# THE FRÖLICHER SPECTRAL SEQUENCE AND THE $\partial\bar{\partial}$ LEMMA

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For any complex manifold $X$, we have the Frölicher spectral sequence, which computes the de Rham cohomology of a complex manifold in terms of the $\partial$ and $\bar{\partial}$ cohomology. On the $E_0$ page, it is given by the Dolbeault cohomology of the bundles $\Omega^{p,0}$, i.e. the item in the $(p,q)$ position is the bundle $\Omega^{p,q}$ of $(p,q)$ forms, and the differential on $E_0$ is just $\bar{\partial}$. For example, a small section of the $E_0$ page would be:

$$
\begin{array}{c|cccc}
2 & \Omega^{0,2} & \Omega^{1,2} & \Omega^{2,2} & \Omega^{3,2} \\[2pt]
& \bar{\partial}\uparrow & \bar{\partial}\uparrow & \bar{\partial}\uparrow & \bar{\partial}\uparrow \\[2pt]
1 & \Omega^{0,1} & \Omega^{1,1} & \Omega^{2,1} & \Omega^{3,1} \\[2pt]
& \bar{\partial}\uparrow & \bar{\partial}\uparrow & \bar{\partial}\uparrow & \bar{\partial}\uparrow \\[2pt]
0 & \Omega^{0,0} & \Omega^{1,0} & \Omega^{2,0} & \Omega^{3,0} \\[2pt]
\hline
& 0 & 1 & 2 & 3
\end{array}
$$

Then in the $E_1$ page, we have $E_1^{p,q}$ is the cohomology in the $(p,q)$ slot in the $E_0$ page, which is just $H^{p,q}(X)$ (abbreviated to $H^{p,q}$) by Hodge theory in the compact case. The differentials going from left to right are the operators $\partial$, which descends to cohomology because $\partial\bar{\partial} = -\bar{\partial}\partial$. A small section of the $E_1$ page would be :

$$
\begin{array}{c|cccc}
2 & H^{0,2} \xrightarrow{\partial} & H^{1,2} \xrightarrow{\partial} & H^{2,2} \xrightarrow{\partial} & H^{3,2} \\[12pt]
1 & H^{0,1} \xrightarrow{\partial} & H^{1,1} \xrightarrow{\partial} & H^{2,1} \xrightarrow{\partial} & H^{3,1} \\[12pt]
0 & H^{0,0} \xrightarrow{\partial} & H^{1,0} \xrightarrow{\partial} & H^{2,0} \xrightarrow{\partial} & H^{3,0} \\[2pt]
\hline
& 0 & 1 & 2 & 3
\end{array}
$$

For a general compact complex manifold $X$, this continues to the $E_2$ page, where the differential "rotates," and the $(p,q)$ slot is the cohomology in the $(p,q)$ slot of the $E_1$ page. The big theorem we want to prove is:

**Theorem 1.1.** *For a compact Kähler manifold X, the Frölicher spectral sequence degenerates at the $E_1$ page, i.e. all the differentials are $0$.*

In other words, we can terminate our spectral sequence computations at $E_1$. Going to the $E_1$ page is easy, since all the computations are done with the operators $\partial$ and $\bar{\partial}$. In practice, continuing on to further pages is difficult. The entire spectral sequence story seems very difficult (and it is), but in the compact Kähler story, it reduces to a simple lemma.

