ČECH COHOMOLOGY, SHEAF COHOMOLOGY, AND LINE BUNDLES

JEFFREY JIANG

1. Čech Cohomology

For the most part, we will consider “sufficiently nice" topological spaces. For the most part, think of a space X as a (smooth, complex) manifold, an analytic space, or a quasi-compact separated scheme if you're feeling adventurous.

Definition 1.1. Let X be a space and $\mathcal{F}$ a sheaf of abelian groups over X. Let $\mathcal{U} = \{U_i\}_{i\in\mathbb{N}}$ be a countable open cover of X that is locally finite, i.e. for any $x \in X$, only finitely many U_i contain x. For $I = \{i_1, \ldots, i_k\}$, let

$$U_I := \bigcap_{i\in I} U_i$$

Then define the ***Čech cochain groups*** of $\mathcal{F}$ for the cover $\mathcal{U}$ by

$$C^k(\mathcal{U}, \mathcal{F}) := \prod_{|I|=k+1} \mathcal{F}(U_I)$$

an element of $C^k(\mathcal{U}, \mathcal{F})$ is called a ***Čech cochain***. For a k-cochain σ, and $I = \{i_0, \ldots i_k\}$, we denote the component of σ over U_I as σ_I or $\sigma_{i_0,\ldots i_k}$.

The Čech cochain groups are equipped with a differential $d : C^k(\mathcal{U}, \mathcal{F}) \to C^{k+1}(\mathcal{U}, \mathcal{F})$ where for $\sigma \in C^k(\mathcal{U}, \mathcal{F})$, the $i_0, \ldots i_{k+1}$ component of $d\sigma$ is given by

$$(d\sigma)_{i_0,\ldots i_{k+1}} = \sum_{j=1}^{p+1} (-1)^j \sigma_{i_0,\ldots,\widehat{i_j},\ldots,i_{k+1}}|_{U_0\cap\cdots\cap U_{k+1}}$$

where $\widehat{i_j}$ denotes that i_j is missing. We have that $d^2 = 0$ for a similar reason that $d^2 = 0$ for singular cohomology, you get repeats of terms with opposite signs due to the omitted index. We denote the kernel of $d : C^i(\mathcal{U}, \mathcal{F}) \to C^{i+1}(\mathcal{U}, \mathcal{F})$ as $Z^i(\mathcal{U}, \mathcal{F})$, and we say that the element are i-cocycles. We denote the image of $d : C^{i-1}(\mathcal{U}, \mathcal{F}) \to C^i(\mathcal{U}, \mathcal{F})$ as $B^i(\mathcal{U}, \mathcal{F})$, and we call the elements i-coboundaries.

Definition 1.2. The ***Čech cohomology groups*** of $\mathcal{F}$ with respect to the cover $\mathcal{U}$, denoted $\check{H}^i(\mathcal{U}, \mathcal{F})$ is the cohomology of the Čech complex

$$0 \xrightarrow{d} C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{d} \cdots$$

i.e. we have

$$\check{H}^i(\mathcal{U}, \mathcal{F}) := \frac{Z^i(\mathcal{U}, \mathcal{F})}{B^i(\mathcal{U}, \mathcal{F})}$$

Definition 1.3. Given an open cover $\mathcal{U} = \{U_i\}$, a ***refinement*** of $\mathcal{U}$ is another open cover $\mathcal{V} = \{V_j\}$ such that every V_j is contained in some U_i. If $\mathcal{V}$ is a refinement of $\mathcal{U}$, we write $\mathcal{V} < \mathcal{U}$.

If $\mathcal{V} < \mathcal{U}$, then we know we can find a map $\varphi : \mathbb{N} \to \mathbb{N}$ such that $V_i \subset U_{\varphi}(i)$. Consequently, we can restrict sections over U_i to sections over $V_{\varphi(i)}$, so this induces a chain map $\rho_\varphi : C^k(\mathcal{U}, \mathcal{F}) \to C^k(\mathcal{V}, \mathcal{F})$ where

$$(\rho_\varphi(\sigma))_{i_0,\dots,i_k} = \sigma_{\varphi(i_0),\dots\varphi(i_k)}|_{U_{i_0}\cap\cdots\cap U_{i_k}}$$

This map commutes with the differentials for $C^\bullet(\mathcal{U}, \mathcal{F})$ and $C^\bullet(\mathcal{V}, \mathcal{F})$, so it descends to homomorphisms $\check{H}^i(\mathcal{U}, \mathcal{F}) \to \check{H}^i(\mathcal{V}, \mathcal{F})$. It can be shown that a different choice of chain map ρ_ψ for $\psi : \mathbb{N} \to \mathbb{N}$ is chain homotopic to ρ_φ, so the induced maps on cohomology are independent of our choice of φ.

Definition 1.4. The ***Čech cohomology groups*** of a sheaf $\mathcal{F}$ over X is the limit over refinements

$$\check{H}^i(X, \mathcal{F}) := \lim_{\mathcal{V}<\mathcal{U}} \check{H}^i(\mathcal{V}, \mathcal{F})$$

i.e. the quotient disjoint union $\amalg_{\mathcal{U}} \check{H}^i(\mathcal{U}, \mathcal{F}$ over all refinements, where we identify $\sigma \in \check{H}^i(\mathcal{U}, \mathcal{F}$ and $\tau \in \check{H}^i(\mathcal{V}, \mathcal{F})$ if $\mathcal{V}$ refines $\mathcal{U}$ and $\rho(\sigma) = \tau$ under the map induced on cohomology by the refinement.

This definition of the Čech cohomology groups is essentially useless for computation. It's true power comes from the following theorem, due to Leray.

