SHEAF COHOMOLOGY, LINE BUNDLES, AND DIVISORS

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Contents

1. Čech Cohomology

For the most part, we will consider "sufficiently nice" topological spaces. For the most part, think of a space X as a (smooth, complex) manifold, an analytic space, or a quasi-compact separated scheme if you're feeling adventurous.

Definition 1.1. Let X be a space and $\mathcal{F}$ a sheaf of abelian groups over X. Let $\mathcal{U} = \{U_i\}_{i\in\mathbb{N}}$ be a countable open cover of X that is locally finite, i.e. for any $x \in X$, only finitely many U_i contain x. For $I = \{i_1, \ldots, i_k\}$, let

$$U_I := \bigcap_{i\in I} U_i$$

Then define the ***Čech cochain groups*** of $\mathcal{F}$ for the cover $\mathcal{U}$ by

$$C^k(\mathcal{U}, \mathcal{F}) := \prod_{|I|=k+1} \mathcal{F}(U_I)$$

an element of $C^k(\mathcal{U}, \mathcal{F})$ is called a ***Čech cochain***. For a k-cochain σ, and $I = \{i_0, \ldots i_k\}$, we denote the component of σ over U_I as σ_I or $\sigma_{i_0,\ldots i_k}$.

The Čech cochain groups are equipped with a differential $d : C^k(\mathcal{U}, \mathcal{F}) \to C^{k+1}(\mathcal{U}, \mathcal{F})$ where for $\sigma \in C^k(\mathcal{U}, \mathcal{F})$, the $i_0, \dots i_{k+1}$ component of $d\sigma$ is given by

$$(d\sigma)_{i_0,\dots i_{k+1}} = \sum_{j=1}^{p+1} (-1)^j \sigma_{i_0,\dots,\widehat{i_j},\dots,i_{k+1}}|_{U_0\cap\cdots\cap U_{k+1}}$$

where $\widehat{i_j}$ denotes that i_j is missing. We have that $d^2 = 0$ for a similar reason that $d^2 = 0$ for singular cohomology, you get repeats of terms with opposite signs due to the omitted index. We denote the kernel of $d : C^i(\mathcal{U}, \mathcal{F}) \to C^{i+1}(\mathcal{U}, \mathcal{F})$ as $Z^i(\mathcal{U}, \mathcal{F})$, and we say that the element are i-cocycles. We denote the image of $d : C^{i-1}(\mathcal{U}, \mathcal{F}) \to C^i(\mathcal{U}, \mathcal{F})$ as $B^i(\mathcal{U}, \mathcal{F})$, and we call the elements i-coboundaries.

Definition 1.2. The ***Čech cohomology groups*** of $\mathcal{F}$ with respect to the cover $\mathcal{U}$, denoted $\check{H}^i(\mathcal{U}, \mathcal{F})$ is the cohomology of the Čech complex

$$0 \xrightarrow{d} C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{d} \cdots$$

i.e. we have

$$\check{H}^i(\mathcal{U}, \mathcal{F}) := \frac{Z^i(\mathcal{U}, \mathcal{F})}{B^i(\mathcal{U}, \mathcal{F})}$$

Definition 1.3. Given an open cover $\mathcal{U} = \{U_i\}$, a ***refinement*** of $\mathcal{U}$ is another open cover $\mathcal{V} = \{V_j\}$ such that every V_j is contained in some U_i. If $\mathcal{V}$ is a refinement of $\mathcal{U}$, we write $\mathcal{V} < \mathcal{U}$.

If $\mathcal{V} < \mathcal{U}$, then we know we can find a map $\varphi : \mathbb{N} \to \mathbb{N}$ such that $V_i \subset U_\varphi(i)$. Consequently, we can restrict sections over U_i to sections over $V_{\varphi(i)}$, so this induces a chain map $\rho_\varphi : C^k(\mathcal{U}, \mathcal{F}) \to C^k(\mathcal{V}, \mathcal{F})$ where

$$(\rho_\varphi(\sigma))_{i_0,\dots,i_k} = \sigma_{\varphi(i_0),\dots\varphi(i_k)}|_{U_{i_0}\cap\cdots\cap U_{i_k}}$$

This map commutes with the differentials for $C^\bullet(\mathcal{U}, \mathcal{F})$ and $C^\bullet(\mathcal{V}, \mathcal{F})$, so it descends to homomorphisms $\check{H}^i(\mathcal{U}, \mathcal{F}) \to \check{H}^i(\mathcal{V}, \mathcal{F})$. It can be shown that a different choice of chain map ρ_ψ for $\psi : \mathbb{N} \to \mathbb{N}$ is chain homotopic to ρ_φ, so the induced maps on cohomology are independent of our choice of φ.

Definition 1.4. The ***Čech cohomology groups*** of a sheaf $\mathcal{F}$ over X is the limit over refinements

$$\check{H}^i(X, \mathcal{F}) := \lim_{\mathcal{V}<\mathcal{U}} \check{H}^i(\mathcal{V}, \mathcal{F})$$

i.e. the quotient disjoint union $\amalg_{\mathcal{U}} \check{H}^i(\mathcal{U}, \mathcal{F}$ over all refinements, where we identify $\sigma \in \check{H}^i(\mathcal{U}, \mathcal{F}$ and $\tau \in \check{H}^i(\mathcal{V}, \mathcal{F})$ if $\mathcal{V}$ refines $\mathcal{U}$ and $\rho(\sigma) = \tau$ under the map induced on cohomology by the refinement.

This definition of the Čech cohomology groups is essentially useless for computation. It's true power comes from the following theorem, due to Leray.

Theorem 1.5 (***Leray***)**.** *Suppose $\mathcal{U} = \{U_i\}$ is an open cover of X that is* ***acyclic*** *with respect to the sheaf $\mathcal{F}$, i.e. for $|I| > 1$ and any i,*

$$\check{H}^i(U_I, \mathcal{F}) = 0$$

Then

$$\check{H}^i(\mathcal{U},\mathcal{F}) = \check{H}^i(X,\mathcal{F})$$

The intuition to keep in mind for an acyclic cover is the notion of a good cover in differential geometry. On a smooth manifold M, there exists a covering of M by open sets $\{U_i\}$ such that any nonempty intersections are contractible, which is done by taking geodesic balls around each point in M. Since the homotopical information on each U_i is trivial, the only nontrivial topological information in the cohomology of M comes from how the sets are glued together to form M. As with exact sequences of chain complexes, short exact sequences of sheaves give long exact sequences in sheaf cohomology.

Theorem 1.6. *Let*

$$0 \longrightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \longrightarrow 0$$

be an exact sequence of sheaves over X. Then this induces a long exact sequence in cohomology:

$$\begin{tikzcd}
0 \arrow[r] & \check{H}^0(X,\mathcal{E}) \arrow[r, "\alpha^*"] & \check{H}^0(X,\mathcal{F}) \arrow[r, "\beta^*"] & \check{H}^0(X,\mathcal{G}) \arrow[dll, "\delta"'] \\
& \check{H}^1(X,\mathcal{E}) \arrow[r, "\alpha^*"] & \check{H}^1(X,\mathcal{F}) \arrow[r, "\beta^*"] & \check{H}^1(X,\mathcal{G}) \arrow[dlll, "\delta"'] \\
 & & \cdots & \cdots
\end{tikzcd}$$

Proof. We first define the maps $\alpha^* : \check{H}^i(X,\mathcal{E}) \to \check{H}^i(X,\mathcal{F})$ and $\beta^* : H^i(X,\mathcal{F}) \to \check{H}^i(X,\mathcal{G})$, and will then define the connecting homomorphism $\delta : \check{H}^i(X,\mathcal{E}) \to \check{H}^{i+1}(X,\mathcal{G})$. Given an open cover $\mathcal{U} = \{U_i\}$ of X, the sheaf morphism α gives for each open set U_i a homomorphism $\alpha(U_i) : \mathcal{E}(U_i) \to \mathcal{F}(U_i)$, which induces a chain map $C^\bullet(\mathcal{U},\mathcal{F}) \to C^\bullet(\mathcal{U},\mathcal{G}))$. Since the maps $\alpha(U_i)$ commute with restriction maps, this chain map commutes with the differentials, so it descends to a map on cohomology $\check{H}^\bullet(\mathcal{U},\mathcal{F}) \to \check{H}^\bullet(\mathcal{U},\mathcal{F})$, which, after taking the limit over refinements or choosing $\mathcal{U}$ to be simultaneously acyclic for $\mathcal{E}$ and $\mathcal{F}$, gives us the induced map $\alpha^* : \check{H}^\bullet(X,\mathcal{F}) \to \check{H}^\bullet(X,\mathcal{F})$. The map β^* is defined similarly.

The construction of the connecting homomorphism mirrors the construction for singular (or de Rham) cohomology. We represent an element of $\check{H}^i(X,\mathcal{G})$ with a cocycle $\sigma \in Z^i(\mathcal{U},\mathcal{G})$ with respect to some open cover $\mathcal{U}$. By surjectivity of β, by potentially passing to a refinement, we can write $\sigma = \beta(\tau)$ for some $\tau \in C^i(\mathcal{U},\mathcal{F})$. Then since the induced map on chains commutes with the differentials, we have that $d\beta(\tau) = \beta(d\tau) = 0$, since τ is a cocycle. Therefore, $d\tau$ is in the kernel of β^*, so we can write $d\tau = \alpha(\eta)$ for some $\eta \in C^{i+1}(\mathcal{U},\mathcal{F})$. We then define $\delta(\sigma)$ to be the class of η in the limit. We note that this is independent of our choice of τ, since any other choice of preimage of σ differs by an element of the form $\alpha(e)$ for some cocycle e by exactness. Then since α commutes with the differentials, we get $d(\tau + \alpha(e)) = d\tau + d\alpha(e) = d\tau + \alpha(de) = d\tau$. ∎

We make one observation about Čech cohomology

Proposition 1.7. *The 0^{th} Čech cohomology group is isomorphic to the space of global sections, i.e.*

$$\check{H}^0(\mathcal{U},\mathcal{F}) \cong \Gamma(X,\mathcal{F})$$

Proof. The 0^{th} cohomology is just the kernel of the map $d : C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})$. For a 0-cochain σ, we have that

$$(d\sigma)_{ij} = \sigma_j|_{U_i \cap U_j} - \sigma_i|_{U_i \cap U_j}$$

We then claim that the map $\Gamma(X, \mathcal{F}) \to \ker d$ sending a section σ to the cocycle $\tilde{\sigma}$ defined by

$$\tilde{\sigma}_i = \sigma|_{U_i}$$

is bijective. It is surjective, since any 0-cocycle contained in the kernel is a collection of local sections that agrees on intersections, which is exactly a global section of $\mathcal{F}$. In addition, it is injective, since if a section restricts to 0 on every open set, it is the zero section. ■

2. Čech Cohomology on $\mathbb{CP}^n$

We now compute Čech cohomology for various sheaves over $\mathbb{CP}^n$. The main objects of interest are the line bundles $\mathcal{O}(k)$. We have that $\mathbb{CP}^n$ admits a nice cover $\mathcal{U} = \{U_i\}$ where U_i is the open set where the coordinate z_i does not vanish. We will soon show that this covering is acyclic for the structure sheaf $\mathcal{O}_{\mathbb{CP}^n}$, and since all line bundles are trivial on the open sets in this cover, this means that the covering will acyclic for any line bundle $\mathcal{O}(k)$. So it suffices to compute Čech cohomology with respect to this cover. In particular, we note that for any line bundle $\mathcal{O}(k) \to \mathbb{CP}^n$, the transition functions ψ_{ij} for $\mathcal{O}(k)$ are $\psi_{ij}(\ell) = (z_j/z_i)^k$. We already know one cohomology group:

Theorem 2.1. *The 0^{th} cohomology group $\check{H}^0(\mathbb{CP}^n, \mathcal{O}(k))$ is isomorphic to the space $\mathbb{C}[x_0, \ldots x_n]_k$ of homogeneous degree k polynomials in the variables $x_0, \ldots, x_n$.*

We compute the rest of the cohomology now, which we do in stages.

Theorem 2.2. *For any $i > n$, we have $\check{H}^n(\mathcal{U}, \mathcal{O}(k)) = 0$.*

Proof. The cover of $\mathbb{CP}^i$ by the U_i has cardinality $n + 1$. Therefore, $C^{n+1}(\mathcal{U}, \mathcal{O}(k)) = 0$. ■

To compute the rest of the cohomology groups, we first prove a lemma characterizing local sections of $\mathcal{O}(k)$.

Lemma 2.3. *Let $\pi : \mathbb{C}^{n-1} - \{0\} \to \mathbb{CP}^n$ be the usual projection sending $z \in C^{n-1}$ to* span $\{z\}$. *Then the space of sections $\mathcal{O}(k)(U)$ is isomorphic to the space of homogeneous of degree k holomorphic functions $f : \pi^{-1}(U) \to \mathbb{C}$ i.e.*

$$f(tz_0, \ldots, tz_n) = t^k f(z_0, \ldots, z_n)$$

Proof. Let $\sigma \in \mathcal{O}(k)(U)$ be a section. Then the set $\{U \cap U_i\}$ is an open cover of U, so σ is determined by its restrictions $\sigma_i := \sigma_i|_{U \cap U_i}$. Since the bundle $\mathcal{O}(d)$ is trivial over the U_i, the local sections σ_i can be identified with holomorphic functions $U \cap U_i \to \mathbb{C}$ with the compatibility condition

$$\sigma_i([z_0 : \ldots : z_n]) = \left(\frac{z_j}{z_i}\right)^k \sigma_j([z_0 : \ldots : z_n])$$

We then give maps in both directions. Given a section $\sigma \in \mathcal{O}(k)(U)$, define the function f_σ by

$$f_\sigma(z_0, \ldots z_n) = z_i^k \sigma_i(\pi(z_0, \ldots z_n))$$

We must verify that this is well-defined, i.e. it is independent of our choice of i. We compute

$$\begin{aligned} f_\sigma(z_0, \dots z_n) &= z_i^k \sigma_i(\pi(z_0, \dots z_n)) \\ &= \left(\frac{z_j}{z_i}\right)^k z_i^k \sigma_j(\pi(z_0, \dots z_n)) \\ &= z_j^k \sigma_k(\pi(z_0, \dots z_n)) \end{aligned}$$

So this determines a well defined function on $\pi^{-1}(U)$. In addition, it is visibly homogeneous of degree k, since the σ_i are constant on lines and z_i^k is homogeneous of degree k. To show this is an isomorphism, we provide an inverse. Given a homogeneous function f of degree k on $\pi^{-1}(U)$, define the section σ_f locally by

$$(\sigma_f)_i([z_0 : \dots : z_n]) = \frac{f(z_0, \dots z_n)}{z_i^k}$$

then to show that this defines a section, we must show that they agree on intersections using the transition functions. We compute

$$\left(\frac{z_i}{z_j}\right)^k (\sigma_f)_i|_{U \cap U_i \cap U_j}([z_0 : \dots : z_n]) = \left(\frac{z_i}{z_j}\right)^k \frac{f(z_0, \dots, z_n)}{z_i^k} = \frac{f(z_0, \dots z_n)}{z_j^k} = (\sigma_f)_j$$

The two mappings provided are visibly inverses, since one is essentially multiplication by z_j^k and the other is essentially division by z_j^k. ■

Over intersections of the distinguished open sets U_i, the sections have a particularly nice form. Under the projection $\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{CP}^n$, the preimage of U_I for $I = \{i_0, \dots i_d\}$ is just $\mathbb{C}^{n+1}$ minus the coordinate axes $z_{i_j} = 0$. By taking power series, a holomorphic function on $\pi^{-1}(U_I)$ is given by Laurent series where the z_{i_j} can appear in negative degree. Being homogeneous of degree d implies that all the terms in the series expansion must be homogeneous of degree k, where the degree of $(z_k)^a/(z_{i_j})^b$ is $a - b$. Consequently, all such holomorphic functions must be polynomials in $\mathbb{C}[z_0, \dots z_n, z_{i_0}^{-1}, \dots z_{i_d}^{-1}]$ of degree k.

3. Čech Cohomology and Line Bundles

Let $\mathcal{U} = \{U_i\}$ be a covering of X, and $\mathcal{F}$ a sheaf of abelian groups over X. Then with respect to this cover, a Čech 2-cocycle $\sigma \in Z^2(\mathcal{U}, \mathcal{F})$ is defined by the equation

$$0 = (d\sigma)_{ijk} = \sigma_{jk}|_{U_i \cap U_j \cap U_k} - \sigma_{ik}|_{U_i \cap U_j \cap U_k} + \sigma_{ij}|_{U_i \cap U_j \cap U_k}$$

Written multiplicatively (and omitting the restriction), this becomes

$$1 = \sigma_{jk}\sigma_{ik}^{-1}\sigma_{ij}$$

which, since the group is abelian, is equivalent to

$$\sigma_{ik} = \sigma_{ij}\sigma_{jk}$$

which looks exactly like a cocycle condition for transition functions of a line bundle. Recall that given a holomorphic line bundle $\pi : L \to X$, we have local trivializations – we can find a cover $\mathcal{U} = \{U_i\}$ with maps $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$ such that

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, "\varphi_i"] \arrow[dr] & & U_i \times \mathbb{C} \arrow[dl] \\ & U_i & \end{tikzcd}$$

commutes, where the maps to U_i are the projections. Therefore, if we consider the map $\varphi_i \circ \varphi_k^{-1} : U_i \cap U_j \times \mathbb{C} \to U_i \cap U_j \times \mathbb{C}$, we have that $\varphi(x, \lambda) = (x, \psi_{ij}(x)(\lambda))$, where the functions $\psi_{ij} : U_i \cap U_j \to \mathrm{GL}_1\mathbb{C}$ are holomorphic. The ψ_{ij} are called the ***transition functions*** of the line bundle L.

Proposition 3.1. *The transition functions ψ_{ij} satisfy the following conditions*

(1) $\psi_{ij}\psi_{ji} = 1$ (*i.e. the constant function* $x \mapsto 1$)
(2) $\psi_{ij}\psi_{jk} = \psi_{ik}$.

