

A (NOT SO) QUICK SURVEY OF HODGE THEORY

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CONTENTS

1. Linear Algebra	1
2. Complex Manifolds	4
3. Cohomology of Complex Manifolds.	6
4. Kähler Manifolds	8
5. Laplacians and Harmonic Forms	10
6. The Hodge Decomposition	15
7. Hodge Theory on Kähler Manifolds	16
8. The Lefschetz Decomposition	21

1. LINEAR ALGEBRA

Definition 1.1. Let V be a finite dimensional \mathbb{R} -vector space. An *almost complex structure* on V is a linear map $J : V \rightarrow V$ satisfying $J^2 = \text{id}_V$. An *almost complex vector space* is tuple (V, J) , where V is a finite dimensional \mathbb{R} -vector space equipped with an almost complex structure J .

The data of an almost complex structure J is equivalent to giving V the structure of a complex vector space, where we define $(a + bi) \cdot v = av + bJv$. Because of this, we may call J a *complex structure*. We use the name *almost* complex structure to emphasize the differences between the analogous constructions in the nonlinear world of manifolds. Note that since an almost complex structure is equivalent to a complex structure, this immediately implies that V is even dimensional.

One way to think about an almost complex structure is through the geometric interpretation of complex multiplication. If we regard \mathbb{C} as a 2-dimensional vector space over \mathbb{R} , multiplication by i corresponds to a rotation by $\pi/2$ in the counterclockwise direction, and multiplication by $-i$ corresponds to a rotation by $\pi/2$ in the clockwise direction. From this we see that a choice of a square root of -1 comes with a choice of clockwise or counterclockwise. This implies that every complex vector space is canonically oriented as a real vector space. Given a \mathbb{C} -vector space V with \mathbb{C} -basis $\{z_1, \dots, z_n\}$, we have that the ordered basis $(z_1, iz_1, \dots, z_n, iz_n)$ defines a positively oriented basis for V over \mathbb{R} .

Definition 1.2. Let V be any \mathbb{R} -vector space. The *complexification* of V is the complex vector space $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$. The complexification $V_{\mathbb{C}}$ is naturally a complex vector space, where for $\lambda \in \mathbb{C}$, the action of λ on a homogeneous element $\mu \otimes v$ is given by

$$\lambda \cdot (\mu \otimes v) = \lambda\mu \otimes v$$

The complexification is a different way of obtaining a complex vector space from a real one. The complexification $V_{\mathbb{C}}$ is a vector space of twice the dimension of V as a vector space over \mathbb{R} . In addition, there is a natural inclusion $V \hookrightarrow V_{\mathbb{C}}$ given by $v \mapsto 1 \otimes v$. Complexification is an instance of *extension of scalars* – every element of $V_{\mathbb{C}}$ is of the form $av + biw$ where $a, b \in \mathbb{R}$, and $v, w \in V$. Therefore, we will denote the element $(a + bi) \otimes v \in V_{\mathbb{C}}$ by $av + biw$, and we have a direct sum decomposition $V_{\mathbb{C}} = V \oplus iV$. Given a linear map $T : V \rightarrow W$ of \mathbb{R} -vector spaces, we can extend T to a complexified map $T_{\mathbb{C}} : V_{\mathbb{C}} \rightarrow W_{\mathbb{C}}$, where $T_{\mathbb{C}}(av + biw) = aTv + biTw$. In other words, $T_{\mathbb{C}} = \text{id}_{\mathbb{C}} \otimes T$. In this way, we see that complexification defines a covariant functor $\text{Vect}_{\mathbb{R}} \rightarrow \text{Vect}_{\mathbb{C}}$ from the category of \mathbb{R} -vector spaces to the category of \mathbb{C} -vector spaces.

A natural question to ask is how an almost complex structure interacts with the process of complexification. Let (V, J) be an almost complex vector space. Then the complexified map $J_{\mathbb{C}} : V_{\mathbb{C}} \rightarrow V_{\mathbb{C}}$ squares to -1 , and admits eigenvalues $\pm i$. For example, consider $V = \mathbb{C}$. We then can then make the natural identification of \mathbb{C} with the almost complex vector space (\mathbb{R}^2, J) , where J is given by the matrix

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

using the ordered \mathbb{R} basis $(1, i)$. Then $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}^2$, and $J_{\mathbb{C}}$ is given by the matrix

$$J_{\mathbb{C}} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

in the ordered \mathbb{R} basis $(1, i, 1, i)$, where the first pair elements are elements of one copy of \mathbb{C}^2 , and the second pair of elements are elements of a separate copy of \mathbb{C}^2 , giving an \mathbb{R} basis for $\mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$. The matrix $J_{\mathbb{C}}$ clearly has eigenvalues $\pm i$, with the i -eigenspace being spanned by the vectors

$$\begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{i}{2} \\ 0 \\ -\frac{i}{2} \\ 0 \end{pmatrix}$$

and the $-i$ eigenspace being the span of

$$\begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{i}{2} \\ 0 \\ -\frac{i}{2} \end{pmatrix}$$

The case for general V is similar, and amounts to fixing an \mathbb{R} basis of the form $\{v_1, Jv_1, \dots, v_n, Jv_n\}$ for (V, J) . The decomposition of $V_{\mathbb{C}}$ into the $\pm i$ -eigenspaces of $J_{\mathbb{C}}$ gives a direct sum decomposition

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$$

where $V^{1,0}$ denotes the i -eigenspace, and $V^{0,1}$ denotes the $-i$ -eigenspace. Note that in the bases we chose above for \mathbb{C}^2 , complex conjugation is given by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and determines an isomorphism of complex vector spaces $V^{1,0} \cong \overline{V^{0,1}}$ and vice versa. In addition, complex conjugation picks out a distinguished subspace of $V_{\mathbb{C}}$ isomorphic to V – the 1-eigenspace. Elements of this eigenspace are called *real* (much like how the \mathbb{R}^n is left invariant under complex conjugation in \mathbb{C}^n).

An almost complex structure J on V induces a dual map $J^* : V^* \rightarrow V^*$. By functoriality of taking dual spaces, $(J^*)^2 = -\text{id}_{V^*}$, giving V^* the structure of an almost complex vector space via J^* . Explicitly, given a linear functional $\alpha \in V^*$, we have that the action of J^* on α is given by

$$(J^*\alpha)(v) = \alpha(Jv)$$

for every $v \in V$. We then get an analogous decomposition of the complexified dual space $V_{\mathbb{C}}^* = \text{Hom}_{\mathbb{R}}(V, \mathbb{C})$ as

$$V_{\mathbb{C}}^* = (V^*)^{1,0} \oplus (V^*)^{0,1}$$

into the $\pm i$ eigenspaces of J^* , and have very natural interpretations in terms of V . The subspace $(V^*)^{1,0}$ consists of the α such that $\alpha(Jv) = i\alpha(v)$. We have a natural pairing $(V^*)^{1,0} \otimes V^{1,0} \rightarrow \mathbb{C}$ given by

$$\langle \alpha, v \rangle = \alpha(v)$$

which is nondegenerate, establishing the two vector spaces as dual to each other (as complex vector spaces). A similar statement holds for $V^{0,1}$ and $(V^*)^{0,1}$, putting them in duality as well. Another perspective to take is that the condition that $\alpha(Jv) = i\alpha(v)$ is equivalent to α being *complex linear* with respect to the complex structure J on V and i on \mathbb{C} , giving us an isomorphism $(V^*)^{1,0} \cong \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$. Similarly, we have that $(V^*)^{0,1} \cong \text{Hom}_{\mathbb{C}}(V, \overline{\mathbb{C}})$.

We then consider the effects of complexification on another linear algebraic construction – the exterior algebra. Given a finite dimensional \mathbb{R} -vector space V , we have that the exterior algebra $\Lambda^{\bullet}V$ has a natural \mathbb{Z} -grading

$$\Lambda^{\bullet}V = \bigoplus_{k=0}^n \Lambda^k V$$

Upon complexification, we have a canonical isomorphism $(\Lambda^{\bullet}V)_{\mathbb{C}} \cong \Lambda^{\bullet}V_{\mathbb{C}}$, where the right hand side is the exterior algebra of $V_{\mathbb{C}}$ as a *complex* vector space (i.e. $\Lambda^0 V_{\mathbb{C}} \cong \mathbb{C}$, not \mathbb{R}), and $\Lambda^{\bullet}V$ is canonically embedded in $\Lambda^{\bullet}V_{\mathbb{C}}$ as the subspace invariant under complex conjugation. Further assume that we have an almost complex structure $J : V \rightarrow V$. This gives a direct sum decomposition $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$, which then induces a decomposition

$$\Lambda^k V_{\mathbb{C}} \cong \bigoplus_{p+q=k} \Lambda^p V^{1,0} \otimes_{\mathbb{C}} \Lambda^q V^{0,1}$$

giving the complexified exterior algebra a bigrading via the subspaces

$$\Lambda^{p,q}V := \Lambda^p V^{1,0} \otimes_{\mathbb{C}} \Lambda^q V^{0,1}$$

elements of $\Lambda^{p,q}$ are said to have **bidegree** (p, q) . We now make several simple observations regarding the bigrading of $\Lambda^\bullet V_{\mathbb{C}}$, which stem from our observations regarding the decomposition $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$.

- (1) From the fact that $\overline{V^{1,0}} = V^{0,1}$, we get that $\overline{\Lambda^{p,q}V} = \Lambda^{q,p}V$.
- (2) Given $\omega \in \Lambda^{p,q}V$ and $\eta \in \Lambda^{s,t}V$, we have that $\omega \wedge \eta \in \Lambda^{p+s,q+t}V$.

2. COMPLEX MANIFOLDS

We now transfer our linear algebraic knowledge to complex manifolds. Let $U \subset \mathbb{C}$ be an open subset, and $z = x + iy$ the usual coordinate function on U . Considered as a subset of \mathbb{R}^2 , for any point $p \in U$, the tangent space $T_p U$ is spanned by the coordinate vectors $\partial_x|_p$ and $\partial_y|_p$. If we then complexify, the $\partial_x|_p$ and $\partial_y|_p$ also form a basis for $(T_p U)_{\mathbb{C}}$ as a *complex* vector space. Consequently, we can also form a basis for $(T_p U)_{\mathbb{C}}$ with the vectors

$$\begin{aligned} \frac{\partial}{\partial z} \Big|_p &:= \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \\ \frac{\partial}{\partial \bar{z}} \Big|_p &:= \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \end{aligned}$$

which we abbreviate as $\partial_z|_p$ and $\partial_{\bar{z}}|_p$ respectively. Doing this for all p , we obtain vector fields ∂_z and $\partial_{\bar{z}}$ on all of U , where a function $f \in C^\infty(U, \mathbb{C})$ is holomorphic if and only if $\partial_{\bar{z}}f = 0$. Functions f annihilated by ∂_z are **antiholomorphic**, i.e. \bar{f} is holomorphic. These vector fields give a framing for the complexified tangent bundle $TU_{\mathbb{C}}$, i.e. an isomorphism $TU_{\mathbb{C}} \rightarrow U \times \mathbb{C}^2$.

