THE RISING SEA: CATEGORIES AND SHEAVES

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These are some notes + exercises I've compiled working through the first 2 chapters of Ravi Vikhil's *The Rising Sea*, with the main purpose being to gain some familiarity and comfort with categories and sheaves.

1. Category Theory

Definition 1.1. A ***category*** $\mathscr{C}$ is a collection[1] of ***objects***, denoted $\mathsf{Ob}(\mathscr{C})$ and a collection[2] ***morphisms*** $\mathsf{Hom}(A, B)$[3] for every pair of objects $A, B \in \mathsf{Ob}(\mathscr{C})$ satisfying the following axioms:

(1) Given morphisms $f : A \to B$ and $g : B \to C$, there is a unique map $g \circ f : A \to C$ that makes the following diagram commute

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

(2) For every object $A \in \mathsf{Ob}(\mathscr{C})$, there exists an ***identity morphism*** $\mathrm{id}_A \in \mathsf{Hom}(A, A)$ such that for any morphisms $f : A \to B$ and $g : C \to A$, we have that $\mathrm{id}_A \circ f = f$ and $g \circ \mathrm{id}_A = g$

A morphism $f : A \to B$ is an ***isomorphism*** is there exists a morphism $g : B \to A$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$. We then call g the ***inverse*** to f. Isomorphisms $A \to A$ are called ***automorphisms*** of A.

Example 1.2. The category of sets, often denoted Set has sets as its objects, and maps of sets as its morphisms.

Example 1.3. Vector spaces over a field $\mathbb{F}$ also form a category, denoted $\mathsf{Vec}_{\mathbb{F}}$, where the objects are $\mathbb{F}$-vector spaces, and the objects are $\mathbb{F}$-linear maps.

Exercise 1.4. Let A be an object of a category $\mathscr{C}$. Show that the automorphisms of $\mathsf{Hom}(A, A)$ form a group, called the ***Automorphism group*** of A. Show that two isomorphisc objects in $\mathscr{C}$ have isomorphic automorphism groups.

Proof. Verifying that $\mathrm{Aut}(A)$ is a group is mainly an exercise in definition chasing. Associativity comes from the category axioms, the identity element is the identity morphism, and inverses exist by the definition of the automorphism group.

For the second part, let $A, B \in \mathsf{Ob}(\mathscr{C})$ be isomorphic, with $f : A \to B$ an isomorphism. We then define a map $f^* : \mathrm{Aut}(B) \to \mathrm{Aut}(A)$ where $f^*\sigma = f^{-1} \circ \sigma \circ f$. This is clearly a group homomorphism, and is an isomorphism with inverse given by $(f^{-1})^*$. ■

Now that we have categories, the natural thing to study are maps of categories.

Definition 1.5. A (covariant)***functor*** $\mathcal{F}$ from a category $\mathscr{C}$ to another category $\mathscr{D}$ assigns each object $A \in \mathsf{Ob}(\mathscr{C})$ an object $\mathcal{F}(A) \in \mathsf{Ob}(\mathscr{D})$ and to each map $f : A \to B$ in $\mathscr{C}$ a map $\mathcal{F}(f) : \mathcal{F}(A) \to \mathcal{F}(B)$ such that $\mathcal{F}$ respects composition, i.e.

$$\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$$

a ***contravariant functor*** can be defined similarly, except it reverses the arrows, i.e. $\mathcal{F}(f)$ is now a map $B \to A$, rather than $A \to B$.

Example 1.6. For mathematical objects that are just sets with extra structure (e.g. vector spaces, groups, rings, etc), the ***forgetful functor*** is the functor that takes each object to its underlying set, and each map to itself (thought of as a map of sets). Categories that admit a forgetful functor into Set are called ***concrete categories***.

