LINE BUNDLES ON $\mathbb{CP}^n$

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Definition 1.1. ***Complex projective space***, denoted $\mathbb{CP}^n$ is the set of 1-dimensional subspaces of $\mathbb{C}^{n+1}$.

Given a line $\ell \in \mathbb{CP}^n$, it can be recovered from any nonzero vector $v \in \ell$ by taking the span of v. If $v \in \ell$ has coordinates

$$v = (z_0, \ldots, z_n)$$

we denote the line ℓ with the notation

$$\ell = [z_0 : \ldots : z_n]$$

where it is understood that the coordinates in the square brackets are only determined up to scaling, since λv determines the same line as v for any $\lambda \in \mathbb{C}^\times$. These are called ***homogeneous coordinates*** for $\mathbb{CP}^n$.

Proposition 1.2. *$\mathbb{CP}^n$ can be endowed with the structure of an n-dimensional complex manifold.*

Proof. It suffices to provide a covering of $\mathbb{CP}^n$ by charts with holomorphic transition functions. Define the set $U_i \subset \mathbb{CP}^n$ by

$$U_i := \{[z_0 : \cdots z_n] \ : \ z_i \neq 0\}$$

note that this is well defined, since if the i^{th} component of a vector $v \in \ell$ is 0, then the i^{th} component of λv will be as well. Then define the maps

$$\begin{aligned} \varphi_i : U_i &\to \mathbb{C}^n \\ [z_0 : \ldots : z_n] &\mapsto \left(\frac{z_0}{z_i}, \ldots, \widehat{\frac{z_i}{z_i}}, \ldots \frac{z_n}{z_i} \right) \end{aligned}$$

where $\widehat{\frac{z_i}{z_i}}$ denotes the fact that the term is missing. We note that this is well defined since we have that $z_i \neq 0$ for any line in U_i. The maps φ_i are bijections, with inverses given by

$$\begin{aligned} \varphi_i^{-1} : \mathbb{C}^n &\to U_i \\ (z_1, \ldots z_n) &\mapsto [z_1 : \ldots : z_{i-1} : 1 : z_i \ldots : z_n] \end{aligned}$$

i.e. we insert a 1 in the i^{th} slot in homogeneous coordinates. We then check the transition functions. We compute for $i \neq j$

$$(\varphi_j \circ \varphi_i^{-1})(z_1, \ldots z_n) = \begin{cases} \left(\frac{z_1}{z_{j-1}}, \ldots, \frac{z_{i-1}}{z_{j-1}}, \frac{1}{z_{j-1}}, \frac{z_i}{z_{j-1}}, \ldots, \widehat{\frac{z_{j-1}}{z_{j-1}}}, \ldots, \frac{z_n}{z_{j-1}} \right) & j > i \\ \left(\frac{z_1}{z_j}, \ldots, \widehat{\frac{z_j}{z_j}}, \ldots \frac{z_{i-1}}{z_j}, \ldots, \frac{1}{z_j}, \frac{z_i}{z_j}, \ldots, \frac{z_n}{z_j} \right) & j < i \end{cases}$$

which is visibly holomorphic, since in either case, the functions $1/z_j$ or $1/z_{j-1}$ are holomorphic if z_j or z_{j-1} are nonzero. ∎

Definition 1.3. Let X be a complex manifold. A ***holomorphic line bundle*** over X is a complex manifold L equipped with a holomorphic map $\pi : L \to X$ such that for every $x \in X$, the fiber $L_x := \pi^{-1}(x)$ has the structure of a 1-dimensional complex vector space. In addition, there exists an open set U about x and a map $\psi : \pi^{-1}(U) \to U \times \mathbb{C}$ such that $\psi|_{L_x}$ is a linear map and the following diagram commutes

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & U \times \mathbb{C} \arrow[dl, "\pi_U"] \\
& U &
\end{tikzcd}$$

where the map $\pi_U : U \times \mathbb{C} \to U$ is projection onto the first factor. The maps ψ are called ***local trivializations***.

One example of a holomorphic line bundle is the product $X \times \mathbb{C} \to X$, where the bundle projection is just projection onto the first factor, which is called the ***trivial bundle***. A bundle isomorphic to the trivial bundle is said to be ***trivial*** or ***trivializable***. One way to tell if a line bundle $L \to X$ is trivial is to provide a global nonvanishing section $\sigma : X \to L$. This determines a global trivialization of L in the following way: at each x, we have that $\sigma(x) \neq 0$ determines a basis for the fiber L_x. Therefore, for any element v of the fiber L_x, we can map it to $(x, \lambda) \in X \times \mathbb{C}$, where λ is the component of v in the basis $\{\sigma(x)\}$ for L_x. This determines an bundle isomorphism $L \cong X \times \mathbb{C}$.