The second condition is often called the ***cocycle condition****, in reference to the identity we derived above for the defining property of a Čech cocycle.*

Proof. Consider the map $\varphi_i \circ \varphi_j^{-1} \circ \varphi_j \circ \varphi_i^{-1} = \mathrm{id}$. We compute under the action on a general element (x, λ)

$$(x, \lambda) \xrightarrow{\varphi_j \circ \varphi_i^{-1}} (x, \psi_{ji}(x)(\lambda)) \xrightarrow{\varphi_i \circ \varphi_j^{-1}} (x, \psi_{ij}(x)\psi_{ji}(x)(\lambda))$$

Therefore, we have that $\psi_{ij}(x)\psi_{ji}(x) = 1$ for all x, showing the first property. For the second property, we do the same thing. Consider the function $\varphi_i \circ \varphi_j^{-1} \circ \varphi_j \circ \varphi_k = \varphi_i \circ \varphi_k^{-1}$. Then fore (x, λ), we compute the action of this function to be

$$(x, \lambda) \xrightarrow{\varphi_j \circ \varphi_k^{-1}} (x, \psi_{jk}(x)(\lambda)) \xrightarrow{\varphi_i \circ \varphi_j^{-1}} (x, \psi_{ij}(x)\psi_{jk}(x)(\lambda))$$

So we get that $\psi_{ij}(x)\psi_{jk}(x) = \psi_{ik}(x)$ for all x. ■

Under the Čech differential, the image of a Čech 0-cochain σ is given by

$$(d\sigma)_{ij} = \sigma_j - \sigma_i$$

written multiplicatively, this becomes

$$(d\sigma)_{ij} = \sigma_j \sigma_i^{-1}$$

In the same spirit, we translate this to a statement regarding transition functions of a line bundle.

Proposition 3.2. *Let $\pi L \to X$ be a holomorphic line bundle where the transition functions ψ_{ij} with respect to a cover $\{U_i\}$ satisfy the* ***coboundary condition****, i.e. there exist holomorphic functions $\sigma_i : U_i \to \mathrm{GL}_1\mathbb{C}$ such that*

$$\psi_{ij} = \sigma_j \sigma_i^{-1}$$

Then L is a trivial line bundle.

Proof. It suffices to provide a nonvanishing section $X \to L$. A section $s : X \to L$ is equivalent to functions $s_i : U_i \to \mathbb{C}$ with the compatibility condition

$$s_i = \psi_{ij} s_j$$

define the s_i by $s_i = \sigma_i^{-1}$. Then they satisfy the compatibility condition, since

$$\psi_{ij}\sigma_j = \sigma_j \sigma_i^{-1}\sigma_j^{-1} = \sigma_i^{-1} = s_i$$

then since the σ_i are functions to $\mathrm{GL}_1\mathbb{C} = \mathbb{C}^\times$, they glue to a global nonvanishing section, so L is isomorphic to the trivial line bundle $X \times \mathbb{C}$. ■

Recall that isomorphism classes of line bundles over X form a group under tensor product, where the inverse of a line bundle L is the dual bundle L^*. Given line bundles $L, L' \to X$ and an open cover $\mathcal{U} = \{U_i\}$ of X in which both L and L' are trivialized over the U_i (for instance, a good cover of X), let ψ_{ij} be the transition functions for L and let φ_{ij} be the transition functions for L'. Then the transition functions for $L \otimes L'$ are $\psi_{ij}\varphi_{ij}$, and the transition functions for L^* are given by φ_{ij}^{-1}.

Theorem 3.3. *Let X be a complex manifold, and $\mathcal{O}_X$ its sheaf of holomorphic functions. Then let $\mathcal{O}_X^\times$ be the sheaf of invertible functions, which is a sheaf of abelian groups under multiplication. Then we have a group isomorphism*

$$\check{H}^1(X, \mathcal{O}_X^\times) \cong \mathrm{Pic}(X)$$

Proof. Fix a good cover $\mathcal{U} = \{U_i\}$ for X. Since all the sets and their nonempty intersections are contractible, we have that $\check{H}^i(U_i, \mathcal{F}) = 0$ for all $i > 0$ where $\mathcal{F}$ is the sheaf of sections of any line bundle. Since all the U_i are contractible, we also have that any line bundle over U_i is trivial, so it admits transition functions ψ_{ij} with respect to this cover. As shown above, the functions ψ_{ij} exactly define a Čech 1-cocycle, and any Čech coboundary defines a trivial bundle. In addition, we have that the the transition functions of a tensor product are exactly the products of the transition functions. Putting everything together, this tells us that the mapping $L \mapsto \{\psi_{ij}\}$ sending a line bundle to the cocycles determined by its transition functions is a bijective group homomorphism. ■

4. Divisors

Given a complex manifold X, a complex hypersurface $Y \subset X$ (also called an analytic hypersurface) is locally cut out a single holomorphic function, i.e. there is an open cover $\{U_i\}$ of X such that $Y \cap U_i$ is the vanishing locus of a single holomorphic function g_i. We say that the functions g_i are ***defining functions*** for Y. The hypersurface Y is said to be ***irreducible*** if at any $p \in Y$, the defining function g for Y in a neighborhood about p is irreducible in the local ring $\mathcal{O}_{X,p}$. If the defining function for a hypersurfaces is not irreducible, then it can be written as a product of irreducible functions g_i in $\mathcal{O}_{X,p}$, in which case we can write Y as the union of the irreducible hypersurfaces determined by the g_i.

On the complex line $\mathbb{C}$, the irreducible hypersurfaces are just points $p \in \mathbb{C}$, which have defining functions $f(z) = (z - p)$. Recall that all meromorphic functions on $\mathbb{C}$ are of the form $p(z)/q(z)$ for polynomials $p(z), q(z) \in \mathbb{C}[z]$. A great deal of the study of meromorphic functions on $\mathbb{C}$ revolves around studying their zeroes and poles, which is something that divisors will encapsulate for higher dimensional complex manifolds.

Definition 4.1. Let $Y \subset X$ be an irreducible hypersurface. Let $f \in \mathcal{K}(X)$ be a meromorphic function, and fix a point $y \in Y$. Then Y is locally cut out by an irreducible holomorphic function $g \in \mathcal{O}_{X,y}$. The ***order*** of f at y, denoted $\mathrm{ord}_{Y,y}(f)$ is the smallest integer n such that there exists an invertible holomorphic function $h \in \mathcal{O}_{X,y}$ such that $f = g^n h$.

We note that this definition is independent of our choice of defining function, since any two irreducible defining functions for Y differ by a unit in $\mathcal{O}_{X,y}$. In addition, if Y is irreducible, it is also independent of our choice of point $y \in Y$, since if a holomorphic function g is irreducible in $\mathcal{O}_{X,y}$, then it is also irreducible in $\mathcal{O}_{X,y'}$ for any point y' sufficiently close. Therefore, we can simply write $\mathrm{ord}_Y(f) := \mathrm{ord}_{Y,y}(f)$ for any point $y \in Y$, provided that Y is an irreducible hypersurface. The intuition behind the definition is that the order of f at y should be the degree to which the function vanishes at the point y, or the degree of the pole at y

Example 4.2. Let $X = \mathbb{C}$, and $Y = \{p\}$, so Y has locally (in fact globally) defining function $g(z) = (z-p)$. Then let f be the meromorphic function

$$f(z) = \frac{(z-p)^k}{q(z)}$$

where $q(z)$ is a polynomial that does not vanish at p. Then the order of f at p is, as you would expect, k. The role of h is played by the locally invertible function $1/q(z)$, which is a unit in the local ring $\mathcal{O}_{\mathbb{C},p}$. If instead we have

$$f(z) = \frac{q(z)}{(z-p)^k}$$

for a polynomial $q(z)$ that does not vanish at p, then the order of f at y is $-k$, where the role of h is played by $q(z)$.

A simple, but important observation is that order is additive on products.

Proposition 4.3. *Let $Y \subset X$ be an irreducible hypersurface with locally defining function g at $y \in Y$. Let $f_1, f_2 \in \mathcal{K}(X)$ be meromorphic functions. Then*

$$\mathrm{ord}_Y(f_1 f_2) = \mathrm{ord}_Y(f_1) + \mathrm{ord}_Y(f_2)$$

Some more key observations are that for a nonvanishing holomorphic function f, we have that $\mathrm{ord}_Y(f) = 0$ for any irreducible hypersurface $Y \subset X$, and that $\mathrm{ord}_Y(f) = 0\mathrm{ord}_Y(1/f)$.

Proof. In the local ring $\mathcal{O}_{X,y}$, we have that

$$f_1 = g^{\mathrm{ord}_Y(f_1)} h_1 \qquad f_2 = g^{\mathrm{ord}_Y(f_2)} h_2$$

Therefore, we have that

$$f_1 f_2 = g^{\mathrm{ord}_Y(f_1) + \mathrm{ord}_Y(f_2)} h_1 h_2$$

The function $h_1 h_2$ does not vanish at y, and we have that $\mathrm{ord}_Y(f_1) + \mathrm{ord}_Y(f_2)$ must be the smallest integer for $f_1 f_2$, since $\mathrm{ord}_Y(f_1)$ and $\mathrm{ord}_Y(f_2)$ are minimal for f_1 and f_2 respectively. ∎

Definition 4.4. A ***divisor*** D on X is a formal integral combination

$$D = \sum a_i [Y_i]$$

where Y_i is an irreducible hypersurface for all i. The group of divisors on X is denoted $\mathrm{Div}(X)$. The divisor D is said to be an ***effective divisor*** if the a_i are all nonnegative.

The intuition for a divisor is that it should interpreted as a prescription for zeroes and poles of a meromorphic function. For example, in $\mathbb{C}$, you should think of the divisor

$$D = p_1 + 4p_2 - 4p_3 - 6p_4$$

as corresponding the the meromorphic function

$$f(z) = \frac{(z-p_1)(z-p_2)^4}{(z-p_3)^4(z-p_4)^6}$$

Of course, this isn't exactly true or precise. For example, we could multiply the numerator or denominator by nonvanishing holomorphic functions and then obtain a function with the same poles and zeroes as f. From this perspective, and effective divisor is a divisor that only prescribes zeroes.

Definition 4.5. Let $f \in \mathcal{K}(X)$ be a meromorphic function. Then the ***zero divisor*** of f is the divisor

$$(f)_0 = \sum_{\mathrm{ord}_Y(f) > 0} \mathrm{ord}_Y(f)[Y]$$

where the sum is taken over all irreducible hypersurfaces $Y > 0$. Likewise, the ***pole divisor*** of f is defined to be the divisor

$$(f)_\infty = \sum_{\mathrm{ord}_Y(f) < 0} \mathrm{ord}_Y(f)[Y]$$

The ***associated divisor*** of f is the divisor

$$(f) = (f)_0 - (f)_\infty = \sum_Y \mathrm{ord}_Y(f)$$

This makes precise what we mean by divisors being prescriptions for poles and zeroes, but the correspondence between meromorphic functions and divisors is far from bijective, as we noted above. Our observation was that if any two meromorphic functions differed by a nonvanishing holomorphic function, then they will define the same divisor. This turns out to be the only redundancy.

Theorem 4.6. *Let X be a complex manifold. Then let $\mathcal{K}_X^\times$ denote the sheaf of nonzero meromorphic functions, and $\mathcal{O}_X^\times \subset \mathcal{K}_X^\times$ the sheaf of nonvanishing holomorphic functions, which are both sheaves of abelian groups under multiplication. Then we have an isomorphism*

$$\Gamma(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times) \to \mathrm{Div}(X)$$

Proof. An global section σ of the quotient sheaf $\mathcal{K}_X^\times / \mathcal{O}_X^\times$ is a collection of pairs (U_i, f_i) of meromorphic functions $f_i \in \mathcal{K}_X^\times(U_i)$ such that on the intersections $U_i \cap U_j$, their quotient f_i / f_j is an element of $\mathcal{O}_X^\times$, i.e. a nonvanishing holomorphic function over $U_i \cap U_j$. Therefore, over the intersection, we have that

$$\mathrm{ord}_Y(f_i) - \mathrm{ord}_Y(f_j) = \mathrm{ord}_Y(f_i / f_j) = 0$$

for any irreducible hypersurface Y. Therefore, we have that $\mathrm{ord}_Y(f_i) = \mathrm{ord}_Y(f_j)$. Therefore, the order $\mathrm{ord}_Y(\sigma)$ of σ is well-defined for any irreducible hypersurface $Y \subset X$, so we get a well-defined map

$$\begin{aligned}\Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times) &\to \mathrm{Div}(X)\\ \sigma &\mapsto (\sigma) := \sum_Y \mathrm{ord}_Y(\sigma)[Y]\end{aligned}$$

To show that this mapping is bijective, we provide an inverse map. Let $D = \sum_i a_i[Y_i] \in \mathrm{Div}(X)$. Then we can find an open cover $\{U_i\}$ of X where in each open set U_j, any irreducible hypersurface Y_i intersecting U_j nontrivially is the vanishing locus of a holomorphic function g_{ij}, which is unique up to multiplication by an element in $\mathcal{O}_X^\times(U_j)$. Then we claim that the functions $f_j := \prod_i g_{ij}$ glue to a global section of $\mathcal{K}_X^\times/\mathcal{O}_X^\times$, i.e. on any $U_j \cap U_k$, we have that f_j/f_k is a nonvanishing holomorphic function. However, on any intersection $U_j \cap U_k$, the functions g_{ij} and g_{ik} define the same irreducible hypersurface Y_i, so they differ by multiplication by a nonvanishing holomorphic function. Therefore, the f_j glue to a global section.

It is easily verified that these two constructions are inverses, so the maps are bijective. Furthermore, the map is a group homomorphism since order is additive on products of meromorphic functions, so it is a group isomorphism. ■

Remark. For complex manifolds, we have that global sections of the quotient sheaf and divisors are isomorphic, but in the algebro-geometric setting, this is not true without some smoothness assumptions. In algebraic geometry, elements of $\Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$ are called ***Cartier divisors***, while the elements of what we called $\mathrm{Div}(X)$ are called ***Weil divisors***.

One of the big punchlines for divisors is their correspondence with holomorphic line bundles over X, which we can make explicit using the isomorphism with $\Gamma(X, \mathcal{K}_X^\times/O_X^\times)$. In the proof we gave above, we noted that a global section of $\mathcal{K}_X^\times/O_X^\times$ can be seen as a collection of meromorphic functions f_i on an open cover $\{U_i\}$, such that on intersections, the function f_i/f_j is a nonvanishing holomorphic function. In addition, it is easily seen that the functions satisfy the cocycle condition, so they can be used to define transition functions for a line bundle, which we call $\mathcal{O}(D)$. The mapping $D \mapsto \mathcal{O}(D)$ is clearly a group homomorphism, so it defines a natural map $\mathrm{Div}(X) \to \mathrm{Pic}(X)$. In particular, this shows that line bundles are closely related to the codimension 1 geometry of the complex manifold X.

Like line bundles, divisors also pull back, but there is some more subtlety when working with divisors from the perspective of Weil divisors. Given a holomorphic map $f : X \to Y$ and a hypersurface $D \subset Y$, we would like to define the pullback $f^*[D]$ of the divisor $[D]$ to be $[f^{-1}(D)]$, and then extend linearly to all divisors. In most cases, this is fine, and $f^{-1}(D)$ is still codimension 1, since locally it will be given by the vanishing of $f \circ g$ where g is some locally defining function for D. However, if the image of X is contained in D, then $f^{-1}(D) = X$, which isn't codimension 1! One way to fix this is to restrict our attention to maps with dense image. The other is to just be careful when pulling back a divisor along a map. The other issue involves irreducibility. In general, given an irreducible hypersurface $D \subset Y$, the inverse image $f^{-1}(D)$ need not be irreducible. This can be solved by defining

$f^*[D]$ to be the divisor corresponding to the sum $\sum_{n_i} n_i[Y_i]$ where n_i is the exponent of the locally definining function for $[Y_i]$ in a factorization of $f \circ g$, where g is a locally defining function for D. Alternatively, one can instead use the perspective of Cartier divisors. Given a section of $\mathcal{K}_Y^\times/\mathcal{O}_Y^\times$, we can write it in terms of an open cover by a collection of pairs $\{(U_i, f_i)\}$ for open sets U_i and $f_i \in \mathcal{K}_Y$ whose quotients are holomorphic on $U_i \cap U_j$. Then we define the pullback of this divisor to be the Cartier divisor $\{(f^{-1}(U_i), f \circ f_i)\}$. It is easy to verify that this coincides with the definition in terms of Weil divisors when that definition makes sense. Furthermore, unpacking the definition of the line bundle $\mathcal{O}(D)$ corresponding to a Cartier divisor $D \in \Gamma(Y, \mathcal{K}_Y^\times/\mathcal{O}_Y^\times)$, we find that

$$\mathcal{O}(f^*D) \cong f^*\mathcal{O}(D)$$

whenever the pullback divisor f^*D makes sense.

Given an irreducible hypersurface D, we get a line bundle $\mathcal{O}(D)$ – the correspondence the other way comes in the form of sections. Given a holomorphic line bundle $L \to X$ and a global section $s \in \Gamma(X, L)$, we define a divisor. $Z(s)$ as follows : fix an open cover $\{U_i\}$ such that L admits local holomorphic trivializations $\varphi_i : L|_{U_i} \to U_i \times \mathbb{C}$ with transition functions ψ_{ij}. This gives holomorphic functions $\varphi_i \circ s : U_i \to \mathbb{C}$, which together define a Cartier divisor $\{(U_i, \varphi_i \circ s)\}$ which we define to be $Z(s)$. It is easy to verify that this definition is independent of our choice of trivialization.

Proposition 4.7.

(1) *Let $s \in \Gamma(X, L)$ be a nonzero section. Then $\mathcal{O}(Z(s)) \cong L$.*
(2) *For an effective divisor D, there exists a nonzero section s of $\mathcal{O}(D)$ such that $Z(s) = D$.*

Proof.