We get a similar description of the complexified cotangent bundle $T^*U_{\mathbb{C}}$, which is framed by the covector fields

$$\begin{aligned} dz &= \frac{1}{2} (dx + idy) \\ d\bar{z} &= \frac{1}{2} (dx - idy) \end{aligned}$$

which are easily verified to be the duals of ∂_z and $\partial_{\bar{z}}$ respectively. For an open subset $U \subset \mathbb{C}^n$ with complex coordinates $z^i = x^i + iy^i$, we have analogous vector fields $\partial/\partial z^i$ and $\partial/\partial \bar{z}^i$ and covector fields dz^i and $d\bar{z}^i$.

Definition 2.1. An **almost complex manifold** is the data of a smooth manifold X and a smooth section $J \in \Gamma_X(\text{End}(TX))$ such that $J^2 = -\text{id}_{TX}$.

Definition 2.2. A **complex manifold** is a smooth manifold X such that there is an open cover $\{(U_\alpha, \varphi_\alpha)\}$ of X by charts U_α and diffeomorphisms $\varphi : U_\alpha \rightarrow V_\alpha$ to open sets $V_\alpha \subset \mathbb{R}^{2n} \cong \mathbb{C}^n$ such that the transition maps $\varphi_\alpha \circ \varphi_\beta^{-1}$ are holomorphic.

Definition 2.3. Let X be a complex manifold. A complex valued function $f \in C^\infty(X, \mathbb{C})$ is *holomorphic* if for any holomorphic chart (φ, U) , we have that $f \circ \varphi^{-1}$ is holomorphic. The ring of holomorphic functions over X is denoted \mathcal{O}_X .

Unlike the linear case, the notion of an almost complex manifold and a complex manifold are *not* equivalent. Let (X, J) be an almost complex manifold. For a point $x \in X$, we have that J_x is an almost complex structure on the tangent space $T_x X$, which gives a decomposition of the complexified tangent space

$$T_x X_{\mathbb{C}} = T_x^{1,0} X \oplus T_x^{0,1} X$$

into the $\pm i$ -eigenspaces of complexified J_x . Doing this over all points x , we get a decomposition of the complexified tangent bundle

$$TX_{\mathbb{C}} = T^{1,0} X \oplus T^{0,1} X$$

into the $\pm i$ -eigenbundles of J .

Suppose X is a complex manifold. Then every tangent space $T_x X$ inherits an almost complex structure coming from multiplication by i in \mathbb{C}^n , giving an almost complex structure J on X . To show this explicitly, let $z^i = x^i + iy^i$ be local holomorphic coordinates for X . Then the coordinate functions $(x^1, y^1, \dots, x^n, y^n)$ form a coordinate system for X as a *real* manifold, and the action of J in these coordinates is simply given by

$$\begin{aligned} \frac{\partial}{\partial x^i} &\mapsto \frac{\partial}{\partial y^i} \\ \frac{\partial}{\partial y^i} &\mapsto -\frac{\partial}{\partial x^i} \end{aligned}$$

and the subbundle $T^{1,0} X \subset TX_{\mathbb{C}}$ is spanned by the $\partial/\partial z^i$, and $T^{0,1} X$ is spanned by the $\partial/\partial \bar{z}^i$. The composition

$$TX \hookrightarrow TX_{\mathbb{C}} \longrightarrow T^{1,0} X$$

consisting of the inclusion followed by projecting then defines an isomorphism of the real tangent bundle with $T^{1,0} X$, which is called the *holomorphic tangent bundle*. Explicitly, this map is given in local coordinates by

$$\begin{aligned} \frac{\partial}{\partial x^i} &= \frac{\partial}{\partial z^i} + \frac{\partial}{\partial \bar{z}^i} \mapsto \frac{\partial}{\partial z^i} \\ \frac{\partial}{\partial y^i} &= i \frac{\partial}{\partial z^i} - i \frac{\partial}{\partial \bar{z}^i} \mapsto i \frac{\partial}{\partial z^i} \end{aligned}$$

and the $\partial/\partial z^i$ along with the $i\partial/\partial z^i$ form a local frame for the holomorphic tangent bundle $T^{1,0} X$.

Given an almost complex manifold (X, J) , a natural question to ask is whether or not X admits the structure of a complex manifold where J is multiplication by i . The answer to this question is a celebrated theorem of Newlander and Nirenberg

Theorem 2.4 (Newlander-Nirenberg). *An almost complex manifold (X, J) is a complex manifold (where J is multiplication by i) if and only if the subbundle $T^{1,0}X \subset TX_{\mathbb{C}}$ is integrable in the sense of Frobenius, i.e.*

$$[T^{1,0}X, T^{1,0}X] \subset T^{1,0}X$$

via complex conjugation, this is equivalent to $T^{0,1}X$ being integrable.

Intuitively, the existence of local holomorphic coordinates amounts to finding an integral submanifold for $T^{1,0}X$, since if X was complex, $T^{1,0}X$ would arise as the holomorphic tangent bundle of X .

3. COHOMOLOGY OF COMPLEX MANIFOLDS.

Let X be a complex manifold, and let J denote the canonical almost complex structure on X . Applying our linear algebraic knowledge pointwise, we get a decomposition of the complexified cotangent bundle

$$T^*X_{\mathbb{C}} = (T^*X)^{1,0} \oplus (T^*X)^{0,1}$$

which then gives a decomposition of the space $\Omega_X^k(\mathbb{C})$ of complex valued differential k -forms as

$$\Omega_X^k(\mathbb{C}) = \bigoplus_{p+q=k} \Omega_X^{p,q}(\mathbb{C})$$

where $\Omega_X^{p,q}(\mathbb{C})$ denotes the space of (p, q) -forms, i.e. sections of $\Lambda^{p,q}T^*X$, and the direct sum is taken as modules over $C^\infty(X, \mathbb{C})$. Out of all the $\Lambda^{p,q}T^*X$, there is a distinguished bundle.

Definition 3.1. Let X be a complex manifold of complex dimension $\dim_{\mathbb{C}} X = n$ (i.e. $\dim X = 2n$). The *canonical bundle* of X , denote K_X , is the complex vector bundle

$$K_X = \Lambda^{n,0}T^*X$$

Let d denote the de Rham differential $\Omega_X^k \rightarrow \Omega_X^{k+1}$ after complexifying it to obtain a map $\Omega_X^k(\mathbb{C}) \rightarrow \Omega_X^{k+1}(\mathbb{C})$. In local holomorphic coordinates $\{z^i\}$, we have that the dz^i form a basis for the space of smooth $(1, 0)$ -forms over the ring $C^\infty(M, \mathbb{C})$, and the $d\bar{z}^i$ form a basis for the space of smooth $(0, 1)$ -forms. Therefore, given $f \in C^\infty(X, \mathbb{C})$, we can uniquely write $df \in \Omega_X^1(\mathbb{C})$ as

$$df = \frac{\partial f}{\partial z^i} dz^i + \frac{\partial f}{\partial \bar{z}^i} d\bar{z}^i$$

where we use Einstein summation notation. Given an arbitrary (p, q) form α , it has the local coordinate expression

$$\alpha = \sum_{p,q} \alpha_{p,q} dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q}$$

for smooth $\alpha_{p,q} \in C^\infty(X, \mathbb{C})$. Then we have that

$$\begin{aligned} d\alpha = & \left(\sum_{p,q} \frac{\partial \alpha_{p,q}}{\partial z^i} dz^i \wedge dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{i_1} \dots \wedge d\bar{z}^{i_q} \right) \\ & + \left(\sum_{p,q} \frac{\partial \alpha_{p,q}}{\partial \bar{z}^i} d\bar{z}^i \wedge dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{i_1} \dots \wedge d\bar{z}^{i_q} \right) \end{aligned}$$

which shows that $d\alpha$ is a $(p+1, q+1)$ form, i.e. d is of bidegree $(1, 1)$. This expression for $d\alpha$ allows us to decompose d as a sum $d = \partial + \bar{\partial}$, where

$$\begin{aligned} \partial\alpha &= \sum_{p,q} \frac{\partial \alpha_{p,q}}{\partial z^i} dz^i \wedge dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{i_1} \dots \wedge d\bar{z}^{i_q} \\ \bar{\partial}\alpha &= \sum_{p,q} \frac{\partial \alpha_{p,q}}{\partial \bar{z}^i} d\bar{z}^i \wedge dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{i_1} \dots \wedge d\bar{z}^{i_q} \end{aligned}$$

from the definition, it is clear that ∂ is of bidegree $(1, 0)$ and $\bar{\partial}$ is of bidegree $(0, 1)$.

Proposition 3.2.

- (1) $\partial^2 = 0$
- (2) $\bar{\partial}^2 = 0$
- (3) $\partial\bar{\partial} + \bar{\partial}\partial = 0$

Proof. We know that $d = \partial + \bar{\partial}$ satisfies $d^2 = 0$. Therefore, we have that

$$d^2 = (\partial + \bar{\partial})^2 = \partial^2 + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\partial}^2$$

we then note that the bidegrees of all the components. We have:

- (1) ∂^2 has bidegree $(2, 0)$.
- (2) $\partial\bar{\partial} + \bar{\partial}\partial$ has bidegree $(1, 1)$
- (3) $\bar{\partial}^2$ has bidegree $(0, 2)$

in order for d^2 to vanish, all the terms of different bidegrees must vanish, proving all three parts of the proposition. ■

The operators ∂ and $\bar{\partial}$ satisfy many of the same identities as d . Another helpful identity to note is that

$$\partial\alpha = \overline{\bar{\partial}\bar{\alpha}}$$

Definition 3.3. Let X be a complex manifold. A *holomorphic vector bundle* over X is a complex vector bundle $\pi : E \rightarrow X$ such that the total space E is a complex manifold and the projection π is a holomorphic map.

Example 3.4. For a complex manifold X , the complex vector bundles $\Lambda^{q,0}T^*X$ are all holomorphic vector bundles, which is easily seen using local trivializations defined by local holomorphic coordinates.

Note again that a holomorphic vector bundle is *not* a complex vector bundle (for example, a complex vector bundle can be odd dimensional if the base space is odd dimensional).