Suppose we have a holomorphic line bundle $\pi : L \to X$, and two local trivializations $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$ and $\psi_j : \pi^{-1}(U_j) \to U_j \to \mathbb{C}$. Then if we consider the map $(\psi_i \circ \psi_j^{-1})$ after appropriately restricting and shrinking the domain and codomains, if we fix a $x \in U_i \cap U_j$, the map $(\psi_i \circ \psi_j^{-1})|_{\{x\}\times\mathbb{C}}$ is a linear function from $\{x\} \times \mathbb{C}$ to itself. Therefore, the maps $(\psi_i \circ \psi_j)^{-1}$ determines holomorphic maps $\psi_{ij} : U_i \cap U_j \to \mathrm{GL}_1\mathbb{C}$. Holomorphicity of the ψ_{ij} comes from the fact that L is a holomorphic line bundle. We can go the other direction as well. One way to construct line bundles is to use trivial bundles over sets in an open cover, and to glue them together by specifying the transition functions.

Theorem 1.4 (***The gluing construction***)**.** *Let X be a complex manifold, and $\{U_i\}$ an open cover of X. Then for each $U_i \cap U_j$, let $\psi_{ij} : U_i \cap U_j \to \mathrm{GL}_1\mathbb{C}$ be holomorphic maps satisfying the* ***cocycle condition***

$$\psi_{ij}\psi_{jk} = \psi_{ik}$$

Then the set

$$L = \coprod_i U_i \times \mathbb{C}/\sim$$

where the equivalence relation $\sim$ is defined by

$$(x, \lambda) \sim (y, \mu) \iff x = y \text{ and there exist } i, j \text{ such that } \psi_{ij}(x)(\lambda) = \mu$$

equipped with the natural projection $\pi : L \to X$ is a holomorphic line bundle.

With respect to a cover $\{U_i\}$, the pairs (U_i, ψ_{ij}) are called ***cocycles***, though the set U_i is sometimes omitted, and the transition functions themselves are called coycles. The gluing construction tells us that the cocycles can almost determine the line bundle up to isomorphism . In particular, if we have that all the cocycles are given by the constant map $\psi_{ij}(x) = \lambda$, then we know that the bundle is trivial. In a similar vein, if we have two bundles $L, L' \to X$, and we know that their transition functions differ by a constant, then we can conclude that L and L' are isomorphic as holomorphic line bundles. In more generality, it is true that if the transition functions differ by something called a coboundary, then the bundles are isomorphic.

Definition 1.5. Let $\pi : L \to X$ be a holomorphic line bundle. A ***local section*** is a holomorphic map $\sigma : U_i \to L$ from an open set $U \subset X$ such that $\pi \circ \sigma = \mathrm{id}_U$. If $U = X$, then σ is said to be a ***global section***. If it is not specified whether a section is local or global, it is implicitly assumed to be global.

Remark. The idea of a section is to generalize the notion of a function. A section of the trivial bundle $X \times \mathbb{C}$ is just a holomorphic function on X. However, sections of nontrivial holomorphic line bundles are maps into holomorphically varying complex lines, and could possibly fail to exist.

The gluing construction gives an alternate characterization of sections of a line bundle.

Proposition 1.6. *Let $\pi : L \to X$ be a holomorphic line bundle, and $\{U_i\}$ an open cover of X in which L is trivialized over each U_i with transition functions ψ_{ij}. Then the data of a section $\sigma : X \to L$ is equivalent to the data of holomorphic functions $\sigma_i : U_i \to \mathbb{C}$ with the compatibility condition $\sigma_i = \psi_{ij}\sigma_j$.*

Proof. Given a section $\sigma : X \to L$, by composing with the trivialization $\pi^{-1}(U_i) \to U_i \times \mathbb{C}$, we obtain smooth functions $\sigma_i : U \to \mathbb{C}$, and the fact that they satisfy the compatibility condition can be easily checked. In the other direction, given a a collection of compatible maps σ_i, we can define σ by specifying σ on each U_i to be the map σ_i composed with the inverse of the local trivialization. The fact that these local definitions glue to a well-defined holomorphic map $\sigma : X \to L$ is easily checked to be exactly the condition that they are compatible. ■

