YANG-MILLS ON RIEMANN SURFACES

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1. PRELIMINARY SETUP

To discuss the Yang-Mills functional, we must first fix some data. The setup will consist of the following ingredients

(1) A compact manifold M.
(2) A compact connected Lie group G.
(3) A principal G-bundle $P \to M$.

With this data, we have two associated bundles

$$\operatorname{Ad} P := P \times_G G$$
$$\mathfrak{g}_P := P \times_G \mathfrak{g}$$

Where the action of G on G is by conjugation, and the action of G on $\mathfrak{g}$ is the adjoint action. We note that these bundles both contain additional structure – $\operatorname{Ad} P$ is a bundle of groups (not a principal bundle), and $\mathfrak{g}_P$ is a bundle of Lie algebras. The space of sections $\Gamma(M, \operatorname{Ad} P)$ has a natural group structure given by pointwise multiplication, and is called the ***gauge group*** $\mathscr{G}(P)$. Likewise, the space of sections $\Gamma(M, \mathfrak{g}_P)$ has a natural Lie algebra structure given by the pointwise Lie bracket, and can be naturally identified with the Lie algebra of $\mathscr{G}(P)$, as we shall see. An alternate characterization of these spaces of sections comes from a general characterization of sections of associated bundles

Proposition 1.1. *We have natural correspondences*

$$\Gamma(M, \operatorname{Ad} P) \longleftrightarrow \left\{ f : P \to G \ : \ f(p \cdot g) = g^{-1} f(p) g \right\}$$
$$\Gamma(M, \mathfrak{g}_P) \longleftrightarrow \left\{ f : P \to \mathfrak{g} \ : \ f(p \cdot g) = \operatorname{Ad}_{g^{-1}} f(p) \right\}$$

From the above correspondence we a group isomorphism $\mathscr{G}(P) \to \operatorname{Aut}(P)$, where $\operatorname{Aut}(P)$ denotes the group of G-equivariant diffeomorphisms $\varphi : P \to P$ such that the following diagram commutes :

$$\begin{tikzcd} P \arrow[rr, "\varphi"] \arrow[dr] & & P \arrow[dl] \\ & M & \end{tikzcd}$$

The isomorphism is given by mapping a G-equivariant map $f : P \to g$ to the automorphism $\varphi_f : P \to P$ defined by $\varphi_f(p) = p \cdot f(p)$. In a similar fashion, we get a Lie algebra isomorphism of $\Gamma(M, \mathfrak{g}_P)$ with the vertical vector fields on P, which lets us explicitly realize $\Gamma(M, \mathfrak{g}_P)$ as the Lie algebra of the gauge group. Regarding a section $\phi \in \Gamma(M, \mathfrak{g}_P)$ as a G-equivariant map $P \to \mathfrak{g}$, we get a bundle automorphism $\exp(\phi)$ defined by $p \mapsto p \cdot \exp(\phi(p))$. From this perspective, we see that $\Gamma(M, \mathfrak{g}_P)$ can be identified with the G-invariant vertical vector fields on P, which are the infinitesimal generators of the action of $\mathscr{G}(P)$ on P.

In the the case of $\Gamma(M, \mathfrak{g}_P)$, we can extend the correspondence to the spaces of $\mathfrak{g}_P$ valued forms. The kernel of the differential of the projection $P \to M$ gives a subbundle of TP, which has a natural identification with the trivial bundle $\underline{\mathfrak{g}} = P \times \mathfrak{g}$. Then we can identify the space of sections of $\Lambda^k T^*M \otimes \mathfrak{g}_P$, (i.e. the space $\Omega^k_M(\mathfrak{g}_p)$ of $\mathfrak{g}_p$ valued k-forms with a subspace of the space $\Omega^k_P(\mathfrak{g})$ of $\mathfrak{g}$-valued k-forms ω on P satisfying :

(1) $R_g^* \omega = \mathrm{Ad}_{g^{-1}} \omega$, where $R_g : P \to P$ denotes the right action of $g \in G$.
(2) $\iota_\xi \omega = 0$ for any $\xi \in \mathfrak{g}$, where ι denotes interior multiplication, and we identify ξ with the constant vector field ξ under the identification of the vertical space with $\mathfrak{g}$.

We have a maps

$$\begin{aligned} \Omega^p_M(\mathfrak{g}_p) \otimes \Omega^q_M(\mathfrak{g}_P) &\to \Omega^{p+q}_M(\mathfrak{g}_p \otimes \mathfrak{g}_p) \\ (\omega_1 \otimes \xi_1) \otimes (\omega_2 \otimes \xi_2) &\mapsto (\omega_1 \wedge \omega_2) \otimes (\xi_1 \otimes \xi_2) \end{aligned}$$

From now on, we will usually omit the tensor symbol for $\mathfrak{g}_P$-valued forms in favor of juxtaposition, i.e. we write $\omega\xi$ instead of $\omega \otimes \xi$. Using the Lie bracket, we then get

$$\begin{aligned} \Omega^p_M(\mathfrak{g}_p) \otimes \Omega^q_M(\mathfrak{g}_P) &\to \Omega^{p+q}_M(\mathfrak{g}_p) \\ \omega \otimes \eta &\mapsto [\omega, \eta] \end{aligned}$$

