REPRESENTATIONS OF $\mathfrak{sl}_3\mathbb{C}$

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The representation theory of $\mathfrak{sl}_3\mathbb{C}$ is more complex than the situation with $\mathfrak{sl}_2\mathbb{C}$, and involves generalizing some of the tools used to analyze the irreducible representations of $\mathfrak{sl}_2\mathbb{C}$. However, this will develop a relatively general framework for understanding the representations of semisimple Lie algebras.

Recall that the key piece for understanding the irreducible representations of $\mathfrak{sl}_2\mathbb{C}$ was the basis H, X, and Y, where H was diagonzlizable and satisfied the commutation relations

$$[H,X] = 2X \qquad [H,Y] = -2Y \qquad [X,Y] = H$$

for the higher dimensional case, we will not have such a basis anymore. The idea will be to replace the matrix H with an abelian subalgebra $\mathfrak{h}$. The reasoning here is that commuting matrices preserve each other's eigenspaces, so they are simultaneously diagonalizable.

Definition 1.1. Given a representation V and a subalgebra $\mathfrak{h} \subset \mathfrak{sl}_3\mathbb{C}$, a vector $v \in V$ is an ***eigenvector*** for $\mathfrak{h}$ if for all $H \in \mathfrak{h}$, v is an eigenvector for H.

Note that the eigenvalues for a eigenvector v need not be the same for different H. Instead, we have that $Hv = \lambda(H)v$ for some $\lambda \in \mathfrak{h}^*$. Therefore, the generalization of the eigenspace decomposition for an representation of $\mathfrak{sl}_2\mathbb{C}$ is a decomposition $V = \oplus_\lambda V_\lambda$ where λ ranges over a finite subset of $\mathfrak{h}^*$. We also want to generalize the commutation relations from $\mathfrak{sl}_2\mathbb{C}$. We see that from before, X and Y are eigenvectors of $\mathrm{ad}(H)$, with eigenvalues 2 and -2 repsectively. When we replace H with $\mathfrak{h}$, we see we want to find a decomposition of $\mathfrak{sl}_3\mathbb{C}$ as

$$\mathfrak{sl}_3\mathbb{C} = \mathfrak{h} \oplus \left(\bigoplus_\alpha V_\alpha\right)$$

where each V_α is an eigenspace for $\mathrm{ad}(\mathfrak{h})$, and again, the α form a finite subset of $\mathfrak{h}^*$. This procedure will also be used in the general case as well.

When we specialize to $\mathfrak{sl}_3\mathbb{C}$, it turns out that an ideal choice of $\mathfrak{h}$ is the subalgebra of diagonal matrices in $\mathfrak{sl}_3\mathbb{C}$. Let L_i denote the linear functionals such that $L_i(A) = A^i_i$. Then the condition that the matrices in $\mathfrak{sl}_3\mathbb{C}$ are traceless implies the dual $\mathfrak{h}^*$ is given by linear combinations a^iL_i where we quotient by the relation $L_1 + L_2 + L_3 = 0$. We then want to find eigenvectors of $\mathrm{ad}(\mathfrak{h})$. To do this, let D denote an arbitrary diagonal matrix in $\mathfrak{h}$, and $M \in \mathfrak{sl}_3\mathbb{C}$. Then DM is the matrix where $(DM)^i_j = D^i_iM^i_j$ (i.e. the i^{th} row is multiplied by D^i_i), and MD is the matrix where $(MD)^i_j = D^j_jM^i_j$ (i.e. the j^{th} column of MD is multiplied by D^j_j). Therefore, the $(i,j)^{th}$ component of the commutator $[D,M]^i_j$ is given by

$$[D,M]^i_j = (DM)^i_j - (MD)^i_j = D^i_iM^i_j - D^j_jM^i_j = (D^i_i - D^j_j)M^i_j$$

Therefore, for a matrix to be and eigenvector of $\mathrm{ad}(D)$ for all $D \in \mathfrak{h}$, we need all but a single entry to be 0. To see this, we note that we can pick a matrix D such that the multipliers $(D^i_i = D^j_j)$ for the M^i_j component are all different, so M can only be an eigenvector for $\mathrm{ad}(D)$ if all but one of the entries is 0. Then the elementary matrices E_{ij} with a 1 in the (i,j) give an eigenspace decomposition for $\mathfrak{sl}_3\mathbb{C}$, and the action of $ad(D)$ on E_{ij} will have eigenvalue $L_i(D) - L_j(D)$. This gives us that the eigenspace generated by E_{ij} will have "eigenvalue" $L_i - L_j \in \mathfrak{h}^*$.