Definition 1.7. The ***tautological bundle*** over $\mathbb{CP}^n$, denoted $\mathcal{O}(-1)$, is the bundle where the fiber over $\ell \in \mathbb{CP}^n$ is the line ℓ, i.e. as a set,

$$\mathcal{O}(-1) := \{(\ell, v) \ : \ \ell \in \mathbb{CP}^n, v \in \ell\}$$

Proposition 1.8. $\mathcal{O}(-1)$ *actually is a holomorphic line bundle.*

Proof. Let $\pi : \mathcal{O}(-1) \to \mathbb{CP}^n$ denote the bundle projection. It suffices to produce local trivializations and then check that the transitions functions are holomorphic. Let $\varphi_i : \mathbb{C}^n \to U_i$ be the charts on $\mathbb{CP}^n$ defined earlier. These coordinates induce trivializations $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$ as follows:

Let $\ell = [z_0 : \ldots : z_n] \in \mathbb{CP}^n$. Then

$$\varphi(\ell) = \left(\frac{z_0}{z_i}, \ldots, \frac{\widehat{z_i}}{z_i}, \ldots \frac{z_n}{z_i}\right)$$

this determines a vector $v_\ell^i \in \ell$ via the mapping

$$\left(\frac{z_0}{z_i}, \ldots, \frac{\widehat{z_i}}{z_i}, \ldots \frac{z_n}{z_i}\right) \mapsto v_\ell := \left(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, 1, \frac{z_{i+1}}{z_i}, \ldots \frac{z_n}{z_i}\right)$$

Therefore, we can define the map ψ_i by declaring $\psi_i(\ell, v_\ell^i) = (\ell, 1)$ and extending linearly to the rest of the fiber over ℓ. This clearly makes the diagram

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, "\psi_i"] \arrow[dr, "\pi"'] & & U_i \times \mathbb{C} \arrow[dl, "\pi_{U_i}"] \\ & U_i & \end{tikzcd}$$

commute, so we now check the transition functions. To do this, we compute the action of $(\psi_i \circ \psi_j^{-1})$ on $(\ell, 1) \in U_i \cap U_j \times \mathbb{C}$. We compute, letting $\ell = [z_0 :, \dots : z_n]$

$$\begin{aligned}(\psi_i \circ \psi_j^{-1})(\ell, 1) &= \psi_i \left(\ell, \left(\frac{z_0}{z_j}, \dots \frac{z_{j-1}}{z_j}, 1, \frac{z_{j+1}}{z_j}, \dots z_n z_j \right) \right) \\ &= \left(\ell, \frac{z_i}{z_j} \right)\end{aligned}$$

where the last equality comes from the fact that the component of the vector

$$v_\ell^j := \left(\frac{z_0}{z_j}, \dots \frac{z_{j-1}}{z_j}, 1, \frac{z_{j+1}}{z_j}, \dots \frac{z_n}{z_j} \right)$$

with respect to the basis $\{v_\ell^i\}$ for ℓ where

$$v_\ell^i := \left(\frac{z_0}{z_i}, \dots \frac{z_{i-1}}{z_i}, 1, \frac{z_{i+1}}{z_i}, \dots \frac{z_n}{z_i} \right)$$

is just z_i/z_j. Therefore, we have that the transition functions ψ_{ij} are given by

$$\psi_{ij}(\ell) = \left(\frac{z_i}{z_j} \right)$$

where $\ell = [z_0 : \dots : z_n]$. Since these functions are all holomorphic, this shows that $\mathcal{O}(-1)$ is a holomorphic line bundle. ■

To discuss all of the line bundles over $\mathbb{CP}^n$, we need to discuss how to construct new line bundles from existing ones.

Proposition 1.9 (***New bundles from old***)**.** *Let $L, L' \to X$ be holomorphic line bundles over X. Then the following linear algebraic operations, done fiberwise, define another holomorphic line bundle.*

(1) *$L \otimes L' \to X$, where the fiber over x is $L_x \otimes L'_x$.*
(2) *$L^* \to X$, where the fiber over x is the dual space L_x^*.*

Proof. It suffices to provide local trivializations and check the transition functions. By potentially shrinking sets, we can find an open cover $\{U_i\}$ of X such that both L and L' are trivialized over the U_i. Let φ_i and φ'_i denote the local trivializations for L and L' respectively, and let ψ_{ij}, ψ'_{ij} denote their respective transition functions.

