REPRESENTATION THEORY AND TOPOLOGICAL QUANTUM FIELD THEORIES

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1. Bordism and TQFTs

Definition 1.1. Let X and Y be n-dimensional closed manifolds (i.e. compact and without boundary). A ***bordism*** from X to Y is an $n+1$ dimensional manifold M such that the boundary is diffeomorphic to the disjoint union $X \coprod Y$. The ***Bordism category*** Bord_n is the category where the objects are closed n-dimensional manifolds, and the morphisms are bordisms between them.

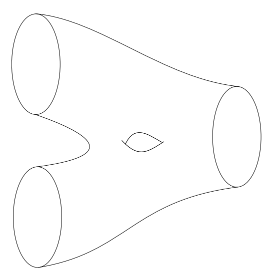

Figure 1. An example of a bordism $S^1 \coprod S^1 \to S^1$.

We make a distinction between the *incoming* and *outgoing* manifolds. In the bordism drawn above, we think of the two circles as the incoming manifolds, and single circle as the outgoing manifold. One way to think of a bordism is the time evolution of the incoming manifold to the outgoing one (think of the figure above extending outwards from the two circles and growing towards the single circle over some fixed amount of time).

The category of bordisms comes with a natural product – the disjoint union $\coprod$, where given two n-manifolds X and Y, we obtain a new n-manifold $X \coprod Y$. This operation is *symmetric*, i.e. there is a natural isomorphism $X \coprod Y \to Y \coprod X$, and gives the set of objects the structure of a commutative monoid. We say that $(\mathsf{Bord}_n, \coprod)$ is a ***symmetric monoidal category***. Another important example of a symmetric monoidal category is the category $\mathsf{Vect}_{\mathbb{F}}$ of finite dimensional $\mathbb{F}$-vector spaces, where the operation is the tensor product $\otimes$.

Definition 1.2. An $n+1$-dimensional ***topological quantum field theory (TQFT)*** is a symmetric monoidal functor $Z : (\mathsf{Bord}_n, \coprod) \to (\mathsf{Vect}_{\mathbb{C}}, \otimes)$, i.e. a functor $\mathsf{Bord}_n \to \mathsf{Vect}_{\mathbb{C}}$ satisfying:

(1) $Z(\varnothing) = \mathbb{C}$
(2) $Z(X \coprod Y) = Z(X) \otimes Z(Y)$

Note that the conditions on Z essentially state that it is a homomorphism of symmetric monoidal categories, hence the name. Note that the empty set is vacuously a manifold of *any* dimension. While this sounds silly at first, it ends up being very important, as it is the unit element under the disjoint union. In addition, we can interpret a closed manifold M as a bordism $\varnothing \to \varnothing$. Applying

Z to M, we get a linear map $\mathbb{C} \to \mathbb{C}$, which is just multiplication by a complex number λ. The number is called the ***partition function*** of M.

While the definition might seem somewhat abstract, it is very grounded in physical motivation. The incoming manifold of a bordism can be thought of as a space or system at an initial time, and the outgoing manifold can be thought of as the end state after undergoing the time evolution specified by the bordism. The functor Z then assigns to the initial and final states a state space. The fact that the state space of a disjoint union is the tensor product of the state spaces also matches the physical model.

2. Dijkgraaf-Witten Theory With Finite Gauge Group

We now define a specific TQFT, which is a toy model developed by Dijkgraaf and Witten. Fix a finite group G. For a fixed manifold M, let $\mathsf{Bun}_G(M)$ denote the category of principal G-bundles over M. Any morphism $P \to Q$ of principal G-bundles is an isomorphism, so this category is a ***groupoid***. For any given groupoid $\mathcal{G}$, we let $\pi_0(\mathcal{G})$ denote the set of isomorphism classes of objects in $\mathcal{G}$. Given a basepoint $x \in M$, we let $\mathsf{Bun}_G(M)$ denote the category of pointed G-bundles over M, which are pairs (P, p) where $P \to M$ is a principal G-bundle, and $p \in P$ is an element of the fiber P_x over x.