[1] Loosely speaking; there's some set-theoretic issues here, but it's not that important for us

[2] Again, ignoring set-theoretic problems

[3] Vikhil uses Mor, but we'll use the more standard notation of Hom

Example 1.7 (***The functor of points***)**.** For a category $\mathscr{C}$, fix an object $A \in \mathsf{Ob}(\mathscr{C})$. We use this to define the functor $\mathcal{F}_A : \mathscr{C} \to \mathsf{Set}$, where for $B \in \mathsf{Ob}(\mathscr{C})$, we let $\mathcal{F}(B) = \mathsf{Hom}(A, B)$ and for $f : B \to C$, we let $\mathcal{F}_A(f) : \mathsf{Hom}(A, B) \to \mathsf{Hom}(A, C)$ be the map where $\mathcal{F}_A(f)(g) = f \circ g$.

Like functions, we can compose functors $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ and $\mathcal{G} : \mathscr{B} \to \mathscr{C}$ to obtain $\mathcal{G} \circ \mathcal{F} : \mathscr{A} \to \mathscr{C}$, where for $A \in \mathsf{Ob}(\mathscr{A})$, we have that $\mathcal{G} \circ \mathcal{F}(A) = \mathcal{G}(\mathcal{F}(A))$, and the same thing for morphisms in $\mathscr{A}$. Also like functions, we have notions of injectivity and surjectivity.

Definition 1.8. A covariant functor $\mathcal{F} : \mathscr{C} \to \mathscr{D}$ is ***faithful*** if the induced map $\mathsf{Hom}_{\mathscr{C}}(A, B) \to \mathsf{Hom}_{\mathscr{D}}(\mathcal{F}(A), \mathcal{F}(B))$ is injective and ***full*** if it surjective.

Example 1.9. The forgetful functor $\mathsf{Vec}_k \to \mathsf{Set}$ is not full, since there are more maps of sets than there are maps of vector spaces, since an element of $\mathsf{Hom}_{\mathsf{Set}}(V, W)$ need not be linear. However, the “inclusion" functor $i : \mathsf{Ab} \to \mathsf{Grp}$ from abelian groups into groups is full. Given two abelian groups G and H, we clearly have that $\mathsf{Hom}_{\mathsf{Ab}}(G, H) = \mathsf{Hom}_{\mathsf{Grp}}(G, H)$, so we have that i is a faithful functor as well.

Functors in a lot of ways act just like functions, but there's some more things we can do with them. For example, it some sense, we can have maps between functors.

Definition 1.10. Given covariant functors $\mathcal{F}, \mathcal{G} : \mathscr{A} \to \mathscr{B}$, a ***natural transformation*** $\mathcal{F} \to \mathcal{G}$ assigns each object $A \in \mathsf{Ob}(\mathscr{A})$ a morphism $m_A : \mathcal{F}(A) \to \mathcal{G}(A)$, such that for every morphism $f : A \to B$ in $\mathscr{A}$, the following diagram commutes

$$\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "m_A"'] & \mathcal{F}(B) \arrow[d, "m_B"] \\
\mathcal{G}(A) \arrow[r, "\mathcal{G}(f)"'] & \mathcal{G}(B)
\end{tikzcd}$$

a ***natural isomorphism*** is when all the given morphisms m_A are isomorphisms.

Example 1.11. A trivial example of a natural transformation between $\mathcal{F} \to \mathcal{G}$ assigns the zero map $z : \mathcal{F}(A) \to \mathcal{G}(A)$ for every object A. Such a zero map only exists in some categories, such as Vec_k and Grp.

This gives a rigorous definition for the words “canonical" and “natural" that get thrown around a bit too often. The prototypical example for this idea is the natural isomorphism from a finite dimensional vector space to its double dual.

Exercise 1.12. Let $\mathcal{D} : \mathsf{Vec}_k \to \mathsf{Vec}_k$ be the functor that maps each k-vector space V to its dual space V^* of linear functions $V \to k$ and maps a linear map $A : V \to W$ to its pullback (sometimes called the transpose) $A^* : W^* \to V^*$. Show that the double dual functor $\mathcal{D} \circ \mathcal{D}$ is naturally isomorphic to the identity functor id.

