(G, X)-STRUCTURES

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1. The Statement of Local Rigidity

Before discussing (G, X)-structures, we first state the result we plan on working towards, which is local rigidity for semisimple Lie groups. Fix a semisimple lie group G and $\Gamma \subset G$ an irreducible lattice Heuristically, Γ being irreducible means that Γ is not locally a product of lattices. More explicity, it is defined as follows: in the case that G is simply connected, G being semisimple implies that it is a product $G = \prod_i G_i$ of simple Lie groups G_i. Then $\Gamma \subset G_i$ being irreducible simply means that it is not a product $\Gamma = \prod_i \Gamma_i$ of lattices $\Gamma_i \subset G_i$. When G isn't simply connected, Γ being irreducible means that the preimage of Γ under the universal covering $\widetilde{G} \to G$ is irreducible.Then let $\mathcal{R}(\Gamma, G)$ denote the space of maps $\Gamma \to G$ with the topology of pointwise convergence.

Definition 1.1. A lattice Γ is ***locally rigid*** if there is a neighborhood U of the inclusion map $\Gamma \hookrightarrow G$ in $\mathcal{R}(\Gamma, G)$ such that any other homomorphism $\Gamma \to G$ contained in U is conjugate to the inclusion.

Our goal will be to prove the following statement (at least for the rank 1 case)

Theorem 1.2. *Under some conditions on G and Γ, if $\rho \in \mathcal{R}(\Gamma, G)$ is sufficiently close to the inclusion $\Gamma \hookrightarrow G$, then $\rho(\Gamma)$ is a lattice in G and $\rho|_\Gamma$ is an isomorphism, i.e. Γ is locally rigid.*

The proof of this will use the notion of (G, X) structures, where G is the semisimple Lie group, and X is a symmetric space.

2. (G, X)-Structures

Fix once and for all a manifold X with a transitive action of a Lie group G by real analytic homeomorphisms. The idea of a (G, X) structure is to use X as a "model geometry," from which we can construct other manifolds that locally "look like" X, and have symmetries that "look like the action" of G.

Definition 2.1. A (G, X)***-manifold*** is a manifold M that admits an open cover $\{U_i\}$ with charts $\varphi_i : U_i \to X$ (i.e. diffeomorphisms onto their images) such that the transition functions $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_j \cap U_j) \to \varphi_i(U_i)$ are given by restrictions of the actions of group elements $g_{ij} \in G$. Two such charts will be called ***compatible***. Given a smooth manifold M, a (G, X)-stucture on M is a choice of ***atlas***, which is a maximal set of compatible charts $\{\varphi_i : U_i \to X\}$ giving M the structure of a (G, X)-manifold.

The discussion of the (G, X)-structure on a manifold M involves fiber bundles and connections on them.

Definition 2.2. Let M and F be smooth manifolds. A ***fiber bundle*** with model fiber F over M is a smooth manifold E equipped with a submersion $\pi : E \to M$ such that

(1) The fibers $E_p := \pi^{-1}(p)$ are diffeomorphic to F
(2) For every point p, there eixsts a ***local trivialization*** of E, i.e. a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ for a neighborhood U of p such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl] \\ & U & \end{tikzcd}$$

where the map $U \times F \to U$ is projection onto the first factor.

We refer to E as the ***total space*** and M as the ***base space***.

For this discussion, we will be concerned with fiber bundles with model fiber X, so "fiber bundle" without further modifiers will mean "fiber bundle with model fiber X." Our goal will be to relate the (G, X)-structure on a manifold M to a fiber bundle equipped with the data of a connection. Given any fiber bundle $\pi : E \to M$, we can find an open cover $\{U_i\}$ of M such that the fiber bundle $E|_{U_i} := \pi^{-1}(U_i)$ admits local trivializations $\varphi_i : E|_{U_i} \to U_i \times X$. If we consider the maps $\varphi_i \circ \varphi_j^{-1}$, then we have that they are given in components by the mapping

$$(u, x) \mapsto (u, [\psi_{ij}(u)](x))$$

for ***transition functions*** $\psi_{ij} : U_i \cap U_j \to \mathrm{Diff}(X)$. We can recover the original bundle $E \to M$ by gluing these bundles together using the transition functions ψ_{ij}, which is well defined because they agree in triple intersections and satisfy a cocycle condition.

