SPIN GEOMETRY CONFERENCE COURSE

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Week 1

Exercise 1.1. Prove $SL_n(\mathbb{R})$ and $O(n)$ are manifolds

Exercise 1.2. What is the "shape" of $SL_2(\mathbb{R})$?

Exercise 1.3. Prove that

$$O(2) = \left\{ \begin{pmatrix} \cos\theta - \sin\theta \\ \sin\theta \cos\theta \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} \cos\theta \sin\theta \\ \sin\theta - \cos\theta \end{pmatrix} \right\}$$

The first set consists of rotations and the second set consists of reflections. Which rotations commute? Which reflections commute? Do reflections commute with reflections?

Exercise 1.4. Investigate $O(3)$. What is it's "shape?"

Week 2

Exercise 2.1. What is the derivative of det?

Exercise 2.2. Explore the exponetial map $\mathfrak{sl}_2(\mathbb{R}) \to SL_2(\mathbb{R})$

Exercise 2.3. Prove that every element of $O(n)$ can be written as the composition of at most n reflections about hyperplanes in $\mathbb{R}^n$.

Proof. We do this by induction. For $n = 1$, this is obvious, since $O(1) \cong \pm 1$. The assuming that this holds for dimension $n-1$, Let $A \in O(n)$, and let $v \in \mathbb{R}$. We want to construct a hyperplane reflection R such that $RAv = v$, which is obtained by taking R to be the hyperplane reflection about the bisector of v and Av. More explicitly, take R to be the hyperplane reflection about the vector

$$\frac{Av - v}{\|Av - v\|}$$

which is given by the equation

$$Rw = w - 2\frac{\langle Av - v, v\rangle}{\langle Av - v, Av - v\rangle}(Av - v)$$

Computing its action on v, we get

$$\begin{aligned} Rv &= v - 2\frac{\langle Av - v, v\rangle}{\langle Av - v, Av - v\rangle}(Av - v) \\ &= v - \frac{2\langle Av, v\rangle - 2\langle v, v\rangle}{2\langle v, v\rangle - 2\langle Av, v\rangle}(Av - v) \\ &= v + Av - v \\ &= Av \end{aligned}$$

Then since R is its own inverse (being a reflection), we have that $RAv = v$, so RAv fixes v and its orthogonal complement. ∎

TODO. Add motivation for $A_n^{\pm}$

Definition 2.4. Define $A_n^{\pm}$ to be the unital algebra generated by $\mathbb{R}^n$ such that $\xi^2 = \pm 1$. and $\xi\eta =?\ \eta\xi$. Determine the sign of $\eta\xi$. Explore these algebras. Find $A\pm_1$, $A_2^{\pm}$ .... What are they isomorphic to? Can you identify $O(n)$ as a subgroup?

Week 3

Exercise 3.1. Classify the algebras A_n^+ (we messed these up week 2).

Exercise 3.2. Prove that

$$\{e_{i_1}e_{i_2}\dots e_{i+k} \mid 1 \leq i_1 < i_2 < \dots < i_k \leq n\}$$

is a basis for $A_n^\pm$.

Exercise 3.3. Modify the isomorphisms found for A_n^- by choosing $\mathbb{Z}/2\mathbb{Z}$ gradings for the domains and codomains such that the isomorphisms are now isomorphisms as superalgebras.

Exercise 3.4. Construct a tensor product for super vector spaces and superalgebras.

Exercise 3.5. Explore the "shape" of the group

$$G = \langle v \mid \|v\| = 1\rangle \subset (A_n^-)^\times$$

and the nature of the surjection $G \twoheadrightarrow O(n)$. What is the kernel of this map?

Week 4

Exercise 4.1. Define $\varphi : A_n^\pm \to A_n^\pm$ by $\varphi(v) = -v$ and $\varphi(vw) = (wv)$ (i.e. φ reverses products), and extending linearly to sums. Does $\varphi(x)\cdot x$ define a norm on $A_n^\pm$?

Exercise 4.2. Let $(A, \|\|)$ be a normed $\mathbb{R}$-algebra such that $\|ab\| \leq \|a\|\,\|b\|$ for all $a, b \in A$. Show that the multiplicative units form an open subset.