Theorem 1.5 (***Leray***)**.** *Suppose $\mathcal{U} = \{U_i\}$ is an open cover of X that is* ***acyclic*** *with respect to the sheaf $\mathcal{F}$, i.e. for $|I| > 1$ and any i,*

$$\check{H}^i(U_I, \mathcal{F}) = 0$$

Then

$$\check{H}^i(\mathcal{U}, \mathcal{F}) = \check{H}^i(X, \mathcal{F})$$

The intuition to keep in mind for an acyclic cover is the notion of a good cover in differential geometry. On a smooth manifold M, there exists a covering of M by open sets $\{U_i\}$ such that any nonempty intersections are contractible, which is done by taking geodesic balls around each point in M. Since the homotopical information on each U_i is trivial, the only nontrivial topological information in the cohomology of M comes from how the sets are glued together to form M. As with exact sequences of chain complexes, short exact sequences of sheaves give long exact sequences in sheaf cohomology.

Theorem 1.6. *Let*

$$0 \longrightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \longrightarrow 0$$

be an exact sequence of sheaves over X. Then this induces a long exact sequence in cohomology:

$$\begin{tikzcd} 0 \arrow[r] & \check{H}^0(X,\mathcal{E}) \arrow[r, "\alpha^*"] & \check{H}^0(X,\mathcal{F}) \arrow[r, "\beta^*"] & \check{H}^0(X,\mathcal{G}) \arrow[dll, "\delta"'] \\ & \check{H}^1(X,\mathcal{E}) \arrow[r, "\alpha^*"] & \check{H}^1(X,\mathcal{F}) \arrow[r, "\beta^*"] & \check{H}^1(X,\mathcal{G}) \arrow[dlll, "\delta"'] \\ {} & & \cdots & \cdots \end{tikzcd}$$

Proof. We first define the maps $\alpha^* : \check{H}^i(X,\mathcal{E}) \to \check{H}^i(X,\mathcal{F})$ and $\beta^* : H^i(X,\mathcal{F}) \to \check{H}^i(X,\mathcal{G})$, and will then define the connecting homomorphism $\delta : \check{H}^i(X,\mathcal{E}) \to \check{H}^{i+1}(X,\mathcal{G})$. Given an open cover $\mathcal{U} = \{U_i\}$ of X, the sheaf morphism α gives for each open set U_i a homomorphism $\alpha(U_i) : \mathcal{E}(U_i) \to \mathcal{F}(U_i)$, which induces a chain map $C^\bullet(\mathcal{U},\mathcal{F}) \to C^\bullet(\mathcal{U},\mathcal{G}))$. Since the maps $\alpha(U_i)$ commute with restriction maps, this chain map commutes with the differentials, so it descends to a map on cohomology $\check{H}^\bullet(\mathcal{U},\mathcal{F}) \to \check{H}^\bullet(\mathcal{U},\mathcal{F})$, which, after taking the limit over refinements or choosing $\mathcal{U}$ to be simultaneously acyclic for $\mathcal{E}$ and $\mathcal{F}$, gives us the induced map $\alpha^* : \check{H}^\bullet(X,\mathcal{F}) \to \check{H}^\bullet(X,\mathcal{F})$. The map β^* is defined similarly.

The construction of the connecting homomorphism mirrors the construction for singular (or de Rham) cohomology. We represent an element of $\check{H}^i(X,\mathcal{G})$ with a cocycle $\sigma \in Z^i(\mathcal{U},\mathcal{G})$ with respect to some open cover $\mathcal{U}$. By surjectivity of β, by potentially passing to a refinement, we can write $\sigma = \beta(\tau)$ for some $\tau \in C^i(\mathcal{U},\mathcal{F})$. Then since the induced map on chains commutes with the differentials, we have that $d\beta(\tau) = \beta(d\tau) = 0$, since τ is a cocycle. Therefore, $d\tau$ is in the kernel of β^*, so we can write $d\tau = \alpha(\eta)$ for some $\eta \in C^{i+1}(\mathcal{U},\mathcal{F})$. We then define $\delta(\sigma)$ to be the class of η in the limit. We note that this is independent of our choice of τ, since any other choice of preimage of σ differs by an element of the form $\alpha(e)$ for some cocycle e by exactness. Then since α commutes with the differentials, we get $d(\tau + \alpha(e)) = d\tau + d\alpha(e) = d\tau + \alpha(de) = d\tau$. ∎

We make one observation about Čech cohomology

Proposition 1.7. *The 0^{th} Čech cohomology group is isomorphic to the space of global sections, i.e.*

$$\check{H}^0(\mathcal{U},\mathcal{F}) \cong \Gamma(X,\mathcal{F})$$

Proof. The 0^{th} cohomology is just the kernel of the map $d : C^0(\mathcal{U},\mathcal{F}) \to C^1(\mathcal{U},\mathcal{F})$. For a 0-cochain σ, we have that

$$(d\sigma)_{ij} = \sigma_j|_{U_i \cap U_j} - \sigma_i|_{U_i \cap U_j}$$

We then claim that the map $\Gamma(X,\mathcal{F}) \to \ker d$ sending a section σ to the cocycle $\tilde{\sigma}$ defined by

$$\tilde{\sigma}_i = \sigma|_{U_i}$$

is bijective. It is surjective, since any 0-cocycle mapping in the kernel is a collection of local sections that agrees on intersections, which is exactly a global section of $\mathcal{F}$. In addition, it is injective, since if a section restricts to 0 on every open set, it is the zero section. ∎

