Diff Geo Lecture 1

Q: Multivariable Calculus / Analysis? | Good reference - John Lee, Introduction to Smooth Manifolds
Texed up notes - want?

Given a function $F: \mathbb{R}^n \to \mathbb{R}^m$, we can write it in components $F = (F_1 \dots F_m)$ $\quad F_i: \mathbb{R}^n \to \mathbb{R}$

If F is <u>differentiable</u>, its <u>derivative</u> DF_p is the best linear approximation at $p \in \mathbb{R}^n$

$$DF_p = \begin{pmatrix} \frac{\partial F_1}{\partial x^1}(p) & \cdots & \frac{\partial F_1}{\partial x^n}(p) \\ \vdots & & \\ \frac{\partial F_m}{\partial x^1} & \cdots & \frac{\partial F_m}{\partial x^n} \end{pmatrix}$$

Given $\quad \mathbb{R}^n \xrightarrow{F} \mathbb{R}^m \xrightarrow{G} \mathbb{R}^k$

$$D(G \circ F)_p = DG_{F(p)} \circ DF_p \quad \text{- \underline{The chain rule}}$$

<u>Def</u>: $U, V \subset \mathbb{R}^n$ open, $F: U \to V$ is a <u>diffeomorphism</u> if F is smooth, bijective, and has a smooth inverse

<u>Note</u>) $x \mapsto x^3$ is a smooth bijection, but is not a diffeomorphism

<u>Def</u>: $F: U \to V$ is a <u>local diffeomorphism</u> if for all $p \in U$ $\exists$ nbhd $U_p \subset U$ s.t. $F|_{U_p}$ is a diffeomorphism onto its image

E.g. The polar transform
$(r, \theta) \mapsto (r\cos\theta, r\sin\theta)$ is a local diffeomorphism

Thm (Inverse Function Theorem) Given smooth $F: U \to V$, F is a local diffeomorphism at $p \iff DF_p$ is an isomorphism

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Einstein Summation Convention

- If an index appears on top and on bottom, there is a summation, i.e

Linear Combinations $v^i e_i \rightsquigarrow \sum_i v^i e_i$

Matrix Multiplication $(AB)^i_j = A^i_k B^k_j$

Def A n-dimensional topological manifold is a 2^{nd} Countable Hausdorff space M with an open cover $\{U_\alpha\}$ and maps $\varphi_\alpha : U_\alpha \to V_\alpha$ where φ_a is a homeomorphism $U_\alpha \to V_\alpha \subset \mathbb{R}^n$

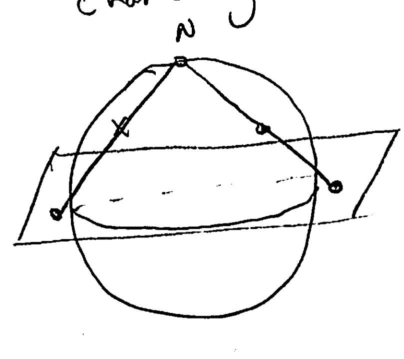
We call $(U_\alpha, \varphi_\alpha)$ a chart / local coordinate system (abuse notation, U_α

We can write $\varphi = (x^1 \ldots x^n)$ $x^i : M \to \mathbb{R}$; the x^i are local coordinate functions.

