LOCAL COHOMOLOGY

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Definition 1.1. An R-module I is ***injective*** if for every injection $M \hookrightarrow N$, any homomorphism $M \to I$ extends to a homomorphism $N \to I$.

$$\begin{tikzcd} M \arrow[r, hook] \arrow[dr] & N \arrow[d, dashed] \\ & I \end{tikzcd}$$

Equivalently, the functor $\mathsf{Hom}_R(\cdot, I)$ is exact.

Over a PID, injective modules have a relatively nice characterization.

Proposition 1.2. *Let R be a PID and M an R-module. Then M is injective if and only if it is **divisible**, i.e. given a non-zero divisor (nzd) x and $u \in M$, there exists $v \in M$ such that $xv = u$.*

Corollary 1.3.

(1) *$\mathbb{Q}$ is an injective $\mathbb{Z}$-module.*
(2) *Let k be a field. Then $k(x)$ is an injective $k[x]$-module.*

Definition 1.4. Let $I \subset R$ be an ideal, and M an R module. The ***sections of M supported on $V(I)$*** is the submodule

$$\Gamma_I(M) := \{m \in M \ : \ I^n m = 0 \text{ for some } n\}$$

Recall that for a ring R, the sets $V(I) = \{P \in \operatorname{Spec} R \ : \ I \subset P\}$ are the closed sets in the Zariski topology on R, and an R-module can be thought of as a sheaf over $\operatorname{Spec} R$ (e.g. sections of a vector bundle). In this perspective, the elements of M are the global sections, and the elements of $\Gamma_I(M)$ tell us the elements supported on the closed set $V(I)$. Using Γ_I, we can define a functor from $\mathsf{Mod}_R \to \mathsf{Mod}_R$ where $M \mapsto \Gamma_I(M)$, and given $\varphi : M \to N$, $\Gamma_I(\varphi)$ is just φ restricted to the submodule $\Gamma_I(M)$. From this it is clear that Γ_I is left-exact, since it clearly preserves injective maps.

Definition 1.5. The ***local cohomology modules*** $H^i_I(M)$ are obtained from the right derived functors of Γ_I, i.e. $H^i_I(M) = R^i\Gamma_I(M)$.

The idea here is that $H^i_I(M)$ should be like the relative cohomology with respect closed set $V(I)$, we'll explore this analogy more later.

The modules $H^i_I(M)$ can be computed by taking an injective resolution

$$0 \longrightarrow M \longrightarrow I^1 \longrightarrow \cdots$$

and taking the cohomology of the complex

$$0 \longrightarrow \Gamma_I(M) \longrightarrow \Gamma_I(I^1) \longrightarrow \cdots$$

As with other derived functors, we will always have $\Gamma_I(M) = H^0_I(M)$.

Example 1.6. Let $I = (p) \subset \mathbb{Z}$. Then we have the injective resolution

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

So we want to compute the cohomology of the sequence

$$0 \longrightarrow \Gamma_{(p)}(\mathbb{Z}) \longrightarrow \Gamma_{(p)}(\mathbb{Q}) \longrightarrow \Gamma_{(p)}(\mathbb{Q}/\mathbb{Z}) \longrightarrow 0$$

We note that $\Gamma(p)$ simply picks out the submodule of elements of p^n torsion. Therefore, we have that $\Gamma_{(p)}(\mathbb{Z}) = H^0_{(p)}(\mathbb{Z}) = 0$. In addition, since $\mathbb{Q}$ is torsion-free, we also have that $\Gamma_{(p)}(\mathbb{Q}) = 0$. Therefore, we get that $H^1_{(p)}(\mathbb{Z}) = \Gamma_{(p)}(\mathbb{Q}/\mathbb{Z})$. By factoring the numerator and denominator of a rational number into products of prime powers, we get the identification $H^1_{(p)}(\mathbb{Z}) = \mathbb{Z}[p^{-1}]/\mathbb{Z}$.