We note that this is *not* skew-symmetric, instead, given $\omega \in \Omega^p_M(\mathfrak{g}_P)$ and $\eta \in \Omega^q_M(\mathfrak{g}_P)$, we have

$$[\omega, \eta] = (-1)^{pq+1}[\eta, \omega]$$

For any semisimple Lie group G (in particular, for any compact Lie group G), we have an inner product $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ that is invariant under the adjoint action (e.g. the Killing form). Invariance under the adjoint action gives us

$$\begin{aligned}\langle [\xi_1, \xi_2], \xi_3 \rangle &= \langle [-\xi_2, \xi_1], \xi_3 \rangle \\ &= \frac{d}{dt}\bigg|_{t=0} \langle \mathrm{Ad}_{\exp(-t\xi_2)} \xi_1, \xi_3 \\ &= \frac{d}{dt}\bigg|_{t=} \langle \mathrm{Ad}_{\exp(t\xi_2)} \mathrm{Ad}_{\exp(-t\xi_2)} \xi_1, \mathrm{Ad}_{\exp(t\xi_2)} \xi_3 \rangle \\ &= \langle \xi_1, [\xi_2, \xi_3] \rangle\end{aligned}$$

Fixing one such inner product induces a fiber product on the trivial bundle $P \times \mathfrak{g}$, and invariance guarantees that this descends to a fiber product on $\mathfrak{g}_P$. This give us pairings

$$\begin{aligned}\Omega^p_M(\mathfrak{g}_P) \otimes \Omega^q_M(\mathfrak{g}_P) &\to \Omega^{p+q}_M \\ \omega \otimes \eta &\mapsto \langle \omega, \eta \rangle\end{aligned}$$

which also satisfies the identity

$$\langle [\omega, \eta], \xi \rangle = \langle \omega, [\eta, \xi] \rangle$$

We again note that this is not symmetric or skew symmetric, and instead behaves like the wedge product, i.e. for $\omega \in \Omega^p_M(\mathfrak{g}_P)$ and $\eta \in \Omega^q_M(\mathfrak{g}_P)$, we have

$$\langle \omega, \eta \rangle = (-1)^{pq} \langle \eta, \omega \rangle$$

which can be seen by writing $\omega = \omega^i \xi_i$ and $\eta = \eta^i \xi_i$ for an orthonormal basis $\{\xi_i\}$ for $\mathfrak{g}$. We then fix an orientation and Riemannian metric on M, which gives us a Hodge star operator $\star : \Omega^p_M \to \Omega^{n-p}_M$ and a Riemannian volume form dV_g. The Hodge star extends to $\mathfrak{g}_P$-valued k-forms, where given $\omega \in \Omega^p_M$ and $\xi \in \Gamma(M, \mathfrak{g}_P)$, we define $\star(\omega\xi) = (\star\omega)\xi$. Then given $\omega_1\xi_1, \omega_2\xi_2 \in \Omega^p_M(\mathfrak{g}_P)$, we have

$$\langle \omega_1\xi_1, \star\omega_2\xi_2 \rangle = \langle \omega_1, \omega_2 \rangle_g \langle \xi_1, \xi_2 \rangle$$

where $\langle \cdot, \cdot \rangle_g$ denotes the fiber metric on $\Lambda^p T^*M$ induced by g. This gives us an inner product on each $\Omega^p_M(\mathfrak{g}_P)$ defined by

$$(\theta, \varphi) = \int_M \langle \theta, \star\varphi \rangle$$

Which gives us the L_2 norm on $\Omega^p_M(\mathfrak{g}_P)$ with $\|F\|^2_{L^2} = (F, F)$.

Definition 1.2. A connection on a principal bundle $\pi : P \to M$ is a choice of G-invariant splitting of the exact sequence of vector bundles over P

$$0 \longrightarrow \underline{\mathfrak{g}} \longrightarrow TP \longrightarrow \pi^*TM \longrightarrow 0$$

i.e. a distribution $H \subset TP$ such that

(1) $(R_g)_* H_p = H_{p \cdot g}$
(2) $H \oplus \underline{\mathfrak{g}} = TP$

Equivalently, it is the data of a $\mathfrak{g}$-valued 1-form $A \in \Omega^1_P(\mathfrak{g})$ satisfying

(1) $R^*_g A = \mathrm{Ad}_{g^{-1}} A$
(2) $\iota_\xi A = \xi$ for all $\xi \in \mathfrak{g}$.

Note in particular that by a dimension count, we have that $\pi_*|_H : H \to TM$ is an isomorphism. This implies that given a tangent vector v at x and a point $p \in P$ in the fiber over x, we get a unique horizontal lift $\tilde{v} \in H_p$. For a fixed principal G-bundle $\pi : P \to M$, we let $\mathscr{A}(P)$ denote the space of all connections on P, which is an affine space over $\Omega^1_M(\mathfrak{g}_P)$. The connection form A on P induces an exterior covariant derivative on any associated vector bundle $E = P \times_G V$ arising from a linear representation $\rho : G \to \mathrm{GL}(V)$. Let $\dot{\rho} : \mathfrak{g} \to \mathrm{End}(V)$ be the derivative of ρ at the identity. Then the exterior covariant derivative is given by

$$\begin{aligned} d_A : \Omega^p_M(E) &\to \Omega^{p+1}_M(E) \\ \psi &\mapsto d\psi + \dot{\rho}(A) \wedge \psi \end{aligned}$$

In particular, we get an exterior covariant derivative on $\mathfrak{g}_P$, which is given by

$$d_A\psi = d\psi + [A, \psi]$$

Proposition 1.3. *Let $\phi \in \Omega^p_M(\mathfrak{g}_P)$ and $\psi \in \Omega^q_M(\mathfrak{g}_P)$. Then*

$$d\langle \phi, \psi \rangle = \langle d_A\phi, \psi \rangle + (-1)^p \langle \phi, d_A\psi \rangle$$