Instead, it is a complex bundle with extra structure. Let $\sigma : X \rightarrow E$ be a *smooth* section. In a local holomorphic trivialization of E , we can write σ as

$$\sigma = (\sigma^1, \dots, \sigma^k)$$

for smooth complex-valued functions σ^i , where k is the rank of E as a complex vector bundle. In this trivialization, we can apply the operator $\bar{\partial}$ component-wise to get

$$\bar{\partial}\sigma = (\bar{\partial}\sigma^1, \dots, \bar{\partial}\sigma^k)$$

and this local definition glues together to give a well defined operator $\bar{\partial}_E : \Omega_X^{0,0}(E) \rightarrow \Omega_X^{0,1}(E)$, where $\Omega_X^{p,q}(E)$ denotes the space of sections $\Gamma_X(T^{p,q}X \otimes_{\mathbb{C}} E)$. We then note that the holomorphic sections $X \rightarrow E$ are exactly those annihilated by $\bar{\partial}_E$. Since $\bar{\partial}_E$ is defined locally in terms of $\bar{\partial}$, we immediately see that $\bar{\partial}_E^2 = 0$, giving us the **Dolbeault complex** of E .

$$\Omega_X^{0,0}(E) \xrightarrow{\bar{\partial}_E} \Omega_X^{0,1}(E) \xrightarrow{\bar{\partial}_E} \dots \xrightarrow{\bar{\partial}_E} \Omega_X^{0,n}(E) \xrightarrow{\bar{\partial}_E} 0$$

The **Dolbeault cohomology groups** of E are the cohomology groups of the Dolbeault complex, and are denoted $H^p(X, E)$. In the case $E = \Lambda^{0,q}T^*X$, we have that $\bar{\partial}_E = \bar{\partial}$, and we denote these cohomology groups by

$$H^{p,q}(X) = H^p(X, \Lambda^{0,q}T^*X)$$

4. KÄHLER MANIFOLDS

Kähler manifolds will form a special class of manifolds where Hodge theory will prove especially useful. Before diving into the world of Kähler manifolds, we do some prerequisite linear algebra, which is closely tied to Hermitian geometry.

Definition 4.1. Let V be a complex vector space. A **Hermitian form** on V is a bilinear map $h : \bar{V} \otimes_{\mathbb{C}} V \rightarrow \mathbb{C}$ satisfying $h(v, w) = \overline{h(w, v)}$.

Another way to define a Hermitian form is as a **sesquilinear** form $h : V \otimes_{\mathbb{C}} V \rightarrow \mathbb{C}$, i.e. conjugate linear in the first term, and linear in the second term, along with the Hermitian symmetry condition $h(v, w) = \overline{h(w, v)}$. In a basis for V , a Hermitian form is given by a Hermitian matrix H , i.e. $H^\dagger = H$.

Given a Hermitian form h on a complex vector space V , we can decompose h into its real and imaginary parts. Let $g = \Re(h)$ and $\omega = \Im(h)$, so we have $h = g + i\omega$.

Proposition 4.2. Let $h = g + i\omega$ be a hermitian form. Then we have

- (1) g is symmetric
- (2) ω is skew-symmetric

Proof.

- (1) Since h is Hermitian, we have that $h(v, w) = \overline{h(w, v)}$, which have the same real part. Therefore, $g(v, w) = g(w, v)$, so g is symmetric
- (2) We again use the fact that $h(v, w) = \overline{h(w, v)}$. Since conjugation reverses the sign on the imaginary component, we immediately get that $\omega(v, w) = -\omega(w, v)$.

■

Remark. The decomposition of h into its real and imaginary components is one way of seeing the useful identity

$$U_n = O_{2n} \cap Sp_{2n} \cap GL_n \mathbb{C}$$

where we regard all of the groups as subgroups of $GL_{2n} \mathbb{R}$. Any matrix preserving the standard Hermitian form on \mathbb{C}^n must necessarily preserve both the real and imaginary components, so it must preserve the standard Euclidean inner product on $\mathbb{R}^{2n} \cong \mathbb{C}^n$ and the standard symplectic form on $\mathbb{R}^{2n} \cong \mathbb{C}$.

Given a complex vector space V equipped with a hermitian metric h , let J denote the natural almost complex structure induced by multiplication by i . Writing $h = g + i\omega$, we see that ω and g are intimately related, i.e.

- (1) $g(v, Iw) = \omega(v, w)$
- (2) $\omega(v, Iw) = g(v, w)$

both of which come from the fact that h is compatible with J in the sense that $h(Jv, Jw) = h(v, w)$. Finally, we note that ω can be naturally identified with a real element of $(V^*)^{1,1}$.

Definition 4.3. Let h be a Hermitian form, and $\omega = \Im(h)$. Then ω is called the *Kähler form* of h .

We now move back to the world of manifolds. Let (X, J) be an almost complex manifold. The almost complex structure gives each tangent space $T_x X$ the structure of a complex vector space, so it makes sense to discuss a Hermitian form h_x on $T_x X$.

Definition 4.4. A *Kähler manifold* is an almost complex manifold (X, J) with a smoothly varying Hermitian metric (i.e. a positive definite Hermitian form) h where J is integrable and the Kähler form ω obtained from h is closed.

From the decomposition $h = g + i\omega$, we see that a Kähler manifold is a melting pot of several compatible structures, namely the Riemannian metric g , the almost complex structure J , and the symplectic form ω .

From the basic definition, we can already make some surprisingly deep statements about Kähler manifolds.

Proposition 4.5. Let X be a Kähler manifold with Hermitian form $h = g + i\omega$. Then the volume form of X with respect to the Riemannian metric g is $\omega^n / n!$, where $n = \dim_{\mathbb{C}} X$.

Proof. It suffices to verify this pointwise for a fixed point $x \in X$. The Riemannian volume form dV_g is uniquely characterized by the fact that $(dV_g)_x(e_1, \dots, e_{2n}) = 1$ for an oriented orthonormal basis $\{e_1, \dots, e_{2n}\}$ for $T_x X$. Arrange local holomorphic coordinates $\{z^i\}$ about x such that at x , the Hermitian form h_x is given by the identity matrix, i.e.

$$h_x \left(\left. \frac{\partial}{\partial z^i} \right|_x, \left. \frac{\partial}{\partial z^j} \right|_x \right) = \delta_{ij}$$

Let ∂_i denote $\partial/\partial z^i|_x$. We then have that $\{\partial_1, i\partial_1, \dots, \partial_n, i\partial_n\}$ is an oriented orthonormal \mathbb{R} -basis for $T_x X$ with respect to g , so it suffices to evaluate ω_x^n on this basis, and verify that it evaluates to $n!$. To do this, we use a lemma.

Lemma 4.6. *In local holomorphic coordinates $\{z^i\}$, the Kähler form ω is given by*

$$\omega = \frac{i}{2} \sum_i H_j^i dz^i \wedge d\bar{z}^j$$

where $H_j^i = h(\partial_i, \partial_j)$.

The proof is an easy consequence of ω being of type $(1,1)$ and the fact that it is the imaginary component of h .

In the coordinates we specified above, we have that H_x is the identity matrix, so the lemma gives us

$$\omega_x = \frac{i}{2} \sum_i dz^i \wedge d\bar{z}^i$$

we note that when we wedge ω_x with itself n times, we constantly pick up duplicates of the dz^i and $d\bar{z}^i$, so only one term survives, leaving us with

$$\omega_x^n = \frac{i}{2} \left(dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \right) = dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n = \frac{dV_g}{n!}$$

where $z^i = x^i + iy^i$. We use the convention that $v \wedge w = \text{Alt}(v \otimes w)$ without the normalizing factor, which explains the extra term of $n!$. ■

Proposition 4.7. *Let X be a Kähler manifold. Then all the even degree de Rham cohomology groups $H_{dR}^{2n}(X, \mathbb{R})$ are nontrivial.*

Proof. The fact that $\omega^n = dV_g/n!$ implies that $\int_M \omega^n = \int_M dV_g = \text{Vol}(M) > 0$, so by Stokes' theorem, ω^n cannot be exact. From this, we can also conclude that ω^k for any k cannot be exact, for if $\omega^k = d\alpha$, then we would have that

$$\omega^n = \omega^{n-k} \wedge d\alpha = d(\omega^{n-k} \wedge \alpha)$$

Therefore, since ω is a degree 2 form, we must have that all the even dimensional cohomology groups are nontrivial. ■

This shows that the Kähler condition is quite rigid. A special property of Kähler manifolds is that the Hermitian form h is constant up to second order terms, which is an important ingredient in the Kähler identities. Because of this, many of the local computations on Kähler manifolds reduce to computation on the standard Hermitian metric on \mathbb{C}^n .

5. LAPLACIANS AND HARMONIC FORMS

We start in the most general case, that of a compact Riemannian manifold (X, g) . For the remainder of these notes, all manifolds will be compact and boundaryless, unless denoted otherwise.

Definition 5.1. Let (X, g) be a Riemannian manifold. Then the vector bundles $\Lambda^k T^*X$ inherit natural fiber metrics (\cdot, \cdot) , which are defined by the set

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} : 1 \leq i_1 < i_2 < \cdots < i_k\}$$

being an orthonormal basis for $\Lambda^k T_x^* X$ where $\{e_i\}$ is an orthonormal basis for $T_x^* X$. This then introduces an L^2 **inner product** $(\cdot, \cdot)_{L^2}$ on the space Ω_X^k of differential k -forms, where we define

$$(\alpha, \beta)_{L^2} = \int_X (\alpha, \beta) dV_g$$

where (α, β) denotes the smooth function $x \mapsto (\alpha_x, \beta_x)$.

Definition 5.2. The *Hodge star* operator, denoted \star (though many choose to denote it $*$), is a bundle homomorphism $\star : \Lambda^k T^* X \rightarrow \Lambda^{n-k} T^* X$ defined by the property such that for any $\alpha, \beta \in \Lambda^k T^* X$

$$\alpha \wedge \star \beta = (\alpha, \beta) dV_g$$

this property uniquely characterizes \star .

Proposition 5.3. \star exists.

Proof. The defining property uniquely characterizes \star , so it suffices to specify a local definition, which will then glue to a smooth bundle homomorphism. Let E_i be a local orthonormal frame for $T^* X$. Then we know that the set of k -fold wedge products of the E_i with increasing indices forms a basis for each $\Lambda^k T_x^* X$. In addition, we know that $E_1 \wedge \cdots \wedge E_n = dV_g$. Putting this all together, We must have that for $E_I = E_{i_1} \wedge \cdots \wedge E_{i_k}$, $\star E_I = E_J$, with $J = \{j_{i_1}, \dots, j_{i_{n-k}}\}$ where the tuple $(i_1, \dots, i_k, j_1, \dots, j_k)$ is an even permutation of $(1, \dots, n)$. We have then specified \star on an orthonormal basis. ■

One thing to note is that \star is an isomorphism, since it maps orthonormal frames to orthonormal frames. It also satisfies the useful identity

$$\star^2 \alpha = (-1)^{k(n-k)} \alpha$$

where $\alpha \in \Omega_X^k$.

Complexification allows us to obtain analogous operations on complex valued forms on an almost complex manifold (X, J) , where we extend g and its induced fiber metrics on the $\Lambda^k T^* X$ to a Hermitian metric h on $TX_{\mathbb{C}}$ and fiber metrics on $\Lambda^k T^* X_{\mathbb{C}}$. Likewise, we complexify the Hodge star \star , which gives the identity

$$(\alpha, \beta) dV_g = \alpha \wedge \overline{\star \beta}$$

We then get an analogous L^2 metric on complex valued forms, defined by

$$(\alpha, \beta)_{L^2} = \int_X \alpha \wedge \overline{\star \beta} dV_g$$

The Hodge star operators allows us to define formal adjoints to d , ∂ , and $\bar{\partial}$ with respect to the L^2 metrics.