**Theorem 1.2** (*The $\partial\bar{\partial}$ lemma*)**.** *Let X be a Kähler manifold, and $\eta$ a complex k-form that is $\partial$ and $\bar{\partial}$-closed. Then if $\eta$ is d, $\partial$, or $\bar{\partial}$-exact, there exists a form $\xi$ such that $\eta = \partial\bar{\partial}\xi$.*

The proof of this lemma requires the following results from Hodge Theory:

**Theorem 1.3** (*Comparison of the Laplacians*)**.** *Let X be a compact Kähler manifold. Then*

$$\Delta = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$$

*where $\Delta$, $\Delta_\partial$, and $\Delta_{\bar{\partial}}$ are the Laplacians*

$$\Delta = dd^* + d^*d$$
$$\Delta_\partial = \partial\partial^* + \partial^*\partial$$
$$\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$$

The proof of this theorem requires certain commutation relations to hold, called the *Kähler identities*. This identity is not true in general for an arbitrary compact complex manifold. The other result we need is

**Theorem 1.4** (*The Hodge Decomposition*)**.** *Any complex valued form $\alpha \in \Omega^{p,q}$ can be written as*

$$\alpha = \beta + \Delta\gamma$$

*where $\beta$ is harmonic, i.e. $\Delta\beta = 0$.*

This theorem is true for general compact complex manifolds, not just Kähler manifolds.

*Proof of the $\partial\bar{\partial}$ lemma.* The proof for all three cases is much the same, so we do just the one where $\eta = \bar{\partial}\alpha$ is $\bar{\partial}$-exact. By the Hodge decomposition, we write $\alpha = \beta + \Delta\gamma$, with $\beta$ harmonic. Since $\Delta = 2\Delta_{\bar{\partial}}$, and $\beta$ is $\Delta_{\bar{\partial}}$-harmonic if and only if $\bar{\partial}\beta = \bar{\partial}^*\beta = 0$, we have that $\bar{\partial}\beta = 0$. Then using Lemma 7.8, we compute

$$\begin{aligned}
\eta &= \bar{\partial}\alpha \\
&= \bar{\partial}(\beta + \Delta\gamma) \\
&= \bar{\partial}\beta + 2\bar{\partial}(\Delta_\partial\gamma) \\
&= 0 + 2\bar{\partial}(\partial\partial^*\gamma + \partial^*\partial\gamma) \\
&= 2\bar{\partial}\partial\partial^*\gamma - 2\partial^*\bar{\partial}\partial\gamma \\
&= -2\partial\bar{\partial}\partial^*\gamma - 2\partial^*\bar{\partial}\partial\gamma
\end{aligned}$$

Then since $\eta$ is $\partial$-closed, we have that $\partial^*\bar{\partial}\partial\gamma$ must also be $\partial$-closed. By orthogonality of the image of $\partial^*$ with the kernel of $\partial$, we have that $\partial^*\bar{\partial}\partial\gamma = 0$, so $\eta = -2\partial\bar{\partial}\partial^*\gamma = 2\bar{\partial}\partial\partial^*\gamma$, so letting $\xi = \partial^*\gamma$, we are done. $\blacksquare$

We now use this to prove Theorem 1.1.

*Proof of 1.1.* We want to show that all the differentials on the $E_1$ page are 0, i.e. for a co-homology class $[\alpha] \in H^{p,q}$, $[\partial\alpha] = 0$. Since $[\alpha]$ is a Dolbeault cohomology class, we know that $\alpha$ is $\bar\partial$-closed. Therefore, $\partial\alpha$ is both $\bar\partial$ and $\partial$ closed, since $\partial$ and $\bar\partial$ anticommute. Then by the $\partial\bar\partial$ lemma, we have that $\partial\alpha = \partial\bar\partial\eta$ for some $\eta$. Therefore, using the fact that $\partial$ and $\bar\partial$ commute one final time, we find that $\partial\alpha$ is $\bar\partial$-exact, i.e. $[\partial\alpha] = 0$ ∎

The idea of the Frölicher spectral sequence is that we can compute the cohomology of $d = \partial + \bar\partial$ in terms of the cohomology of $\partial$ and $\bar\partial$ themselves. However, for general complex manifolds, this is hard, since the differentials and cohomology beyond the $E_1$ page become much more complicated than the ones on the $E_0$ and $E_1$ pages, which are all objects we are familiar with, and are relatively easy to compute with. What Theorem 1.1 tells us is that in the case that $X$ is compact Kähler, this is enough.