YANG-MILLS ON RIEMANN SURFACES

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Contents

1. Preliminary Setup

To discuss the Yang-Mills functional, we must first fix some data. The setup will consist of the following ingredients

(1) A compact manifold M.
(2) A compact connected Lie group G.
(3) A principal G-bundle $P \to M$.

With this data, we have two associated bundles

$$\begin{aligned} \operatorname{Ad} P &:= P \times_G G \\ \mathfrak{g}_P &:= P \times_G \mathfrak{g} \end{aligned}$$

Where the action of G on G is by conjugation, and the action of G on $\mathfrak{g}$ is induced by the differential of conjugation. We note that these bundles both contain additional structure – $\operatorname{Ad} P$ is a bundle of groups (not a principal bundle), and $\mathfrak{g}_P$ is a bundle of Lie algebras. The space of sections $\Gamma(M, \operatorname{Ad} P)$ has a natural group structure given by pointwise multiplication, and is called the ***gauge group*** $\mathscr{G}(P)$. Likewise, the space of sections $\Gamma(M, \mathfrak{g}_P)$ has a natural Lie algebra structure given by the pointwise Lie bracket, and can be naturally identified with the Lie algebra of $\mathscr{G}(P)$. An alternate characterization of these spaces of sections comes from a general characterization of sections of associated bundles

Proposition 1.1. *We have natural correspondences*

$$\begin{aligned} \Gamma(M, \operatorname{Ad} P) &\longleftrightarrow \left\{ f : P \to G \ : \ f(p \cdot g) = g^{-1} f(p) g \right\} \\ \Gamma(M, \mathfrak{g}_P) &\longleftrightarrow \left\{ g : P \to \mathfrak{g} \ : \ f(p \cdot g) = \operatorname{Ad}_{g^{-1}} f(p) \right\} \end{aligned}$$

From the above correspondence we a group isomorphism $\mathscr{G}(P) \to \operatorname{Aut}(P)$, where $\operatorname{Aut}(P)$ denotes the group of G-equivariant diffeomorphisms $\varphi : P \to P$ such that the following diagram commutes :

$$\begin{tikzcd} P \arrow[rr, "\varphi"] \arrow[dr] & & P \arrow[dl] \\ & M & \end{tikzcd}$$

The isomorphism is given by mapping a G-equivariant map $f : P \to g$ to the automorphism $\varphi_f : P \to P$ defined by $\varphi_f(p) = p \cdot f(p)$. In the the case of $\Gamma(M, \mathfrak{g}_P)$, we can extend this to the spaces of $\mathfrak{g}_P$ valued forms. The kernel of the differential of the projection $P \to M$ gives a subbundle ofTP, which has a natural identification with the trivial bundle $\underline{\mathfrak{g}} = P \times \mathfrak{g}$. Then we can identify the space of sections of $\Lambda^k T^*M \otimes \mathfrak{g}_P$, (i.e. the space $\Omega^k_M(\mathfrak{g}_p)$ of $\mathfrak{g}_p$ valued k-forms with a subspace of the space $\Omega^k_P(\mathfrak{g})$ of $\mathfrak{g}$-valued k-forms ω on P satisfying :

(1) $R_g^*\omega = \mathrm{Ad}_{g^{-1}}\,\omega$, where $R_g : P \to P$ denotes the right action of $g \in G$.
(2) $\iota_\xi \omega = 0$ for any $\xi \in \mathfrak{g}$, where ι denotes interior multiplication, and we identify ξ with the constant vector field ξ under the identification of the vertical space with $\mathfrak{g}$.

We have a maps

$$\Omega^p_M(\mathfrak{g}_p) \otimes \Omega^q_M(\mathfrak{g}_P) \to \Omega^{p+q}_M(\mathfrak{g}_p \otimes \mathfrak{g}_p)$$
$$(\omega_1 \otimes \xi_1) \otimes (\omega_2 \otimes \xi_2) \mapsto (\omega_1 \wedge \omega_2) \otimes (\xi_1 \otimes \xi_2)$$

From now on, we will usually omit the tensor symbol for $\mathfrak{g}_P$-valued forms in favor of juxtaposition, i.e. we write $\omega\xi$ instead of $\omega \otimes \xi$. Using the Lie bracket, we then get

$$\Omega^p_M(\mathfrak{g}_p) \otimes \Omega^q_M(\mathfrak{g}_P) \to \Omega^{p+q}_M(\mathfrak{g}_p)$$
$$\omega \otimes \eta \mapsto [\omega, \eta]$$

