$ as $S^1 \times S^1$.

## Manifolds: January 23, 2017

As the name suggests, a diffeomorphism between manifolds fulfils the same role as an isomorphism between groups and rings and homeomorphism for topological spaces.

*Remark.* From this point onwards, all functions are smooth unless stated otherwise

**Definition 2.3.** If there exists a diffeomorphism between two subsets $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$, The we say $X$ and $Y$ are **diffeomorphic**, denoted $X \simeq Y$

**Definition 2.4.** Given a function $f : X \to \mathbb{R}^m$, define

$$\text{graph}\, f = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m |\ y = f(x)\}$$

**Theorem 2.5.** graph $f \simeq X$.

*Proof.* Define

$$\begin{aligned} g : X &\to \operatorname{graph} f \\ x &\mapsto (x, f(x)) \end{aligned}$$

which is clearly a smooth bijection, since all the coordinate functions are smooth (namely, id and $f$), so we can extend them using id and the extension $\tilde{f}$ for $f$. Note that projection functions are smooth, and that

$$\pi_1 \circ g = \mathrm{id}_{\mathbb{R}^n}$$

Consequently, the restriction of $\pi_1$ to the graph of $X$ is $g^{-1}$, so $g$ is a diffeomorphism, and $X \simeq$ graph $f$. ∎

Note that being diffeomorphic is a strictly stronger condition than being homeomorphic, since differentiable functions are always continuous. However, homeomorphic spaces need not be diffeomorphic. As an example, consider the following example.

**Example 2.6.** Let $X$ denote the union of the positive $x$ and $y$ axes in $\mathbb{R}^2$. Then $X$ is not diffeomorphic to $\mathbb{R}$.

*Proof.* Suppose it were, i.e. there exists some diffeomorphism $f : \mathbb{R} \to X$. WLOG say that $f(0) = (0, 0)$. Then there exists a smooth $g : X \to \mathbb{R}$ such that $g \restriction_X = f^{-1}$. Then for some neighborhood of $(0, 0)$, we have $g \circ f = \mathrm{id}_{\mathbb{R}}$. Then by the chain rule that $dg_{(0,0)} \circ df_0 = \mathrm{id}_{\mathbb{R}}$. Consequently, $df_0$ must be injective. We then note that this means the linear transformation $df_0$ is non-singular, so

$$df_0 = \begin{pmatrix} \frac{df_1}{dx} \\ \frac{df_2}{dx} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We note that the derivative in this case is just some velocity vector, and that for some $(x, 0)$ close to $(0, 0)$, $df(x, 0) = (a, 0)$ and that for some $(0, y)$ close to $(0, 0)$, $df(0, y) = (0, b)$, Conseuqently, $f'$ cannot be continuous, so $f$ is not smooth. ∎

Finally, we define a manifold

**Definition 2.7.** A subset $X \subset \mathbb{R}^n$ is a ***k*-dimensional manifold** if for all $x \in X$, there exists some neighborhood $V$ of $x$, an open set $U \subset \mathbb{R}^k$, and a diffeomorphism $\psi : V \to U$. We then say that $\psi$ is a **local coordinate system** and $\psi^{-1}$ is a **local parameterization**. In addition, the ordered pair $(V, \psi)$ is called a **coordinate chart**

Thus the one of the propositions above actually postulates that for any smooth function $f : \mathbb{R}^n \to \mathbb{R}^m$, graph $f$ is an $n$-manifold. As a familiar example,

**Example 2.8.** The $n$-sphere, denoted $S^n$ is a $n$-manifold.

*Proof.* We can parameterize $S^1$ (the circle) with the map $\phi(\theta) = (\cos\theta, \sin\theta)$, which is clearly smooth. By restricting $\phi$ to the domains $(-\pi, \pi)$ and $(0, 2\pi)$ we completely cover $S^1$.

In general, for $S^k$ we can use the maps

$$\begin{aligned} &\phi_{i\pm} : B_1(0) \to \mathbb{R}^{k+1} \\ &(x_1, \dots, x_k) \mapsto (x_1, \dots x_{i-1}, \pm\sqrt{1 - \sum x_i^2}, \dots x_{k+1}) \end{aligned}$$

where $B_1(0) \subset \mathbb{R}^k$. Doing this covers $S^k$ with $2(K + 1)$ coordinate charts, where each coordinate has two functions with $\pm\sqrt{1 - \sum x_i}$ in the corresponding coordinate. ∎

## The Tangent Space: Janulary 25, 2017

To begin, let's make few remarks on manifolds. For some notation, if a manifold $X$ is $n$-dimensional, we may refer to it as $X^n$. Don't confuse this with the cartesian product of $X$ with itself however, but from context, it's usually easy to determine the intent. Last lecture, we've defined what a manifold is. Given two manifolds, there's a natural way to make another.

**Proposition 2.9.** *For manifolds $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$, $X \times Y \subset \mathbb{R}^{n+m}$ is also a manifold. In addition, if $X$ is a $k$-manifold and $Y$ is an $\ell$-manifolds, $X \times Y$ is a $k + \ell$-manifold.*

**Example 2.10.** The 2-Torus $T^2 = S^1 \times S^1$ is a manifold. In general, the $k$-Torus is the cartesian product of $k$ copies of $S^1$.

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So far, we've only defined what it means to do differentiation on open subsets of Euclidean space. Given that Spivak's book is titled Calculus on *Manifolds*, we'd like to be able to generalize this to an arbitrary manifold, which is not necessarily an open subset of $\mathbb{R}^n$. So given some $n$-manifold $X$ and a point $a \in X$ with a neighborhood $W_a$, we know there exists an open set $U$ and a diffeomorphism $\varphi : U \to w_a$ Then $d\varphi_a : \mathbb{R}^k \to \mathbb{R}^n$ is linear.

**Definition 2.11.** For a manifold $X$, a point $x \in X$, and a parameterization $\varphi$, the **Tangent Space** of $x$, denoted $T_xX$ is defined to be

$$T_xX = \operatorname{Im} d\varphi_x$$

Note that this definition requires us to choose some local parameterization $\varphi$. However, the tangent space should be independent of our choice of parameterization. So given another parameterization $\psi : V \to W'_a$, we get the commutative diagram

$$\begin{tikzcd} V \arrow[r, "\psi"] \arrow[d, dashed, "h"'] & X \\ U \arrow[ur, "\varphi"'] & \end{tikzcd}$$

We wish to define $h$ such that this diagram commutes, i.e. $h = \varphi^{-1} \circ \psi$. With some appropriate shrinking of $U$ and $V$, we can say that $\phi(U) = \psi(V)$. Then for $b \in V$ and $a = h(b) \in U$, by the chain rule,

$$d\psi_b = d\varphi_a \circ dh_b$$

We then note that $h$ is a diffeomorphism (being the composition of diffeomorphisms), so $dh_b$ is an isomorphism. Consequently, $d\varphi_a$ is surjective, so $d\psi_b(V) \subset d\phi_a(U)$. Similarly, we note that

$$d\varphi_a = d\psi_b \circ dh_a^{-1}$$

so $d\varphi_a(U) \subset d\psi_b(V)$. So the definition is indeed independent of our choice of parameterization. In addition, the dimension of $T_xX$ is what you would expect it to be.

**Theorem 2.12.** $\dim T_xX = \dim X$

*Proof.* Given a parameterization $\phi : U \to V$ for some open subset $V \subset X$, we know that $\phi^{-1}$ is smooth, so there exists some $W \supset \phi(U)$ and $\Phi : W \to \mathbb{R}^k$ that extends $\phi^{-1}$. Therefore,

$$\begin{aligned} &\Phi \circ \phi = \mathrm{id}_U \\ \Longrightarrow\ & d\Phi_{\phi(a)} \circ d\phi_a = \mathrm{id}_{\mathbb{R}^k} \\ \Longrightarrow\ & d\phi_a \text{ is injective} \end{aligned}$$

and since $d\phi$ by definition is surjective onto $T_aX$, if follows that $d\phi$ is an isomorphism from $\mathbb{R}^k$ to $T_aX$, so the dimensions match. ∎

Intuitively, the tangent space $T_xX$ is a subspace of the ambient $\mathbb{R}^n$ such $x+T_xX$ is a good linear approximation of $X$ at $x$. Vectors in the tangent space can be geometrically interpreted as tangent vectors through $X$ starting from $x$ i.e. for some $v \in T_xX$, we often see it geometrically as $v + x$. To further build intuition, let's consider some examples

**Example 2.13.** A **curve** is a smooth function $\gamma : U \to \mathbb{R}^n$ for some open subset $U \subset \mathbb{R}$. Then $d\gamma_a : \mathbb{R} \to \mathbb{R}^n$ is a good linear approximation of $\gamma$ at $a$. As a matrix, if we evaluate $d\gamma_a$ at the basis vector 1,

$$d\gamma_a(1) = \begin{pmatrix} \gamma_1'(a) \\ \gamma_2(a) \\ \vdots \\ \gamma_n(a) \end{pmatrix}$$

We define the **velocity vector** of $\gamma$ at $a$, denoted $\gamma'(a)$ to be $d\gamma)a(1)$. We note that $d\gamma_a(1)$ forms the basis for the tangent space of graph $\gamma$.

**Example 2.14.** Given a smooth $f : \mathbb{R}^2 \to \mathbb{R}$, we know that the graph of $f$ is parameterized by the function $g(x) = (x, f(x))$. As a matrix,

$$g(x, y) = \begin{pmatrix} x \\ y \\ f(x, y) \end{pmatrix}$$

Note that $g : \mathbb{R}^2 \to \mathbb{R}^3$ where

$$dg_{(a,b)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial x}(a, b) & \frac{\partial f}{\partial y}(a, b) \end{pmatrix}$$

Evaluating on the standard basis, then we find that

$$dg_{(a,b)}(e_1) = \begin{pmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x}(a, b) \end{pmatrix} \qquad dg_{(a,b)}(e_2) = \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial y}(a, b) \end{pmatrix}$$

Geometrically, evaluating on a basis vector $e_i$ "slices" the graph of $f$ by the plane spanned by the $e_3$ and $e_i$, and gives us the tangent vector to the curve defined by $f$. Then note that these two vectors then span the tangent space of graph $f$.

We want to define what it means for a function $f : X \to Y$ to be differentiable. Recall that our current defintions only work for open subsets of $\mathbb{R}^n$, so they're not enough here. Since differentiation is all about linear approximation, it seems like the tangent space is a natural way to do this. So given $f : X \to Y$, we would expect $df_x$ (whatever it is) to be a linear map from $T_xX \to T_{f(y)}Y$. Taking local parameterizations $\phi : U \to X$ and $\psi : V \to Y$, we get the commutative diagram (after modifying $U$ and $V$ such that the domains and codomains line up)

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, dashed, "h"'] & V \arrow[u, "\psi"'] \end{tikzcd}$$

where $h = \psi^{-1} \circ f \circ \phi$. We'd hope that we'd have some sort of chain rule for a map between manifolds here, but we'd naïvely expect

$$df_x = d\psi \circ dh \circ d\phi^{-1}$$

which is the only real possibility here, and this will turn out to be the case. We'll verify and prove the chain rule for manifolds in the next lecture.

## The Chain Rule for Manifolds: January 27, 2017

*Remark.* For a manifold $X$, $x \in X$, we can always choose a parameterization $\phi$ such that $\phi(0) = x$, which we will adopt as convention.

Following our naïve expectation,

**Definition 2.15.** Given a smooth map of manifolds $f : X \to Y$, and local parameterizations $\phi$ and $\psi$ for $x \in X$ and $f(x)$ respectively, define the differential of $f$ at $x$ to be

$$df_x = d\psi_0 \circ dh_0 \circ (d\phi_0)^{-1}$$

where $h = \psi^{-1} \circ f \circ \phi$.

We first want to check that $df_x$ actually makes sense (i.e. it maps from tangent spaces to tangent spaces). To do that, consider the diagram

$$\begin{tikzcd}
T_xX \arrow[r, dashed, "df_x"] & T_{f(x)}Y \\
\mathbb{R}^k \arrow[u, "d\phi_0"] \arrow[r, "dh_0"'] & \mathbb{R}^l \arrow[u, "d\psi_0"']
\end{tikzcd}$$

So far, our defintions is doing what we want it to do.

We hope that this definition of the derivative satisfies the chain rule, but before we do that, we want to verify that this is actually well defined i.e. $df_x$ is independednt of our choice of parameterization. To do this consider the following commutative diagram.

$$\begin{tikzcd}
X \arrow[rrr, "f"] & & & Y \\
& U' \arrow[ul, "\phi'"'] \arrow[dl, "\alpha"] \arrow[r, "h'"] & V' \arrow[ur, "\psi'"] \arrow[dr, "\beta"'] & \\
U \arrow[uu, "\phi"] \arrow[rrr, "h"'] & & & V \arrow[uu, "\psi"']
\end{tikzcd}$$

So given a map of manifolds $f$, we have two different parameterizations for neighborhoods of $x$, $\phi : U \to X$ and $\phi' : U' \to X$. Likewise, we have two different parameterizations of neighborhoods of $f(x)$. Finally, being open sets in Euclidean space, there exist diffeomorphisms $\alpha, \beta$ from $U' \to U$ and $V' \to V$ respectively. For each set of parameterizations, we get $h, h'$ as induced maps between their coresponding open sets. Just following the diagrams, we find that as per the defintion,

$$df_x = d\psi_0 \circ dh_0 \circ d\phi_x^{-1}$$

$$df'_x = d\psi'_0 \circ dh'_0 \circ d\phi'^{-1}_x$$

We want to show that $df_x = df'_x$.

We then note that $\psi' = \psi \circ \beta$, and $\phi' = \phi \circ \alpha$ and since these are just maps of Euclidean spaces (the tangent space is just a subspace of $\mathbb{R}^l$), we can apply the chain rule we already know to find

$$\begin{aligned} d\psi' &= d\psi \circ d\beta \\ d\phi' &= d\phi \circ d\alpha \\ \implies d\phi'^{-1} &= d\alpha^{-1} \circ d\phi^{-1} \end{aligned}$$

So we conclude that

$$df'_x = d\psi_0 \circ d\beta_0 \circ dh'_0 \circ d\alpha_0^{-1} \circ d\phi_x^{-1}$$

and since the diagram commutes, this is the same things as $df_x$.

With that out of the way, let's finally prove the chain rule. For the preliminary setup, for manifolds $X, Y, Z$, we have two functions $f : X \to Y$, and $g : Y \to Z$, parameterizations $\phi : U \to X$, $\psi : V \to Y$, and $\eta : W \to Z$, for neighborhoods of $x, f(x)$ and $g(f(x))$ respectively, and induced maps $j = \eta^{-1} \circ g \circ \psi$ and $h = \psi^{-1} \circ f \circ \phi$. Altogether, this gives us the commutative diagram

$$\begin{tikzcd}
& V \arrow[d, "\psi"] & \\
& Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "g\circ f"'] & & Z \\
U \arrow[u, "\phi"] \arrow[rr, "j\circ h"'] \arrow[uuur, bend left, "h"] & & W \arrow[u, "\eta"'] \arrow[uuul, bend right, "j"']
\end{tikzcd}$$

Just following the diagram, from our definition of the derivative of a map of manifolds, we compute that

$$d(g \circ f) = d\eta_0 \circ d(j \circ h)_0 \circ d\phi_x^{-1}$$

Noting that $j \circ h$ is a map of Euclidean spaces, we have the regular chain rule at our disposal. This gives us

$$d(g \circ f) = d\eta_0 = dj_0 \circ dh_0 \circ d\phi_x^{-1}$$

While it seems a bit weird, we can take a little detour through $\psi$, giving us

$$d(g \circ f) = d\eta_0 \circ dj_0 \circ d\psi_{f(x)}^{-1} \circ d\psi_0 \circ dh_0 \circ d\phi_x^{-1}$$

However, this gives us something really nice. Notice that for the first half of the expression,

$$dg_{f(x)} = d\eta_0 \circ dj_0 \circ d\psi_{f(x)}^{-1}$$

and

$$df_x = d\psi_0 \circ dh_0 \circ d\phi_x^{-1}$$

Therefore, we find the chain rule

**Theorem 2.16** (**The Chain Rule**). *For $f : X \to Y$ and $g : Y \to Z$ smooth maps of manifolds,*

$$d(g \circ f)_x = dg_{f(x)} \circ df_x$$

There's an alternative way to define the derivative, which gives the same thing. Given a manifold $X \subset \mathbb{R}^m$, and a map $f : X \to \mathbb{R}^n$, we know we can extend $f$ to a function $F$ on an open set $U$ of $\mathbb{R}^m$, such that $F \restriction_{U \cap X} = f$. Then $dF \restriction_{T_x X} = df_x$.

*Remark.* The above characterization of $df_x$ works for arbitrary manifolds in the codomain, not just $\mathbb{R}^n$.

To make this concrete, consider the example

**Example 2.17.** Given the inclusion map $\iota : Y \hookrightarrow \mathbb{R}^n$, we note that id extends $\iota$. Hence $d\iota_x = id \restriction_{T_x Y}$, which is the inclusion map from the tangent space into $\mathbb{R}^n$.

As of now, all of the concepts we've discussed so far regarding differentiation on Euclidean space have their generalized counterparts when it comes to manifolds. We will find the same to be true for the Inverse Function Theorem.

but first,

**Definition 2.18.** If $X$ and $Y$ are smooth manifolds of dimension $k$, $f : X \to Y$ is a **local diffeomorphism** *at $x$* if there exists an neighborhood $U$ of $x$ and $V$ of $f(x)$ such that $f$ maps $U$ diffeomorphically onto $V$ i.e. $f \restriction_U : U \to V$ is a diffeomorphism. If $f$ is a local diffeomorphism at every $x$, we call it a local diffeomorphism.

**Theorem 2.19** (**The Inverse Function Theorem**)**.** *Given a smooth map of manifolds $f : X \to Y$, $f$ is a local diffeomorphism at $x$ iff $df_x$ is an isomorphism of tangent spaces.*

*Proof.* For parameterizations $\phi$ and $\psi$, consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "h"'] & V \arrow[u, "\psi"'] \end{tikzcd}$$

On the level of derivatives, this gives us

$$\begin{tikzcd} T_x X \arrow[r, "df_x"] & T_{f(x)} Y \\ \mathbb{R}^k \arrow[u, "d\phi"] \arrow[r, "dh"'] & \mathbb{R}^l \arrow[u, "d\psi"'] \end{tikzcd}$$

Where $h$ is again the induced map $\psi^{-1} \circ f \circ \phi$. We then note that since $\phi$ and $\psi$ are diffeomorphisms, $d\phi$ and $d\psi$ are isomorphisms. Consequently, by how we've defined $df_x$, we know this diagram commutes, so $df_x$ is an isomorphism $\iff$ $dh$ is an isomorphism. Note that this implies that $k = l$. Finally, since $U, V$ are subsets of Euclidean space, we can use the Inverse Function Theorem we already know to prove that $h$ is a local diffeomorphism. Consequently, since the above diagram also commutes based on how we've defined $h$, $f$ is also a local diffeomorphism. ■

This gives us an interesting way to reformulate the Inverse Function Theorem. Note that since $f$ is a local diffeomorphism, for suitably small neighborhoods of $x$ and $f(x)$, $f$ is a diffeomorphism between these neighborhoods. Consequently the open sets $U$ and $V$ that parameterize these neighborhoods are diffeomorphic to each other. Therefore, we can parameterize the neighborhood about

$f(x)$ with $U$ as well. This gives us

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "\mathrm{id}"'] & U \arrow[u, "\psi"'] \end{tikzcd}$$

In other words, $f$ is a local diffeomorphism $\iff$ there exist parameterizations $\phi$ and $\psi$ such that when $f$ is pulled back to Euclidean space, it just appears to be the identity. This means that locally, $f$ is about as nice as it can get.

---

The Inverse Function Theorem tells us when the local behavior of a function is reasonably nice, but can only really be used when the dimensions of the manifolds are the same. This motivates us to ask what the best local behavior for functions between manifolds of different dimensions.

**Definition 2.20.** A map of manifolds $f : X \to Y$ is an **immersion** at $x \in X$ if $df_x$ is injective. We say $f$ is an immersion if it is an immersion at every $x \in X$.

*Remark.* If $f$ is an immersion then $\dim X \leq \dim Y$.

We note that there is a very simple (almost trivial) example of an immerison. For reasons we'll quickly learn, we call it the **canonical immersion**.

**Definition 2.21.** For $k \leq l$, the canonical immersion is a function

$$\begin{aligned} \iota : \mathbb{R}^k &\to \mathbb{R}^l \\ (x_1, \dots, x_k) &\mapsto (x_1, \dots, x_k, 0, \dots, 0) \end{aligned}$$

In fact, up to diffeomorphism, all immersions are just the canonical immersion, so in some sense, the canonical immersion is the *only* immersion.

**Theorem 2.22.** *If $f : X \to Y$ is an immersion at $x \in X$, then there exist local parameterizations at $x$ such that when pulled back to a map beween $U$ and $V$, $f$ is just the canonical immersion.*

*Proof.* Let $\dim X = k$ and $\dim Y = l$. Then consider the commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \\ & W \arrow[ur, "\psi'"] \arrow[dr, "H"] & \\ U \arrow[uu, "\phi"] \arrow[ur, "\iota"] \arrow[rr, "h"'] & & V \arrow[uu, "\psi"'] \end{tikzcd}$$

Again, $h$ is the induced map. We first note that by definition,

$$df_x = d\psi_0 \circ dh_0 \circ d\phi_0^{-1}$$

Then since $\phi, \psi$ are diffeomorphisms, $d\phi_0, d\psi_0$ are isomorphisms and $df_x$ is injective, $dh_0$ is also injective. Consequently, there exists a basis for $\mathbb{R}^k$ such that $dh_0 = \left(\frac{I_k}{0}\right)$, i.e. the identity matrix for $\mathbb{R}^k$ stacked on top of a $k - l \times k$ block of zeroes. We let $W$ be an open set in $U \times \mathbb{R}^{l-k}$, and define

$$\begin{aligned} H : U \times \mathbb{R}^{l-k} &\to \mathbb{R}^l \\ (x, z) &\mapsto h(x) + (0_1, 0, \dots 0_k, z) \end{aligned}$$

We then note that $dH_0 = \mathrm{id}_{\mathbb{R}^l}$, so by the Inverse Function Theorem, $H$ is a local diffeomorphism at 0. We then note that $h = H \circ \iota$. We then define $\psi' : W \to Y$ such that $\psi' = \psi \circ H$. Consequently, since $H$ and $\psi$ are local diffeomorphisms, $\psi'$ is a local diffeomorphism, and after appropriate shrinking of $U$ and $V$ to line up domains and codomains, we have the follwing commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ U \arrow[u, "\phi"] \arrow[r, "\iota"'] & V \arrow[u, "\psi\circ H"'] \end{tikzcd}$$

So $f$ is equivalent to $\iota$. ■

## Not All Immersions are Nice: January 30, 2017

*What I said there wasn't exactly correct. We'll say that's an alternative fact*

So given an immersion, it's image need not be a manifold, so the local knowledge we have regarding derivatives isn't enough here. So we'd like to add some global *topological* restrictions on what are immersions behave.

**Definition 2.23.** A continuous map $f : X \to Y$ between metric spaces is **proper** if the preimage of every compact set is compact.

*Remark.* If $X$ is compact, then every continuous $f : X \to Y$ is proper.

**Proposition 2.24.** *A proper map is closed i.e. the image of any closed set is closed*

*Proof.* Given a proper $f : X \to Y$, and $C \subset X$ closed, we want to show that $f(C)$ is closed i.e.e given a sequence $(y_i) \subset f(C)$ such that $(y_i) \to y \in Y$, we want to show that $y \in f(C)$. We first note that each $y_i = f(x_i)$ for some $x_i \in X$. We then note that $K = \{y_i\} \cup \{y\}$ is compact. Then since $f$ is proper, $f^{-1}(K) \supset \{x_i\}$ is compact. Then we have the subsequence $(x_i) \subset f^{-1}(K) \subset C$, which converges to $x$ since it's contained in a compact set. Then since $C$ is closed, $x \in C$. Then by the continuity of $f$, $(f(x_i) \to f(x))$, so $(y_i) \to f(x)$. ■

**Corollary 2.25.** *An injective proper map is a homeomorphism is onto its image*

With the general topology established, lets go back to the world of smooth manifolds, and we claim that these additional constraits will give us the nicer functions we desire.

**Definition 2.26.** For manifolds $X, Y$, an embedding is a smooth, injective, and proper immersion $f : X \to Y$

**Theorem 2.27.** *If $f : X \to Y$ is an embedding, then $f(X)$ is a submanifold of $Y$, and is a closed subset of $Y$.*

*Proof.* First. since $f$ is proper, it is closed, so $f(X)$ is closed. Also, since $f$ is injective and closed, $f$ is open as well. In order to show $f(X)$ is a manifold, for every $x \in X$, we want to find a local parameterization for $f(x)$. Let $\phi : U \to X$ be a local paremeterization at $x$. Then $f(\phi(U))$ is open in $f(x)$. We claim that $f \circ \phi$ is a local parameterization for $f(x)$. Since $f$ is an immersion, it is injective, and therefore a diffeomorphism onto its image. Then $f \circ \phi^{-1}$ is a composition of diffeomorphisms, and consequently, a diffeomorphsim. ■

We also have the converse to the above theorem. Given a closed submanifold $X \subset Y$, the inclusion map $X \overset{\iota}{\hookrightarrow} Y$ is an embedding.