Proof. We compute

$$\begin{aligned} \langle d_A\phi, \psi \rangle + (-1)^p \langle \phi, d_A\psi \rangle &= \langle d\phi, \psi \rangle + \langle [A, \phi], \psi \rangle + (-1)^p (\langle \phi, d\psi \rangle + \langle \phi, [A, \psi] \rangle) \\ &= \langle d\phi, \psi \rangle + \langle [A, \phi], \psi \rangle + (-1)^p (\langle \phi, d\psi \rangle + \langle [\phi, A], \psi \rangle) \\ &= \langle d\phi, \psi \rangle + \langle [A, \phi], \psi \rangle + (-1)^{2p+1} \langle [A, \phi], \psi + (-1)^p \langle \phi, d\psi \rangle \\ &= \langle d\phi, \psi \rangle + (-1)^p \langle \phi, d\psi \rangle \end{aligned}$$

Writing $\phi = \phi^i \xi_i$ and $\psi = \psi^i \xi_i$ in an orthonormal basis $\{\xi_i\}$ for $\mathfrak{g}$, this becomes

$$\begin{aligned} \langle d_A\phi, \psi \rangle + (-1)^p \langle \phi, d_A\psi \rangle &= \sum_i d\phi^i \wedge \psi^i + (-1)^p \phi^i \wedge d\psi^i \\ &= d\langle \phi, \psi \rangle \end{aligned}$$

■

Given any distribution $E \subset TP$, we get a Frobenius tensor $\phi_E : E \otimes E \to TP/E$ given by $X \otimes Y \to [X, Y] \mod E$ where we extend X and Y to local vector fields. The Frobenius tensor should be thought of as the obstruction to the existence of an integral submanifold for the distribution E. In the case of a connection H on a principal bundle $P \to M$, we can extend to all of TP by first projecting onto H, and have an identification of $TP/H \cong \underline{\mathfrak{g}}$, and the Frobenius tensor is given by $X \otimes Y \mapsto A([X, Y])$, where A is the connection 1-form, and is called the ***curvature form*** of the connection, and is denoted F_A. In terms of differential forms, we have that for horizontal vectors ξ_1, ξ_2 on TP,

$$dA(\xi_1, \xi_2) = \xi_1 A(\xi_2) - \xi_2 A(\xi_1) - A([\xi_1, \xi_2])$$

The fact that ξ_1 and ξ_2 are horizontal implies that they are in the kernel of A, which gives us $dA(\xi_1, \xi_2) = -F_A(\xi_1, \xi_2)$. We also know that F_A vanishes on vertical vectors, and since

$A(X) = X$ for $X \in \mathfrak{g}$, we get that

$$dA + \frac{1}{2}[A, A] = -F_A$$ [1]

It can be shown that F_A transforms by the adjoint action under pullback, and vanishes on vertical vectors, so it descends to a $\mathfrak{g}_P$-valued 2-form on the base manifold M.

Another thing to note is that there is a natural action of the gauge group $\mathscr{G}(P)$ on the space of connections $\mathscr{A}(P)$. Interpreting the elements of $\mathscr{G}(P)$ as bundle automorphisms $\varphi : P \to P$ and elements of $\mathscr{A}(P)$ as $\mathfrak{g}$-valued 1-forms A on P, the action is simply pullback, $(\varphi, A) \mapsto \varphi^* A$. To show that this defines an action, we must check that $\varphi^* A$ satisfies the conditions

(1) $R_g^* \varphi^* A = \mathrm{Ad}_{g^{-1}} \varphi^* A$
(2) $\iota_\xi \varphi^* A = \xi$ for all $\xi \in \mathfrak{g}$.

Which are all simple consequences of the G-equivariance of φ and the transformation law for A. For a specific formula, let $\varphi : P \to P$ be an element of the gauge group, and let $g_\varphi : P \to G$ be its associated G-equivariant map. Then

$$\varphi^* A = \mathrm{Ad}_{g_\varphi^{-1}} A + g_\varphi^* \theta$$

where $\theta \in \Omega^1_G(\mathfrak{g})$ denotes the ***Maurer-Cartan form***

$$\theta_g(v) = (dL_{g^{-1}})_g(v)$$

which satisfies the ***Maurer-Cartan equation***

$$d\theta + \frac{1}{2}[\theta, \theta] = 0$$

Proposition 1.4. *Let $A \in \mathscr{A}(P)$ be a connection and $\varphi : P \to P$ an element of $\mathscr{G}(P)$ with associated G-equivariant map $g_\varphi : P \to G$. Then*

$$F_{\varphi^* A} = \mathrm{Ad}_{g_\varphi^{-1}} F_A$$

Proof. Using the transformation law for $\varphi^* A$ we compute

$$\begin{aligned}
F_{\varphi^* A} &= d(\mathrm{Ad}_{g_\varphi^{-1}} A + g_\varphi^* \theta) + \frac{1}{2}[\mathrm{Ad}_{g_\varphi^{-1}} A + g_\varphi^* \theta, \mathrm{Ad}_{g_\varphi^{-1}} A + g_\varphi^* \theta] \\
&= \mathrm{Ad}_{g_\varphi^{-1}} dA + g_\varphi^* d\theta + \frac{1}{2}\Big([\mathrm{Ad}_{g_\varphi^{-1}} A, \mathrm{Ad}_{g_\varphi^{-1}} A] + [\mathrm{Ad}_{g_\varphi^{-1}} A, g_\varphi^* \theta] + [g_\varphi^* \theta, \mathrm{Ad}_{g_\varphi^{-1}}] + [g_\varphi^* \theta, g_\varphi^* \theta]\Big) \\
&= \mathrm{Ad}_{g_\varphi^{-1}} dA + \frac{1}{2}[\mathrm{Ad}_{g_\varphi^{-1}} A, \mathrm{Ad}_{g_\varphi^{-1}} A]]
\end{aligned}$$

Where we use skew-symmetry and the Maurer-Cartan equation. ■

We similarly compute the infinitesimal action of the Lie algebra $\Gamma(M, \mathfrak{g}_P)$.

