A QUICK SURVEY OF HODGE THEORY

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Contents

1. Linear Algebra

Definition 1.1. Let V be a finite dimensional $\mathbb{R}$-vector space. An ***almost complex structure*** on V is a linear map $J : V \to V$ satisfying $J^2 = \mathrm{id}_V$. An ***almost complex vector space*** is tuple (V, J), where V is a finite dimensional $\mathbb{R}-$vector space equipped with an almost complex structure J.

The data of an almost complex structure J is equivalent to giving V the structure of a complex vector space, where we define $(a + bi) \cdot v = ab + bJv$. Because of this, we may call J a ***complex structure***. We use the name *almost* complex structure to emphasize the differences between the analogous constructions in the nonlinear world of manifolds. Note that since an almost complex structure is equivalent to a complex structure, this immediately implies that V is even dimensional.

One way to think about an almost complex structure is through the geometric interpretation of complex multiplication. If we regard $\mathbb{C}$ as a 2-dimensional vector space over $\mathbb{R}$, multiplication by i corresponds to a rotation by $\pi/2$ in the counterclockwise direction, and multiplication by $-i$ corresponds to a rotation by $\pi/2$ in the clockwise direction. From this we see that a choice of a square root of -1 comes with a choice of clockwise or counterclockwise. This implies that every complex vector space is canonically oriented as a real vector space. Given a $\mathbb{C}$-vector space V with $\mathbb{C}$-basis $\{z_1, \ldots, z_n\}$, we have that the ordered basis $(z_1, iz_1, \ldots z_n, iz_n)$ defines a positively oriented basis for V over $\mathbb{R}$.

Definition 1.2. Let V by any $\mathbb{R}$-vector space. The ***complexification*** of V is the complex vector space $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$. The complexification $V_{\mathbb{C}}$ is naturally a complex vector space, where for $\lambda \in \mathbb{C}$, the action of λ on a homogeneous element $\mu \otimes v$ is given by

$$\lambda \cdot (\mu \otimes v) = \lambda\mu \otimes v$$

The complexification is a different way of obtaining a complex vector space from a real one. The complexification $V_{\mathbb{C}}$ is a vector space of twice the dimension of V as a vector space over $\mathbb{R}$. In addition, there is a natural inclusion $V \hookrightarrow V_{\mathbb{C}}$ given by $v \mapsto 1 \otimes v$. Complexification is an instance of *extension of scalars* – every element of $V_{\mathbb{C}}$ is of the form $av + biw$ where $a, b \in \mathbb{R}$, and $v, w \in V$. Therefore, we will denote the element $(a + bi) \otimes v \in V_{\mathbb{C}}$ by $av + biv$, and we have a direct sum decomposition $V_{\mathbb{C}} = V \oplus iV$. Given a linear map

$T : V \to W$ of $\mathbb{R}$-vector spaces, we can extend T to a complexified map $T_{\mathbb{C}} : V_{\mathbb{C}} \to W_{\mathbb{C}}$, where $T_{\mathbb{C}}(av + biw) = aTv + biTv$. In other words, $T_{\mathbb{C}} = \mathrm{id}_{\mathbb{C}} \otimes T$. In this way, we see that complexification defines a covariant functor $\mathsf{Vect}_{\mathbb{R}} \to \mathsf{Vect}_{\mathbb{C}}$ from the category of $\mathbb{R}$-vector spaces to the category of $\mathbb{C}$-vector spaces.

A natural question to ask is how an almost complex structure interacts with the process of complexification. Let (V, J) be an almost complex vector space. Then the complexified map $J_{\mathbb{C}} : V_{\mathbb{C}} \to V_{\mathbb{C}}$ squares to -1, and admits eigenvalues $\pm i$. For example, consider $V = \mathbb{C}$. We then can then make the natural identification of $\mathbb{C}$ with the almost complex vector space $(\mathbb{R}^2, J)$, where J is given by the matrix

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

using the ordered $\mathbb{R}$ basis $(1, i)$. Then $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}^2$, and $J_{\mathbb{C}}$ is given by the matrix

$$J_{\mathbb{C}} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

in the ordered $\mathbb{R}$ basis $(1, i, 1, i)$, where the first pair elements are elements of one copy of $\mathbb{C}^2$, and the second pair of elements are elements of a separate copy of $\mathbb{C}^2$, giving an $\mathbb{R}$ basis for $\mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$. The matrix $J_{\mathbb{C}}$ clearly has eigenvalues $\pm i$, with the i-eigenspace being spanned by the vectors

$$\begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{i}{2} \\ 0 \\ -\frac{i}{2} \\ 0 \end{pmatrix}$$

and the $-i$ eigenspace being the span of

$$\begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{i}{2} \\ 0 \\ -\frac{i}{2} \end{pmatrix}$$

The case for general V is similar, and amounts to fixing an $\mathbb{R}$ basis of the form $\{v_1, Jv_1, \ldots v_n, Jv_n\}$ for (V, J). The decomposition of $V_{\mathbb{C}}$ into the $\pm i$-eigenspaces of $J_{\mathbb{C}}$ gives a direct sum decomposition

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$$

where $V^{1,0}$ denotes the i-eigenspace, and $V^{0,1}$ denotes the $-i$-eigenspace. Note that in the bases we chose above for $\mathbb{C}^2$, complex conjugation is given by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and determines an isomorphism of complex vector spaces $V^{1,0} \cong \overline{V^{0,1}}$ and vice versa.

An almost complex structure J on V induces a dual map $J^* : V^* \to V^*$. By functoriality of taking dual spaces, $(J^*)^2 = -\operatorname{id}_{V^*}$, giving V^* the structure of an almost complex vector space via J^*. Explicitly, given a linear functional $\alpha \in V^*$, we have that the action of J^* on α is given by

$$(J^*\alpha)(v) = \alpha(Jv)$$

for every $v \in V$. We then get an analogous decomposition of the complexified dual space$V^*_{\mathbb{C}} = \mathsf{Hom}_{\mathbb{R}}(V, \mathbb{C})$ as

$$V^*_{\mathbb{C}} = (V^*)^{1,0} \oplus (V^*)^{0,1}$$

into the $\pm i$ eigenspaces of J^*, and have very natural interpretations in terms of V. The subspace $(V^*)^{1,0}$ consists of the α such that $\alpha(Jv) = i\alpha(v)$. We have a natural pairing $(V^*)^{1,0} \otimes V^{1,0} \to \mathbb{C}$ given by

$$\langle \alpha, v \rangle = \alpha(v)$$

which is nondegenerate, establishing the two vector spaces as dual to each other (as complex vector spaces). A similar statement holds for $V^{0,1}$ and $(V^*)^{0,1}$, putting them in duality as well. Another perspective to take is that the condition that $\alpha(Jv) = i\alpha(v)$ is equivalent to α being *complex linear* with respect to the complex structure J on V and i on $\mathbb{C}$, giving us an isomorphism $(V^*)^{1,0} \cong \mathsf{Hom}_{\mathbb{C}}(V, \mathbb{C})$. Similarly, we have that $(V^*)^{0,1} \cong \mathsf{Hom}_{\mathbb{C}}(V, \overline{\mathbb{C}})$

2. Differential Forms