Example

1) $\mathbb{R}^n$ - $(\mathbb{R}^n, id)$ is a chart

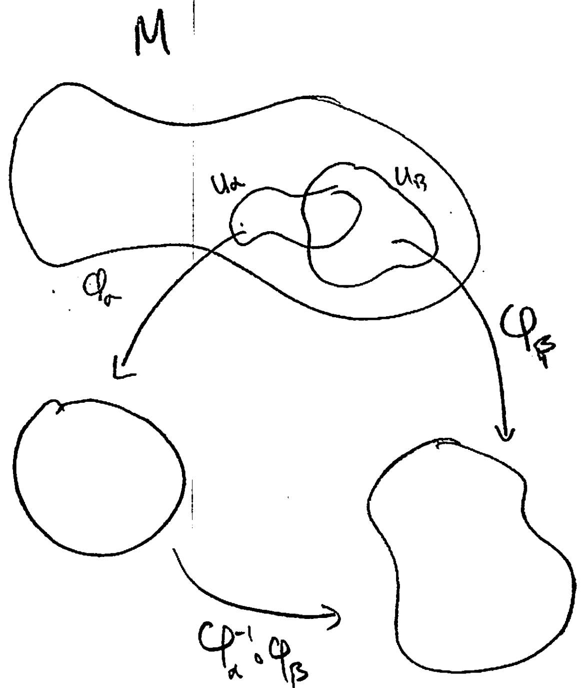
2) $S^2 = \{(x,y,z) \mid x^2+y^2+z^2 = 1\}$ charts given by stereographic projection



$\varphi_N(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$

In order for smoothness to make sense, needs to be independent of local coordinates - don't want a function to "look smooth" in one chart and not smooth in another.

Charts need to be **compatible** $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are compatible if $\varphi_\beta^{-1} \circ \varphi_\alpha$ and $\varphi_\alpha^{-1} \circ \varphi_\beta$ are smooth in the usual sense



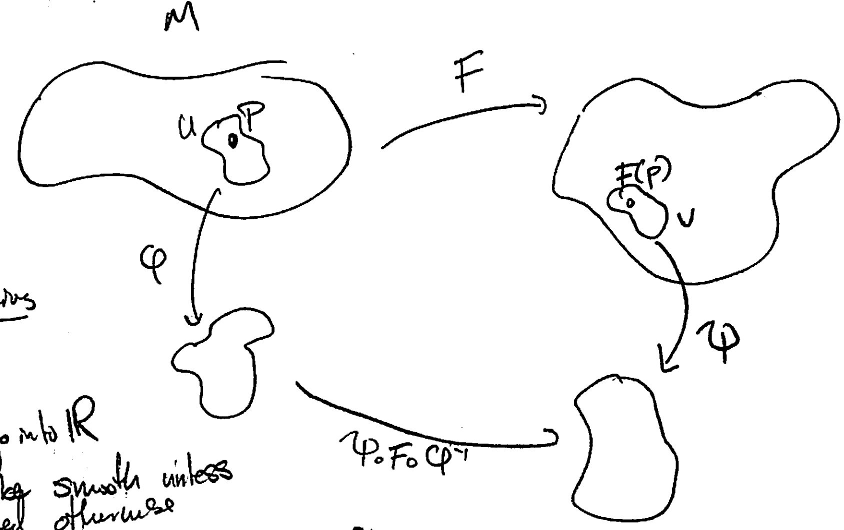
Def: An **atlas** for a topological manifold M is a collection $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ of smoothly compatible charts covering M.

$\mathcal{A}$ is **maximal** if it is not properly contained in any other atlas

Def: A **smooth manifold** is the data of a topological manifold M and a maximal atlas $\mathcal{A}$ for M.

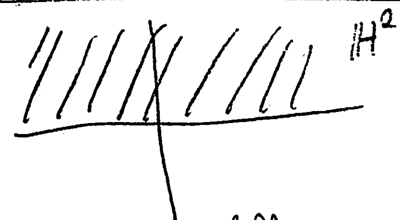
- Don't worry too much about atlases - not used much in practice and usually omitted - we'll just say "let M be a smooth manifold"

Def: For smooth mflds M, N, a map $F: M \to N$ is <u>smooth</u> if $\forall p \in M$, there exist charts (U, φ) and (V, ψ) of p, $F(p)$ respectively s.t. $\psi \circ F \circ \varphi^{-1}$ is smooth in the usual sense.



Rmk: It's common practice to distinguish b/w <u>functions</u> and <u>maps</u>. maps are b/w mflds, functions go into $\mathbb{R}$.
Also, all maps/functions will be smooth unless stated otherwise.