Example 1.7. Let $R = k[x]$ and $I = (x)$, where k is a field. We have the injective resolution

$$0 \longrightarrow R \longrightarrow k(x) \longrightarrow k(x)/R \longrightarrow 0$$

So we want to compute the cohomology of

$$0 \longrightarrow \Gamma_{(x)}(R) \longrightarrow \Gamma_{(x)}(k(x)) \longrightarrow \Gamma_{(x)}(k(x)/R) \longrightarrow 0$$

Since R and $k(x)$ are both domains, $\Gamma_{(x)}(R) = \Gamma_{(x)}(k(x)) = 0$, so we have that $H^0_{(x)}(R) = 0$ and $H^1_{(x)}(R) = \Gamma_{(x)}(k(x)/R)$. Which is similar to the case with $\mathbb{Q}/\mathbb{Z}$. Factoring the numerators and denominators, into products of irreducibles, we get an identification $H^1_{(x)}(R) = k[x, x^{-1}]/k[x]$.

In general, injective modules are hard to write down, so we want an easier way to compute local cohomology. In the case that R is Noetherian, we can do this using Koszul complexes. For $x \in R$, construct the complex

$$K^\bullet(x, R) := \; 0 \longrightarrow R \longrightarrow R_x \longrightarrow 0$$

where R is in degree 0, and the map $R \to R_x$ is the localization map. Given multiple elements $x_1, \dots x_n \in R$, we let $K^\bullet(x_1, \dots x_n, R) := K^\bullet(x_1, R) \otimes \cdots \otimes K^\bullet(x_n, R)$. We then define $K^\bullet(x_1, \dots x_n, M) := K^\bullet(x_1, \dots x_n, R) \otimes M$. We denote the cohomology of this complex by $H^i(x_1, \dots x_n, M)$.

Proposition 1.8. *Let R be Noetherian, and $I \subset R$ an ideal with $\sqrt{I} = \sqrt{(x_1, \dots x_n)}$. Then*

$$H^i_I(M) = H^i(x_1, \dots x_n, M)$$

This can be interpreted geometrically as follows. The complement of $V(I)$ is covered by the open sets $D(x_i) = \{P \in \operatorname{Spec} R \;:\; x_i \notin P\}$, where $x \in I$. The sections of M over an open set $D(x)$ are given by the localized module $M_x = R_x \otimes_R M$. Then the complex $K^\bullet(x_1, \dots x_n, M)$ is the Čech complex for the open covering $\{D(x_i)\}$. This shows that what we are doing is computing the sheaf cohomology of M restricted to the complement of

$V(I)$. In addition, this perspective sheds light on how $H^i_I(M)$ behaves like relative cohomology. If we look at the first terms in the Koszul complex, we have

$$0 \longrightarrow M \longrightarrow \bigoplus_{x \notin \sqrt{I}} M_x \longrightarrow \bigoplus_{x,y \notin \sqrt{I}} M_{x,y} \longrightarrow \cdots$$

the kernel of $\bigoplus_{x \notin \sqrt{I}} R_x \to \bigoplus_{x,y \notin \sqrt{I}} R_{x,y}$ consists of tuples of elements of M_x that map to the same element in $M_{x,y}$, which should be thought of as sections agreeing on intersections. The image of the first map should be thought of as sections that extend over $V(I)$ to a global section. In this way, we see that this is analogous to relative cohomology with respect to the closed subspace $V(I)$.

The Koszul complex perspective also allows us to prove results about base change.

Proposition 1.9. *Let R be Noetherian, $I \subset R$ an ideal, and M an R-module.*

(1) *Let $\varphi : R \to S$ be ring homomorphism such that S is a flat R-module. Then*

$$H^i_I(M) \otimes_R S \cong H^i_{IS}(M \otimes_R S)$$

In particular, taking local cohomology commutes with localization and completion.

(2) *For any ring homomorphism $\varphi : R \to S$ and an S-module N,*

$$H^i_I(N) \cong H^i_{IS}(N)$$

Proof. The first claim is easy. Let $I = (x_1, \dots x_n)$ We have that

$$K^\bullet(x_1, \dots x_n, M) \otimes_R S = K^\bullet(\varphi(x_1), \dots \varphi(x_n), M \otimes_R S)$$

Then since S is flat, tensoring with S is exact, so it commutes with taking cohomology.