2. Čech Cohomology on $\mathbb{CP}^n$

We now compute Čech cohomology for various sheaves over $\mathbb{CP}^n$. The main objects of interest are the line bundles $\mathcal{O}(k)$. We have that $\mathbb{CP}^n$ admits a nice cover $\mathcal{U} = \{U_i\}$ where U_i is the open set where the coordinate z_i does not vanish. We will soon show that this covering is acyclic for the structure sheaf $\mathcal{O}_{\mathbb{CP}^n}$, and since all line bundles are trivial on the open sets in this cover, this means that the covering will acyclic for any line bundle $\mathcal{O}(k)$. So it suffices to compute Čech cohomology with respect to this cover. In particular, we note that for any line bundle $\mathcal{O}(k) \to \mathbb{CP}^n$, the transition functions ψ_{ij} for $\mathcal{O}(k)$ are $\psi_{ij}(\ell) = (z_j/z_i)^k$. We already know one cohomology group:

Theorem 2.1. *The 0^{th} cohomology group $\check{H}^0(\mathbb{CP}^n, \mathcal{O}(k))$ is isomorphic to the space $\mathbb{C}[x_0, \dots x_n]_k$ of homogeneous degree k polynomials in the variables $x_0, \dots, x_n$.*

We compute the rest of the cohomology now, which we do in stages.

Theorem 2.2. *For any $i > n$, we have $\check{H}^n(\mathcal{U}, \mathcal{O}(k)) = 0$.*

Proof. The cover of $\mathbb{CP}^i$ by the U_i has cardinality $n+1$. Therefore, $C^{n+1}(\mathcal{U}, \mathcal{O}(k)) = 0$. ∎

To compute the rest of the cohomology groups, we first prove a lemma characterizing local sections of $\mathcal{O}(k)$.

Lemma 2.3. *Let $\pi : \mathbb{C}^{n-1} - \{0\} \to \mathbb{CP}^n$ be the usual projection sending $z \in C^{n-1}$ to* span $\{z\}$. *Then the space of sections $\mathcal{O}(k)(U)$ is isomorphic to the space of homogeneous of degree k holomorphic functions $f : \pi^{-1}(U) \to \mathbb{C}$ i.e.*

$$f(tz_0, \dots, tz_n) = t^k f(z_0, \dots, z_n)$$

Proof. Let $\sigma \in \mathcal{O}(k)(U)$ be a section. Then the set $\{U \cap U_i\}$ is an open cover of U, so σ is determined by its restrictions $\sigma_i := \sigma_i|_{U \cap U_i}$. Since the bundle $\mathcal{O}(d)$ is trivial over the U_i, the local sections σ_i can be identified with holomorphic functions $U \cap U_i \to \mathbb{C}$ with the compatibility condition

$$\sigma_i([z_0 : \dots : z_n]) = \left(\frac{z_j}{z_i}\right)^k \sigma_j([z_0 : \dots : z_n])$$

We then give maps in both directions. Given a section $\sigma \in \mathcal{O}(k)(U)$, define the function f_σ by

$$f_\sigma(z_0, \dots z_n) = z_i^k \sigma_i(\pi(z_0, \dots z_n))$$

We must verify that this is well-defined, i.e. it is independent of our choice of i. We compute

$$\begin{aligned} f_\sigma(z_0, \dots z_n) &= z_i^k \sigma_i(\pi(z_0, \dots z_n)) \\ &= \left(\frac{z_j}{z_i}\right)^k z_i^k \sigma_j(\pi(z_0, \dots z_n)) \\ &= z_j^k \sigma_k(\pi(z_0, \dots z_n)) \end{aligned}$$

So this determines a well defined function on $\pi^{-1}(U)$. In addition, it is visibly homogeneous of degree k, since the σ_i are constant on lines and z_i^k is homogeneous of degree k. To show this is an isomorphism, we provide an inverse. Given a homogeneous function f of degree k on $\pi^{-1}(U)$, define the section σ_f locally by

$$(\sigma_f)_i([z_0 : \ldots : z_n]) = \frac{f(z_0, \ldots z_n)}{z_i^k}$$

then to show that this defines a section, we must show that they agree on intersections using the transition functions. We compute

$$\left(\frac{z_i}{z_j}\right)^k (\sigma_f)_i|_{U \cap U_i \cap U_j}([z_0 : \ldots : z_n]) = \left(\frac{z_i}{z_j}\right)^k \frac{f(z_0, \ldots, z_n)}{z_i^k} = \frac{f(z_0, \ldots z_n)}{z_j^k} = (\sigma_f)_j$$

The two mappings provided are visibly inverses, since one is essentially multiplication by z_j^k and the other is essentially division by z_j^k. ■

Over intersections of the distinguished open sets U_i, the sections have a particularly nice form. Under the projection $\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{CP}^n$, the preimage of U_I for $I = \{i_0, \ldots i_d\}$ is just $\mathbb{C}^{n+1}$ minus the coordinate axes $z_{i_j} = 0$. By taking power series, a holomorphic function on $\pi^{-1}(U_I)$ is given by Laurent series where the z_{i_j} can appear in negative degree. Being homogeneous of degree d implies that all the terms in the series expansion must be homogeneous of degree k, where the degree of $(z_k)^a/(z_{i_j})^b$ is $a - b$. Consequently, all such holomorphic functions must be polynomials in $\mathbb{C}[z_0, \ldots z_n, z_{i_0}^{-1}, \ldots z_{i_d}^{-1}]$ of degree k.