Definition 5.4. Define the operators

- (1) $d^* : \Omega_X^k \rightarrow \Omega_X^{k-1}$
- (2) $\partial^* : \Omega_X^{p,q}(\mathbb{C}) \rightarrow \Omega_X^{p-1,q}(\mathbb{C})$
- (3) $\bar{\partial}^* : \Omega_X^{p,q}(\mathbb{C}) \rightarrow \Omega_X^{p,q-1}(\mathbb{C})$

by the formulas

$$(1) d^* = (-1)^k \star^{-1} d \star$$

$$(2) \partial^* = -\star \bar{\partial} \star$$

$$(3) \bar{\partial}^* = -\star \partial \star$$

Proposition 5.5. *The operators d^* , ∂^* , and $\bar{\partial}^*$ are formally adjoints to their respective operators with respect to the appropriate L^2 metrics, i.e.*

$$(d\alpha, \beta)_{L^2} = (\alpha, d^* \beta)_{L^2}$$

$$(\partial\alpha, \beta)_{L^2} = (\alpha, \partial^* \beta)_{L^2}$$

$$(\bar{\partial}\alpha, \beta)_{L^2} = (\alpha, \bar{\partial}^* \beta)_{L^2}$$

Proof. We prove the case for d and $\bar{\partial}^*$. The proof for ∂^* is near identical to the one for $\bar{\partial}^*$. We know that the exterior derivative is an **anti-derivation** i.e. it satisfies

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k d\beta$$

where $k = |\alpha|$ denotes the degree of α . In particular, this implies that for a k -form α and a $k+1$ -form β , we have that

$$d(\alpha \wedge \star\beta) = d\alpha \wedge \star\beta + (-1)^k \alpha \wedge d\star\beta$$

From Stokes' theorem, we have that

$$\int_X d(\alpha \wedge \star\beta) = \int_{\partial X} \alpha \wedge \star\beta = 0$$

since $\partial X = \emptyset$. Therefore, we have that

$$\int_X d\alpha \wedge \star\beta = \int_X (-1)^{k+1} \alpha \wedge d\star\beta$$

we further note that the left hand side is the definition of $(d\alpha, \beta)_{L^2}$. We also compute

$$\begin{aligned} (\alpha, d^* \beta)_{L^2} &= \int_X \alpha \wedge \star d^* \beta \\ &= \int_X \alpha \wedge ((-1)^{k+1} \star \star^{-1} d\star\beta) \\ &= \int_X (-1)^{k+1} \alpha \wedge d\star\beta \end{aligned}$$

proving the desired equality.

For $\bar{\partial}^*$, we have a similar proof. Let α be a complex k -form, and β a complex $k+1$ -form. Then we have

$$(\bar{\partial}\alpha, \beta)_{L^2} = \int_X \alpha \wedge \star \bar{\beta}$$

again, we have that

$$\bar{\partial}(\alpha \wedge \star \bar{\beta}) = \bar{\partial}\alpha \wedge \star \bar{\beta} + (-1)^k \bar{\partial}(\star \bar{\beta})$$

the integral of the left hand side over all of X is 0 as a consequence of Stokes' theorem, which gives us

$$\int_X \bar{\partial}\alpha \wedge \star \bar{\beta} = \int_X (-1)^{k+1} \alpha \wedge \bar{\partial}(\star \bar{\beta}) = \int_X (-1)^{k+1} \alpha \wedge \bar{\partial} \star \bar{\beta}$$

The left hand side is $(\bar{\partial}\alpha, \beta)_{L^2}$. We claim (as was the case with d^*), that the right hand side is equal to $(\alpha, \bar{\partial}^* \beta)$. We compute

$$\begin{aligned} (\alpha, \bar{\partial}^* \beta)_{L^2} &= \int_X \alpha \wedge \overline{\star \bar{\partial}^* \beta} \\ &= \int_X \alpha \wedge \overline{\star(-\star \bar{\partial} \star \beta)} \\ &= - \int_X \alpha \wedge \overline{\star^2 \bar{\partial} \star \beta} \\ &= (-1)^{k(n-k)+1} \int_X \alpha \wedge \overline{\bar{\partial} \star \beta} \end{aligned}$$

We note that since X is a complex manifold, n is even, so this is equal to

$$(-1)^{k^2+1} \int_X \alpha \wedge \overline{\bar{\partial} \star \beta} = (-1)^{k+1} \int_X \alpha \wedge \overline{\bar{\partial} \star \beta} = (\bar{\partial}\alpha, \beta)_{L^2}$$

where we use the fact that k and k^2 have the same parity. ■

A similar story holds for the $\bar{\partial}$ operator on a holomorphic vector bundle $E \rightarrow X$. Suppose $\dim_{\mathbb{C}} X = n$, so $\dim X = 2n$, and let $E \rightarrow X$ be a holomorphic vector bundle equipped with a Hermitian metric. Let ω denote the Kähler form of X . Then $\omega^n \in \Omega_X^{n,n}$ determines a trivialization $\Lambda^{n,n} T^* X \rightarrow X \times \mathbb{C}$, which introduces a pairing

$$\begin{aligned} \Lambda^{0,q} T^* X \otimes \Lambda^{n,n-q} T^* X &\rightarrow \mathbb{C} \\ \alpha \otimes \beta &\mapsto \lambda \end{aligned}$$

where $\lambda \in \mathbb{C}$ is the unique number where $\alpha \wedge \beta = \lambda \omega^n$. This pairing is nondegenerate, which establishes the duality

$$\Lambda^{0,q} T^* X \cong (\Lambda^{n,n-q} T^* X)^*$$

tensoring with the holomorphic vector bundle E , we get the duality

$$\Lambda^{0,q} T^* X \otimes E \cong \Lambda^{n,n-q} T^* X \otimes E^*$$

Furthermore, the Hermitian metrics on both E and X gives an antilinear isomorphism

$$\Lambda^{0,q} T^* X \otimes E \rightarrow (\Lambda^{0,q} T^* X \otimes E)^*$$

putting these together, we get an antilinear isomorphism

$$\Lambda^{0,q} T^* X \otimes E \rightarrow (\Lambda^{0,q} T^* X \otimes E)^* \rightarrow \Lambda^{n,n-q} T^* X$$

which we denote \star_E . As the notation suggests, \star_E is the analogue of the Hodge star operator. Let $\langle \cdot, \cdot \rangle$ denote the Hermitian metric on E . Then given E -valued forms $\alpha, \beta \in \Omega_X^{0,q}(E)$, we get a smooth function $X \rightarrow \mathbb{C}$ where $x \mapsto \langle \alpha_x, \beta_x \rangle$. The Hermitian metric on X induces a Riemannian metric, which then gives a volume form $dV_g \in \Omega_X^{2n}$. This gives an L_2 metric $(\cdot, \cdot)_{L^2}$ on $\Omega_X^{0,q}(E)$ defined by

$$(\alpha, \beta)_{L^2} = \int_X \langle \alpha, \beta \rangle dV_g$$

as before, this inner product is related to \star_E via the identity

$$(\alpha, \beta)_{L^2} = \int_X \alpha \wedge \star_E \beta$$

We first clarify the notation $\alpha \wedge \star_E \beta$. Given $\beta \in \Omega_X^{0,q}(E)$, we have that $\star_E \beta \in \Omega_X^{n,n-q}(E^*)$. Wedging with α gives us a form in $\Omega_X^{n,n}(E \otimes E^*)$, and contracting E with E^* gives us an element of $\Omega_X^{n,n}$, which we can integrate over X .

We then use \star_E to define an adjoint to $\bar{\partial}_E$.

Definition 5.6. Define $\bar{\partial}_E^* : \Omega_X^{0,q}(E) \rightarrow \Omega_X^{0,q-1}(E)$ by

$$(-1)^q \star_E^{-1} \bar{\partial}_{K_X \otimes E^*} \star_E$$

Proposition 5.7. $\bar{\partial}_E^*$ is the formal adjoint of $\bar{\partial}_E$ with respect to $(\cdot, \cdot)_{L^2}$.

The proof is very similar to the ones we performed above.

With these adjoint operators, we construct Laplacian operators.

Definition 5.8. Define the Laplacians Δ , Δ_{∂} , $\Delta_{\bar{\partial}}$, and Δ_E by

$$\begin{aligned} \Delta &= dd^* + d^*d \\ \Delta_{\partial} &= \partial\bar{\partial}^* + \bar{\partial}^*\partial \\ \Delta_{\bar{\partial}} &= \bar{\partial}\partial^* + \partial^*\bar{\partial} \\ \Delta_E &= \bar{\partial}_E\bar{\partial}_E^* + \bar{\partial}_E^*\bar{\partial}_E \end{aligned}$$

Forms that are annihilated by their respective Laplacian operator are said to be *harmonic*. The subspace of harmonic forms is denoted

- (1) $\mathcal{H}^k(X)$ for Δ .
- (2) $\mathcal{H}^{q,0}(X)$ for Δ_{∂} .
- (3) $\mathcal{H}^{0,q}(X)$ for $\Delta_{\bar{\partial}}$.
- (4) $\mathcal{H}^{0,q}(X, E)$ for Δ_E .

We first prove some helpful properties of these Laplacian operators

Proposition 5.9.

$$(\alpha, \Delta\beta)_{L^2} = (d\alpha, d\beta)_{L^2} + (d^*\alpha, d^*\beta)_{L^2}$$

and the analogous statement for all the other Laplacians.

Proof. We compute

$$\begin{aligned} (\alpha, \Delta\beta)_{L^2} &= (\alpha, dd^*\beta + d^*d\beta)_{L^2} \\ &= (\alpha, dd^*\beta)_{L^2} + (\alpha, d^*d\beta)_{L^2} \\ &= (d^*\alpha, d^*\beta) + (d\alpha, d\beta) \end{aligned}$$

where the last equality comes from the fact that d and d^* are adjoints. The proof is the same for all the other Laplacians. ■

Proposition 5.10. A k -form α is harmonic if and only if $d\alpha = d^*\alpha = 0$, and the analogous statements for ∂ , $\bar{\partial}$, and $\bar{\partial}_E$.

Proof. This is an immediate corollary of the previous proposition, along with the fact that $(\cdot, \cdot)_{L^2}$ is positive definite. ■

Corollary 5.11. *The Laplacians are formally self-adjoint, i.e.*

$$(\Delta\alpha, \beta)_{L^2} = (\alpha, \Delta\beta)_{L^2}$$

and a similar statement for the other Laplacians.

We will use the following properties of the Laplacian operators, but will not prove them.

Proposition 5.12.

- (1) *The Laplacian operators defined above are elliptic.*
- (2) *For an elliptic differential operator $D : E \rightarrow F$ between vector bundles equipped with fiber metrics of the same rank over a compact manifold X , the kernel of D is finite dimensional, the image of D is closed and of finite codimension, and there is an orthogonal decomposition*

$$\Gamma_X(E) = \ker D \oplus \operatorname{Im} D^*$$

where $D^ : F \rightarrow E$ is the adjoint of D with respect to the L^2 metrics on E and F*

6. THE HODGE DECOMPOSITION

The Hodge decomposition is an immediate corollary of the above proposition and the fact that the Laplacian is self-adjoint.