We note that an algebra element $a \in A$ determines a linear map $L_a : A \to A$ by left multiplication, i.e. $L_a(b) = ab$. By fixing a basis for A as a vector space, we get an assignment $a \mapsto M_a$, where M_a is the matrix for L_a in this basis. We claim that an element $a \in A$ is a unit iff $\det M_a \neq 0$. To see this, we note that if L_a is not invertible, then a certainly cannot be, since otherwise $L_{a^{-1}}$ would be an inverse. For the other direction, we note that if a is not a unit, then L_a is not surjective, since 1_A is not in the image. We then claim that this mapping $a \mapsto M_a$ is continuous. Do do this, define a norm on the space of linear maps on A by

$$\|M\| = \sup_{v\in A} \frac{\|Mv\|}{\|v\|}$$

Then given $a, b \in A$, we compute

$$\begin{aligned}
\|M_{a-b}\| &= \sup_{v\in A} \frac{\|(a-b)v\|}{\|v\|} \\
&\leq \sup_{v\in A} \frac{\|a-b\|\,\|v\|}{\|v\|} \\
&\leq \|a-b\|
\end{aligned}$$

So as $b \to a$, we have that $\|M_{a-b}\| \to 0$ as well, so this mapping is continuous. Therefore, the mapping $a \mapsto \det M_a$ is then continuous, which makes the group of units $A^\times$ an open set, being the preimage of the open set $\mathbb{R} - \{0\}$.

One thing to note is that the argument we use to show that $a \mapsto M_a$ is continuous works with any norm such that $\|ab\| \leq c\,\|a\|\,\|b\|$ for any constant c. Therefore, we have a small lemma regarding finite dimensional algebras with an inner product.

Lemma. *Let A be an n-dimensional algebra with inner product $\langle\cdot,\cdot\rangle$, and let $\|\cdot\|$ denote the norm induced by the inner product $\|x\|^2 = \langle x, x\rangle$. Then for all $xy \in A$, we have*

$$\|xy\| \leq n^5\,|\Gamma|\,\|a\|\,\|b\|$$

where Γ denotes the structure constant of maximal magnitude with respect to a fixed orthonormal basis.

Proof. Fix a basis $\{e_i\}$ for A, and let c_{ij}^k denote the structure constants where

$$e_i e_j = c_{ij}^k e_k$$

Then let $x = a^i e_i$ and $y = b^j e_j$. We then compute

$$\begin{aligned}
\|xy\|^2 &= \langle a^i b^j e_i e_j, a^\ell b^m e_\ell e_m \rangle \\
&= \langle a^i b^j c_{ij}^k e_k, a^\ell b^m c_{\ell m}^n e_n \rangle \\
&= a^i a^\ell b^j b^m c_{ij}^k c_{\ell m}^n \langle e_k, e_n \rangle \\
&\leq a^i a^\ell b^j b^m \Gamma^2 n \\
&\leq n^{5/2} \Gamma^2 \|a\| \|b\|
\end{aligned}$$

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Because of this, we have that for the Clifford algebras $A_n^\pm$, the mapping from algebra elements to linear maps on the algebra is continuous, regardless of our choice of inner product. We can then use this to define a nicer norm on the Clifford algebras. First fix an arbitrary inner product and denote the induced norm $\|\cdot\|_1$. Then define

$$\|a\| = \sup_{v \in A_n^\pm} \frac{\|av\|_1}{\|v\|_1}$$

which gives us a submultiplicative norm, so the group of units is an open subset.

We now want to prove that G is a topological group. To do this, it suffices to show that multiplication and inversion are continuous on $A_n^\pm$. For multiplication, fix $c, d \in A$, and suppose we have $a, b \in A$ such that

$$\|a - c\| < \varepsilon \qquad \|b - d\| < \varepsilon$$

for small $\varepsilon > 0$. Then we have

$$\begin{aligned}
\|ab - cd\| &= \|ab - ad + ad - cd\| \\
&= \|a(b-d) + (a-c)d\| \\
&\leq \|a(b-d)\| + \|(a-c)d\| \\
&\leq \|a\| \|b-d\| + \|a-c\| \|d\| \\
&\leq (\|a\| + \|d\|)\varepsilon \\
&\leq (\|a - c + c\| + \|d\|)\varepsilon \\
&\leq (\|a-c\| + \|c\| + \|d\|)\varepsilon \\
&\leq (\varepsilon + \|c\| + \|d\|)\varepsilon
\end{aligned}$$

so multiplication is continuous at (c, d).