Proposition 1.5. *The vector field corresponding to $\phi \in \Gamma(M, \mathfrak{g}_P)$ is $A \mapsto d_A \phi \in \Omega^1_M(\mathfrak{g}_P)$*

[1] Our convention for the sign of the curvature is opposite from many other conventions, which usually sets $F_A = dA + \frac{1}{2}[A, A]$

Proof. We compute the vector field at a connection $A \in \mathscr{A}(P)$ to be

$$\begin{aligned}
&\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp(t\phi)^{-1}} A + \exp(t\phi)^*\theta = -[\phi, A] + \left.\frac{d}{dt}\right|_{t=0} (dL_{\exp(-t\phi)} d(\exp(t\phi))) \\
&= [A, \phi] + \left(\left.\frac{d}{dt}\right|_{t=0} dL_{\exp(-t\phi)}\right) d(\exp(0)) + dL_{\exp(0)} \left(\left.\frac{d}{dt}\right|_{t=0} d(\exp(t\phi))\right) \\
&= [A, \phi] + d\phi \\
&= d_A\phi
\end{aligned}$$

where for the third equality we use the product rule, and in the fourth equality we use the fact that $\exp(0) = \mathrm{id}$ and that the derivative of $\exp(t\phi)$ as $t \to 0$ is ϕ. ■

For any other connection $A + \eta$ with $\eta \in \Omega^1_M(\mathfrak{g}_P)$, a quick computation yields

$$F_{A+\eta} = F_A + \frac{1}{2}[\eta, \eta] + d_A\eta$$

From this description, we can relate the curvature F_A with the covariant derivative. Note that for the line of connections $A + t\eta$, we have that

$$\left.\frac{d}{dt}\right|_{t=0} F_{a+t\eta} = \left.\frac{d}{dt}\right|_{t=0} F_A + \frac{t^2}{2}[\eta, \eta] + td_A\eta = d_A\eta$$

So $d_A\eta$ measures the infinitesimal change of the curvature F_A in the direction η.

2. The Yang-Mills Functional

With the setup done, we have the ingredients necessary to define the Yang-Mills functional.

Definition 2.1. The ***Yang-Mills functional*** is the map $L : \mathscr{A}(P) \to \mathbb{R}$ given by

$$L(A) = \|F_A\|^2_{L^2} = \int_M \langle F_A, \star F_A\rangle$$

We immediately see that the Yang-Mills equations are invariant under $\mathscr{G}(P)$ in the following sense – if we have any gauge transformation φ with associated map $g_\varphi : P \to G$, we have that $L(\varphi^* A) = L(A)$, which follows immediately from the invariance of $\langle \cdot, \cdot \rangle$ and the transformation law for curvature.

Our goal now will be to find the Euler-Lagrange equations for the Yang-Mills functional by computing the first and second variations. Using the Hodge star operator, we construct the formal adjoint with respect to the inner product $d_A^* : \Omega^p_M(\mathfrak{g}_P) \to \Omega^{p-1}_M(\mathfrak{g}_P)$ in the same manner as for classical Hodge theory on a Riemannian manifold. Explicitly, the formula on p-forms is given by

$$d_A^* = (-1)^{n(p+1)+1} \star d_A \star$$

where $n = \dim M$. We then compute the first variation of L.

Proposition 2.2 (***The First Variation***)**.** *For a local extremum $A \in \mathscr{A}(P)$ of the Yang-Mills functional, we have*

$$d_A \star F_A = 0$$

*The local extremum connection A is then called a **Yang-Mills connection**, and the space of Yang-Mills connections is denoted $\mathscr{A}_{\mathrm{YM}}(P)$.*

Proof. Consider a variation $A + t\eta$ with $t \in \mathbb{R}$ and $\eta \in \Omega^1_M(\mathfrak{g}_P)$. We have that the curvature is given by

$$F_{A+t\eta} = F_A + \frac{t^2}{2}[\eta, \eta] + t d_A \eta$$

This then gives us

$$\begin{aligned} \|F_{A+t\eta}\|_{L^2} &= \int_M \langle F_{A+t\eta}, F_{A+t\eta} \rangle \\ &= \int_M \langle F_A + \frac{t^2}{2}[\eta, \eta] + t d_A \eta, \star(F_A + \frac{t^2}{2}[\eta, \eta] + t d_A \eta) \rangle \end{aligned}$$

Expanding this out, we get that the term that is linear in t is

$$\int_M \langle F_A, \star d_A \eta \rangle + \langle d_A \eta, \star F_A \rangle = 2(F_A, d_A \eta)$$

where we use symmetry of $(\cdot, \cdot)$. Since A is extremal, we have that this term must vanish, giving us that $(F_A, d_A \eta) = (d_A^* F, \eta) = 0$ for every η. Then since we have (up to sign) $d_A^* = \star d_A \star$, and $\star$ is an isomorphism, this implies $d_A \star F_A = 0$. ■