3. Čech Cohomology and Line Bundles

Let $\mathcal{U} = \{U_i\}$ be a covering of X, and $\mathcal{F}$ a sheaf of abelian groups over X. Then with respect to this cover, a Čech 2-cocycle $\sigma \in Z^2(\mathcal{U}, \mathcal{F})$ is defined by the equation

$$0 = (d\sigma)_{ijk} = \sigma_{jk}|_{U_i \cap U_j \cap U_k} - \sigma_{ik}|_{U_i \cap U_j \cap U_k} + \sigma_{ij}|_{U_i \cap U_j \cap U_k}$$

Written multiplicatively (and omitting the restriction), this becomes

$$1 = \sigma_{jk}\sigma_{ik}^{-1}\sigma_{ij}$$

which, since the group is abelian, is equivalent to

$$\sigma_{ik} = \sigma_{ij}\sigma_{jk}$$

which looks exactly like a cocycle condition for transition functions of a line bundle. Recall that given a holomorphic line bundle $\pi : L \to X$, we have local trivializations – we can find a cover $\mathcal{U} = \{U_i\}$ with maps $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$ such that

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, "\varphi_i"] \arrow[dr] & & U_i \times \mathbb{C} \arrow[dl] \\ & U_i & \end{tikzcd}$$

commutes, where the maps to U_i are the projections. Therefore, if we consider the map $\varphi_i \circ \varphi_k^{-1} : U_i \cap U_j \times \mathbb{C} \to U_i \cap U_j \times \mathbb{C}$, we have that $\varphi(x, \lambda) = (x, \psi_{ij}(x)(\lambda))$, where the functions $\psi_{ij} : U_i \cap U_j \to \mathrm{GL}_1\mathbb{C}$ are holomorphic. The ψ_{ij} are called the ***transition functions*** of the line bundle L.

Proposition 3.1. *The transition functions ψ_{ij} satisfy the following conditions*

(1) $\psi_{ij}\psi_{ji} = 1$ (*i.e. the constant function* $x \mapsto 1$)
(2) $\psi_{ij}\psi_{jk} = \psi_{ik}$.

The second condition is often called the ***cocycle condition****, in reference to the identity we derived above for the defining property of a Čech cocycle.*

Proof. Consider the map $\varphi_i \circ \varphi_j^{-1} \circ \varphi_j \circ \varphi_i^{-1} = \mathrm{id}$. We compute under the action on a general element (x, λ)

$$(x, \lambda) \xrightarrow{\varphi_j \circ \varphi_i^{-1}} (x, \psi_{ji}(x)(\lambda)) \xrightarrow{\varphi_i \circ \varphi_j^{-1}} (x, \psi_{ij}(x)\psi_{ji}(x)(\lambda))$$

Therefore, we have that $\psi_{ij}(x)\psi_{ji}(x) = 1$ for all x, showing the first property. For the second property, we do the same thing. Consider the function $\varphi_i \circ \varphi_j^{-1} \circ \varphi_j \circ \varphi_k = \varphi_i \circ \varphi_k^{-1}$. Then fore (x, λ), we compute the action of this function to be

$$(x, \lambda) \xrightarrow{\varphi_j \circ \varphi_k^{-1}} (x, \psi_{jk}(x)(\lambda)) \xrightarrow{\varphi_i \circ \varphi_j^{-1}} (x, \psi_{ij}(x)\psi_{jk}(x)(\lambda))$$

So we get that $\psi_{ij}(x)\psi_{jk}(x) = \psi_{ik}(x)$ for all x. ■

Under the Čech differential, the image of a Čech 0-cochain σ is given by

$$(d\sigma)_{ij} = \sigma_j - \sigma_i$$

written multiplicatively, this becomes

$$(d\sigma)_{ij} = \sigma_j \sigma_i^{-1}$$

In the same spirit, we translate this to a statement regarding transition functions of a line bundle.

Proposition 3.2. *Let $\pi L \to X$ be a holomorphic line bundle where the transition functions ψ_{ij} with respect to a cover $\{U_i\}$ satisfy the* ***coboundary condition****, i.e. there exist holomorphic functions $\sigma_i : U_i \to \mathrm{GL}_1\mathbb{C}$ such that*

$$\psi_{ij} = \sigma_j \sigma_i^{-1}$$

Then L is a trivial line bundle.

Proof. It suffices to provide a nonvanishing section $X \to L$. A section $s : X \to L$ is equivalent to functions $s_i : U_i \to \mathbb{C}$ with the compatibility condition

$$s_i = \psi_{ij} s_j$$

define the s_i by $s_i = \sigma_i^{-1}$. Then they satisfy the compatibility condition, since

$$\psi_{ij}\sigma_j = \sigma_j \sigma_i^{-1} \sigma_j^{-1} = \sigma_i^{-1} = s_i$$

then since the σ_i are functions to $\mathrm{GL}_1\mathbb{C} = \mathbb{C}^{\times}$, they glue to a global nonvanishing section, so L is isomorphic to the trivial line bundle $X \times \mathbb{C}$. ■

Recall that isomorphism classes of line bundles over X form a group under tensor product, where the inverse of a line bundle L is the dual bundle L^*. Given line bundles $L, L' \to X$ and an open cover $\mathcal{U}_\rangle$ of X in which both L and L' are trivialized over the U_i (for instance, a good cover of X), let ψ_{ij} be the transition functions for L and let φ_{ij} be the transition functions for L'. Then the transition functions for $L \otimes L'$ are $\psi_{ij}\varphi_{ij}$, and the transition functions for L^* are given by φ_{ij}^{-1}.