Proposition 2.1. *There is a bijective correspondence*

$$\mathsf{Hom}(\pi_1(M, x), G) \longleftrightarrow \mathsf{Bun}_G(M, x)$$

Proof. Given a pointed bundle $(P, p) \to (M, x)$, we obtain a map $\varphi : \pi_1(M, x) \to G$ as follows: For a homotopy class of a loop $\sigma : I \to M$ based at x, we can lift σ to a path $\tilde{\sigma}$ on P starting at p. Then the endpoint $\tilde{\sigma}(1)$ is an element of the fiber P_x, so it can be uniquely written as $p \cdot g$ for some $g \in G$. Then defining $\varphi(\sigma) = g$ gives the desired homomorphism (though one should check that this construction is well defined on homotopy classes) called the ***holonomy*** of the bundle. This map is also called the ***monodromy***. In the other direction, given a homomorphism $\varphi : \pi_1(M, x) \to G$, we want to construct a pointed bundle $(P, p) \to (M, x)$ with holonomy φ. The manifold M admits a universal cover $(\widetilde{M}, \tilde{x})$, which is a principal $\pi_1(M, x)$-bundle over M. The homomorphism φ induces a left action of $\pi_1(M, x)$ on G, so we construct the associated bundle

$$P = \widetilde{M} \times_{\pi_1(M,x)} G$$

where we choose the basepoint of P to be $p = [\tilde{x}, e]$. It is easy to verify that the holonomy of this bundle is φ, and it is also easy to verify that the holonomy of this bundle is indeed φ, giving us the desired bijection. ∎

The group G acts on the category $\mathsf{Bun}_G(M, x)$ by permuting basepoints. Given a pointed bundle $(P, p) \to (M, x)$ and a group element $g \in G$, the action of g on (P, p) is the bundle $(P, p \cdot g)$. In addition, g acts on maps $(P, p) \to (Q, q)$ by precomposition. Taking the quotient by this group action, we obtain the groupoid $\mathsf{Bun}_G(M)$, since quotienting by the group action amounts to forgetting the basepoint. It is also useful to see the group action from the perspective of homomorphisms $\varphi : \pi_1(M, x) \to G$, which we can determine by studying how the holonomy of a bundle changes when we permute its basepoint. Let φ be the holonomy of a bundle $(P, p) \to (M, x)$, and let $\gamma : I \to M$ be a loop based at x. Then let $\widetilde{\gamma}$ denote its lift to a loop based at p. By the uniqueness of path lifting, the lift of γ to the a loop based at $p \cdot g$ is the loop $\widetilde{\gamma}_g(t) = \widetilde{\gamma(t)} \cdot g$. We the compute

$$\begin{aligned} \widetilde{\gamma}_g(1) &= \widetilde{\gamma}(1) \cdot g \\ &= p \cdot \varphi[\gamma] \cdot g \\ &= p \cdot g \cdot g^{-1} \cdot \varphi[\gamma] \cdot g \end{aligned}$$

So the holonomy φ transforms under the group action by conjugation, i.e. the holonomy φ_g of $(P, p \cdot g) \to (M, x)$ is $g^{-1} \cdot \varphi \cdot g$.

We now have the necessary tools to define the TQFT. Given a closed n-manifold M, define $Z(M) = \mathbb{C}_G(M)$, where $\mathbb{C}_G(M)$ denotes the vector space generated by complex valued functions on $\pi_0(\mathsf{Bun}_G(M))$. Then given a bordism X between manifolds M, and N, we need to produce a map $Z(X) : \mathbb{C}_G(M) \to \mathbb{C}_G(N)$. We do so as follows. We have inclusions $M \hookrightarrow X$ and $N \hookrightarrow X$ of M and N into ∂X, giving us the diagram

$$\begin{tikzcd} & X & \\ M \arrow[ur] & & N \arrow[ul] \end{tikzcd}$$

the inclusion maps induce maps $\mathsf{Bun}_G(X) \to \mathsf{Bun}_G(M)$ and $\mathsf{Bun}_G(X) \to \mathsf{Bun}(Y)$ by pullback (also called restriction), where we map a bundle $P \to X$ to the pullback bundle $P|_M$ and $P|_N$ respectively, where we identify M and N with the images under the inclusion into X, giving the diagram

$$\begin{tikzcd} & \mathsf{Bun}_G(X) \arrow[dl] \arrow[dr] & \\ \mathsf{Bun}_G(M) & & \mathsf{Bun}_G(N) \end{tikzcd}$$