In our situation, we have a distinguished group of symmetries of X coming from the action of G, which can be interpreted as a group homomorphism $G \to \mathrm{Diff}(X)$, so the fiber bundle $E \to M$ we want to construct from the (G, X)-structure on M should have transition functions valued in G. Furthermore, we will impose the condition that the transition functions ψ_{ij} are *constant*, so the second components of the maps $\varphi_i \circ \varphi_j^{-1}$ are given by the action of a single group element $g_{ij} \in G$. We also want a notion of compatibility with the charts of the (G, X)-structure. Specifically, we want the domains U_i of the local trivilizations to be the domains of charts $U_i \to X$, so we must be able to trivialize the bundle E over the charts in the atlas defining the (G, X)-structure. Given a chart $\kappa : U \to X$ such that E is trivialized over U, i.e. there is a local trivialization $\varphi_i : E|_{U_i} \to U_i \times X$. Then we get a section $s : U \to E|_{U_i}$, where $(\varphi_i \circ s)(p) = (p, \kappa(p))$, which embeds U in $E|_{U_i}$, and is given as the graph of κ under this choice of trivialization. Our next restriction is to require that two compatible charts $\kappa : U \to X$ and $\kappa' : U' \to X$ with intersecting domains have sections $s : U \to E|_U$ and $s' : U \to E|_{U'}$ that agree on $U \cap U'$. Putting everything together, given a (G, X)-manifold M, we can construct a fiber bundle $E \to M$ where a point of E is an equivalence class $[p, \kappa, x]$ where p is a point of M, $\kappa : U \to X$ is a chart containing p, and x is a point of X, where $[p, \kappa, x] = [p', \kappa', x']$ if $p = p'$ and there exists a group element $g \in G$ such that $\kappa = g \circ \kappa'$ and $x' = gx$ on some neighborhood containing p.

For those in the know, the requirement that the transition functions are constant suggests that the fiber bundles we constructed is analogous to a ***local system***, which has an identification with vector bundles equipped with a flat connection. A similar story will hold here. To discuss this, we will use Ehresmann's notion of a connection on a fiber bundle, which generalizes the notion for vector bundles to general fiber bundles.

Let $\pi : E \to M$ be a fiber bundle. This determines a rank n (where $n := \dim X$) distribution $V \subset TE$ (i.e. a rank n vector subbundle of $TE \to E$.) called the ***vertical distribution***, where the fiber V_e over $e \in E$ is $\ker d\pi_e$. Geometrically, one can identify V_e as the tangent vectors in T_eE that are tangent to the fiber $E_{\pi(e)}$.

Definition 2.3. A ***connection*** on a fiber bundle $E \to M$ is a choice of complementary subbundle to V, i.e. a distribution $H \subset TE$ such that $TP = V \oplus H$. The distribution H is also referred to as the ***horizontal distribution***.

The horizontal distribution should be thought of as distinguished "manifold directions" on E, which is complementary to the "fiber directions" specified by the vertical distribution. By a dimension count, we get that $d\pi_e|_{H_e} : H_e \to T_{\pi(e)}M$ is an isomorphism, so given $e \in E$, every tangent vector $v \in T_{\pi(e)}M$ has a unique ***horizontal lift*** to a tangent vector $\tilde{v} \in H_e \subset T_eE$. This the allows us to uniquely lift paths on M going through $\pi(e)$ to a path passing through e by taking horizontal lifts of velocity vectors, giving us a vector field, and then taking the integral curve passing through e. Using path lifting, we can then define the ***holonomy*** of a connection. Fix a point $p \in M$, and let γ be a loop based at p. Then for a point $e \in E_p$, we can lift γ to a path $\widetilde{\gamma}$ based starting at e, whose endpoint is another point $e' \in E_p$. Doing this for all points $e \in E_p$, we obtain a diffeomorphism $E_p \to E_p$ (for the inverse, do the same construction with the reversed loop) called the ***holonomy*** of loop γ.

We then want to restrict our attention to a specific subset of connections that are compatible with the action of G on X.