(1) The local trivializations for $L \otimes L$ are just given by $\varphi_i \otimes \varphi'_i$, where if $\varphi_i(\ell) = (x, \lambda)$ and $\varphi'_i(\ell) = (x, \lambda')$, we have

$$(\varphi_i \otimes \varphi'_i)(\ell) = (x, \lambda \otimes \lambda')$$

the transition functions are given by $\psi_{ij} \otimes \psi'_{ij}$, where

$$(\psi_{ij} \otimes \psi'_{ij})(\ell) = \psi_{ij}(\ell) \otimes \psi'_{ij}(\ell)$$

where the $\otimes$ in the right hand side denotes the standard notion of the tensor product of matrices/linear maps.

(2) The trivializations for L^* are just the trivializations for φ_i for L composed with the isomorphism $\mathbb{C} \to \mathbb{C}^*$. The transition functions are given by $(\psi_{ij}^{-1})^T$, where T denotes the transpose (*not* the conjugate transpose), where the inverse comes into play due to the contravariance of taking dual spaces. In the case of the line bundle, transposing a 1×1 matrix is pointless, so the transition functions are ψ_{ij}^{-1} (i.e. inverting the matrix, not inverting the function ψ_{ij}).

■

Proposition 1.10. *For a holomorphic line bundle $L \to X$, The bundle $L \otimes L^*$ is isomorphic to the trivial bundle $X \times \mathbb{C}$.*

Proof. We know for a vector space V that $V^* \otimes V$ is isomorphic to $\mathrm{End}(V)$. Therefore, the bundle $L \otimes L$ is isomorphic to the bundle $\mathrm{End}(L)$, where the fiber over $x \in X$ is the space of endomorphisms of L_x. To show that this is trivial, it suffices to provide a global nonvanishing section, σ, which we can do by defining $\sigma(x) = \mathrm{id}_{L_x}$. ■

Definition 1.11. For a complex manifold X, the ***Picard group*** of X, denoted $\mathrm{Pic}(X)$ is the group of isomorphism classes of holomorphic line bundles over X, where the group operation is given by the tensor product. The operation is associative, since tensor products are associative, and the trivial bundle is the identity with respect to the tensor product. Finally, for any bundle $L \in \mathrm{Pic}(X)$, it has an inverse given by the dual bundle L^*.

We now restrict our attention again to $\mathbb{CP}^n$.

Definition 1.12. Define the bundle $\mathcal{O}(1) := \mathcal{O}(-1)^*$. It is often referred to as the ***hyperplane bundle***. The define the bundle $\mathcal{O}(k) := \mathcal{O}(1)^{\otimes^k}$ and $\mathcal{O}(-k) := \mathcal{O}(-1)^{\otimes^{-k}}$ for $k \in \mathbb{Z}^{>0}$. We let $\mathcal{O}(0) := \mathbb{CP}^n \times \mathbb{C}$.

A fact that we will not prove is that all line bundles over $\mathbb{CP}^n$ are of the form $\mathcal{O}(k)$ for some integer k. Using our observations in Proposition 1.9, we can identify the transition functions for the line bundles $\mathcal{O}(k)$ with respect to the local trivializations defined for $\mathcal{O}(-1)$ in the open cover $\{U_i\}$.

For the following discussion, fix $\ell \in \mathbb{CP}^n$ with $\ell = [z_0 : \ldots : z_n]$. Then since the transition functions for the dual bundle are the inverses of the transition functions, we have that the transition functions for $\mathcal{O}(1)$ are $\psi_{ij}(\ell) = z_j/z_i$. Then for the tensor powers, the transition functions for $\mathcal{O}(k)$ where $k \geq 0$ are $\psi_{ij}(\ell) = (z_j/z_i)^{\otimes^k}$, and using the identification $\mathbb{C}^{\otimes^k}$ with $\mathbb{C}$, under the mapping

$$\lambda_1 \otimes \cdots \otimes \lambda_k \mapsto \lambda_1 \lambda_2 \cdots \lambda_k$$

this is the same as $\psi_{ij}(\ell) = z_j^k/z_i^k$. Likewise, the transition functions for $\mathcal{O}(-k)$ are $\psi_{ij}(\ell) = z_i^k/z_j^k$. By noting how transition functions behave under tensor product, we see that for integers $d, k \in \mathbb{Z}$ we have that

$$\mathcal{O}(d) \otimes \mathcal{O}(k) \cong \mathcal{O}(d+k)$$

If we assume the fact that all line bundles over $\mathbb{CP}^n$ are isomorphic to some $\mathcal{O}(k)$, this shows that $\operatorname{Pic}(\mathbb{CP}^n) \cong \mathbb{Z}$ via the mapping $\mathcal{O}(k) \mapsto k$.