Corollary 6.1 (The Hodge decomposition). *For a compact Riemannian manifold (X, g) , there is an orthogonal decomposition*

$$\Omega_X^k = \mathcal{H}^k(X) \oplus \operatorname{Im} \Delta$$

Corollary 6.2. *The canonical map $\mathcal{H}^k(X) \rightarrow H_{dR}^k(X)$ mapping a harmonic form α to its de Rham cohomology class is an isomorphism*

Proof. Let $\beta \in \Omega_X^k$ be a closed form. By the Hodge decomposition, we can write

$$\begin{aligned} \beta &= \alpha + \Delta\eta \\ &= \alpha + dd^*\eta + d^*d\eta \end{aligned}$$

with $\alpha \in \mathcal{H}^k(X)$ and some $\eta \in \Omega_X^k$. Since α is harmonic, we have that $d\alpha = 0$, and we also have that $ddd^*\eta = 0$ since $d^2 = 0$, so for β to be closed, we must have $dd^*d\eta = 0$ as well. Therefore, $[\beta] = [\alpha]$, so the map $\alpha \mapsto [\alpha]$ is surjective. For injectivity, let $\alpha \in \mathcal{H}^k(X)$ with $[\alpha] = 0$, i.e. α is exact. Since α is harmonic, we have that $d\alpha = d^*\alpha = 0$. Since the kernel of d^* is orthogonal to the image of d (which is a general fact about adjoint linear transformations), we conclude that $\alpha = 0$, so the map is injective, and therefore an isomorphism. ■

As expected, we have a similar situation for the Δ_E -harmonic forms and the sheaf cohomology $H^q(X, E)$ for a holomorphic vector bundle E , which is naturally identified with the Dolbeault cohomology groups $H^{0,q}(X, E)$. A near identical proof gives us that the map $\mathcal{H}^{0,q}(X, E) \rightarrow H^q(X, E)$ is an isomorphism.

The Hodge decomposition gives us easy proofs of various duality theorems.

Theorem 6.3 (Poincaré Duality). *For a compact manifold X , the pairing $H_{dR}^k(X) \otimes H_{dR}^{n-k}(X) \rightarrow \mathbb{R}$ defined by*

$$\langle [\alpha], [\beta] \rangle = \int_X \alpha \wedge \beta$$

is nondegenerate, giving an isomorphism $H_{dR}^k(X) \cong (H_{dR}^{n-k}(X))^*$.

Proof. Fix a Riemannian metric g for X , which allows us to construct the Laplacian Δ and the spaces $\mathcal{H}^k(X)$, which are isomorphic to the de Rham cohomology groups $H_{dR}^k(X)$. We use a lemma, which is a result from a simple (but somewhat tedious) computation.

Lemma 6.4. *The Hodge star \star commutes with Δ .*

In particular, the lemma implies that if $\alpha \neq 0$ is harmonic, then $\star\alpha$ is also a nonzero harmonic form. Therefore, it suffices to show that $\langle [\alpha], [\star\alpha] \rangle \neq 0$ for $\alpha \neq 0$. But

$$\langle [\alpha], [\star\alpha] \rangle = \int_X \alpha \wedge \star\alpha = (\alpha, \alpha)_{L^2} \neq 0$$

■

A little extra work gets the holomorphic vector bundle analogue, which is *Serre duality*. Given a holomorphic vector bundle $E \rightarrow X$, there is a natural pairing

$$H^q(X, E) \otimes H^{n-q}(X, E^* \otimes K_X) \rightarrow H^q(X, K_X) \rightarrow \mathbb{C}$$

where the first map contracts the E and E^* factors, and the second map is given by integration over X . Once we make the proper identification of this pairing with Dolbeault cohomology, we can use the same technique we used for proving Poincaré duality to prove that this pairing is nondegenerate. In Dolbeault cohomology, let $\alpha \in H^{0,q}(X, E)$ and $\beta \in H^{0,n-q}(X, E^* \otimes K_X)$. Then we can take the exterior product $\alpha \wedge \beta \in H^{0,n}(X, E \otimes E^* \otimes K_X)$, which is obtained by class of the product of two representatives. Contracting the E and E^* factors gives us an element of $H^{0,n}(X, K_X)$, which is canonically identified with $H^{2n}(X, \mathbb{C}) = H^{n,n}(X)$. Integration then gives the pairing.

Theorem 6.5 (Serre duality). *The pairing*

$$H^q(X, E) \otimes H^{n-q}(X, E^* \otimes K_X) \rightarrow \mathbb{C}$$

defined above is nondegenerate.

Proof. The proof is similar to the one of Poincaré duality. From the discussion above, we prove the pairing with Dolbeault cohomology. Fix Hermitian metrics on E and X , giving us the operator Δ_E and the spaces $\mathcal{H}^{0,q}(X, E)$ of harmonic forms, which are isomorphic to the cohomology groups $H^{0,q}(X, E) = H^q(X, E)$. As before, we use (without proof) that \star_E commutes with Δ_E to deduce that for α harmonic, $\star_E\alpha$ is also harmonic. Therefore, it again suffices to show that for α harmonic and nonzero, the pairing $\langle [\alpha], [\star\alpha] \rangle$ is nonzero. However, the pairing is just $(\alpha, \star\alpha)_{L^2}$, which is nonzero since α is. ■

7. HODGE THEORY ON KÄHLER MANIFOLDS

We now restrict to the case where X is a compact Kähler manifold.

Definition 7.1. Let X be a compact Kähler manifold with Kähler form $\omega \in \Omega_X^2(\mathbb{C})$. Then define the *Lefschetz operator* $L : \Omega_X^k(\mathbb{C}) \rightarrow \Omega_X^{k+2}(\mathbb{C})$ by

$$L\alpha = \omega \wedge \alpha$$

Let $\Lambda : \Omega_X^k(\mathbb{C}) \rightarrow \Omega_X^{k-2}(\mathbb{C})$ denote the adjoint to L with respect to the metric on each fiber, i.e. for all $x \in X$ we have that

$$\langle L_x \alpha_x, \beta_x \rangle_x = \langle \alpha_x, \Lambda_x \beta_x \rangle_x$$

Proposition 7.2.

$$\Lambda = \star^{-1} L \star = (-1)^k \star L \star$$

Proof. Let dV_g be the volume form for the Riemannian metric induced by the Hermitian form on X . Then by definition, we have that

$$\langle L\alpha, \beta \rangle dV_g = \langle \alpha, \Lambda\beta \rangle dV_g$$

Expanding this out, we get

$$\omega \wedge \alpha \wedge \star \beta = \alpha \wedge \star \Lambda \beta$$

Then we have that

$$\alpha \wedge \star \star^{-1} L \star \beta = \alpha \wedge L \star \beta = \alpha \wedge \omega \star \beta = \omega \wedge \alpha \wedge \star \beta$$

where the last equality comes from the fact that ω is of even degree. Then by uniqueness of the adjoint, we conclude that Λ is of the desired form. ■

The operator Λ satisfies very important commutation relations with the operators ∂ and $\bar{\partial}$.

Theorem 7.3 (The Kähler Identities).

$$[\Lambda, \partial] = i\bar{\partial}^* \quad [\Lambda, \bar{\partial}] = -i\partial^*$$

Proof. Since the Kahler metric is constant to second order, and the operators in question only use the metric to first order, it suffices to prove the Kähler identities for the standard Hermitian metric on \mathbb{C}^n . The two equalities are equivalent by passing to the complex conjugate, so it suffices to verify one of them, which we choose to be $[\Lambda, \bar{\partial}] = -i\partial^*$. Since the Hodge star \star and Λ are both 0-th order operators, and ∂ and $\bar{\partial}$ are first order, the operators $[\Lambda, \bar{\partial}]$ and $-i\partial^*$ are both first order operators. Furthermore, both operators annihilate constant coefficient forms, so the 0th order terms are trivial. Therefore, it suffices to verify the first order terms (i.e. the symbols). are the same. The symbol of d is the map $\alpha \mapsto \varepsilon_\alpha$, where $\varepsilon_\alpha : \Omega_X^k(\mathbb{C}) \rightarrow \Omega_X^k(\mathbb{C})$ is defined by $\varepsilon_\alpha(\beta) = \alpha \wedge \beta$. The decomposition $d = \partial + \bar{\partial}$ then gives us the symbol of $\bar{\partial}$ is the mapping $\alpha \mapsto \varepsilon_{\alpha^{0,1}}$, where $\alpha^{0,1}$ is the projection of α onto antiholomorphic component. Therefore, we may assume that α is type $(0,1)$. Then the symbol of $-i\partial^* = i \star \bar{\partial} \star$ is $i \star \varepsilon_\alpha \star$. Since Λ is order 0, we are reduced to verifying the identity

$$[\Lambda, \varepsilon_\alpha] = i \star \varepsilon_\alpha \star$$

by linearity, we can further assume that α is one of the basis elements $d\bar{z}^i$ for the standard coordinates $\{z^i\}$ for \mathbb{C}^n , furthermore, we may assume that $\alpha = d\bar{z}^1$. We then prove a series of lemmas.

Lemma 7.4. *We have that*

$$\star \varepsilon_{d\bar{z}^1} \star = 2\iota_1$$

where $\iota_1 \beta = \beta(\partial/\partial z^1)$.

Proof. We have that $d\bar{z}^1 = dx^1 - idy^1$, so we compute on those components separately. It also suffices to verify on the basis $dx^I \wedge dy^J$ for $I = \{i_1, \dots, i_p\}$ and $J = \{j_1, \dots, j_q\}$ where $p + q = k$. We note that if $1 \notin I$, then $\star(dx^I \wedge dy^J)$ will contain dx^1 as a factor, so it is annihilated by $\varepsilon_{d\bar{z}^1}$. Otherwise, $\star(dx^I \wedge dy^J)$ will not contain a factor of dx^1 , so after

checking signs, $\star^{-1}\varepsilon_{dx^1}(dx^I \wedge dy^J)$ will be $dx^I \wedge dy^J$ except the factor of dx^1 is missing. Therefore, we have that

$$\begin{aligned}\star\varepsilon_{d\bar{z}^1}\star &= \star(\varepsilon_{dx^1} - i\varepsilon_{dy^1})\star \\ &= \iota_{dx^1} - i\iota_{dy^1} \\ &= 2\iota_1\end{aligned}$$

■

Lemma 7.5.