Theorem 3.3. *Let X be a complex manifold, and $\mathcal{O}_X$ its sheaf of holomorphic functions. Then let $\mathcal{O}_X^\times$ be the sheaf of invertible functions, which is a sheaf of abelian groups under multiplication. Then we have a group isomorphism*

$$\check{H}^1(X, \mathcal{O}_X^\times) \cong \mathrm{Pic}(X)$$

Proof. Fix a good cover $\mathcal{U} = \{U_i\}$ for X. Since all the sets and their nonempty intersections are contractible, we have that $\check{H}^i(U_i, \mathcal{F}) = 0$ for all $i > 0$ where $\mathcal{F}$ is the sheaf of sections of any line bundle. Since all the U_i are contractible, we also have that any line bundle over U_i is trivial, so it admits transition functions ψ_{ij} with respect to this cover. As shown above, the functions ψ_{ij} exactly define a Čech 1-cocyle, and any Čech coboundary defines a trivial bundle. In addition, we have that the the transition functions of a tensor product are exactly the products of the transition functions. Putting everything together, this tells us that the mapping $L \mapsto \{\psi_{ij}\}$ is a bijective group homomorphism. ∎

4. Divisors

5. Sheaf Cohomology

Over a complex manifold X, we have many different cohomology theories at our disposal:

(1) The singular cohomology groups $H^i_{\mathrm{sing}}(X, \mathbb{Z})$.
(2) The de Rham cohomology groups $H^i_{dR}(X)$.
(3) The Dolbeault cohomology groups of a holomorphic vector bundle $H^i_{\bar{\partial}}(X, E)$.
(4) The Čech cohomology groups of a sheaf $\check{H}^i(X, \mathcal{F})$.

We want to compare these various cohomology theories. To do so, we show that many of these cohomology theories are computing the same thing : sheaf cohomology.

Remark. While we might explicitly work with sheaves of abelian groups, the following discussion is applicable to sheaves of $\mathcal{O}_X$ modules, C^∞ modules, etc.

Definition 5.1. The ***global sections functor*** $\Gamma(X, \cdot)$ is a functor $\mathsf{Ab}(X) \to \mathsf{Ab}$ of the category of sheaves of abelian groups over X to the category of abelian groups, where given a sheaf of abelian groups $\mathcal{F}$,

$$\Gamma(X, \mathcal{F}) := \mathcal{F}(X)$$

The functor is left-exact, i.e. given an exact sequence of sheaves

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}$$

we get an exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{E}) \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{G})$$

However, the functor is not right exact, which is due to the local definitions of injectivity and surjectivity. The sheaf axiom guarantees that being injective on stalks implies that a sheaf morphism is injective on sections, but it does not imply the same thing for surjectivity. As an example, let $\mathcal{Z}^k$ be the sheaf of closed smooth k-forms, and let B^k be the sheaf of exact k-forms. Then the inclusion $\mathcal{B}^k \hookrightarrow \mathcal{Z}^k$ is surjective, since every closed k-form is exact in a sufficiently small neighborhood. However, if $H^k_{dR}(X) \neq 0$, then $\Gamma(X, \mathcal{Z}^k) \to \Gamma(X, \mathcal{B}^k)$ is not surjective.

Definition 5.2. The ***sheaf cohomology groups*** $H^i(X, \mathcal{F})$ of a sheaf $\mathcal{F}$ over X are the right derived functors of the global sections functor applied to $\mathcal{F}$

$$H^i(X, \mathcal{F}) := R^i\Gamma(X, \mathcal{F})$$

i.e, we take an injective resolution $\mathcal{I}^\bullet = \left\{\mathcal{I}^j\right\}$ of $\mathcal{F}$

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \cdots$$

and apply $\Gamma(X, \cdot)$ term-wise to the sequence to get

$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{I}^0) \longrightarrow \Gamma(X, \mathcal{I}^1) \longrightarrow \cdots$$

and then compute the cohomology of this sequence.

We note that in order for these right derived functors to be defined, there need to be *enough injectives*. We say that an abelian category $\mathcal{A}$ has enough injectives if every object $A \in \mathrm{Ob}(\mathcal{A})$ admits an injective map $A \hookrightarrow I$ into an injective object I, where an injective object is defined to be an object I where given any map $X \to I$ and an injection $X \hookrightarrow I$, there exists a map $Y \to Q$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[rr, hook] \arrow[dr] & & Y \arrow[dl, dashed] \\ & Q & \end{tikzcd}$$

Alternatively, the pullback map $\mathrm{Hom}(Y, Q) \to \mathrm{Hom}(X, Q)$ is surjective. We will take the following results on faith:

Theorem 5.3. *The categories* Ab, $\mathsf{Ab}(X)$, $\mathsf{Mod}_{\mathcal{O}_X}$, *and* Mod_{C^∞} *have enough injectives.*

Like with the definition of the Čech cohomology groups as limits, the definition in terms of injective resolutions is practically useless computationally, since injective sheaves are hard to write down and difficult to find in the wild. The name of the game here is to find a nicer class of resolutions we can take.

Definition 5.4. Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact additive functor. An object $A \in \mathrm{Ob}(\mathcal{A})$ is ***acyclic*** for the functor F if $R^iF(A) = 0$ for all $i > 0$.

Proposition 5.5. *Let* $F : \mathcal{A} \to \mathcal{B}$ *be an additive functor, and* $A \in \mathrm{Ob}(\mathcal{A})$. *Then let* $A \to M^\bullet$ *be a resolution of* A *by* F*-acyclic objects. Then* $R^iF(A)$ *is isomorphic to the* i^{th} *cohomology of the complex* $F(M^\bullet)$.

Proof. Let $d^0 : M^0 \to M^1$. Then let $B = \operatorname{coker} d^0$ by exactness of

$$0 \longrightarrow A \longrightarrow M^0 \xrightarrow{d^0} M^1$$

we get a short exact sequence

$$0 \longrightarrow A \longrightarrow M^0 \longrightarrow B \longrightarrow 0$$

where the map $M^0 \to B$ is the composition of d^0 with the natural map $M^1 \to B$. We then take injective resolutions $A \to I^\bullet$, $M^0 \to J^\bullet$, and $B \to K^\bullet$. The maps $A \to M^0$ and $M^0 \to B$ induce a short exact sequence of chain maps between resolutions by using the defining property of injective objects, giving us