A natural question to ask is which of these bundles admit global sections. We work through the case of $\mathcal{O}(-1)$ explicitly.

Theorem 1.13. *The bundle $\mathcal{O}(-1)$ admits no nonzero global holomorphic sections, i.e.*

$$\Gamma_{\mathbb{CP}^n}(\mathcal{O}(-1)) = 0$$

We give two proofs of this fact.

Proof 1. The line bundle naturally lives as a subbundle of the trivial bundle $\mathbb{CP}^n \times \mathbb{C}^{n+1}$, since each element $\ell \in \mathbb{CP}^n$ is a line in $\mathbb{C}^{n+1}$. Therefore, a global holomorphic section $\sigma : \mathbb{CP}^n \to \mathcal{O}(-1)$ is also a global holomorphic section $\mathbb{CP}^n \to \mathbb{CP}^n \times \mathbb{C}^{n+1}$ by composing with the inclusion $\mathcal{O}(-1) \hookrightarrow \mathbb{CP}^n \times \mathbb{C}^{n+1}$. However, a a holomorphic section of $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ is just a holomorphic map $\mathbb{CP}^n \to \mathbb{C}^{n+1}$. Then since $\mathbb{CP}^n$ is compact and connected, the map is constant by the maximum principle for holomorphic maps. However, $\sigma(\ell) \in \mathbb{C}^{n+1}$ must be a point in ℓ for every ℓ, and the only point that lies in all lines in $\mathbb{C}^{n+1}$ is 0. Therefore, $\sigma = 0$. ■

Proof 2. The transition functions for $\mathcal{O}(-1)$ are given by $\psi_{ij}(\ell) = z_i/z_j$ with respect to the standard open cover $\{U_i\}$ of $\mathbb{CP}^n$. Since $\mathcal{O}(-1)$ is trivialized over the U_i, the data of a section σ is the same as specifying a function $f_i : U_i \to \mathbb{C}$ with a compatibility condition on the intersections specified by the transition functions ψ_{ij}. To determine this compatibility condition, we have that by composing with the chart maps $\varphi_i^{-1} : \mathbb{C}^n \to U_i$, the function f_i on the intersection $U_i \cap U_j$ can be interpreted as a holomorphic function of the variables $x_k := z_k/z_i$ (where x_i is missing). Likewise, f_j can be interpreted as a holomorphic function of the variable $y_k := z_k/z_j$ (where y_j is missing). The compatibility condition is that these two functions are related by the transition function ψ_{ij}, i.e. at the point $\ell = [z_0 : \ldots z_n] \in U_i \cap U_j$,

$$f_i(x_0, \ldots, \widehat{x_i}, \ldots, x_n) = \psi_{ij}(\ell) f_j(y_1, \ldots, \widehat{y_j}, \ldots, y_n)$$

But since $\psi_{ij}(\ell) = z_i/z_j = y_i = 1/x_j$, this condition is asking that

$$f_i(x_0, \ldots, \widehat{x_i}, \ldots, x_n) = \frac{f_j(y_1, \ldots y_{i-1}, 1/x_j, y_{i+1}, \ldots y_n)}{x_j}$$

However this is not possible for any nonzero functions f_i and f_j. To see this, we can fix all the variables other than x_j and $y_i = 1/x_j$, and take series expansions of f_i and f_j. Then the expansion for f_i would be a power series in x_j and the expansion for f_j would be a Laurent series in x_j with only negative degrees of x_j (i.e. a power series in $1/x_j$), and the only way for the identity above to hold on the level of series is for $f_i = f_j = 0$, since holomorphic functions are determined by their series expansions. ■

However, not all is lost!

Theorem 1.14. *The bundle $\mathcal{O}(1)$ admits global sections, and*

$$\Gamma_{\mathbb{CP}^n}(\mathcal{O}(1)) \cong \mathbb{C}^{n+1}$$

We again give two proofs of this fact, mirroring our proofs for Theorem 1.13 to illustrate the differences between $\mathcal{O}(-1)$ and $\mathcal{O}(1)$, which are a bit subtle at first.

