SYMPLECTIC GEOMETRY AND KÄHLER MANIFOLDS: AN INTRODUCTION

JEFFREY JIANG

Linear Algebra

As typical when studying smooth manifolds, we first look at the linear algebra that will be non-linearized. To discuss Kälher manifolds (and consequently, Kähler vector spaces), we will need to assemble several mutually compatible structures – an inner product, a symplectic form, and a complex structure.

Definition 0.1. An ***inner product space*** is a vector space V equipped with a bilinear map $g : V \times V \to V$ that is symmetric and positive definite. In the case that V is a complex vector space, we replace the bilinear condition with the ***sesquilinear*** condition – conjugate linear in the first term, and linear in the second term. In this case, g is often referred to as a ***hermitian inner product***.

You're probably familiar with these, so we won't delve into them.

Definition 0.2. A ***symplectic vector space*** is a vector space V equipped with a nondegenerate skew symmetric bilinear form $\omega : V \times V \to V$.

Nondegeneracy here means that if for some v, $\omega(v, w) = 0$ for all $w \in V$, then $v = 0$. In other words, the form ω gives an isomorphism $V \to V^*$ via the mapping

$$v \mapsto \omega(v, \cdot)$$

We note that the positive definite condition on an inner product g implies nondegeneracy, so that also defines an analogous isomorphism. An equivalent formulation of nondegeneracy for ω is that the wedge product $\omega \wedge \ldots \wedge \omega$ with itself $n/2$ times gives a nonzero volume form for V.

Example 0.3. $\mathbb{R}^{2n}$ with coordinates (x_i, y_i) comes with a symplectic structure given by

$$\tilde{\omega} = \sum_i dx^i \wedge dy^i$$

which can be expressed in coordinates by

$$\tilde{\omega}(v, w) = v^T \Omega\, w$$

where

$$\Omega = \begin{pmatrix} 0 & \mathrm{id}_{\mathbb{R}^n} \\ -\mathrm{id}_{\mathbb{R}^n} & 0 \end{pmatrix}$$

In some sense, this is the *only* symplectic vector space, which should be made clear very soon.

Theorem 0.4. *Every symplectic vector space (V, ω) admits a* ***symplectic basis*** *$\{e_i, f_i\}$ where*

$$\omega(e_i, e_j) = 0 \qquad \omega(e_i, f_i) = \delta_{ij}$$

Corollary 0.5. *Every symplectic vector space is* ***symplectomorphic*** *to $(\mathbb{R}^{2n}, \tilde{\omega})$. That is, there exists a linear isomorphism $\varphi : (V, \omega) \to (\mathbb{R}^{2n}, \tilde{\omega})$ where $\varphi^*\tilde{\omega} = \omega$.*

Proof. Let $\{f_i, g_i\}$ be a symplectic basis for V on $\mathbb{R}^n$ and consider the map $\varphi : V \to \mathbb{R}^{2n}$ given by mapping $f_i \mapsto e_i$ and $g_i \mapsto e_{n+i}$ ∎

Definition 0.6. A ***complex structure*** on a vector space V is an automorphism $J : V \to V$ such that $J^2 = -\mathrm{id}_V$

Given a complex structure J and an $\mathbb{R}$-vector space V, we can make V a $\mathbb{C}$-vector space by defining the action of $i \in \mathbb{C}$ by $i \cdot v = Jv$. We can then extend this to arbitrary complex numbers $\alpha + \beta i$ by $(\alpha + \beta i) \cdot v = \alpha v + \beta J v$.

Remark. Might want to skip this part – not enough time.

If you know some linear algebra, you might know that there is another way to turn a $\mathbb{R}$-vector space into a complex one.

Definition 0.7. Given an $\mathbb{R}$-vector space V, define its complexification $V_{\mathbb{C}}$ as $V \otimes_{\mathbb{R}} \mathbb{C}$.