## Submersions: February 1, 2017

*Remark.* For convenience, unless specified otherwise, manifolds $X$ and $Y$ will be $k$ and $\ell$-dimensional respectively.

So immersions are in some sense, functions with the nicest local behavior if the domain is or lower dimension than the codomain. Conversely, there's an analogous concept for when the domain is of higher dimension.

**Definition 2.28.** A smooth map $f : X \to Y$ is a **submersion** at $x \in X$. if $df_x$ is surjective. $f$ is a submersion if it is a submersion at every $x \in X$.

*Remark.* If $f : X \to Y$ is an submersion, this implies $\dim X \geq \dim Y$.

Just like with immersions, we have a canonical submersion (which again will turn out to be the *only* submersion- up to diffeomorphism of course).

**Definition 2.29.** For $k \geq \ell$, the **canonical submmersion** is the function

$$\begin{aligned} \pi : \mathbb{R}^k &\to \mathbb{R}^\ell \\ (x_1, \dots, x_k) &\mapsto (x_1, \dots x_\ell). \end{aligned}$$

Note that the canonical submersion is the projection functions on the first $\ell$ coordinates. Again, we have an analogous theorem.

**Theorem 2.30** (**Local Submersion Theorem**)**.** *If $f : X \to Y$ is a submerison at $x \in X$, then there exist local parameterizations at $x$ and $f(x)$ such that $f$ is equivalent to the canonical submersion.*

*Proof.* Given a submersion $f : X \to Y$, is a submersion at $x \in X$, then there exists local paramterizations $\phi, \psi$ for $x, f(x)$ respectively such that $\phi(0) = x$ and $\psi(0) = f(x)$. On the level of derivatives, we get the commutative diagram

$$\begin{tikzcd} T_xX \arrow[r, "df"] & T_{f(x)}Y \\ \mathbb{R}^k \arrow[u, "d\phi"] \arrow[r, "dg"'] & \mathbb{R}^\ell \arrow[u, "d\psi"'] \end{tikzcd}$$

Since the diagram commutes, we note that $dg$ is surjective iff $df$ is, and consequently, $dg$ is a surjection. Therefore, there exist bases for $\mathbb{R}^k$ and $\mathbb{R}\ell$ such that $dg_0$ is the matrix given in block form by $(I_k|0)$ i.e. the identity matrix for $\mathbb{R}^k$ stacked next to a block of zeroes. Then pick a basis for $\mathbb{R}^\ell$, and pull them back to $\mathbb{R}^k$ using $dg_0^{-1}$. Then this gives us $\ell$ linearly independent vectors. If we then append onto this set a basis for $\ker dg_0$, we are given a basis for $\mathbb{R}^k$. Then define

$$\begin{aligned} G : U &\to \mathbb{R}^\ell \times \mathbb{R}^{k-\ell} \\ a &\mapsto (g(a), a_{\ell+1}, \dots a_k) \end{aligned}$$

Then by the Inverse Function Theorem, $dG_a = I_{\mathbb{R}^k}$, so $G$ is a local diffeomorphism. So $\phi' = \phi \circ G^{-1}$ is a parameterization for $x$, and we note that $\pi \circ G = g$, so we have that $f$ is equivalent to $\pi$. All in all, we get the commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] & & Y \\
& U' \subset \mathbb{R}^k \arrow[ul, "\phi'"'] \arrow[dr, "\pi"] & \\
U \arrow[uu, "\phi"] \arrow[ur, "G"] \arrow[rr, "g"'] & & V \arrow[uu, "\psi"']
\end{tikzcd}$$

■

However, unlike immersions, submersions are somewhat hard to find. However, if we loosen our definitons a bit, we can find some more of them.

**Definition 2.31.** Fpr a smooth $f : X \to Y$, a point $y \in Y$ is called a **regular value** if $\forall x \in f^{-1}(y)$, $f$ is a submersion at $x$. Otherwise, we say $y$ is a **critical value**.

**Definition 2.32.** For a manifold $X$, and a submanifold $Y \subset X$, $\dim X - \dim Y$ is called the **codimension** of $Y$.

Then we get something analogous to the embeding theorem for immersions.

**Theorem 2.33** (**The Preimage Theorem**). *If $y$ is a regular value of $f : X \to Y$, then $f^{-1}(y)$ is a manifold with $\dim f^{-1}(y) = k - \ell$.*

*Proof.* At $x \in f^{-1}(y)$, $f$ is a submersion. Therefore, by the local submersion theorem, we have a neighborhood of $x$ and a parameterization $\phi$ such that $f$ is equivalent to the canonical submersion. Then $U' = \mathbb{R}^{k-\ell} \cap U$ is open in $\mathbb{R}^{k-\ell}$ be definition of the subspace topology, and the function $\phi \restriction_{U'}$ is a local parameterization for $f^{-1}(y)$. ■

That being said, this doesn't mean that preimages of critical values aren't submanifolds. Consider some $f : X \to \mathbb{R}$ with 0 as a regular value, and $x \in f^{-1}(0)$. Then let define $s : \mathbb{R} \to \mathbb{R}$ such that $s(x) = x^2$. Then $(s \circ f)^{-1} = f^{-1} \circ s^{-1}$, so $(s \circ f)^{-1}(0) = f(0)$. However, 0 is not a regular value of this function. since $d(s \circ f)_x = ds_0 \circ df_0 = 0 \circ df_x = 0$. Therefore preimages can stil be submanifolds, even if they aren't preimages of regular values.

**Proposition 2.34.** *For a regular value $y$, $T_x f^{-1}(y) = \ker df_x$*

*Proof.* Consider the inclusion map

$$f^{-1}(y) \overset{\iota}{\hookrightarrow} X \xrightarrow{f} Y$$

clearly, the composition $f \circ \iota : f^{-1}(y) \to Y$ is just the constant function that maps everything to $y$. Therefore, $d(f \circ \iota)_x = 0$, and by the chain rule, is equal to $df_x \circ \iota : T_x f^{-1}(y) \to T_y Y$, since the derivative of the inclusion map is the inclusion map of the tangent space. Consequently, $T_x f^{-1}(y) \subset \ker df_x$, but since they are of the same dimension, we conclude that $T_x f^{-1}(y) = \ker df_x$. ■

## More Submersions: February 3, 2017

For a quick aside into linear algebra, given a vector space $V$, we have its associated **dual space** $V^*$, the space of linear functionals of $V$. Given a basis $v_1, \ldots, v_n$, we can get the associated dual basis $v_1^*, \ldots, v_n^*$ such that

$$v_i^*(v_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Likewise, for tangent spaces, we have their associated dual spaces.

**Definition 2.35.** For a manifold $X$, the **cotangent space** at $x \in X$, denoted $T_x^*X$ is the dual space of $T_xX$.

While less relavant now, this will be of utter importance later, when we talk about integration on manfolds.

Recall that the Preimage Theorem tells us when the premiage of a point is a submanifold of the domain. One might like to ask when given a collection of smooth maps $g_i : X \to R$, that their common zeroes form a submanifold. We know that if 0 is a regular value of $g = (g_1, \dots, g_n)$, that $g^{-1}(0)$ is a submanifold of codimension $n$. We note that since each $g_i$, its derivative $dg_{i_x}$ is a linear function from $T_xX$ to $\mathbb{R}$, i.e. an element of the cotangent space. We then note that

$$dg_{i_x} = \begin{pmatrix} dg_1 \\ \vdots \\ dg_n \end{pmatrix}$$

is surjective precisely when the $dg_i$ are linearly independent. If this is the case, we say the functions $g_1, \dots g_n$ and **independent** at $x$.

**Proposition 2.36** (**A Reworded, More Specific Preimage Theorem**)**.** *For $n$ smooth functions $g_i : X \to R$, if they are independent at their common zeroes, the set $Z$ of common zeroes is a submanifold of $X$ with codimension $n$*

## (Almost) Varieties and Transversality: February 6, 2017

*As per usual, everything in sight is smooth and every set is a manifold*

*Remark.* One might note that all this talk of vanishing sets sounds a lot like algebraic varieties.

So now we know that $m$ independent functions "cut out" a submanifold of codimension $m$. Intuitively, we can think of each function as removing a degree of freedom i.e. a dimension, since they are all independent. However a natural question to ask is if any submanifold $Z \subset X$ can somehow be seen as the common zero set of a collection of functions. Sadly, this isn't always true, but if we impose some stricter requirements, we can still get what Guillemin and Pollack call "partial converses."

**Theorem 2.37** (**Partial Converse**)**.** *If $Z \subset X$ is the preimage of a regular value under a function $f : X \to Y$, then for some neighborhood, $Z$ is cut out by independent functions.*

*Proof.* Let $y$ denote the regular value, and $U_y \subset Y$ a neighborhood of $y$. Then we know there exists an open set $W \subset \mathbb{R}^\ell$ and a diffeomorphsim $h : U \to W_y$ such that $h(y) = 0$. Then define $g = h \circ f$. Then $g^{-1}(0) = f^{-1} \circ h^{-1}(0) = f^{-1}(y)$. Consequently, 0 is a regular value of $g$. We note that $g$ is composed of coordinate functions $g = (g_1, \dots, g_\ell)$, which are all independent since $dg$ is surjective. Consequently, we can say that $Z$ is cut out by $g_1 \dots g_\ell$. ∎

We also have another partial converse. Even though the converse fails at a global level, locally, everything looks fine.

**Theorem 2.38** (**Another Partial Converse**)**.** *Every submanifold is locally cut out by independent functions*

*Proof.* Let $Z \subset X$ a submanifold of codimension $\ell$, and let $z \in Z$ be arbitrary. We want to find $\ell$ independent functions such that on some neighborhood $W_z \subset X$ of $z$, $Z \cap W$ is the common vanishing set of these functions. By the Local Immersion Theorem, if we consider the inclusion map $Z \hookrightarrow X$, this is equivalent to the canonical immersion. Consequently, in local coordinates $Z$ is exactly the set where the last $\ell$ functions of the canonical immerison vanish. ∎

So the vanishing sets of functions are very important (at least locally) at determining submanifolds, and we can interpret many submanifolds as level sets of functions, or alternatively, the graph of a function.

**Theorem 2.39** (**Implicit Function Theorem**). *For $U \subset \mathbb{R}^n \times \mathbb{R}^m$ and a smooth function $f : U \to \mathbb{R}^m$, such that $f(0) = 0$. Then if $df_0 \restriction_{0\times\mathbb{R}^m}$ is an isomorphism, there exist neighborhoods $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ such that $f^{-1}(0) \cap (V \times W)$ is the graph a smooth function $\varphi : V \to W$, i.e. $\varphi$ is the unique solution to $f(x, \varphi(x)) = 0$.*

*Proof.* Trivial. Actually, will be added later after I do it for homework. ∎

So now we have a better idea of when the primages of points are submanifolds. But given $f : X \to Y$, with $Z \subset Y$ a submanifold, when do we know when $f^{-1}(Z)$ is a submanifold of $X$? Using the local immersion theorem we have that any $z \in Z$ has a neighborhood $U$ such that $Z \cap U$ is cut out by independent functions $g_1, \dots, g_k$, with each $g_i : U \to \mathbb{R}$. Then $f^{-1}(Z) = (g \circ f)^{-1}(0) = (f^{-1} \circ g^{-1})(0)$. Then we know this a manifold when 0 is a regular value, i.e. when $d(g \circ f) = dg_z \circ df_{f^{-1}(z)}$ is surjective. Clearly this is surjective when $df$ is, but since we already know that $dg_z$ is surjective, this is overkill. Recall that $T_zZ = \ker dg_z$. Since we know that $dg_z$ is surjective, for 0 to be a regular value it is sufficient for $\operatorname{Im} df_{f^{-1}(z)} + T_zZ = T_zY$. So as long as $df_{f^{-1}(z)}$ hits everything outside the kernel, we are fine. Note that this isn't a direct sum. What the notation means is that any vector in $T_zY$ can be expressed as a linear combinations of vectors is $T_zZ$ and $\operatorname{Im} df_{f^{-1}(z)}$.

This is actually a very powerful idea. So much so in fact, that we choose to grace it with the honor of a definition.

**Definition 2.40.** Given a function $f : X \to Y$, with $Z \subset Y$ a submanifold, we say $f$ is **transversal** to $Z$ if $\forall x \in f^{-1}(Z)$, $\operatorname{Im} df_x + T_zZ = T_zY$. We denote this $f \pitchfork Z$.

A special, but important case is when $f$ is the inclusion map $\iota : X \to Z$. Suppose that $\iota \pitchfork Z$. This is true precisely when $\forall x \in X \cap Z$, $T_xX + T_xZ = T_xY$. We then say the manifolds $X$ and $Z$ are transversal, again denoted $X \pitchfork Z$.

*Remark* 2.41. We note by a dimension counting argument that $\operatorname{codim} f^{-1}(Z) = \operatorname{codim} Z$

## Transversaility on a Bus: February 8, 2017

*It's hard to draw quadratic tangency while riding a bus. However, I can draw a transversal intersection without even looking at the board.*

First a few remarks on some of the consequences of transversality.

**Proposition 2.42.**
*(1) $f \pitchfork Z \iff f^{-1}(Z)$ is a manifold with with codimension equal to $\operatorname{codim} Z$*
*(2) $T_xf^{-1}(Z) = (df_x)^{-1}(T_{f(x)}Z)$.*

In some sense, maps that are transversal are still transversal after some "wiggling." Referring to the quote, imagine a parabola tangent to a line. If we wiggle the parabola just a little, it might not be intersecting the line anymore, or it might be multiple intersections. This kind of intersection isn't very stable, as small perturbations can drastically change what kidn of behavior we are observing at the point of intersection. This motivates us to develop some more formalism.

**Definition 2.43.** Given two maps $f_0, f_1 : X \to Y$, we say these maps are **homotopic** if there exists a function $F : X \times I \to Y$ (where $I$ is the unit interval $[0, 1] \subset \mathbb{R}$), such that for all $x \in X$, $F(x, 0) = f_0(x)$ and $f(x, 1) = f_1(x)$. We call $F$ a **homotopy** of $f_0, f_1$.

A way to think of homotopy is a family of smooth functions that are all smoothly deformed into each other. A good geometric way to think of this is a cylinder of $X$, stacked upon eachother, indexed by some time $t$, and at the cross section at some time $t$, we have a map to $Y$.

## Homotopy and Stability: February 10, 2017

So homotopy gives us a way to "wiggle" our functions into a family of functions that are smooth deformations of our original function. As one would expect, homotopy defines an equivalence relation on the set of all functions between two manifolds $X$ and $Y$. As we have been studying properties of functions, we'd like to know which ones remain unchanged by the "wiggling" that homotopy does.

**Definition 3.1.** A property $P$ of smooth maps $f : X \to Y$ is **stable** if whenever $f_0$ satisfies $P$ and $\{f_t\}_{t \in I}$ is a homotopy of $f$, there exists an $\varepsilon > 0$ such that $\forall\ t < \varepsilon$, $f_t$ also satisfies the property $P$.

**Theorem 3.2.** *For $f : X \to Y$ with $X$ compact, the following properties are stable*

1. *local diffeomorphisms*
2. *immersions*
3. *submersions*
4. *maps transversal to some fixed, closed submanifold $Z \subset Y$*
5. *embeddings*
6. *diffeomorphisms*

*Proof.* We note that (1) follows form $(2), (3)$, since local diffeomorphism are special cases of immersions and submersions. so we first prove (2).

Given and immersion $f_0 : X \to Y$, and a homotopy $F = \{f_t \mid t \in I\}$, we want to find a $\varepsilon$ such that for any $t \leq \varepsilon$, $f_t$ is an immersion as well. In other words, $\forall x \in X$, we want to find a neighborhood of $(x, 0) \in X \times I$ on which $df_x$ is injective. Then this neighborhood would contain an open set of the form $U_x \times [0, \varepsilon_x)$, where $U_x$ is open in $X$. Then all the $U_x$ would cover $X$, and by compactness give us a finie subcover, from which we can pick the minimal $\varepsilon_x$. To do this, we first choose local coordinates. Then $df_t$ is given by a matrix, so $df_t$ is injective if and only if the matrix contains a $K \times K$ submatrix with nonzero determinant (where $k = \dim X$). We know $(df_t)_x$ is continuous on $X \times I$, since $f$ is smooth. Then since det is a polynomial in it's entries, $\det(df_t)_x$ is continuous in the variables $x, t$. Consequently, since $(df_0)_x$ is injective and det is continuous, we know that $(df_t)_x$ is injective in a neighborhood of $t$, and we are done. Proving (2) is almost idenitical, since being injective/ surjective is equivalent to being full rank (depending on dimenions of domains and codomains). We then prove (4).

Again, we want to show that transversality holds in a neighborhood of $(x, 0)$. We note that since $Z$ is closed, $f^{-1}(Z)$ is closed, so every point of the complement has an open set that misses the preimage $f^{-1}(Z)$, so transversality holds vacuously there. Again we work locally. For $x \in f^{-1}(Z)$ we know it has a neighborhood that is locally cut out by independent functions $g_1 \dots g_\ell$. Then $f_t \pitchfork \mathbb{Z}$ if and only if 0 is a regular value of the composition $g \circ f_t$. This then reduces to a near identical problem to (2).

We now prove (5). Since $X$ is compact, all smooth maps are proper, so it is sufficient to show that $f$ is an injective immersion. We know immersions are stable, so again, it's sufficient to show that in some neighborhoodof $(x, 0)$, $f_t$ is injective. Then define

$$\begin{aligned} G : X \times I &\to Y \times I \\ (x, t) &\mapsto (f(x), t) \end{aligned}$$

Then we wish to prove this by contradiction. If such an $\varepsilon$ did not exist, the $\forall \varepsilon > 0$, $\exists\, t \in (0, \varepsilon)$ such that $f_t$ is not injective. Then we can make a sequence $(t_i)$ that converges to 0 such that $f_{t_i}$ is not injective. This gives us correspoinding $p_i, q_i \in X$ such that $f_{t_i}(q_i) = f_{t_i}(p_i)$. Since $X$ is compact, these both have convergent subsequences, so we can assume that $(p_i) \to p$ and $(q_i) \to q$. Then we have

$$(p_i, t_i) \to (p, 0)$$
$$(p_i, t_i) \to (p, 0)$$

Then since $G$ is continuous, we have that

$$G(p_i, t_i) \to G(p, 0)$$
$$G(q_i, t_i) \to G(q, 0)$$

But since $f(p_i) = f(q_i)$ and $f_0$ is injective, this implies that $p = q$. Then since $f_0$ is injective, if we look at $dG_{(x,0)}$ in local coordinates, it is given by some matrix of the form

$$\begin{pmatrix} (df_0)_x & ? \\ 0 \dots & \dots 0, 1 \end{pmatrix}$$

Which implies that $G$ is an immersion at $(x, 0)$, so $G$ must be injective in a neighborhood of $(x, 0)$, a contradiction. ■

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## A Brief Foray into the Terrors of Analysis, or There and Back Again: Februrary 13, 2017

*You might be worried that all sets have measure zero ...*

Returning back to regular values, we know that regular values give us nice submanifolds as preimages. We have the theorem

**Theorem 4.1** (**Sard's Theorem**)**.** *For a smooth $f : X \to Y$, "almost every" point of $Y$ is a regular value of $f$*

"Almost every" is a very vague term, so we'd like to rigorously define what that means. Unfortunately, it leads us into a short detour into the dangerous lands of analysis and measure theory. Fortunately, we don't need to fully develop measure theory, as the only bit we need is the concept of a measure zero set.

**Definition 4.2.** A set $A \subset \mathbb{R}^n$ has **measure** 0 is $\forall \varepsilon > 0$, $A$ can be covered by *countably many* boxes $R_i$ such that

$$\sum_{i \in \mathbb{N}} \mathrm{Vol}(R_i) < \varepsilon$$

*Remark.* For notational compactness, as well as a grudging head nod to analysis, we'll often denote that $A$ has measure 0 by $\mu(A) = 0$.

*Remark.* Another comment- boxes are exactly what you expect them to be, namely

$$\prod_{i=1}^{n} [a_i, b_i]$$

and their volume is also exactly as expected

$$\mu(R) = \prod_{i=1}^{n} (b_i - a_i)$$

We then state some properties of measure zero sets

**Proposition 4.3.** *For a countable collection $\{A_i\}_{i\in\mathbb{N}}$, where each $A_i$ has measure zero, $\cup A_i$ also has measure zero.*

*Proof.* Since each $A_i$ has measure zero, given any $\varepsilon > 0$, we can find a covering by boxes such that $\mu(A_i) < \frac{\varepsilon}{2^i}$. Then $\cup A_i$ is covered by all these covers and the total volume of the complete cover is less that $\sum \frac{\varepsilon}{2^i} < \varepsilon$. ■

**Proposition 4.4.** *If $A$ contains a non-empty open subset, then $\mu(A) < 0$*

*Proof.* Since $A$ has a non empty open subset, we must necessarily be able to find a box into this open set. This box then has some nonzero volume, so the entire set must have some nonzero volume as well. (Check the appendix of Guillemin and Pollack for a proof that the measure of a box is invariant among coverings). ■

*Proof.* For $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}^n$ smooth, if $A \subset U$ has measure zero, then so does $f(A)$. ■

*Remark.* We note that the hypotheses of the dimensions being the same and $f$ being smooth are both neccessary. For example, if we have different dimensions , say $U \subset \mathbb{R}^k$ for $k > n$, then if we just consider the projection of the the $x$-axis onto $\mathbb{R}$, we have a measure zero set being projected onto all of $\mathbb{R}$, which is clearly not measure zero. Smoothness is also necessary, since we know of space-filling curves like the Hilbert Curve, which is a continuous mapping of the unit interval onto the unit square in $\mathbb{R}^2$.

Having defined measures in Euclidean space, as we have done before, we generalize this to arbitrary manifolds.

**Definition 4.5.** A subset $A \subset X$ for some $k$-dimensional manifold $X$ has measure 0 iff for every local parameterizations $\varphi : U \to X$, $\varphi^{-1}(A)$ has measure zero in $\mathbb{R}^k$.

One might wonder if the choice of parameterization affects the measure of a set. Well clearly, if it weren't well defined, we wouldn't have made it a defintion! Still, we prove this.

**Proposition 4.6.** *The measure of a set $A \subset X$ is invarient under different parameterizations*

*Proof.* To prove this, we note that it is sufficient to show that every point in $A$ has a local parameterization $\phi$ to a neighborhood $V$ such that $\phi^{-1}(A \cap V)$ has measure zero.

We then cover $A$ with contably many local parameterizations (which we can do since $\mathbb{R}^n$ is $2^{nd}$ countable. Then if we look at any parameterization $\psi$ of a neighborhood containing all of $A$, we can cover $\psi^{-1}(A)$ with a countable union of measure zero sets that we get from the local parameterisations of the individual neighborhoods of the points. We can then compose diffeomorphism onto the domain of $\psi$, and since smooth maps preserve measure zero sets, we can cover $\psi^{-1}(A)$ with a countable collection of measure zero sets. ■

This allows us to state Sard's Theorem in a more precise manner.

**Theorem 4.7** (**Sard's Theorem**)**.** *For any smooth $f : X \to Y$ for manifolds $X, Y$, the set of critical values has measure zero in $Y$.*

**Corollary 4.8.** *The set of regular values of $f$ is dense in $Y$. In face, given a countable set of functions $\{f_i\}_{i\in\mathbb{N}}$, each with a set of critical values $C_i$, $\cap C_i$ is dense in $Y$.*

**Definition 4.9.** For $f : X \to Y$, $x \in X$ is a **regular point** iff $df_x$ is surjective. Otherwise, we say $x$ is a **critical point**.

Note the distinction between regular/critical *points* versus *values*. While critical *values* form a measure zero subset of $Y$, critical *points* need not form a measure zero set in $X$.

With the analysis out of the way, we can return to the world of smooth manifolds. This nasty measure business will return however, when we discuss integration on manifolds.

---

We've talked about the tangent space of a manifold at a point $x \in X$, denoted $T_xX$. We then created the derivative of a smooth map of manifolds $f : X \to Y$, as a map of tangent spaces $df_x : T_xX \to T_{f(x)}Y$. However, at every point, the tangent space is different. How on earth would one go about talking about second derivatives, or even continuity of the derivative of a function $f$? To do this, we want to "glue" together all the tangent spaces into a single object, while remembering where each tangent space originated from.

**Definition 5.1.** For a manifold $X \subset \mathbb{R}^N$, the **tangent bundle of** $X$, denoted $TX$ is defined as

$$TX = \{(x, v) \mid x \in Xv \in T_xX\}$$

Essentially what we've done is a disjoint union of tangent spaces $T_xX$, marking each tangent space with the point it originated from. Another way to think of it is in the context of an affine space, where each $T_xX$ is a vector space with $x$ as the origin. We can then think of the "0 Section", the set of all points $(x, 0)$ as a copy of $X$ sitting inside the tangent bundle. So the tangent bundle is a much larger object that encodes both the tangent spaces and the manifold itself into a single entity.