(1) Fix an open cover $\{U_i\}$ with trivializations $\varphi_i : L|_{U_i} \to U_i \times \mathbb{C}$ and transition functions ψ_{ij}. We want to show that the divisor $Z(s) = \{(U_i, f_i)\}$ (where $f_i := \varphi_i \circ s$) defines a line bundle isomorphic to L. The line bundle defined by $Z(s)$ is the line bundle given by the cocycle $\{(U_i, f_i/f_j)\}$. We then note that since the f_i came from a section s, they satisfy the compatibility condition $f_i = \psi_{ij} f_j$, so $f_i/f_j = \psi_{ij}$.
(2) For an effetive divisor D, the associated Cartier divisor $\{(U_i, f_i)\}$ has the property that the f_i are all holomorphic, since they will be products of holomorphic functions with irreducible germs at some $p_i \in U_i$. The line bundle $\mathcal{O}(D)$ is given by the cocycle $\{(U_i, f_i/f_j)\}$. Then by construction, the f_i glue together to a global section of $\mathcal{O}(D)$, since we have $f_i = f_i/f_j \cdot f_j$.

■

We note that the preceding discussion relies on the existence of a nonzero global holomorphic section of L, which may or may not exist. We also note that the correspondence only concerns effective divisors. We can generalize this to line bundles with no global sections with the idea of a ***meromorphic section*** of a line bundle L, i.e. a global section of the sheaf $\mathcal{K}_X \otimes_{\mathcal{O}_X} L$. Much like holomorphic sections, a meromorphic section can be written in a trivializing open cover $\{U_i\}$ of X as a collection of meromorphic functions $(f_i/g_i) \in \mathcal{K}_X(U_i)$ satisfying the compatibility condition $(f_i/g_i) = \psi_{ij}(f_j/g_j)$, where the ψ_{ij} are the transition functions of L. Then given a meromorphic section s, we can define a divisor in a similar manner as $Z(s)$, except the Cartier divisor we obtain consists of meromorphic functions over the U_i rather than just holomorphic functions. This gives us the following generalization of the previous proposition:

Proposition 4.8.

(1) *Let $s \in \Gamma(X, \mathcal{K}_X \otimes_{\mathcal{O}_X} L)$ be a global meromorphic section of L, and let $D(s)$ be the divisor associated to s. Then $\mathcal{O}(D(s)) \cong L$.*

(2) *For any divisor D, there exists a nonzero meromorphic section s of L such that $D(s) = D$.*

The proof of this is near identical. One thing to note is that given a meromorphic section s, we get that $D(1/s) = -D$.

Using these propositions, we can get a better understanding of the sheaf of sections of the line bundle $\mathcal{O}(D)$ Let $D \subset X$ be a divisor. Let $\mathcal{L}(D) \subset \mathcal{K}_X)$ denote the subsheaf of meromorphic functions on X such that $D + (f) \geq 0$. If we write $D = \sum_i a_i V_i$ with $a_i \in \mathbb{Z}$ and V_i irreducible hypersurfaces, the functions in $\mathcal{L}(D)$ are exactly the meromorphic functions f that are holomorphic on the complement of the V_i, and satisfy $\mathrm{ord}_{V_i}(f) \geq -a_i$. Working locally, suppose g_i is a locally defining function for V_i. Then if a_i is positive, then this condition says that $fg_i^{a_i}$ is holomorphic. If a_i is negative, than this condition says that $f/g_i^{a_i}$ is holomorphic. Then let $s_0 \in \Gamma(X, \mathcal{K}_X \otimes_{\mathcal{O}_X} \mathcal{O}(D))$ be a nonzero global meromorphic section with $D(s_0) = D$. Then given a holomorphic section s of $\mathcal{O}(D)$, we get a meromorphic function $f_s \in \Gamma(X, \mathcal{K}_X)$ defined locally by $f_s = s/s_0$. We then have that $(f_s) = D(s) - D(s_0) \geq -D$, so $f_s \in \mathcal{L}(D)$. In the other direction, we can obtain a holomorphic section of $\mathcal{O}(D)$ from a meromorphic function $f \in \mathcal{L}(D)$ by taking fs_0. Furthermore, these mappings are clearly inverses of each other, giving us an isomorphism of sheaves $\mathcal{L}(D) \cong \mathcal{O}(D))$. Similarly, multiplication with $1/s_0$ gives an isomorphism $\mathcal{L}(-D) \cong \mathcal{O}(-D)$. Under this identification, we note that multipying a section of $\mathcal{O}(-D)$ with s_0 yields a meromorphic function which has no poles, and vanishes along the hypersurfaces appearing in D to the specified orders. In particular, if D is a hypersurface, then we get an exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}(D)|_D \longrightarrow 0$$

where the first map is multiplication by s_0, and the second map is restricting sections from X to D, and we are implicitly making the identifications $\mathcal{L}(D) \cong \mathcal{O}(D)$ and $L(-D) \cong \mathcal{O}(-D)$. In fact, we get an exact sequence for any holomorphic vector bundle E. If we let $\mathcal{E}$ denote the sheaf of holomorphic sections of E, then we get an exact sequence of sheaves

$$0 \longrightarrow \mathcal{E} \otimes \mathcal{O}(-D) \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}|_D \longrightarrow 0$$

where again the first map is multiplication by s_0 and the second map is restriction.

5. The Adjunction Formula

The relationship between line bundles and the codimension 1 geometry of X can also be seen through the Adjunction Formula, which relates the normal bundle of a hypersurface $S \subset X$ (a line bundle!) with the divisor $[S]$ it defines.

Definition 5.1. Let $S \subset X$ be a complex submanifold. Then the ***normal bundle*** of S, denoted N_S, is the quotient vector bundle

$$N_S := T^{1,0}X|_S / T^{1,0}S$$

Where $T^{1,0}X$ and $T^{1,0}S$ denote the holomorphic tangent bundles of X and S respectively. The ***conormal bundle*** of S is the dual bundle N_S^*.

From standard linear algebra, we can regard elements of N_S^* as tangent covectors that vanish on the tangent space of S.

In the case that S is a hypersurface (i.e complex codimension 1), then we have that S defines a divisor $[S] \in \mathrm{Div}(X)$. In addition, N_S is a line bundle. One would hope that the divisor $[S]$ and N_S are closely related by the correspondence between divisors and line bundles. This turns out to be the case!

Proposition 5.2.

$$N_S^* \cong \mathcal{O}(-S)|_S$$

Proof. Let $\{U_\alpha\}$ be an open cover of X such that S is locally defined by holomorphic functions $f_\alpha \in \mathcal{O}_X(U_\alpha)$. Then the line bundle $\mathcal{O}(S)$ is given by the Čech cocycle $\{g_{\alpha\beta}\}$ where

$$g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}$$

Since f_α vanishes along $V \cap U_\alpha$, the differential df_α vanishes on the tangent vectors of $V \cap U_\alpha$ tangent to V, which is to say that it defines a local section of the conormal bundle N_S^* over the open set $V \cap U_\alpha$. Furthermore, since S is nonsingular, df_α is nonzero over every point of $V \cap U_\alpha$. Over $V \cap U_\alpha \cap U_\beta$, we have

$$\begin{aligned} df_\alpha &= d(g_{\alpha\beta} f_\beta) \\ &= f_\beta dg_{\alpha\beta} + g_{\alpha\beta} df_\beta \\ &= g_{\alpha\beta} df_\beta \end{aligned}$$

where we use the fact that f_β vanishes on $V \cap U_\beta$. Therefore, we have that we can interpret the $\{df_\alpha\}$ as local sections of $N_S^* \otimes \mathcal{O}(S)|_S$ which glue together to form a global nonvanishing sections of $N_S^* \otimes \mathcal{O}(S)|_S$, which tells us that $N_S^* \otimes \mathcal{O}(S)|_S$ is a trivializable line bundle. Therefore, we have that $N_S^* \cong \mathcal{O}(-S)|_S$. ∎

This gives us a particularly nice description of the canonical bundle $K_S := (T^{n,0}S)^*$ of a hypersurface S in terms of the canonical bundle of X.

Proposition 5.3. *Let $S \subset X$ be a hypersurface. Then*

$$K_S \cong (K_X \otimes \mathcal{O}(S))|_S$$

Proof. By the definition of the normal bundle, we have the exact sequence of vector bundles over S

$$0 \longrightarrow T^{1,0}S \longrightarrow T^{1,0}X|_S \longrightarrow N_S \longrightarrow 0$$

Dualizing, this becomes

$$0 \longrightarrow N_S^* \longrightarrow (T^{1,0}X|_S)^* \longrightarrow (T^{1,0}S)^* \longrightarrow 0$$

From this, a linear algebra fact gives us that the determinant line of $(T^{1,0}X|_S)^*$ is isomorphic to the tensor product of the determinant line bundle of the other two, i.e.

$$K_X|_S \cong K_S \otimes N_S^*$$

where we use the fact that N_S^* is a line bundle to conclude that it is its own determinant line bundle. Then since $N_S^* \cong \mathcal{O}(-S)$, we get that

$$K_X|_S \otimes \mathcal{O}(S)|_S \cong K_S$$

■

This is particularly useful for hypersurfaces in $\mathbb{CP}^n$, since we know that the canonical bundle of $\mathbb{CP}^n$ is $\mathcal{O}(-n-1)$ and we also can work with the conormal bundle in the form of an ideal sheaf. Indeed, the discussion above can be done on the levels of sheaves of sections, where we identify the sheaf of sections of N_S^* with the sheaf $\mathcal{I}/\mathcal{I}^2$, where $\mathcal{I}$ denotes the ideal sheaf of holomorphic functions vanishing along S.

6. Sheaf Cohomology

Over a complex manifold X, we have many different cohomology theories at our disposal:

(1) The singular cohomology groups $H^i_{\text{sing}}(X, \mathbb{Z})$.
(2) The de Rham cohomology groups $H^i_{dR}(X)$.
(3) The Dolbeault cohomology groups of a holomorphic vector bundle $H^i_{\bar{\partial}}(X, E)$.
(4) The Čech cohomology groups of a sheaf $\check{H}^i(X, \mathcal{F})$.

We want to compare these various cohomology theories. To do so, we show that many of these cohomology theories are computing the same thing : sheaf cohomology.

Remark. While we might explicitly work with sheaves of abelian groups, the following discussion is applicable to sheaves of $\mathcal{O}_X$ modules, C^∞ modules, etc.

Definition 6.1. The ***global sections functor*** $\Gamma(X, \cdot)$ is a functor $\mathsf{Ab}(X) \to \mathsf{Ab}$ of the category of sheaves of abelian groups over X to the category of abelian groups, where given a sheaf of abelian groups $\mathcal{F}$,

$$\Gamma(X, \mathcal{F}) := \mathcal{F}(X)$$

The functor is left-exact, i.e. given an exact sequence of sheaves

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}$$

we get an exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{E}) \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{G})$$

However, the functor is not right exact, which is due to the local definitions of injectivity and surjectivity. The sheaf axiom guarantees that being injective on stalks implies that a sheaf morphism is injective on sections, but it does not imply the same thing for surjectivity. As an example, let $\mathcal{Z}^k$ be the sheaf of closed smooth k-forms, and let $\mathcal{B}^k$ be the sheaf of exact k-forms. Then the inclusion $\mathcal{B}^k \hookrightarrow \mathcal{Z}^k$ is surjective, since every closed k-form is exact in a sufficiently small neighborhood. However, if $H^k_{dR}(X) \neq 0$, then $\Gamma(X, \mathcal{Z}^k) \to \Gamma(X, \mathcal{B}^k)$ is not surjective.

Definition 6.2. The ***sheaf cohomology groups*** $H^i(X, \mathcal{F})$ of a sheaf $\mathcal{F}$ over X are the right derived functors of the global sections functor applied to $\mathcal{F}$

$$H^i(X, \mathcal{F}) := R^i\Gamma(X, \mathcal{F})$$

i.e, we take an injective resolution $\mathcal{I}^\bullet = \left\{\mathcal{I}^j\right\}$ of $\mathcal{F}$

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \cdots$$

and apply $\Gamma(X, \cdot)$ term-wise to the sequence to get

$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{I}^0) \longrightarrow \Gamma(X, \mathcal{I}^1) \longrightarrow \cdots$$

and then compute the cohomology of this sequence.

We note that in order for these right derived functors to be defined, there need to be *enough injectives*. We say that an abelian category $\mathcal{A}$ has enough injectives if every object $A \in \mathrm{Ob}(\mathcal{A})$ admits an injective map $A \hookrightarrow I$ into an injective object I, where an injective object is defined to be an object I where given any map $X \to I$ and an injection $X \hookrightarrow I$, there exists a map $Y \to Q$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[rr, hook] \arrow[dr] & & Y \arrow[dl, dashed] \\ & Q & \end{tikzcd}$$

Alternatively, the pullback map $\mathrm{Hom}(Y, Q) \to \mathrm{Hom}(X, Q)$ is surjective. We will take the following results on faith:

Theorem 6.3. *The categories* Ab, $\mathsf{Ab}(X)$, $\mathsf{Mod}_{\mathcal{O}_X}$, *and* Mod_{C^∞} *have enough injectives.*

Like with the definition of the Čech cohomology groups as limits, the definition in terms of injective resolutions is practically useless computationally, since injective sheaves are hard to write down and difficult to find in the wild. The name of the game here is to find a nicer class of resolutions we can take.

Definition 6.4. Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact additive functor. An object $A \in \mathrm{Ob}(\mathcal{A})$ is ***acyclic*** for the functor F if $R^iF(A) = 0$ for all $i > 0$.

Proposition 6.5. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor, and $A \in \mathrm{Ob}(\mathcal{A})$. Then let $A \to M^\bullet$ be a resolution of A by F-acyclic objects. Then $R^iF(A)$ is isomorphic to the i^{th} cohomology of the complex $F(M^\bullet)$.*

Proof. Let $d^0 : M^0 \to M^1$. Then let $B = \mathrm{coker}(A \to M^0)$. By exactness of

$$0 \longrightarrow A \longrightarrow M^0 \xrightarrow{d^0} M^1$$

we get a short exact sequence

$$0 \longrightarrow A \longrightarrow M^0 \longrightarrow B \longrightarrow 0$$

where the map $M^0 \to B$ is the map d^0, which descends to B by exactness. We then take injective resolutions $A \to I^\bullet$, $M^0 \to J^\bullet$, and $B \to K^\bullet$. The maps $A \to M^0$ and $M^0 \to$

B, which gives a short exact sequence of chain maps between resolutions by using the defining property of injective objects, giving us

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & M^0 \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^0 \arrow[r] \arrow[d] & J^0 \arrow[r] \arrow[d] & K^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^1 \arrow[r] \arrow[d] & J^1 \arrow[r] \arrow[d] & K^1 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

which gives us a long exact sequence in cohomology. Noting that the cohomology of the respective sequences are just the right derived functors of A, M^0, and B, we get the long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & R^0F(A) \arrow[r] & R^0F(M^0) \arrow[r] & R^0F(B) \arrow[dll] \\
 & R^1F(A) \arrow[r] & R^1F(M^0) \arrow[r] & R^1F(B) \arrow[dll] \\
 & R^2F(A) \arrow[r] & R^2F(M^0) \arrow[r] & R^2F(B) \arrow[dll] \\
 & & \cdots & \cdots
\end{tikzcd}$$

Then using the fact that M^0 is F-acyclic, along with the fact that $R^0F = F$, we get that this long exact sequence is actually

$$\begin{tikzcd}
0 \arrow[r] & F(A) \arrow[r] & F(M^0) \arrow[r] & F(B) \arrow[dll] \\
 & R^1F(A) \arrow[r] & 0 \arrow[r] & R^1F(B) \arrow[dll] \\
 & R^2F(A) \arrow[r] & 0 \arrow[r] & R^2F(B) \arrow[dll] \\
 & & \cdots & \cdots
\end{tikzcd}$$

which gives us isomorphisms $R^iF(B) \to R^{i+1}F(A)$ for $i > 0$, as well as an isomorphism $R^1F(A) \cong \operatorname{coker} F(M^0) \to F(B)$. To compute the cokernel of that map, we note that B admits the resolution

$$0 \longrightarrow B \longrightarrow M^1 \longrightarrow \cdots$$

where the map $B \to M^1$ is the map induced by d^0, using exactness of the acyclic resolution. Then since F is left-exact, we get that

$$0 \longrightarrow F(B) \longrightarrow F(M^1) \longrightarrow \cdots$$

is exact, giving us that $F(B)$ is isomorphic to the kernel of $F(M^1) \to F(M^2)$. We then note that the image of $F(M^0) \to F(M^1)$ is contained in $F(B)$, regarded as the kernel of $F(M^1) \to F(M^2)$. We then get

$$R^1F(A) \cong \frac{\ker(F(M^1) \to F(M^2))}{\operatorname{Im}(F(M^1) \to F(M^2))} = H^1(F(M^\bullet))$$

Then to get the isomorphism for $R^2F(A)$, we note that since $R^1F(B) \cong R^2F(A)$, we can play the same game using the resolution of B to compute $R^1F(B)$ to be $H^2(F(M^\bullet))$, and then inductively repeat the process with the cokernel of $M^1 \to M^2$ to get $R^3F(A)$ and so on. ■

Something that is silly to observe, but useful.

Proposition 6.6. *Injective objects are F-acyclic.*

Proof. Let I be injective. Then take the injective resolution $0 \to I \to I \to 0$ where the map is the identity map. ■

Therefore, to compute right derived functors, it suffices to find acyclic resolutions. This gets us one step closer to finding nice resolutions for computing sheaf cohomology.

Definition 6.7. A sheaf $\mathcal{F}$ over X is ***flasque*** (also called ***flabby***) if the restriction maps are surjective.

Most sheaves in nature aren't flasque. However, flasque sheaves are useful for giving us acyclic resolutions. To do this, we'll need some lemmas.

Lemma 6.8. *Let*

$$0 \longrightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \longrightarrow 0$$

be a short exact sequence where $\mathcal{E}$ is flasque. Then for any open set U, the sequence

$$0 \longrightarrow \mathcal{E}(U) \xrightarrow{\alpha(U)} \mathcal{F}(U) \xrightarrow{\beta(U)} \mathcal{G}(U) \longrightarrow 0$$

is exact.