Proof 1. Since $\mathcal{O}(-1)$ is a subbundle $\mathcal{O}(-1) \hookrightarrow \mathbb{CP}^n \times \mathbb{C}^{n+1}$, we have that the dual bundle $\mathcal{O}(-1)$ is a subbundle $\mathcal{O}(-1) \hookrightarrow (\mathbb{C}^{n+1})^*$. Again, any holomorphic section of $\mathcal{O}(1)$ is a holomorphic function $\mathbb{CP}^n \to \mathbb{C}^{n+1}$, which is constant by the maximum principle. However, any linear functional $\omega \in (\mathbb{C}^{n+1})^*$ determines a linear functional on all the lines $\ell \in \mathbb{CP}^n$ by taking the restriction $\omega|_\ell$. Therefore, the space of sections is $(\mathbb{C}^{n+1})^* \cong \mathbb{C}^{n+1}$. ■

Proof 2. The transition functions for $\mathcal{O}(1)$ are given by $\psi_{ij}(\ell) = z_j/z_i$. Again, using the trivialization of $\mathcal{O}(1)$ in the standard open cover $\{U_i\}$ for $\mathbb{CP}^n$, we have that the data of a section $\mathbb{CP}^n \to \mathcal{O}(1)$ is equivalent to the data of holomorphic functions $f_i : U_i \to \mathbb{C}$, where in $U_i \cap U_j$, we again interpret f_i as a holomorphic function of the variables $x_k := z_k/z_i$ (where x_i is missing) and f_j as a holomorphic function of the variables $y_k := z_k/z_j$ (where y_j is missing). However, we now have that the transition functions are given by

$$\psi_{ij} = z_j/z_i = x_j = 1/y_i$$

We then have that the compatibility condition prescribed by the transition function is

$$f_i(x_1, \dots, \widehat{x_i}, \dots, x_n) = x_j f_j(y_1, \dots y_{i-1}, 1/x_j, y_{i+1}, \dots y_n)$$

We again take the series expansions of f_i and f_j in the variables x_j and $1/x_j$ (holding the rest fixed) and we find that if the identity were to hold, f_j must be at most degree 1 in y_i, i.e. at least degree -1 in x_j, and f_i must be at most degree 1 in x_j, since f_j is holomorphic as a function of $y_i = 1/x_j$ and f_i is holomorphic as a function of x_j. Doing this over all pairs f_i and f_j, we find that the compatibility conditions force the function f_i to be a polynomial of at most degree 1 in all the $x_1, \dots \widehat{x_i}, \dots x_n$, and a choice of f_i determines all the other functions f_j, since they must have the same coefficients (rearranged appropriately) as f_i. Therefore, the space of sections is isomorphic to the subspace of $\mathbb{C}[x_1, \dots, \widehat{x_i}, \dots x_n]$ consisting of linear polynomials in the variables $x_0, \dots, \widehat{x_i} x_n$, which is isomorphic to $\mathbb{C}^{n+1}$, or more suggestively, remembering that $x_i := z_i/z_j$, the subspace of the space

$$\mathbb{C}\left[\frac{z_1}{z_i}, \dots \frac{\widehat{z_i}}{z_i} \dots, \frac{z_n}{z_i}\right]$$

of linear polynomials in the variables $z_1/z_i, \dots \widehat{z_i/z_i}, \dots, z_n/z_i$, which is isomorphic to the space of degree 1 homogeneous polynomials in the $z_0, \dots z_n$ via multiplication by z_i. ■

Using similar methods, we can determine the sections for $\mathcal{O}(k)$ for any $k \in \mathbb{Z}$, using either proof method. If $k > 0$, we have that $\mathcal{O}(-k)$ admits no nonzero holomorphic sections, which can be verified by checking the transition functions, or noting that $\mathcal{O}(-k)$ embeds as a subbundle of $\mathbb{CP}^n \times (\mathbb{C}^{n+1})^{\otimes^k}$, and repeating the maximum principle argument. In the case of $\mathcal{O}(k)$, we are in a similar situation as $\mathcal{O}(1)$, where by looking at how the transition functions interact with the series expansions of functions, we have that the space of sections is isomorphic to the space of polynomials in n variables of degree $\leq k$, which by multiplying through by a the k^{th} power of a variable, is isomorphic to the space of homogeneous polynomials in $n+1$ variables of degree k. In fact, we can give a very explicit description of all local sections of the bundles $\mathcal{O}(d)$.