With the tangent space established, we define a global derivative, one that isn't restricted to the derivative at a point like we had earlier.

**Definition 5.2.** Given a smooth map of manifolds $f : X \to Y$, we define the derivative

$$df(x, v) = (f(x), df_x(v))$$

Also important to note, the tangent bundle itself is a manifold.

**Theorem 5.3.** *$TX$ is a manifold of dimension $2 \dim X$.*

**Lemma 5.4.** *For an open subset $U \subset \mathbb{R}^k$, $TU = U \times \mathbb{R}^k$.*

*Proof.* Consier the inclusion map $\iota : U \to \mathbb{R}^k$. The tangent space to $U$ is $\mathbb{R}^k$, so the derivative $dU_x = \mathrm{id}_{\mathbb{R}^k}$. Then $\operatorname{Im} dU_x = \mathbb{R}^k$ ■

*Proof.* Given $(x, v) \in TX$, we want to find a local parameterization of a neighborhood of this point. Since $X$ is a manifold , we have a local parameterization $\phi : U \to X$ for $x$. Then $d\phi$ is a function from $TU$ to $TX$. We note that $TU = U \times \mathbb{R}^k$, which is open in $\mathbb{R}^{2k}$ under the subspace topology. $d\phi$ is smooth since $d\phi(x, v) = (\phi(x), d\phi_x(v))$, of which both components are smooth functions. Then $d\phi$ in matrix form takes the form

$$\begin{pmatrix} d\phi_x & 0 \\ ? & d\phi_x \end{pmatrix}$$

So it is injective since $\phi$ and $d\phi$ are. Then $\operatorname{Im} d\phi = TX \cap (\phi(U) \times \mathbb{R}^N$, which is open in the product topology on $X \times \mathbb{R}^N$. Then by the local immesion theorem $d\phi^{-1}$ is smooth, so $d\phi$ is a local parameterization of a neighborhood of $(x, v)$. ■

## Embedding Manifolds: Februrary 15, 2017

So after constructing the tangent bundle of a manifold, we defined a more general form of the derivative. Now we can talk about the function $df$ rather than having to specity the point $df_x$. With that construction, we'd hope it still behaves like we expect derivatives do, namely, the chain rule.

**Proposition 5.5** (**An even more general chain rule**). *For maps of manifolds $X \xrightarrow{g} Y \xrightarrow{f} Z$, $d(f \circ g) = df \circ dg$.*

*Proof.* On the level of derivatives, we have

$$\begin{tikzcd} TX \arrow[r, "dg"] \arrow[rr, bend right, "d(f\circ g)"'] & TY \arrow[r, "df"] & TZ \end{tikzcd}$$

If we then compute $d(f \circ g)$, we get

$$\begin{aligned} d(f \circ g)(x, v) &= ((f \circ g(x), d(f \circ g)_x(v)) \\ &= ((f \circ g(x), df_{g(x)} \circ dg_x(v)) \\ &= (df \circ dg)(x, v) \end{aligned}$$

■

So as we expect, the chain rule remains intact. If you want to look deeper, the chain rule defines a functor from smooth manifolds to their tangent bundles. So there is some sort of intrinsic connection from a manifold to its tangent bundle (one that is also independent of embedding) since a diffeomorphism of manifolds translates to a diffeomorphism of tangent bundles through the derivative and chain rule.

So having developed some heavier machinery through measure theory, Sard's Theorem, and tangent bundles, we have the tools necessary to prove a theorem about embedding manifolds into Euclidean space.

**Theorem 5.6** (**Whitney Embedding Theorem**). *Every k-manifold can be embedded in $\mathbb{R}^{2k+1}$*

Techincally this has been reduced down to $2k$, but there's a lot more work to prove that one. Also note that this is strictly an upper bound. One notes that the 2-torus is perfectly embeddable in $\mathbb{R}^3$, though the theorem only guarantees it to be embdeddable in $\mathbb{R}^5$ (techincally 4). Things vary on a case by case basis, but this theorem imposes an upper bound on the necessary amount of dimensions. To prove this, we first prove a lemma

**Lemma 5.7.** *Every k-manifold $X$ has an injective immersion into $\mathbb{R}^{2k+1}$.*

*Remark.* We note that since all smooth maps with a compact domain are proper, this lemma actually proves the embdedding theorem for all compact manifolds.

*Proof.* As a brief sketch of what we want to do, given $X \subset \mathbb{R}^N$ embedding in some big ambient Euclidean space with $N > 2k+1$, we want to cut down the dimensions by finding a linear projection onto $\mathbb{R}^{2k+1}$ that restricts to an injective immerison on $X$.

So given an injective immersion $f : X \to \mathbb{R}^M$, if $M > 2k+1$, we want to find a vector $a$ such that for the projection $\pi : \mathbb{R}^M \to a^\perp$ onto $a^\perp$ (the orthogonal complement of $a$), $\pi \circ f$ is also an injective immersion- noting that $a \perp \simeq \mathbb{R}^{M-1}$. We then define the functions

$$\begin{aligned} h : X \times X \times \mathbb{R} &\to \mathbb{R}^M \\ (x, y, t) &\mapsto t(f(x) - f(y)) \\ g : TX &\to \mathbb{R}^m \\ (x, v) &\mapsto df_x(v) \end{aligned}$$

Note that $X \times X \times \mathbb{R}$ is of dimension $2k+1$. So what $h$ is doing is finding the secant lines between points of $X$. This in some sense tells us the "bad" directions we don't want to project along, since doing so would collapse multiple points onto the same point, which would destroy injectivity. For the other function $g$, what it's doing is looking for tangent vectors that get collpased down, again telling us the "bad" directions to project onto, since collapsing tangent vectors would ruin our immersion. Then by Sard's Theorem, the critical values of both $f$ and $g$ form a set of measure zero, so we can find some regular value of both. We then note that since the dimensions are strictly smaller that $M$, that this regular value must not lie in the image of either, since it's impossible to have a surjection from a smaller dimensional space onto a larger dimesional space. This gives us the vector we want to project along. We then proceed with induction until we've reduced the dimension sufficiently. ■

## The Whitney Embedding Theorem and Partitions of Unity: February 17, 2017

*If you just imagine an infinite skyscraper…*

To fully prove the Whitnet Embedding Theorem, we need one more piece of machinery.

**Definition 5.8.** For a function $f : X \to \mathbb{R}$ (where $X$ is an arbitrary set), the **support** of $f$, denoted $\operatorname{supp} f$ is defined to be

$$\operatorname{supp} f = \overline{\{x \in X \mid f(x) \neq 0\}} = \overline{f^{-1}(\mathbb{R} - \{0\}}$$

where the overline denotes the closure of the set.

**Definition 5.9.** Let $X \subset \mathbb{R}^N$ (In this case $X$ can be an arbitrary subset, not necessarily a manifold). Then let $\{U_\alpha\}$ be an open cover of $X$ (where each $U_\alpha$ is open on the subspace topology on $X$). Then a **partition of unity** subordinate to $\{U_\alpha\}$ is a sequence of functions $(\theta_i)$ that satisfy the following properties:

(1) $\forall x \in X, \theta_i(x) \in [0, 1]$
(2) $\forall x \in X$, there exists a neighborhood of $x$ where all but finitely many $\theta_i(x)$ are nonzero
(3) $\operatorname{supp} \theta_i \subset U_\alpha$ for some $\alpha$.
(4) $\forall x \in X$, $\sum \theta_i(x) = 1$.

So we've defined what a partition of unity is, but it isn't immediately obvious that such a collection of functions actually exists.

**Proposition 5.10.** *Partitions of unity exist.*

*Proof.* We first extend the open sets $U_\alpha$ to sets $W\alpha \supset U_\alpha$ where every $W\alpha$ is open in the ambient Euclidean space. Then for every $x \in X$, there exits a ball $B_x \subset W_\alpha$ containing $x$, and a function $\eta : \mathbb{R}^N \to \mathbb{R}$ such that $\eta$ is positive in $B$ and 0 outside of $B$ (We note that $\eta$ is just a bump function defined with the help of the functon $e^{-\frac{1}{x^2}}$, see exercise 18 in Guillemin and Pollack 1.1). If $X$ is compact, then we are done, since we can cover $X$ with finitely many open sets of the form $\eta_i^{-1}(0, \infty)$, and define $\theta_i$ as

$$\theta_i(x) = \frac{\eta_i(x)}{\sum \eta_i(x)}$$

since the denominator would be a finite sum.

In the noncompact case, we note the above definition still makes sense if the sum in the denominator is infinite if all bum finitely many of the terms are 0. We can do this by constructing a "compact cover" of nested closed balls $K_i$, which eventually covers the set $W = \bigcup W_\alpha$. Then we note that the collection of open balls in $\mathbb{R}^N$ (the ambient space) whose closures are contained in at least one $W_\alpha$ form an open cover of $W$. We then contruct $\eta_i$ for each one of these balls (note WLOG we have countably many balls since $\mathbb{R}^N$ is $2^{nd}$ countable) such that $\eta_i$ is positive in the ball and 0 outside. We then inductively make more functions for our sequence. For $i \geq 3$, we have the compact set $K_i - \operatorname{Int} K_{i-1}$ is contained in the open set $W - K_{i-2}$, so the open balls with closure contained in $W - K_{i-2}$ and containedin some $W\alpha$ form an open cover of $K_i - \operatorname{Int} K_{i-2}$. Then pass to a finite subcover (by compactness) and add bump functions to our collection $(\eta_i)$ where we define a bump function on each ball. Then we can define $\theta_i$ as done above. ■

One last thing we need to prove,

**Theorem 5.11.** *For any manifold $X$, there exists a proper (positive) map $\rho : X \to \mathbb{R}$.*

*Proof.* Let $\{U_\alpha\}$ be a cover where each $U_\alpha$ has compact closure, and let $(\theta_i)$ be a partition of unity subordinate to this cover. We claim the function

$$\rho(x) = \sum i\theta_i(x)$$

is the function we want. To prove this, we note it is sufficient to show that $\rho^{-1}([-j, j])$ is compact. If $x \in \rho^{-1}([-j, j])$, we have that $\rho(x) \leq j$, which implies that some $\theta_i(x)$ is nonzero (think of $\rho$ is a weighted sum). Then $x \in \operatorname{supp}\theta_i \subset \overline{U_\alpha}$ for some $\alpha$. Then $\rho^{-1}([-j, j])$ is a closed subset of a compact set, i.e. compact, so $\rho$ is proper. ■

We now provide a proof sketch of the Whitney Theorem

*Proof.* From previously, we have that we have an injective immersion $f : X \to \mathbb{R}^{2k+1}$, and as proven above, we can construct a proper map $\rho : X \to \mathbb{R}$. Since $\mathbb{R}^{2k+1}$ is diffeomorphic to an open ball (consider the diffeomorphism $x \mapsto \frac{x}{1-\|x\|}$), WLOG say that $\operatorname{Im} f \subset B^{2k+1}$ for an open ball in $\mathbb{R}^{2k+1}$. we then define $F : X \to \mathbb{R}^{2k+1}$ such that

$$F(x) = (f(x), \rho(x))$$

Then we claim that if we pick a "direction" not in the dimension defined by $\rho$ we get a proper map. The actual proof here is a bit involved, and not really in the flavor we care to address in a differential topology class, so I'll defer to Guillemin and Pollack here. ■

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So far, all of our manifolds have lived in some ambient Euclidean space. One might notice that some things are a bit inconvenient if we have to reference the ambient space, since sometimes our proofs might rely on the number of coordinates we have in the space we're embedded in. We'd like to free ourselves from this definition for convenience's sake and talk about manifolds in a more abstract manner. We then begin the next part of our journey with a definition.

**Definition 6.1.** A **topological $k$-manifold** is a topological space $X$ that is Hausdorff and $2^{nd}$ countable such that for any $x \in X$, there exists a *homemorphism* $\phi : U \to V$, where $U$ is open in $\mathbb{R}^k$ and $V$ is open in $X$.

Note the emphasis on homeomorphism means that not all topological manifolds admit a smooth structure (and can actually be quite nasty). Fret not, for we will quickly define a smooth structure on these manifolds to make them nicely behaved.

## Atlases and Smooth Structure: February 20, 2017

*Days since using Zorn's Lemma: 0*

So last lecture we finally released ourselves from the shackles of ambient Euclidean space and defined what a topological manifold was. Being a differential topology course, we want to know what it means for functions between topological manifolds to be smooth. In order to do this, we're going to need to add some additional structure to a manifold

**Definition 6.2.** An **Atlas** (sometimes called a **smooth structure**) $\mathcal{A}$ for a manifold $X$ is a collection of charts

$$\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$$

(Recall that a **chart** is an ordered pair $(U, \phi)$ where $U$ is open in $\mathbb{R}^k$ and $\phi : U \to V$ is a homeomorphism of an open $U \subset \mathbb{R}^k$ onto of a neighborhood $V$ of some $x \in X$) such that

(1) $\{U_\alpha\}$ is an open cover of $X$.
(2) For any two homeomorphisms $\phi_\alpha, \phi_\beta$, the composition

$$\phi_\beta \circ (\phi_\alpha)^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is a diffeomorphism.
(3) $\mathcal{A}$ is maximal, i.e. all charts compatible with (1) and (2) are included in $\mathcal{A}$ (We invoke Zorn's Lemma here).

With that, we return to the realm of differential topology

**Definition 6.3.** A **smooth manifold** is an ordered pair $(X, \mathcal{A})$ where $X$ is a topological manifold and $\mathcal{A}$ is an atlas of $X$.

With an atlas, we can finally say what it means for a function to be smooth in this new context

**Definition 6.4.** A map of smooth manifolds $f : X \to Y$ is **smooth** if it "looks smooth in the charts"

What we mean by "looks smooth in the charts is that when we pull back to get a function between Euclidean spaces, the corresponding map is smooth. In other words, given a function $f : X \to Y$, we have the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"'] & Y \arrow[d, "\psi"] \\ U \arrow[r, "h"'] & V \end{tikzcd}$$

Where $h = \psi \circ f \circ \phi^{-1}$ (after appropriate shrinking of domains). Then $f$ is smooth if $h$ is smooth. We then define diffeomorphism the same way as before with our new definition of smooth (smooth with smooth inverse).

We then define a way to "glue" two manifolds together.

**Definition 6.5.** Given two $k$-manifolds $X, Y$, we define the **connected sum** of $X$ and $Y$, denoted $X\#Y$, by first choosing some $x \in X$ and $y \in Y$ and charts $\phi, \phi\prime$ for neighborhoods of $x$ and $y$ respectively such that $\phi(x) = 0$ and $\phi\prime(y) = 0$. We then use the gluing map

$$\psi(x) = \phi'^{-1}\left(\frac{\phi(x)}{\|\phi(x)\|^2}\right)$$

While for topological manifolds, it's sufficient just to identify the boundaries of the open sets, we want to preserve the smooth structure here, so we actually give a diffeomorphism for the entire neighborhoods, which preserves property (2) of the atlases.

*Note.* To give more intution, recall that the connected sum in the topological case is cutting out a ball of $X$ and $Y$ and identifying the boundaries. In order to preserve the smooth structure (making sure the atlases of $X$ and $Y$ are compatible to form $X\#Y$), we go a little further. In a sense what we did was identify a small strip with a diffeomorphism so we can preserve the smooth structure. (Imagine sleeving one ball over another).

## Projective Spaces: February 22, 2017

Given this new ambient-space-independent way of defining a manifold, we'd like to talk about some manifolds that are easier to represent in this way.

**Definition 6.6. Real projective space** $\mathbb{R}\mathbf{P}^k$ is defined to be

$$\mathbb{R}\mathbf{P}^k = \left\{\text{1-d subspaces of } \mathbb{R}^{k+1}\right\}$$

Another more useful characterization is as the quotient

$$\mathbb{R}\mathbf{P}^k = (\mathbb{R}^{k+1} - \{0\})/\sim$$

Where $x \sim y$ iff $x = \lambda y$ for some $\lambda \in \mathbb{R}$.
**Complex projective space** $\mathbb{C}\mathbf{P}^k$ is defined similarly.

We then define "coordinates" on $\mathbb{R}\mathbf{P}^k$ called **homogeneous coordinates** with the notation

$$[x_0 :, \dots : x_k]$$

to denote the equivalence class of $(x_0, \dots, x_k) \in \mathbb{R}^{k+1}$. However, this isn't a great way to talk about elements of $\mathbb{R}\mathbf{P}^k$, since these homogeneous coordinates are only well-defined up to a nonzero scalar. As the notation suggests, $\mathbb{R}\mathbf{P}^k$ is a $k$-manifold, so we'll construct parameterizations and a smooth structure for $\mathbb{R}\mathbf{P}^k$.

Doing things a big backwards, we do some setup that essentitally defines the smooth structure first.

Define **Affine Coordinates** on $\mathbb{R}\mathbf{P}^k$ via the functions (for $i \in \{1, 2, \dots k\}$)

$$\begin{aligned} \phi_i : \mathbb{R}^k &\to \mathbb{R}\mathbf{P}^k \\ (x_1, \dots, x_k) &\mapsto [x_1 : x_2 : \dots x_i : 1 : x_{i+1} : \dots x_k] \end{aligned}$$

Note that for $\mathbb{R}^k$ we start indexing at 1 and for $\mathbb{R}\mathbf{P}^k$ we start indexing at 0, so the image of a point $x$ under $\phi_i$ is just the homogeneous coordinates of $x$ with a 1 inserted into the $i^{th}$ slot to give us $k+1$ coordinates in total. As we'll see shortly, this will define a topology, and in fact gives us an atlas for $\mathbb{R}\mathbf{P}^k$. Each $\phi_i$ is injective since different point will give different ratios, and then define the **affine patch** $U_i$ to be

$$U_i = \operatorname{Im} \phi_i = \{[x_0 : \dots : x_k \mid x_i \neq 0\}$$

Then $\bigcup U_i = \mathbb{R}\mathbf{P}^k$, and we claim that these form an atlas. To check that this is an atlas, we need to check the transition maps. Given two affine patches $U_i$ and $U_j$, we have two functions (soon to be parameterizations) $\phi_i$ and $\phi_j$, each from $\mathbb{R}^k$. We want to check that $\psi_{ij} = \phi_j^{-1} \circ \phi_i$ is a diffeomorphism. WLOG say that $j > i$ (the other case is similar). Then

$$\begin{aligned} \psi_{ij}(x_1, \dots x_k) &= \phi_j^{-1}([x_1 : \dots x_i : 1 : \dots : x_k]) && \text{The 1 is in position } i \\ &= \phi_j^{-1}\left(\left[\frac{x_1}{x_j} : \dots : \frac{1}{x_j} : \dots : 1 : \dots \frac{x_k}{x_j}\right]\right) && \text{The 1 is in position } j \\ &= \frac{1}{x_j}(x_1, \dots 1, \dots x_k) && \text{The 1 is in position } i \end{aligned}$$

Note that since the $j^{th}$ coordinate of the homogenoeous coordinates is nonzero by how $U_j$ is defined, so we'll never divide by 0. Being linear, $\psi_i j$ is clearly smooth, and it's not hard to verify that it's a diffeomorphism. So provided that these $U_i$ are open sets in the topology we have yet to define on $\mathbb{R}\mathbf{P}^k$. we've just constructed an atlas.

To construct the topology, we'll just use the quotient topology where $U \subset \mathbb{R}\mathbf{P}^k$ is open iff $\phi_i^{-1}(U)$ is open for every $i$. Then the sets $U_i$ are open (almost tautologically), and we can think of $\mathbb{R}\mathbf{P}^k$ as gluing together $k+1$ copies of $\mathbb{R}^k$. The topological properties of being $2^{nd}$ countable and Hausdorff follow quite easily from the fact that $\mathbb{R}^k$ is, and we have the local Euclidean structure from the $\phi_i$, so $\mathbb{R}\mathbf{P}^k$ is a smooth manifold. We then have the canonical projection

$$\begin{aligned}\pi : \mathbb{R}^{k+1} - \{0\} &\to \mathbb{R}\mathbf{P}^k \\ (x_0, \dots x_k) &\mapsto [x_0 : x_1 : \dots : x_k]\end{aligned}$$

which is actually a submersion (to be done for HW) We then note that $\mathbb{R}\mathbf{P}^k$ is compact, since the image of the $k$-sphere $S^k \subset \mathbb{R}^{k+1}$ under the projection man $\pi$ is $\mathbb{R}\mathbf{P}^k$ and $S^k$ is compact, and $\pi$ is continuous.

## The Tangent Bundle Revisited: Februrary 24, 2017

Now that we've established what a smooth manifold is in the abstract sense, we need to recreate how differentiation and tangent spaces work in this new context. To do this, we're going to work a bit backwards from how we developed it in the previous context- opting to start with defining the tangent bundle, then deriving the tangent spaces.

Given some smooth manifold $X$ with $\dim X = k$, we know every point $x \in X$ is contained in some chart $(\phi_\lambda, V_\lambda)$ (Recall that $\phi_\lambda$ is a homeomorphism and $V_\lambda$ is an open subset of $\mathbb{R}^k$) that parameterizes some neighborhood $U_\lambda \subset X$ containing $x$. We know what it means for these neighborhoods $V_\lambda$ to have a tangent bundle, it's just $V_\lambda \times \mathbb{R}^k$! One might want to just take the disjoint union of these tangent bundles and call it a day, but there's some finer details we need to iron out. Nontheless, we are almost there. Let

$$W = \bigsqcup_{\lambda \in \Lambda} V_\lambda \times \mathbb{R}^k$$

Then define the equivalence relation $\sim$ such that for $w, w' \in W$, $w \sim w'$ iff there exits some transition map between charts $\psi_{ab}$ such that $d\psi_{ab}(w') = w$. In other words, we're identifying points in the intersections of the charts so we don't have the deal with the confusion of how a point of the tangent bundle will look depending on the chart we choose. It's fairly simple to check that this indeed is an equivalence relation.

**Proposition 6.7.** *$\sim$ is an equivalence relation, i.e. $\sim$ is*

*(1) Reflexive*
*(2) Symmetric*
*(3) Transitive*

*Proof.* (1) Since for any point $v \in W$, $v$ lies in the same chart as itself, so $\psi_{aa} = \text{id}$, so $d\psi = \text{id} \implies d\psi(v) = v$.

(2) We know $\psi_{ab}^{-1} = \psi_{ba}$, so $d\psi_{ab}^{-1} = d\psi_{ba}$, so if $w \sim w'$, $w' \sim w$.

(3) Given some other chart $(\phi_\gamma, U_\gamma)$, suppose $w_1 \sim w_2$ and $w_2 \sim w_3$, with $w_3 \in V_\gamma \times \mathbb{R}^k$. Then we know that $d\psi_{b\gamma}(w_2) = w_3$. If we then look at the transition map $\psi_{a\gamma}$, we find that by the chain rule,

$$\begin{aligned}d\psi_{a\gamma} &= d(\psi_{ab} \circ \psi_{b\gamma}) \\ &= d\psi_{ab} \circ d\psi_{b\gamma}\end{aligned}$$

consequently, we get that

$$\begin{aligned}d\psi_{a\gamma}(w_3) &= d\psi_{ab}(d\psi_{b\gamma}(w_3)) \\ &= d\psi_{ab}(w_2) \\ &= w_1\end{aligned}$$

So $w_1 \sim w_3$.

∎

Then we define the **tangent bundle** of $X$ to be $W/\sim$. This gives us canonical maps

$$\eta_\lambda : V_\lambda \times \mathbb{R}^k \to TX$$
$$x \mapsto [x]$$

Since all the transition maps $\psi_{ab}$ are diffeomorphisms, they are injective, so $d\psi_{ab}$ is injective. Consequently, all $\eta_\lambda$ are injective. Therefore, we get a well defined projection

$$\pi : TX \to X$$

where $\pi \restriction_{U_\lambda} = \phi_\lambda \circ \pi_1 \circ \eta_\lambda^{-1}$. Using this projection map, we can now define the **tangent space** $T_xX$ at a point $x$, to be the inverse projection $\pi^{-1}(x)$. (Note that we referred to this as a vector bundle earlier, now it makes a bit more sense, since the preimage of every point under the projection map is a vector space. If you want to be even fancier, this is just a special case of a **fiber bundle**, where the fibers are vector spaces.)

Having defined the tangent bundle $TX$, we now have to go through the tedious process of "topologizing" the thing. We'll use the quotient topology here

**Definition 6.8.** A set $U \subset TX$ is **open** iff for all $\lambda$, $\eta^{-1}(U)$ is open in $V_\lambda \times \mathbb{R}^k$.

One can check that the maps $\eta_\lambda$ are in fact homeomorphisms, and this takes care of the locally Euclidean part of the definition of a manifold. $TX$ is also $2^{nd}$ countable, since we can cover $X$ with countably many charts, and push forward a countable number of countable bases for each $V_\lambda \times \mathbb{R}^k$ to $TX$ through $\eta_\lambda$, giving us a countable basis for $TX$. Likewise for Hausdorff, depending on if two points lie in the same chart or not, we can separate them with open sets in $V_\lambda \times \mathbb{R}^k$ or in $X$ (since both are Hausdorff), and then we can push these open sets back into $TX$ either with $\eta_\lambda$ or with $\pi$. We then note that the collection

$$\mathcal{A} = \{(\eta_\lambda, V_\lambda)\}$$

actually defines an atlas on $TX$, so $TX$ is a smooth manifold, since all the transition maps $d\psi_{ab}$ are smooth (being linear).