Proof. By left-exactness of taking sections, it suffices to show that $\beta(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective. Let $\sigma \in \mathcal{G}(U)$. Since β is a surjective sheaf morphism, for any $x \in U$, the induced map on stalks $\beta_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective, which implies that there exists a sufficiently small neighborhood $V_x \subset U$ of x where $\beta(V_x) : \mathcal{F}(V_x) \to \mathcal{G}(V_x)$ is surjective, so we can find $\tau_x \in \mathcal{F}(V_x)$ such that $\beta(V_x)(\tau) = \sigma|_{V_x}$. Then let $y \in U$ such that $V_x \cap V_y \neq \varnothing$, and let $\tau_y \in \mathcal{F}(V_y)$ such that $\beta(V_y)(\tau_y) = \sigma|_{V_y}$. Then we know that $\tau_x|_{V_x \cap V_y} - \tau_y|_{V_x \cap V_y} \in \ker \beta(V_x \cap V_y)$, so it is the image of an element $k \in \mathcal{E}(V_x \cap V_y)$. Since $\mathcal{E}$ is flasque, we know that we can lift k to an section $\chi_{x,y} \in \mathcal{E}(V_x)$. Then the element $\tau_x' = \tau_x - \alpha(V_x)(\chi_{x,y})$ still maps to $\sigma|_{V_x}$ under $\beta(V_x)$ by exactness, and restricts to τ_y on $V_x \cap V_y$. Therefore, τ_x' and τ_y glue to a section over $V_x \cup V_y$ that maps to $\sigma|_{V_x \cup V_y}$. We then find a maximal pair (W, τ) such that $\tau \in \mathcal{F}(W)$ and $\beta(W)(\tau) = \sigma|_W$. Then we must necessarily have $W = U$, since otherwise, we can find another open subset V of U and a section φ over V mapping to $\sigma|_V$, and extend τ to a section over $W \cup V$, since if $W \cap V \neq \varnothing$, we can use our above argument, and otherwise, no work needs to be done. Either way, finding such a U and φ contradicts maximality of (W, τ). ∎

Lemma 6.9. *Let*

$$0 \longrightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \longrightarrow 0$$

be a short exact sequence of sheaves where $\mathcal{E}$ and $\mathcal{F}$ are flasque. Then $\mathcal{G}$ is flasque.

Proof. Let $V \subset U$. Then given $\sigma \in \mathcal{G}(V)$, we want to show that there exits $\widetilde{\sigma} \in \mathcal{G}(U)$ that restricts to σ. Since $\mathcal{E}$ is flasque, we have that

$$0 \longrightarrow \mathcal{E}(V) \xrightarrow{\alpha(V)} \mathcal{F}(V) \xrightarrow{\beta(V)} \mathcal{G}(V) \longrightarrow 0$$

is exact, so we can find a section $\tau \in \mathcal{F}(V)$ with $\beta V(\tau) = \sigma$. Then since $\mathcal{F}$ is flasque, this lifts to an element $\widetilde{\tau} \in \mathcal{F}(U)$. Then taking $\widetilde{\sigma} = \beta(U)(\widetilde{\tau})$ gives us the desired section, since the properties of a sheaf morphism implies that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\beta(U)"] \arrow[d] & \mathcal{G}(U) \arrow[d] \\
\mathcal{F}(V) \arrow[r, "\beta(V)"'] & \mathcal{G}(V)
\end{tikzcd}$$

∎

Proposition 6.10. *Flasque sheaves are acyclic for the global sections functor $\Gamma(X, \cdot)$.*

Proof. Let $\mathcal{F}$ be a flasque sheaf. We first embed $\mathcal{F}$ into an injective flasque sheaf $\mathcal{I}$. Since Ab has enough injectives, we can embed each stalk $\mathcal{F}_x$ into an injective group I_x. Then define the sheaf $\mathcal{I}$ by

$$\mathcal{I}(U) = \prod_{x \in U} I_x$$

and the restriction maps are the projection maps $\prod_{x \in U} I_x \to \prod_{x \in V} I_x$. Since these maps are surjective, $\mathcal{I}$ is flasque. In addition, it is injective by construction. Then $\mathcal{F}$ embeds into $\mathcal{I}$ by composing the inclusions $\mathcal{F}(U) \hookrightarrow \prod_{x \in U} \mathcal{F}_x \hookrightarrow \prod_{x \in V} I_x$ Then let $\mathcal{G}$ be the cokernel of $\mathcal{F} \hookrightarrow \mathcal{I}$, giving us the exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{G} \longrightarrow 0$$

taking resolutions of $\mathcal{F}$, $\mathcal{I}$ and $\mathcal{G}$ gives us a long exact sequence in cohomology

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,\mathcal{F}) \arrow[r] & H^0(X,\mathcal{I}) \arrow[r] & H^0(X,\mathcal{G}) \arrow[dll] \\
& H^1(X,\mathcal{F}) \arrow[r] & H^1(X,\mathcal{I}) \arrow[r] & H^1(X,\mathcal{G}) \arrow[dll] \\
& H^2(X,\mathcal{F}) \arrow[r] & H^2(X,\mathcal{I}) \arrow[r] & H^2(X,\mathcal{G}) \arrow[dll] \\
& & \cdots & \cdots
\end{tikzcd}$$

The since H^0 is just $\Gamma(X, \cdot)$ and $\mathcal{I}$ is injective, this becomes

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}(X) \arrow[r] & \mathcal{I}(X) \arrow[r] & \mathcal{G}(X) \arrow[dll] \\
& H^1(X,\mathcal{F}) \arrow[r] & 0 \arrow[r] & H^1(X,\mathcal{G}) \arrow[dll] \\
& H^2(X,\mathcal{F}) \arrow[r] & 0 \arrow[r] & H^2(X,\mathcal{G}) \arrow[dll] \\
& & \cdots & \cdots
\end{tikzcd}$$

Since $\mathcal{F}$ is flasque, we have that

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{I}(X) \longrightarrow \mathcal{G}(X) \longrightarrow 0$$

is exact, so this implies that $H^1(X,\mathcal{F}) = 0$. In addition, for $i > 0$, we get isomorphisms $H^i(X,\mathcal{G}) \to H^{i+1}(X,\mathcal{F})$. To get that $H^2(X,\mathcal{F}) = 0$, we note that $\mathcal{G}$ is flasque, so we can repeat the argument to get that $H^1(X,\mathcal{G}) = H^2(X,\mathcal{F})$, and then continue for the other cohomology groups. ■

As a consequence, it suffices to find resolutions by flasque sheaves to compute sheaf cohomology. The good news here is that every sheaf admits a canonical acyclic resolution, the ***Godement resolution***. For a sheaf $\mathcal{F}$, let $\mathcal{F}_{\text{God}}$ be the sheaf $U \mapsto \prod_{x \in U} \mathcal{F}_x$, which is clearly flasque. Then we construct a flasque resolution for $\mathcal{F}$ as follows : embed $\mathcal{F} \hookrightarrow \mathcal{F}_{\text{God}}$, and let $\mathcal{G}^1$ be the cokernel. Then let the next sheaf in the sequence be $\mathcal{G}^1_{\text{God}}$, where the map $\mathcal{F}_{\text{God}} \to \mathcal{G}^1_{\text{God}}$ is the quotient map $\mathcal{F}_{\text{God}} \to \mathcal{G}^1$ composed in the the inclusion $\mathcal{G}^1 \hookrightarrow \mathcal{G}^1_{\text{God}}$. Then take $\mathcal{G}^2$ to be the cokernel of $\mathcal{F}_{\text{God}} \to \mathcal{G}^1_{\text{God}}$, and continue with $\mathcal{G}^2_{\text{God}}$ and so on. Pictorially, the Godement construction is constructed as follows:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}_{\mathrm{God}} \arrow[r] \arrow[dr] & \mathcal{G}^1 \arrow[r] \arrow[d] & 0 & & \\
& & & \mathcal{G}^1_{\mathrm{God}} \arrow[r] \arrow[dr] & \mathcal{G}^2 \arrow[r] \arrow[d] & 0 & \\
& & & & \mathcal{G}^2_{\mathrm{God}} \arrow[r] \arrow[dr] & \mathcal{G}^3 \arrow[r] \arrow[d] & 0 \\
& & & & & \ddots &
\end{tikzcd}$$

Once more, this resolution is computationally useless, but it serves the purpose of showing that every sheaf admits a resolution by flasque sheaves. With this in hand, we can finally show that we can compute sheaf cohomology with a nice class of sheaves.

Definition 6.11. Let $\mathcal{A}$ be a sheaf of rings over X such that $\mathcal{A}$ admits ***partitions of unity***, i.e. for any open cover $\{U_i\}$ of X, there exist global sections $f_i \in \mathcal{A}(X)$ such that $\sum_i f_i = 1$ and f_i is supported in U_i, and where over any particular open set, all but finitely many of the f_i are 0. Then a sheaf of $\mathcal{A}$-modules is a ***fine sheaf***.

These are sheaves we care about, and appear in nature.

Example 6.12. Let M be a smooth manifold. Then the sheaf C^∞ of smooth functions admits partitions of unity. Therefore, Mod_{C^∞} consists of fine sheaves.

The punchline is that fine sheaves are acyclic, which gives us a source of reasonable sheaves with which to build resolutions.

Theorem 6.13. *Fine sheaves are acyclic with respect to* $\Gamma(X, \cdot)$.

Proof. Let $\mathcal{F}$ be a fine sheaf – a sheaf of modules over a sheaf of rings $\mathcal{A}$ that admits partitions of unity. Then by taking the Godement resolution of $\mathcal{F}$, we get an injective resolution of flasque $\mathcal{A}$-modules.

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \xrightarrow{d^0} \mathcal{I}^1 \xrightarrow{d^1} \cdots$$

Then since flasque sheaves are acyclic, we get that we can compute the sheaf cohomology of $\mathcal{F}$ as

$$H^i(X, \mathcal{F}) = \frac{\ker d^{i+1}(X) : \mathcal{I}^i(X) \to \mathcal{I}^{i+1}(X)}{\operatorname{Im} d^i(X) : \mathcal{I}^{i-1}(X) \to \mathcal{I}^i(X)}$$

Then let $\alpha \in \ker d^{i+1}(X)$, by exactness, we know that locally, in a sufficiently small open cover $\{U_j\}$, the map $d^i(U_j)$ is surjective, so we can find $\beta_j \in \mathcal{I}^{i-1}(U_j)$ such that $d^i(V_j)(\beta_j) = \alpha|_{U_j}$. Then $f_i\beta_i$ determines a global section that is equal to $f_j\beta_j$ on U_i and 0 elsewhere, and we get that $\sum_j f_j\beta_j$ maps to $\sum_j f_j\alpha|_{U_j} = \alpha$ under $d^i(X)$. Therefore, the sequence is exact on global sections for $i > 0$, so $\mathcal{F}$ is acyclic. ■

This tells us that many of the sheaves we know about, like sheaves of smooth sections of a vector bundle are trivial from the perspective of sheaf cohomology.

7. Extensions and H^1

For a sheaf of $\mathcal{O}_X$-modules (where X is a scheme or complex manifold) $\mathcal{F}$, the first sheaf cohomology group $H^1(X, \mathcal{F})$ has a fairly explicit geometric interpretation.

Definition 7.1. Let $\mathcal{E}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. An ***extension*** of $\mathcal{E}$ by $\mathcal{G}$ is a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

In other words, it is the data of a sheaf $\mathcal{F}$ containing $\mathcal{E}$ as a subsheaf such that the quotient is $\mathcal{G}$.

Proposition 7.2. *Let $\mathcal{E}$ be a sheaf of $\mathcal{O}_X$-modules. Then there is a bijective correspondence*

$$\{\textit{Extensions of } \mathcal{E} \textit{ by } \mathcal{O}_X\} \longleftrightarrow H^1(X, \mathcal{E})$$

Proof. We provide maps in both directions. In one direction, let

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

be an extension of $\mathcal{E}$ by $\mathcal{O}_X$. The long exact sequence in sheaf cohomology gives a boundary map $\delta : H^0(X, \mathcal{O}_X) \to H^1(X, \mathcal{E})$. So we can associate an extension to the element $\delta(1) \in H^1(X, \mathcal{E})$.

In the other direction, let $\beta \in H^1(X, \mathcal{E})$. Fix an injective sheaf $\mathcal{I}$ with an injection $\mathcal{E} \hookrightarrow \mathcal{I}$, and let $\mathcal{Q}$ denote the quotient sheaf $\mathcal{I}/\mathcal{E}$. This gives us us the short exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{I} \longrightarrow \mathcal{Q} \longrightarrow 0$$

The long exact sequence in sheaf cohomology gives us

$$H^0(X, \mathcal{Q}) \longrightarrow H^1(X, \mathcal{E}) \longrightarrow 0$$

where we use the fact that $\mathcal{I}$ is injective to deduce that $H^1(X, \mathcal{I}) = 0$. Therefore, we know that β is the image of some $\widetilde{\beta} \in H^0(X, \mathcal{Q}) = \mathrm{Hom}(\mathcal{O}_X, \mathcal{Q})$. Taking the fiber product, we get the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E} \arrow[r] \arrow[d, equal] & \mathcal{I} \times_{\mathcal{Q}} \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d, "\widetilde{\beta}"] & 0 \\
0 \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{Q} \arrow[r] & 0
\end{tikzcd}$$

We can then take the extension to be the upper row. ■

In the case where $\mathcal{E}$ is locally free (e.g. the sheaf of sections of a holomorphic vector bundle), then any extension of $\mathcal{E}$ by $\mathcal{O}_X$ is locally free. For instance, we can fix a good (or affine) cover, in which case the short exact sequence defined by the extension splits.

8. Comparison of Cohomology Theories

The fact that many of the sheaves we encounter naturally have trivial sheaf cohomology might come as a surprise, since we know we can extract topological data from these sheaves. The reason for this is that they provide good resolutions of other sheaves with nontrivial sheaf cohomology.

Proposition 8.1 (Poincaré Lemma)**.** *Every closed smooth k-form ω is locally exact, i.e. for a sufficiently small U, we have that $\omega|_U = d\eta$ for some $k-1$-form η.*

Proposition 8.2 ($\overline{\partial}$-Poincaré Lemma)**.** *Every $\overline{\partial}$-closed smooth (p,q)-form ω is locally $\overline{\partial}$-exact*

Corollary 8.3. *The de Rham complex*

$$A^0(X) \xrightarrow{d} A^1(X) \xrightarrow{d} \cdots$$

is an exact sequence of sheaves.

The constant sheaf $\underline{\mathbb{R}}$ of locally constant real-valued functions naturally lives as a subsheaf of $A^0(X)$, and we know that this exactly the kernel of $d : A^0(X) \to A^1(X)$. This tells us that the inclusion $0 \to \underline{\mathbb{R}} \to A^\bullet(X)$ is a resolution of $\underline{\mathbb{R}}$, called the ***de Rham resolution***. Furthermore, since all the $A^i(X)$ are sheaves of C^∞-modules, they are fine, so the resolution is a resolution of $\underline{\mathbb{R}}$ by acyclic sheaves. Therefore, we get the isomorphisms

$$H^i(X, \underline{\mathbb{R}}) \cong H^i_{dR}(X)$$

A similar story holds for the sheaf cohomology of the sheaf of sections of a holomorphic vector bundle $E \to X$. The $\overline{\partial}$-Poincaré lemma implies that the Dolbeault complex

$$\mathcal{A}^0(E) \xrightarrow{\overline{\partial}_E} \mathcal{A}^1(E) \xrightarrow{\overline{\partial}_e} \cdots$$

of sheaves of smooth sections of $(\Lambda^i T^*X)_{\mathbb{C}} \otimes E$ is an exact sequence, since $\overline{\partial}_E$ is defined locally in terms of the operator $\overline{\partial}$ on X. Then since the kernel of $\overline{\partial}_E : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ is exactly the sheaf $\mathcal{E}$ of holomorphic sections of E, we get that $0 \to \mathcal{E} \to \mathcal{A}^\bullet(E)$ is an acyclic resolution of $\mathcal{E}$, which gives us isomorphisms

$$H^i(X, \mathcal{E}) \cong H^i_{\overline{\partial}}(X, E)$$

9. The First Chern Class

On any complex manifold X, we have the following exact sequence of sheaves:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \longrightarrow 0$$

where exp is the sheaf morphism where a function $f \in \mathcal{O}_X(U)$ is mapped to the function e^f. This gives us the long exact sequence in sheaf cohomology

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,\mathbb{Z}) \arrow[r] & H^0(X,\mathcal{O}_X) \arrow[r] & H^0(X,\mathcal{O}_X^\times) \arrow[dll] \\
& H^1(X,\mathbb{Z}) \arrow[r] & H^1(X,\mathcal{O}_X) \arrow[r] & H^1(X,\mathcal{O}_X^\times) \arrow[dll, "c_1"'] \\
& H^2(X,\mathbb{Z}) \arrow[r] & \cdots &
\end{tikzcd}$$

We are particularly interested in the boundary map $H^1(X, \mathcal{O}_X^\times) \to H^2(X, \mathbb{Z})$, which we called c_1. Recall that we have an isomorphism $H^1(X, \mathcal{O}_X^\times) \cong \mathrm{Pic}(X)$.

Definition 9.1. The ***first Chern class*** of a line bundle L is its image under the map c_1, using the canonical identification $H^1(X, \mathcal{O}_X^\times) \cong \mathrm{Pic}(X)$.

The first Chern class will prove to be a powerful invariant of line bundles, and our goal will be to understand these cohomology classes in terms of the geometry of the manifold X. We first note some properties of c_1 that can be immediately deduced from the definition and our knowledge of the group structure on $\mathrm{Pic}(X)$.

Proposition 9.2.

(1) $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$
(2) $c_1(L^*) = -c_1(L)$.
(3) *For a map* $F : X \to Y$, *we have* $c_1(F^*L) = F^*c_1(L)$.

The first two observations are immediate, and the last one follows from functoriality of cohomology.

Using some Hodge theory, we can already learn something about $\mathbb{CP}^n$ using the first Chern class.