Proposition 2.3 (***The Second Variation***)**.** *At a Yang-Mills connection $A \in \mathscr{A}(P)$, we have*

$$d_A^* d_A \eta + \star[\eta, \star F_A] = 0$$

Proof. We differentiate the first variational equation with respect to t, i.e. we compute

$$\left.\frac{d}{dt}\right|_{t=0} d^*_{A+t\eta} F_{A+t\eta}$$

We expand out

$$\begin{aligned} d^*_{A+t\eta} F_{A+t\eta} &= \pm \star d_{A+t\eta} \star F_{A+t\eta} \\ &= \pm \left(\star d_A \star \left(F_A + t d_A \eta + \frac{t^2}{2}[\eta, \eta] \right) + t \star \left[\eta, \star \left(F_A + t d_A \eta + \frac{t^2}{2}[\eta, \eta] \right) \right] \right) \end{aligned}$$

Taking the term linear in t yields

$$\pm \left(\star d_A \star d_A \eta + \star[\eta, \star F_A] \right)$$

Giving us that at an extremal connection A, we have

$$d_A^* d_A \eta + \star[\eta, \star F_A] = 0$$

■

The second variation should be thought of as the Hessian to the Yang-Mills functional, which will allow us to apply Morse theory techniques to the space of connections.

3. The $U(1)$ Case

We first restrict to the special case $G = U(1)$. In this case, the Lie algebra is abelian, so the adjoint action of $U(1)$ on $\mathfrak{u}(1)$ is trivial, giving us that $\mathfrak{g}_P$ is a trivial bundle. Identifying $\mathfrak{u}(1)$ with $\mathbb{R}$, we can then identify $\mathfrak{u}(1)$ valued forms on P with ordinary differential forms. Likewise, using triviality of $\mathfrak{g}_P$, we can identify $\mathfrak{g}_P$-valued forms with ordinary differential forms on M. The vertical bundle in this case is a trivial line bundle over P, and there is a unique $U(1)$-invariant vertical vector field on P, which on each fiber restricts to the vector field dual to the Maurer-Cartan formθ. Then given a connection A on P, we have that $dA = \pi^* F_A$, since $[A, A] = 0$. This immediately tells us that F_A is closed, since d commutes with pullback. Furthermore, for any other connection $A + \eta$, we have that

$$F_{A+\eta} = F_A + \frac{1}{2}[\eta, \eta] + d_A \eta$$

Then since $d_A = d$ and $[\eta, \eta] = 0$, this gives us that $F_{A+\eta} = F_A + d\eta$, which tells us that the cohomology class of F_A is independent of our choice of A. Using our sign convention, this is equal to $-2\pi i c_1(P)$. In addition, it tells us that every representative of the curvature class can be realized as the curvature of some connection. Furthermore, in this situation, the Yang-Mills functional reduces to the standard Hodge theory picture, since the L^2 norm will coincide with the L^2 norm on differential forms. Therefore, a Yang-Mills connection on P is equivalent to finding the unique connection that minimizes the L^2 norm in the cohomology class $2\pi i c_1(P)$. By standard Hodge theory, there exists a unique harmonic representative of the curvature class, and the Yang-Mills connections for P are a torsor over the space Z^1_P of closed 1-forms. In total, this gives us the fibration

$$\begin{tikzcd} Z^1_M \arrow[r] & \mathscr{A}_{\mathrm{YM}}(P) \arrow[d] \\ & 2\pi i c_1(P) \end{tikzcd}$$

In the flat case $c_1(P) = 0$, the Yang-Mills connections are just flat connections, which are parameterized by conjugacy classes of homomorphisms $\pi_1(M) \to U(1)$.

With this information, we can construct the Yang-Mills moduli space $\mathscr{A}_{\mathrm{YM}}(P)/\mathscr{G}(P)$ in this simple case. Since the conjugation action is trivial, the bundle $\mathrm{Ad}(P)$ is a trivial bundle, so the gauge group is just $\mathscr{G}(P) = \mathrm{Map}(M, U(1))$. Given $f : M \to U(1)$, the action of f on a connection A is given by

$$A \mapsto A + \pi^* f^* \theta$$

The gauge group acts on Z^1_M in the same way, so if we fix some reference Yang-Mills connection A_0 to identify $\mathscr{A}_{\mathrm{YM}}(P)$ with Z^1_M, the group actions are identified. Therefore, it suffices to compute the quotient of $Z^1_M/\mathscr{G}(P)$. To compute this quotient, we do it in two steps, first quotienting by the identity component $\mathscr{G}_0(P)$, and then quotienting by the component group $\pi_0\mathscr{G}(P)$. The components of $\mathscr{G}(P)$ are simply the homotopy classes of maps $M \to S^1$, so $\mathscr{G}_0(P)$ consists of all nullhomotopic maps $M \to S^1$. Let dx denote the standard form on $\mathbb{R}$. Then any nullhomotopic map $f \in \mathscr{G}_0(P)$ lifts to a function $\tilde{f} : M \to \mathbb{R}$ that exponentiates to f. Since the Maurer-Cartan form on S^1 pulls back to dx, we find that