Definition 2.4. Let M be a smooth manifold (not necessarily with a (G, X)-structure). A (G, X)***-connection*** on M is the data of

(1) A fiber bundle $E \to M$ with model fiber X.
(2) A section $s : M \to E$.
(3) A connection on E such that the holonomy about any closed curve γ is given by the action of an element of G.

A (G, X)-connection is ***flat*** if the horizontal distribution is integrable (e.g. closed under Lie brackets), equivalently, if any contractible closed curve has trivial holonomy. For convenience, we might say "let $E \to M$ be a (G, X)-connection", and leave the section and connection implicit.

From our definition, we conveniently get the following:

Proposition 2.5. *Let M be a (G, X)-manifold, and $E \to M$ the fiber bundle we constructed above using the (G, X)-structure. Then E can be equipped with a connection that gives it the structure of a flat (G, X)-connection.*

Proof. In the construction we made earlier, we got a distinguished section $s : M \to E$ given locally by taking charts $\kappa : U \to X$, and embedding $U \to E$ by taking the graph of κ with respect to a local trivialization of E over U. Explicitly, using the description of points of E

as equivalence classes of the form $[m, \kappa, x]$ with $m \in M$, $\kappa : U \to X$ a chart, and $x \in X$, the section s is given by

$$s(m) = [m, \kappa, \kappa(m)]$$

for any choice of chart κ about m. Furthermore, we have a foliation of E by leaves diffeomorphic to M, where a leaf of the foliation is given by the points $\{[m, \kappa, x] \ : \ m \in M\}$. This gives rise to a horizontal distribution H by taking the space H_e at a point $e \in E$ to be the tangent space of the leaf passing through e, and H is integrable since it admits integral submanifolds, namely the leaves of the foliation. Therefore, H is a flat connection. The final thing we need to check is that the holonomy is given by the action of G, and this follows from the equivalence relation we used to define E, namely identifying points that were related by the action of G. Having identified all of the pieces, this gives E the structure of a flat (G, X)-connection. ■

The punchline is that the converse is also true!

Proposition 2.6. *Let M be manifold equipped with a flat (G, X)-connection $E \to M$. Then M can be equipped with the structure of a (G, X)-manifold such that $E \to M$ is the bundle coming from the construction described above.*

Proof. The proof of this should be familiar to those familiar with principal bundles and local systems. Given a flat (G, X)-connection $E \to M$, we have that the holonomy of a loops γ is determined by its homotopy class, since nullhomotopic maps have trivial holonomy. Therefore, the map taking a homotopy class $[\gamma] \in \pi_1(M, m_0)$ its holonomy $g \in G$ defines a group homomorphism $\pi_1(M, m_0) \to G$, which then determines an action of $\pi_1(M, m_0)$ on X. Then if we let $\widetilde{M}$ denote the universal cover of M, we can take the associated bundle

$$\widetilde{M} \times_{\pi_1(M)} X = (\widetilde{M} \times X)/\pi_1(M)$$

where the action on the product is the diagonal action. The associated bundle is a fiber bundle with model fiber X. Furthermore, it comes equipped with a natural connection, where the horizontal distribution is obtained by taking the image of $T\widetilde{M} \subset T(\widetilde{M} \times X)$ under the differential of the quotient map $\widetilde{M} \times X \to \widetilde{M} \times_{\pi_1(M)} X$. Furthermore, this connection is easily seen to be flat, since $T\widetilde{M} \subset T(\widetilde{M} \times X)$ is integrable, integrability of a distribution is a local property, and the fact that the quotient map is a covering map. The holonomy of this connection then corresponds exactly with the original homomorphism $\pi_1(M) \to G$, up to conjugation on G. Finally, one can show that the bundle $\widetilde{M} \times_{\pi_1(M)} X$ with the connection defined above is isomorphic to the (G, X)-connection $E \to M$, which allows us to identify a section $s : M \to \widetilde{M} \times_{\pi_1(M)} X$ corresponding to the section $M \to E$. From this, we want to recover an atlas of charts on M, which we can do by taking an evenly covered neighborhood of a point $m_0 \in M$, which necessarily trivializes the associated bundle $\widetilde{M} \times_{\pi_1(M)} X$, and then taking the second component of the section s with repect to this trivialization to obtain a chart about m_0. ■