Note that adding a complex structure to a vector space is *not* the same as complexifying it. If we find some $J \in GL(V)$ such that $J^2 = -\operatorname{id}_V$, we haven't changed the dimension of V, but complexifying V doubles its dimension over $\mathbb{R}$. Despite this, the concepts are quite similar. If we have a vector space V with complex structure J, then if we complexify V, then J extends to a map $V_{\mathbb{C}} \to V_{\mathbb{C}}$, where $J(v + iw) = Jv + iJw$. This gives a decomposition

$$V_{\mathbb{C}} = V^+ \oplus V^-$$

where $V^+ = \{v + iJv : v \in V\}$ and $V^- = \{v - iJv : v \in V\}$, and we get an isomorphism of $\mathbb{C}$-vector spaces $V \to V^{\pm}$ where $v \mapsto v \pm iJv$, using the fact that $J^2 v = -v$.

Now that we have defined all of these, we would like to define when these structures are compatible, and see what results

Definition 0.8. A complex structure J is ***compatible*** with a symplectic form ω if $\omega(Jv, Jw) = \omega(v, w)$.

An analogous definition is used for an inner product g.

Theorem 0.9. *Given a vector space V with complex structure J, given a J-compatible inner product g, we obtain a symplectic form ω where $\omega(v, w) = g(Jv, w)$. Likewise, given a J-compatible symplectic form ω, we obtain an inner product g where $g(v, w) = \omega(Jv, w)$. Symplectic forms/inner products obtained in this manner are said to be* ***compatible*** *with the other.*

Definition 0.10. A vector space V is ***Kähler*** if it has compatible structures J, g, ω.

One final way to obtain these compatible structures is to start with a Hermitian vector space (V, h), and let $g(v, w) = \operatorname{Re} h(v, w)$ and $\omega(v, w) = -\operatorname{Im} h(v, w)$.

Kähler Manifolds

With the linear algebra set up, we can move on to talking about the manifolds. We'll first translate the linear algebraic concepts to ones on a manifold.

Definition 1.11. A ***Riemannian metric*** on a smooth manifold M is a smooth symmetric positive definite 2-tensor field $g : M \to \mathcal{T}^2(M)$.

Smooth here mean one of several equivalent things:

(1) The component functions of the matrix $g_{ij} = \langle \partial_i, \partial_j \rangle$ are smooth in any chart.
(2) g is a smooth map from M into the tensor bundle $\mathcal{T}^2(M)$.
(3) For any vector fields $X, Y \in \mathfrak{X}(M)$, the function given by $g(X, Y)$ is smooth.

We note that a Riemannian metric on a manifold is analogous to an inner product on a vector space – it a smooth assignment of an inner product to each tangent space $T_p M$. Also, we have that g_p being an inner product on $T_p M$ implies that it is nondegenerate, so g induces an isomorphism $TM \to T^*M$.

Definition 1.12. A ***symplectic forn*** on a smooth manifold M is a smooth nondegenerate closed skew-symmetric 2-form $\omega : M \to \Lambda^2(M)$.

Here closed means that the exterior derivative $d\omega = 0$. A symplectic form makes every tangent space $T_p M$ a symplectic vector space.

Example 1.13. GIven any manifold M, we can define a symplectic structure on its cotangent bundle T^*M as follows. Define the ***tautological*** **1*-form*** $\alpha \in \Omega^1(T^*M)$ by $\alpha_(p, \xi) = d\pi_p^* \xi$ where $\pi : T^*M \to M$ is the projection. Then the 2-form $\omega = d\alpha$ is symplectic. If (p^i) denote coordinates on M, and (p^i, q^i) denote the induced coordinates given by the local trivialization for T^*M, we have that

$$\omega = \sum_i dq^i \wedge dp^i$$

which looks exactly like the standard form!

As it turns out, every symplectic manifold has the same local structure, what this means is that every symplectic manifold (M, ω) is locally symplectomorphic to $\mathbb{R}^{2n}$ (now thought of as a manifold, rather than a vector space). The same does not hold for Riemannian manifolds. It is not the case that every Riemannian manifold is locally isometric to $\mathbb{R}^n$ with the standard Euclidean metric.