Now fully equipped with the tangent bundle, we can talk about differentiation in the abstract setting. Given a smooth function $f : X \to Y$ (where $X$ and $Y$ are now abstract manifolds without an ambient Euclidean space), consider a point $p \in TX$ and its corresponding point in $X$, $\pi(x)$. Looking at $f$ in a local coordinate chart of $\pi(x)$, we get the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\phi_\lambda"'] & Y \arrow[d, "\phi_\gamma"] \\
V_\lambda \arrow[r, "h"'] & U_\gamma
\end{tikzcd}$$

Where $h$ is the unique function that makes this diagram commute i.e.

$$h = \phi_\gamma \circ f \circ \phi_\lambda^{-1}$$

On the level of derivatives, we have

$$\begin{tikzcd}
TX \arrow[r, dashed, "df"] & TY \\
V_\lambda \times \mathbb{R}^k \arrow[u, "\eta_\lambda"] \arrow[r, "dh"'] & V_\gamma \times \mathbb{R}^k \arrow[u, "\eta_\gamma"']
\end{tikzcd}$$

We then use this diagram to define

$$df = \eta_\lambda^{-1} \circ dh \circ \eta_\gamma$$

It's easy to verify that this definition is in fact, well defined. To check this, if we choose another neighborhood containing $\pi(x)$, we can use the chain rule to find that we get the same definition. Using this definition, we now define

$$df_x = df \restriction_{T_x X}$$

## A Conclusion to our Adventures with the Abstract: Fevruary 27, 2017

*It's like we made a bubble at every point.*

Having set up the machinery for differentiation in the abstract sense, we recover all of the concepts we've defined before (e.g. immersions, submersions, transversality, etc.), by simply pulling our problem into Euclidean space using the charts. This works fine since most of the concepts defiend before are local, so we can actually deal with them in charts. Somewhat important for our next theorem is that partitions of unity also generalize to the abstract case.

**Theorem 6.9 (Generalized Whitney Embedding).** *Every $k$-manifold $X$ embeds into $\mathbb{R}^k$*

*Proof.* We note that we've already proven this for manifolds lying in an ambient space, so it's sufficeint to show that any *topological* manifold admits an embedding into any Euclidean space $\mathbb{R}^N$, since we can then apply the theorem we already know.

For simplicity's sake, we'll mostly deal with the compact case here, so we want to find an injective immersion into $\mathbb{R}^N$ (properness is already taken care of here). Assuming our manifold is compact, we know we can find a partition of unity $(\theta_i)$ for $1 < i < m$ subordinate to a finite covering by charts $\{(\phi_i, U_i)\}$. We can reduce to the case where the indices of the $\theta_i$ and charts $(\phi_i, U_i)$ match. Then define $\psi_i : X \to \mathbb{R}^{k+1}$ by

$$\psi_i(x) = \begin{cases} (\theta_i(x)\phi_i(x), \theta_i(x)) & x \in U_i \\ 0 & x \notin U_i \end{cases}$$

Then define

$$\begin{aligned} \psi : X &\to \mathbb{R}^{m(k+1)} \\ x &\mapsto (\psi_1(x), \ldots, \psi_m(x)) \end{aligned}$$

We claim that $\psi$ is the function we're looking fore (an injective immersion).

For injectivity, suppose $\psi(x) = \psi(y)$ then we know $\forall i$, $\theta_i(x) = \theta_i(y)$, so we know for some $i$, $x, y$ are in the same chart $U_i$. Then since $\theta_i(x) = \theta_i(y)$, we cancel on both sides to conclude that $\phi_i(x) = \phi(y)$, so $x = y$ since $\phi$ is injective (being a homeomorphism).

To show that $\psi$ is an immersion, we want to show $d\psi_x$ is injective for every $x \in X$. This is where our new definition of differentiation is necessary. We note that it's suffcient to look at each $\psi_i$ to show that it is an immersion. Define the function $\tilde{\theta_i} = \theta_i \circ \phi_i^{-1}$ Then define

$$\begin{aligned} \psi &= \psi \circ \phi_i^{-1} \\ \implies \tilde{\psi}(x) &= (x\tilde{\theta}_i(x), \tilde{\theta}_i(x)) \end{aligned}$$

so this should have $2m + 1$ coordinate functions. Assuming $\phi_i(x) = a$, we compute (in "block" form)

$$d\tilde{\psi}_a = \begin{pmatrix} \theta_i\tilde{(a)}\delta_{ij} + a\frac{\partial \tilde{\theta_i}}{\partial x_j}(a) \\ \frac{\partial \tilde{\theta_i}}{\partial x_j}(a) \end{pmatrix}$$

After some row reduction, we'll find the top rows (excluding the bottom one) to be a scalar multiple of the identity, so this is injective, so $d\psi_a$ is an immersion. ■

Having completed our journey into the abstract, we now return to manifolds embedded in Euclidean space. Our notion of a manifold earlier is a bit restrictive, so we're going to add another class of manifolds to our arsenal.

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**Definition 7.1.** The **half-plane** is the set $\mathbb{H}^k \subset \mathbb{R}^k = \{(x_1, \dots, x_k) \mid x_k \geq 0\}$

**Definition 7.2.** A **manifold with boundary** is a subset $X \subset \mathbb{R}^N$ such that for all $x \in X$, there exits an open $V \subset X$ containing $x$ and an open set $U \subset \mathbb{H}^k$ of the half-plane such that there exists a diffeomorphism $\phi : U \to V$.

**Definition 7.3.** The **boundary** of $X$, denoted $\partial X$, is defined to be

$$\partial X = \left\{ x \in X \mid \exists\ (\phi_\alpha, U_\alpha) \text{ such that } \phi_\alpha(x) \in \mathbb{R}^{k-1} \right\}$$

*Remark.* In this case $\mathbb{R}^{k-1} \subset \mathbb{R}^k$ is just all the points with $k^{th}$ coordinate equal to 0.

We define the **interior** of $X$, denoted $\operatorname{Int} X$, to be $X - \partial X$.

*Note.* The interior and boundary here don't necessarily coincide with the point-set definitions, (though they do when the codimension of the manifold we are discussing in the ambient Euclidean space is 0) we usually don't have much use for the point-set definition here, so we'll use Guillemin and Pollack's convention.

## Manifolds with Boundary: March 1, 2017

So we know what it means to be smooth for points not on the boundary, (it's just the definition we had before), but for points on the boundary, what does it mean to be smooth? For $x \in X$ such that for a parameterization $\phi$ of a neighborhood of $x$ with $\phi(a) = x$, we say $\phi$ is smooth at $a$ if there exists a smooth extension to a smooth $\phi'$ on all of Euclidean space (not just half-space).

We the define $d\phi_a = d\phi'_a$. We note that for any given $\phi$, there exist many different possible extensions of $\phi$. However, these all yield the same result. To show this, consider a sequence of points converging to $a$. Then $d\phi$ agrees with $d\phi'$ at all these points, so they must converge to the same thing by continuity of all the partials. Then we define $T_x X = \operatorname{Im} \phi$ as before, and define derivatives similarly.

One might wonder why $\partial X$ is even well defined at all- that is, how do we know that if $x$ is the image of a point in $\mathbb{R}^{k-1} \subset \mathbb{H}^k$, that there does not exist some other parameteriztion of $x$ such that $x \in \operatorname{Int} X$?

**Lemma 7.4.** *If there exists a parameterization $\psi : V \to X$ such that $\psi(a) = x$ with $a \in \mathbb{R}^{k-1}$, then every paramterization $\phi : U \to X$, has that $\phi^{-1}(x) = b$ for $b \in \mathbb{R}^{k-1}$.*

*Proof.* Suppose not. Then there exists some $\psi$ such that $\psi(a) = x$ with $a \in \mathbb{R}^{k-1}$ and there also exists some $\phi$ such that $\phi(b) = x$ with $b \notin \mathbb{R}^{k-1}$. Then after appropriate shrinking of domains and codomains, we have that $h = \psi^{-1} \circ \phi$ is a local diffeomorphism. But then by the inverse function theorem, we have that $h$ maps some neighborhood of $\mathbb{R}^k$ diffeomorphically onto a neighborhood of a point on $\mathbb{R}^{k-1}$, a contradiction, since these sets are not diffeomorphic. ∎

**Corollary 7.5.**
*(1) If $X$ is a manifold (without boundary) then $\partial X = \varnothing$. Alternatively, $X = \operatorname{Int} X$*
*(2) $\operatorname{Int} X$ is a manifold*

There's more here about the boundary though. Not only do we know that the interior of a manifold is itself a manifold, we also know that the boundary is a manifold with dimension one less than the interior.

**Proposition 7.6.** *If $X$ is a manifold with boundary, then $\partial X$ is a $k-1$ manifold without boundary.*

*Proof.* Given $x \in \partial x$, let $\phi : U \to X$ be a parameterization of a neighborhood of $x$ with $\phi(a) = x$, where $a \in \mathbb{R}^{k-1}$. Then $\phi \restriction_{U \cap \mathbb{R}^{k-1}}$ is a parameterization of an open subset of $\partial X$, which implies that $\partial X$ is a $k-1$ manifold without boundary. ■

A few more notes,

(1) $T_x \partial X \subset T_x X$ with codimension 1.
(2) If $X$ is a manifold with boundary and $Y$ is a manifold without boundary (i.e. $\partial Y = \varnothing$), then $X \times Y$ is a manifold with boundary with $\partial(X \times Y) = \partial X \times Y$.

We establish some more notation,

**Definition 7.7.** Given a smooth map of manifolds $f : X \to Y$, define $\partial f = f \restriction_{\partial X}$. Consequently, using the inclusion map of $\partial X$ into $X$ and then chain rule, we also get that for $x \in \partial X$,

$$d_x(\partial f) = df_x \restriction_{T_x \partial X}$$

.

We have an easy way to find manifolds with boundary- identifying them as zero sets of smooth functions

**Theorem 7.8.** *Given a manifold $X$ such that $\partial X = \varnothing$, and a smooth function $\pi : X \to \mathbb{R}$ such that $0$ is a regular value, the subset $S = \pi^{-1}([0, \infty))$ is a manifold with boundary, with $\partial S = \pi^{-1}(0)$.*

*Proof.* We know $(0, \infty)$ is an open set of $\mathbb{R}$, so $\pi^{-1}(0, \infty)$ is an open subset of $X$, and consdequently, a submanifold with codimension 0. Then since 0 is a regular value, we have that by the local submersion theorem, $\pi$ is locally given by the canonical submersion, which implies that the last coordinate is mapped to 0, i.e. $\pi^{-1}(0)$ "looks like" half-space. ■

## More Manifolds with Boundary: March 3, 2017

*I wrote $f$ when I meant to write $g$ ... let's say $f = g$*

**Theorem 7.9.** *Given $f : X \to Y$ with $Z \subset Y$ with $X$ a manifold with boundary and $\partial Y = \partial Z = \varnothing$, suppose $f$ and $\partial f$ are both transverse to $Z$. Then*

*(1) $f^{-1}(Z)$ is a manifold with boundary*
*(2) $\partial f^{-1}(Z) = f^{-1}Z \cap \partial X$*
*(3) $f^{-1}(Z) \pitchfork \partial X$.*

*Proof.* We first note that for $\operatorname{Int} X$ (a manifold without boundary), we already know that $f^{-1}(Z) \cap \operatorname{Int} X$ is a manifold with codimesion equal to $\operatorname{codim} Z$. Therefore, it is sufficient to just deal with the points in $f^{-1}(Z) \cap \partial X$. Let $x \in f^{-1}(Z) \cap \partial X$. Let $\ell = \operatorname{codim} Z$. We then know that $Z$, being a submanifold, is locally cut out by independent functions

$$\phi = (\phi_1 \dots \phi_\ell)$$

Then 0 is a regular value for $\phi \circ f$. Then for $U \subset \mathbb{H}^k$ open, with local parameterization $g : U \to X$ (WLOG assume $g(0) = x$), let

$$h = \phi \circ f \circ g : U \to \mathbb{R}^\ell$$

So our problem now translates to the following:

(1) $h^{-1}(0)$ is a manifold with boundary
(2) $\partial h^{-1}(0) = h^{-1}(0) \cap \mathbb{R}^{k-1}$
(3) $h^{-1}(0) \pitchfork \mathbb{R}^{k-1}$

Since $h$ is a composition of smooth functions, $h$ is smooth, so $h$ extends in a neighborhood of 0 to a smooth $\tilde{h}$ defined on some open set in $\mathbb{R}^k$. From transversality of $f$, we have that $d\tilde{h}_0$ is surjective, so by definition $dh_0$ is as well. Then $\tilde{h}^{-1}(0)$ is a boundaryless submanifold of $\mathbb{R}^k$. Since $dh_0$ is surjective, we have that $d(h\restriction_{\mathbb{R}^{k-1}})_0$ is as well. Then

$$\dim\ker d\tilde{h}_0 = \dim\ker(d\tilde{h}_0\restriction_{\mathbb{R}^{k-1}}) + 1$$

Therefore, we have that $\tilde{h}^{-1}(0) \pitchfork \mathbb{R}^{k-1}$. Then let $\pi : \mathbb{R}^k \to \mathbb{R}$ be projection onto the last coordinate. Then 0 is a regular value of $\pi\restriction_{\tilde{h}^{-1}(0)}$, and we apply the theorem from last lecture. ■

Sard's Theorem also generalizes nicely

**Theorem 7.10** (**Sard's Theorem Revisited**). *Given $f : X \to Y$ with $\partial Y = \varnothing$, almost every $y \in Y$ is a regular value of both $f$ and $\partial f$.*

*Proof.* A regular point of $\partial f$ is automatically a regular point of $f$. We then note that $\partial f$ and $f\restriction_{\operatorname{Int} X}$ are both maps of boundaryless manifolds, so we can apply the Sard's theorem we already know to conclude that the union of their critical values is a set of measure 0 in $Y$. ■

We now state without proof

**Theorem 7.11** (**Classification of compact 1-manifolds**). *Every connected compact 1-manifold is diffeomorphic to $[0, 1]$ or $S^1$.*

**Corollary 7.12.** *If $X$ is a compact connected 1-manifold, then $\partial X$ has an even number of points.*

**Theorem 7.13.** *Let $X$ be a compact manifold. Then there exists no* ***retraction*** *$g : X \to \partial X$. i.e. no function $g$ such that $\partial g = \mathrm{id}$.*

*Proof.* By Sard's Theorem, let $x \in \partial X$ be a regular value of both $g$ and $\partial g$. Then $g^{-1}(x)$ is a compact submanifold with dimension 1. Then $\partial g^{-1}(x) = g^{-1}(x) \cap \partial X$, but in order for the boundary to consist of an even number of points there must be another point in this intersection, contradicting that $\partial g = \mathrm{id}$. ■

**Theorem 7.14** (**Brouwer Fixed Point Theorem**). *Every smooth $f : B^k \to B^k$ (the unit ball) has a fixed point.*

*Proof.* Suppose otherwise, then define $g : B^k \to \partial B^k$ where $g(x)$ is the point on the line between $x$ and $f(x)$ that intersects $\partial B^k$. Then we have that $g$ is a retraction provided that it is smooth. I'll defer to Guillemin and Pollack here for the proof of smoothness. ■

## Return to Transversality: March 6, 2017

We first generalize the notion of homotopy a bit. Recall a homotopy $F$ of a smooth function $f : X \to Y$ is a function $F : X \times I \to Y$, where $F(x, t) = f_t(x)$. Now we generalize to let $F : X \times S \to Y$ where $S$ is a manifold with $\partial S = \varnothing$. Similarly, $F(x, s) = f_s(x)$.

**Theorem 7.15** (**Transversality Theorem**). *Given $F : X \times S \to Y$ with $Z \subset Y$ a submanifold, and only $X$ allowed to have boundary, suppose $F, \partial F \pitchfork Z$. Then for almost every $s \in S$, $f_s, \partial f_s \pitchfork Z$.*

*Proof.* Since $F \pitchfork Z$ we know that $W = F^{-1}(Z)$ is a submanifold of $X$ with

$$\partial W = F^{-1}(Z) \cap \partial(X \times S)$$

Let $\pi : X \times S \to S$ be the projection onto the $s$-coordinate. If we show that when $s \in S$ is a regular value for $\pi\restriction_W$, that $f_s \pitchfork Z$, in addition to showing that when $s$ is a regular value for $\partial\pi\restriction_{\partial W}$ that $\partial f_s \pitchfork W$, we will have proven the theorem (after application of Sard). We first show this for $f_s$.

Suppose $f_s(x) = z \in Z$. Since $F \pitchfork Z$, we know that

$$dF_{(x,s)}\left(T_{(x,s)}(X \times S)\right) + T_z Z = T_z Y$$

Then given any vector $a \in T_y Y$, we know there exists $b \in T_{(x,s)}(X \times S)$ such that $dF_{(x,s)}(b) - a \in T_z Z$. To show transversality, we wish to find $v \in T_x X$ such that $d(f_s)_x(v) - a \in T_z Z$. We know that

$$T_{(x,s)}(X \times S) = T_x X \times T_s S$$

So $b = (w, e)$ for $w \in T_x X$ and $e \in T_s S$. If $e = 0$, we note that we are done. Otherwise, we use the projection $\pi$ onto the $S$ coordinate. We have since $s$ is a regular value of $\pi$, that $d\pi_{(x,s)} : T_{(x,s)} \to T_s S$ is surjective. Then there exists some vector of the form $(u, e) \in T_{(x,s)} W$. Then the vector $v = w - u$, is the vector we desire. The proof for $\partial f_s$ is the exact same. ∎

The Transversality Theorem is very powerful, as it tells us that given any smooth function, we can wiggle it a little to get a transversal mapping. As an example, if the codomain is $\mathbb{R}^m$, transversality is *generic*. Given any submanifold $Z \subset \mathbb{R}^m$, we can let $S = B^m$, and define $F(x, s) = f(x) + s$. the transversaility theorem tells us that for *any* submanifold $Z$ we can translate our map by an arbirarily small vector $s$ such that $f_s$ is transversal to $Z$.

## A Sea of Constructions for a Single Theorem: March 8, 2017

We state the theorem first, then discuss some consequences and develop the necessary machinery.

**Theorem 7.16** (**The $\varepsilon$- Neighborhood Theorem**)**.** *Given $Y \subset \mathbb{R}^m$ and a smooth $\varepsilon : Y \to (0, \infty)$, let*

$$Y^\varepsilon = \{w \in \mathbb{R}^m \mid \exists\ y \in Y \text{ s.t. } d(y, w) < \varepsilon(y)\}$$

*Then for $\varepsilon(y)$ sufficiently small for each $y \in Y$¡ we have that there exists a submersion $\pi : Y^\varepsilon \to Y$ such that $\pi \restriction_Y = \mathrm{id}_Y$. If $Y$ is compact, then $\varepsilon$ can be taken to be a constant map.*

**Corollary 7.17.** *For $f : X \to Y$ with $\partial Y = \varnothing$, there exists a homotopy $F : X \times S \to Y$ such that for all $x \in X$, the map $f(x, s)$ is a submersion (note that this is for fixed $x$ and only $s$ varies. Also, for all $x \in \partial X$, the map $\partial F(x, s)$ is also a submersion (also for a fixed $x$).*

*Proof.* We first embed $Y$ into $\mathbb{R}^m$. Then let $F(x, s) = \pi(f(x) + \varepsilon(f(s)s)$ where $\pi$ and $\varepsilon$ are the ones given from the $\varepsilon$-Neighborhood Theorem. Then since $\pi$ restricts to the identity, we have that $F(x, 0) = f(x)$. Consequently, for fixed $x$, we have that $F(x, s)$ is a submersion from $S \to Y^\varepsilon$. ∎

Another consequence,

**Theorem 7.18.** *For any smooth $f : X \to Y$ and boundary-less submanifold $Z \subset Y$, there exists a smooth map $g : X \to Y$ homotopic to $f$ such that $g \pitchfork Z$ and $\partial g \pitchfork Z$.*

*Remark.* We've been pretty lax on what we mean by homotopic. In this case, this is the more stringent definition using $I$ instead of $S$.

*Proof.* By the corollary and the Transversality theorem, we have that $f_s \pitchfork Z$ and $\partial f_s \pitchfork Z$ for almost every $s \in S$. Every $f_s$ is homotopic to $f$ via the homotopy $(x, t) \mapsto F(x, ts)$ ∎

Now we develop some machinery.

**Definition 7.19.** Give $Y \subset \mathbb{R}^m$ with $y \in Y$, define the **normal space** of $y$, denoted $N_y Y$ to be the orthogonal complement of $T_y Y$, i.e.

$$N_y Y = (T_y Y)^\perp$$

**Definition 7.20.** The **Normal Bundle** of $Y$, denote $NY$ is given by

$$NY = \{(y, v) \mid y \in Y, v \in N_y Y\}$$

We then need a lemma

**Lemma 7.21.** *Given a linear transformation $A : R^m \to \mathbb{R}^k$, the function*

$$A^T : \mathbb{R}^l \to (\ker A)^\perp$$

*is an isomorphism if $A^T$ is surjective.*

*Proof.* GIven $v \in \mathbb{R}^m$ and $w \in \mathbb{R}^k$, we have that

$$\langle Av, w\rangle = (Av)^T w = v^T A^T w = \langle v, A^T w\rangle$$

Then if $v \in \ker A$, for all $w \in \mathbb{R}^k$, we have that

$$\langle v, A^T w\rangle = 0$$

which implies that $A^T w \perp v$. Then suppose $A^T w = 0$. Then $\langle Av, w\rangle = 0$, for any $v \in \mathbb{R}^m$. So if $A$ is surjective, then $w = 0$, so $A^T$ is an isomorphism onto $(\ker A)^\perp$. ■

We need one more theorem here

**Theorem 7.22.** *$NY$ is a manifold, and the standard projection $\sigma : NY \to Y$ is a submersion*

*Proof.* We know $Y$ is locally cut out by independent functions. Using them as coordinate functions defines $\phi : \tilde{U} \to \mathbb{R}^\ell$ with $Y \cap \tilde{U} = \phi^{-1}(0)$ where $\tilde{U}$ is a neighborhood of $y \in Y$. Let $U = Y \cap \tilde{U}$. Then $\forall y \in U$, $d\phi_y : \mathbb{R}^m \to \mathbb{R}^l$ is surjective, with $\ker d\phi_y = T_yY$. Then $(d\phi_y)^T$ is an isomorphism onto $(\ker d\phi_y)^\perp$. But $\ker d\phi_y = T_yY$, so $(d\phi_y)^T$ is an isomorphism onto $N_yY$. Then define

$$\psi U \times \mathbb{R}^k \to NU$$
$$(y, v) \mapsto (y, (d\phi_y)^T(v))$$

We note that $\psi$ is injective, and

$$d\phi_{(y,v)} = \begin{pmatrix} I & 0 \\ ? & (d\phi_y)^T \end{pmatrix}$$

Which is an isomorphism, so $\psi$ is a local parameterization, so $\sigma \circ \psi$ is a submersion, where $\sigma$ is the standard projection $NY \to Y$. ■

## The Fruits of our Labor: March 10, 2017

We now prove the $\varepsilon$-Neighbhorhood Theorem

*Proof.* Define $hNY \to \mathbb{R}^m$ where $h(y, v) = y + v$. We note that $\mathbb{R}^m = T_yY \oplus N_yY$ and $T_y(NY) = \mathbb{R}^m$. Then

$$dh_{(y,0)} : T_y(NY) \to T_y\mathbb{R}^m$$

is an isomorphism. Then $h$ is a local diffeomorphism for every $y \in Y \times \{0\}$. Then we know that there exits a neighborhood $U$ of $Y \times \{0\}$ such that $h \restriction_U$ is a diffeomorphism onto a neighborhood of $Y$. and $h(U)$ contains some $Y^\varepsilon$. Then we define $\pi = \sigma \circ h^{-1} : Y^\varepsilon \to Y$ is the function we desire. ■

We now use this theorem to prove another important theorem. But first, a definition.

**Definition 7.23.** Given $f : X \to Y$ with $Z \subset Y$ a submanifold and $C \subset X$ a arbitrary subset, we say $f$ **transverse along** $C$ if $\operatorname{Im} df_x + T_{f(x)}Z = T_{f(x)}Y$ for all $x \in C \cap f^{-1}(z)$.

**Theorem 7.24** (**Extension Theorem**). *Given $f : X \to Y$ $Z \subset Y$ a closed submanifold and $C \subset X$ a closed subset with $\partial Y = \partial Z = \varnothing$, suppose that $f \pitchfork Z$ along $C$ and $\partial f \pitchfork Z$ along $\partial X \cap C$. Then $\exists$ $g : X \to Y$ homotopic to $f$ such that $g \pitchfork Z$, $\partial G \pitchfork Z$, and $g = f$ on a neighborhood of $C$.*

Before we prove this, we state an immediate corollary

**Corollary 7.25.** *Given $f : X \to Y$ with $Z \subset Y$ a closed submanifold, if $\partial f \pitchfork Z$, then there exists $g$ homotopic to $f$ such that $g \pitchfork Z$ and $\partial g = \partial f$.*

We now prove the theorem.

