SPIN GEOMETRY CONFERENCE COURSE

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Week 1

Exercise 1.1. Prove $SL_n(\mathbb{R})$ and $O(n)$ are manifolds

Exercise 1.2. What is the "shape" of $SL_2(\mathbb{R})$?

Exercise 1.3. Prove that

$$O(2) = \left\{ \begin{pmatrix} \cos\theta - \sin\theta \\ \sin\theta \cos\theta \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} \cos\theta \sin\theta \\ \sin\theta - \cos\theta \end{pmatrix} \right\}$$

The first set consists of rotations and the second set consists of reflections. Which rotations commute? Which reflections commute? Do reflections commute with reflections?

Exercise 1.4. Investigate $O(3)$. What is it's "shape?"

Week 2

Exercise 2.1. What is the derivative of det?

Exercise 2.2. Explore the exponetial map $\mathfrak{sl}_2(\mathbb{R}) \to SL_2(\mathbb{R})$

Exercise 2.3. Prove that every element of $O(n)$ can be written as the composition of at most n reflections about hyperplanes in $\mathbb{R}^n$.

Proof. We do this by induction. For $n = 1$, this is obvious, since $O(1) \cong \pm 1$. The assuming that this holds for dimension $n - 1$, Let $A \in O(n)$, and let $v \in \mathbb{R}$. We want to construct a hyperplane reflection R such that $RAv = v$, which is obtained by taking R to be the hyperplane reflection about the bisector of v and Av. More explicitly, take R to be the hyperplane reflection about the vector

$$\frac{Av - v}{\|Av - v\|}$$

which is given by the equation

$$Rw = w - 2\frac{\langle Av - v, v\rangle}{\langle Av - v, Av - v\rangle}(Av - v)$$

Computing its action on v, we get

$$\begin{aligned} Rv &= v - 2\frac{\langle Av - v, v\rangle}{\langle Av - v, Av - v\rangle}(Av - v) \\ &= v - \frac{2\langle Av, v\rangle - 2\langle v, v\rangle}{2\langle v, v\rangle - 2\langle Av, v\rangle}(Av - v) \\ &= v + Av - v \\ &= Av \end{aligned}$$

Then since R is its own inverse (being a reflection), we have that $RAv = v$, so RAv fixes v and its orthogonal complement. ∎

TODO insert motivation of $A_n^{\pm}$

Definition 2.4. Define $A_n^{\pm}$ to be the unital algebra generated by $\mathbb{R}^n$ such that $\xi^2 = \pm 1$. and $\xi\eta =? \eta\xi$. Determine the sign of $\eta\xi$. Explore these algebras. Find $A\pm_1$, $A_2^{\pm}$ .... What are they isomorphic to? Can you identify $O(n)$ as a subgroup?

Week 3

Exercise 3.1. Classify the algebras A_n^+ (we messed these up week 2).

Exercise 3.2. Prove that

$$\{e_{i_1}e_{i_2}\dots e_{i+k} \mid 1 \le i_1 < i_2 < \dots < i_k \le n\}$$

is a basis for $A_n^{\pm}$.

Exercise 3.3. Modify the isomorphisms found for A_n^- by choosing $\mathbb{Z}/2\mathbb{Z}$ gradings for the domains and codomains such that the isomorphisms are now isomorphisms as superalgebras.

Exercise 3.4. Construct a tensor product for super vector spaces and superalgebras.

Exercise 3.5. Explore the "shape" of the group

$$G = \langle v \mid \|v\| = 1\rangle \subset (A_n^-)^\times$$

and the nature of the surjection $G \twoheadrightarrow O(n)$. What is the kernel of this map?

Week 4

Exercise 4.1. Define $\varphi : A_n^{\pm} \to A_n^{\pm}$ by $\varphi(v) = -v$ and $\varphi(vw) = (wv)$ (i.e. φ reverses products), and extending linearly to sums. Does $\varphi(x) \cdot x$ define a norm on $A_n^{\pm}$?

Exercise 4.2. Let $(A, \|\|)$ be a normed $\mathbb{R}$-algebra such that $\|ab\| \le \|a\|\,\|b\|$ for all $a, b \in A$. Show that the multiplicative units form an open subset.

