**THE YONEDA LEMMA**

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Let $\mathcal{C}$ be a category. Any object $X \in \mathcal{C}$ determines a covariant functor $h_X : \mathcal{C} \to \mathsf{Set}$ where given an object $Y \in \mathcal{C}$, we let $h_X(Y) = \mathrm{Map}_{\mathcal{C}}(X, Y)$. Given a morphism $f : Y \to Z$, we let $h_X(f) : \mathrm{Map}_{\mathcal{C}}(X, Y) \to \mathrm{Map}_{\mathcal{C}}(X, Z)$ be defined by $h_X(f)(\varphi) = f \circ \varphi$. This defines a contravariant functor $h : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}, \mathsf{Set})$ assigning each object $X$ the functor $h_X$, and to each morphism $f : X \to Y$ a natural transformation $h_Y \to h_X$ that maps a morphism $\varphi \in \mathrm{Map}_{\mathcal{C}}(Y, Z)$ to $\varphi \circ f \in \mathrm{Map}_{\mathcal{C}}(X, Z)$ for all objects $Z \in \mathcal{C}$.

**Theorem 0.1** (***The Yoneda Lemma***)**.** *The functor* $h : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}, \mathsf{Set})$ *is fully faithful, i.e. for any objects* $X, Y \in \mathcal{C}$*, the map* $\mathrm{Map}_{\mathcal{C}}(X, Y) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathsf{Set})}(h_Y, h_X)$ *is bijective.*

*Proof.* Let $X, Y \in \mathcal{C}$. Suppose two functions $f, g : X \to Y$ induce the same natural transformation $h_Y \to h_X$. The mappings $\mathrm{Map}_{\mathcal{C}}(Y, Y) \to \mathrm{Map}_{\mathcal{C}}(X, Y)$ given by precomposition with $f$ and $g$ agree, so in particular, we must have that $\mathrm{id}_Y \circ f = \mathrm{id}_Y \circ g$, so $f = g$. This shows that the map $\mathrm{Map}_{\mathcal{C}}(X, Y) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathsf{Set})}(h_Y, h_X)$ is injective. For surjectivity, we want to show that any natural transformation $\eta : h_Y \to h_X$ is given by precomposition by some morphism $X \to Y$. Let $\eta_Y : \mathrm{Map}_{\mathcal{C}}(Y, Y) \to \mathrm{Map}_{\mathcal{C}}(X, Y)$ be the map given by $\eta$. Then we claim that $\eta_Y$ is given by precomposition with the morphism $\eta_Y(\mathrm{id}_Y)$. Let $\varphi : X \to Y$ be a morphism. Then $\eta$ being a natural transformations gives us the commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(Y, Y) \arrow[d, "\eta_Y"'] & \mathrm{Map}_{\mathcal{C}}(Y, Y) \arrow[l, "\varphi\circ(-)"'] \arrow[d, "\eta_Y"] \\
\mathrm{Map}_{\mathcal{C}}(X, Y) & \mathrm{Map}_{\mathcal{C}}(X, Y) \arrow[l, "\varphi\circ(-)"]
\end{tikzcd}$$

Starting with $\mathrm{id}_Y$ in the top right we get

$$\eta_Y(\varphi \circ \mathrm{id}_Y) = \varphi \circ \eta_Y(\mathrm{id}_Y)$$

which is the desired result. Then given any $Z \in \mathcal{C}$, we want to show that $\eta_Z$ is also given by precomposition with $\eta_Y(\mathrm{id}_Y)$. Given a morphism $\varphi : Y \to Z$, we again get the diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}}(Y, Z) \arrow[d, "\eta_Z"'] & \mathrm{Map}_{\mathcal{C}}(Y, Y) \arrow[l, "\varphi\circ(-)"'] \arrow[d, "\eta_Y"] \\
\mathrm{Map}_{\mathcal{C}}(X, Z) & \mathrm{Map}_{\mathcal{C}}(X, Y) \arrow[l, "\varphi\circ(-)"]
\end{tikzcd}$$

Then the fact that this commutes tells us that

$$\eta_Z(\varphi \circ \mathrm{id}_Y) = \varphi \circ \eta_Y(\mathrm{id}_Y)$$

which is the desired result. Therefore, the map $\mathrm{Map}_{\mathcal{C}}(X, Y) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathsf{Set})}(h_Y, h_X)$ is surjective ■