Theorem 1.15. *Let $\pi : \mathbb{C}^{n-1} - \{0\} \to \mathbb{CP}^n$ be the usual projection sending $z \in C^{n-1}$ to* $\operatorname{span}\{z\}$*. Then the space of sections $\mathcal{O}(k)(U)$ is isomorphic to the space of homogeneous of degree k holomorphic functions $f : \pi^{-1}(U) \to \mathbb{C}$ i.e.*

$$f(tz_0, \ldots, tz_n) = t^k f(z_0, \ldots, z_n)$$

Proof. Let $\sigma \in \mathcal{O}(k)(U)$ be a section. Then the set $\{U \cap U_i\}$ is an open cover of U, so σ is determined by its restrictions $\sigma_i := \sigma_i|_{U \cap U_i}$. Since the bundle $\mathcal{O}(d)$ is trivial over the U_i, the local sections σ_i can be identified with holomorphic functions $U \cap U_i \to \mathbb{C}$ with the compatibility condition

$$\sigma_i([z_0 : \ldots : z_n]) = \left(\frac{z_j}{z_i}\right)^k \sigma_j([z_0 : \ldots : z_n])$$

We then give maps in both directions. Given a section $\sigma \in \mathcal{O}(k)(U)$, define the function f_σ by

$$f_\sigma(z_0, \ldots z_n) = z_i^k \sigma_i(\pi(z_0, \ldots z_n))$$

We must verify that this is well-defined, i.e. it is independent of our choice of i. We compute

$$\begin{aligned} f_\sigma(z_0, \ldots z_n) &= z_i^k \sigma_i(\pi(z_0, \ldots z_n)) \\ &= \left(\frac{z_j}{z_i}\right)^k z_i^k \sigma_j(\pi(z_0, \ldots z_n)) \\ &= z_j^k \sigma_k(\pi(z_0, \ldots z_n)) \end{aligned}$$

So this determines a well defined function on $\pi^{-1}(U)$. In addition, it is visibly homogeneous of degree k, since the σ_i are constant on lines and z_i^k is homogeneous of degree k. To show this is an isomorphism, we provide an inverse. Given a homogeneous function f of degree k on $\pi^{-1}(U)$, define the section σ_f locally by

$$(\sigma_f)_i([z_0 : \ldots : z_n]) = \frac{f(z_0, \ldots z_n)}{z_i^k}$$

then to show that this defines a section, we must show that they agree on intersections using the transition functions. We compute

$$\left(\frac{z_i}{z_j}\right)^k (\sigma_f)_i|_{U \cap U_i \cap U_j}([z_0 : \ldots : z_n]) = \left(\frac{z_i}{z_j}\right)^k \frac{f(z_0, \ldots, z_n)}{z_i^k} = \frac{f(z_0, \ldots z_n)}{z_j^k} = (\sigma_f)_j$$

The two mappings provided are visibly inverses, since one is essentially multiplication by z_j^k and the other is essentially division by z_j^k. ∎

There is another, more geometric way to interpret why the space of global sections of the bundle $\mathcal{O}(k)$ is the space of degree k homogeneous polynomials, which echoes the discussion in the first proofs we gave for Theorem 1.13 and 1.14. A degree k homogeneous polynomial $p \in \mathbb{C}[z_0, \ldots, z_n]_k$ determines a linear map $(\mathbb{C}^{n+1})^{\otimes^k} \to \mathbb{C}$, so the constant maps $\mathbb{CP}^n \to \mathbb{C}[z_0, \ldots, z_n]_k$ mapping all of $\mathbb{CP}^n$ to a homogeneous polynomial p are exactly the holomorphic sections of the trivial bundle $\mathbb{CP}^n \times [(\mathbb{C}^{n+1})^{\otimes^k}]^*$. Restricting these linear maps to $\ell^{\otimes^k} \subset (\mathbb{C}^{n+1})^{\otimes^k}$ for $\ell \in \mathbb{CP}^n$ then determines a section of $\mathcal{O}(k)$.

As a final remark, note that while we started by working with complex geometry and holomorphic maps, we were quickly reduced to studying polynomials, which hints at the closely intertwined nature of complex and algebraic geometry.