The second claim is also easy. This follows from

$$\begin{aligned} K^\bullet(x_1, \dots x_n, N) &= K^\bullet(x_1, \dots x_n, R) \otimes_R N \\ &= K^\bullet(x_1, \dots x_n, R) \otimes_R S \otimes_S N \\ &= K^\bullet(\varphi(x_1), \dots \varphi(x_n), S) \otimes_R N \\ &= K^\bullet(\varphi(x_1), \dots \varphi(x_n), N) \end{aligned}$$

The first equality comes from the definition of the Koszul complex. The second comes from the definition of the R-module structure, the third comes from the first part, and the last comes from the the definition of the R-module structure as well as the definition of the Koszul complex. ∎

For another geometric perspective, there's a "Mayer-Vietoris"-like sequence in local cohomology. To do this, we need another perspective for local cohomology over Noetherian rings. Given a Noetherian ring R and an R-module M, define the ideal quotients

$$0 :_M I^n := \{m \in M \ : \ I^n m = 0\}$$

Then we have that Γ_I is the direct limit of the $0 :_M I^n$, which it can be identified with the quotient

$$\bigoplus_{n \in \mathbb{Z}^{\geq 0}} (0 :_M I^n) / \sim$$

where we identify $x \in (0 :_M I^n) \in \bigoplus_{n \in \mathbb{Z}^{\geq 0}} (0 :_M I^n)$ with its copies lying in $0 :_M I^m$ for $m < n$. Heuristically, it is best to think of the limit as the "union" $\cup(0 :_M I^n)$. Furthermore, we can make the identification $0 :_M I^n \cong \mathsf{Hom}_R(R/I^n, M)$, where we identify $m \in 0 :_M I^n$ with the homomorphism determined by $1 \mapsto m$. This gives us

$$\Gamma_I(M) = \varinjlim \mathsf{Hom}_R(R/I^n, M)$$

Then if we take an injective resolution

$$0 \longrightarrow M \longrightarrow E^1 \longrightarrow \cdots$$

and apply Γ_I, we get

$$0 \longrightarrow \varinjlim \mathsf{Hom}(R/I^n, M) \longrightarrow \varinjlim \mathsf{Hom}(R/I^n, E^1) \longrightarrow \cdots$$

By abstract nonsense, we can commute the limit taking cohomology, giving us the identification $H^i_I(M) = \varinjlim \mathrm{Ext}^i_R(R/I^n, M)$. In fact, we see that we can do this for a slightly wider class of collections of ideals. Suppose we had some collection $\{J_n\}$ that is ***cofinal*** with respect to the collection $\{I^n\}$, i.e. given I^n, it contains some J_m. Then the limits $\lim_{\longrightarrow} R/I^n$ and $\lim_{\longrightarrow} R/J_n$ coincide, so we can compute local cohomology with the J_n as well.

This interpretation of local cohomology gives us a "Mayer-Vietoris" sequence.

Theorem 1.10. *Let $I, J \subset R$ be ideals in a Noetherian ring R. Then there is a long exact sequence in local cohomology*

$$\begin{tikzcd} 0 \arrow[r] & H^0_{I+J}(M) \arrow[r] & H^0_I(M) \oplus H^0_J(M) \arrow[r] & H^0_{I\cap J}(M) \arrow[dll] \\ & H^1_{I+J}(M) \arrow[r] & H^1_I(M) \oplus H^1_J(M) \arrow[r] & H^1_{I\cap J}(M) \arrow[dll] \\ & \cdots & & \end{tikzcd}$$

Proof. The ideals $\{I^n + J^n\}$ are cofinal with respect to $\{(I+J)^n\}$ and the $\{I^n \cap J^n\}$ are cofinal with $\{(I \cap J)^n\}$. We have a short exact sequence

$$0 \longrightarrow R/(I^n \cap J^n) \longrightarrow R/I^n \oplus R/J^n \longrightarrow R/(I^n + J^n) \longrightarrow 0$$