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & M^0 \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^0 \arrow[r] \arrow[d] & J^0 \arrow[r] \arrow[d] & K^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^1 \arrow[r] \arrow[d] & J^1 \arrow[r] \arrow[d] & K^1 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

which gives us a long exact sequence in cohomology. Noting that the cohomology of the respective sequences are just the right derived functors of A, M^0, and B, we get the long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & R^0F(A) \arrow[r] & R^0F(M^0) \arrow[r] & R^0F(B) \arrow[dll] \\
 & R^1F(A) \arrow[r] & R^1F(M^0) \arrow[r] & R^1F(B) \arrow[dll] \\
 & R^2F(A) \arrow[r] & R^2F(M^0) \arrow[r] & R^2F(B) \arrow[dll] \\
 & & \cdots & \cdots
\end{tikzcd}$$

Then using the fact that M^0 is F-acyclic, along with the fact that $R^0F = F$, we get that this long exact sequence is actually

$$\begin{tikzcd}
0 \arrow[r] & F(A) \arrow[r] & F(M^0) \arrow[r] & F(B) \arrow[dll] \\
& R^1F(A) \arrow[r] & 0 \arrow[r] & R^1F(B) \arrow[dll] \\
& R^2F(A) \arrow[r] & 0 \arrow[r] & R^2F(B) \arrow[dll] \\
& {} & \cdots & \cdots
\end{tikzcd}$$

which gives us isomorphisms $R^iF(B) \to R^{i+1}F(A)$ for $i > 0$, as well as an isomorphism $R^1F(A) \cong \operatorname{coker} F(M^0) \to F(B)$. To compute the cokernel of that map, we note that B admits the resolution

$$0 \longrightarrow B \longrightarrow M^1 \longrightarrow \cdots$$

where the map $B \to M^1$ is the map induced by d^0, using exactness of acyclic resolution. Then since F is left-exact, we get that

$$0 \longrightarrow F(B) \longrightarrow F(M^1) \longrightarrow \cdots$$

is exact, giving us that $F(B)$ is isomorphic to the kernel of $F(M^1) \to F(M^2)$. Therefore, we get that

$$R^1F(A) \cong \frac{\ker(F(M^1) \to F(M^2))}{\operatorname{Im}(F(M^1) \to F(M^2))} = H^1(F(M^\bullet))$$

Then to get the isomorphism for $R^2F(A)$, we note that since $R^1F(B) \cong R^2F(A)$, we can play the same game using the resolution of B to compute $R^1F(B)$ to be $H^2(F(M^\bullet))$, and then inductively repeat the process with the cokernel of $M^1 \to M^2$ to get $R^3F(A)$ and so on. ■

Something that is silly to observe, but useful.

Proposition 5.6. *Injective objects are F-acyclic.*

Proof. Let I be injective. Then take the injective resolution $0 \to I \to I \to 0$ where the map is the identity map. ■

Therefore, to compute right derived functors, it suffices to find acyclic resolutions. This gets us one step closer to finding nice resolutions for computing sheaf cohomology.

Definition 5.7. A sheaf $\mathcal{F}$ over X is ***flasque*** (also called ***flabby***) if the restriction maps are surjective.

Most sheaves in nature aren't flasque. However, flasque sheaves are useful for giving us acyclic resolutions. To do this, we'll need some lemmas.

Lemma 5.8. *Let*

$$0 \longrightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \longrightarrow 0$$

be a short exact sequence where $\mathcal{E}$ is flasque. Then for any open set U, the sequence

$$0 \longrightarrow \mathcal{E}(U) \xrightarrow{\alpha(U)} \mathcal{F}(U) \xrightarrow{\beta(U)} \mathcal{G}(U) \longrightarrow 0$$

is exact.

Proof. By left-exactness of taking sections, it suffices to show that $\beta(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective. Let $\sigma \in \mathcal{G}(U)$. Since β is a surjective sheaf morphism, for any $x \in U$, the induced map on stalks $\beta_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective, which implies that there exists a sufficiently small neighborhood $V_x \subset U$ of x where $\beta(V_x) : \mathcal{F}(V_x) \to \mathcal{G}(V_x)$ is surjective, so we can find $\tau_x \in \mathcal{F}(V_x)$ such that $\beta(V_x)(\tau) = \sigma|_{V_x}$. Then let $y \in U$ such that $V_x \cap V_y \neq \varnothing$, and let $\tau_y \in \mathcal{F}(V_y)$ such that $\beta(V_y)(\tau_y) = \sigma|_{V_y}$. Then we know that $\tau_x|_{V_x \cap V_y} - \tau_y|_{V_x \cap V_y} \in \ker \beta(V_x \cap V_y)$, so it is the image of an element $k \in \mathcal{E}(V_x \cap V_y)$. Since $\mathcal{E}$ is flasque, we know that we can lift k to an section $\chi_{x,y} \in \mathcal{E}(V_x)$. Then the element $\tau_x' = \tau_x - \alpha(V_x)(\chi_{x,y})$ still maps to $\sigma|_{V_x}$ under $\beta(V_x)$ by exactness, and restricts to τ_y on $V_x \cap V_y$. Therefore, τ_x' and τ_y glue to a section over $V_x \cup V_y$ that maps to $\sigma|_{V_x \cup V_y}$. We then find a maximal pair (W, τ) such that $\tau \in \mathcal{F}(W)$ and $\beta(W)(\tau) = \sigma|_W$. Then we must necessarily have $W = U$, since otherwise, we can find another open subset V of U and a section φ over V mapping to $\sigma|_V$, and extend τ to a section over $W \cup V$, since if $W \cap V \neq \varnothing$, we can use our above argument, and otherwise, no work needs to be done. Either way, finding such a U and φ contradicts maximality of (W, τ). ∎

Lemma 5.9. *Let*

$$0 \longrightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \longrightarrow 0$$

be a short exact sequence of sheaves where $\mathcal{E}$ and $\mathcal{F}$ are flasque. Then $\mathcal{G}$ is flasque.