For any semisimple Lie group G (in particular, for any compact Lie group G), we have an inner product $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ that is invariant under the adjoint action (e.g. the Killing form). Fixing one such inner product induces a fiber product on the trivial bundle $P \times \mathfrak{g}$, and invariance guarantees that this descends to a fiber product on $\mathfrak{g}_P$. This give us pairings

$$\Omega^p_M(\mathfrak{g}_P) \otimes \Omega^q_M(\mathfrak{g}_P) \to \Omega^{p+q}_M$$
$$\omega \otimes \eta \mapsto \langle \omega, \eta \rangle$$

We then fix an orientation and Riemannian metric on M, which gives us a Hodge star operator $\star : \Omega^p_M \to \Omega^{n-p}_M$ and a Riemannian volume form dV_g. The Hodge star extends to $\mathfrak{g}_P$-valued k-forms, where given $\omega \in \Omega^p_M$ and $\xi \in \Gamma(M, \mathfrak{g}_P)$, we define $\star(\omega\xi) = (\star\omega)\xi$. Then given $\omega_1\xi_1, \omega_2\xi_2 \in \Omega^p_M(\mathfrak{g}_P)$, we have

$$\langle \omega_1\xi_1, \star\omega_2\xi_2 \rangle = \langle \omega_1, \omega_2 \rangle_g \langle \xi_1, \xi_2 \rangle$$

where $\langle \cdot, \cdot \rangle_g$ denotes the fiber metric on $\Lambda^p T^*M$ induced by g. This gives us an inner product on each $\Omega^p_M(\mathfrak{g}_P)$ defined by

$$(\theta, \varphi) = \int_M \langle \theta, \star\varphi \rangle$$

Which gives us the L_2 norm on $\Omega^p_M(\mathfrak{g}_P)$ with $\|F\|^2_{L^2} = (F, F)$.

Definition 1.2. A connection on a principal bundle $\pi : P \to M$ is a choice of G-invariant splitting of the exact sequence of vector bundles over P

$$0 \longrightarrow \underline{\mathfrak{g}} \longrightarrow TP \longrightarrow \pi^* TM \longrightarrow 0$$

i.e. a distribution $H \subset TP$ such that

(1) $(R_g)_* H_p = H_{p \cdot g}$
(2) $H \oplus \underline{\mathfrak{g}} = TP$

Equivalently, it is the data of a $\mathfrak{g}$-valued 1-form $A \in \Omega^1_P(\mathfrak{g})$ satisfying

(1) $R^*_g A = \mathrm{Ad}_{g^{-1}} A$
(2) $\iota_\xi A = \xi$ for all $\xi \in \mathfrak{g}$.

Note in particular that by a dimension count, we have that $\pi_*|_H : H \to TM$ is an isomorphism. This implies that given a tangent vector v at x and a point $p \in P$ in the fiber over x, we get a unique horizontal lift $\tilde{v} \in H_p$. For a fixed principal G-bundle $\pi : P \to M$, let $\mathscr{A}(P)$ denote the space of all connections on P, which is an affine space over $\Omega^1_M(\mathfrak{g}_P)$.

Given any distribution $E \subset TP$, we get a Frobenius tensor $\phi_E : E \otimes E \to TP/E$ given by $X \otimes Y \to [X, Y] \mod E$ where we extend X and Y to local vector fields. The Frobenius tensor should be thought of as the obstruction to the existence of an integral submanifold for the distribution E. In the case of a connection H on a principal bundle $P \to M$, we can extend to all of TP by first projecting onto H, and have an identification of $TP/H \cong \underline{\mathfrak{g}}$, and the Frobenius tensor is given by $X \otimes Y \mapsto A([X, Y])$, where A is the connection 1-form, and is called the ***curvature form*** of the connection, and is denoted F_A. In terms of differential forms, we have that for horizontal vectors ξ_1, ξ_2 on TP,

$$dA(\xi_1, \xi_2) = \xi_1 A(\xi_2) - \xi_2 A(\xi_1) - A([\xi_1, \xi_2])$$

The fact that ξ_1 and ξ_2 are horizontal implies that they are in the kernel of A, which gives us that

$$dA + \frac{1}{2}[A, A] = -F_A$$ [1]

It can be shown that F_A transforms by the adjoint action under pullback, and vanishes on vertical vectors, so it descends to a $\mathfrak{g}_P$-valued 2-form on the base manifold M.

Another thing to note is that there is a natural action of the gauge group $\mathscr{G}(P)$ on the space of connections $\mathscr{A}(P)$. Interpreting the elements of $\mathscr{G}(P)$ as bundle automorphisms $\varphi : P \to P$ and elements of $\mathscr{A}(P)$ as $\mathfrak{g}$-valued 1-forms A on P, the action is simply pullback, $(\varphi, A) \mapsto \varphi^* A$. To show that this defines an action, we must check that $\varphi^* A$ satisfies the conditions

(1) $R^*_g \varphi^* A = \mathrm{Ad}_{g^{-1}} \varphi^* A$
(2) $\iota_\xi \varphi^* A = \xi$ for all $\xi \in \mathfrak{g}$.