*Proof.* We want to show that if $f \pitchfork Z$ along $C$, then $\exists$ $U$ an open neighborhood of $C$ such that $f \pitchfork Z$ along $U$ (and similarly for $\partial f$). Given $x \in C$, if $x \notin f^{-1}(Z)$, then $\exists$ a neighborhood of $x$ disjoint from $f^{-1}(Z)$, since $Z$ is closed. Then the condition hold vacuously.

On the other hand, if $x \in f^{-1}(Z)$, we cut out $Z$ locally by the function $\phi$. Then $f \pitchfork Z$ at $x$ implies that $\phi \circ f$ is a submersion at $x$, so by the local submersion theorem, we have that $\phi \circ f$ is a submersion in a neighborhood, so $f \pitchfork Z$ in a neighborhood as well. Then we cover $C$ by open sets where $f \pitchfork Z$ in these open sets, and we take $U$ to be the union of all of these open sets. We do the same for $\partial f$. Then (from an old HW), we have that there exists a $\gamma : X \to [0,1]$ where near $C$, $\gamma(x) = 0$ and $\gamma(x) = 1$ outside of $C$. Let $\tau = \gamma^2$. Then $d\tau_x = 2\gamma(x)d\gamma_x = 0$ whenever $\tau = 0$. Then define $m : X \times S \to X \times S$ where $m(x,s) = (x, \tau(x)s)$. Let $F : X \times S \to Y$ be as in the Transversality Theorem. Then $F\partial F$ are submersions and $F(x,0) = f(x)$. Let $G = F \circ m$. We claim $G$ is the function we desire. First we show $G \pitchfork Z$. Fix some $x \in X$. If $\tau(x) \neq= 0$, $m$ is a local diffeomprhism, so $G$ is a submersion at $x$. If $\tau(x) = 0$, then $dm_{(x,s)}(v,w) = (v,0)$, since in local coordinates, $dm_{(x,s)}$ is given by the matrix

$$dm_{(x,s)} = \begin{pmatrix} I & 0 \\ 0 & \tau(x)I \end{pmatrix}$$

Then we have that

$$\begin{aligned} dG_{(x,s)}(v,w) &= (dF_{x,0} \circ dm_{(x,s)}(v,w) \\ &= dF_{(x,0)}(v,0) = df_x(v) \end{aligned}$$

We know $f \pitchfork Z$, so we then conclude that $G \pitchfork Z$. A near identical argument holds of $\partial G$. Then by the Transversality Theorem for almost every $s \in S$, we have $g_s \pitchfork Z$ and $\partial g_s \pitchfork Z$. Then $g_s$ $f$ and $g_s \restriction_{U'} = f \restriction_{U'}$ for some neighborhood $U' \supset C$. ■

## Mod 2 Intersection Theory: March 20, 2017

**Definition 8.1.** A manifold $X$ is **simply connected** if every smooth map $f : S^1 \to X$ is homotopic to a constant map.

Recall that $S^n$ is simply connected. We actually have another example of a simply connected space, $\mathbb{C}\mathbf{P}^k$!, though it shouldn't be too surprising, since we know that $\mathbb{C}\mathbf{P}^1$ is diffeomorphic to $S^2$.

**Theorem 8.2.** *$\mathbb{C}\mathbf{P}^k$ is simply connected*

*Proof.* Given a smooth map $f : S^1 \to \mathbb{C}\mathbf{P}^k \supset \mathbb{C}\mathbf{P}^{k-1}$, we have after an arbitrarily small homotopy that $f \pitchfork \mathbb{C}\mathbf{P}^{k-1}$. Therefore, we have that $f(S^1) \subset \mathbb{C}\mathbf{P}^K - \mathbb{C}\mathbf{P}^{k-1}$ by a simple count of dimension. We recall that this set difference is just one of the charts of $\mathbb{C}\mathbf{P}^k$!, so we know that $\mathbb{C}\mathbf{P}^k - \mathbb{C}\mathbf{P}^{k-1} \simeq \mathbb{C}^k$, but we know $\mathbb{C}^k$ is contractible, so we are done. ■

We want to talk about the geometric notions of intersections of manifolds, but to do so we have to have specific conditions which are satisfied. We make some definitions to avoid excessive repitition.

**Definition 8.3.** For two submanifodls $X \subset Y$ and $Z \subset Y$, we say that $X$ and $Z$ have **complimentary dimension** if $\dim X + \dim Z = \dim Y$.

**Definition 8.4.** Given $f : X \to Y$ with a submanifold $Z$, we say that the tuple $(f, X, Y, Z)$ is **suitable for intersection theory** if we have that $\partial X = \partial Y = \partial Z = \varnothing$, $X$ is compact, $Z \subset Y$ is closed, and that $X$ and $Z$ have complimentary dimension.

We note that complimentary dimensions mean that the set $f^{-1}(Z)$ is discrete, i.e. every point has a neighborhood locally diffeomorphic to $\mathbb{R}^0$. Then compactness of $X$ gives us that the intersection is a finite point set, so we can go about talking about the number of intersection points

**Definition 8.5.** Suppose $(f, X, Y, Z)$ is suitable for intersection theory. Then we define the **mod 2 intersection number** of $f$ with $Z$, denoted $I_2(f, Z)$ to be

$$I_2(f, Z) = \left|f^{-1}(Z)\right| \mod 2$$

**Proposition 8.6.** *If $f_0, f_1 \pitchfork Z$, $f_0$ is homotopic to $f_1$, and everything is suitable for intersection theory, we have that*

$$I_2(f_0, Z) = I_2(f_1, Z)$$

*Proof.* Let $F$ be a homotopy of $f_0$ to $f_1$. By the Extension Theorem, we can assume WLOG $F \pitchfork Z$ (Otherwise, we just homotope it a small amount without changing $f_0, f_1$, noting that $f_0$ and $f_1$ define $\partial F$). Then we know that $F^{-1}(Z)$ is a compact 1-manifold, and $\partial F^{-1}(Z) = f_0^{-1}(Z) \cup f_1^{-1}(Z)$. We also know that $\left|\partial F^{-1}(Z)\right|$ is even, since every compact 1-manifold must have an even number of boundary points. Consequently, we have that

$$I_2(F, Z) = I_2(f_0, Z) + I_2(F_1, Z)$$

■

The transversality condition required for $f$ in order to define the intersection number is a bit restrictive. Since we just proved that homotopic maps preserve intersection number, using the transversality theorem we have a nice way to define the intersection number for any map $f$ suitable for intersection theory- even if it isn't transverse to the submanifold $Z$.

**Definition 8.7.** If $(f, X, Y, Z)$ is suitable, but $f$ is not transverse to $Z$, then we know thee exists some $g \sim f$ such that $g \pitchfork Z$. We then define

$$I_2(f, Z) = I_2(g, Z)$$

We note this is well-defined by the previous proposition. Much like we how we can talk about functions or manifolds being transverse to some submanifold for transversality, we can do the same here for intersection number

**Definition 8.8.** Given $X, Z \subset Y$ suitable, we define

$$I_2(X, Z) = I_2(\iota, Z)$$

Where $\iota : X \to Y$ is the inclusion map.

## More Intersection Theory: March 22, 2017

We first note that unless specified otherwise, everything in sight is suitable for intersection theory.

**Theorem 8.9 (Boundary Theorem).** *Suppose $X = \partial W$ for some compact manifold $W$, and we have a smooth map $f : X \to Y$ that extends to some smooth $F : W \to Y$. Then $I_2(f, Z) = 0$ for any submanifold $Z \subset Y$.*

*Proof.* First homotope $F$ into some function $G$ such that $G \pitchfork Z$. Then $f = \partial F = \partial G$. Then $I_2(f, Z) = I_2(\partial G, Z)$, since $f \sim \partial G$. But then $I_2(\partial G, Z) = \left|\partial G^{-1}(Z)\right| \mod 2$, which must be even. ■

**Theorem 8.10.** *Given $f : X \to Y$ with $X$ compact, $Y$ connected, $\partial X = \partial Y = 0$, and $\dim X = \dim Y$, we have that $I_2(f, \{y\})$ is independent of our choice of $y \in Y$. Then we denote $I_2(f, \{y\}) = \deg_2(f)$.*

*Proof.* Suppose $f \pitchfork \{y\}$. Then $y$ is a regular value. By the Stack of Records Theorem (See Guillemin and Pollack exercise 1.4.7), there exists a nieghborhood $U$ of $y$ such that every point in $f^{-1}(y)$ has a nieghborhood mapped diffeomorphically onto $U$. Therefore, $I_2(f, \{y\})$ is constant for any fixed $z \in U$. But then this holds for any $y \in Y$ since the intersection is homotopy invariant, and we can homotope any constant function to any other constant function just by translation. Then we get two subsets

$$\{y \in Y \mid I_2(f, \{y\}) = 0\}$$
$$\{y \in Y \mid I_2(f, \{y\}) = 1\}$$

which are both open, disjoint and cover $Y$. But since $Y$ is connected, we conclude one of these sets must be empty and the other must be all of $Y$. ∎

**Corollary 8.11.** *(1)* $\deg_2(f)$ *is homotopy invariant*
*(2) If $f$ extends to $F : W \to Y$ for a compact $W$ with $\partial W = X$, then* $\deg_2(f) = 0$.

With this we can prove some cool things, like half of the Fundamental Theorem of Algebra! I'll defer this application of intersection theory to Guillemin and Pollack.

## Orientation: March 24, 2017

Proving half of the Fundamental Theorem of Algebra is cool, but allows us to already notice the shortcomings of mod 2 intersection theory- namely that we waste a lot of the information (pretty much half) by reducing the intersection numberm modulo 2. To get more, we need to develop oriented intersection theory. As always with manifolds, we start with the linear case.

**Definition 8.12.** Suppose $V$ is a vector space with bases $\beta, \beta'$. Then the change of basis matrix $A$ has nonzero determinant. We say that $\beta \sim \beta'$ if $\det A > 0$.

It's not hard to see that this is an equivalence relation with exactly two equivalence classes (just use the multiplicative property of the determinant).

*Remark.* We note the above statement is technically a lie, since a 0-dimensional vector space has a single unique basis (namely $\varnothing$), so there is only one equivalence class. This isn't a particularly interesting examply though, and is yet another example of the empty set ruining everyone's fun.

**Definition 8.13.** An **orientation** on $V$ is an assignment of $+1$ to one equivalence class and $-1$ to the other. Bases in the former class are said to be **positively oriented** and the bases in the latter are said to be **negatively oriented** for obvious reasons.

We note that for oriented vector spaces $V, W$, and an isomorphism $T : V \to W$, that equivalence classes get mapped to equivalence classes. We call $T$ **orientation preserving** if positively oriented bases are mapped to positively oriented bases. In the opposite case, we say that $T$ is **orientation reversing**. We now generalize the concept of orientation to manifolds.

**Definition 8.14.** Given a manifold $X$, an **orientation** on $X$ is a smooth choice of orientation for each tangent space $T_xX$.

One might be a bit confused as to what smooth means in this contest. We say this assignment is smooth if for all $x$ there exists a parameterization $\phi : U \to X$ of a neighborhood of $X$ such that $d\phi_u$ preserves orientation for all $u \in U$. We note that this might not always be possible, so if such an assigment exists, we say that the manifold $X$ is **orientable**.

**Definition 8.15.** A local diffeomorphism $\phi$ **preserves orientation** if each $d\phi_x$ does.

**Theorem 8.16.** *If $X$ is orientable and connected, it has exaclt two orientations, with one of tem being the oppositve of the other.*

*Proof.* Given two orientations on $X$, consider $x \in X$. Choose local parameterizations $\phi$ and $\psi$ preserving the first and second orientation respectively. Then $d(\psi^{-1} \circ \phi)_U$ is an isomorphism from $\mathbb{R}^k \to \mathbb{R}^k$. Then if we consider $\operatorname{sign}\det d(\psi^{-1} \circ \phi)_u$, we have that this is constant on a neighborhood of 0 by continuity of det and the partial derivatives. Then the sign tells us whether or not the diffeomorphism $\psi^{-1} \circ \phi$ is orientation reversing or preseving on an open subset of $X$. Connecteness then implies that one set must be empty and the other must be all of $X$. ■

**Definition 8.17.** Given an oriented manifold $X$, denote the **opposite orientation** of $X$ as $-X$ (somtimes $\overline{X}$ in the literature, but this can be confusing with complex manifolds, and also when discussing closures).

Given two oriented manifolds, we also have a canonical way to assign an orientation on their product. We note that assigning an orientation amounts to orienting the tangent spaces, so we should first come up with a way or orienting the direct sum of two vector spaces

**Definition 8.18.** Given two oriented vector spaces $V, W$, pick two oriented bases $\alpha = (v_1, \dots, v_k), \beta = (w_1, \dots, w_\ell$ for $V$ and $W$ respectively. Then this defines an ordered basis on the direct sum $V \oplus W$.

$$\Gamma = (\alpha \times 0_W, 0_V \times \beta)$$

Where $0_V$ and $0_W$ are the zero vectors for their respective vector spaces. We then define the orienation of $\Gamma$ to be $\operatorname{sign}\Gamma = (\operatorname{sign}\alpha)(\operatorname{sign}\beta)$.

We note that under this definition, $V \oplus W$ may not inherit the same oreintation as $W \oplus V$. Since the bases of the two vector spaces differ up to permutation of entries, we have that the orientation on $W \oplus V = (-1)^{k\ell}\ V \oplus W$.

**Definition 8.19.** Given oriented manifolds $X, Y$ with one of them not having boundary, define the **product orientation** on $X \times Y$ with the direct sum orientation on $T_{(x,y)}(X \times Y) = T_xX \oplus T_yY$.

## Feeling Disoriented: March 27, 2017

Since we know the orientation of the product of vector spaces is determined by the orientation of its two factors, this gives us a natural way of orienting the boundary of a manifold with boundary

**Definition 8.20.** For an oriented manifold $X$ the **boundary orientation** on $\partial X$ is defined such that for any ordered basis $\beta = (v_1, \dots, v_{k-1})$ for $T_x\partial X$, we define $\operatorname{sign}\beta = \operatorname{sign}(\vec{n}(x), v_1, \dots, v_{k-1})$, where $\vec{n}(x)$ is the outward normal vector at $x$. For the definition of $\vec{n}(x)$, we refer to excercises 2.1.7 and 2.1.8 in Guillemin and Pollack.

To help digest the idea of orienting manifolds, we give some examples

**Example 8.21.** Consider the closed unit ball $B^n \subset \mathbb{R}^n$. $\mathbb{R}^n$ is canonically oriented with the standard ordered basis $(e_1, \dots e_n)$, and since $T_xB^n - T_x\mathbb{R}^n$, we inherit an orientation on $B^n$. Then $S^{n-1} = \partial B^n$ inherits the boundary orientation. To better understand this concretely, consider the point $p = (1, 0) \in \partial B^2$. We have that $\vec{n}(p) = e_1$ and $T_x\partial B^2$ is canonically the $y$-axis, defining the oirientation of $T_p\partial B^2$. Then the canonical orientatino of $S^1$ is where the "positive" direction is counterclockwise.

If we consider $-I$, sometimes called the anitpodal map, we note that $\det -I = (-1)^n$. So we note that $-I$ preserves orientation of $\mathbb{R}^n$ if and only if $n$ is even. Consequently, if it preserves the inherited orientation on $B^n$ and $S^{n-1}$ as well. This gives us a way to define an orientation on $\mathbb{R}\mathbf{P}^k$ for $k$ odd. If $k$ is even $\mathbb{R}\mathbf{P}^k$ is non-orientable.

Now let $X$ be an oriented manifold such that $\partial X = \varnothing$. Then if we consider $I \times X$, then $I \times X$ inherits the product orientation. $\partial(I \times X) = \{1\} \times X \cup \{0\} \times X$. Then given a basis $\beta$ for $T_xX$, we give the canonical orientation on $T_x\partial X$ with the basis $((1,0), 0\times\beta)$. This gives us the positive orientation on $\{1\}\times X$ and the opposite on $\{0\}\times X$, so as an oriented manifold,

$$\partial X = \{1\}\times X \cup \{0\}\times X$$

We note that we can identify $I$ as $\{p\}\times I$ for some singleton set. Then we have that the boundary is negatively oriented at 0 and positivly oriented at 1.

**Corollary 8.22.** *For any compact* 1*-manifold* $X$*,* $\partial X$ *is finite, and the signs sum to* 0*.*

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## Oriented Intersection Theory: March 29, 2017

Now that we've defined what an orientation is on a manifold, we can return to the topic of intersection theory, this time over $\mathbb{Z}$ instead of $\mathbb{Z}/2$. This should be more powerful, as we aren't throwing out "half" the information. But first, one more thing

**Definition 8.23.** For a smooth function $f : X \to Y$, with $Z \subset Y$ a submanifold with $\partial Y = \partial Z = 0$, with $X, Y, Z$ all oriented, and $f, \partial f \pitchfork Z$, let $S = f^{-1}(Z)$, which is a submanifold of $X$ by transversality. Then this gives us a way to define the **preimage orientation** on $S$. For $x \in \operatorname{Int} S$, we have $T_xS \subset T_xX$, and we can choose $H \subset T_xX$ complimentary to $T_xX$ such that $H \oplus T_xS = T_xX$. Consequently, $df_x \restriction_H$ is injective, and consequently, an isomorphism onto its image, since $T_xS = df_x^{-1}(T_{f(x)}Z)$ and $H \cap T_xS = \{0\}$. By transversality, we get that $df_x(H) \oplus T_{f(x)}Z = T_{f(x)}Y$. We already know that $T_{f(x)}Y$ and $T_{f(x)}Z$ are oriented, so this determins an orientation on $df_x(H)$ such that the direct sum orientation matchs the orientation we already have. This gives us an orientation on $H$, so then this determines an orientation on $S$. We do the same thing for $\partial S$, but using $\partial X$ instead of $X$.

We note that this gives us two different characterizations of an orientatino on $\partial S$, the one that is the boundary of the preimage, and the other as the preimage orientation given by $(\partial f)^{-1}$. We'd hope that these are the same, but unfortunately, this isn't true in general

**Theorem 8.24.** *With the same setup as above,* $\partial(f^{-1}(Z)) = (-1)^{\operatorname{codim} Z}(\partial f)^{-1}(Z)$*.*

*Proof.* For $x \in \partial S$, we first note that $d(\partial f)_x = df_x \restriction_{T_x\partial X}$. We pick the complimentary $H$ as done above. Then $f$ and $\partial f$ induce the same orientation on $H$. Let $\alpha$ be a positive ordered basis for $H$. For $\partial(f^{-1}(Z))$, we have that give a basis $\beta$ for $T_x\partial S$, $\beta$ is positive if and only if $(\vec{n}(x), \beta)$ is a positive basis for $T_xS$, which is true if and only if $(\alpha, \vec{n}(x), \beta)$ is a positive basis for $T_xX$.

On the other hand, for $(\partial f^{-1})(Z)$, a basis $\beta$ is positive iff $(\alpha, \beta)$ is positive, iff $(\vec{n}, \alpha, \beta)$ is positive. Since $\dim H = \operatorname{codim} Z$, we conclude that

$$\partial(f^{-1}(Z)) = (-1)^{\operatorname{codim} Z}(\partial f)^{-1}(Z)$$

Since we need $\dim H$ transpositions to get the two bases in the same order. ∎

We recall that for compact one dimensional manifolds with boundary, the boundary consists of a finite even set of points, with each point assigned either a $-1$ or 1 by the boundary orientation. For convenience, when we talk about these points, given $p$ in the boundary, the **intersection number** of $p$ is just the number assigned to $p$ via the boundary orientation of the 1-manifold.

**Definition 8.25.** Given $(f, X, Y, Z)$ suitable for intersection theory and $X, Y, Z$ oriented manifolds, we define the **oriented intersection number** of $f$ with $Z$, denoted $I(f, Z)$ to be

$$I(f, Z) = \sum \text{orientation numbers of } f^{-1}(Z)$$

A lot of the the mod 2 intersection theory transfers over to oriented intersection theory with minimal- if any modification. For example,

**Lemma 8.26 (Oriented Boundary Theorem).** *If $W$ is a compact, oriented manifold with $X = \partial W$, if $f : X \to Y$ extends to $F : W \to Y$, then $I(f, Z) = 0$.*

*Proof.* Assume that $F \pitchfork Z$ by the Extnesion Theorem. Then $F^{-1}(Z)$ is an oriented 1-manifold. We note that $f = \partial F$, so $f^{-1}(Z) = (-1)^{\operatorname{codim} Z}\partial(F^{-1}(Z))$. We know that

$$I(f, Z) = (-1)^{\operatorname{codim} Z} \sum \text{orientation numbers of } f^{-1}(Z)$$

but since $f^{-1}(Z)$ is a compact 1-manifold, we know the above sum must be 0 ■

**Corollary 8.27.** *If $f \sim g$, then $I(f, Z) = I(g, Z)$*

*Proof.* We know that $\partial(I \times X) = X_1 \cup -X_0$, so then given a homotopy $F$, we have that $(\partial F)^{-1}(Z) = f_1^{-1}(Z) \cup -f_0^{-1}(Z)$. Therefore,

$$I(F, Z) = I(f, Z) - I(g, Z) = 0$$

, so we conclude the intersection numbers are the same. ■

Given this homotopy invariance, we do the same thing with intersection numbers of maps that aren't transverse to $Z$ simply by defining their intersection number to be the one of a homotopic map that is transverse. This is well defined since homotopy is an equivalence relation.

## What? Mod 2 Intersection is Evolving!: March 31, 2017

*Remark.* When we say suitable for intersection theory in this case, we now append the condition that the manifolds are all oriented.

We recover most of our definitions from mod 2 intersection theory

**Definition 8.28.** Given $f : X \to Y$ with $X$ compact, $Y$ connected, $\partial X = \partial Y = \varnothing$, and both $X$ and $Y$ oriented, we define

$$\deg(f) = I(f, \{y\})$$

For any $y \in Y$.

We note that this is again independent of our choice of $y$. The proof transfers over quite nicely with minimal modification. Likewise, we also have

**Definition 8.29.** If $X \subset Y$ and $Z \subset Y$ are suitable for intersection theory, we define

$$I(X, Z) = I(\iota_x, Z)$$

where $\iota_x : X \to Y$ is inclusion.

We now state a theorem, but postpone the proof for later, as it will require the development of more machinery

**Theorem 8.30.** *For$X, Z \subset Y$ suitable for intersection theory with both $X, Z$ compact, then*

$$I(Z, X) = (-1)^{\dim X \dim Z} I(X, Z)$$

This is obvious if $X \pitchfork Z$, since we have that the sign is positive if and only if $T_xX \oplus T_xZ = T_xY$. However, if $X$ and $Z$ do not intersect transversally, we have some problems. We'd like to be able to define the intersection number of two maps, rather than the intersection of a function with a submanifold. That way, we can homotope the inclusino maps of the two submanifolds at the same time. To do this, we need to do some set up.

**Definition 8.31.** Suppose we have two smooth functions $f : X \to Y$ and $g : Z \to Y$. We say that $f \pitchfork g$ iff

$$\operatorname{Im} df_x + \operatorname{Im} dg_z = T_yY$$

for all pairs $x, z$ such that $f(x) = g(z) = y$.

**Definition 8.32.** If $f \pitchfork Z$ and everything is suitable for intersection theory, with the additional requirement that both $X$ and $Z$ are compact, for each ordered pair $(x, z)$ such that $f(x) = g(z)$, we define the sign of the intersection to be positive iff

$$\operatorname{Im} df_x \oplus \operatorname{Im} dg_z = T_y Y$$

as oriented vector spaces. If $\operatorname{Im} df_x \oplus \operatorname{Im} dg_z = -T_y Y$, we say that the sign is negative.

*Note.* Since everything is suitable for intersection theory, we have that $\dim X + \dim Z = \dim Y$. Consequently, we have that $\operatorname{Im} df_x \oplus \operatorname{Im} dg_z = T_y Y$ only when the dimensions sum to the dimension of $Y$. This requires that $df_x$ and $dg_z$ to be isomorphism onto their respective images.

We then tentatively define the intersection number

$$I(f, g) = \sum \text{orientation numbers}$$

We say *tentatively* because it is not immediately obvious that this is well defined, as we aren't guaranteed that this is acutally a finite sum (though it will turn out that way). On that unnerving note, we're left on a cliffhanger until the next lecuture.

## Transversality Again: April 3, 2017

First, recall the setup we had to discuss the transversality of two maps $f : X \to Y$ and $g : Z \to Y$. We require that there are no boundaries, $\dim X + \dim Z = \dim Y$, and that $X, Z$ are both compact. we defined $f \pitchfork g$ when for all $x, z$ such that $f(x) = g(z) = y$, we have that

$$\operatorname{Im} df_x + \operatorname{Im} dg_z = T_y Y$$

Recall that when this is true, we actually have that this is a direct sum, and that both $df_x$ and $dg_z$ must be injective. Like mathematicians do, we'd like to put the kettle back on the floor and sove the problem from there[1]. Let $\Delta_Y \subset Y \times Y$ be the diagonal of $Y$, i.e.

$$\Delta_Y = \{(y, y) \mid y \in Y\}$$

You can thnk of $\Delta_Y$ as being the graph of $\mathrm{id}_Y$.