We note that an algebra element $a \in A$ determines a linear map $L_a : A \to A$ by left multiplication, i.e. $L_a(b) = ab$. By fixing a basis for A as a vector space, we get an assignment $a \mapsto M_a$, where M_a is the matrix for L_a in this basis. We claim that an element $a \in A$ is a unit iff $\det M_a \neq 0$. To see this, we note that if L_a is not invertible, then a certainly cannot be, since otherwise $L_{a^{-1}}$ would be an inverse. For the other direction, we note that if a is not a unit, then L_a is not surjective, since 1_A is not in the image. We then claim that this mapping $a \mapsto M_a$ is continuous. Do do this, define a norm on the space of linear maps on A by

$$\|M\| = \sup_{v \in A} \frac{\|Mv\|}{\|v\|}$$

Then given $a, b \in A$, we compute

$$\begin{aligned}
\|M_{a-b}\| &= \sup_{v \in A} \frac{\|(a-b)v\|}{\|v\|} \\
&\le \sup_{v \in A} \frac{\|a-b\|\,\|v\|}{\|v\|} \\
&\le \|a-b\|
\end{aligned}$$

So as $b \to a$, we have that $\|M_{a-b}\| \to 0$ as well, so this mapping is continuous. Therefore, the mapping $a \mapsto \det M_a$ is then continuous, which makes the group of units $A^\times$ an open set, being the preimage of the open set $\mathbb{R} - \{0\}$.

One thing to note is that the argument we use to show that $a \mapsto M_a$ is continuous works with any norm such that $\|ab\| \le c\,\|a\|\,\|b\|$ for any constant c. Therefore, we have a small lemma regarding finite dimensional algebras with an inner product.

Lemma. *Let A be an n-dimensional algebra with inner product $\langle \cdot, \cdot \rangle$, and let $\|\cdot\|$ denote the norm induced by the inner product $\|x\|^2 = \langle x, x\rangle$. Then for all $xy \in A$, we have*

$$\|xy\| \le n^5\,|\Gamma|\,\|a\|\,\|b\|$$

where Γ denotes the structure constant of maximal magnitude with respect to a fixed orthonormal basis.

Proof. Fix a basis $\{e_i\}$ for A, and let c_{ij}^k denote the structure constants where

$$e_i e_j = c_{ij}^k e_k$$

Then let $x = a^i e_i$ and $y = b^j e_j$. We then compute

$$\begin{aligned}
\|xy\|^2 &= \langle a^i b^j e_i e_j, a^\ell b^m e_\ell e_m \rangle \\
&= \langle a^i b^j c_{ij}^k e_k, a^\ell b^m c_{\ell m}^n e_n \rangle \\
&= a^i a^\ell b^j b^m c_{ij}^k c_{\ell m}^n \langle e_k, e_n \rangle \\
&\leq a^i a^\ell b^j b^m \Gamma^2 n \\
&\leq n^{5/2} \Gamma^2 \|a\| \|b\|
\end{aligned}$$

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Because of this, we have that for the Clifford algebras $A_n^\pm$, the mapping from algebra elements to linear maps on the algebra is continuous, regardless of our choice of inner product. We can then use this to define a nicer norm on the Clifford algebras. First fix an arbitrary inner product and denote the induced norm $\|\cdot\|_1$. Then define

$$\|a\| = \sup_{v \in A_n^\pm} \frac{\|av\|_1}{\|v\|_1}$$

which gives us a submultiplicative norm, so the group of units is an open subset.

We now want to prove that G is a topological group. To do this, it suffices to show that multiplication and inversion are continuous on $A_n^\pm$. For multiplication, fix $c, d \in A$, and suppose we have $a, b \in A$ such that

$$\|a - c\| < \varepsilon \qquad \|b - d\| < \varepsilon$$

for small $\varepsilon > 0$. Then we have

$$\begin{aligned}
\|ab - cd\| &= \|ab - ad + ad - cd\| \\
&= \|a(b-d) + (a-c)d\| \\
&\leq \|a(b-d)\| + \|(a-c)d\| \\
&\leq \|a\| \|b-d\| + \|a-c\| \|d\| \\
&\leq (\|a\| + \|d\|)\varepsilon \\
&\leq (\|a - c + c\| + \|d\|)\varepsilon \\
&\leq (\|a-c\| + \|c\| + \|d\|)\varepsilon \\
&\leq (\varepsilon + \|c\| + \|d\|)\varepsilon
\end{aligned}$$

so multiplication is continuous at (c, d). For inversion,

Exercise 4.3. An algebra A is called a ***matrix algebra*** if there exists an isomorphism $A \cong \mathrm{End}(V)$ for some vector space V. Which $A_n^\pm$ are matrix algebras?