Another difference is that every manifold admits a Riemannian metric. The same is not true in the symplectic case.

Proposition 1.14. *The only sphere that admits a symplectic structure is S^2*

Proof. We can give S^2 a symplectic structure with a choice of volume form, which is closed and skew symmetric. Then let $n \in 2\mathbb{Z}$ with $n \neq 2$. We claim that there exists no symplectic structure on S^n. We know that $H^2_{dR}(M) = 0$, so every 2-form is exact. This tells us that the $n/2$-fold wedge of ω with itself is also exact, so it is equal to $d\beta$ for some β. Then by Stokes' Theorem

$$\int_{S^n} \omega^{n/2} = \int_{S^n} d\beta == \int_{\varnothing} \beta = 0$$

so $\omega^{n/2}$ is not a volume form. ∎

Definition 1.15. An ***almost complex structure*** on a smooth manifold is smoothly varying J where at each $p \in M$, $J_p^2 : T_pM \to T_pM$ is equal to $-\operatorname{id}_{T_pM}$

Since the endomorphism is acting on a different vector space at each point, what does smooth mean? We have a canonical isomorphism $V^* \otimes V \to \operatorname{End}(V)$, so we identify J as a smoothly varying $(1,1)$ tensor field, and ask that the map to the tensor bundle $J : M \to \mathcal{T}^1_1(M)$ is smooth.

Definition 1.16. A ***complex manifold*** is a smooth manifold with an atlas of charts $\varphi : U \to \mathbb{C}^n$ where the transition maps $\varphi \circ \psi^{-1} : \psi(U) \cap \varphi(V) \to \mathbb{C}^n$ are holomorphic.

As it turns out, an almost complex structure isn't quite enough for our needs – it will need to satisfy some integrability condition. If you know about the Frobenius theorem regarding when vector fields determine a subbundle of the tangent bundle, this is the same concept. Recall that an complex structure on a vector space V induces a decomposition of the complexification

$$V_{\mathbb{C}} = V^+ \oplus V^-$$

Then given a manifold M with complex structure J, we can talk about the complexified tangent bundle, which has fibers $T_pM_{\mathbb{C}}$. Then J gives us an analogous decomposition of each fiber as

$$T_pM_{\mathbb{C}} = T_pM^+ \oplus T_pM^-$$

which then gives us two smoothly varying subspaces of $T_pM_{\mathbb{C}}$. Then the integrability condition translates to these distributions being integrable, i.e closed under Lie bracket. If such a condition is satisfied, then we can endow M with a complex structure.

Definition 1.17. A ***Kähler manifold*** is a complex manifold M with a Riemannian metric g and symplectic form ω such that at each T_pM, g_p and ω_p are compatible with the complex structure given by multiplication by i, i.e.

$$g_p(i \cdot v, w) = \omega(v, w) \qquad \omega_p(i \cdot v, w) = g(v, w)$$

We end with some examples

Proposition 1.18. *Complex projective space $\mathbb{CP}^n$ is Kähler.*

Proof. To see this, we note that the unitary group $U(n+1)$ acts transitively on $\mathbb{CP}^n$ thought of as lines in $\mathbb{C}^{n+1}$, and $\mathbb{CP}^n$ will be both homogeneous and isotropic under this action. Using this, we can define a $U(n+1)$ invariant metric on $\mathbb{CP}^n$, from which we can recover a hermitian form on $\mathbb{CP}^n$, which we can then use the construct the symplectic form. The explicit computations get quite nasty though. ∎

Proposition 1.19. *Any torus $T^{2n} = \mathbb{C}^n / \Lambda^n$ (where $\Lambda^n \cong \mathbb{Z}^{2n}$ is a lattice) is Kähler.*

Proof. The standard hermitian metric on $\mathbb{C}^n$ descends to a hermitian metric on T^{2n}. We can then use this to define the Riemannian metric and symplectic form. ∎