THE CLASSIFYING SPACE OF A PRODUCT GROUP

JEFFREY JIANG

Let $G = H \times K$ be a product group. The goal is to show that the product space $BH \times BK$ has the homotopy type of BG.

Lemma 0.1. *Let $\pi_H : P \to M$ be a principal H bundle and $\pi_l : Q \to M$ a principal K-bundle. Then the pullback bundle $\pi_H^* Q$ is a principal $H \times K$ bundle over M.*

Proof. We have the pullback diagram

$$\begin{tikzcd} \pi_H^* Q \arrow[r] \arrow[d] & Q \arrow[d, "\pi_K"] \\ P \arrow[r, "\pi_H"'] & M \end{tikzcd}$$

Let U be a neighborhood where both P and Q are trivial. Then locally, the pullback diagram becomes

$$\begin{tikzcd} U \times H \times K \arrow[r] \arrow[d] & U \times K \arrow[d] \\ U \times H \arrow[r] & U \end{tikzcd}$$

where all the maps are the obvious projections. This shows local triviality of $\pi_H^* Q$ as a bundle over M, and also shows that the fibers are $H \times K$ torsors, so $\pi_H^* Q$ defines a principal $H \times K$ bundle over M. ■

Proposition 0.2. *Any principal $G = H \times K$ bundle $P \to M$ can be obtained as the pullback of a principal K-bundle by a principal H bundle.*

Proof. The principal $H \times K$-bundle P has free actions of H and K via their inclusions into the product $H \times K$, and the quotient spaces P/K and P/H have natural structures as principal H and K-bundles respectively, where we use the fact that the H-action and K-action on P commute. We then want to show that $P \to M$ is isomorphic to the pullback bundle obtained from the pullback diagram

$$\begin{tikzcd} \pi_H^* P_K \arrow[r] \arrow[d] & P/K \arrow[d] \\ P/H \arrow[r, "\pi_H"'] & M \end{tikzcd}$$

We note that P admits maps to P/K and P/H by quotienting by the actions of K and H respectively, so by the universal property of the pullback, we get a unique map $P \to \pi_H^* P/K$. Furthermore, this map is $H \times K$-equivariant, so it is a map of principal $H \times K$-bundles over M, so it is an isomorphism. ■

In this way, we see that a principal $H \times K$-bundle is equivalent to the data of a principal H-bundle and a principal K-bundle. We know that the data of a principal $H \times K$-bundle is equivalent to a homotopy class of maps $M \to B(H \times K)$, and likewise for H and K. Therefore, using the universal property of the product, this tells us that $BH \times BK$ has the homotopy type of $B(H \times K)$.

In fact, more can be said. Let $\pi : P \to M$ be a principal $H \times K$-bundle, and let $P_H \to M$ and $P_K \to M$ be principal H and K-bundles respectively such that P is isomorphic to the pullback of one along the other. Then a connection on P is equivalent to the data of connections on both P_H and P_K. To see this, we use the characterization of a connection as a splitting of the exact sequence of vector bundles over P

$$0 \longrightarrow \ker \pi_* \longrightarrow TP \longrightarrow \pi^* TM \longrightarrow 0$$

Since the bundle projections $P_H, P_K \to M$ are transverse (since they are both submersions), we have that the tangent space of the fiber product P is the fiber product of the tangent spaces. Therefore, working fiberwise, it suffices to prove the following linear algebra fact

Proposition 0.3. *Let $\alpha : A \to C$ and $\beta : B \to C$ be surjective linear maps, and let $D := A \times_C B$, giving us the pullback diagram*

$$\begin{tikzcd} D \arrow[r, "p_A"] \arrow[d, "p_B"'] \arrow[dr, "p"] & A \arrow[d, "\alpha"] \\ B \arrow[r, "\beta"'] & C \end{tikzcd}$$

Then a splitting of the exact sequence

$$0 \longrightarrow \ker p \longrightarrow D \xrightarrow{p} C \longrightarrow 0$$

is equivalent to the data of splittings of the analogous exact sequences for A and B.

Proof. By the definition of the fiber product, we have that

$$D = \{(a, b) \in A \times B \;:\; \alpha(a) = \beta(b)\}$$

Therefore, if $(a, b) \in \ker p$, we must have that $\alpha(a) = \beta(b) = 0$. This gives us a canonical isomorphism $\ker p \cong \ker \alpha \oplus \ker \beta$ and natural maps $\ker p \to \ker \alpha$ and $\ker p \to \ker \beta$. This gives us the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker \alpha \arrow[r] & A \arrow[r, "\alpha"] & C \arrow[r] & 0 \\ 0 \arrow[r] & \ker p \arrow[r] \arrow[u] \arrow[d] & D \arrow[r, "p"] \arrow[u, "p_A"] \arrow[d, "p_B"'] & C \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \ker \beta \arrow[r] & B \arrow[r, "\beta"] & C \arrow[r] & 0 \end{tikzcd}$$

For one direction, suppose we have a splitting $j_D : C \to D$. Then we let $j_A : C \to A$ be defined by $j_A := p_A \circ j_D$, and we define $j_B : C \to B$ similarly. To see that j_A is a splitting, we have that for $c \in C$

$$\begin{aligned}(\alpha \circ j_A)(c) &= (\alpha \circ p_A \circ j_D)(c)\\ &= (p \circ j_D)(c)\\ &= c\end{aligned}$$

So j_A indeed defines a splitting. The same argument shows that j_B is a splitting.

In the other direction, suppose we are given splittings $j_A : C \to A$ and $j_B : C \to B$. Then C fits into the diagram

$$\begin{tikzcd} C \arrow[r, "j_A"] \arrow[d, "j_B"'] & A \arrow[d, "\alpha"] \\ B \arrow[r, "\beta"'] & C \end{tikzcd}$$

So the universal property of the fiber product guarantees a map $j_D : C \to D$ such that the following diagram commutes:

$$\begin{tikzcd} C \arrow[dr, "j_D"] \arrow[drr, bend left, "j_A"] \arrow[ddr, bend right, "j_B"'] & & \\ & D \arrow[r, "p_A"] \arrow[d, "p_B"'] \arrow[dr, "p"] & A \arrow[d, "\alpha"] \\ & B \arrow[r, "\beta"'] & C \end{tikzcd}$$

Then since $\beta \circ j_B = \mathrm{id}_C$, we have that $p \circ j_D = \mathrm{id}_C$, so j_D defines a splitting. ■