Exercise 4.4. Given a unital associative algebra A and left A-modules M and N, how would you form the direct sum? Can you tensor them? What if A was a super algebra and M, N super vector spaces?

Exercise 4.5. Let V be a vector space and $b : V \times V \to V$ a bilinear form. We want to construct the Clifford algebra $\mathrm{Cliff}(V, b)$ as the "best" associative unital $\mathbb{R}$-algebra generated by V subject to the relation

$$v_1 v_2 + v_2 v_1 = 2b(v_1, b_2)1_A$$

where 1_A denotes the multiplicative unit in A.

We claim that the above relation is equivalent to the relation $v^2 = b(v, v)1_A$. To see this, we first note that the above condition implies this when we take $v_1 = v_2$. Then for the other direction, consider

$$(v_1 + v_2)^2 = v_1 v_2 + v_2 v_1 + v_1^2 + v_2^2$$

We then apply our relation, giving us

$$\begin{aligned} b(v_1+v_2, v_1+v_2) &= v_1v_2 + v_2v_1 + b(v_1,v_1) + b(v_2,v_2) \\ \implies b(v_1+v_2, v_1+v_2) - b(v_1,v_1) - b(v_2,v_2) &= v_1v_2 + v_2v_1 \end{aligned}$$

Then applying polarization, we arrive at the desired identity.

With this, we want to construct $\mathrm{Cliff}(V,b)$ as the unital algebra satisfying our relation and subject to no others (other than bilinearity of multiplication). Therefore, we can consider the quotient of the tensor algebra $\mathcal{T}(V)$ by the ideal $(v^2 - b(v,v))$ to construct $\mathrm{Cliff}(V,b)$. To characterize it, we think of it as the universal such algebra containing V subject to our relation. Since it is subject to no other relations, we expect this object to be *initial*. It should have a map into every other such algebra satisfying this relation. In other words, for every algebra A with an inclusion $j : V \hookrightarrow A$ such that $j(v_1)j(v_2) + j(v_2)j(v_1) = 2b(v_1,v_2)1_A$, we get a unique map $\mathrm{Cliff}(V,b) \to A$ such that the following diagram commutes

$$\begin{tikzcd} V \arrow[d, hook] \arrow[dr, hook, "j"] & \\ \mathrm{Cliff}(V,b) \arrow[r, dashed] & A \end{tikzcd}$$

We claim that this characterizes the Clifford algebra up to unique isomorphism. Let A be another algebra with map $j : V \hookrightarrow A$ satisfying the same property we gave above. From the universal property of $\mathrm{Cliff}(V,b)$, we get a unique map $\mathrm{Cliff}(V,b) \to A$. Likewise, the inclusion $V \hookrightarrow \mathrm{Cliff}(V,b)$ gives us a unique map $A \to \mathrm{Cliff}(V,b)$. We claim that these two maps are inverses. We note that both maps are given by $v \mapsto j(v)$ $j(v) \mapsto v$ and extending to products and sums, so we have that they are inverses.

We now want to verify that the construction $\mathcal{T}(V)/(v^2 - b(v,v))$ satisfies this universal property, i.e. the Clifford algebra exists. Given an algebra A with a map $j : V \hookrightarrow A$ satisfying $j(v)^2 - b(v,v) = 0$, define the map $\mathcal{T}(V) \to A$ by $v \mapsto j(v)$ and extending linearly and to products. Then the ideal $(v^2 - b(v,v))$ lies in the kernel of this map, so the map factors through uniquely through $\mathcal{T}(V)/(v^2 - b(v,v))$, so it satisfies the property we laid out.