*Claim.* $f \pitchfork g \iff f \times g \pitchfork \delta$

Note that this is a reduction to the transversality we already know and love. This should follow from a more statement in linear algebra.

**Lemma 8.33.** *Suppose $U, W$ are subspaces of a vector space $V$. Then*

$$U \oplus W = V \iff (U \times W) \oplus \Delta_V = V \times V$$

The claim follow from the lemma when we tak $U = \operatorname{Im} df_x$, $W = \operatorname{Im} dg_z$, and $V = T_y Y$. We then prove the lemma

*Proof.* We note that for general vector spaces $A, B, C$, by the defintion, we have that

$$A \oplus B = C \iff A \cap B = \{0\} \text{ and } \dim A + \dim B = \dim C$$

We then note that

$$(U \times W) \cap \Delta_V = \{(0, 0)\} \iff U \cap V = \{0\}$$

and that

$$\dim U \times W + \dim \Delta_V = \dim V \times V \iff \dim U + \dim W = \dim V$$

Since $\Delta_V \simeq V$ ■

[1] There's a joke about two mathematicians were walking in the woods and stumbled into a cabin where they found a kettle on the floor. They proceeded to put the kettle on the stove and make some tea. Then they continue on their walk and stumble into another cabin, in which case the kettle is on the stove. They then place the kettle on the floor, thus "reducing the problem to one already soved"

Doing some more linear algebra

**Lemma 8.34.** *Suppose $U \oplus W = V$ as oriented vector spaces. Then*

$$(U \times V) \oplus \Delta_V = (-1)^{\dim W} V \times V$$

*Where $\Delta_V$ is oriented by sending forward a positive basis for $V$ using the obvious isomorphism $V \to \Delta_V$.*

*Proof.* Let $\alpha, \beta$ be two positively oriented ordered bases for $U$ and $W$ respectively. Then we get an ordered basis for $V \times V$(with abuse of notation)

$$(\alpha \times 0, 0 \times \beta, \alpha \times \alpha, \beta \times \beta)$$

which after some reduction gives us

$$(\alpha \times 0, 0 \times \beta, 0 \times \alpha, \beta \times 0)$$

A positive basis for $V \times V$ looks like

$$(\alpha \times 0, \beta \times 0, 0 \times \alpha, 0 \times \beta)$$

Then to get our basis to match with this positive basis, we need to swap the $0 \times \beta$ terms with the $\beta \times 0$ terms, so we need to do $\dim W$ swaps, giving us the $(-1)^{\dim W}$ term. ■

**Corollary 8.35.** *If $f \pitchfork g$, then $I(f, g) = (-1)^{dim Z} I(f \times g, \Delta_Y)$*

Then we get homotopy equivalence again, so we have a well defined notion of the intersection number of two maps, even when the two maps that aren't transverse.

**Theorem 8.36.** *If $f_0 \sim f$ and $g_1 \sim g$, then $I(f_0, g_0) = I(f, g)$.*

We also get the Transversality Theorem translated over

**Theorem 8.37.** *For any $f, g$ suitable for intersection theory, there exists $f' \sim f$ such that $f' \pitchfork g$. In fact, for $F : X \times S \to Y$, for almost every $s \in S$, $f_s \pitchfork g$.*

**Corollary 8.38.** *$I(f, g) = (-1)^{\dim Z \dim X} I(g, f)$.*

## More Intersection Theory: April 5, 2017

*If you're holding a copy of $\mathbb{C}\mathbf{P}^2$ in your hands and look in the mirror, you'd see something very different.*

We have some examples of applications of intersection theory

**Example 8.39.** Suppose $X \subset Y$, are both oriented. Then if $\dim X = \frac{1}{2} \dim Y$, and $\dim Y$ is odd. Then we can talk about the intersection number $I(X, X)$. From the above corollary, we have that

$$\begin{aligned} & I(X, X) = -I(X, X) \\ \implies & I(X, X) = 0 \\ \implies & I_2(X, X) = 0 \end{aligned}$$

However, if either $X$ or $Y$ are not oriented, we can have $I(X, X) \neq 0$. For example, if we consider the central circle of a möbius band, it has intersection number 1 with itself. We use this information to conclude that one of them isn't orientable.

The professor then went on a tangent relating the intersection theory we've been discussing with homology theory and algebraic topology. As I haven't learned either subject, this part of the lecture was lost on me. After the aside, we return to our regularly scheduled programming.

As it turns out, due to the intimate relations intersection theory has with homology theory, we can define several topological constants with intersection thery, with makes it quite useful. We'll define these in terms of the intersection theory we know, rather than the usual way using algebraic topology

**Definition 9.1.** Given $X$ a compact oriented manifold with $\partial X = \varnothing$, let $\Delta_X \subset X \times X$ be the diagonal. Then then **Euler characteristic** of $X$, denoted $\chi(X)$, is defined to be

$$\chi(X) = I(\Delta_X, \Delta_X)$$

Right off the bat, we get a worrisome result.

**Proposition 9.2.** *If* $\dim X$ *is odd, then* $\chi(X) = 0$.

We note that this follows right from the example we did earlier. The Euler Characteristic being 0 for odd dimensional manifolds makes it quite useless in getting any information, and might discourage us into thinking that it will be 0 for all manifolds, but as we'll se shortly, there will be uses for it. We also get another topological constant.

**Definition 9.3.** For a smooth $f : X \to X$, the **Lefschetz number** of $f$, denoted $\mathfrak{L}(f)$, is defined to be

$$\mathfrak{L}(f) = I(\Delta_X, \operatorname{graph} f)$$[2]

## Lefschetz Theory and Hot Fudge: April 7, 2017

*If you imagine pouring hot fudge of this 4-holed donut, this defines a self-diffeomorphism.*

*Remark.* Since we're going to be referring to it often, $\Delta$ will denote the diagonal unless mentioned otherwise. In the case where it isn't obvious, we give it a subscript to indicate which space it lies in- e.g.

$$\Delta_X \subset X \times X$$

Last lecture we defined the Lefschetz Number, which we denoted $\mathfrak{L}(f)$. We first note that $\mathfrak{L}(f)$ is homotopy invariant. Given a map $f : X \to X$, we have the map $g : X \to \operatorname{graph} f$ where $g(x) = (x, f(x))$, which is a diffeomorphism (See the exercise in Guillemin and Pollack). Then a homotopy on $f$ induces an obvious homotopy on $g$. That being said, we'd like to talk about a special class of functions when we discuss Lefschetz Theory.

**Definition 9.4.** A map $f : X \to X$ is a **Lefschetz map** iff $\operatorname{graph} f \pitchfork \Delta$.

Much like transversal maps and maps appropriate for intersection theory, it isn't surprising that Lefschetz maps aren't particularly hard to find

**Theorem 9.5.** *Every* $f$ *is homotopic to a Lefschetz map*

*Proof.* We know we can define $F : X \to S \to X$ with $f_o = f$ and $F$ a submersion. Then if we let $G : X \times S \to X \times X$ by $G(x, s) = (x, F(x, s))$. We then note that $G$ is a submersion since $G = \operatorname{id} \times F$ and $F$ is a submersion. Then by the Transversality Theorem, for almost every $s \in S$, $g_s \pitchfork \Delta$, and we know that $g_s \sim g$. ■

Given a fixed point $x$ of a Lefschetz map, we note that $(x, f(x)) = (x, x)$, and the transversality assumption, along with dimension complimentarity gives us that $\Delta_{T_xX} \oplus T_{(x,x)} \operatorname{graph} f = T_xX \times T_xX$, where $\Delta_{T_xX} \subset T_xX \times T_xX$. In fact, these equalities hold as oriented vector spaces. We note

[2] We note that this differs from the usual definition, which reverses the role of the diagonal and the graph, this gives us an errant sign depending on dimension of our manifold. We'll stick with Guillemin and Pollack's convention here, which will end up being convenient in certain cases

that $T_{(x,x)}\operatorname{graph} f = \operatorname{graph} df_x$ (again from an excercise in Guillemin and Pollack). so we can write this direct sum as

$$\Delta_{T_xX} \oplus \operatorname{graph} df_x = T_xX \times T_xX$$

We then note that almost by definition that these two spaces direct sum into the entire space if their intersection is just $(0,0)$. Hence, this equality holds if and only if $df_x$ has no nonzero fixed points. Rephrased into the language of linear algebra, this is true if and onlf if $df_x$ does not have 1 as an eigenvalue. This motivates a definition.

**Definition 9.6.** For a fixed point $x \in X$ for a Lefschetz map $f : X \to X$, $x$ is a **Lefschetz fixed point** iff 1 is not an eigenvalue of $df_x$.

We note that $f$ is Lefschetz if and only if every fixed point is a Lefschetz fixed point. We then continue our slew of "Lefschetz Definitions"

**Definition 9.7.** The **local Lefschetz number** of a Lefschetz fixed point, denoted $\mathfrak{L}_x(f)$, is the orientation number of $(x,x) \in \Delta_X \cap \operatorname{graph} f$

We then note that for a Lefschetz map

$$\mathfrak{L}(f) = \sum_{\text{fixed points } x} \mathfrak{L}_x(f)$$

**Theorem 9.8.** *A fixed point $x$ is Lefschetz iff $df_x - I$ is an isomorphism. If so, the $\mathfrak{L}_x(f) = +1 \iff df_x - I$ is orientation preserving, and $-1$ otherwise.*

*Proof.* The first part is fairly simple, since $x$ is Lefschetz iff $df_x$ does not have 1 as an eigenvalue, which is true iff $df_x - I$ does not have 0 as an eigenvalue, which is true iff $df_x - I$ is an isomorphism

For the second part, let $(v_1, \ldots, v_k)$ be a positive ordered basis for $T_xX$, and for notational simplicity, let $A = df_x$. Then $((v_1, v_1, \ldots (v_k, v_k))$ is a positive ordered basis for $\Delta$.

$$\mathfrak{L}_x(f) = \operatorname{sign}\left((v_1, v_1) \ldots (v_k, v_k), (v_1, Av_1), \ldots, (v_k, Av_k)\right)$$

We can do some "row reduction" to simplify to an equivalent basis

$$\begin{aligned}
&= \operatorname{sign}((v_1, v_1), \ldots, (v_k, v_k), (0, (A-I)v_1), \ldots (0, (A-I)v_k)) \\
&= \operatorname{sign}((v_1, 0, \ldots, (v_k, ), (0, (A-I)v_1), \ldots (0, (A-I)v_k)) && \text{since A-I is an isomorphism} \\
&= \operatorname{sign}((A-I)v_1, \ldots (A-I)v_k)
\end{aligned}$$

∎

*Remark.* We ended this lecture with some examples and pictures on geometric interpretations of Lefschetz maps and fixed points. I'll reference Guillemin and Pollack for a good exposition (and some great pictures) on some ways of interpreting the information on Lefschetz Theory, on how $f$ locally behaves as "expanding" and "contracting." There's also a fun analogy on pouring hot fudge on a 4-holed donut that I found quite amusing.

## Lefshetz Theory Pt. 2: April 10, 2017

*Remark.* Unless specified otherwise, we're going to assume our maps are "Appropriate for Lefschetz Theory," i.e. $X$ is compact, $\partial X = \varnothing$, and $X$ is oriented.

We'd like an easier way to talk about Lefschetz numbers. To do this, we're going to find a way to extract Lefschetz numbers from fixed points where the intersection with $\Delta$ isn't necessarily transverse. To do this, we reduce to a local question, so we can look at the problem in our local charts. Given some map $f : X \to X$, for a neighborhood of a fixed point $x \in X$, we get the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] & X \\ U \arrow[u, "\phi"] \arrow[r, "g"'] & V \arrow[u, "\phi"'] \end{tikzcd}$$

Where $g = \phi^{-1} \circ f \circ \phi$, and $U \subset V$ is appropriately chosen so that the diagram commutes. A.so, WLOG, we say that $\phi(0) = x$. Then we note that $x = f(x) \iff g(0) = 0$. In addition, $x$ is a Lefschetz fixed point $\iff df_x - I$ is an isomorphism $\iff dg_0 - I$ is an isomorphism, so all the properties carry over into our local parameterization.

The general idea of what we want to do is to take a non-Lefschetz map and perturb it with a homotopy that has some number of Lefschetz fixed points. In essence, we split it into a bunch of Lefschetz points.

**Theorem 9.9** (**Splitting Proposition**). *For a neighborhood $U$ of a fixed point $x$ of a map $f : X \to X$, that contains no other fixed points. Then there exists a homotopy $f_t$ of $f$ such that $f_1$ has only Lefschetz fixed points in $U$ and each $f_t = f$ outside a compact subset of $U$.*

*Proof.* Since this is a local question, we pull back to Euclidean space and assume WLOG that $U \subset \mathbb{R}^k$. Then define the bump function $\rho : U \to [0, 1]$ such that for a compact $K \subset U$, we have that $\rho(x) = 0$ outside of $K$ and $\rho(x) = 1$, on a neighborhood $V \subset K$ of 0. The for $a \in \mathbb{R}^k$ define the homotopy $f_t = f(x) - t\rho(x)a$. We then claim that for $\|a\|$ sufficiently small that for $t \in [0, 1]$, $f_t$ has no fixed points. Outside we note that this is obvious since nothing has changed. Then inside the compact set $K - V$, we have that the function $f(x) - x$ vanishes iff $x$ is a fixed points, so we have that

$$\|f(x) - x\| > c > 0$$

So if we pick $a$ such that $\|a\| < \frac{c}{2}$, we get by the triangle inequality that,

$$\|(f_t(x) - x)\| \geq \|f(x) - x\| - t\rho(x)\,\|a\| > \frac{c}{2}$$

Then by Sard's Theorem, we can choose $a$ arbitrarily close to 0 such that $a$ is a regular value of $f - \mathrm{id}_U$. Then if $y$ is a fixed point of $f_1$, $y \in U$. We note that

$$(df_1)_y = df_y$$

Since it only differs by a constant factor. So then $a$ is a regular value for $df_y$, so all the fixed points are Lefschetz. ∎

This motivates an alternative characterization of an old definition

**Definition 9.10.** Given $f : U \to \mathbb{R}^k$, with an isolated fixed point $x$, define

$$F(z) = \frac{f(z) - z}{\|f(z) - z\|}$$

Which for an appropriately small ball, is defined on $\partial B$. Then

$$\mathfrak{L}_x(f) = \deg F$$

We should check that this agrees with our old definition in the case that $x$ is a Lefschetz fixed point. WLOG we assume that $x$ is 0 after a translation. Then we know that $f(z) = df_0(z) + r(z)$ where $\frac{r(z)}{\|z\|} \to 0$ as $z \to 0$ by the definition of the derivative. Then the function

$$f_t(z) = df_0(z) + tr(z)$$

defines a homotopy from $df_0$ to $f$. We then claim that for a sufficiently small ball $B$, there is no $t$ such that $f_t(z) = z$ on $\partial B$. SInce $df_0 - I$ is an isomorphism, we know $df_0$ has no zero on $\partial B$. Then since $\partial B$ is compact, this implies that $\|df_0 - I\|$ has a minimum $c > 0$ on $\partial B$. Then by linearity, this minimum decreases linearly as the radius of $B$ decreases. However, we have that $\frac{r(z)}{\|r(z)\|}$ decreases faster than any linear function. Our new definition then defines

$$\mathfrak{L}_0(f) = \deg \frac{df_0 - I}{\|df_0 - I\|}$$

which is exactly the sign of $df_0 - I$, so we recover our old definition.

## Vector Fields: April 12, 2017

*If you tried to comb a hairy coconut you'd always get a zero. If you had a hairy donut on the other hand, you could just comb it around... but you probably wouldn't want to eat it.*

Recall that the whole point of the last lecuture was to discuss Lefschetz numbers for fixed points that weren't necessarily Lefschetz. We redeuced it to calculating the degree of a map. This leads us to claim that we have another way to calculate Local Lefschetz Numbers.

For $U$ an open set in $\mathbb{R}^k$, and a function $f : U \to \mathbb{R}^k$ with an isolated fixed point $x$, let $B$ be a ball containing $x$ that contains no other fixed points. Then let $g$ be the map homotopic to $f$ that we obtain from the Splitting Proposition from the last lecture. We then define

$$G : \partial B \to S^k$$
$$y \mapsto \frac{g(y) - y}{\|g(y) - y\|}$$

Then we know $\mathfrak{L}_x(f) = \deg G$. Then let $\{z_i\}$ be the finite collection fixed points of $g$, and choose a collection of balls $\{B_i\}$about the points that are disjoint. The $G$ extends to

$$B' - \bigcup_i \operatorname{Int} B_i$$

So the degree of $G \restriction_{\partial B'}$ is 0. We note that

$$\partial B' - \partial B - \bigcup_i \partial B_i$$

So $\deg G \restriction_{B_i} = \mathfrak{L}_{z_i}(g)$, so we have that

$$\mathfrak{L}_x(f) = \deg G = \sum_{\text{fixed points } z} \mathfrak{L}_z(g)$$

Now that we have a good characterization of Lefschetz Theory, we generalize to arbitrary manifolds. Let $f : X \to X$ with $X$ compact, oriented, and $\partial X = \varnothing$, and assume $f$ has finitely many fixed points. As you'd expect, definining Local Lefschetz Numbers passes to local parameterizations.

**Definition 9.11.** For a fixed point $x$ of $f$, $\phi : U \to X$ a local parameterization of a neighborhood of $x$ such that $\phi(0) = x$,

$$\mathfrak{L}_x(f) = \mathfrak{L}_0(\phi^{-1} \circ f \circ \phi)$$

As per usual, we want to check that this is independent of parameterization. First we pick $f' \sim f$ where $f'$ has only Lefschetz fixed points. Then

$$\begin{aligned}\mathfrak{L}_0(\phi^{-1} \circ f \circ \phi) &= \sum_{\text{fixed points z}} \mathfrak{L}_z(\phi \circ f' \circ \phi) \\ &= \sum_{\text{fixed points } y} \mathfrak{L}_y(f')\end{aligned}$$

While not exactly independent of our choice of $\phi$, all $\phi$ is really telling us is how big our perturbation is. So given two parameteriztions of neighborhoods of $x$, we can just intersect the neighborhoods and then restrict our perturbation to that smaller neighborhood.

✧

We now move to one big application of Lefschetz Theory: Vector Fields.

**Definition 9.12.** A **vector field** on a manifold $X$ is a smooth map $\vec{v} : X \to TX$ such that for the standard projection

$$\pi : TX \to X, \pi \circ v = \mathrm{id}_x$$

We note that this is equivalent to saying that $\vec{v}$ is a smooth assigment of $v \in T_xX$ for every $x \in X$.

There is a very boring example of a vector field on $\mathbb{R}^k$. For every $x \in \mathbb{R}^k$, let $\vec{v}(x) = e_1$. Why bother with such a pedestrian example? For any manifold $X$ with an open set $U \subset X$ that contains no zeroes of the vector field (i.e. $x \in X$ such that $\vec{v}(x) = 0$, there exists a parameteriztion where this vector field looks like the $e_1$ vector field. Therefore, the only interesting parts of a vector field to consider are the zeroes. Like with Lefschetz Theory, we first define things in Euclidean space.

**Definition 9.13.** Given a vector field $\vec{v} : \mathbb{R}^k \to \mathbb{R}^k \times \mathbb{R}^k$ with an isolated zero at 0, let $B$ be a sufficiently small ball about 0 containing no zeroes. Then define the function

$$\begin{aligned}\frac{\vec{v}}{\|\vec{v}\|} &: \partial B \to S^{k-1} \\ x &\mapsto (\pi_2 \circ \vec{v})(x)\end{aligned}$$

Where $\pi_2 : TX \to X$ is projection onto the second component. Then define the **index** of $\vec{v}$ at 0 to be

$$\mathrm{ind}_0(\vec{v}) = \deg \frac{\vec{v}}{\|\vec{v}\|}$$

Now that we've defined something for Euclidean space, we'd like a way to translate a question about a manifold to a question in $\mathbb{R}^k$, to absolutely no one's surprise, we pass to a local paraneterization

**Definition 9.14.** Given a local diffeomorphism $\varphi : X \to Y$ and a vector field $\vec{v}$ on $Y$, we have the **pullback**[3] **vector field**, denoted $\varphi^*\vec{v}$ to be

$$\varphi^*\vec{v} = d\varphi_x^{-1} \circ \circ \pi_2 \vec{v} \circ \varphi$$

So the pullback does exactly what one would expect. For a point $x \in X$, we map it forward to $\varphi(x)$, evaluate the vector field, and then take the inverse image of the tangent vector under the inverse of the derivative. We note that the pushforward of a vector field need not exist unless $\varphi$ is bijective.

[3]Hadani would be proud

**Definition 9.15.** Let $\vec{v}$ be a vector field on a compact, oriented manifold with $\partial X = \varnothing$ with an isolated zero at $x \in X$, let $\varphi : U \to X$ be a paramterization of a neighborhood of $x$ with $\varphi(0) = x$, then define

$$\operatorname{ind}_x \vec{v} = \operatorname{ind}_0 \varphi^* \vec{v}$$

We postpone the proof that this is independent of parameterization. There's a cool theorem we'd like to prove. We'll state it now as the proverbial carrot dangling above our noses

**Theorem 9.16** (**Poincaré-Hopf Theorem**). *If a vector field $\vec{v}$ on a manifold $X$ has finitely many zeroes, then*

$$\sum_{zeroes x} \operatorname{ind}_x \vec{v} = \chi(X)$$

We note an immediate corollary

**Corollary 9.17.** *If $\chi(X) \neq 0$, then every vector field has a zero*

We're going to deviate from Guillemin and Pollack here to discuss another concept that's very similar to a vector field as an alternative way to prove Poincaré-Hopf.

**Definition 9.18.** A **flow** on a manifold $X$ is a 1-parameter family of maps

$$H : X \times \mathbb{R} \to X$$

that satisfies

(1) $h_0 = \mathrm{id}_x$
(2) $h_s \circ h_t = h_{s+t}$[4]

where $h_t(x) = H(x, t)$ for fixed $t$.

We immediately note that $h_t^{-1} = h_{-t}$, so each $h_t$ is in fact a diffeomorphism. Intuitively, one can interpret a flow as a constant vector field, like a river, where if you drop a leaf at a point $x$ and wait some time $t$, the leaf is then at $h_t(x)$. Then if we wait again for some time $s$, then the leaf is at $h_{t+s}(x)$. Alternatively, we could have just dropped the leaf at the position $h_t(x)$, and waited $s$ seconds, which would have the leaf ending at the same position.

Instead of varying $x$ while fixing $t$, we could do the opposite- choosing to fix $x$ to define $\gamma_x$ as a function of $t$. Then we can interpret the second condition on a flow as

$$\gamma_x(s + t) = \gamma_{\gamma_x(t)}(s)$$

So a single point $x$ determines a curve on $X$, which we call a **flow line**. We note that our definition prohibits flow lines from splitting or merging. Unsurprisingly, vector fields and flows are very closely related. One can define a vector field on $X$ simply by differentiating a flow, and as we'll see soon, a vector field also uniquely determines a flow, so the two concepts are practically equivalent.

## More Vector Fields: April 14, 2017

We mentioned last lecture that a flow gives us a vector field via differentiation. We'll make this claim more precise now. Given a flow $H : X \times \mathbb{R} \to X$, for a fixed $x$, we get a flow line $\gamma_x : \mathbb{R} \to X$. Then we define the vector field $\vec{v}$ where $\vec{v}(x) = \gamma_x'(0)$. In essence, we take the flow and we use the velocity to impose a vector field on the manifold. This also works in the opposite direction provided our manifold is compact. If it isn't, we need some compactness assumptions. Given that we satisfy these compactness conditions, we can "Integrate a vector field."

**Theorem 9.19.** *Given a vector field $\vec{v}$ on $X$ such that $\vec{v} = 0$ outside a compact $K \subset \int X$. Then there exists a unique flow $H$ whose velocity vector field is given by $\vec{v}$.*

*Remark.* We note that $h_t = \mathrm{id}$ outside $K$, since it has velocity 0.

[4] We note that this can also be expressed algebraicly as a group action $\mathbb{R} \circlearrowleft X$ of the additive group of the reals on $X$

*Proof.* We note that $\gamma_x'(t) = \gamma_{\gamma_x(t)}(0)$. Therefore, in order for $\vec{v}$ to be the velocity vector field for a flow, the flow lines must satisfy the equation

$$\gamma_x'(t) = \vec{v}(\gamma_x(t))$$

This is a first order ODE (or a system of real valued ODE) with the initial condition that $\gamma_x(0) = x$. Then by the Picard-Lindelöf Existence Uniqueness Theorem, we have in a neighborhood $U_x$ of $x$ that there exists a unique solution $\gamma_x : (-\varepsilon_x, \varepsilon_x) \to X$ that depends smoothly on the initial condition. Then we get a smooth $H_x : U_x \times (-\varepsilon_x, \varepsilon_x) \to X$. Then since $K$ is compact, $\{U_x\}_{x\in X}$ covers $K$, so we can pick $\varepsilon = \min\{\varepsilon_x\}$ giving us a global $H : X \times (-\varepsilon, \varepsilon)$, since thise $\varepsilon$ works for every $U_x$. We note that for $s, t, s+t \in (-\varepsilon, \varepsilon)$ we have

(1) $h_0 = \mathrm{id}$
(2) $h_{s+t} = h_s \circ h_t$

So $H$ satisfies the flow conditions for a small interval of time. Then we can patch together paths in small time intervals to give us $H : X \times \mathbb{R} \to X$. ■

Note that in the above construction, $\vec{v}$ has a zero at $x$ if and only if $h_t$ has a fixed point for all $t$. So now that we have the notion of the vector field of a flow and the flow of a vector field, we'd like to connect the concepts of zeroes of $\vec{v}$ with the Lefschetz Fixed Points of $h_t$.