Theorem 9.3.

$$\mathrm{Pic}(\mathbb{CP}^n) \cong \mathbb{Z}$$

Proof. Hodge theory gives us the Hodge decomposition

$$H^k(\mathbb{CP}^n, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X, \mathbb{C})$$

as well as isomorphisms

$$H^q(\mathbb{CP}^n, \Omega^{p,0}_{\mathbb{CP}^n}) \cong H^{p,q}(\mathbb{CP}^n, \mathbb{C})$$

We know the singular cohomology of $\mathbb{CP}^n$ is given by

$$H^k(\mathbb{CP}^n, \mathbb{C}) \cong \begin{cases} \mathbb{C} & k \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

In particular, if $p + q$ is odd, then we know that $H^{p,q}(X, \mathbb{C}) = 0$. A special case is

$$0 = H^{1,0}(\mathbb{CP}^n, \mathbb{C}) \cong H^1(\mathbb{CP}^n, \Omega^{0,0}_{\mathbb{CP}^n}) = H^1(\mathbb{CP}^n, \mathcal{O}_{\mathbb{CP}^n})$$

Therefore, we have that in the long exact sequence induced by the exponential exact sequence, we have

$$0 \longrightarrow \mathrm{Pic}(\mathbb{CP}^n) \xrightarrow{c_1} \mathbb{Z}$$

which tells us that c_1 is injective. Since the only subgroups of $\mathbb{Z}$ are isomorphic to $\mathbb{Z}$, this gives $\mathrm{Pic}(\mathbb{CP}^n) \cong \mathbb{Z}$. In addition, it tell us that c_1 is a complete isomorphism invariant. Two holomorphic line bundles over $\mathbb{CP}^n$ are isomorphic if and only if they have the same first Chern class. ∎

Given that the first Chern class is defined using sheaves of holomorphic functions, as well as our preliminary observation with $\mathbb{CP}^n$, one might first think that c_1 is an invariant of holomorphic line bundles, but the truth is that it is a slightly coarser invariant – it cannot distinguish between holomorphic line bundles and smooth complex line bundles. To see this, let $C^\infty_{\mathbb{C}}$ denote the sheaf of smooth $\mathbb{C}$-valued functions on X. We note that we also have the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow C^\infty_{\mathbb{C}} \xrightarrow{\exp} (C^\infty_{\mathbb{C}})^\times \longrightarrow 0$$

which similarly induces a long exact sequence in cohomology, yielding a boundary map $\delta : H^1(X, (C^\infty)^\times) \to H^2(X, \mathbb{Z})$. Using functoriality of sheaf cohomology, the inclusions $\mathcal{O}_X \hookrightarrow C^\infty$ and $\mathcal{O}_X^\times \hookrightarrow (C^\infty)^\times$ induce maps on cohomology, giving us the commutative diagram

$$\begin{tikzcd}
H^1(X, C^\infty_{\mathbb{C}}) \arrow[r] & H^1(X, (C^\infty_{\mathbb{C}})^\times) \arrow[r, "\delta"] & H^2(X, \mathbb{Z}) \\
H^1(X, \mathcal{O}_X) \arrow[r] \arrow[u] & H^1(X, \mathcal{O}_X^\times) \arrow[r, "c_1"'] \arrow[u] & H^{(}X, \mathbb{Z}) \arrow[u, equal]
\end{tikzcd}$$

with exact rows. Given a holomorphic line bundle L, we can forget its complex structure, regarding it as a complex line bundle, i.e. an element of $H^1(X, (C^\infty_{\mathbb{C}})^\times)$, and then take the image under δ. Commutativity of the diagram implies that this is the same thing as $c_1(L)$. Since $C^\infty_{\mathbb{C}}$ is a fine sheaf, we have that this diagram is actually

$$\begin{tikzcd}
0 \arrow[r] & H^1(X, (C^\infty_{\mathbb{C}})^\times) \arrow[r, "\delta"] & H^2(X, \mathbb{Z}) \\
H^1(X, \mathcal{O}_X) \arrow[r] \arrow[u] & H^1(X, \mathcal{O}_X^\times) \arrow[r, "c_1"'] \arrow[u] & H^{(}X, \mathbb{Z}) \arrow[u, equal]
\end{tikzcd}$$

which tells us that δ is injective. Therefore, on smooth complex line bundles, the first Chern class is a perfect invariant – two complex line bundles are smoothly isomorphic if and only if they have the same first Chern class. However, this also reveals to us that it cannot distinguish between different holomorphic structures on the same underlying complex line bundle. However, we can use the complex geometry as a tool to compute certain quantities of a holomorphic bundle, which will turn out to be topological invariants of the bundle, independent of the holomorphic structure. Our goal will be to realize the first Chern class of a line bundle as the cohomology class associated to the curvature of a connection. Eventually, we will add a third characterization to the mix by introducing the Atiyah class.

We first discuss the relationship of c_1 with the curvature of a connection on a complex line bundle. We will discuss connections on complex vector bundles in general, and will then specialize to the case of line bundles.

Definition 9.4. Let $E \to X$ be a complex vector bundle, and let $\mathcal{A}^k(E)$ denote the sheaf of smooth E-valued k-forms, i.e. the sheaf of sections of the bundle $\Lambda^k(T^*X)_{\mathbb{C}} \otimes E$, which is a sheaf of $\mathbb{C}^\infty_{\mathbb{C}}$-modules. A ***connection*** ∇ is a $\mathbb{C}$-linear map $\nabla : \mathcal{A}^0 \to \mathcal{A}^1$ where for a smooth function f and local section s, we have the ***Leibniz rule***

$$\nabla(fs) = df \otimes s + f \nabla s$$

A connection can be naturally extended to an operator $\nabla : \mathcal{A}^k \to \mathcal{A}^{k+1}$, where for a k-form α and a local section s of E, we define

$$\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^k \alpha \wedge \nabla s$$

In a similar fashion, the connection ∇ on E induces connection on associated bundles (e.g. tensor powers, duals). In particular, it induces a connection on the bundle $\operatorname{End}(E)$, which by abuse of notation we also denote ∇. Given a section F of $\operatorname{End}(E)$, we have that the action of ∇F on a section s is given by the formula

$$(\nabla F)(s) = \nabla(F(s)) - F(\nabla s)$$

Proposition 9.5. *Let ∇_1 and ∇_2 be two connections on a complex vector bundle $E \to X$. Then $\nabla_1 - \nabla_2$ is an element of $\mathcal{A}^1(\operatorname{End} E)$.*

Proof. We want to show that the difference is $C^\infty_{\mathbb{C}}$-linear. We compute

$$\begin{aligned}(\nabla_1 - \nabla_2)(fs) &= \nabla_1(fs) - \nabla_2(fs)\\ &= df \otimes s + f\nabla_1 s - (df \otimes s + f\nabla_2 s)\\ &= f(\nabla_1 - \nabla_2)(s)\end{aligned}$$

■

Proposition 9.6. *Let $a \in \mathcal{A}^1(\operatorname{End} E)$ and let ∇ be a connection on E. Then $\nabla + a$ is a connection on E, where given a local section s, we define $\nabla + a$ by*

$$(\nabla + a)s = \nabla s + as$$

where given a tangent vector v, we have that a acts on s by $a(v)s$.

Proof. We clearly have that $\nabla + a$ is $\mathbb{C}$-linear. To check that it satisfies the Leibniz rule, we compute

$$\begin{aligned}(\nabla + a)(fs) &= \nabla(fs) + a(fs)\\ &= df \otimes s + f\nabla s + fas\\ &= df \otimes s + f(\nabla + a)(s)\end{aligned}$$

■

Therefore, given a complex vector bundle $E \to X$ equipped with connection, the connection is locally of the form $d + A$, where d is the usual complexified de Rham differential and A is a matrix of 1-forms.

Definition 9.7. Let $E \to X$ be a complex vector bundle equipped with a connection ∇. The ***curvature*** of ∇ is defined to be the map $F_\nabla := \nabla^2 : \mathcal{A}^0(E) \to \mathcal{A}^2(E)$.

One might expect the curvature to be a second order differential operator, but a small miracle happens.

Proposition 9.8. *The curvature transformation F_∇ is $C^\infty_{\mathbb{C}}$-linear, i.e. it defines a global section of $\mathcal{A}^2(\operatorname{End} E)$.*

Proof. We compute for a function f and local section s

$$\begin{aligned}
F_\nabla(fs) &= \nabla(\nabla(fs)) \\
&= \nabla(df \otimes s + f\nabla s) \\
&= d^2 f \otimes s - df \wedge \nabla s + \nabla(f\nabla s) \\
&= -df \wedge \nabla s + df \wedge \nabla s + f\nabla^2 s \\
&= f\nabla^2 s \\
&= fF_\nabla(s)
\end{aligned}$$

■

Theorem 9.9 (***The Bianchi Identity***)**.** *Let F_∇ be the curvature form of a connection on a vector bundle $E \to X$. Then*

$$\nabla(F_\nabla) = 0$$

Proof. Let s be a section of E. By the definition of the induced connection on $\mathrm{End}(E)$, we compute

$$\begin{aligned}
(\nabla(F_\nabla))(s) &= \nabla(F_\nabla(s)) - F_\nabla(\nabla s) \\
&= \nabla^3 s - \nabla^3 s \\
&= 0
\end{aligned}$$

■

Finally we make one more observation

Proposition 9.10. *Let ∇ be a connection on $E \to X$, and $\nabla' = \nabla + A$. Then*

$$F_{\nabla'} = F_\nabla + dA + A \wedge A$$

Proof. We compute locally for a section s,

$$\begin{aligned}
F_{\nabla'} &= (d + A)((d + A)(s)) \\
&= (d + A)(ds + A(s)) \\
&= d^2 s + A(ds) + d(A(s)) + (A \wedge A)(s) \\
&= dA(s) + (A \wedge A)(s)
\end{aligned}$$

■

In the case of a line bundle $L \to X$, much of this discussion simplifies. First of all we have that the bundle $\mathrm{End}(L)$ is a trivial bundle, so we have that every connection is locally of the form $d + A$ for a 1-form A. This allows several small miracles to occur.

Proposition 9.11. *The curvature F_∇ of a connection on a complex line bundle $L \to M$ is a globally defined 2-form in $H^2(X, \mathbb{C})$.*

Proof. We have that locally, $\nabla = d + A$, so we have locally

$$F_\nabla = dA + A \wedge A = dA$$

We then need to show that these local definitions glue. In two trivializing neighborhoods U_i and U_j with transition functions ψ_{ij} and ψ_{ji}, let A_i and A_j denote the connection

$1-$forms over each trivialization. Let s be a local section over $U_i \cap U_j$ that is nonzero and constant when viewed as a section of the trivial bundle $U_i \cap U_j \times \mathbb{C}$. Therefore, we have that Then the local representations s_i and s_j are related by $\psi_{ij} s_j = s_i$. In addition, we have that over U_i, $\nabla s = ds_i + A_i s_i = A_i s_i$, since s_i is a constant function, and we have a similar picture over U_j. We then compute

$$A_i \psi_{ij} s_j = \nabla(\psi_{ij} s_j) = d\psi_{ij} s_j + \psi_{ij} \nabla s_j = d\psi_{ij} s_j + \psi_{ij} A_j s_j$$

Therefore, we have

$$A_i \psi_{ij} = d\psi_{ij} + \psi_{ij} A_j$$

Multiplying both sides by ψ_{ij}^{-1}, moving terms around, we find

$$A_j = A_i - \psi_{ij}^{-1} d\psi_{ij}$$

We then verify that the local descriptions of F_∇ agree on the intersection. We compute

$$\begin{aligned}
F_\nabla &= dA_i \\
&= \psi_{ij}^{-1} d\psi_{ij} \wedge \psi_{ij}^{-1} d\psi_{ij} + dA_i \\
&= \psi_{ij}^{-2} d\psi_{ij} \wedge d\psi_{ij} + dA_i \\
&= -d\psi_{ij}^{-1} \wedge d\psi_{ij} + dA_i \\
&= d(A_i - \psi_{ij}^{-1} d\psi_{ij}) \\
&= dA_j
\end{aligned}$$

Where we use the fact that $d(\psi_{ij}^{-1}) = \psi_{ij}^{-2} d\psi_{ij}$. Therefore, the local descriptions of F_∇ glue to a global 2-form. ∎

In addition, since we have locally that F_∇ is given by dA_i, we have that F_∇ is closed, so it defines a cohomology class in $H^2(X, \mathbb{C})$. This gives us two ways of obtaining a cohomology class from a line bundle. The first is by taking the first Chern class from the exponential exact sequence, and the other is by taking the curvature form of a connection.

Theorem 9.12. *Let $L \to X$ be a complex line bundle, and let ∇ be any connection on L with curvature form F_∇. Then*

$$\left[\frac{i}{2\pi} F_\nabla\right] = c_1(L)$$

Proof. We first give an explicit formula for the Čech cocycle defining $c_1(L)$. Fix a good cover $\mathcal{U} = \{U_i\}$ for X, and let the ψ_{ij} be the transition functions for L with respect to this covering. The ψ_{ij} determine a Čech cocycle in $H^1(X, (C^\infty_{\mathbb{C}})^\times)$. By surjectivity of exp, by passing to a finer cover we may assume that we can find a branch of the logarithm for each ψ_{ij}, so $\log \psi_{ij}$ is well defined, and defines a Čech cocycle in $H^1(X, (C^\infty)^\times)$. Then since the ψ_{ij} satisfy the cocycle condition

$$\psi_{ij} \psi_{jk} \psi_{ik}^{-1} = 1$$

we have that

$$\log \psi_{ij} + \log \psi jk - \log \psi_{ik} \in 2\pi i \mathbb{Z}$$

So we have that

$$\frac{1}{2\pi i}\log\psi_{ij}+\log\psi_{jk}-\log\psi_{ik}\in\mathbb{Z}$$

is the cocycle representing $c_1(L)$. In the natural map $H^2(X,\mathbb{Z})\to H^2(X,\mathbb{C})$, we have that the cocycle has the same formula (up to a sign depending on convention).

Then let ∇ be any connection on L, with local connection forms A_i. We have that F_∇ is a closed 2-form, with local description $F_\nabla = dA_i$. Let $\mathcal{A}^k(X)$ denote the sheaf of smooth k-forms over X, and let $\mathcal{Z}^k(X)$ denote the sheaf of closed smooth k-forms over X. Then we have the exact sequences of sheaves

$$0\longrightarrow \mathcal{Z}^1(X)\longrightarrow \mathcal{A}^1(X)\xrightarrow{d}\mathcal{Z}^2(X)\longrightarrow 0$$

$$0\longrightarrow \mathbb{C}\longrightarrow \mathcal{A}^0(X)\xrightarrow{d}\mathcal{Z}^1(X)\longrightarrow 0$$

which give rise to boundary homomorphisms $\delta_1 : H^0(X,\mathcal{Z}^2(X))\to H^1(X,\mathcal{Z}^2(X))$ and $\delta_2 : H^1(X,\mathcal{Z}^1(X))\to H^2(X,\mathbb{C})$ respectively. The fact that the local definitions $F_\nabla = dA_i$ glue to a global form is exactly the statement that the dA_i form a Čech cocycle in $H^0(X,\mathcal{Z}^2(X))$, i.e. $dA_i - dA_j = 0$ for all i,j with $U_i\cap U_k\neq\varnothing$. Following the definition of the boundary homomorphism, this implies that it is the image of the cocycle determined by the A_i under d, and then applying the Čech differential we get that $\delta_1(F_\nabla)$ is the cocycle $\{A_j - A_i\}$. Then recall from our earlier calculation that $A_j = A_i - \psi_{ij}^{-1}d\psi_{ij}$, we get that

$$A_j - A_i = -\psi_{ij}^{-1}d\psi_{ij}$$

which is exactly $-d\log\psi_{ij}$. Then tracing though the definition of the boundary homomorphism δ_2, we have that $\delta_2(A_j - A_i)$ is the preimage of the Čech differential applied to the cocycle $\log\psi_{ij}$, which is exactly $-(\log\psi_{ij}+\log\psi_{jk}-\log\psi_{ik})$. Putting everything together, we find that

$$c_1(L)=\frac{1}{2\pi i}\log\psi_{ij}+\log\psi_{jk}-\log\psi_{ik}=\left[\frac{i}{2\pi}F_\nabla\right]$$

∎

10. The Chern Connection

Definition 10.1. Let $E\to X$ be a complex vector bundle. A ***Hermitian structure*** on E is the data of a smoothly varying hermitian form[1] $H(x)$ on each fiber E_x. If E is a complex vector bundle equipped with a Hermitian structure, we call it a ***Hermitian vector bundle***.

Hermitian structures always exist due to convexity of the space of Hermitian forms. Therefore, we can define Hermitian forms locally in neighborhoods where the bundle E is trivial, and then use a partition of unity to glue them together. Like in the case of Riemannian geometry, we can ask for a connection to be compatible with a Hermitian structure.

[1] Our convention will be that Hermitian forms are complex linear in the first argument and conjugate linear in the second argument.

Definition 10.2. Let $E \to X$ be a Hermitian vector bundle. A connection ∇ on E is a ***Hermitian connection*** if for sections s_1, s_2, we have

$$dh(s_1, s_2) = h(\nabla s_1, s_2) + h(s_1, \nabla s_2)$$

In the complex world, we can also impose a holomorphic structure on vector bundles, and we can also ask for a compatibility condition with the holomorphic structure.

Definition 10.3. Let $E \to X$ be a holomorphic structure. A connection ∇ on E is said to be ***compatible with the holomorphic structure*** if the $(0,1)$ component $\nabla^{0,1}$ is equal to $\overline{\partial}_E$.

This leads us to a natural question : given a holomorphic vector bundle $E \to X$ equipped with a Hermitian structure h, can we find a connection ∇ that is compatible both with the Hermitian structure and the holomorphic structure?