Then given a function $f \in \mathbb{C}_G(M)$ on principal bundles over M, we need to produce a function $Z(X)(f) \in \mathbb{C}_G(N)$. To do so, we need to define one more object. Let $Q \to N$ be a principal bundle over N. We then construct a groupoid $\mathcal{G}_Q$, where the objects are pairs (P, φ), where $P \to X$ is a principal bundle, and φ is an isomorphism $P|_N \to Q$, and the morphisms from $(P, \varphi) \to (P', \varphi')$ are bundle morphisms $\psi : P \to P'$ such that the diagram

$$\begin{tikzcd} P|_N \arrow[rr, "\psi"] \arrow[dr, "\varphi"'] & & P'|_N \arrow[dl, "\varphi'"] \\ & Q & \end{tikzcd}$$

commutes, where we pullback the map ψ using the inclusion map $N \hookrightarrow X$. We now construct the linear map $Z(X) : \mathbb{C}_G(M) \to \mathbb{C}_G(N)$. Let $f \in \mathbb{C}_G(M)$, Then define $Z(X)(f) \in \mathbb{C}_G(N)$ by

$$Z(X)(f)(Q) = \sum_{(P,\varphi) \in \pi_0(\mathcal{G}_Q)} \frac{f(P|_M)}{|\operatorname{Aut}(P, \varphi)|}$$

While a bit opaque, you can think of this as a sort of categorified Fourier transform. Given a function over M, we pull it back to a function over X, convolve with some sort of kernel (where we sum instead of integrate because everything is finite), and then push down to N. Such push-pull constructions arise often as a sort of generalized Fourier transform.

3. Dijkgraaf-Witten Theory in Two Dimensions

We now restrict our attention to the 2-dimensional version of the TQFT Z we constructed above. First, we recall some of the details of the classification of $2d$-TQFTs, which are in bijection with ***commutative Frobenius algebras***. It is well known that the only connected closed 1-manifold is the circle S^1, up to diffeomorphism. Therefore, all the objects in the category Bord_n (again, up to diffeomorphism) are just disjoint unions of a finite number of circles. Using the fact that $Z(M \coprod N) = Z(M) \otimes Z(N)$, this means we can interpret the morphisms in Bord_n as operations on $Z(S^1)$, which will form the underlying vector space of our commuative Frobenius algebra. To classify the algebra, it suffice to know the following list of bordisms and the operations they induce on $Z(S^1)$

(1) The ***pair of pants*** P is a bordism $S^1 \coprod S^1 \to S^1$. Applying the functor Z, we get a ***multiplication*** $Z(P) : Z(S^1) \otimes Z(S^1) \to Z(S^1)$.

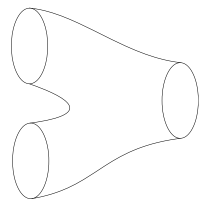

FIGURE 2. A pair of pants

(2) We can flip the pair of pants around to get a bordism $S^1 \to S^1 \coprod S^1$, which then induces a map $Z(S^1) \to Z(S^1) \otimes Z(S^1)$ called ***comultiplication***.

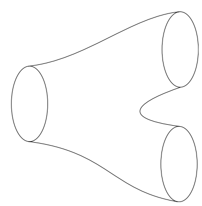

FIGURE 3. The pair of pants, flipped around

(3) The ***cap*** C is a bordism $S^1 \to \varnothing$. Applying the functor Z, we get a linear map $Z(C) : Z(S^1) \to Z(\varnothing) = \mathbb{C}$, called the ***trace***.

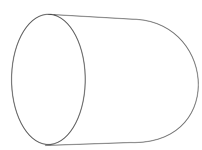

FIGURE 4. A cap

(4) By flipping the cap around, we get a bordism $\varnothing \to \mathbb{S}^1$, called the ***cocap***, which gives us a map $\mathbb{C} \to Z(S^1)$. The identity element then determines the identity of $Z(S^1)$ under the multiplication by the pair of pants.

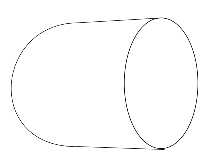

FIGURE 5. A cocap

The operations induced by the bordisms above are essentially the defining ingredients for a Frobenius algebra (along with some relations they satisfy, which we will not prove).