$f^*\theta = d\tilde{f}$. Therefore, the action of $\mathscr{G}_0(P)$ on Z^1_M is given by the addition of exact 1-forms, giving us that the quotient $Z^1_M/\mathscr{G}_0(P)$ is $H^1(M,\mathbb{R})$. Then since S^1 is a $K(\mathbb{Z},1)$, we know that homotopy classes of maps $M \to S^1$ are in bijection with $H^1(M,\mathbb{Z})$. Therefore, by quotieting by $\mathscr{G}_0(P)$ and putting everything together, we get isomorphism

$$\mathscr{A}_{\mathrm{YM}}(P)/\mathscr{G}(P) \cong Z^1_M/\mathscr{G}(P) \cong \mathrm{Jac}(M) := H^1(X,\mathbb{R})/H^1(X,\mathbb{Z})$$

which is isomorphic to a torus $\mathbb{T}^{b_1(M)}$. However, we note that in general, this isomorphism is highly non-canonical – to make the identification of $\mathscr{A}_{\mathrm{YM}}(P)$ with Z^1_M, we need to fix a reference connection. In general, there is no canonical choice of reference except in the case where P is a trivial bundle, in which case the trivial connection defines a canonical reference connection. As we'll see soon, this reflects the fact that $\mathscr{A}_{YM}(P)$ is a $\mathrm{Jac}(M)$-torsor when P is a $U(1)$-bundle.

4. Yang-Mills Over a Riemann Surface

We now restrict our attention to when M is a orientable surface, with genus $g > 0$. Let $Q \to M$ be a principal $U(1)$ bundle with $c_1(Q) = 1$, i.e.

$$\frac{1}{2\pi i}\int_M c_1(Q) = 1$$

Then fix a Riemannian metric on M with volume form ω such that $\int_M \omega = 1$, and Yang-Mills connection A on Q. Since $c_1(Q) = 1$, we have that $[c_1(Q)] = [\omega]$. Furthermore, since $\star\omega = 1$ and $-2\pi i c_1(Q) = [F_A]$, we get that the curvature of A must be equal to $-2\pi i\omega$ to minimize the Yang-Mills functional. Similarly, for any other $U(1)$-bundle P, the curvature of a Yang-Mills connection must be $-2\pi i c_1(P)\omega$. Then let $\widetilde{M} \to M$ be the universal cover of M. Since the genus of M is at least 1, $\widetilde{M}$ is contractible, so the pullback of Q along the covering projection gives us a trivial $U(1)$ bundle over $\widetilde{M}$

$$\begin{array}{ccc} \widetilde{M} \times U(1) & \longrightarrow & Q \\ \downarrow & & \downarrow \\ \widetilde{M} & \longrightarrow & M \end{array}$$

Then we have a covering map $\widetilde{M} \times \mathbb{R} \to U(1)$, using the usual covering $\mathbb{R} \to U(1)$, giving a principal $\mathbb{R}$-bundle over $\widetilde{M}$. Then if we consider the composite map $\widetilde{M} \times \mathbb{R} \to \widetilde{M} \to M$, this is a fiber bundle over M. Furthermore, since the action of $\mathbb{R}$ on on $\widetilde{M} \times \mathbb{R}$ commutes with the $\pi_1(M)$ action on $\widetilde{M}$, we know that this is a principal bundle with structure group $\Gamma_{\mathbb{R}}$, where $\Gamma_{\mathbb{R}}$ is a central extension of $\pi_1(M)$ by $\mathbb{R}$. Let J denote the element of $\mathbb{R} \subset \Gamma_{\mathbb{R}}$ corresponding to $1 \in \mathbb{R}$. Then consider M as the quotient of the $2g$-gon. The holonomy about the path traversed by the boundary is exactly the product $\prod_i [a_i, b_i]$ of the commutators of representatives of generators of $\pi_1(M)$, and has holonomy equal to 2π, which follows from the fact that the holonomy about any loop bounding a disk is equal to the integral of the curvature, and the fact that $c_1(Q)$ is represented by the curvature class of any connection, divided by $2\pi i$. Therefore, if we consider the pullback connection on $\widetilde{M} \times U(1)$, the holonomy about the lifts of the boundary path to $\widetilde{M}$ will also be 2π, which then lifts to translation by 1 in the bundle $\widetilde{M} \times \mathbb{R}$. This gives the relation that $\prod_i [a_i, b_i] = J$, which gives us a presentation of the group $\Gamma_{\mathbb{R}}$. Since $\pi_1(M)$ is discrete, $\widetilde{M}$ is a flat bundle, so the pullback connection on $\widetilde{M} \times U(1)$ still has curvature $-2\pi i\omega$, and the curvature also remains unchanged after lifting to $\widetilde{M} \times \mathbb{R}$.

Suppose we have a homomorphism $\rho : \Gamma_{\mathbb{R}} \to G$ to a compact group G. This then gives us an associated bundle $P = (\widetilde{M} \times \mathbb{R}) \times_{\Gamma_{\mathbb{R}}} G$, which is a principal G bundle In addition to ρ, we get a Lie algebra homomorphism $\dot{\rho} : \mathbb{R} \to \mathfrak{g}$. Using this, $\dot{\rho}(A) \in \Omega^1_P(\mathfrak{g})$ determines a connection on P, which has curvature $\dot{\rho}(F_A)$. Furthermore, we have that

$$\dot{\rho}(d_A \star F_A) = d_{\dot{\rho}(A)} \star \dot{\rho}(F_A)$$

which tells us that $\dot{\rho}(A)$ is a Yang-Mills connection on P. The main theorem is that every Yang-Mills connection on every principal bundle arises in this way.

Theorem 4.1. *The above construction gives a bijective correspondence*

$$\mathsf{Hom}(\Gamma_{\mathbb{R}}, G)/G \longleftrightarrow \{G\text{-bundles } P \to M \text{ equipped with a Yang-Mills connection } A\}/\mathscr{G}(P)$$

Where the action of G on $\mathsf{Hom}(\Gamma_{\mathbb{R}}, G)$ is by conjugation.