$$L \circ \iota_1 = \iota_1 \circ L - \frac{i}{2}\varepsilon_{d\bar{z}^1}$$

Proof. Let $\eta \in \Omega_X^k(\mathbb{C})$. Then we compute

$$\begin{aligned}(L \circ \iota_1)(\eta) &= (\omega \wedge \iota_1\eta) \\ &= \frac{i}{2} \sum_i dz^i \wedge d\bar{z}^i \wedge \iota_1\eta\end{aligned}$$

on the other hand, we have

$$\begin{aligned}(\iota_1 \circ L - \frac{i}{2}\varepsilon_{d\bar{z}^1})(\eta) &= \frac{i}{2} \left(\sum_i \iota_1(dz^i \wedge d\bar{z}^i \wedge \eta) - d\bar{z}^1 \wedge \eta \right) \\ &= \frac{i}{2} \left(\sum_i (\iota_1(dz^i \wedge d\bar{z}^i) \wedge \eta + dz^1 \wedge d\bar{z}^i \wedge \iota_1\eta) - d\bar{z}^1 \wedge \eta \right)\end{aligned}$$

where the computation on the second line comes from the identity

$$\iota_1(\alpha \wedge \eta) = \iota_1(\alpha) \wedge \eta + (-1)^{|\alpha|} \alpha \wedge \iota_1\eta$$

with $\alpha = dz^i \wedge d\bar{z}^i$. We note that the only i for which $\iota_1(dz^i \wedge d\bar{z}^i) \neq 0$ is when $i = 1$, in which case

$$\iota_1(dz^1 \wedge d\bar{z}^1) = \frac{1}{2}d\bar{z}^1$$

the equality then immediately follows. ■

Lemma 7.6.

$$\star L \star^{-1} = \star^{-1} L \star$$

Proof. Since the dimension of X is even dimensional, we have that $\star^{-1}\beta = (-1)^{|\beta|} \star \beta$. Since ω is a two form, wedging with ω preserves the parity of the degree. The lemma immediately follows. ■

We then compute $[\Lambda, \varepsilon_{d\bar{z}^1}]$ to be

$$\begin{aligned}[\Lambda, \varepsilon_{d\bar{z}^1}] &= \Lambda \varepsilon_{d\bar{z}^1} - \varepsilon_{d\bar{z}^1} \Lambda \\ &= \star^{-1} L \star \varepsilon_{d\bar{z}^1} - \varepsilon_{d\bar{z}^1} \star^{-1} L \star \\ &= 2 \star^{-1} L \circ \iota_1 \star^{-1} - \varepsilon_{d\bar{z}^1} \star L \star^{-1}\end{aligned}$$

where the last equality uses the fact that $\star \varepsilon_{d\bar{z}^1} = 2\iota_1 \star^{-1}$ (an immediate consequence of Lemma 7.4), and lemma 7.6. We then apply Lemma 7.5 to obtain

$$\begin{aligned} 2 \star^{-1} L \circ \iota_1 \star^{-1} - \varepsilon_{d\bar{z}^1} \star L \star^{-1} &= 2 \star^{-1} \left(\iota_1 \circ L - \frac{i}{2} \varepsilon_{d\bar{z}^1} \right) \star^{-1} - 2 \star^{-1} \iota_1 \circ L \star^{-1} \\ &= 2 \star^{-1} \iota_1 \circ L \star^{-1} - i \star^{-1} \varepsilon_{d\bar{z}^1} \star^{-1} - 2 \star^{-1} \iota_1 \circ L \star^{-1} \\ &= -i \star^{-1} \varepsilon_{d\bar{z}^1} \star^{-1} \end{aligned}$$

a quick verification with signs shows that this is equal to $i \star \varepsilon_{d\bar{z}^1} \star$, which is what we wanted to show. ■

The Kähler identities establish an important relationship between the Laplacians Δ , Δ_∂ , and $\Delta_{\bar{\partial}}$.

Theorem 7.7. *Let X be a Kähler manifold. Then the Laplacians satisfy*

$$\Delta_\partial = \Delta_{\bar{\partial}} = \frac{1}{2} \Delta$$

Proof. We first prove a short lemma.

Lemma 7.8.

$$\partial^* \bar{\partial} = -\bar{\partial} \partial^*$$

Proof. We compute

$$\begin{aligned} \partial^* \bar{\partial} &= -i[\Lambda, \bar{\partial}] \bar{\partial} \\ &= (-i\Lambda \bar{\partial} + i\bar{\partial} \Lambda) \bar{\partial} \\ &= -i\Lambda \bar{\partial} \bar{\partial} + i\bar{\partial} \Lambda \bar{\partial} \\ &= i\bar{\partial} \Lambda \bar{\partial} \end{aligned}$$

On the other hand, we compute

$$\begin{aligned} -\bar{\partial} \partial^* &= -i\bar{\partial}[\Lambda, \partial] \\ &= -i\bar{\partial} \Lambda \partial + i\bar{\partial} \partial \Lambda \\ &= -i\bar{\partial} \Lambda \bar{\partial} \end{aligned}$$

■

Using the decomposition $d = \partial + \bar{\partial}$ and the definition $\Delta = dd^* + d^*d$, we compute

$$\begin{aligned}
\Delta &= (\partial + \bar{\partial})(\partial^* + \bar{\partial}^*) + (\partial^* + \bar{\partial}^*)(\partial + \bar{\partial}) \\
&= (\partial + \bar{\partial})(\partial^* - i[\Lambda, \partial]) + (\partial^* - i[\Lambda, \partial])(\partial + \bar{\partial}) \\
&= (\partial + \bar{\partial})(\partial^* - i\Lambda\partial + i\partial\Lambda) + (\partial^* - i\Lambda\partial + i\partial\Lambda)(\partial + \bar{\partial}) \\
&= \partial\partial^* + \bar{\partial}\bar{\partial}^* - i\partial\Lambda\partial - i\bar{\partial}\Lambda\partial + i\bar{\partial}\partial\Lambda + \partial^*\partial + \partial^*\bar{\partial} - i\Lambda\partial\bar{\partial} + i\partial\Lambda\partial + i\partial\Lambda\bar{\partial} \\
&= \partial\partial^* + \bar{\partial}\bar{\partial}^* - i\bar{\partial}\Lambda\partial + i\bar{\partial}\partial\Lambda + \partial^*\partial + \partial^*\bar{\partial} - i\Lambda\partial\bar{\partial} + i\partial\Lambda\bar{\partial} \\
&= \partial\partial^* + \partial^*\partial - i\bar{\partial}\Lambda\partial + i\bar{\partial}\partial\Lambda - i\Lambda\partial\bar{\partial} + i\partial\Lambda\bar{\partial} \\
&= \Delta_\partial + i\partial[\Lambda, \bar{\partial}] + i[\Lambda, \bar{\partial}]\partial \\
&= \Delta_\partial + \partial\partial^* + \partial^*\partial \\
&= 2\Delta_\partial
\end{aligned}$$

the proof for $\Delta_{\bar{\partial}}$ is near identical. ■

Corollary 7.9. *The Laplacian Δ preserves the bigrading on $\Omega_X^{\bullet, \bullet}$, i.e. for a (p, q) form α , the form $\Delta\alpha \in \Omega_X^{p, q}$.*

Corollary 7.10. *A complex k -form $\alpha \in \Omega_X^k(\mathbb{C})$ is Δ -harmonic if and only if its components $\alpha^{p, q} \in \Omega_X^{p, q}$ are all Δ -harmonic.*

Corollary 7.11 (Hodge decomposition for Kähler manifolds).

$$\mathcal{H}^k(X) = \bigoplus_{p+q=k} \mathcal{H}^{p, q}(X)$$

We also get a decomposition of the cohomology $H^k(X, \mathbb{C})$, but this isn't as immediate as one would think. The naïve idea is to use the isomorphism $\mathcal{H}^K(X) \rightarrow H^k(X, \mathbb{C})$, and then obtain the decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p, q}(X, \mathbb{C})$$

where $H^{p, q}(X, \mathbb{C})$ is defined to be the image of $\mathcal{H}^{p, q}(X)$ under the natural map. However, the space of harmonic forms depends on the choice of Hermitian metric, so a priori there could be different decompositions depending on the metric. However, this is not the case.

Theorem 7.12. *The decomposition*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p, q}(X, \mathbb{C})$$

induced by the Hodge decomposition does not depend on the choice of metric.

Proof. We let $K^{p, q} \subset H^k(X, \mathbb{C})$ denote the space of cohomology classes that admit a representative of type (p, q) . Clearly, $K^{p, q}$ does not use the metric, so if we prove the inclusion $K^{p, q} \subset H^{p, q}(X, \mathbb{C})$, we are done. Given a class $[\eta] \in K^{p, q}$, by the definition of $K^{p, q}$, we may assume η is a closed form of type (p, q) . By the Hodge decomposition, we write $\eta = \alpha + \Delta\beta$, with α a harmonic form. By projecting onto $\Omega_X^{p, q}$, we have that $\eta = \alpha^{p, q} + \Delta\beta^{p, q}$, and $\alpha^{p, q}$ is still harmonic by Corollary 7.11. Expanding this out, we

have $\eta = \alpha^{p,q} + dd^*\beta^{p,q} + d^*d\beta^{p,q}$. Then since η is closed, we have that $d\eta = 0$. Since $d\alpha^{p,q} = 0$ and $ddd^*\beta^{p,q} = 0$, we must have that $dd^8d\beta^{p,q} = 0$, and by orthogonality of $\ker d$ with $\text{Im } d^*$, we have that $d^*d\beta^{p,q} = 0$. Therefore, $\eta = \alpha^{p,q} + dd^*\beta^{p,q}$, where $\alpha^{p,q}$ is a harmonic (p, q) form, which is in $H^{p,q}(X, \mathbb{C})$ by definition. Therefore, $[\eta] = [\alpha^{p,q}]$, proving the inclusion. ■

Since $H^{p,q}(X, \mathbb{C}) = K^{p,q}$, we note that this implies $H^{p,q}(X, \mathbb{C}) = \overline{H^{q,p}(X, \mathbb{C})}$, since we clearly have that $K^{p,q} = \overline{K^{q,p}}$.

Theorem 7.13. *Let X be a Kähler manifold. Then the odd dimensional cohomology groups $H^{2k+1}(X, \mathbb{C})$ are even dimensional*

Proof. By the Hodge decomposition, we have

$$H^{2k+1}(X, \mathbb{C}) = \bigoplus_{p+q=2k+1} H^{p,q}(X, \mathbb{C}) = \bigoplus_{i=0}^{2k+1} H^{i,2k+1-i}(X, \mathbb{C}) = \bigoplus_{i=0}^k H^{i,2k+1-i}(X, \mathbb{C}) \oplus \overline{H^{i,2k+1-i}(X, \mathbb{C})}$$

and since $\dim V = \dim \overline{V}$, this implies that $H^k(X, \mathbb{C})$ is even dimensional. ■

We now prove an extremely useful result.