Let $\mathbb{H}^n = \{(x^1 \ldots x^n) \in \mathbb{R}^n \mid x^n \geq 0\}$



Def: A <u>manifold with boundary</u> is a 2^{nd} countable Hausdorff space X covered by charts $(U_\alpha, \varphi_\alpha)$ where $\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{H}^n$.
A point $p \in M$ is a <u>boundary point</u> if $\exists (U_\alpha, \varphi_\alpha)$ s.t. $\varphi_\alpha(p) = (x^1 \ldots x^n)$ with $x^n = 0$. The boundary ∂X is the set of boundary points, and the interior is $X - \partial X$.

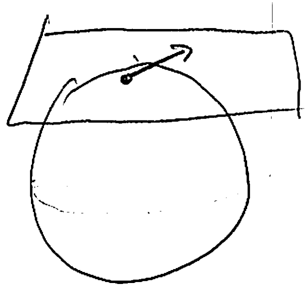
$\mathbb{H}^n$ is a mfld w/ boundary with $\partial \mathbb{H}^n = \{(x^1 \ldots x^n) \mid x^n = 0\} \cong \mathbb{R}^{n-1}$

<u>Things to think about</u>

1) ∂X well-defined? If $\varphi(p) \in \partial \mathbb{H}^n$, and ψ is another chart containing p, why must $\psi(p) \in \partial \mathbb{H}^n$? (Use inverse function thm).

2) ∂X is a $(n-1)$-mfld w/o boundary, why? Give charts.

Intuitively, tangent vectors at p are arrows based at p, tangent spaces are planes attached to p.



Not intrinsic - can be defined this way, but relies on an embedding. Want an intrinsic notion of tangent vector

Motivation

Let $\{e_i\}$ be the standard basis for $\mathbb{R}^n$. Then $v \in \mathbb{R}^n$ can be written as $v = v^i$; The directional derivative in the v direction is

$$D_v\big|_p = v^i \frac{\partial}{\partial x^i}\Big|_p \qquad D_v\big|_p f = v^i \frac{\partial f}{\partial x^i}(p)$$

$D_v|_p$ satisfies the Leibniz rule $D_v|_p fg = f(p) D_v|_p g + g(p) D_v f|_p$

Def For $p \in \mathbb{R}^n$, A derivation at p is a linear map
$D: C^\infty(\mathbb{R}^n) \to \mathbb{R}$ satisfying the Leibniz rule
$C^\infty(\mathbb{R}^n) = \{\text{smooth } f: \mathbb{R}^n \to \mathbb{R}\}$

Thm: There is a linear isomorphism $\{\text{Derivations at } p\} \leftrightarrow \{\text{Tangent vectors at } p\}$
$D_v \longleftarrow v$

This characterization works intrinsically for mflds!

Def: For a smooth mfld M, $p \in M$. The tangent space at p is $T_pM = \{\text{Derivations } D: C^\infty(M) \to \mathbb{R} \text{ at } p\}$

The intuition should be the same -- the tangent space is the best linear approximation to M

A smooth map $F: M \to N$ induces maps of tangent spaces

How should $dF_p : T_pM \to T_{F(p)}N$ act on $v \in T_pM$?

Let $g \in C^\infty(N)$. Then $dF_p(v)g = v(g \circ F)$

Another notation is F_* (the pushforward)

What do tangent vectors / derivatives look like in local coords?
Should look like the standard picture in $\mathbb{R}^n$

Let M be a smooth mfld, and $x^1 \dots x^n$ local coordinates about $p \in M$
Define the <u>coordinate vectors</u> as the derivations $\frac{\partial}{\partial x^i}\Big|_p$ defined by

$$\frac{\partial}{\partial x^i}\Big|_p f = \frac{\partial (f \circ \varphi^{-1})}{\partial x^i}\Big|_{\varphi(p)} \leftarrow \text{Literally the partial derivative operator}$$