TODO. Show inversion is continuous

Exercise 4.3. An algebra A is called a ***matrix algebra*** if there exists an isomorphism $A \cong \mathrm{End}(V)$ for some vector space V. Which $A_n^\pm$ are matrix algebras?

We explore what it means for $A_n^\pm$ to be a matrix algebra. We have that the isomorphism $\varphi : A_n^\pm \to \mathrm{End}(V)$ induces a Clifford module structure on V, where the action on V is exactly $a \cdot v = \varphi(a)v$. What does φ being an isomorphism imply about the module structure on V? We note that there cannot exist any invariant subspaces of V under that action, since there always exists an endomorphism of V that moves a subspace off of itself. Therefore, there cannot exist any $A_n^\pm$-submodules of V. Therefore, for $A_n^\pm$ to be a matrix algebra, there certainly must exist an irreducible $A_n^\pm$-module V.

From the universal property we laid out below, we have that out map $A_n^\pm \to \mathrm{End}(V)$ is equivalent data to a map $j : (V, b) \to \mathrm{End}(V)$ satisfying the relation $j(v)^2 = \pm b(v, v)\, \mathrm{id}_V$

Exercise 4.4. Given a unital associative algebra A and left A-modules M and N, how would you form the direct sum? Can you tensor them? What if A was a super algebra and M, N super vector spaces?

Exercise 4.5. Let V be a vector space and $b : V \times V \to V$ a bilinear form. We want to construct the Clifford algebra $\mathrm{Cliff}(V, b)$ as the "best" associative unital $\mathbb{R}$-algebra generated by V subject to the relation

$$v_1 v_2 + v_2 v_1 = 2b(v_1, b_2) 1_A$$

where 1_A denotes the multiplicative unit in A.

We claim that the above relation is equivalent to the relation $v^2 = b(v, v)1_A$. To see this, we first note that the above condition implies this when we take $v_1 = v_2$. Then for the other direction, consider

$$(v_1 + v_2)^2 = v_1 v_2 + v_2 v_1 + v_1^2 + v_2^2$$

We then apply our relation, giving us

$$\begin{aligned} & b(v_1 + v_2, v_1 + v_2) = v_1 v_2 + v_2 v_1 + b(v_1, v_1) + b(v_2, v_2) \\ \implies & b(v_1 + v_2, v_1 + v_2) - b(v_1, v_1) - b(v_2, v_2) = v_1 v_2 + v_2 v_1 \end{aligned}$$

Then applying polarization, we arrive at the desired identity.

With this, we want to construct $\mathrm{Cliff}(V, b)$ as the unital algebra satisfying our relation and subject to no others (other than bilinearity of multiplication). Therefore, we can consider the quotient of the tensor algebra $\mathcal{T}(V)$ by the ideal $(v^2 - b(v, v))$ to construct $\mathrm{Cliff}(V, b)$. To characterize it, we think of it as the universal such algebra containing V subject to our relation. Since it is subject to no other relations, we expect this object to be *initial*. It should have a map into every other such algebra satisfying this relation. In other words, for every algebra A with an inclusion $j : V \hookrightarrow A$ such that $j(v_1)j(v_2) + j(v_2)j(v_1) = 2b(v_1, v_2)1_A$, we get a unique map $\mathrm{Cliff}(V, b) \to A$ such that the following diagram commutes

$$\begin{tikzcd} V \arrow[d, hook] \arrow[dr, hook, "j"] & \\ \mathrm{Cliff}(V, b) \arrow[r, dashed] & A \end{tikzcd}$$

In other words, the data of a map $\mathrm{Cliff}(V, b) \to A$ is equivalent to a map $j : V \to A$ satisfying the relation we want.

We claim that this characterizes the Clifford algebra up to unique isomorphism. Let A be another algebra with map $j : V \hookrightarrow A$ satisfying the same property we gave above. From the universal property of $\mathrm{Cliff}(V, b)$, we get a unique map $\mathrm{Cliff}(V, b) \to A$. Likewise, the inclusion $V \hookrightarrow \mathrm{Cliff}(V, b)$ gives us a unique map $A \to \mathrm{Cliff}(V, b)$. We claim that these two maps are inverses. We note that both maps are given by $v \mapsto j(v)$ $j(v) \mapsto v$ and extending to products and sums, so we have that they are inverses.