**Theorem 9.20.** *Let $\vec{v}$ be a vector field on $\mathbb{R}^k$ with a neighborhood $U$ of $0$ in which $0$ is the unique zero of $\vec{v}$ inside of $U$. If on a neighborhood of $0$, $h_t$ for $t < \varepsilon$ for a sufficiently small $\varepsilon$, $h_t$ has no fixed points except for $0$, then*

$$\mathrm{ind}_0\,\vec{v} = \mathfrak{L}_0(h_t)$$

*for all $t < \varepsilon$.*

*Proof.* For sufficiently small $t$ and $z$, we have that $h_t(z) = \gamma_z(t)$. By Taylor's theorem, we expand this to get

$$h_t(z) = \gamma_z(0) + t(\gamma_z'(0) + t^2 r(t, z)$$

for some second order remainder term $r(t, z)$. But this is equal to

$$z + t(\vec{v}(z) + tr(t, z)$$

which implies that

$$h_t(z) - z = t(\vec{v}(z) + tr(t, z))$$

Then

$$\mathfrak{L}_0(h_t) = \deg \frac{\vec{v}(z) + tr(t,z)}{\|\vec{v}(z) + tr(t,z)\|}$$

and if we restrict to a sufficiently small ball, we have that

$$deg \frac{\vec{v}(z) + tr(t,z)}{\|\vec{v}(z) + tr(t,z)\|} = \deg \frac{\vec{v}(z)}{\|\vec{v}(z)\|} = \mathrm{ind}_0\,\vec{v}$$

■

Immediately we recover something we put off proving.

**Corollary 9.21.** *The index of the zero of a vector field is well defined (i.e. independent of parameterization) since Lefschetz numbers are independent of parameterization*

This extra machinery of flows lets us prove Poincaré-Hopf fairly easily now. Recall the statement

**Theorem 9.22** (**Poincaré-Hopf Theorem**). *If a vector field $\vec{v}$ on a manifold $X$ has finitely many zeroes, then*

$$\sum_{zeroes x} \mathrm{ind}_x\,\vec{v} = \chi(X)$$

*Proof.* Given a compact $X$ with $\partial X = \varnothing$, we have that

$$\sum_{\vec{v}(x)=0} \operatorname{ind}_x \vec{v} = \sum_{\text{fixed points} x} \mathfrak{L}_x(h_t)$$

For a sufficiently small $t$. Therefore we have that

$$\sum_{\vec{v}(x)=0} \operatorname{ind}_x \vec{v} = \sum_{\text{fixed points} x} \mathfrak{L}(h_t) = \mathfrak{L}(\mathrm{id}) = \chi(X)$$

Since $h_t \sim \mathrm{id}$. To see this, we let $t$ go to 0, this determines the homotopy. ∎

## Tensors: April 17, 2017

*That's a brilliant proposition, except I didn't define what that product was*
*-Dan Freed*

Professor Gompf is gone this week, so we'll be learning differential forms from Professor Freed.

As with all things differential, we should understand the linear case first. We're going to first provide some motivation for why we would want such a thing. If we consider integration in $\mathbb{R}^2$, we could have two different coordinate systems $u, v$ and $x, y$ for $\mathbb{R}^2$, which are related by a diffeomorphism

$$\begin{aligned} \varphi : \mathbb{R}^2 &\to \mathbb{R}^2 \\ (u, v) &\mapsto (x(u, v), y(u, v)) \end{aligned}$$

Let $V$ be an open set in $\mathbb{R}^2$ and $U = \varphi(V)$. Then given a function $f : U \to V$, we want to interpret what

$$\int_U f \; dx \; dy$$

means in our other coordinate system, for the pullback function $\varphi^* f = f \circ \varphi$. From multivariable calculus, we have that the change of coordinate integral is given by

$$\int_V \varphi^* f \left| \det d\varphi_{(u,v)} \right| \; du \; dv$$

Since intuitively, these coordinate systems have different volumes corresponding to the parallelogram spanned by their bases, we need the determinant to rescale the area accordingly. This hints at the idea that we aren't really integrating functions, but rather volumes, so we'd like the develop the "Linear Algebra of Volume Functions."

For concreteness, we'll work in 2-dimensions to lay out what we want, and then generalize to arbitrary (but finite) dimensions. Given a 2 dimensional vector space $V$ over $\mathbb{R}$, we want an "area function" $\alpha : V \times V \to \mathbb{R}$ that takes in two vectors and spits out the area of the parallelogram spanned by them. Some properties we would like $\alpha$ to satisfy are:

(1) $\alpha(v, v) = 0$
(2) $\alpha(v_1 + w, v_2) = \alpha(v_1, v_2) + \alpha(w, v_2)$
(3) $\alpha(\lambda v_1, v_2) = \lambda \alpha(v_1, v_2)$

These are fairly intuitive conditions to impose since a single vector has no area, and areas of parallelograms sum if we scale or add vectors. We note an immediate consequence of our conditions

$$\begin{aligned} 0 &= \alpha(v+w, v+w) \\ &= \alpha(v+v) + \alpha(w+v) + \alpha(v+w) + \alpha(w,w) \\ &= 0 + \alpha(w,v) + \alpha(v,w) + 0 \\ \implies \alpha(v,w) &= -\alpha(w,v) \end{aligned}$$

So the order matters, which should remind us a lot of orientations., and implies that we should really be measuring *signed* volumes. We'll see soon that this defines something called an *alternating* 2*-form.* Noting that these are just bilear forms on $V$, this defines a vector space, and this motivates us to try to define

$$\Lambda^2 V^* = \{\text{Space of “area funtions”}\}$$

and more generally, $\Lambda^p V^*$ as a vector space of volume functions on $\mathbb{R}^p$.

**Definition 10.1.** Let $V$ be an $\mathbb{R}$-vector space, and $p \in \mathbb{Z}^{\geq 0}$. A $p$-linear function on $V$ is a function

$$T : \underbrace{V \times V \times \ldots \times V}_{p \text{ times}} \to \mathbb{R}$$

that satisfies

$$T(v_1, \ldots, \lambda v_i + \mu w, \ldots, v_p) = \lambda T(v_1, \ldots, v_i, \ldots, v_p) + \mu T(v_1, \ldots, w, \ldots, v_p)$$

In otherwords, if we fix all $p-1$ variables except for the $i^{th}$ variable, we get a linear function in the $i^{th}$ variable.

**Definition 10.2.** For a finite dimensional $V$, let $\bigotimes^p V^*$ denote the vector space of $p$-linear functions

We confidently state that these functions indeed form a vector space, and will suggestively refer to elements of this vector space as $p$-forms

*Remark.*

(1) $\bigotimes^1 V^* = V^*$
(2) We define $\bigotimes^0 V^*$ to be $\mathbb{R}$.

Given a $p$-form and a $q$-form, we can construct a $p+q$-form

**Definition 10.3.** Given $T \in \bigotimes^p V^*$, and $T' \in \bigotimes^q V^*$, define the **tensor product** of $T$ and $T'$, denoted $T \otimes T' \in \bigotimes^{p+q} V^*$ to bethe function that satisfies

$$(T \otimes T')(v_1, \ldots, v_p, v_{p+1}, \ldots v_{p+q}) = T(v_1 \ldots, v_p) T'(v_{p+1}, \ldots, v_{p+q})$$

**Proposition 10.4.** *Given a finite dimensional vector space $V$, with basis $\{e_1 \ldots v_n\}$, let $\{e^1, \ldots, e^n\}$ be the corresponding dual basis for $V^*$ where*

$$e^i(e_j) = \delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

*Then $\{e^{i_1} \otimes \ldots \otimes e^{i_p} \mid 1 \leq i_j \leq n \forall j\}$ (i.e. all tensor products of dual basis elements that need not be unique) form a basis for $\bigotimes^p V^*$, so $\dim \bigotimes^p V^* = n^p$.*

*Proof.* Given $v_1, \ldots v_p \in V$, where

$$v_j = \sum_{i=1}^n \alpha_{ij} e_i$$

we'll introduce a more compact Einstein notation where

$$\sum_i^n \alpha_{ij} e_i = \alpha^i_j e_i$$

Then for $T \in \bigotimes^p V^*$

$$\begin{aligned} T(v_1, \ldots, v_p) &= T(\alpha^{i_1}_j e_{i_1}, \ldots, \alpha^{i_p}_j e_{i_p}) \\ &= (\alpha^{i_1}_i \ldots \alpha^{i_p}_p) T(e_{i_1}, \ldots e_{i_p}) \\ &= T(e_{i_1}, \ldots, e_{i_p})(e^{i_1} \otimes \ldots \otimes e^{i_p})(v_1, \ldots, v_p) \end{aligned}$$

So the set is a spanning set. For linear independence, we just evaluate on a basis.

*Remark* (**Einstein Summation Notation**). Einstein summation notation is helpful with tensors since the sums become exponentially large. In this case, when we have the expression

$$\alpha^i_j e_i$$

We are implictly summing over $i$, as the index moves from a superscript to a subscript. In addition, when variables are left "free," there's an implicit summation over the entire indexing set, and when there are multiple such "free variables," (like in the last line of the proof above) we are summing over all permutations of those indexing sets

■

## Exterior Algebra: April 19, 2017

*The vectors span a . . . yes, a "parallelo"*
*-Dan Freed*

**Definition 10.5.** An **Algebra** over $\mathbb{R}$ is a tuple $(A, 1, \mu)$ where

(1) A is a $\mathbb{R}$-vector space
(2) $1 \in A$
(3) $\mu : A \times A \to A$

that satisfies the following conditions

(1) $\mu$ is bilinear
(2) $\mu(1, a) = \mu(a, 1) = a$
(3) $\mu$ is associative, i.e.

$$\mu(\mu(a, b), c) = \mu(a, \mu(b, c))$$

**Definition 10.6.** A **$\mathbb{Z}$-grading** on an $\mathbb{R}$-Algebra $A$ is a direct sum decomposition

$$A = \bigoplus_{q \in \mathbb{Z}} A^q$$

such that $\mu(A^q, A^p) \subset A^{q+p}$.

A $\mathbb{Z}$-graded algebra is **anticommutative** if

$$\mu(a, b) = (-1)^{ab} \mu(b, a)$$

*Remark.* We can use an arbitrary abelian group to grade an algebra replacing $p, q$ with group elements, and $q + p$ with the group operation on said elements.

**Definition 10.7.** An element of $A^q$ is called **homogeneous** and we say it has degree $q$.

Last time, we constructed $\bigotimes^p V^*$ as the space of $p$-linear functions on $V$. This allows us to create

$$\bigotimes V^* = \bigoplus_{q=0}^{\infty}\left[\bigotimes^{q} V^*\right]$$

Which is a $\mathbb{Z}$-graded vector space. With the tensor product as we defined earlier, this forms a $\mathbb{Z}$-graded alegebra over $\mathbb{R}$. We note that this algebra isn't commutative however. There's a very important subset (a subspace in fact) of tensors that we'd like to consider

**Definition 10.8.** A tensor $T \in \bigotimes^p V^*$ is **alternating** if

$$T(v_1, \dots, v_p) = 0$$

if $v_i = v_j$ for some $i \neq j$. We then let

$$\Lambda^p V^* = \left\{\alpha \in \bigotimes^{p} V^* \mid \alpha \text{ is alternating}\right\}$$

The suggestive notation implies that this is the space of "signed area functions" that we're looking for. Then if we have an arbitrary $p$-tensor, we'd like the find a natural way to convert it into an alternating tensor. In other words, we want to find a projection function, which we will call Alt onto the subspace of alternating tensors. In other words, we want

$$\mathrm{Alt} : \bigotimes^{p} V^* \twoheadrightarrow \Lambda^p V^*$$

Where $\mathrm{Alt} \circ \iota = \mathrm{id}_{\Lambda^p V^*}$ ($\iota$ is inclusion $\Lambda^p V^* \hookrightarrow \bigotimes^P V^*$). To do this, we need to recall some group theory.

For $S_q$ the group of permutations on the set $\{1, \dots q\}$, we have a homomorphism

$$\mathrm{sgn} : S_q \to \{\pm 1\}$$
$$\sigma \mapsto \begin{cases} 1 & \sigma \text{ is odd} \\ -1 & \sigma \text{ is even} \end{cases}$$

Where a permutation is even if it can be written as the product of an even number of transpositions. For notational convenience, we will write $\mathrm{sgn}(\sigma)$ as $(-1)^\sigma$. Then for $T \in \bigotimes^p V^*$, define

$$T^\sigma(v_1 \dots, p) = T(v_{\sigma(1)}, \dots, v_{\sigma(p)})$$

We note that if $T$ is alternating $T^\sigma = (-1)^\sigma T$ and $T^{\tau\circ\sigma} = ((-1)^\sigma)^\tau$. We then define

$$\mathrm{Alt}(T) = \frac{1}{q!}\sum_{\sigma \in S_p} (-1)^\sigma T^\sigma$$

Note that given any linear combination of alternating tensors is also an alternating tensors, so we are very close to defining a anticommutative $\mathbb{R}$-Algebra. However, the tensor product of two alternating tensors need not be alternating. this motivates us to define a "multiplication" on alternating tensors as the map $\wedge$ where given $\alpha_1 \in \bigotimes^k V^*$ and $\alpha_2 \in \bigotimes^\ell V^*$,

$$\alpha_1 \wedge \alpha_2 = \frac{(k+\ell)!}{k!\ell!}\,\mathrm{Alt}(\alpha_1 \otimes \alpha_2)$$ [5]

**Proposition 10.9.**

*(1) If $T$ is alternating,* $\mathrm{Alt}(T) = T$

*(2) $\wedge$ is associative*

[5] We note that this differs from the convention used by Guillemin and Pollack. The factorials end up making defining volumes much simpler. As Freed says, "There's a choice to be made here, and this is the right one."

*(3)* $\wedge$ *is anticommutative*

One can varify this by straight computation. I might put them up here if I find the time to run through the calculations myself. Here are some other good properties of the wedge product to check.

**Proposition 10.10.**
*(1)* $(\omega_1 + \omega_2) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta$
*(2)* $\eta \wedge (\omega_1 + \omega_2) = \eta \wedge \omega_1 + \eta \wedge \omega_2$
*(3) For* $a \in \mathbb{R}$, $a\eta \wedge \omega = \eta \wedge a\omega = a(\eta \wedge \omega)$
*In other words,* $\wedge$ *satsifies the linearity properties we expect.*

Recall that given a basis $\{e_1 \dots e_n\}$ for $V$, we get a basis $\{e^1, \dots, e^n\}$ for $V^*$, and that $\{e^{i_1} \dots, e^{i_p}\}$ forms a basis for $\bigotimes^p V^*$. Then given any $p$-tensor $T$, we could write it as a sum

$$\sum_{i_1=1}^{n} \dots \sum_{i_p=1}^{n} T(e_{i_1} \dots e_{i_p})(e^{i_1} \otimes \dots \otimes e^{i_p})$$

But since $T$ is alternating, this eventually simplifies down to

$$\sum_{1 \le i_1 \le i_2 \dots \le i_p} T(e^{i_1}, \dots, e^{i_p} \sum_{\sigma \in S_p} (-1)^{\sigma}(e^{\sigma(i_1)}, \dots, e^{\sigma(i_p)})$$

which is equal to

$$p! \sum_{1 \le i_1 \dots \le i_p} T(e_{i_1}, \dots, e_{i_p})(e^{i_1} \wedge \dots \wedge e^{i_p})$$

**Corollary 10.11.** $\{e^{i_1} \wedge \dots \wedge e^{i_p}\}$ *is a basis for* $\Lambda^p V^*$*, with* $\dim \Lambda^p V^* = \binom{n}{p}$ *where* $n = \dim V$*.*

Then we can define the **exterior algebra** on $V$ as the $\mathbb{Z}$-graded $\mathbb{R}$-Algebra

$$\bigwedge V^* = \bigoplus_{p=0}^{n} \Lambda^p V^*$$

we note that $\dim \bigwedge V^* = 2^n$. From the formula, we note that our formula says that $\dim \Lambda^n = 1$, so it is spanned by a single alternating tensor. In the case that $V = \mathbb{R}^n$, this acutally is something we're very familiar with. If we pick the tensor in $\Lambda^n \mathbb{R}^n *$ that evaluates the standard ordered basis $(e_1, \dots e_n)$ to 1, we acutally get our old friend det! In general, we don't have a canonical basis for an abstract vector space $V$, so we don't have a distingushed alternating tensor to pick out. In this case, we call $\Lambda^n V^*$ the **Determinant Line**, and this will be the tensor that takes in $n$ vectors and spit out a signed volume. We note that a choice of tensor will also induce an orientation (as well as a notion of volume) on $V$ by feeding in basis vectors, much like how we used det to determine an orientation on $\mathbb{R}^n$.

## Pullbacks: April 21, 2017

*Any questions? ..."How was your week?" ...It was good, how was yours?*

For concreteness, a quick example

**Example 10.12.** Given two alternating 2-forms $\alpha_1$ and $\alpha_2$, we have that

$$(\alpha_1 \wedge \alpha_2)(v_1, v_2) = \frac{4!}{4}[\alpha_1(v_1)\alpha_2(v_2) - \alpha_1(v_2)\alpha_2(v_1)]$$

In $\mathbb{R}^n$ we have several different ways to compute the determinant. One formula that ends up being quite usefull is the **Leibnitz Formula**, where given $v_1 \dots v_n \in \mathbb{R}^n$, we can write each vector $v_j$ as a linear combination $A^i_j e_i$ of basis vectors. Then we can put these vectors into a matrix as columns, i.e.

$$A = \begin{pmatrix} A^1_1 & \dots & A^1_n \\ \vdots & \dots & \vdots \\ A^n_1 & \dots & A^n_n \end{pmatrix}$$

Then

$$\det(v_1 \dots v_n) = \det A = \sum_{\sigma \in S_n} (-1)^\sigma A^{\sigma(1)}_1 A^{\sigma(2)}_2 \dots A^{\sigma(n)}_n$$

---

Suppose we have two vector spaces $V$ and $W$. Then if we have forms on $W$, we'd like to use these to define forms on $V$. To start, given a linear transformation $T : V \to W$ and a linear functional $\omega^* \in W^*$, we can use this to define a linear functional on $V$ in a natural way through a **pullback**

$$T^* : V \to W$$
$$\omega^* \mapsto T^*\omega^*$$

Where $T^*\omega^*(v) = \omega^*(Tv) = (\omega^* \circ T)(v)$. We can generalize this to pullback $p$ forms in $\Lambda^P W^*$, and even stronger, to pull back the entire exterior algebra $\bigwedge W^*$. Note that in this case, we have some more structure on $\bigwedge W^*$ in addition to the vector space structure. Namely, $\bigwedge W^*$ is an algebra equipped with the multiplication operation $\wedge$, so we want this preserve wedge products as well, i.e. for the pullback map $\bigwedge^* T^*$, we have

$$\bigwedge^* T^*(a \wedge b) = \bigwedge^* T^*(a) \wedge \bigwedge^* T^*(b)$$

We note that this actually uniquely determines the map, as we have that every $q$-form $\omega \in \Lambda^p V^*$ can be expressed as a sum of $\alpha_1 \wedge \dots \wedge \alpha_p$ hwere each $\alpha_i \in V^*$. In addition, this pullback is **contravariant**, i.e. given maps

$$V \xrightarrow{T} W \xrightarrow{S} X$$

we get the corresponding picture in the dual spaces

$$V^* \xleftarrow{T^*} W^* \xleftarrow{S^*} X$$

So we have that $(S \circ T)^* = T^* \circ S^*$.

**Theorem 10.13.** *Given a vector space $V$ with $\dim V = n$, given $T : V \to V$, the pullback $T^* : \Lambda^* n V^* \to \Lambda^n V^*$ is just scalar multiplication by $\det T$.*

*Proof.* Let $e_1 \dots e_n$ be a basis for $F$, $\omega \in \Lambda^* n V^*$, and let $T$ be given by the matrix $A$ i.e.

$$Te_j = A^i_j e_i$$

Then

$$\begin{aligned} T^*(\omega)(e_1 \dots e_n) &= \omega(Te_1 \dots Te_n) \\ &= \omega(A^{i_1}_j e_1 \dots A^{i_n}_n e_n) \\ &= A^{i_1}_1 A^{i_2}_2 \dots A^{i_n}_n \omega(e_{i_1}, \dots, e_{i_n}) \\ &= \sum_{\sigma \in S_n} (-1)^\sigma A^{\sigma(i_1)}_1 A^{\sigma(i_2)}_2 \dots A^{\sigma(i_n)}_n \omega(e_1, \dots, e_n) \\ &= (\det T)\ \omega(e_1 \dots e_n) \end{aligned}$$

■

## Differential Forms: April 24, 2017

With the requisite linear algebra done, we return to the world[6] of smooth manifolds.

**Definition 10.14.** A **differential $p$-form** on a smooth manifold $X$ is a function $\omega$ such that for all $x \in X$, $\omega(x) \in \Lambda^p(T_x^* X)$.

*Remark.* The concepts of addition, scalar multiplication, and wedge product carry over as pointwise operations e.e.

$$(\omega \wedge \eta)(x) = \omega(x) \wedge \eta(x)$$

We note that we canonically identify $\Lambda^0(T_x^* X) = \mathbb{R}$, so 0-forms on $X$ are just real valued functions $f : X \to \mathbb{R}$. Also, given a smooth function $\varphi : X \to \mathbb{R}$, we have that its derivative at a point $x$, $d\varphi_x$ is a linear map to $\mathbb{R}$, which takes in vectors and spits out real numbers. This coincidentally sounds a lot like what a 1-form should be, and unsurprisingly, the assignement $x \mapsto d\varphi_x$ defines a smooth 1-form on $X$, which we denote $d\varphi$, called the **differential of** $\varphi$.

Note that on $\mathbb{R}^k$, we have the standard coordinate functions $x_1 \dots x_k : \mathbb{R}^k \to \mathbb{R}$, where $x_i(v_1 \dots v_k) = v_i$. We can "differentiate" these functions, yielding 1-forms $dx_1 \dots dx_k$. Then for $z \in \mathbb{R}^k$, we know that $T_z \mathbb{R}^k = \mathbb{R}^k$, and when we evaluate $d(x_i)_z(e_j) = \delta^i_j$, so $\{dx_i\}$ is just the dual basis $\left\{e^i\right\}$ for $\mathbb{R}^k$. We know that the collection

$$\left\{dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p} \mid 1 \le i_1 \le i_2 \le \ldots \le i_p \le k\right\}$$

forms a basis for $\Lambda^p \mathbb{R}^{k*}$, giving us

**Proposition 10.15.** *Every $p$-form $\omega$ on an open set $U \subset \mathbb{R}^k$ can be uniquely expressed as a a sum over increasing indices*

$$\omega = \sum f_I dx_I$$

**Example 10.16.** For $U \subset \mathbb{R}^k$ open and a smooth function $\varphi : U \to \mathbb{R}$,

$$d\varphi = \sum_{i=1}^{k} \frac{\partial \varphi}{\partial x_i} dx_i$$

Now we want to be able to talk about pullbacks of forms. Given a smooth $f : X \to Y$ and a $p$-form $\omega$ on $Y$, we can use $f$ to help us define a $p$-form on $X$, which we denote $f^*\omega$ and is defined in an analogous manner to the pullback in the linear case

$$f^*\omega(x) = df_x^* \omega(f(x))$$

As a refresher, $df_x^*$ is the **transpose map** of $df_x$, and is a map $T^*_{f(x)} Y \to T^*_x X$, and the pullback is given by

$$df_x^* \omega(f(x))(v_1 \dots v_p) = \omega(df_x(v_1), \dots df_x(v_p))$$

giving us a $p$-form on $X$. We note that for a 0-form $\omega Y \to \mathbb{R}$

$$f^*\omega = \omega \circ f$$

**Proposition 10.17 (Homework).** *For $\phi : Y \to \mathbb{R}$, $f : X \to Y$,*

$$f^* d\phi = d(f^*\phi)$$

Using this, let's do an example of a pullback

[6] He won't even say the words "category!"