<u>Fact</u>. The ∂_i form a basis for T_pM

Given $F: M \to N$, with $dF_p : T_pM \to T_{F(p)}N$ is given in local coords by a matrix; which? (Look at how $dF_p(\partial_i)$ acts on $g \in C^\infty(N)$)

<u>Change of coordinates</u>

Let $\varphi = (x^1 \dots x^n)$ $\psi = (y^1 \dots y^n)$ be two coordinate systems about p

Then $$\frac{\partial}{\partial x^i}\Big|_p = \frac{\partial y^j}{\partial x^i}(\varphi(p)) \frac{\partial}{\partial y^j}\Big|_p \quad \left(\text{Look at } d(\psi \circ \varphi^{-1})_p \left(\frac{\partial}{\partial x^i}\Big|_p\right)\right)$$

Looks like the chain rule!

If $v = v^i \frac{\partial}{\partial x^i}\Big|_p$, $V = \tilde{V}^j \frac{\partial}{\partial y^j}\Big|_p$ where $\tilde{V}^j = v^i \frac{\partial y^j}{\partial x^i}(\varphi(p))$

Same vector different coordinates

The Tangent Bundle

Def For a smooth manifold M, the tangent bundle is

$$TM = \coprod_{p \in M} T_pM = \{(p,v) \mid p \in M,\ v \in T_pM\}$$

It comes with a natural map $\pi : TM \to M$

by $\pi(p,v) = p$

So far just a set, but we can make TM a smooth mfld

Thm: TM is a smooth mfld

Proof: We define a basis for the topology on TM, and these will also be our charts.

For $p \in M$ let (U_p, φ), $\varphi = (x^1 \ldots x^n)$ be a chart.

Then for any point $q \in U_p$ $\{\partial_i|_q\}$ is a basis for T_qM

$\Rightarrow$ any pair $(q,v) \in \pi^{-1}(U)$ can be written uniquely as $(x^1(q) \ldots x^n(q), v^1 \ldots v^n)$

where $V = v^i \partial_i$ this defines a bijection $\Phi_p : \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^n$

Then the $\{(\pi^{-1}(U_p), \Phi_p)\}$ are smoothly compatible, and determine the topology + smooth structure, making TM a smooth manifold.

Vector Bundles

Def: Let M be a smooth manifold. A <u>vector bundle</u> over M is the data of a smooth manifold E and a smooth map $\pi : E \to M$ s.t.

1) π is surjective
2) Each fiber $\pi^{-1}(p)$ has the structure of a real vector space
3) For all $p \in M$, $\exists\, U \ni p$ and a diffeomorphism Φ s.t.

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\Phi"] \arrow[dr, "\pi"'] & U \times \mathbb{R}^K \arrow[d, "P_1"] \\ & U \end{tikzcd}$$

commutes.

Φ is a <u>local trivialization</u>

Intuitively, a vector bundle is a smooth family of vector spaces parameterized by M.

<u>Examples</u>

1) $M \times V$ for a fixed v.s. V - the <u>trivial bundle</u>
2) the tangent bundle TM
3) $\mathbb{RP}^n$ is the space of lines in $\mathbb{R}^{n+1}$ the <u>tautological bundle</u> E is the vector bundle whose the fiber over $l \in \mathbb{RP}^n$ is ... l

Same can be done with Grassmanians $Gr_k(\mathbb{R}^n)$.

Bundles pullback

TS^n is not trivial

Def Let $\pi : E \to M$ be a vector bundle. A <u>local section</u> is a map $\sigma : U \to E$ s.t. $\pi \circ \sigma = id_U$. If $U = M$, σ is a <u>global section</u>

Think of sections as a smooth assignment of a vector at every point.

Irreducible modules over $A = M_n\mathbb{R} \oplus M_n\mathbb{R}$ are isomorphic to either
$\mathbb{R}^n$ with the left factor acting trivially
or $\mathbb{R}^n$ with the right factor acting trivially

The two are clearly irreducible, since A acts transitively

1) Nonisomorphic?
2) Rest are isomorphic to one or the other?