We now want to verify that the construction $\mathcal{T}(V)/(v^2 - b(v, v))$ satisfies this universal property, i.e. the Clifford algebra exists. Given an algebra A with a map $j : V \hookrightarrow A$ satisfying $j(v)^2 - b(v, v) = 0$, define the map $\mathcal{T}(V) \to A$ by $v \mapsto j(v)$ and extending linearly and to products. Then the ideal $(v^2 - b(v, v))$ lies in the kernel of this map, so the map factors through uniquely through $\mathcal{T}(V)/(v^2 - b(v, v))$, so it satisfies the property we laid out.

Week 5

Exercise 5.1. Continue thinking about what makes an algebra a matrix algebra, either for a regular vector space, or for a super vector space.

Exercise 5.2. Is $\mathbb{H}$ isomorphic to an endomorphism algebra $\mathrm{End}(V)$ for some V? Does $\mathbb{H}$ admit an irreducible module?

The quaternions $\mathbb{H}$ are not an endomorphism algebra because they are a division algebra – there's too many invertible elements! It does admit an irreducible module however, namely itself. To show that it is irreducible, we show that it cannot have any proper invariant subspaces. To show this, we note that for any $q, p \in \mathbb{H}$, we have an element that when multiplied on the left with q gives us p, namely pq^{-1}.

Exercise 5.3. For an algebra A, an A-module M is indecomposable if it can be expressed as the direct sum $M = M_1 \oplus M_2$ of submodules M_1 and M_2. Any algebra acts on itself via left multiplication, which we will call the left regular representation. Is the left regular representation of a Clifford algebra indecomposable?

Question. *How many irreducible representations are there for any given Clifford algebra* $\mathrm{Cliff}(V, b)$*. How many indecomposable ones?*

Question. *From before, we have a map* $G \to SO(n)$ *where* G *is the group generated by unit vectors in* $\mathbb{R}^n$*. Extending this to an algebra map* $C_{-n} \to \mathrm{End}(R_n)$*, is this representation irreducible?*

Question. *Does Schur's lemma hold for Clifford modules? Does Maschke's theorem hold?*

Clearly, an irreducible module is indecomposable, so this raises the question – is the left regular representation for a Clifford algebra irreducible?

Exercise 5.4. Given an A-module M and a B-modules N, how would you realize $M \otimes N$ as a $A \otimes B$ module? What if A and B were superalgebras?

We have that a module V over an algebra R is equivalent to giving an algebra homomorphism $R \to \mathrm{End}(V)$. Therefore, we have algebra homomorphisms $\varphi : A \to \mathrm{End}(M)$ and $\psi : B \to \mathrm{End}(N)$. Then we can use these maps to define a map $A \times B \to \mathrm{End}(M) \otimes \mathrm{End}(N)$ where $(a, b) \mapsto \varphi(a) \otimes \psi(v)$, which descends to the tensor product. Composing with the canonical isomorphism $\mathrm{End}(M) \otimes \mathrm{End}(N) \to \mathrm{End}(M \otimes N)$ then gives us the $A \otimes B$-module structure we desire on $M \otimes N$.

More explicitly, this more or less does what you expect, where the action of an element $a \otimes b$ is given by

$$(a \otimes b) \cdot (m \otimes n) = am \otimes bn$$

and extending linearly to sums of algebra elements.

Exercise 5.5. Using only the universal property of the Clifford algebra,

(1) Show the map $\iota : V \to \mathrm{Cliff}(V, b)$ is injective
(2) Carefully state what it means for the $\mathrm{Cliff}(V, b)$ to be unique, and prove it
(3) Can you obtain the $\mathbb{Z}/2\mathbb{Z}$ grading?
(4) How do you get maps between Clifford algebras?