**Example 10.18.** For open sets $U \subset \mathbb{R}^K$ and $V \subset \mathbb{R}^\ell$,with coordinate functions $x_i$ and $y_i$ respectively, given $f : V \to U$, $f$ can be written in terms of its coordinate functions

$$f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_k \end{pmatrix}$$

Let's say we want to pull back the 1-form $dx_i$

$$\begin{aligned} f^*(dx_i) &= d(f^*x_i) = d(x_i \circ f) \\ &= df_i \\ &= \sum_{j=1}^{\ell} \frac{\partial f_i}{\partial x_j} dy_j \end{aligned}$$

We note that at some point $p$, this is the transpose map $(df_p)^T$.

The above example actually generalizes quite nicely, using linearity of the pullback. Given a $p$-form $\omega$ we know that $\omega$ can be written as, with strictly inreasing indexing set $I = \{i_1 \dots i_p\}$ and functions $a_I$

$$\omega = \sum a_I dx_I$$

Then the pullback is given by

$$\begin{aligned} f^*(dx_i) &= \sum f^* a_I f^* dx_I \\ &= \sum (a_I \circ f) df_i \end{aligned}$$

*Remark* 10.19. As we get to integration, the thing that's going to vital for is is pulling back volume forms. Suppose $K = \ell = P$. Then let $\omega$ be a $p$-form. Then the pullback

$$f^*(dx_i \wedge \ldots \wedge dx_l) = \det(df)(dy_1 \wedge \ldots \wedge dy_k)$$

where $\det(df)(x) = \det(df_x)$

Note that we never specified anything has to be smooth (though we usually assume everything is). Let's make this precise before we go back to our convention of everything in sight being smooth.

**Definition 10.20.** A $p$-form $\omega$ on $X$ is **smooth** if for every local parameterization $\phi$, when we pull back $\omega$ to Euclidean space

$$\phi^*\omega = \sum f_I dx_I$$

each $f_I$ is smooth. The space of smooth $p$-forms on $X$ is the denoted $\Omega^p(X)$.

As with most definitions, we probably want to check that it's well-defined, i.e. independent of our choice of parameterization. Suppose we have another parameterization $\psi$. Then

$$\psi^*\omega = (\phi \circ \phi^{-1} \circ \psi)^*\omega = (\phi^{-1} \circ \psi)^* \phi^* \omega$$

We note $\phi^*\omega$ is smooth, and $\phi^{-1} \circ \psi$ is a diffeomorphism, so $\omega$ looks smooth here as well.

## Integration: April 26, 2017

*Engineers just cut it up into pieces and integrate over those pieces. We'll use something more precise . . . a partition of unity*

Now that we've established what a differential form is, we can begin discussing integration. Recall the change of variables theorem from multivariable calculus

**Theorem 11.1** (**Change of Variables**). *Given a diffeomorphism $f : V \to U$ between open subsets of $\mathbb{R}^k$ with coordiantes $x_i$ and $y_i$ respectively, and a smooth function $a : U \to R$,*

$$\int_U a\ dx_1\ dx_2 \dots dx_k = \int_V (a \circ f)\, |\det df|\ \ dy_1\ dy_2\ \ \dots dy_k$$

*Where* $\det df(x) = \det df_x$.

The determinant term in this integral is telling us how much the diffeomorphism $f$ is stretching volumes so we can rescale the integral correctly. This hints that we aren't really integrating functions on $\mathbb{R}^k$, but that we are integrating forms. As usual, we start with Euclidean space, this time with a pretty elementary defintion

**Definition 11.2.** For an open set $U \subset \mathbb{R}^k$ and $\omega \in \Omega^k(\mathbb{R}^k)$, we know that

$$\omega = a dx_1 \wedge \dots \wedge dx_k$$

For some smooth function $f : U \to \mathbb{R}$. Then define the integral

$$\int_U \omega = \int_U a\ dx_1 \dots dx_k$$

Extending this to pullbacks, we have that if $f : V \to U$ is an orientation preserving diffeomorphism,

$$\int_V f^*\omega = \int_U \omega$$

and if $f$ is orientation reversing

$$\int_V f^*\omega = -\int_U \omega$$

Note that this works exactly we want it to, since the pullback encodes the volume change from the determinant.

Fortunately, that's about all the work we need to do in $\mathbb{R}^k$. We now generalize to manifolds. Let $X$ be an oriented $k$-dimensional manifold (possibly with boundary). Then for suppose $\omega \in \Omega^k(X)$ has support contained in a compact subset of $\operatorname{Im}\phi$ for some orientation preserving local parameterization. We then define

$$\int_X \omega = \int_U \phi^*\omega$$

So we now know how to integrate forms that are nonzero on small chuncks of our manifold. We should check that this is independent of parameterization. Let $\psi : V \to X$ be another local parameterization containing the support of $\omega$. Then (after appropriate shrinking of $V$ and $U$ to get the domains to line up)

$$\begin{aligned}\int_V \psi^*\omega &= \int_U (\psi^{-1} \circ \phi)^*\phi^*\omega \\ &= \int_U (\psi \circ \psi^{-1} \circ \phi)^*\omega \\ &= \int_U \phi^*\omega\end{aligned}$$

We'd like the define the integral for a wider class of forms, ones whose support need not be contained in a single chart. As usual, we use a partition of unity to patch things together. With $\omega$ defined as it is above, let $\{\rho_\alpha\}$ be any partition of unity on $X$. We then claim that

$$\int_X \omega = \int_X \sum_\alpha \rho_\alpha \omega$$

We note that the local finiteness property of the partition of the unity lets us move the sum outside of the integral, since integration is linear, giving us

$$\sum_\alpha \int_X \rho_\alpha \omega$$

Which is again a finite sum by local finiteness. Then for *any* form $\eta \in \Omega^k(X)$, let $\{\rho_\alpha\}$ be a partition of unity subordinate to a cover of $X$ by local parameterizations. Then define

$$\int_X \eta = \sum_\alpha \int \rho_\alpha \eta$$

We check that this is independent of our choice of partition of unity. let $\{\theta_\beta\}$ be another partition of unity subordinate to the open cover. Then

$$\begin{aligned}
\sum_\alpha \int_X \rho_\alpha \eta &= \sum_\alpha \sum_\beta \int_X \theta_\beta \rho_\alpha \eta \\
&= \sum_\beta \sum_\alpha \int_X \theta_\beta \rho \alpha \eta \\
&= \sum_\beta \int_X \theta_\beta \eta \\
&= \int_X \eta
\end{aligned}$$

In addition, this integral is preserved across orientation preserving diffeomorphisms i.e.

$$\int_X f^*\omega = \int_{f(X)} \omega$$

So $k$-forms are meant to be integrated over $k$-manifolds. But what do we do with all the $p$-forms on $X$ for $p < k$? As you'd expect, these are for integrating over $p$-dimensional subanifolds. Let $Z \subset X$ be a $p$-dimensional submanifold. and $\omega \in \Omega^p(X)$, and let $\iota : Z \hookrightarrow X$ the inclusion map. Then we call $\iota^*\omega$ the restriction of $\omega$ to $Z$, and define

$$\int_Z \omega = \int_Z i^*\omega$$

**Example 11.3 (A Line Integral).** Let $\gamma : [0,1] \to \mathbb{R}^n$ be an embedding. Then let $\omega \in \Omega^1(\mathbb{R}^n)$ be a 1-form defined in an open set containing $\operatorname{Im}\gamma$. We know that

$$\omega = \sum_{i=1}^N a_i dx_i$$

for smooth functions $a_i : \mathbb{R}^n \to \mathbb{R}$. Then

$$\begin{aligned}\int_{\operatorname{Im}\gamma} \omega &= \int_{[0,1]} \gamma^*\omega \\ &= \sum_i \int_{[0,1]} \gamma^* a_i \; dx_i \\ &= \int\limits_0^1 \sum (a_i \circ \gamma) d\gamma_i \\ &= \int\limits_0^1 a_i \gamma(t) \frac{d\gamma_i}{dt} \; dt\end{aligned}$$

If we let

$$\vec{v} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \omega^T$$

Then this integral is

$$\int_0^1 \vec{v}(t) \cdot \gamma'(t) \; dt$$

Which is the classical line integral $\oint$.

## The Exterior Derivative: April 28, 2017

We saw a more modern formulation of the line integral at the end of the last lecture. There's another construction we can reformulate

**Example 11.4** (**A Surface Integral**). Let $S \subset \mathbb{R}^3$ be some smooth surface, realized as the graph of some smooth functions $G : \mathbb{R}^2 \to \mathbb{R}$. Then let $\omega$ be a smooth 2-form defined near $S$. We know we can parameterize $S$ with

$$\begin{aligned}h : U &\to \mathbb{R}^2 \\ (x_1, x_2) &\mapsto (x_1, x_2, G(x_1, x_2))\end{aligned}$$

and $\omega$ can be expressed as

$$\omega = f_1 dx_2 \wedge dx_3 + f_2 dx_3 \wedge dx_1 + f_3 dx_1 \wedge dx_2$$

for smooth real valued functions $f_1, f_2, f_3$. Noting that integration is linear over addition of forms, we compute the pullback each term in the sum seperately.

$$\begin{aligned}h^*(dx_1 \wedge dx_2) &= dh_1 \wedge dh_2 \\ &= dx_1 \wedge dx_2 && \text{since } h_1 = x_1 \text{ and } h_2 = x_2\end{aligned}$$

Next, we compute

$$\begin{aligned}h^*(dx_2 \wedge dx_3) &= dh_2 \wedge dh_3 \\ &= dx_2 \wedge dG \\ &= dx_2 \wedge \left( \frac{\partial G}{\partial x_1} dx_1 + \frac{\partial G}{\partial x_2} dx_2 \right)\end{aligned}$$

We note that when we distribute the $dx_2$, we get $dx_2 \wedge dx_1 = -dx_1 \wedge dx_2$ on the first term, and $dx_2 \wedge dx_2 = 0$ on the second term, so in total this gives us that

$$h^*(dx_2 \wedge dx_3) = -\frac{\partial G}{\partial x_1} dx_1 \wedge dx_2$$

We finally compute

$$\int_S \omega = \int_U h^*\omega = \int_U \left( (f_1 \circ h)\frac{-\partial G}{\partial x_1} + (f_2 \circ h)\frac{\partial G}{\partial x_1} + f_3 \circ h \right) dx_1 \wedge dx_2$$

In more classical notation, this is the integral

$$\int_U \vec{F} \cdot \vec{n}\ dx_1 dx_2$$

where $\vec{n}$ is the unit normal. In classical vector calculus, this measures the flux of the vector field flowing out of the surface.

So given a function $f$, we know how to take the differential of the function, namely $df$, which assigns to each point $x$ the linear functional $df_x$. So we can think of $d$ as some sort of function $\Omega^0(X) \to \Omega^1(X)$. This generalizes quite naturally to arbitrary $p$-forms. We start with the Euclidean case once more, and then generalize to manifolds.

Let $U \subset \mathbb{R}^k$ be an open set and $\omega \in \Omega^p(X)$ a $p$-form on $U$. We know that $\omega$ can be expressed in terms of our standard differential forms as

$$\omega = \sum_I a_I dx_I$$

For some multi-index $I = \{i_1, \dots i_p\}$ and smooth functions $a_I$. We note that we can also view the $a_I$ as 0-forms, and multiplication by $a_I$ is actually the wedge product. In a more suggestive notation, we have that

$$\omega = \sum_I a_I \wedge dx_I$$

We then define $d : \Omega^p(U) \to \Omega^{p+1}(U)$ in the natural way where we let

$$d\omega = \sum_I da_I \wedge dx_I$$

**Theorem 11.5.**

*(1)* $\dot{(}\omega \wedge \eta) = d\omega \wedge d\eta$
*(2) For* $\omega \in \Omega^p(U)$, $\eta \in \Omega^q(U)$, $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p\ \omega \wedge d\eta$
*(3)* $d(d\omega) = 0$

*Also $d$ is the unique collection of operators that satisfies these conditions and agree with the $d$ we already have $\Omega^0(U) \to \Omega^1(U)$.*

**Corollary 11.6.** *For a diffeomoprhism $g : V \to U$,*

$$g^* \circ d = d \circ g^*$$

With the Euclidean case covered, we generalize to manifolds.

**Definition 11.7.** For a manifold $X$ and a $p$-form $\omega \in \Omega^p(X)$, define the **exterior derivative** of $\omega$, denoted $d\omega$, where for any local param $\phi : U \to X$, on $\phi(U)$,

$$d\omega = (\phi^{-1})^* d(\phi^*\omega)$$

We check that this construction is independent of our choice of parameterization. If $\psi$ is another local parameterization,

$$\begin{aligned}(\psi^{-1})^* d(\psi^*\omega) &= (\psi^{-1}\circ\phi\circ\phi^{-1})^* d(\phi\circ\phi^{-1}\circ\psi)^*\omega \\ &= (\phi^{-1})^*(\psi^{-1}\circ\phi)^* d(\phi^{-1}\circ\psi)^*\phi^*\omega\end{aligned}$$

and since $\phi^{-1}\circ\psi$ is a diffemorphism, this is exactly $(\phi^{-1})^* d\phi^*\omega$.

## Stokes' Theorem: May 1, 2017

We note the the above corollary applies to general smooth functions on manifolds, not just diffeomorphisms between open sets. We'll state this here for completeness

**Corollary 11.8.** *For a smooth function $g: X \to Y$, $g^*\circ d = d\circ g^*$.*

So for a smooth manifold $X$ where $\dim X = k$, if we have a smooth $\omega \in \Omega^{k-1}(X)$, there are two operations we can do. We can either integrate $\omega$ over the boundary, or we can take $d\omega$ and integrate it over the entire manifold. While this might not seem that familiar, it should be something you've done many times over during your mathematical career. For some motivation, we return to some classical vector calculus examples.

**Example 11.9.**

(1) For an open set $U \subset \mathbb{R}^k$ and a smooth function $f \in \Omega^0(U)$, we have that

$$df = \sum_{i=0}^{k} \frac{\partial f}{\partial x_i} dx_i$$

We can interpret this as a vector (techinically a covector), namely

$$\begin{pmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_k}\end{pmatrix}$$

Which in classical vector calculus is just the gradient $\nabla f$

(2) In the case $U \subset \mathbb{R}^3$, and $\omega \in \Omega^1(U)$, we can write $\omega$ as

$$\omega = f_1\ dx_1 + f_2\ dx_2 + f_3\ dx_3$$

We then compute

$$d\omega = \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)\ dx_1\wedge dx_2 + \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right) dx_3\wedge dx_1 + \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) dx_2\wedge dx_3$$

If we let the differences of paritals attached to each term be $G_1, G_2$, and $G_3$ respectively and $\vec{G} = (G_1, G_2, g_3)$, we have that $\vec{G} = \nabla\times F$, otherwise known as curl $\vec{F}$

(3) Again, if $U \subset \mathbb{R}^3$, this time suppose $\omega \in \Omega^2(U)$. Then we know $\omega$ can be written as

$$\omega = f_1 dx_2\wedge dx_3 + f_2 dx_3\wedge dx_1 + f_3 dx_1\wedge dx_2$$

Then if we compute $d\omega$, we get

$$d\omega = \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\right) dx_1\wedge dx_2\wedge dx_3$$

Which again has a classical name, namely the divergence of the vector field $\vec{F}$

Coming back to our remark on what we do with $k-1$-forms on $X$, we have a very powerful theorem that generalizes many vector calculus identites, and can be seen in some sense as a generalization of the Fundamental Theorem of Calculus.

**Theorem 11.10** (**Stokes' Theorem**). *For $X$ a $k$-dimensional compact oriented manifold with boundary, and $\omega \in \Omega^{k-1}(X)$,*

$$\int_{\partial X} \omega = \int_X d\omega$$

*Proof.* For a partition of unity $\{\rho_\alpha\}$ subordinate to a cover of $X$ by charts, write

$$\omega = \sum \rho_\alpha \omega$$

WLOG, we take $\operatorname{supp} \omega \subset \phi(U)$ where $\phi : U \to X$ is an orientation preserving local parameterization. Then there are two cases to consider

**Case 1: $U$ is open in $\mathbb{R}^k$**

In this case $\phi(U) \cap \partial X = \varnothing$. Therefore

$$\int_{\partial X} \omega = 0$$

Let $\eta = \phi^* \omega$. Then

$$\eta = \sum_{i=1}^{k} (-1)^{i-1} f_i dx_1 \wedge \ldots \wedge \overline{dx_i} \wedge \ldots \wedge dx_k$$

Where $\overline{dx_i}$ indicates that $dx_i$ is missing from the wedge product. Then

$$d\eta = \sum \frac{\partial f_i}{\partial x_i} \, dx_1 \wedge \ldots \wedge dx_k$$
$$\implies \int_U d\eta = \sum \int_{\mathbb{R}^k} \frac{\partial f_i}{\partial x_i} \, dx_1 \ldots dx_k$$

We note that the right hand side is an iterated integral over $\mathbb{R}^k$, and since $\omega$ has compact support, we can find $[-a, a] \subset \mathbb{R}$ such that $\operatorname{supp} f_i \subset [-a, a]$. Then each iteration of the integral will be

$$\int_{-} a^a \frac{\partial f_i}{\partial x_i} \, dx_i = f(a) - f(-a) = 0 - 0$$

**Case 2: $U$ is open in $\mathbb{H}^k$**

We do the same thing, giving us that

$$\int_X d\omega = \sum \int_{\mathbb{H}^k} \frac{\partial f_i}{\partial x_i} \, dx_1 \ldots dx_k$$

As before, we have for $i < k$

$$\int_{\mathbb{R}} \frac{\partial f_i}{\partial x_i} = 0$$

This gives us that

$$\int_X d\omega = \int_{\mathbb{R}^{k-1}} \left( \int\limits_0^\infty \frac{\partial f_k}{\partial x_k} \, dx_k \right) dx_1 \ldots dx_{k-1}$$
$$= -\int_{\mathbb{R}^{k-1}} f_k(x_1 \ldots x_{k-1}, 0) \, dx_1 \ldots dx_{k-1}$$

Then if we compute

$$\int_{\partial X} \omega + (-1)^{k-1} \int_{\partial \mathbb{H}^k} f_k(x_1 \ldots x_{k-1}, 0) \, dx_1 \ldots dx_{k-1} = \int_X d\omega$$

Since $\partial \mathbb{H}^k = (-1)^k \mathbb{R}^{k-1}$. ∎

## Stokes' Theorem Continued: May 3, 2017

Last lecture we proved the generalized Stokes' Theorem. This generalizes some classical theorems.

**Example 11.11.**

(1) Let $W \subset \mathbb{R}^3$ be a 3-dimensional compact submanifold, $\omega \in \Omega^2(W)$. We have that $\omega$ can be interpreted as a vector field $\vec{F}$. Then

$$\int_W \text{div } \vec{F} \; dx \; dy \; dz = \int_{\partial W} \vec{F} \cdot \vec{n} \; dA$$

Which is know as the **Divergence Theorem**, or sometimes **Gauss' Theorem**.

(2) For $\Sigma \subset \mathbb{R}^3$ an oriented compact surface, $\partial\Sigma = S^1$ parameterized by $\gamma : [a, b] \to \partial\Sigma$, $\omega \in \Omega^1(U)$ for a neighborhood $U \supset \Sigma$,

$$\int_\Sigma (\text{curl } \vec{F}) \cdot \vec{n} \; dA = \int_a^b \vec{F}(\gamma(t)) \cdot \gamma'(t) \; dt$$

Which is the original **Stokes' Theorem**

**Definition 11.12.** A $p$-form $\omega \in \Omega^p(X)$ is **closed** if $d\omega = 0$

Note that this implies that every top degree form on a manifold is closed. Another thing to note, for a connected manifold $X$, a real valued function (i.e. a 0-form) is closed $\iff$ $f$ is a constant function. In the case that $X$ isn't connected, a function is closed if it is constant in each connected component, though it can take different values on each component. Also, this implies that the pullback $f^*\omega$ of a closed form is closed, since $d(f^*\omega) = f^*(d\omega) = 0$.

**Theorem 11.13.** *Given compact and oriented manifolds $X$ and $W$ with $\partial W = X$ and $\dim X = k$, a function $f : X \to Y$ and $\omega \in \Omega^k(Y)$ a closed form, if $f$ extends to $F : W \to Y$, then $\int_X f^*\omega = 0$.*

*Proof.*

$$\int_X f^*\omega = \int_W d(F^*\omega) = \int_W F^*(d\omega) = \int_W 0 = 0$$

■

**Corollary 11.14.** *For homotopic maps $f_0, f_1 : X \to Y$ with $X$ compact oriented and $\partial X = \varnothing$, and $\omega \in \Omega^k(Y)$ closed,*

$$\int_X f_0^*\omega = \int_X f_1^*\omega$$

*Proof.* Since $f_0$ and $f_1$ are homotopic, there exists a homotopy $F : X \times I \to Y$. Then from the previous theorem, we have that

$$\int_{\partial(X \times I)} F^*\omega = 0$$

We then note that $\partial(X \times I) = X \sqcup -X$, so

$$\int_{\partial(X \times I)} F^*\omega = \int_X f_0^*\omega + \int_{-X} f_1^*\omega = 0$$

And since integrating over the negatively oriented manifold reverses the sign of the integral, this gives us

$$\int_X f_0^*\omega = \int_X f_1^*\omega$$

■

Note that the above proof works for cobordisms as well.

**Example 11.15.** Consider the manifold $S^2 \times S^2$, which contains two submanifolds $S^2 \times \{p\}$ and $\{p\} \times S^2$. This gives us two "inclusions" $f_1 : S^2 \to S^2 \times \{p\}$ and $f_2 : S^2 \to \{p\} \times S^2$. We proved with intersection theory earlier that $f_0$ is not homotopic to $f_1$. Using what we just proved, we get another proof.

Choose $\omega \in \Omega^2(S^2)$ such that $\int_{S^2} \omega > 0$. Note that since $\omega$ is a top degree form, $d\omega = 0$. Then for the projection functions $\pi_1, \pi_2 : S^2 \times S^2 \to S^2$, we get forms $\pi_1^* \omega$ and $\pi_2^* \omega$ defined on $S^2 \times S^2$. Pulling them back again, we get forms $f_1^* pi_1^* \omega$ and $f_2^* \pi_1^* \omega$ on $S^2$. We compute, using the contravariance of pullback and the fact that $d\omega = 0$,

$$f_1^* \pi_1^* \omega = (\pi_1 \circ f_1)^* \omega = \omega \qquad \text{Since } \pi_1 \circ f_1 = \mathrm{id}$$
$$f_2^* \pi_1^* \omega = (\pi_1 \circ f_2)^* \omega = 0 \qquad \text{Since } \pi_1 \circ f_2 \text{ is a constant map}$$

Then integrating gets us

$$\int_{S^2} f_1^* \pi_1^* \omega = \int_S^2 \omega > 0$$
$$\int_{S^2} f_2^* \pi_1^* \omega = 0$$

So $f_1$ and $f_2$ could not have been homotopic.

The End: May 5, 2017

**Theorem 11.16.** *Given $X$ and $Y$ compact oriented $k$-manifolds where $\partial X = \partial Y = \varnothing$, and $Y$ begin connected, if we have $f : X \to Y$ and $\omega \in \Omega^k(Y)$,*

$$\int_X f^* \omega = (\deg f) \int_Y \omega$$

*Proof.* From the Stack of Records theorem, if we have a regular value $y$ and a neighborhood $U$ of $y$, then for each $p_i \in f^{-1}(y)$, there exists a neighborhood $V_i$ of $p_i$ that is mapped diffeomorphically on $U$ by $f$. Also, under the preimage orientation, we have that $p$ gets an orientation number based on whether or not that diffeomorphism is orientation preserving or not. There are two cases to consider here.

**Case 1:** supp $\omega \subset U$
In this case, we have that

$$\begin{aligned}\int_X f^* \omega &= \sum_i \int_{V_i} f^* \omega \\ &= \sum_i \mathrm{sign}(p_i) \int_U \omega \\ &= (\deg f) \int_U \omega = \deg f \int_Y \omega\end{aligned}$$

**Case 2**
Since $Y$ is connected, it is homogeneous, i.e. $\forall z \in Y$, there exits a diffeomorphism $h_z : Y \to Y$ where $h_z(y) = z$ with $h_z$ isotopic to $\mathrm{id}_Y$ (i.e. homotopic through diffeomorphisms). Then $\deg h_z = \deg \mathrm{id}_Y \implies h_z$ preserves orientation. Then since $Y$ is compact, we can cover $Y$ with finitely

many $h_{z_i}(U)$. Using a partition of unity subordinate to this cover, we reduce to the case where supp $\omega \subset h_{z_i}(U)$. Then

$$\int_X f^*\omega = \int_X (h_{z_i} \circ f)^*\omega$$

SInce $h_{z_i}$ is homotopic to $\mathrm{id}_Y$ and $\omega$ is closed, this gives us that $f$ and $h_{z_i} \circ f$ are homotopic as well. Then we compute using the fact that $h_{z_i}$ is orientation preserving and the proof of the previous case,

$$\begin{aligned}\int_X (h_{z_i} \circ f)^*\omega &= \int_X f^*(h_{z_i}^*\omega) \\ &= (\deg f) \int_Y h_{z_i}^*\omega \\ &= (\deg f) \int_Y \omega\end{aligned}$$

∎

**Definition 11.17.** A differential form $\omega \in \Omega^p(X)$ is **exact** if $\omega = d\eta$ for some $\eta \in \Omega^{p-1}(X)$.