Proof. Let $V \subset U$. Then given $\sigma \in \mathcal{G}(V)$, we want to show that there exits $\widetilde{\sigma} \in \mathcal{G}(U)$ that restricts to σ. Since $\mathcal{E}$ is flasque, we have that

$$0 \longrightarrow \mathcal{E}(V) \xrightarrow{\alpha(V)} \mathcal{F}(V) \xrightarrow{\beta(V)} \mathcal{G}(V) \longrightarrow 0$$

is exact, so we can find a section $\tau \in \mathcal{F}(V)$ with $\beta V(\tau) = \sigma$. Then since $\mathcal{F}$ is flasque, this lifts to an element $\widetilde{\tau} \in \mathcal{F}(U)$. Then taking $\widetilde{\sigma} = \beta(U)(\widetilde{\tau})$ gives us the desired section, since the properties of a sheaf morphism implies that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\beta(U)"] \arrow[d] & \mathcal{G}(U) \arrow[d] \\
\mathcal{F}(V) \arrow[r, "\beta(V)"'] & \mathcal{G}(V)
\end{tikzcd}$$

∎

Proposition 5.10. *Flasque sheaves are acyclic for the global sections functor $\Gamma(X, \cdot)$.*

Proof. Let $\mathcal{F}$ be a flasque sheaf. We first embed $\mathcal{F}$ into an injective flasque sheaf $\mathcal{I}$. Since Ab has enough injectives, we can embed each stalk $\mathcal{F}_x$ into an injective group I_x. Then define the sheaf $\mathcal{I}$ by

$$\mathcal{I}(U) = \prod_{x \in U} I_x$$

and the restriction maps are the projection maps $\prod_{x \in U} I_x \to \prod_{x \in V} I_x$. Since these maps are surjective, $\mathcal{I}$ is flasque. In addition, it is injective by construction. Then $\mathcal{F}$ embeds into $\mathcal{I}$ by composing the inclusions $\mathcal{F}(U) \hookrightarrow \prod_{x \in U} \mathcal{F}_x \hookrightarrow \prod_{x \in V} I_x$ Then let $\mathcal{G}$ be the cokernel of $\mathcal{F} \hookrightarrow \mathcal{I}$, giving us the exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{G} \longrightarrow 0$$

taking resolutions of $\mathcal{F}, \mathcal{I}$ and $\mathcal{G}$ gives us a long exact sequence in cohomology

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,\mathcal{F}) \arrow[r] & H^0(X,\mathcal{I}) \arrow[r] & H^0(X,\mathcal{G}) \arrow[dll] \\
& H^1(X,\mathcal{F}) \arrow[r] & H^1(X,\mathcal{I}) \arrow[r] & H^1(X,\mathcal{G}) \arrow[dll] \\
& H^2(X,\mathcal{F}) \arrow[r] & H^2(X,\mathcal{I}) \arrow[r] & H^2(X,\mathcal{G}) \arrow[dll] \\
& {} & \cdots & \cdots
\end{tikzcd}$$

The since H^0 is just $\Gamma(X, \cdot)$ and $\mathcal{I}$ is injective, this becomes

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}(X) \arrow[r] & \mathcal{I}(X) \arrow[r] & \mathcal{G}(X) \arrow[dll] \\
& H^1(X,\mathcal{F}) \arrow[r] & 0 \arrow[r] & H^1(X,\mathcal{G}) \arrow[dll] \\
& H^2(X,\mathcal{F}) \arrow[r] & 0 \arrow[r] & H^2(X,\mathcal{G}) \arrow[dll] \\
& {} & \cdots & \cdots
\end{tikzcd}$$

Since $\mathcal{F}$ is flasque, we have that

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{I}(X) \longrightarrow \mathcal{G}(X) \longrightarrow 0$$

is exact, so this implies that $H^1(X, \mathcal{F}) = 0$. In addition, for $i > 0$, we get isomorphisms $H^i(X, \mathcal{G}) \to H^{i+1}(X, \mathcal{F})$. To get that $H^2(X, \mathcal{F}) = 0$, we note that $\mathcal{G}$ is flasque, so we can repeat the argument to get that $H^1(X, \mathcal{G}) = H^2(X, \mathcal{F})$, and then continue for the other cohomology groups. ■

As a consequence, it suffices to find resolutions by flasque sheaves to compute sheaf cohomology. The good news here is that every sheaf admits a canonical acyclic resolution, the ***Godement resolution***. For a sheaf $\mathcal{F}$, let $\mathcal{F}_{\text{God}}$ be the sheaf $U \mapsto \prod_{x \in U} \mathcal{F}_x$, which is clearly flasque. Then we construct a flasque resolution for $\mathcal{F}$ as follows : embed $\mathcal{F} \hookrightarrow \mathcal{F}_{\text{God}}$, and let $\mathcal{G}^1$ be the cokernel. Then let the next sheaf in the sequence be $\mathcal{G}^1_{\text{God}}$, where the map $\mathcal{F}_{\text{God}} \to \mathcal{G}^1_{\text{God}}$ is the quotient map $\mathcal{F}_{\text{God}} \to \mathcal{G}^1$ composed in the the inclusion $\mathcal{G}^1 \hookrightarrow \mathcal{G}^1_{\text{God}}$. Then take $\mathcal{G}^2$ to be the cokernel of $\mathcal{F}_{\text{God}} \to \mathcal{G}^1_{\text{God}}$, and continue with $\mathcal{G}^2_{\text{God}}$ and so on. Pictorially, the Godement construction is constructed as follows:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}_{\text{God}} \arrow[r] \arrow[dr] & \mathcal{G}^1 \arrow[r] \arrow[d] & 0 & & \\
& & & \mathcal{G}^1_{\text{God}} \arrow[r] \arrow[dr] & \mathcal{G}^2 \arrow[r] \arrow[d] & 0 & \\
& & & & \mathcal{G}^2_{\text{God}} \arrow[r] \arrow[dr] & \mathcal{G}^3 \arrow[r] \arrow[d] & 0 \\
& & & & & \ddots &
\end{tikzcd}$$

Once more, this resolution is computationally useless, but it serves the purpose of showing that every sheaf admits a resolution by flasque sheaves. With this in hand, we can finally show that we can compute sheaf cohomology with a nice class of sheaves.