We first state a refined version of the universal property of $\mathrm{Cliff}(V, b)$. For a vector space V with symmetric bilinear form $b : V \times V \to \mathbb{R}$, the ***Clifford algebra*** is the data of a unital associative algebra $\mathrm{Cliff}(V, b)$ and a map $\iota : V \to \mathrm{Cliff}(V, b)$ such that for any algebra A with a linear map $j : V \to A$ satisfying $j(v)^2 = b(v, v)$, we get a unique algebra homomorphism $\mathrm{Cliff}(V, b) \to A$ such that the following diagram commutes

$$\begin{tikzcd} V \arrow[d, "\iota"'] \arrow[dr, "j"] & \\ \mathrm{Cliff}(V,b) \arrow[r] & A \end{tikzcd}$$

(1) Consider the map $j : V \to \mathcal{T}(V)/(v^2 - b(v, v))$ induced by the inclusion map $V \hookrightarrow \mathcal{T}(V)$. We note that this map is injective, and satisfies the relation we need to get a map $\varphi : \mathrm{Cliff}(V, b) \to \mathcal{T}(V)/(v^2 - b(v, v)$ such that $j = \varphi \circ \iota$. Therefore, ι must be injective.
(2) For uniqueness, suppose (A, j) is another algebra satisfying the universal property of the Clifford algebra. Then we claim that there is a unique isomorphism $\mathrm{Cliff}(V, b) \to A$ such that

$$\begin{tikzcd} V \arrow[d, "\iota"'] \arrow[dr, "j"] & \\ \mathrm{Cliff}(V,b) \arrow[r] & A \end{tikzcd}$$

From the universal property of $\mathrm{Cliff}(V, b)$, we get a unique map $\varphi : \mathrm{Cliff}(V, b) \to A$, and since A satisfies the universal property, the map ι gives us a unique map $\psi : A \to \mathrm{Cliff}(V, b)$. We claim that these maps are inverses (and consequently, isomorphisms). We note that $\psi \circ \varphi : \mathrm{Cliff}(V, b) \to \mathrm{Cliff}(V, b)$ satisfies $(\psi \circ \varphi)(\iota(v)) = \iota(v)$. so it must be the unique map that makes

$$\begin{tikzcd} V \arrow[d, "\iota"'] \arrow[dr, "\iota"] & \\ \mathrm{Cliff}(V,b) \arrow[r] & \mathrm{Cliff}(V,b) \end{tikzcd}$$

commute. We note that the identity map on $\mathrm{Cliff}(V, b)$ satisfies this property, so by uniqueness, we must have $\psi \circ \varphi = \mathrm{id}$. Repeating the argument with j and A, we conclude that $\varphi \circ psi = \mathrm{id}$, so they are inverses.

(3) To obtain the grading, we want to show that the assignment $(V, b) \to \mathrm{Cliff}(V, b)$ is functorial. Given vector spaces V, W with symmetric bilinear forms b_V, b_W respectively, and a linear map $T : V \to W$ satisfying $b_W(Tv_1, Tv_2) = b_V(v_1, v_2)$ (i.e. $T^* b_W = b_V$). Then we claim we get an induced algebra homomorphism $T_* : \mathrm{Cliff}(V, b_V) \to \mathrm{Cliff}(W, b_W)$ such that

$$\begin{tikzcd}
(V, b_V) \arrow[r, "T"] \arrow[d, "\iota_V"'] & (W, b_W) \arrow[d, "\iota_W"] \\
\mathrm{Cliff}(V, b_V) \arrow[r, "T_*"'] & \mathrm{Cliff}(W, b_W)
\end{tikzcd}$$

commutes. We note that the data of a map $\mathrm{Cliff}(V, b_V) \to \mathrm{Cliff}(W, b_W)$ is equivalent to the data of a linear map $j : V \to \mathrm{Cliff}(W, b_W)$ satisfying $j(v)^2 = b_V(v, v)$. In this case, let $j = \iota_W \circ T$. We note that $(\iota_W \circ T)(v)^2 = b_W(Tv, Tv)$. Then since we have that $T^* b_W = b_V$, we have that this is equal to $b_V(v, v)$, so we do indeed get an induced map $T_* : \mathrm{Cliff}(V, b_V) \to \mathrm{Cliff}(W, b_W)$, which is uniquely defined by the rule

$$T_*(\iota_V(v)) = \iota_W(Tv)$$

We claim that this is functorial, i.e. given $T : (V, b_V) \to (W, b_W)$ and $L : (W, b_W) \to (X, b_X)$, we have that $(L \circ T)_* = L_* \circ T_*$. This follows from looking at the diagram