[1] Our convention for the sign of the curvature is opposite from many other conventions, which usually sets $F_A = dA + \frac{1}{2}[A, A]$

Which are all simple consequences of the G-equivariance of φ and the transformation law for A. For a specific formula, let $\varphi : P \to P$ be an element of the gauge group, and let $g_\varphi : P \to G$ be its associated G-equivariant map. Then

$$\varphi^* A = \operatorname{Ad}_{g_\varphi^{-1}} A + g_\varphi^* \theta$$

where $\theta \in \Omega^1_G(\mathfrak{g})$ denotes the ***Maurer-Cartan form***

$$\theta_g(v) = (dL_{g^{-1}})_g(v)$$

which satisfies the ***Maurer-Cartan equation***

$$d\theta + \frac{1}{2}[\theta, \theta] = 0$$

Proposition 1.3. *Let $A \in \mathscr{A}(P)$ be a connection and $\varphi : P \to P$ an element of $\mathscr{G}(P)$ with associated G-equivariant map $g_\varphi : P \to G$. Then*

$$F_{\varphi^* A} = \operatorname{Ad}_{g_\varphi^{-1}} F_A$$

Proof. Using the transformation law for $\varphi^* A$ we compute

$$\begin{aligned}
F_{\varphi^* A} &= d(\operatorname{Ad}_{g_\varphi^{-1}} A + g_\varphi^* \theta) + \frac{1}{2}[\operatorname{Ad}_{g_\varphi^{-1}} A + g_\varphi^* \theta, \operatorname{Ad}_{g_\varphi^{-1}} A + g_\varphi^* \theta] \\
&= \operatorname{Ad}_{g_\varphi^{-1}} dA + g_\varphi^* d\theta + \frac{1}{2}\Big([\operatorname{Ad}_{g_\varphi^{-1}} A, \operatorname{Ad}_{g_\varphi^{-1}} A] + [\operatorname{Ad}_{g_\varphi^{-1}} A, g_\varphi^* \theta] + [g_\varphi^* \theta, \operatorname{Ad}_{g_\varphi^{-1}}] + [g_\varphi^* \theta, g_\varphi^* \theta]\Big) \\
&= \operatorname{Ad}_{g_\varphi^{-1}} dA + \frac{1}{2}[\operatorname{Ad}_{g_\varphi^{-1}} A, \operatorname{Ad}_{g_\varphi^{-1}} A]]
\end{aligned}$$

Where we use skew-symmetry and the Maurer-Cartan equation. ■

2. The Yang-Mills Functional

With the setup done, we have the ingredients necessary to define the Yang-Mills functional.

Definition 2.1. The ***Yang-Mills functional*** is the map $L : \mathscr{A}(P) \to \mathbb{R}$ given by

$$L(A) = \|F_A\|^2_{L^2} = \int_M \langle F_A, F_A \rangle$$

Remark. The bilinear form $\langle \cdot, \cdot \rangle$ should be thought of as a symplectic form on $\mathscr{A}(P)$, and the mapping $A \mapsto F_A$ should be thought of as the moment map to some action of $\mathscr{G}(P)$. In this context, the Yang-Mills functional is the norm-square of the moment map.

We immediately see that the Yang-Mills equations are invariant under $\mathscr{G}(P)$ in the following sense – if we have any gauge transformation φ with associated map $g_\varphi : P \to G$, we have that $L(\varphi^* A) = L(A)$, which follows immeidately from the invariance of $\langle \cdot, \cdot \rangle$ and the transformation law for curvature.

Our goal now will be to find the Euler-Lagrange equations for the Yang-Mills functional by computing the first and second variations. The connection form A on P induces an exterior covariant derivative on any associated vector bundle $E = P \times_G V$ arising from a linear representation $\rho : G \to \mathrm{GL}(V)$. Let $\dot{\rho} : \mathfrak{g} \to \mathrm{End}(V)$ be the derivative of ρ at the identity. Then the exterior covariant derivative is given by

$$d_A : \Omega^p_M(E) \to \Omega^{p+1}_M(E)$$
$$\psi \mapsto d\psi + \dot{\rho}(A) \wedge \psi$$

In particular, we get an exterior covariant derivative on $\mathfrak{g}_P$, which is given by

$$d_A\psi = d\psi + [A, \psi]$$

Using the Hodge star operator, we construct the formal adjoint with respect to the inner product $d_A^* : \Omega^p_M(\mathfrak{g}_P) \to \Omega^{p-1}_M(\mathfrak{g}_P)$ in the same manner as for classical Hodge theory on a Riemannian manifold. Explicitly, the formula on p-forms is given by

$$d_A^* = (-1)^{n(p+1)+1} \star d_A \star$$

where $n = \dim M$.