Proof. There are several things here to show. First, we must show that for any compact group G, any Yang-Mills connection on an associated bundle comes from such a homomorphism. Furthermore, we also must show that for any compact group G, every principal G-bundle can be realized as an associated bundle of $\widetilde{M} \times \mathbb{R} \to M$.

We first tackle the first claim. Since M is 2-dimensional, we have that $\star F_A \in \Omega^0_M(\mathfrak{g}_P)$, so we may regard it as a G-equivariant map $P \to \mathfrak{g}$, i.e. $\star F_A(p \cdot g) = \mathrm{Ad}_{g^{-1}} \star F_A(p)$. This then tells us that the image of $\star F_A$ is exactly one orbit of $\mathfrak{g}$ under the adjoint action. Fix a nonzero element $X \in \mathfrak{g}$ lying in the image of $\star F$, and then consider the preimage $P_X := (\star F_A)^{-1}(X)$. Let $G_X \subset G$ be the stabilizer of X under the adjoint action. Then G_X acts on P_X, since given any $p \in P_X$, and $g \in G_X$, we have that $\star F_A(p \cdot g) = \mathrm{Ad}_{g^{-1}} F_A(p) = X$. This action is clearly transitive and free, so P_X defines a reduction of structure group from G to G_X, giving us a bundle isomorphism $P_X \times_{G_X} G \cong P$. Furthermore, since $d_A \star F = 0$, we have that $\star F$ is constant in the horizontal directions, so the differential of $\star F$ vanishes in the horizontal directions, so the horizontal distribution is contained in the tangent bundle of P_X. Therefore, the connection A on P restricts to a Yang-Mills connection on P_X (which we also denote A). This restricted connection has the property that $\star F_A$ is the constant map with value $X \in \mathfrak{g}$, so F_A is just the 2-form $X \otimes \omega \in \Omega^2_M(\mathfrak{g}_{P_X})$. This tells us that every Yang-Mills connection on any bundle P arises from such a connection on the reduced bundle P_X for some $X \in \mathfrak{g}$.

Then suppose we have a homomorphism $\rho : \Gamma_{\mathbb{R}} \to G$ with derivative $\dot{\rho} : \mathbb{R} \to \mathfrak{g}$. The image of $1 \in \mathbb{R}$ under $\dot{\rho}$ determines an element $X_\rho \in \mathfrak{g}$. Centrality of $\mathbb{R}$ then implies that the image of $\Gamma_{\mathbb{R}}$ preserves X_ρ under the adjoint action, so we may regard ρ as a homomorphism $\Gamma_{\mathbb{R}} \to G_{X_\rho}$. Combining this observation with the previous one, we can then reduce to the case where X is preserved by all of G (i.e. $G = G_X$, which is equivalent to X lying in the center of $\mathfrak{g}$).

To complete the proof of the first claim, we want to reduce to the cases where G is either a torus or a semisimple group. This follows from the fact that any compact group G arises as $H \times_D S$, where H is a maximal torus and $S = [G, G]$ is the maximal connected semisimple subgroup, and $D = H \cap S$ is a finite subgroup of the center of S. Quotienting by D, we get a finite covering $G \to \overline{G} = \overline{H} \times \overline{S}$, where $\overline{H} = H/D$ and $\overline{S} = S/D$. We then claim that we can reduce to the case where the structure group is $\overline{G}$. To see this, we note that if we quotient P by the action of D to obtain $\overline{P}$, we get a finite sheeted covering $P \to \overline{P}$. Since this covering is a local diffeomorphism, we get an identification $TP \cong \pi^* T\overline{P}$ where $\pi : P \to \overline{P}$ is the covering projection. Therefore, we get that the horizontal distribution on P induces a horizontal distribution on $\overline{P}$, and conversely, we get that a connection on $\overline{P}$ lifts to a horizontal distribution on P, so we may reduce to the case $P = \overline{P}$. In addition, since any map $\Gamma_{\mathbb{R}} \to G$ gives a map $\overline{\rho} : \Gamma_{\mathbb{R}} \to \overline{G}$ by composition with the quotient map. Then since we assume that $\mathring{\rho}(1) = X$ is central, we have that the image of $\mathbb{R} \subset \Gamma_{\mathbb{R}}$ is a 1-parameter subgroup of G lying in the center. Furthermore, we have that $\mathbb{Z}$ is contained in the commutator subgroup $[\Gamma_{\mathbb{R}}, \Gamma_{\mathbb{R}}]$, so its image under ρ is central in G and lies in the maximal semisimple $S = [G, G]$, so $\rho(\mathbb{Z}) \subset D$. Therefore, the map $\overline{\rho}$ descends to a map $\Gamma_{\mathbb{R}}/\mathbb{Z} \to \overline{G}$. Then since $\Gamma_{\mathbb{R}}/\mathbb{Z} \cong U(1) \times \pi_1(M)$. Then since $\overline{G}$ is the product group $\overline{H} \times \overline{S}$, we have that $\overline{\rho}$ is given by a pair of homomorphisms $\alpha : U(1) \times \pi_1(M) \to \overline{H}$ and $\beta : U(1) \times \pi_1(M) \to \overline{S}$. Furthermore, since $\overline{S}$ is semisimple, it has a finite center, so centrality of X implies that β is trivial on the $U(1)$ factor, so it is actually a map $\beta : \pi_1(M) \to \overline{S}$, which is exactly the data of a principal $\overline{S}$-bundle with flat connection, which in particular, is a $\overline{S}$-bundle with Yang-Mills connection.