Definition 5.11. Let $\mathcal{A}$ be a sheaf of rings over X such that $\mathcal{A}$ admits ***partitions of unity***, i.e. for any open cover $\{U_i\}$ of X, there exist global sections $f_i \in \mathcal{A}(X)$ such that $\sum_i f_i = 1$ and f_i is supported in U_i, and where over any particular open set, all but finitely many of the f_i are 0. Then a sheaf of $\mathcal{A}$-modules is a ***fine sheaf***.

These are sheaves we care about, and appear in nature.

Example 5.12. Let M be a smooth manifold. Then the sheaf C^∞ of smooth functions admits partitions of unity. Therefore, Mod_{C^∞} consists of fine sheaves.

The punchline is that fine sheaves are acyclic, which gives us a source of reasonable sheaves with which to build resolutions.

Theorem 5.13. *Fine sheaves are acyclic with respect to* $\Gamma(X, \cdot)$.

Proof. Let $\mathcal{F}$ be a fine sheaf – a sheaf of modules over a sheaf of rings $\mathcal{A}$ that admits partitions of unity. Then by taking the Godement resolution of $\mathcal{F}$, we get an injective resolution of flasque $\mathcal{A}$-modules.

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \xrightarrow{d^0} \mathcal{I}^1 \xrightarrow{d^1} \cdots$$

Then since flasque sheaves are acyclic, we get that we can compute the sheaf cohomology of $\mathcal{F}$ as

$$H^i(X, \mathcal{F}) = \frac{\ker d^{i+1}(X) : \mathcal{I}^i(X) \to \mathcal{I}^{i+1}(X)}{\operatorname{Im} d^i(X) : \mathcal{I}^{i-1}(X) \to \mathcal{I}^i(X)}$$

Then let $\alpha \in \ker d^{i+1}(X)$, by exactness, we know that locally, in a sufficiently small open cover $\{U_j\}$, the map $d^i(U_j)$ is surjective, so we can find $\beta_j \in \mathcal{I}^{i-1}(U_j)$ such that $d^i(V_j)(\beta_j) = \alpha|_{U_j}$. Then $f_i\beta_i$ determines a global section that is equal to $f_j\beta_j$ on U_i and 0 elsewhere, and we get that $\sum_j f_j\beta_j$ maps to $\sum_j f_j\alpha|_{U_j} = \alpha$ under $d^i(X)$. Therefore, the sequence is exact on global sections for $i > 0$, so $\mathcal{F}$ is acyclic. ■

This tells us that many of the sheaves we know about, like sheaves of smooth sections of a vector bundle are trivial from the perspective of sheaf cohomology.

6. Comparison of Cohomology Theories

The fact that many of the sheaves we encounter naturally have trivial sheaf cohomology might come as a surprise, since we know we can extract topological data from these sheaves. The reason for this is that they provide good resolutions of other sheaves with nontrivial sheaf cohomology. U

Proposition 6.1 (Poincaré Lemma). *Every closed smooth k-form ω is locally exact, i.e. for a sufficiently small U, we have that $\omega|_U = d\eta$ for some $k-1$-form η.*

Proposition 6.2 ($\overline{\partial}$-Poincaré Lemma). *Ever closed smooth (p,q)-form ω is locally $\overline{\partial}$-exact*

Corollary 6.3. *The de Rham complex*

$$A^0(X) \xrightarrow{d} A^1(X) \xrightarrow{d} \cdots$$

is an exact sequence of sheaves.

The constant sheaf $\underline{\mathbb{R}}$ of locally constant real-valued functions naturally lives as a subsheaf of $A^0(X)$, and we know that this exactly the kernel of $d : A^0(X) \to A^1(X)$. This tells us that the inclusion $0 \to \underline{\mathbb{R}} \to A^\bullet(X)$ is a resolution of $\underline{\mathbb{R}}$, called the ***de Rham resolution***. Furthermore, since all the $A^i(X)$ are sheaves of C^∞-modules, they are fine, so the resolution is a resolution of $\underline{\mathbb{R}}$ by acyclic sheaves. Therefore, we get the isomorphisms

$$H^i(X, \underline{\mathbb{R}}) \cong H^i_{dR}(X)$$

A similar story holds for the sheaf cohomology of the sheaf of sections of a holomorphic vector bundle $E \to X$. The $\overline{\partial}$-Poincaré lemma implies that the Dolbeault complex

$$\mathcal{A}^0(E) \xrightarrow{\overline{\partial}_E} \mathcal{A}^1(E) \xrightarrow{\overline{\partial}_e} \cdots$$

of sheaves of smooth sections of $(\Lambda^i T^*X)_{\mathbb{C}} \otimes E$ is an exact sequence, since $\overline{\partial}_E$ is defined locally in terms of the operator $\overline{\partial}$ on X. Then since the kernel of $\overline{\partial}_E : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ is exactly the sheaf $\mathcal{E}$ of holomorphic sections of E, we get that $0 \to \mathcal{E} \to \mathcal{A}^\bullet(E)$ is an acyclic resolution of $\mathcal{E}$, which gives us isomorphisms

$$H^i(X, \mathcal{E}) \cong H^i_{\overline{\partial}}(X, E)$$

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