For any other connection $A + \eta$ with $\eta \in \Omega^1_M(\mathfrak{g}_P)$, a quick computation yields

$$F_{A+\eta} = F_A + \frac{1}{2}[\eta, \eta] + d_A\eta$$

This allows us to compute the first variation of L.

Proposition 2.2 (***The First Variation***)**.** *For a local extremum $A \in \mathscr{A}(P)$ of the Yang-Mills functional, we have*

$$d_A \star F_A = 0$$

*The local extremum connection A is then callled a **Yang-Mills connection**.*

Proof. Consider a variation $A + t\eta$ with $t \in \mathbb{R}$ and $\eta \in \Omega^1_M(\mathfrak{g}_P)$. We have that the curvature is given by

$$F_{A+t\eta} = F_A + \frac{t^2}{2}[\eta, \eta] + td_A\eta$$

This then gives us

$$\begin{aligned} \|F_{A+t\eta}\|_{L^2} &= \int_M \langle F_{A+t\eta}, F_{A+t\eta} \rangle \\ &= \int_M \langle F_A + \frac{t^2}{2}[\eta, \eta] + td_A\eta, \star(F_A + \frac{t^2}{2}[\eta, \eta] + td_A\eta \rangle \end{aligned}$$

Expanding this out, we get that the term that is linear in t is

$$\int_M \langle F_A, \star d_A\eta \rangle + \langle d_A\eta, \star F_A \rangle = 2(F_A, d_A\eta)$$

where we use symmetry of $(\cdot, \cdot)$. Since A is extremal, we have that this term must vanish, giving us that $(F_A, d_A\eta) = (d_A^* F, \eta) = 0$ for every η. Then since we have (up to sign) $d_A^* = \star d_A \star$, and $\star$ is an isomorphism, this implies $d_A \star F_A = 0$. ■

Proposition 2.3 (***The Second Variation***)**.** *At a Yang-Mills connection $A \in \mathscr{A}(P)$, we have*

$$d_A^* d_A\eta + \star[\eta, \star F_A] = 0$$

Proof. We differentiate the first variational equation with respect to t, i.e. we compute

$$\left.\frac{d}{dt}\right|_{t=0} d^*_{A+t\eta} F_{A+t\eta}$$

We expand out

$$\begin{aligned} d^*_{A+t\eta} F_{A+t\eta} &= \pm \star d_{A+t\eta} \star F_{A+t\eta} \\ &= \pm \left(\star d_A \star \left(F_A + t d_A \eta + \frac{t^2}{2}[\eta, \eta] \right) + t \star \left[\eta, \star \left(F_A + t d_A \eta + \frac{t^2}{2}[\eta, \eta] \right) \right] \right) \end{aligned}$$

Taking the term linear in t yields

$$\pm \left(\star d_A \star d_A \eta + \star [\eta, \star F_A] \right)$$

Giving us that at an extremal connection A, we have

$$d_A^* d_A \eta + \star [\eta, \star F_A] = 0$$

■

3. A Symplectic Viewpoint

4. Yang-Mills Over a Riemann Surface

We now restrict our attention to when M is a surface, i.e. a 2 dimensional real manifold. The Hodge star operator maps $\Omega^1_M \to \Omega^1_M$, and satisfies $\star^2 = -\operatorname{id}$, which induces an almost complex structure on M, giving us a decomposition $\Omega^1_M(\mathbb{C}) = \Omega^{1,0}_M(\mathbb{C}) \oplus \Omega^{0,1}_M(\mathbb{C})$ into the $\pm i$ eigenspaces of the complexified Hodge star. The operator $\overline{\partial} := \pi^{0,1} \circ d$ (where $\pi^{0,1}$ denotes projection onto $\Omega^{0,1}_M(\mathbb{C})$) satisfies $\overline{\partial}^2 = 0$ since by dimension reasons, $\Omega^{0,2}_M(\mathbb{C}) = 0$, so the induced almost complex structure is integrable by the Newlander-Nirenberg theorem. The same argument with projection onto $\Omega^{1,0}_M(\mathbb{C})$ gives an operator ∂ satisfying $\partial^2 = 0$, and we get a decomposition $d = \partial + \overline{\partial}$. Then given a principal bundle $P \to M$, We get a similar decomposition for $\Omega^1_M(\mathfrak{g}_P)$ after complexification giving a decomposition $d_A = \partial_A + \overline{\partial}_A$ for any connection $A \in \mathscr{A}(P)$.