CHAPTER

1

Principal Bundles

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Why Fiber Bundles?

Suppose we want to study some space M, which for our purposes is a smooth manifold. One way to study M is to study functions $M \to F$ for some fixed target space F, e.g. a manifold , $\mathbb{R}$, or a vector space. This is a perfectly good method of studying M, but is sometimes not enough. More often then not, we want to study "function" from M into a vector space that varies over the base manifold M. For example, a vector field $X \in \mathfrak{X}(M)$ is not really a map $M \to \mathbb{R}^n$, it is an assignment to each $p \in M$ a vector in T_pM. In this way, we are led to the study of a smoothly parameterized family of vector spaces – the tangent bundle TM. This leads us to define a fiber bundle.

Definition 0.1. *Let M and F be a smooth manifolds. A* ***fiber bundle*** *over M with model fiber F is the data of a smooth manifold E with a smooth surjective map $\pi : E \to M$ such that for each $p \in M$, there exists an open set U and a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ such that*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "p_1"] \\ & U & \end{tikzcd}$$

where $p_1 : U \times F \to U$ is projection onto the first factor. The map $\varphi : \pi^{-1} \to U \times F$ is called a ***local trivialization****. The space E is called the* ***total space****, while the manifold M is called the* ***base space****. We often omit naming the map π and denote the fiber $\pi^{-1}(x)$ by E_x.*

This definition captures the notion of a family of manifolds diffeomorphic to F that are smoothly parameterized by the base space M. We also have a notion of a morphism between bundles.

Definition 0.2. *Let $\pi_E E \to M$ and $\pi_F : F \to M$ be fiber bundles with model fiber X. A* ***bundle homomorphism*** *is the data of a smooth map $\varphi : E \to F$ such that the diagram*

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "\pi_E"'] & & F \arrow[dl, "\pi_F"] \\ & M & \end{tikzcd}$$

commutes.

Our original motivation for thinking about fiber bundles was for a generalized notion of a function. To this end, we specify a special class of maps associated to a fiber bundle.

Definition 0.3. *Let $\pi : E \to M$ be a a fiber bundle with model fiber F. A* ***local section*** *of $\pi : E \to M$ is a smooth map $\sigma : U \to E$ such that $\pi \circ \sigma = \mathrm{id}_U$ for some open set $U \subset M$. If $U = M$, we call σ a* ***global section****. Equivalently, it is a smooth assignment to each $p \in U$ a point in the fiber E_p. We denote the space of sections over an open set U as $\Gamma_U(E)$.*

A section of $E \to M$ can be thought of as our desired generalization of a function. A map $M \to F$ is the same data as a section of the trivial bundle $M \times F \to M$. However, not every fiber bundle is trivial – there can be a nontrivial "twisting." An example of this is the Möbius band. Can you see why this bundle over S^1 is not isomorphic to the trivial bundle $S^1 \times [0, 1]$?

We are especially interested in two special classes of fiber bundles that carry additional structure – the fibers of a vector bundle carry the extra structure of a vector space, and the fibers of a principal bundle have the extra structure of a G-torsor for a Lie group G.

Definition 0.4. *A* ***vector bundle of rank*** *k is a fiber bundle $E \to M$ such that each fiber E_x has the structure of a k-dimensional vector space (usually over $\mathbb{R}$ or $\mathbb{C}$). A* ***vector bundle homomorphism*** *is a bundle homomorphism that restricts to a linear map on each fiber.*

Vector bundles form a familiar family of fiber bundles, as tangent bundles, cotangent bundles, and their associated tensor bundles are all vector bundles.