Applying $\mathsf{Hom}(\cdot, M)$, we get a long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathsf{Hom}(R/(I^n \cap J^n, M) \arrow[r] & \mathsf{Hom}(R/I^n, M)\mathsf{Hom}(R/J^n, M) \arrow[r] & \mathsf{Hom}(R/(I^n + J^n), M) \arrow[dll] \\ & \mathrm{Ext}^1_{\mathrm{R}}(R/(I^n \cap J^n), M) \arrow[r] & \mathrm{Ext}^1_R(R/I^n, M)\mathrm{Ext}^1_R(R/J^n, M) \arrow[r] & \mathrm{Ext}^1_R(R/(I^n + J^n), M) \arrow[dll] \\ & \cdots & & \end{tikzcd}$$

taking the limit then completes the proof. ■

Geometrically, recall

$$V(I+J) = V(I) \cap V(J)$$
$$V(I \cap J) = V(I) \cup V(J)$$

So, just like Mayer-Vietoris for topological spaces, this allows us to compute local cohomology supported in the union $V(I) \cup V(J)$ by knowing the local cohomology on $V(I)$, $V(J)$, and the intersection. In particular, if $I + J = R$, we have that $\Gamma_{I+J}(M) = \Gamma(M)$, so this allows us to express the sheaf cohomology of M in terms of local cohomology.

Definition 1.11. Let M be an R-module. An ***associated prime*** of M is a prime ideal $P \subset R$ of the form $P = \mathrm{Ann}(m)$ for some $m \in M$.

We can recover the old notion of associated prime for an ideal I by taking the associated primes of R/I.

2. Matlis Duality

Definition 2.1. Let N be a submodule of M. M is ***essential*** over N if every nonzero submodule of M intersects N nontrivially.

In some sense, injective modules are "large." On the other hand, essential modules are "small" in the sense that they are close to a submodule.

Definition 2.2. A maximal essential extension of an R-module M is the same as a minimal injective module containing M. In fact, any essential extension that is injective is maximal. Such a module is called the ***injective hull*** of M, and we denote the injective hull of an R-module M by $E_R(M)$.

Theorem 2.3 (***Matlis***)**.** *Let R be Noetherian.*

(1) *An R-module E is an indecomposable injective if and only if E is of the injective hull $E_R(R/P)$ for some $P \in \operatorname{Spec} R$.*
(2) *Every finitely generated submodule of $E_R(R/P)$ has P as the unique associated prime.*
(3) *Every injective module is a direct sum of indecomposable injective modules.*

From now on, we will assume that (R, m) is a Noetherian local ring. We let k denote the residue field R/m, and we let E denote the injective hull of k, thought of as an R-module.

Definition 2.4. Let M be an R-module. The ***Matlis dual*** of M, denoted $M^\vee$, is the module $M^\vee := \mathsf{Hom}_R(M, E)$.

Proposition 2.5. *Let M and N be R modules. Then*

$$\mathrm{Tor}_i^R(M, N)^\vee \cong \mathrm{Ext}_R^i(M, N^\vee)$$

Proof. Let

$$0 \longrightarrow M \longrightarrow F^1 \longrightarrow \cdots$$

be a free resolution of M. Then $\mathrm{Tor}_i^R(M, N)$ is computed as the homology of the complex

$$0 \longrightarrow M \otimes_R N \longrightarrow F^1 \otimes_R N \longrightarrow \cdots$$

Taking Matlis duals of each term, which commutes with taking homology since E is injective, so $\mathrm{Tor}_i^R(M,N)^\vee$ is the homology of the complex

$$0 \longrightarrow \mathsf{Hom}_R(M \otimes_R N, E) \longrightarrow \mathsf{Hom}_R(F^1 \otimes_R N, E) \longrightarrow \cdots$$

Applying Tensor-Hom adjunction, this becomes

$$0 \longrightarrow \mathsf{Hom}_R(M, \mathsf{Hom}_R(N,E)) \longrightarrow \mathsf{Hom}_R(F^1, \mathsf{Hom}_R(N,E)) \longrightarrow \cdots$$

And the homology of this sequence is exactly $\mathrm{Ext}_R^i(M, N^\vee)$. ■

If we assume that M is finitely generated, we get a similar dual statement where

$$\mathrm{Ext}_R^i(M,N)^\vee \cong \mathrm{Tor}_i^R(M, N^\vee)$$

Theorem 2.6 (***Matlis duality***)**.** *Let $\hat{R} = \varprojlim R/m^n$ be the completion of R with respect to the maximal ideal m. Then we have*

(1) *Any Artinian R-module is isomorphic to a submodule of E^r for some r.*
(2) *Taking Matlis duals gives a bijective correspondence*

$$\{\textit{finitely generated } \hat{R}\textit{-modules}\} \longleftrightarrow \{\textit{Artinian } R\textit{-modules}\}$$

In addition, $(N^\vee)^\vee = N$.