Theorem 7.14 (The $\partial\bar{\partial}$ lemma). *Let X be a Kähler manifold, and η a complex k -form that is ∂ and $\bar{\partial}$ -closed. Then if η is d , ∂ , or $\bar{\partial}$ -exact, there exists a form ξ such that $\eta = \partial\bar{\partial}\xi$.*

Proof. The proof for all three cases is much the same, so we do just the one where $\eta = \bar{\partial}\alpha$ is $\bar{\partial}$ -exact. By the Hodge decomposition, we write $\alpha = \beta + \Delta\gamma$, with β harmonic. Since $\Delta = 2\Delta_{\bar{\partial}}$, and β is $\Delta_{\bar{\partial}}$ -harmonic if and only if $\bar{\partial}\beta = \bar{\partial}^*\beta = 0$, we have that $\bar{\partial}\beta = 0$. Then using Lemma 7.8, we compute

$$\begin{aligned} \eta &= \bar{\partial}\alpha \\ &= \bar{\partial}(\beta + \Delta\gamma) \\ &= \bar{\partial}\beta + 2\bar{\partial}(\Delta_{\partial}\gamma) \\ &= 0 + 2\bar{\partial}(\partial\bar{\partial}^*\gamma + \partial^*\partial\gamma) \\ &= 2\bar{\partial}\partial\bar{\partial}^*\gamma - 2\bar{\partial}^*\bar{\partial}\partial\gamma \\ &= -2\bar{\partial}\bar{\partial}\partial^*\gamma - 2\bar{\partial}^*\bar{\partial}\partial\gamma \end{aligned}$$

Then since η is ∂ -closed, we have that $\partial^*\bar{\partial}\partial\gamma$ must also be ∂ -closed. By orthogonality of the image of ∂^* with the kernel of ∂ , we have that $\partial^*\bar{\partial}\partial\gamma = 0$, so $\eta = -2\bar{\partial}\bar{\partial}\partial^*\gamma = 2\bar{\partial}\partial\bar{\partial}^*\gamma$, so letting $\xi = \partial^*\gamma$, we are done. ■

8. THE LEFSCHETZ DECOMPOSITION

A Kähler manifold X with Kähler form ω admits another decomposition from the Lefschetz operators L and Λ .

Proposition 8.1. *Let X be a Kähler manifold of complex dimension n . Then*

$$[L, \Lambda] = (k - n) \text{id}_{\Omega_X^k}$$

Proof. Since both L and Λ are order 0, it suffices to verify this on $\Lambda^k T^*X$, so we may as well assume that $X = \mathbb{C}^n$ with the standard Hermitian form and Kahler metric ω . Define

$$A_i := \frac{i}{2} \varepsilon_{dz^i \wedge d\bar{z}^i}$$

so $L = \sum_i A_i$ and $\Lambda = \sum_i \star^{-1} A_i \star$. Recall that in the proof of the Kähler identities, we proved the result

$$\star \varepsilon_{d\bar{z}^1} \star = 2\iota_1$$

on \mathbb{C}^n , where

$$\iota_1(\alpha)(v_1, \dots, v_{k-1}) = \alpha\left(\frac{\partial}{\partial z^1}, v_1, \dots, v_{k-1}\right)$$

we note that this is true for any dz^i . It then follows that $\star^{-1} \varepsilon_{d\bar{z}^i} \star = (-1)^{k+1} 2\iota_i$. Similarly, we also have that $\star^{-1} \varepsilon_{dz^i} \star = (-1)^{k+1} 2\iota_{\bar{i}}$ where

$$\iota_{\bar{i}}(\alpha)(v_1, \dots, v_{k-1}) = \alpha\left(\frac{\partial}{\partial \bar{z}^i}, v_1, \dots, v_{k-1}\right)$$

From this, we conclude that

$$\star^{-1} A_i \star = -2i \iota_{\partial_i \wedge \bar{\partial}_i}$$

where $\partial_i := \partial/\partial z^i$ and $\bar{\partial}_i := \partial/\partial \bar{z}^i$. We then have

$$[L, \Lambda] = \sum_{i,j} [A_i, \star^{-1} A_j \star]$$

If $i \neq j$, then A_i and $\star^{-1} A_j \star$ commute, so this reduces to

$$[L, \Lambda] = \sum_i [A_i, \star^{-1} A_i \star] = \sum_i [\varepsilon_{dz^i \wedge d\bar{z}^i}, \iota_{\partial_i \wedge \bar{\partial}_i}]$$

to compute this, we make a small digression. Let $M = \{i_1, \dots, i_m\} \subset \{1, \dots, n\}$, and let $\omega_M \in \Omega_X^{2m}$ be the form

$$\omega_M = dz^{i_1} \wedge d\bar{z}^{i_1} \wedge \dots \wedge dz^{i_m} \wedge d\bar{z}^{i_m}$$

any k form can be written as a linear combinations of the forms

$$\omega_{A,B,M} = dz^A \wedge d\bar{z}^B \wedge \omega_M$$

where A, B , and M are mutually disjoint, and $|A| + |B| + 2|M| = k$. Fix such sets A, B , and M . Let $J \subset \{1, \dots, n\}$ be the complement of A, B , and M . We then make several observations:

- (1) If $i \notin J$, then $dz^i \wedge d\bar{z}^i \wedge \omega_{A,B,M} = 0$.
- (2) If $i \notin M$, then $\iota_{\partial_i \wedge \bar{\partial}_i}(\omega_{A,B,M}) = 0$.
- (3) If $i \in J$, then $\iota_{\partial_i \wedge \bar{\partial}_i}(dz^i \wedge d\bar{z}^i \wedge \omega_{A,B,M}) = \omega_{A,B,M}$.
- (4) If $i \in M$, then $\varepsilon_{dz^i \wedge d\bar{z}^i}(\iota_{\partial_i \wedge \bar{\partial}_i})(\omega_{A,B,M}) = \omega_{A,B,M}$.

From these observations, we get that

$$[L, \Lambda](\omega_{A,B,M}) = (|M| - |J|)\omega_{A,B,M}$$

we know that $|J| = n - |A| - |B| - |M|$. Therefore,

$$|M| - |J| = (k - |A| - |B| - |M|) - (n - |A| - |B| - |M|) = (k - n)$$

therefore,

$$[L, \Lambda](\omega_{A,B,M}) = (k - n)\omega_{A,B,M}$$

and since the $\omega_{A,B,M}$ span, this implies the desired result. \blacksquare

Corollary 8.2. *The exterior algebra of every cotangent space T_x^*X of a Kähler manifold X admits the structure of $\mathfrak{sl}_2\mathbb{C}$ -representation.*

Proof. Let $V = T_x^*X$, and let $\Pi : \Lambda^\bullet V \rightarrow \Lambda^\bullet V$ be the map where $\Pi|_{\Lambda^k V} = (k - n)\text{id}_{\Lambda^k V}$. We want to compute the commutation relations:

- (1) $[\Pi, L]$
- (2) $[\Pi, \Lambda]$
- (3) $[L, \Lambda]$

Let $\omega \in \Lambda^2 V$ be the Kähler form. we compute, for $\eta \in \Lambda^k V$,

$$\begin{aligned} [\Pi, L]\eta &= \Pi(\omega \wedge \eta) - \omega \wedge \Pi\eta \\ &= (k + 2 - n)\omega \wedge \eta - (k - n)\omega \wedge \eta \\ &= 2\omega \wedge \eta \\ &= 2L\eta \end{aligned}$$

next, we compute

$$\begin{aligned} [\Pi, \Lambda]\eta &= \Pi\Lambda\eta - \Lambda\Pi\eta \\ &= (k - 2 - n)\Lambda\eta - (k - n)\Lambda\eta \\ &= -2\Lambda\eta \end{aligned}$$

where we use the fact that Λ decreases the degree of a form by 2. The third commutation is what we just computed, giving us the commutation relations

- (1) $[\Pi, L] = 2L$
- (2) $[\Pi, \Lambda] = -2\Lambda$
- (3) $[L, \Lambda] = \Pi$

The Lie algebra $\mathfrak{sl}_2\mathbb{C}$ admits a basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

which satisfy the commutation relations

$$[H, Y] = 2Y \quad [H, X] = -2X \quad [X, Y] = H$$

which are exactly the same relations as we found for Π, L , and Λ . Therefore, the mapping $H \mapsto \Pi, Y \mapsto L$, and $X \mapsto \Lambda$ determines a Lie algebra homomorphism $\mathfrak{sl}_2\mathbb{C} \rightarrow \text{End } \Lambda^\bullet V$, which gives $\Lambda^\bullet V$ the structure of a $\mathfrak{sl}_2\mathbb{C}$ -representation. The standard picture of an $\mathfrak{sl}_2\mathbb{C}$ representation is that of string of beads, representing eigenspaces of H , and with X and Y being raising and lowering operators, moving up and down the string. Applying this picture to $\Lambda^\bullet V$, we get

$$\begin{array}{ccccccc} & & \xrightarrow{L} & & \xrightarrow{L} & & \xrightarrow{L} & & \xrightarrow{L} & & \\ \cdots & & \Lambda^{k-2}V & & \Lambda^k V & & \Lambda^{k+2}V & & \cdots \\ & & \xleftarrow{\Lambda} & & \xleftarrow{\Lambda} & & \xleftarrow{\Lambda} & & \xleftarrow{\Lambda} & & \end{array}$$

\blacksquare

Definition 8.3. A k -form $\alpha \in \Omega_k X$ is *primitive* if $L\alpha = 0$. The space of primitive k -forms is denoted P^k

Theorem 8.4 (*The Lefschetz decomposition on forms*). *There is an orthogonal direct sum decomposition*

$$\Omega_X^k = \bigoplus_{i \geq 0} L^i P^{k-2i}$$

Proof. Since L is order 0, it suffices to verify this fiberwise. Let $V = T_x^* X$, and we abuse notation by letting L denote the restriction of L to a $\Lambda^\bullet V$, and we also let P^k denote the primitive elements of $\Lambda^\bullet V$. Therefore, we want to prove the fiberwise decomposition

$$\Lambda^k V = \bigoplus_{i \geq 0} L^i P^{k-2i}$$

As we have just noted, $\Lambda^\bullet V$ is a finite dimensional $\mathfrak{sl}_2 \mathbb{C}$ -representation, so we use some facts from the representation theory of $\mathfrak{sl}_2 \mathbb{C}$. We use two main facts:

- (1) Every finite dimensional $\mathfrak{sl}_2 \mathbb{C}$ representation is a direct sum of irreducible representations.
- (2) An $\mathfrak{sl}_2 \mathbb{C}$ representation V is irreducible if and only if each eigenspace in the decomposition of V into the eigenspaces of H (as illustrated above) is 1-dimensional.

We first prove an important lemma.