We are left to consider the homomorphism α, which amounts to understanding Yang-Mills connections when structure group is a torus $U(1) \times \cdots \times U(1)$. By passing to associate bundles, this is equivalent to passing to a direct sum of Hermitian line bundles, each equipped with a Yang-Mills connection.

For the second part, we use the fact that for a group G, we have that principal G-bundles over the surface M are classified by $H^2(M, \pi_1(G)) = \pi_1(G)$. In our case, $\overline{G} = \overline{H} \times \overline{S}$, so we have

$$\pi_1(G) = \pi_1(\overline{H}) \oplus \pi_1(\overline{S})$$

By restricting to a copy of $U(1)$, the map α determines a class in $\pi_1(\overline{H}) \cong \mathbb{Z}^n$, which can be thought of as choosing the first Chern class for each direct summand of a vector bundle. For β, we need to do some additional work. The homomorphism $\beta : \pi_1(M) \to \overline{S}$ defines a group action of $\pi_1(M)$ on $\overline{S}$, which can be lifted to an action of $\Gamma_{\mathbb{Z}}$ on the universal cover of $\overline{S}$. The image of the central element $J \in \Gamma_{\mathbb{Z}}$ in the universal cover is an element of the center (since we assumed that $G = G_X$), which is equivalent to an element of $\pi_1(\overline{S})$, since $\overline{S}$ is the quotient of the universal cover by a subgroup of the center. ■

5. The Symplectic Viewpoint

Even though the space $\mathscr{A}(P)$ is infinite dimensional, it has enough structure to be viewed as a symplectic "manifold," but we will gloss over the formal details. Because $\mathcal{A}(P)$ is an affine space over $\Omega^1_M(\mathfrak{g}_P)$, we may work with it as a manifold, where the tangent space at any point is $\Omega^1_M(\mathfrak{g}_P)$. We will be relatively cavalier with the details, though all we are doing can be made formal by passing to Sobolev completions of spaces of sections.

In the case that M is a Riemann surface, then the Hodge star $\star : \Omega^1_M(\mathfrak{g}_P) \to \Omega^1_M(\mathfrak{g}_P)$ can be viewed as a complex structure on $\mathscr{A}(P)$. In addition, after fixing a Riemannian metric on M, we get a trivialization $\Omega^2_M \cong \mathbb{R}$ using the natural orientation induced by the complex structure. This allows us to view the pairing

$$(\omega, \eta) \mapsto \int_M \langle \omega, \eta \rangle$$

as a symplectic form on $\mathscr{A}(P)$. In addition, these structures are visibly compatible, which gives $\mathscr{A}(P)$ a Kähler structure.

Recall that if we have a symplectic left action of a group G on a symplectic manifold (M, ω) we get an induced map $\mathfrak{g} \to \mathfrak{X}(M)$ mapping ξ to the vector field X_ξ defined by

$$(X_\xi)_p = \frac{d}{dt}\bigg|_{t=0} \exp(t\xi) \cdot p$$

The action is ***Hamiltonian*** if for all $\xi \in \mathfrak{g}$ there exists a function $H_\xi : M \to \mathbb{R}$ called a ***Hamiltonian function*** such that the vector field X_ξ satisfies the identity

$$\omega_p((X_\xi)_p, v) = (dH_\xi)_p v$$

for all points $p \in M$ and tangent vectors $v \in T_pM$, and the mapping $\xi \mapsto H_\xi$ is G-equivariant with respect to the right actions $\xi \cdot g = \mathrm{Ad}_{g^{-1}} \xi$ and $f \cdot g = f \circ L_g$, where $L_g : M \to M$ is the symplectomorphism determined by left multiplication by g. Given a Hamiltonian action of G on M, a ***moment map*** for the action is a map $\mu : M \to \mathfrak{g}^*$ such that for any $p \in M$ and $\xi \in \mathfrak{g}$, we have

$$H_\xi(p) = \mu(p)((X_\xi)_p)$$

We claim that the action of $\mathscr{G}(P)$ on $\mathscr{A}(P)$ is Hamiltonian. We first note that the action is symplectic, since $\langle \cdot, \cdot \rangle$ is Ad-invariant, and the action of a gauge transformation φ on a tangent vector $\eta \in \Omega^1_M(\mathfrak{g}_P)$ is by $\varphi \cdot \eta = \mathrm{Ad}_{g_\varphi^{-1}} \eta$ where $g_\varphi : P \to G$ is the associated G-equivariant map. To show that the action is Hamiltonian, we note that each $\phi \in \Omega^0_M(\mathfrak{g}_P)$ determines a map $H_\phi : \mathscr{A}(P) \to \mathbb{R}$ given by

$$H_\phi(A) = \int_M \langle F_A, \phi \rangle$$

and the mapping $\phi \mapsto H_\phi$ is clearly $\mathscr{G}(P)$-equivariant. We then claim that mapping $\mu(A) = F_A$ defines a moment map. We see this, we first note that the map is $\mathscr{G}(P)$-equivariant by our formulas for how the connection and its curvature transform under a gauge transformation. We then compute

$$\begin{aligned} d(H_\phi)_A(\psi) &= \frac{d}{dt}\bigg|_{t=0} \int_M \langle F_{A+t\psi}, \phi \rangle \\ &= \int_M \langle d_A\psi, \phi \rangle \\ &= \int_M d\langle \psi, \phi \rangle - \int_M \langle \psi, d_A\phi \rangle \\ &= \int_M \langle d_A\phi, \psi \rangle \end{aligned}$$

Then noting that $d_A\phi$ is the vector field determined by ϕ, this shows that $\mu(A) = F_A$ is a moment map for the action.