Theorem 10.4 (***The Chern connection***)**.** *Let $E \to X$ be a Hermitian holomorphic vector bundle. There exists a unique Hermitian connection ∇, called the **Chern connection** that is compatible with the holomorphic structure.*

Proof. The proof has a similar flavor to other proofs of existence and uniqueness of global objects. We use the compatibility conditions to determine local conditions that the connection must satisfy, and show that it uniquely specifies the connection. Therefore, the local definitions glue to a global one. In a local trivialization of E, we have that any connection ∇ is of the form $d + A$ for a matrix valued 1-form A. In addition, we have that the Hermitian structure gives a Hermitian form $H(x)$ at each point x. Let $\{E_i\}$ the local frame for E, where each E_i is just the constant section $x \mapsto e_i$. Then we get a coordinate representation of the Hermitian form H. $H^i_j(x) = h(E_i, E_j)$. We clearly have that $dE_i = 0$, since all the sections are constant, so $\nabla E_i = AE_i$. The the condition that ∇ is Hermitian gives us

$$dH^i_j = dH(E_i, E_j) = h(AE_i, E_j) + h(E_i, AE_j)$$

Writing the matrix A in coordinates A^i_j, this gives us

$$dH^i_j = A^k_i h(E_k, E_j) + h(E_i, E_\ell)\overline{A^\ell_j} = A^k_i H^k_j + H^i_\ell \overline{A^\ell_j}$$

which tells us that we must have that

$$dH = A^T H + H\overline{A}$$

We then consider the restrictions imposed on ∇ by the condition that ∇ is compatible with the holomorphic structure. Decomposing $\nabla = \nabla^{1,0} + \nabla^{0,1}$, we want $\nabla^{0,1} = \overline{\partial}_E$. In our local picture, we have that $\nabla = \partial + \overline{\partial} + A$, and $\overline{\partial}_E$ is just $\overline{\partial}$. Therefore, we must have that $A^{0,1} = 0$, so $A = A^{1,0}$. Decomposing $d = \partial + \overline{\partial}$ and applying it to our Hermitian condition, we find that

$$\partial H + \overline{\partial} H = A^T H + H\overline{A}$$

comparing the bidegrees then gives

$$\partial H = A^T H$$
$$\overline{\partial} H = H\overline{A}$$

The second equation then tells us that

$$\overline{H}^{-1}\partial\overline{H} = A$$

which gives a unique characterization of the matrix A, as well as a local definition, showing existence and uniqueness of the Chern connection. ■

Unsurprisingly, if we have a connection compatible with a Hermitian or holomorphic structure, then the corresponding curvature forms are also compatible.

Proposition 10.5.

(1) *Let $E \to X$ be a Hermitian vector bundle with Hermitian connection ∇. Then F_∇ is locally given by a skew-Hermitian matrix of 2-forms. Globally, we have*

$$h(F_\nabla s_1, s_2) + h(s_1, F_\nabla s_2) = 0$$

(2) *Let $E \to X$ be a holomorphic vector bundle. Then F_∇ has no $(0,2)$ part, i.e. $F_\nabla = F_\nabla^{2,0} + F_\nabla^{1,1}$.*

Proof.

(1) Locally E is isomorphic to the trivial bundle with the standard Hermitian structure, so $\nabla = d + A$ for a skew-Hermitian matrix A. Then using the local formula

$$F_\nabla = dA + A \wedge A$$

we want to show that $F_\nabla^\dagger = -F_\nabla$. We compute

$$\begin{aligned} F_\nabla^\dagger &= d(A^\dagger) + (A \wedge A)^\dagger \\ &= d(A^\dagger) - A^\dagger \wedge A^\dagger \end{aligned}$$

The fact that $(A \wedge A)^\dagger = -A^\dagger \wedge A^\dagger$ is slightly tricky, and comes from the skew symmetry of the wedge product. Explicitly in components, we have

$$\begin{aligned} ((A \wedge A)^\dagger)^i_j &= (\overline{A} \wedge \overline{A})^j_i \\ &= \overline{A}^j_k \wedge \overline{A}^k_i \\ &= -\overline{A}^i_k \wedge \overline{A}^k_j \\ &= -(A^\dagger \wedge A^\dagger)^i_j \end{aligned}$$

We the complete the computation, using the fact that A is skew-Hermitian

$$\begin{aligned} F_\nabla^\dagger &= d(A^\dagger) - A^\dagger \wedge A^\dagger \\ &= -dA - A \wedge A \\ &= -F_\nabla \end{aligned}$$

(2) Locally, we may write $\nabla = d + A$ Since ∇ is compatible with the holomorphic structure, we also have that $A = A^{1,0}$. We have that locally

$$F_\nabla = dA + A \wedge A$$

Splitting $d = \partial + \overline{\partial}$, we get

$$F_\nabla = (\partial + \overline{\partial})(A) + A \wedge A = \overline{\partial}A + \partial A + A \wedge A$$

and since A is type $(1,0)$, we get that $\overline{\partial}A$ is type $(1,1)$ and $\partial A + A \wedge A$ is type $(2,0)$. ■

Putting these together, we get a result on the type of the curvature of a Chern connection.

Proposition 10.6. *Let ∇ be the Chern connection for a Hermitian holomorphic vector bundle $E \to X$. Then F_∇ is type $(1,1)$.*

Proof. Since ∇ is Hermitian, we have that $F_\nabla^\dagger = -F_\nabla$. Since ∇ is compatible with the holomorphic structure, we have that $F_\nabla = F_\nabla^{2,0} + F_\nabla^{1,1}$. Then consider

$$F_\nabla^\dagger = (F_\nabla^{2,0})^\dagger + (F_\nabla^{1,1})^\dagger$$

We have that $(F_\nabla^{2,0})^\dagger$ is type $(0,2)$ and $(F_\nabla^{1,1})^\dagger$ is type $(1,1)$. Therefore, in order to have $F_\nabla^\dagger = -F_\nabla$, we must have that $(F_\nabla^{1,1})^\dagger = -F_\nabla^{1,1}$ and $F_\nabla^{2,0} = 0$ by a simple type check. ■

This yields an important result regarding the curvature of of Chern connection.

Theorem 10.7. *The curvature form F_∇ of a Chern connection on a holomorphic vector bundle $E \to X$ is $\overline{\partial}_E$-closed, so it defines a Dolbeault cohomology class $[F_\nabla] \in H^1(X, \Omega_X \otimes \mathrm{End}(E))$, where Ω_X denotes the bundle of holomorphic 1-forms.*

Proof. Applying the Bianchi Identity to the curvature form $F_\nabla = F_\nabla^{1,1}$, we find

$$0 = \nabla(F_\nabla)^{1,2} = \overline{\partial}_E F_\nabla$$

where we use the fact that the induced connection on $\mathrm{End}(E)$ is also compatible with the holomorphic structure. ■

This gives some intuition behind the Chern connection. For an arbitrary connection on a holomorphic vector bundle E, we need not have that the curvature is $\overline{\partial}_E$-closed. This shows that asking for a connection to be a Chern connection is a sort of integrability condition. In addition, it gives a generalization of the first Chern class for line bundles to holomorphic bundles of arbitrary rank. In the line bundle case, we did not have to worry about curvature being closed, but in the higher dimensional case, we have to restrict our attention to Chern connections.

Given a holomorphic vector bundle $E \to X$, we now get a recipe for producing Dolbeault cohomology classes in $H^1(X, \Omega_X \otimes \mathrm{End}(E))$. In addition, We can place a Hermitian structure on E, and then compute the curvature of the Chern connection. Naturally, the next question is whether this cohomology class depends on the choice of Hermitian structure.

Theorem 10.8. *The cohomology class $[F_\nabla]$ of the curvature of a Chern connection on a holomorphic vector bundle E is independent on the choice of Hermitian structure.*

Proof. Let ∇ be the Chern connection with respect to some Hermitian structure on E. Then any other connection is of the form $\nabla + A$ with curvature $F_{\nabla+A} = F_\nabla + \nabla A + A \wedge A$. Then if $\nabla' = \nabla + A$ is the Chern connection with respect to a different Hermitian structure, then A must be type $(2,0)$. In addition, we know that both F_∇ and $F_{\nabla'} = F_\nabla + \nabla A + A \wedge A$ are both type $(1,1)$. In order for this to be true, we must have that $\nabla A + A \wedge A$ is of type $(1,1)$. Since $A \wedge A$ is of type $(2,0)$, we get

$$(\nabla A + A \wedge A) = (\nabla A \wedge A)^{1,1} = \nabla A$$

Then since A is type $(1,0)$, the $(1,1)$ part of A is $\nabla^{0,1} A = \overline{\partial}_E A$ since ∇ is compatible with the holomorphic structure. Therefore, we have that $\nabla A + A \wedge A = \overline{\partial}_E A$, giving us

$$F_{\nabla'} = F_\nabla + \overline{\partial}_E A$$

so $[F_{\nabla'}] = [F_\nabla]$. ■

Definition 10.9. Let $E \to X$ be a holomorphic vector bundle of rank k, and $\mathcal{U} = \{U_i\}$ a good cover of X with local trivializations $\varphi_i : E|_{U_i} \to U_i \times \mathbb{C}^k$ and transition functions $\psi_{ij} : U_i \cap U_j \to \mathrm{GL}_k\mathbb{C}$. The ***Atiyah class*** of E, denoted $A(E)$ is the element $A(E) \in H^1(X, \Omega_X \otimes \operatorname{End} E)$ determined by the cocycle $\sigma_{ij} = \varphi_j^{-1} \circ (\psi_{ij}^{-1} d\psi_{ij}) \circ \varphi_j$, where Ω_X denotes the bundle of holomorphic 1-forms.

We first make sense of the formula for the cocycle. A component σ_{ij} of the Čech cocycle σ should be a local section $\sigma_{ij} : U_i \cap U_j \to \Omega_X \otimes \operatorname{End} E$. What this means is that we should be able to feed σ_{ij} the following data:

(1) A point $p \in U_i \cap U_j$
(2) A tangent vector $v \in T_pX$
(3) A vector $w \in E_p$ in the fiber

and we should expect another vector in E_p to be the output. We have that $\varphi(p, w)$ should be a pair $(p, w') \in U_i \cap U_j \times \mathbb{C}^k$. Then since $\psi_{ij} : U_i \cap U_j \to \mathrm{GL}_k\mathbb{C}$, we have that $d(\psi_{ij})_p : T_pX \to M_n\mathbb{C}$. Therefore, we have that $d(\psi_{ij})_p(v)$ is a matrix, which we can then apply to w'. Then we can apply the matrix inverse $(\psi_{ij}(p))^{-1}$ to the result, leaving us with some other pair $(p, w'') \in U_i \cap U_j \times \mathbb{C}^k$, and then applying the inverse of the local trivialization φ_j to this gives us the desired output. The fact that this defines a cocycle essentially boils down to the fact that the ψ_{ij} satisfy the cocycle condition, along with using the chain rule for the differentials $d\psi_{ij}$.

11. Vanishing Theorems

For the following discussion, we will restrict our attention to a compact Kähler manifold X, which will allow us to use some tools from Hodge theory, which we will assume.

Definition 11.1. A ***positive*** $(1,1)$ ***form*** on X is a differential form $\omega \in \mathcal{A}^{1,1}$ such that in local holomorphic coordinates $\{z^i\}$, we have that

$$\omega = \frac{i}{2} \sum_{i,j} h_{ij}(z) dz^i \wedge d\bar{z}^j$$

where h_{ij} is a positive definite Hermitian matrix for all z. Equivalently, ω is a positive form if and only if it represents the Kähler form for some Hermitian metric on X.

Definition 11.2. A holomorphic line bundle $L \to X$ is ***positive*** if there exists a Hermitian metric h on X with Chern connection ∇ such that $(i/2\pi)F_\nabla$ is a positive $(1,1)$ form.

Positivity of a line bundle turns out to only depend on the first Chern class of the line bundle. From our previous discussion, this tells us that it only cares about the topological isomorphism class of the line bundle, and is independent of a holomorphic structure. To show this, we'll need two useful lemmas. The first is a classic lemma from Kähler geometry.

Lemma 11.3 (***The*** $\partial\bar{\partial}$***-lemma***)**.** *Let η be a smooth complex valued form such that η is both ∂ and $\bar{\partial}$ closed. Then if η is d, ∂, or $\bar{\partial}$-exact, then there exists a form ξ such that $\eta = \partial\bar{\partial}\xi$. Furthermore, if η is real (i.e. $\bar{\eta} = \eta$), we may take $i\xi$ to be real.*

The other lemma characterizes curvatures of metric compatible connection on line bundles.

Lemma 11.4. *Let $L \to X$ be a holomorphic line bundle equipped with Hermitian metric h. Locally, h is given by a positive function $h(z)$ on X. Then the curvature F_∇ of the metric compatible connection ∇ is given by the formula*

$$F_\nabla = -\partial\overline{\partial}\log h(z)$$

Proof. This follows from our earlier computations for Chern connections. Locally, the Chern connection ∇ is given by $d + A$ for a complex 1-form A, which from our earlier computations, must be holomorphic and satisfy

$$A = \overline{h}^{-1}\partial\overline{h}$$

However, since h is real, this is the same as $A = h^{-1}\partial h$. The curvature is then given by $F_\nabla = dA = (\partial A + \overline{\partial}A)$. However, we note that by a type check, ∂A is of type $(2,0)$, and since the curvature of the Chern connection is of type $(1,1)$, we have that

$$F_\nabla = \overline{\partial}\left(\frac{\partial h}{h}\right) = \overline{\partial}\partial\log h$$

the desired identity then follows from the fact that ∂ and $\overline{\partial}$ anticommute. ∎

Proposition 11.5. *Let $L \to X$ be a line bundle and let ω be a real closed $(1,1)$ form such that the cohomology class of ω is equal to $c_1(L)$. Then there exists a Hermitian metric on L with metric compatible connection ∇ such that $\omega = (i/2\pi)F_\nabla$.*

Proof. Let h be any Hermitian metric on L, and ∇ the Chern connection for h. We know that the curvature F_∇ satisfies

$$\left[\frac{i}{2\pi}F_\nabla\right] = c_1(L)$$

Then suppose we have another Hermitian metric h' with Chern connection ∇'. We know that we can write h' differs from h by multiplication by a smooth positive function, so we may write $h' = e^\rho h$ for some smooth function ρ. Then the curvature of ∇' satisfies

$$\begin{aligned} F_{\nabla'} &= -\partial\overline{\partial}\log e^\rho h \\ &= -\partial\overline{\partial}(\log e^\rho + \log h) \\ &= -\partial\overline{\partial}\rho + F_\nabla \end{aligned}$$

Which tells us that $F_\nabla = \partial\overline{\partial}\rho + F_{\nabla'}$. Then given a real closed $(1,1)$ form ω with $[(i/2\pi)\omega] = c_1(L)$, if we solve the equation

$$F_\nabla = \partial\overline{\partial}\rho + \omega$$

Then the Chern connection ∇' of the Hermitian form $e^\rho h$ will satisfy $F_\nabla = \partial\overline{\partial}\rho + F_{\nabla'}$, which then implies that $F_{\nabla'} = \omega$. To solve the equation, we note that ω and F_∇ are cohomologous by assumption, so their difference $F_\nabla - \omega$ is d-exact. Furthermore, from the local formula

$$F_\nabla = -\partial\overline{\partial}\log h$$

we see that F_∇ is both ∂ and $\overline{\partial}$-closed. Finally, the fact that ω is type $(1,1)$ and d-closed implies that it must be both ∂ and $\overline{\partial}$-closed by a simple type check, so we get that $F_\nabla -$

ω satisfies the conditions of the $\partial\overline{\partial}$-lemma, giving us that $F_\nabla - \omega = \partial\overline{\partial}\rho$ for some real function ρ. ∎

Recall that on a Kähler manifold X with Kähler form ω, we have the Lefschetz operator $L : \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q+1}$ given by wedging with ω. The operator L, together with its adjoint $\Lambda : \mathcal{A}^{p,q} \to \mathcal{A}^{p-1,q-1}$ satisfy the identities

(1) $[\Lambda, L] = (n - (p+q))\,\mathrm{id}_{\mathcal{A}^{p,q}(X)}$
(2) $[\Lambda, \overline{\partial}] = -i\partial^*$

where n is the complex dimension of X. The operators Λ and L can naturally be extended to vector bundle valued forms, and since Λ and L are both 0^{th} order operators, the first identity still holds for their extended versions. The second identity generalizes in a slightly modified form.

Proposition 11.6 (***Nakano identity***). *Let E be a holomorphic vector bundle equipped with a Hermitian metric with Chern connection ∇. Then*

$$[\Lambda, \overline{\partial}_E] = -i(\nabla^{1,0})^*$$

where $(\nabla^{1,0})^*$ is the adjoint of $\nabla^{1,0}$ with respect to the Hermitian inner product on $\mathcal{A}^{p,q}(E)$, which is explicitly given by the formula

$$(\nabla^{1,0})^* = \overline{\star}_E \nabla^{1,0}_{E^*} \overline{\star}_E$$

where ∇_{E^*} is the Chern connection for the dual bundle equipped with the dual Hermitian form.

Proof. Since it suffices to check in a local trivialization, we may assume that we have taken an orthonormal frame for E. However, this cannot be a holomorphic trivialization, so there will be some complications. We take inventory of the various operators in this trivialization. Since the trivialization was from an orthonormal frame, the operator $\star_E$ agrees with the usual Hodge star, so $\overline{\star}_E = \overline{\star}$. In addition, we can write the Chern connection ∇ as $d + A$ for some skew-Hermitian matrix of 1-forms A. This then gives us

$$\nabla_{E^*} = d + A^\dagger = d - A$$

The $\overline{\partial}_E$-operator is of the form $\overline{\partial} + A^{0,1}$, which is one of the added complications when the trivialization is not holomorphic. If the trivialization were holomorphic, we would have had $\overline{\partial}_E = \overline{\partial}$. Putting things together, we find

$$\begin{aligned}(\nabla^{1,0})^* &= \overline{\star}_E \nabla^{1,0}_{E^*} \overline{\star}_E \\ &= \overline{\star}(\partial - A^{1,0})\overline{\star} \\ &= \partial^* - (A^{1,0})^\dagger\end{aligned}$$

We then compute

$$\begin{aligned}[\Lambda, \overline{\partial}_E] + i(\nabla^{1,0})^* &= [\Lambda, \overline{\partial} + A^{0,1}] + i(\partial^* - (A^{1,0})^\dagger) \\ &= [\Lambda, \overline{\partial}] + [\Lambda, A^{0,1}] + i\partial^* - i(A^{1,0})^\dagger\end{aligned}$$

The ordinary Kähler identity then gives us that this is equal to $[\Lambda, A^{0,1}] - i(A^{1,0})^\dagger$. Since this is an order 0 operator, it suffices to verify that it vanishes when restricted to any fiber of E. However, since we can always pick a smooth local trivialization of E in which $A = 0$, we have that $[\Lambda, A^{0,1}] - i(A^{1,0})^\dagger = 0$, which them implies that

$$[\Lambda, \overline{\partial}_E] = -i(\nabla^{1,0})^*$$

■

We then prove one final lemma.

