FUNDAMENTAL GROUPS OF PRINCIPAL CIRCLE BUNDLES OVER RIEMANN SURFACES

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Let Σ be a Riemann surface with genus $g > 0$, and $\pi : P \to \Sigma$ a principal $U(1)$ bundle over Σ. This gives rise to a long exact sequence of homotopy groups

$$\begin{tikzcd} \cdots \arrow[r] & \pi_2(U(1)) \arrow[r] & \pi_2(P) \arrow[r] & \pi_2(\Sigma) \arrow[dll] & \\ & \pi_1(U(1)) \arrow[r] & \pi_1(P) \arrow[r] & \pi_1(\Sigma) \arrow[r] & 1 \end{tikzcd}$$

Since the genus of Σ is greater than 0, the universal cover of Σ is contractible, which implies that $\pi_2(\Sigma) = 1$, since the covering map induces isomorphisms on higher homotopy groups. Then since $\pi_1(U(1)) \cong \mathbb{Z}$, we get the short exact sequence of groups

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(P) \longrightarrow \pi_1(\Sigma) \longrightarrow 1$$

so the fundamental group of P is an extension of $\pi_1(M)$ by $\mathbb{Z}$.

Any principal $U(1)$-bundle $P \to \Sigma$ can be characterized up to isomorphism by its first Chern class $c_1(P) \in H^2(\Sigma, \mathbb{Z}) \cong \mathbb{Z}$, which can be explicitly realized as the cohomology class

$$c_1(P) = \left[\frac{i}{2\pi}\Omega\right]$$

where Ω is the curvature 2-form of any connection $\Theta \in \Omega^1_P$ on P, where we make the identification of $\mathfrak{u}(1) \cong \mathbb{R}$. Given any connection Θ, we get a $U(1)$-invariant horizontal distribution on TP given by $\ker\Theta$, which allows us to lift paths $\gamma : [0,1] \to \Sigma$ to paths $\widetilde{\gamma}_p : [0,1] \to P$ for every lift p of $\gamma(0)$. In particular, if $\gamma(0) = \gamma(1)$, (i.e. γ is a loop on Σ), the lifts may or may not lift to loops on $P:$ in general, for each lift p of $\gamma(0)$, we have $\widetilde{\gamma}_p(1) = p \cdot \theta_\gamma$ for some $\theta_\gamma \in U(1)$, called the ***holonomy*** associated to the loops γ. Then suppose we have a simple curve γ bounding a disk $D \subset \Sigma$ contained in some open set in which P is trivial. In this case, the curvature can be written locally as dA for some 1-form $A \in \Omega^1_M$, and the holonomy is computed by the integral

$$\int_\gamma A = \int_D dA = \int_D \Omega$$

This can be extended to simple curves that don't necessarily lie in neighborhoods where the bundle is trivial by applying a subdivision argument with a triangulation.

We now return to the situation of computing the fundamental group of P. Recall that the fundamental group of Σ is given by the presentation

$$\langle a_1, b_1, \ldots a_g, b_g \ : \ [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$$

The image of the map $\mathbb{Z} \to \pi_1(P)$ is given by mapping 1 to a curve c traversing once around a fiber of $P \to \Sigma$. In addition, if we view Σ as a quotient of a $2g$-gon with identifications at the boundaries, we have that the other generators of $\pi_1(P)$ will be lifts of the loops on the boundary With a slight abuse of notation, we also denote these lifted loops by $a_i, \ldots b_i$. Furthermore, we may assume that the a_i and b_i are horizontal lifts of the generators on the base with respect to some connection on P. This gives us the generators of for $\pi_1(P)$, namely the a_i, b_i, and the new generator c. For relations, we fix a connection on P and consider the product of commutators $[a_1, b_1] \cdots [a_g, b_g]$, which traverses the boundary of the polygon once. By pushing this into the interior of the polygon by an arbitrarily small amount, we get a simple curve that bounds a disk that covers almost all of Σ. We then have that the holonomy about this curve is the integral of the curvature. Taking the limit as the curve approaches the boundary, we get that this integral becomes 2π times the first Chern class of P, which tells us that the holonomy is traversing $c_1(P)$ times around c. This gives the relation

$$[a_1, b_1] \cdots [a_g, b_g] = c_1(P)c$$

We then claim that the element c is central. To see this, we see that since P is orientable, it restricts to a trivial bundle over the 1-skeleton of Σ. Since the bundle is trivial, the loop c commutes with all the a_i and b_i over the 1-skeleton. Consequently, it commutes in Σ as well.

Putting everything together, we get a characterization of fundamental groups of principal $U(1)$-bundles $P \to \Sigma$: they are central extensions of $\pi_1(M)$ by a cyclic group generated by a single loop c, along with the additional relation that $[a_1, b_1] \cdots [a_g, b_g] = c_1(P)c$.