Proof. For a vector v in a vector space V, define the map $\xi_v : V^* \to k$ by $\xi_v(\omega) = \omega(v)$. The let $\Xi_V : V \to V^{**}$ be the map that sends $v \mapsto \xi_v$. We claim that the morphisms Ξ_V define a natural isomorphism from $\mathcal{D} \circ \mathcal{D}$ to id. We first show that Ξ_V defines an isomorphism. Since V and V^{**} are the same dimension, it suffices to check that Ξ_V has trivial kernel. Suppose $v \mapsto \xi_v = 0$. Then $\omega(v) = 0$ for all $\omega \in V^*$, which is only true when $v = 0$. Therefore, Ξ_V is an isomorphism for every V. Showing that the Ξ_V define a natural isomorphism now amounts to showing that the following diagram commutes

$$\begin{tikzcd}
V \arrow[r, "A"] \arrow[d, "\Xi_V"'] & W \arrow[d, "\Xi_W"] \\
V^{**} \arrow[r, "A^{**}"'] & W^{**}
\end{tikzcd}$$

which amounts to showing that $\Xi_W \circ A = A^{**} \circ \Xi_V$. Let $v \in V$ and $\omega \in W^*$. Then we compute

$$\begin{aligned}
(\Xi_W \circ A)(v)(\omega) &= \Xi_W(Av)(\omega) \\
&= \xi_{Av}(\omega) \\
&= \omega(Av)
\end{aligned}$$

We also compute

$$\begin{aligned}(A^{**} \circ \Xi_V)(v)(\omega) &= A^{**}\xi_v(\omega) \\ &= \xi_v(A^*\omega) \\ &= A^*\omega(v) \\ &= \omega(Av)\end{aligned}$$

So we have given a natural isomorphism $\mathcal{D} \circ \mathcal{D} \to \mathrm{id}$ ■

It's also somewhat easy in this language why there is no natural isomorphism from a vector space to its dual. When you saw it, there was most likely some hand-waving about how the choice of a basis made the isomorphism unnatural, so we can make this rigorous now.

Proposition 1.13. *There exists no natural isomorphism from the dual functor $\mathcal{D}$ to the identity functor* id

Proof. Suppose such a natural isomorphism existed, and assigned each vector space an isomorphism $\varphi_V : V \to \vec{v}^*$ such that the following diagram commutes for any linear map $A : V \to W$.

$$\begin{tikzcd} V \arrow[r, "A"] \arrow[d, "\varphi_V"'] & W \arrow[d, "\varphi_W"] \\ V & W \arrow[l, "A^*"] \end{tikzcd}$$

If we let A be the zero map, we have that $A^* \circ \varphi_W \circ A = 0$, so it cannot be φ_V. ■

One product from category theory that ends up being extremely useful is the notion of a ***universal property***. One example you may have seen before is the universal property of a product– Given two objects A, B in some category $\mathscr{C}$, their product, denoted $A \times B \in \mathsf{Ob}(\mathscr{C})$ is (up to unique isomorphism) the unique object with maps $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$ such that given any pair of maps $f : P \to A$ and $g : P \to B$ factor uniquely through $A \times B$, i.e. there exists a unique map $f \times g : P \to A \times B$ such that the following diagram commutes

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, "f\times g"'] & A \\ B & A \times B \arrow[l, "\pi_B"] \arrow[u, "\pi_A"'] \end{tikzcd}$$

Some other useful objects defined by universal properties

Definition 1.14. An ***initial object*** I in a category $\mathscr{C}$ if there exists exactly one map $I \to A$ for every $A \in \mathsf{Ob}(\mathscr{C})$. A ***final object*** is an object F in which there exactly one map $A \to F$ for every object A. An object Z is a ***zero object*** if it is both an initial and final object.

Exercise 1.15. Show any two initial objects are uniquely isomorphic. Show any two final objects are uniquely isomorphic

Proof. Let A, B be initial objects. We note that the definition of an initial object implies that the only map $A \to A$ and $B \to B$ must be id_A and id_B respectively. Therefore, the unique maps $A \to B$ and $B \to A$ must compose to identity, so they are isomorphisms.