Lemma 11.7. *Let E be a holomorphic vector bundles with a Hermitian metric and Chern connection ∇. Let $(\cdot,\cdot)$ denote the Hermitian inner product on $\mathcal{A}^{p,q}(E)$ induced by the Hermitian metric on E and the Hermitian metric on X. Then for a harmonic form $\alpha \in \mathcal{H}^{p,q}(E)$, we have the following inequalities:*

$$i(F_\nabla \Lambda\alpha, \alpha) \leq 0$$
$$i(\Lambda F_\nabla \alpha, \alpha) \geq 0$$

Proof. Since the Chern connection satisfies $\nabla^{0,1} = \overline{\partial}_E$ and the curvature F_∇ is of type $(1,1)$. we know that

$$F_\nabla = \nabla^{1,0}\overline{\partial}_E + \overline{\partial}_E\nabla^{1,0}$$

We then compute

$$\begin{aligned}
i(F_\nabla\Lambda\alpha,\alpha) &= i(\nabla^{1,0}\overline{\partial}_E\Lambda\alpha,\alpha) + i(\overline{\partial}_E\nabla^{1,0}\Lambda\alpha,\alpha)\\
&= i(\overline{\partial}_E\Lambda\alpha,(\nabla^{1,0})^*\alpha) + i(\nabla^{1,0}\Lambda\alpha,\overline{\partial}_E^*\alpha)\\
&= i(\overline{\partial}_E\Lambda\alpha,(\nabla^{1,0})^*\alpha) + i(\nabla^{1,0}\Lambda\alpha,0)\\
&= -(\overline{\partial}_E\Lambda\alpha, i(\nabla^{1,0})^*\alpha)\\
&= (\overline{\partial}_E\Lambda\alpha,[\Lambda,\overline{\partial}_E]\alpha)\\
&= (\overline{\partial}_E\Lambda\alpha,\Lambda\overline{\partial}_E\alpha) - (\overline{\partial}_E\Lambda\alpha,\overline{\partial}_E\Lambda\alpha)\\
&= -\left\|\overline{\partial}_E\Lambda\alpha\right\|_{L^2}^2 \leq 0
\end{aligned}$$

We perform a similar computation for the other inequalty.

$$\begin{aligned}
i(\Lambda F_\nabla\alpha,\alpha) &= i(\Lambda\nabla^{1,0}\overline{\partial}_E\alpha,\alpha) + i(\Lambda\overline{\partial}_E\nabla^{1,0}\alpha,\alpha)\\
&= i(\Lambda\overline{\partial}_E\nabla^{1,0}\alpha,\alpha)\\
&= i([\Lambda,\overline{\partial}_E]\nabla^{1,0}\alpha,\alpha) + i(\overline{\partial}_E\Lambda\nabla^{1,0}\alpha,\alpha)\\
&= i(-i(\nabla^{1,0})^*\nabla^{1,0}\alpha,\alpha) + i(\Lambda\nabla^{1,0}\alpha,\overline{\partial}_E^*\alpha)\\
&= (\nabla^{1,0}\alpha,\nabla^{1,0}\alpha)\\
&= \left\|\nabla^{1,0}\alpha\right\|_{L^2}^2 \geq 0
\end{aligned}$$

■

With all that done, we are ready to prove the Kodaira-Nakano vanishing theorem.

Theorem 11.8 (***Kodaira-Nakano Vanishing***)**.** *Let $L \to X$ be a positive line bundle over a compact Kähler manifold X of complex dimension n. Then for $p+1 > n$, we have*

$$H^q(X, \Omega_X^{p,0} \otimes L) = 0$$

Proof. From Hodge theory we know that $H^q(X, \Omega_X^{p,0} \otimes L)$ is isomorphic to the space $\mathcal{H}^{p,q}(L)$ of harmonic L-valued (p,q)-forms. Therefore, it suffices to show that no harmonic (p,q)-forms exist when $p+q > n$. Since L is a positive line bundle, it admits a Hermitian metric with such that $(i/2\pi)F_\nabla$ is equal to the Kähler form. Therefore, its action on $\mathcal{A}^{p,q}(L)$ is the same as the Lefschetz operator L. Using the commutation relation

$$[\Lambda, L] = (n - (p+q))\,\mathrm{id}_{\mathcal{A}^{p,q}(L)}$$

we get that

$$\frac{i}{2\pi}([\Lambda, F_\nabla]\alpha, \alpha) = ([\Lambda, L]\alpha, \alpha) = (n-(p+q))\,\|\alpha\|_{L^2}^2$$

However, we know that

$$i([\Lambda, F_\nabla]\alpha, \alpha) = i(\Lambda F_\nabla \alpha, \alpha) - i(F_\nabla \Lambda \alpha, \alpha) \geq 0$$

So if $p+q > n$, we must have that $\|\alpha\|_{L^2}^2 = 0$, i.e. $\alpha = 0$. ■

The lemmas we proved earlier also give rise to another vanshing theorem due to Serre.

Theorem 11.9 (***Serre Vanishing***)**.** *Let $L \to X$ be a positive line bundle. Then for any holomorphic vector bundle $E \to X$, there exists a positive integer m_0 such that*

$$H^q(X, E \otimes L^m) = 0$$

For any $m \geq m_0$.

Proof. We first fix Hermitian metrics on E and L, where the Hermitian metric on L is chosen such that the Chern connection ∇_L satisfies $(i/2\pi)F_{\nabla_L} = \omega$, where ω is the Kähler form of X. We note that an easy calculation gives us that $\nabla_{L^m} = m\omega$. Then let ∇_E denote the Chern connection for E, and let

$$\nabla = \nabla_E \otimes \nabla_{L^m} = \nabla_E \otimes \mathrm{id} + \mathrm{id} \otimes \nabla_{L^m}$$

The curvature of ∇ is then given by

$$F_\nabla = F_{\nabla_E} \otimes \mathrm{id} + m(\mathrm{id} \otimes \omega)$$

Using our lemmas from before, we then compute for $\alpha \in \mathcal{H}^{p,q}(E \otimes L^m)$

$$\begin{aligned} i([\Lambda, F_\nabla]\alpha, \alpha) &= ([\Lambda, F_{\nabla_E}]\alpha, \alpha) + i([\Lambda, mL]\alpha, \alpha) \\ &= i([\Lambda, F_{\nabla_E}]\alpha, \alpha) + (m(n-(p+q)))\,\|\alpha\|_{L^2}^2 \end{aligned}$$

The operator $[\Lambda, F_{\nabla_E}]$ is algebraic, so we can use a fiberwise Cauchy-Schwarz inequality

$$|\langle [\Lambda, F_{\nabla_E}]\alpha, \alpha\rangle| \leq \big\|[\Lambda, F_{\nabla_E}]\big\|\,\|\alpha\|^2$$

where we have chosen our favorite operator norm on $\mathrm{End}(E_x)$ for each fiber and we are computing the inner products and norms fiberwise. Compactness of X then ensures that we can obtain a global bound C such that $C \geq |\langle [\Lambda, F_{\nabla_E}]\alpha, \alpha\rangle|$ Then since $i([\Lambda, F_\nabla]\alpha, \alpha) \geq 0$, we have that when $p+q > n$, a sufficiently large choice of m will make $m(n-(p+q))$ sufficiently negative such that the only we we can have $\alpha = 0$. ■

With the Kodaira-Nakano vanishing theorem and more help from Hodge theory, we have another way to compute the sheaf cohomology of line bundles over $\mathbb{CP}^n$. To do this we note two facts.

(1) $\mathcal{O}(1)$ is a positive line bundle.

(2) $\mathcal{O}(-n-1) \cong K_{\mathbb{CP}^n}$.

The proof of the first fact follows from an easy computation of the curvature. $\mathcal{O}(1)$ embeds into the trivial bundle $\mathbb{CP}^n \times (\mathbb{C}^{n+1})^*$, which gives it a Hermitian metric coming from the obvious one on $(\mathbb{C}^{n+1})^*$. Then it can be shown that the Chern connection of this metric is just the Fubini-Study metric on $\mathbb{CP}^n$ (in fact, the standard definition of the Fubini-Study is secretly this construction). The proof of the second fact amounts to checking the determinants of the Jacobians of the transition functions for the standard open cover of $\mathbb{CP}^n$, and noting that they agree with the transition functions for $\mathcal{O}(-n-1)$. We will also use the following result from Hodge theory.

Theorem 11.10 (***Serre Duality***)**.** *Let X be a compact Kähler manifold of complex dimension n and let $E \to X$ be a holomorphic vector bundle. Then*

$$H^{n-q}(X, E^* \otimes K_X) \cong H^q(X, E)^*$$

We then apply the Kodaira-Nakano vanishing theorem to deduce which cohomology groups are 0. For $m > 0$, the vanishing theorem tells us that for $p + q > n$, we have $H^q(\mathbb{CP}^n, \Omega^{p,0} \otimes \mathcal{O}(m)) = 0$. Then since $\Omega^{n,0} = K_X = \mathcal{O}(-n-1)$, this tells us that $H^q(\mathbb{CP}^n, \mathcal{O}(m)) = 0$ for $q > 0$ and $m \geq -n$. Serre duality then gives a similar vanishing result for $K_X \otimes \mathcal{O}(m) = \mathcal{O}(-m-n-1)$. Putting everything together we get that

$$H^q(\mathbb{CP}^n, \mathcal{O}(m)) = \begin{cases} 0 & 0 < q < n \\ 0 & q = 0, m < 0 \\ 0 & q = n, m > -n-1 \end{cases}$$

Recall that the global sections of the line bundle $\mathcal{O}(m)$ are given by

$$H^0(\mathbb{CP}^n, \mathcal{O}(m)) = \mathbb{C}[z_0, \dots z_n]_m$$

for $m \geq 0$, are are 0 otherwise. Serre duality then tells us that

$$H^n(X, \mathcal{O}(-m-n-1)) \cong H^0(X, \mathcal{O}(m))^*$$

Therefore, we get that if $m \geq 0$,

$$H^q(\mathbb{CP}^n, \mathcal{O}(m)) = \begin{cases} \mathbb{C}[z_0, \dots z_n]_m & q = 0 \\ 0 & \text{otherwise} \end{cases}$$

In the case that $m < 0$, we have

$$H^q(\mathbb{CP}^n, \mathcal{O}(m)) = \begin{cases} \mathbb{C}[z_0, \dots z_n]^*_{n-m+1} & q = n \\ 0 & \text{otherwise} \end{cases}$$

12. Maps to Projective Space

Another application of the Kodaira-Nakano vanishing theorem is the Kodaira embedding theorem. To discuss this, we will need to first discuss two things:

(1) The relationship between line bundles and maps to $\mathbb{CP}^n$.
(2) Blow-ups, which allow us to turn points into divisors, which allows us to pass through the divisor-line bundle correspondence.

Suppose we are given a holomorphic line bundle $L \to X$ over a complex manifold X. Then given a section $s : X \to L$ and an open cover $\{U_i\}$ such that $L|_{U_i}$ is trivial, the section can be identified with a collection of holomorphic functions $s_i : U_i \to \mathbb{C}$ such that $s_i = \psi_{ij} s_j$, where the ψ_{ij} are the transition functions of L with respect to the cover $\{U_i\}$.

Proposition 12.1. *Suppose $s_0, \dots s_N$ are sections of a holomorphic line bundle $L \to X$ such that for any $x \in X$, we have $s_i(x) \neq 0$ for some i. Then the map $\varphi : X \to \mathbb{CP}^N$ defined in local trivializations by the mapping*

$$x \mapsto [s_0(x) : \dots : s_N(x)]$$

is well defined and satisfies $\varphi^* \mathcal{O}_{\mathbb{CP}^N}(1) \cong L$.

Proof. We first verify that φ is well-defined. let $\{U_\alpha\}$ be an open cover trivializing L, with transition functions $\psi_{\alpha\beta}$. Then let $s_{i,\alpha} : U_\alpha \to \mathbb{C}$ the functions corresponding to the s_i with respect to these local trivializations, so $s_{i,\alpha} = \psi_{\alpha\beta} s_{i,\beta}$. We note that over any U_α, the map we specified is well defined, since at least one $s_{i,\alpha}$ is nonzero by assumption. We then need to verify that the maps specified on the U_α agree on the intersections $U_{\alpha\beta}$, which amounts to the fact that for $x \in U_{\alpha\beta}$ we have

$$[s_{i,\alpha}(x) : \dots : s_{N,\alpha}(x)] = [\psi_{\alpha\beta}(x) s_{i,\beta}(x) : \dots : \psi_{\alpha\beta} s_{N,\beta}(x)]$$

since homogeneous coordinates are only defined up to scaling. This also shows that the map is independent of our choice of trivializations.

To show that $\varphi^* \mathcal{O}_{\mathbb{CP}^N} \cong L$, it suffices to show that $\varphi^* \mathcal{O}_{\mathbb{CP}^N} \cong \mathcal{O}(Z(s_0))$. Conside the hyperplane divisor H on $\mathbb{CP}^N$, corresponding to the hyperplane $\mathbb{CP}^{N-1}$ defined by the vanishing of the section z_0 of $\mathcal{O}(1)$. By intersecting the hyperplane with the image of φ_L, we get a divisor D on the image of φ_L corresponding to restriction of $\mathcal{O}(1)$. Furthermore, we observe that $\varphi^* D = Z(s_0)$ by construction, since s_0 is the first component of the map φ_L. Therefore, $\varphi^* \mathcal{O}_{\mathbb{CP}^n} \cong \mathcal{O}(Z(s_0)) \cong L$. ■

If the s_i simultaneously vanish, we do not get a map $X \to \mathbb{CP}^N$, but we do get a map $X/\mathrm{Bs}(s_0, \dots s_N) \to \mathbb{CP}^N$, where $\mathrm{Bs}(s_0, \dots s_N)$ is called the ***base locus*** of the sections s_i, and is defined to be the subset of X where the s_i simultaneously vanish.

We collect some terminology concerning this discussion regarding line bundles and maps to projective space.

Definition 12.2. Let $L \to X$ be a holomorphic line bundle.

(1) A ***linear system*** of L is a subspace of $\Gamma(X, L)$. The ***complete linear system*** is the entire space $\Gamma(X, L)$.
(2) If there exist sections $s_0, \dots s_N$ of L with $\mathrm{Bs}(s_0, \dots s_N) = \varnothing$, then L is ***globally generated***.

Given a linear system $S \subset \Gamma(X, L)$ of L, upon fixing a basis $s_0, \dots s_N$ of sections for S, we get a map $X - \mathrm{Bs}(s_0, \dots s_N) \to \mathbb{CP}^N$. While the map depends on this choice of basis, we see from the formula for the associated map that the map associated to any other basis is related by the projective linear automorphism of $\mathbb{CP}^N$ corresponding to the change of basis matrix.

Definition 12.3. A line bundle $L \to X$ is ***very ample*** if the complete linear system $\Gamma(X, L)$ defines a map to $\mathbb{CP}^N$ for some N. The bundle L is said to be ***ample*** if some tensor power L^k for $k > 0$ of L is very ample.

13. Blowing Up

One use for blowing up is to transform higher codimenision information into codimension one information, i.e. given some submanifold $Y \subset X$, the blow up along Y will be another complex manifold $\widehat{X}$ equipped with a map $\widehat{X} \to X$ such that the fiber over Y is codimension one, which we can then use to define a divisor. The general case will be modeled on the blow up of a linear subspace $\mathbb{C}^m \subset \mathbb{C}^n$.

Definition 13.1. The ***blow up*** of $\mathbb{C}^n$ along $\mathbb{C}^m \subset \mathbb{C}^n$ is the space

$$\mathrm{Bl}_{\mathbb{C}^m}(\mathbb{C}^n) := \left\{(\ell, z) \,:\, \ell \in \mathbb{CP}^{n-m-1}, z \in \mathbb{C}^m + \ell\right\}$$

where we interpret ℓ as a line in the complementary subspace $\mathbb{C}^{n-m}$ of $\mathbb{C}^m$.

The blow up $\mathrm{Bl}_{\mathbb{C}^m}(\mathbb{C}^n)$ comes equipped with a pair of maps, one to $\mathbb{CP}^{n-m-1}$ and one to $\mathbb{C}^n$, given by forgetting one of the components. The map $\mathrm{Bl}_{\mathbb{C}^m}(\mathbb{C}^n) \to \mathbb{CP}^{n-m-1}$ has the structure of a holomorphic $\mathbb{C}^{m+1}$-bundle over $\mathbb{CP}^{n-m-1}$. The other map $\pi : \mathrm{Bl}_{\mathbb{C}^m}(\mathbb{C}^n) \to \mathbb{C}^n$ is the one we asked for above. To see why, we first observe that for any $z \notin \mathbb{C}^m$, the span ℓ_z of s is an element of $\mathbb{CP}^{n-m-1}$, and the point (ℓ_z, z) is the unique point in the fiber $\pi^{-1}(z)$. On the other hand, if $z \in \mathbb{C}^m$, then the fiber above z is a copy of $\mathbb{CP}^{n-m-1}$, since z lies in the span of $\mathbb{C}^m + \ell$ for every $\ell \in \mathbb{CP}^{n-m-1}$. To generalize this to manifolds, it's useful to note that we identified the normal bundle of $\mathbb{C}^m \subset \mathbb{C}^n$ with $\mathbb{C}^m \times \mathbb{C}^{n-m}$, and then the fiber bundle $\pi^{-1}(\mathbb{C}^m)$ is the projectivized normal bundle of $\mathbb{C}^m$. Another imporant thing to note is that $\pi^{-1}(\mathbb{C}^m)$ is $n-1$ dimensional, i.e. codiemension one, accomplishing the goal we set out with.

An important special case is when $m = 0$, i.e. we are blowing up $0 \in \mathbb{C}^n$. In this case, the poins of $\mathrm{Bl}_0(\mathbb{C}^n)$ are described by pairs (ℓ, z) where $\ell \in \mathbb{CP}^n$ and $z \in \ell$. Then the fiber of π over a nonzero point $v \in \mathbb{C}^n$ is the span of v, and the fiber over 0 is $\mathbb{CP}^{n-1}$. Thinking of the line bundle $\mathcal{O}(-1)$ as a subbundle of the trivial vector bundle $\mathbb{CP}^{n-1} \times \mathbb{C}^n$, we see that the blow up $\mathrm{Bl}_0(\mathbb{C}^n)$ is isomorphic to the total space of $\mathcal{O}(-1)$, and the map π is given by the map $\mathcal{O}(-1) \to \mathbb{C}^n$ by restricting the projection $\mathbb{CP}^{n-1} \times \mathbb{C}^n \to \mathbb{C}^n$ to $\mathcal{O}(-1)$.