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "\iota_V"'] & W \arrow[r, "L"] \arrow[d, "\iota_W"] & X \arrow[d, "\iota_X"] \\
\mathrm{Cliff}(V, b_V) \arrow[r, "T_*"'] & \mathrm{Cliff}(W, b_W) \arrow[r, "L_*"'] & \mathrm{Cliff}(X, b_x)
\end{tikzcd}$$

With that out of the way, we can use the functoriality to obtain the $\mathbb{Z}/2\mathbb{Z}$ grading. Let (V, b) be a vector space V with symmetric bilinear form b, and consider the map $-\mathrm{id}_V$, where $v \mapsto -v$. Then we note that

$$(-\mathrm{id}_V^* b)(v, w) = b(-v, -w) = b(v, w)$$

so it induces an algebra map $\varphi : \mathrm{Cliff}(V, b) \to \mathrm{Cliff}(V, b)$ which is defined by the property that $\varphi(\iota(v)) = -\iota(v)$. Then let the even subspace of $\mathrm{Cliff}(V, b)$ be the subspace spanned by elements a such that $\varphi(a) = a$, and let the odd subspace of $\mathrm{Cliff}(V, b)$ be the one spanned by elements a such that $\varphi(a) = -a$.

(4) This is answered by the functoriality of $(V, b) \mapsto \mathrm{Cliff}(V, b)$. We get maps between Clifford algebras when we have linear maps that pullback the bilinear forms.

Exercise 5.6. Give a canonical isomorphism $\mathrm{End}(V) \otimes \mathrm{End}(W) \to \mathrm{End}(V \otimes W)$ in the case of regular vector spaces and super vector spaces.

Let $A \in \mathrm{End}(V)$ and $B \in \mathrm{End}(W)$. Then define the map

$$\begin{aligned} A \otimes B : V \otimes W &\to V \otimes W \\ v \otimes w &\mapsto Av \otimes Bw \end{aligned}$$

This defines a bilinear mapping $\mathrm{End}(V) \times \mathrm{End}(W) \to \mathrm{End}(V \otimes W)$, so it factors uniquely to map $\varphi : \mathrm{End}(V) \otimes \mathrm{End}(W) \to \mathrm{End}(V \otimes W)$. We claim that this defines an isomorphism. Suppose $A \otimes B$ defines the zero map $V \otimes W \to V \otimes W$. Then if we let $\{v_i\}$ and $\{w_i\}$ denote bases for V and W respectively, we know that $\{v_i \otimes w_j\}$ is a basis for $V \otimes W$. Then for every pair i, j, we have that

$$\begin{aligned} & (A \otimes B)(v_i \otimes w_j) = 0 \\ \iff & Av_i \otimes Bw_j = 0 \\ \iff & Av_i = 0 \text{ or } Bw_j = 0 \\ \implies & A \otimes B = 0 \end{aligned}$$

So the kernel is trivial, and the map is injective. By checking dimension, we see that this now implies that φ is an isomorphism.

In the case that V and W are super vector spaces, write

$$V = V^0 \oplus V^1$$
$$W = W^0 \oplus W^1$$

We then note that the grading for V and W gives natural gradings for $\mathrm{End}(V)$ and $\mathrm{End}(W)$, where the even subspaces are the ones that preserve the grading and the odd subspaces are the ones that reverse the grading. In other words, a map $T \in \mathrm{End}(V)$ is even if $T(V^i) \subset V^i$ and is odd if $T(V^i) = T(V^{i+1})$, where the addition is done mod 2. In addition, we have that the gradings of V and W induce a grading on $V \otimes W$ where

$$(V \otimes W)^0 = (V^0 \otimes W^0) \oplus (V^1 \otimes W^1)$$
$$(V \otimes W)^1 = (V^0 \otimes W^1) \oplus (V^1 \otimes W^0)$$

Therefore, in the case of V and W being super vector spaces, we want the isomorphism we construct

$$\mathrm{End}(V) \otimes \mathrm{End}(W) \to \mathrm{End}(V \otimes W)$$

to respect this extra structure, which it does.