The same proof applies to final objects. For final objects the unique maps $A \to A$ and $B \to B$ are necessarily the identity maps, and the unique maps $A \to B$ and $B \to A$ must compose to identity. ■

For concreteness, we should determine what these some of objects are

Exercise 1.16. What are the initial/final objects in Set, Ring[4] and Top?

Solution.

[4]Since this book is for algebraic geometry, all rings are commutative and unital. Ring maps are required to map $1 \mapsto 1$

(1) The initial object for Set is the empty set. Given any set S, we have a unique map $\varnothing \to S$ (the empty map). The final object is the single point set $*$. Given any set S, there exists only a single map to $*$, namely the map that takes all of S to the point. We note that the empty set cannot be a final object, since we cannot map a nonempty set to an empty set.
(2) The initial object in Ring is the integers $\mathbb{Z}$. Given any commutative unital ring R (which from now on we'll just say ring), the unique map $\mathbb{Z} \to R$ is entirely determined by the map $1 \mapsto 1$ since 1 generates $\mathbb{Z}$ as an abelian group. The final object is the zero ring $0 = 1$, where the unique map $R \to 0$ is the map that sends everything to 0.
(3) The initial and final objects for Top are the same as for Set. We note that the empty set and a single point set have unique topologies, so we do not need to specify a topology.

■

One reason that universal properties are nice is that they give an extremely helpful way in proving that some object is the one we desire – by showing it satisfies the universal property. Given an arbitrary set P, how can I conclude that it is the product $A \times B$? We can look for an explicit isomorphism, which can be difficult, or we can simply show that it satisfies the same universal property as the product. Another good example of a universal property comes from an important concept in algebraic geometry – the localization of a ring.

Example 1.17 (***Localization of a ring***)**.** Given a ring R and a multiplicative subset $S \subset R$ (a set closed under multiplication that contains 1). We can then define the ring $S^{-1}R$ to be the set of formal fractions $\{r/s : r \in R, s \in S\}$ modulo the equivalence relation $\sim$ where $r/s \sim p/q$ if and only if there exists $t \in S$ such that $t(qr - sp) = 0$[5]. Addition and multiplication of fractions is exactly the same as in $\mathbb{Q}$, namely $r/s + p/q = (rq + ps)/pq$ and $r/s \cdot p/q = rp/sq$. We then have a canonical map $R \to S^{-1}R$ where $r \mapsto r/1$

Exercise 1.18. Show the canonical map $R \to S^{-1}R$ is injective if and only if S contains no zero divisors.

Proof. Suppose there exists two distinct elements p, q that map to the same element, i.e. $p/1 \sim q/1$. This means that there exists some $s \in S$ such that $s(p - q) = 0$, which is true if and only if s is a zero divisor for the nonzero element $p - q$. ■

Exercise 1.19. Show that $S^{-1}R$ satisfies the following universal property: $S^{-1}R$ is initial among R-algebras A where every element of S is mapped to a invertible element. In other words any map $R \to A$ in which every element is mapped to an invertible element factors uniquely through the map $R \to S^{-1}R$.

Proof. Let $\varphi : R \to A$ be such a map where every element of S maps to an invertible element. The we want to find a map $\tilde{\varphi} : S^{-1}R \to A$ such that the following diagram commutes

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "\iota"'] & A \\ S^{-1}R \arrow[ur, "\tilde{\varphi}"'] & \end{tikzcd}$$

where $\iota : R \to S^{-1}R$ is the canonical map. We note that in order for the diagram to commute, we must necessarily have that $\tilde{\varphi}(r/1) = \varphi(r)$ for every $r \in R$. Therefore, the only possible mapping for $\tilde{\varphi}$ must be $\tilde{\varphi}(r/s) = \varphi(r)/\varphi(s)$, which is possible since $\varphi(s)$ is invertible, so we have found our unique map. ■

[5] This might look strange if you compare it to the fractions you know in $\mathbb{Q}$, but special care needs to be taken in the case that R contains zero divisors