We now use this definition as the local model for a blow up of a general submanifold of a complex manifold. Let $Y \subset X$ be a complex submanifold of codimension m. Then locally, there exist holomorphic "slice charts" $\varphi : U \to \mathbb{C}^n$ such that $\varphi(U \cap Y) = \varphi(U) \cap \mathbb{C}^{n-m}$. If we fix such covering of X by such charts $\varphi_i : U_i \to \mathbb{C}^n$, then we can locally define the blow up of X along Y over U_i to be the restriction of the blow up of $\mathbb{C}^m \subset \mathbb{C}^n$ to $\varphi_i(U_i)$. One must show that these local definitions glue to a well defined space with a map $\pi : \mathrm{Bl}_Y(X) \to X$, but we omit this. We note that we have that $\pi^{-1}(Y)$ is the projectivization of the normal bundle of Y in X, since it is true locally. As a hypersurface in $\mathrm{Bl}_Y(X)$, it defines a divisor, which we call the ***exceptional divisor*** of the blow up $\pi : \mathrm{Bl}_Y(X) \to X$. We will need to following facts, but omit the proof.

Proposition 13.2. *The canonical bundle of the blow up of $\pi : \widehat{X} \to X$ along a point $x \in X$ is isomorphic to $\pi^* K_X \otimes \mathcal{O}_{\widehat{X}}((n-1)E)$, where E is the exceptional divisor. Furthermore, $\mathcal{O}_{\widehat{X}}(E)|_E \cong \mathcal{O}(-1)$.*

14. The Kodaira Embedding Theorem

Our goal is to prove the following theorem:

Theorem 14.1 (***Kodaira Embedding***)**.** *A holomorphic line bundle $L \to X$ over a compact complex manifold X is ample if and only if it is positive.*

Recall that L is positive if there exists a Hermitian metric on H such that the curvature of the Chern connection is a Kähler form for some Hermitian metric on X.

One direction of the implication is easy – given an ample line bundle $L \to X$, the very ample line bundle $L^k \to X$ defines an embedding $\varphi_L : X \to \mathbb{CP}^N$ such that L^k is the restriction of $\mathcal{O}(1)$ to X. This gives us a Hermitian metric on L^k such that the curvature of the Chern connection is the restriction of the Fubini-Study metric to X, so L^k is positive. Then since $c_1(L^k) = kc_1(L)$ and any representative $c_1(L)$ can be represented by the curvature of some Hermitian metric, we get that L is positive as well.

Before we prove the Kodaira embedding theorem, we will identify some criteria to determine when a basis of sections $s_0, \dots s_N$ for $\Gamma(X, L)$ determine an embedding. The first is that the base locus $\mathrm{Bs}(s_0, \dots s_N)$ of points where the s_i simultaneously vanish should be empty. One way to phrase this is to say that the map $\Gamma(X, L) \to L_x$ given by evaluating a section at x is surjective. If we let $\mathcal{I}_{\{x\}}$ denote the ideal sheaf of the point $\{x\}$ (i.e. the sheaf of holomorphic functions vanishing at x), the evaluation map comes from the long exact sequence in sheaf cohomology from the short exact sequence of sheaves

$$0 \longrightarrow L \otimes \mathcal{I}_{\{x\}} \longrightarrow L \longrightarrow L_x \longrightarrow 0$$

where we abuse notation and let L_x denote the skyscraper sheaf with stalk L_x at x.

Another requirement for the map to be embedding is that it is injective. We claim that this means for any distinct points $x_1, x_2 \in X$, there must exist a section s of L such that $s(x_1) = 0$ and $s(x_2) \neq 0$. To see this, suppose that there existed so such section. Then for any section s of L, we would either have that $s(x_1) = s(x_2) = 0$ or that both $s(x_1), s_(x_2) \neq 0$, which implies that in any local trivializations about x_1 and x_2, their images under φ_L differ by some nonzero constant, i.e. define the same point in $\mathbb{CP}^N$. Therefore, the map cannot be injective. This is equivalent to the map $H^0(X, L) \to L_{x_1} \oplus L_{x_2}$ given by evaluating a section at x_1 and x_2 being surjective, since we can rescale sections by a complex number λ. This similarly comes from the long exact sequence induced by the short exact sequence of sheaves

$$0 \longrightarrow L \otimes \mathcal{I}_{\{x_1, x_2\}} \longrightarrow L \longrightarrow L_{x_1} \oplus L_{x_2} \longrightarrow 0$$

Finally, for the map to be embedding, it must be an immmersion, since X is compact, it is automatically proper. Therefore, it suffices to verify the the derivative of the induced map is injective. We want to determine a criterion for the derivative of the map $\varphi_L : X \to \mathbb{CP}^N$

to be injective at $x \in X$. Fix a section $s_0 \in \Gamma(X, L)$ such that $s_0(x) \neq 0$, which must exist since L is basepoint free. We can then complete s_0 to a basis $s_0, \dots s_N$ of $\Gamma(X, L)$ such that the $s_i(x) = 0$ for each i. Then the map φ_L (up to the action of some projective linear automorphism of $\mathbb{CP}^N$) is given in a local trivialization by $y \mapsto [s_0(y) : \dots : s_N(y)]$ where we abuse notation by letting s_i denote the function $U \to \mathbb{C}$ that s_i is identified with. Since $s_0(x) \neq 0$, some neighborhood of x is mapped into the subset of $\mathbb{CP}^N$ where the first coordinate is nonzero, so using the standard chart on $\mathbb{CP}^N$, the map φ_L is represented by the map $U \to \mathbb{C}^N$ given by

$$y \mapsto \left(\frac{s_1(y)}{s_0(y)}, \dots, \frac{s_N(y)}{s_0(y)} \right)$$

If we let t_i denote the function s_i/s_0, the transpose of the derivative $d(\varphi_L)_x$ is given by the pullback map sending dz_i to dt_i. Then since the derivative is injective if and only if its transpose is surjective, our criterion for φ_L to be an immersion at x is for the dt_i to span the holomorphic cotangent space $(T_x^{1,0}X)^*$. As with the other criteria, we want to relate this to some short exact sequence of sheaves. Consider the map $\Gamma(X, L \otimes \mathcal{I}_{\{x\}}) \to L_x \otimes (T_x^{1,0}X)^*$ given in a local trivialization by differentiating a section $s \in \Gamma(X, L \otimes \mathcal{I}_{\{x\}})$ at x. This is well defined since the term coming from the Leibniz rule vanishes due to the fact that the section s vanishes at x. This again comes from the long exact sequence coming from the short exact sequence of sheaves

$$0 \longrightarrow L \otimes \mathcal{I}_{\{x\}}^2 \longrightarrow L \otimes \mathcal{I}_{\{x\}} \longrightarrow L_x \otimes (T_x^{1,0}X)^* \longrightarrow 0$$

where we identify the holomorphic cotangent space with $\mathcal{I}_{\{x\}}/\mathcal{I}_{\{x\}}^2$ by taking power series expansions in a neighborhood of x. Furthermore, the functions t_i can be interpreted as sections of $L \otimes \mathcal{I}_{\{x\}}$, and are mapped to the dt_i, interpreting them as sections of $L_x \otimes (T_x^{1,0}X)^*$. Therefore, injectivity of $d(\varphi_L)_x$ is equivalent to surjectivity of the map $\Gamma(X, L \otimes \mathcal{I}_{\{x\}}) \to L_x \otimes (T_x^{1,0}X)^*$

Having identified and reformulated the criteria we need to check, we prove a lemma. Let $\pi : \widehat{X} \to X$ denote the blow up of X at a finite number of distinct points $x_1, \dots, x_\ell$, with corresponding exceptional divisors E_i.

Lemma 14.2. *Let $L \to X$ be a positive line bundle, and $M \to X$ an arbitrary line bundle. Then for any positive integers $n_1, \dots n_\ell$, the line bundle $\pi^*(L^k \otimes M) \otimes \mathcal{O}(-\sum_i n_i E_i)$ on $\widehat{X}$ is positive for k sufficiently large.*

Proof. In neighborhoods U_i of the points x_i, the blow up map $\pi : \widehat{X} \to U$ restricted to $\widehat{U_i} := \pi^{-1}(U_i)$ is given by the blow up of U_i at the point x_i, where we can identify U_i with a subset of $\mathbb{C}^n$, so the $\widehat{U_i}$ is given by the restriction of the projection $\mathcal{O}(-1) \to \mathbb{C}^n$ to U_i. This identifies $\mathcal{O}(E_i)|_{U_i}$ with the pullback of $\mathcal{O}(-1)$ along the map $\widehat{U_i} \to \mathbb{CP}^{n-1}$. pulling back the standard Hermitian form on $\mathcal{O}(1)$, we get a Hermitian metric on $\mathcal{O}(-E_i)|_{U_i}$, and consequently on $\mathcal{O}(-n_i E_i)$ as well. By taking an arbitrary Hermitian metric on $\mathcal{O}(-\sum_i n_i E_i)$ over the complement of the U_i and gluing with a partition of unity, we get a Hermitian metric on all of $\mathcal{O}(-\sum_i n_i E_i)$ where the curvature F_∇ of the Chern connection is positive in a neighborhood of the x_i. This gives us that $\pi * (k\alpha + \beta) + (i/2\pi)F_\nabla$ is positive for any real $(1,1)$ forms α and β with α positive and k sufficiently large. Taking α and β to be representatives for $c_1(L)$ and $c_1(M)$ respectively, we get the desired result, using the fact that L is positive. ■

Using the lemma, we are then ready to prove the Kodaira embedding theorem.

Proof of Kodaira embedding. As mentioned before, one direction is already proven, so all that is left to show is that positivity implies ampleness, i.e. given a positive line bundle $L \to X$, some positive tensor power L^k defines an embedding $\varphi_{L^k} : X \to \mathbb{CP}^N$.

We first show that for k sufficiently large, the line bundle L^k is basepoint-free, i.e. the map $\Gamma(X, L^k) \to L^k_x$ is surjective for every $x \in X$. Fix $x \in X$ and let $\pi : \widehat{X} \to X$ denote the blow up of X along the point x, with exceptional divisor E. Then consider the map $\Gamma(X, L^k) \to \Gamma(\widehat{X}, \pi^* L^k)$ given by pulling back sections along π. Since π is surjective, any two sections whose pullbacks agree must be the same, so the map is injective. If $\dim_{\mathbb{C}} X = 1$, the point x is a divisor, so $\pi : \widehat{X} \to X$ is an isomorphism, and the map $\Gamma(X, L^k) \to \Gamma(\widehat{X}, \pi^* L^k)$ is bijective. Otherwise, the point x is of codimension ≥ 2, so given a section of $\pi^* L^k$, restricting to the complement of X yields a section of $\pi^* L^k$ over $\widehat{X} - E$, which is the same as a section of L^k over $X - \{x\}$, and then Hartog's theorem allows us to extend this to a section of L^k over all of X that pulls back to our original section of $\pi^* L^k$, so the map $\Gamma(X, L^k) \to \Gamma(\widehat{X}, \pi^* L^k)$ is bijective in this case as well. We then note that $\pi^* L^k|_E$ is a trivial bundle, and is isomorphic to the constant sheaf with fiber L^k_x, i.e. $\pi^* L^k|_E \cong \mathcal{O}_E \otimes L^k_x$. We want to show that x is not contained in the base locus of L^k, which amounts to showing that the map $\Gamma(X, L^k) \to L^k_x$ is surjective. From the above observation, it suffices to show that the map $\Gamma(\widehat{X}, \pi^* L^k) \to \Gamma(\widehat{X}, \pi^* L^k|_E)$ is surjective. The restriction map $\Gamma(\widehat{X}, L^k) \to \Gamma(\pi^* L^k|_E) \cong \Gamma(E, \mathcal{O}_E) \otimes L^k_x$ is induced by the short exact sequence of sheaves

$$0 \longrightarrow \pi^* L^k \otimes \mathcal{O}(-E) \longrightarrow \pi^* L^k \longrightarrow \pi^* L^k|_E \longrightarrow 0$$

we want to show that the cokernel of $\Gamma(\widehat{X}, L^k) \to \Gamma(\pi^* L^k|_E)$ is trivial. By exactness of the long exact sequence in sheaf cohomology, we have that the cokernel embeds in $H^1(\widehat{X}, \pi^* L^k \otimes \mathcal{O}(-E))$, so it suffices to show that $H^1(\widehat{X}, \pi^* L^k \otimes \mathcal{O}(-E))$ vanishes. We then use the fact that the canonical bundle of $\widehat{X}$ is given by $K_{\widehat{X}} \cong \pi^* K_X \otimes \mathcal{O}((n-1)E)$ where $n = \dim_{\mathbb{C}} X$, which gives us that

$$\pi^* L^k \otimes K^*_{\widehat{X}} \otimes \mathcal{O}(-E) \cong \pi^*(L^k \otimes K^*_X) \otimes \mathcal{O}(-nE)$$

We then apply the previous lemma with $M = K^*_X$ to conclude that the bundle $\pi^*(L^k \otimes K^*_X) \otimes \mathcal{O}(-nE)$ is positive for k sufficiently large. Therefore by Kodaira vanishing, taking k sufficently large also guarantees that $H^1(\widehat{X}, K_{\widehat{X}} \otimes \pi^*(L^k \otimes K^*_X) \otimes \mathcal{O}(-nE)) = 0$. We then observe that $K_{\widehat{X}} \otimes \pi^*(L^k \otimes K^*_X) \otimes \mathcal{O}(-nE) \cong \pi^* L^k \otimes \mathcal{O}(-E)$, which gives us the desired result that $H^1(\widehat{X}, \pi^* L^k \otimes \mathcal{O}(-E)) = 0$. Therefore, x is not contained in the base locus of L^k for k sufficiently large. We note that the choice of k depends on x. To conclude that there exists a k such that the base locus is empty, we note that we have a map $\Gamma(X, L^k) \to \Gamma(X, L^{2k})$ given by $s \mapsto s \otimes s$. This gives us that the base locus of L^k contains the base locus of L^{2k}, since the section s vanishes if and only if $s \otimes s$ vanishes. This gives us a decreasing chain of closed subsets, which has empty intersection since for any $x \in X$, we have show that there exists some k such that x is not a basepoint of L^k. Then using compactness of X, we have that we can obtain a global bound k such that no x is a basepoint of L^k.

To show that the map is injective, we want to show that the map $\Gamma(X, L) \to \Gamma(L_{\{x_1\}} \oplus L_{\{x_2\}})$ is surjective. This follows from a near identical argument as above, where we replace $\widehat{X}$ with the blow up of X at two points.

Having shown that the map φ_{L^k} (for k sufficiently large) is well-defined and injective, all that is left to show is that the derivative at x is injective for all $x \in X$. Again, let $\pi : \widehat{X} \to X$ denote the blow up of X at x, and let $E = \pi^{-1}(x)$ denote the To this use the fact that the holomorphic sections of $\mathcal{O}(-nE)$ can be identified with the holomorphic functions on $\widehat{X}$ that vanish to n^{th} order along E. Then since $E = \pi^{-1}(x)$, we get an identification of $\mathcal{O}(-nE)$ with the pullback sheaf $\pi^*\mathcal{I}^n_{\{x\}}$. Recall we have the exact sequence

$$0 \longrightarrow \mathcal{I}^2_{\{x\}} \longrightarrow \mathcal{I}_{\{x\}} \longrightarrow (T^{1,0}_x X)^* \longrightarrow 0$$

which, upon twisting with L^k, yields the short exact sequence

$$0 \longrightarrow L^k \otimes \mathcal{I}^2_{\{x\}} \longrightarrow L^k \otimes \mathcal{I}_{\{x\}} \longrightarrow L^k_x \otimes (T^{1,0}_x X)^* \longrightarrow 0$$

where use the fact that $L^k \otimes (T^{1,0}_x X)^* \cong L^k_x \otimes (T^{1,0}_x X)^*$. Taking global sections yields a map $\Gamma(X, L^k \otimes \mathcal{I}_{\{x\}}) \to L^k_x \otimes (T^{1,0}_x X)^*$, and from the preceding discussion, we want to show that this map is surjective. Like before, we do this by showing the cokernel is trivial. To do this, we first note that we have identification $\Gamma(X, L^k \otimes \mathcal{I}_{\{x\}}) \cong \Gamma(\widehat{X}, \pi^* L^k \otimes \mathcal{O}(-E))$. Next, we note that the exception divisor E is just the projectivization of $T^{1,0}_x X$, and $\mathcal{O}_E(-E) \cong \mathcal{O}(1)$. Finally, we note that the global sections of $\mathcal{O}(1)$ over $\mathbb{P}(T^{1,0}_x X)$ is just $(T^{1,0}_x X)^*$. This identification yields an isomorphism $L^k_x \otimes (T^{1,0}_x X)^* \cong L^k_x \otimes \Gamma(E, \mathcal{O}_E(-E))$, and the taking global sections of the restriction map $\pi^* L^k \otimes \mathcal{O}(-E) \to L^k_x \mathcal{O}_E(-E)$ yields the following commutative diagram:

$$\begin{array}{ccc} \Gamma(X, L^k \otimes \mathcal{I}_{\{x\}}) & \longrightarrow & L^k_x \otimes (T^{1,0}_x X)^* \\ \downarrow & & \downarrow \\ \Gamma(\widehat{X}, \pi^* L^k \otimes \mathcal{O}(-E)) & \longrightarrow & L^k_x \otimes \Gamma(E, \mathcal{O}_E(-E)) \end{array}$$

where the vertical maps are isomorphisms. Therefore, to conclude that the cokernel of $\Gamma(X, L^k \otimes \mathcal{I}_{\{x\}}) \to L^k_x \otimes (T^{1,0}_x X)^*$ is trivial it suffices to show that $H^1(\widehat{X}, \pi^* L^k \otimes \mathcal{O}(-2E)) = 0$, which again follows from Kodaira vanishing when k is sufficient large. Furthermore, compactness of X again lets us find a global uniform bound k such that φ_{L^k} is injective for every $x \in X$, which concludes the proof, as we have shown that for k sufficiently large, φ_{L^k} is defined, injective, and an embedding. ∎

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