Lemma 8.5. *Let $r \geq 1$. Then*

$$[L^r, \Lambda] = (r(k - n) + r(r - 1))L^{r-1}$$

Proof. Since the bundles are in question, it suffices to verify injectivity, which we do by induction. For the base case $r = 1$, this is just the statement of proposition 8.1. Then assuming that the statement holds for $r - 1$, we prove the lemma for r . We note that

$$\begin{aligned} [L^r, \Lambda] &= L^r \Lambda - \Lambda L^r \\ &= L(L^{r-1} \Lambda) - \Lambda L^r \\ &= L(L^{r-1} \Lambda - \Lambda L^{r-1}) + L \Lambda L^{r-1} - (\Lambda L)L^{r-1} \\ &= L[L^{r-1}, \Lambda] + [L, \Lambda]L^{r-1} \end{aligned}$$

By assumption, we have that

$$[L^{r-1}, \Lambda] = ((r - 1)(k - n) + (r - 1)(r - 2))L^{r-2}$$

substituting this in, along with the formula from Proposition 8.1 into the above equation, we get

$$\begin{aligned} [L^r, \Lambda] &= L[L^{r-1}, \Lambda] + [L, \Lambda]L^{r-1} \\ &= ((r - 1)(k - n) + (r - 1)(r - 2))L^{r-1} + (k + 2r - 2 - n)L^{r-1} \\ &= (r(k - n) + (r - 1)(r - 2) + 2r - 2)L^{r-1} \\ &= (r(k - n) + r^2 - 3r + 2 + 2r - 2)L^{r-1} \\ &= (r(k - n) + r^2 - r)L^{r-1} \\ &= (r(k - n) + r(r - 1))L^{r-1} \end{aligned}$$

■

We then note that $\Lambda^\bullet V$ admits a primitive v , since we can just pick a nonzero v' , and walk it down the chain by repeatedly applying Λ , which must eventually terminate by finite dimensionality of $\Lambda^\bullet V$. Simple calculations using the commutation relations of $\mathfrak{sl}_2\mathbb{C}$ (equivalently, of L , Λ , and Π) gives that the span of $\{v, Lv, L^2v, \dots\}$ defines a subrepresentation S of $\Lambda^\bullet V$. Additionally, since the $\Lambda^k V$ are all eigenspaces for Π with different eigenvalues, we have that for each q , $L^q v \in S$ spans a 1-dimensional eigenspace for Π restricted to S , so S is an irreducible subrepresentation. We then claim that S^\perp is also a subrepresentation. From the commutation relations we proved earlier, it suffices to prove that S^\perp is invariant under L , Λ , and Π . Since the set $\{v, Lv, L^2v, \dots\}$ forms a basis for S , it suffices to show that for $w \in S^\perp$,

- (1) $\langle \Pi w, L^r v \rangle = 0$
- (2) $\langle \Lambda w, L^r v \rangle = 0$
- (3) $\langle Lw, L^r v \rangle = 0$

The first is immediate, due to the fact that the $\Lambda^k V$ are eigenspaces for Π , and are all orthogonal. The second is also easy, since we have that

$$\langle \Lambda w, L^r v \rangle = \langle w, L^{r+1} v \rangle = 0$$

since $w \in S^\perp$. For the third, we compute

$$\begin{aligned} \langle Lw, L^r v \rangle &= \langle w, \Lambda L^r v \rangle \\ &= \langle w, L^r \Lambda v - [L^r, \Lambda]v \rangle \\ &= \langle w, L^r \Lambda v \rangle - (r(k-n) + r(r-1))\langle w, L^{r-1} v \rangle \\ &= \langle w, L^r \Lambda v \rangle \\ &= 0 \end{aligned}$$

since v is primitive. Therefore, S^\perp defines a subrepresentation. Repeating this process, we get an orthogonal decomposition

$$\Lambda^\bullet V = \bigoplus_k S^k$$

into irreducible subrepresentations S^k , each generated by a primitive form v_k . The v_k are clearly linearly independent (they are orthogonal), and they also span the space $P = \bigoplus_k P^k$ of primitive forms, since if $\alpha \in P$ cannot be written as a linear combination of the P^k , then the representation spanned by $\{\alpha, L\alpha, L^2\alpha, \dots\}$ would not be contained in the direct sum $\bigoplus_k S^k$, contradicting the decomposition of $\Lambda^\bullet V$. Then since each of the subrepresentation is defined to be the span of iterative applications of L to a primitive form v_k , the decomposition immediately follows. ■

We now make several observations related to the Lefschetz decomposition, culminating in the Hard Lefschetz Theorem for differential forms.

Proposition 8.6. *If $k > n = \dim_{\mathbb{C}} X$, then $P^k = 0$.*

Proof. Let $\alpha \in P^k$. By finite dimensionality, we know that there must be some r such that $L^r \alpha = 0$. WLOG, assume that r is minimal. Therefore, since α is primitive, we have that $[L^r, \Lambda] \alpha = 0$. Using Lemma 8.5, we have that

$$\begin{aligned} 0 &= [L^r, \Lambda] \alpha \\ &= (r(k-n) + r(r-1)) L^{r-1} \alpha \end{aligned}$$

Then by assumption, $L^{r-1} \alpha \neq 0$, and $k-n > 0$, so for this to be true, we require that $r = 0$. But this implies that $L^0 \alpha = \text{id } \alpha = 0$, so $\alpha = 0$. ■

Proposition 8.7. *The map $L^{n-k} : P^k \rightarrow \Lambda^{2n-k} T^* X$ is injective for $k \leq n$.*

Proof. Let $\alpha \in P^k$, where $\alpha \neq 0$. Then let $r > 0$ be the minimal integer such that $L^r \alpha = 0$. Then using Lemma 8.5 again, we have that

$$0 = [L^r, \Lambda] \alpha = (r(k-n) + r(r-1)) L^{r-1} \alpha$$

Since by assumption $L^{r-1} \alpha \neq 0$ and $r \neq 0$, we must have that $k-n+r-1 = 0$. Therefore, we have that $n-k = r-1$. Therefore, we must have that $L^{n-k} \alpha = L^{r-1} \alpha \neq 0$, by minimality of r . In addition, we also note that $L^{n-k+1} = L^r$, so $L^{n-k+1} \alpha = 0$. ■

Theorem 8.8 (Hard Lefschetz for forms). *The map $L^{n-k} : \Lambda^k T^* X \rightarrow \Lambda^{2n-k} T^* X$ is an isomorphism for $k \leq n$.*

Proof. We note that from the Lefschetz decomposition, we have that

$$\Lambda^k T^* X = \bigoplus_{i \geq 0} L^i P^{k-2i}$$

Applying L^{n-k} to each factor, we get that

$$\begin{aligned} L^{n-k}(\Lambda^k T^* X) &= \bigoplus_{i \geq 0} L^{n-k} L^i P^{k-2i} \\ &= \bigoplus_{i \geq 0} L^{n-k+i} P^{k-2i} \end{aligned}$$

Applying the Lefschetz decomposition to $\Lambda^{2n-k} T^* X$, we get

$$\Lambda^{2n-k} T^* X = \bigoplus_{i \geq 0} L^i P^{2n-k-2i}$$

Of course, the terms $L^{n-k+i} P^{k-2i}$ all appear in the direct sum decomposition of $\Lambda^{2n-k} T^* X$, but there are other terms in the direct sum decomposition that may a priori be nontrivial. We wish to show that these extra terms are trivial. In particular, these extra terms consist of the $L^i P^{2n-k-2i}$ when $i < n-k$. We note that this implies that $i \geq n - (2n-k-2i) + 1$, since this is equivalent to $-i \geq -n+k+1 \iff i \leq n-k-1 \iff i < n-k$. Then from the observation we made at the end of Proposition 8.7, we have that $L^{n-(2n-k-2i)+1} = 0$ on $P^{2n-k-2i}$, so all these terms are trivial. ■

Remark. The proofs for the Lefschetz decomposition and Hard Lefschetz Theorem go the same for the spaces of complexified forms. We also note that a k -form α is primitive if and only if it is annihilated by L^{n-k+1} .

Since the Kähler form ω is closed, the Lefschetz operator L descends to an operator (which by abuse of notation, we also call L) on the cohomology of X , where

$$\begin{aligned} L : H_{dR}^k(X, \mathbb{R}) &\rightarrow H_{dR}^{k+2}(X, \mathbb{R}) \\ [\eta] &\mapsto [\omega \wedge \eta] \end{aligned}$$

we then want to apply our knowledge of the Lefschetz decomposition and Hard Lefschetz Theorem on the level of forms to the level of cohomology, which relies on a crucial fact.

Theorem 8.9. *The Laplacian Δ commutes with L .*

Proof. We have that $\Delta = 2\Delta_{\partial} = 2(\partial\partial^* + \partial^*\partial)$. We then compute

$$\begin{aligned} [\Delta, L] &= [2(\partial\partial^* + \partial^*\partial), L] \\ &= 2[\partial\partial^*, L] + 2[\partial^*\partial, L] \\ &= 2\partial\partial^*L - 2L\partial\partial^* + 2\partial^*\partial L - 2L\partial^*\partial \end{aligned}$$

we then need a simple computation.

Lemma 8.10.

$$2\partial^*L\partial - 2\partial L\partial^* = 2\partial^*\partial L - 2L\partial\partial^*$$

Proof. We show the equivalent statement that

$$2\partial^*L\partial - 2\partial^*\partial L + 2L\partial\partial^* - 2\partial L\partial^* = 0$$

we compute

$$2\partial^*L\partial - 2\partial^*\partial L + 2L\partial\partial^* - 2\partial L\partial^* = 2\partial^*[L, \partial] + 2[L, \partial]\partial^*$$

we note that $[L, \partial] = 0$, since ω is ∂ -closed and of even degree, and using the fact that ∂ is an antiderivation of bidegree $(1, 0)$. This proves the desired equality. ■

Using the lemma, we can rewrite our previous equality to obtain

$$\begin{aligned} [\Delta, L] &= 2\partial\partial^*L - 2\partial L\partial^* + 2\partial^*L\partial - 2L\partial^*\partial \\ &= 2\partial[\partial^*, L] + 2[\partial^*, L]\partial \end{aligned}$$

From the Kähler identities, we have that $[\partial^*, L] = -i\bar{\partial}$, which anticommutes with ∂ , giving us that $[\Delta, L] = 0$. ■

Corollary 8.11. *Given a harmonic form α , $L\alpha$ is also harmonic.*

The Hodge Theorem then allows us pass to cohomology, using the identification $\mathcal{H}^k(X) \cong H_{dR}^k(X, \mathbb{R})$.

Theorem 8.12 (Hard Lefschetz for cohomology). *The map $L^{n-k} : H_{dR}^k(X, \mathbb{R}) \rightarrow H_{dR}^{2n-k}(X, \mathbb{R})$ is an isomorphism.*

Proof. Using the identification of the spaces of harmonic forms with the cohomology of X , we may identify the Lefschetz operator L on cohomology with the Lefschetz operator on forms restricted to the harmonic forms. From the Hard Lefschetz Theorem on forms, we have that $L^{n-k} : \Lambda^{k,T^*X} \rightarrow \Lambda^{2n-k,T^*X}$ is isomorphism, so it defines an isomorphism $\Omega_X^k \rightarrow \Omega_X^{2n-k}$. Therefore, the restriction to $\mathcal{H}^k(X)$ is injective. By Poincaré duality, we

know that $\mathcal{H}^k(X)$ and $\mathcal{H}^{2n-k}(X)$ are the same dimension, so $L^{n-k} : \mathcal{H}^k(X) \rightarrow \mathcal{H}^{2n-k}(X)$ is an injective map between vector spaces of the same dimension, i.e. an isomorphism. ■

Likewise, the Lefschetz decomposition also descends to the level of cohomology, and the proof is the same as it was for forms.

Theorem 8.13 (*The Lefschetz decomposition on cohomology*). *There is a direct sum decomposition*

$$H_{dR}^k(X, \mathbb{R}) = \bigoplus_{i \geq 0} L^i H^{k-2i}(X, \mathbb{R})_{\text{Prim}}$$

where $H^{k-2i}(X, \mathbb{R})_{\text{Prim}}$ denotes the cohomology classes $[\alpha]$ annihilated by $L^{n-k+2i+1}$ (which we recall is an equivalent formulation of primitive on the level of forms).