CHAPTER

1

Principal Bundles

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Why Fiber Bundles?

Suppose we want to study some space M, which for our purposes is a smooth manifold. One way to study M is to study functions $M \to F$ for some fixed target space F, e.g. a manifold , $\mathbb{R}$, or a vector space. This is a perfectly good method of studying M, but is sometimes not enough. More often then not, we want to study "function" from M into a vector space that varies over the base manifold M. For example, a vector field $X \in \mathfrak{X}(M)$ is not really a map $M \to \mathbb{R}^n$, it is an assignment to each $p \in M$ a vector in T_pM. In this way, we are led to the study of a smoothly parameterized family of vector spaces – the tangent bundle TM. This leads us to define a fiber bundle.

Definition 0.1. *Let M and F be a smooth manifolds. A **fiber bundle** over M with model fiber F is the data of a smooth manifold E with a smooth surjective map $\pi : E \to M$ such that for each $p \in M$, there exists an open set U and a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ such that*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "p_1"] \\ & U & \end{tikzcd}$$

*where $p_1 : U \times F \to U$ is projection onto the first factor. The map $\varphi : \pi^{-1} \to U \times F$ is called a **local trivialization**. The space E is called the **total space**, while the manifold M is called the **base space**. We often omit naming the map π and denote the fiber $\pi^{-1}(x)$ by E_x.*

This definition captures the notion of a family of manifolds diffeomorphic to F that are smoothly parameterized by the base space M. We also have a notion of a morphism between bundles.

Definition 0.2. *Let $\pi_E E \to M$ and $\pi_F : F \to M$ be fiber bundles with model fiber X. A* ***bundle homomorphism*** *is the data of a smooth map $\varphi : E \to F$ such that the diagram*

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "\pi_E"'] & & F \arrow[dl, "\pi_F"] \\ & M & \end{tikzcd}$$

commutes.

Our original motivation for thinking about fiber bundles was for a generalized notion of a function. To this end, we specify a special class of maps associated to a fiber bundle.

Definition 0.3. *Let $\pi : E \to M$ be a a fiber bundle with model fiber F. A* ***local section*** *of $\pi : E \to M$ is a smooth map $\sigma : U \to E$ such that $\pi \circ \sigma = \mathrm{id}_U$ for some open set $U \subset M$. If $U = M$, we call σ a* ***global section****. Equivalently, it is a smooth assignment to each $p \in U$ a point in the fiber E_p. We denote the space of sections over an open set U as $\Gamma_U(E)$.*

A section of $E \to M$ can be thought of as our desired generalization of a function. A map $M \to F$ is the same data as a section of the trivial bundle $M \times F \to M$. However, not every fiber bundle is trivial – there can be a nontrivial "twisting." An example of this is the Möbius band. Can you see why this bundle over S^1 is not isomorphic to the trivial bundle $S^1 \times [0, 1]$?

We are especially interested in two special classes of fiber bundles that carry additional structure – the fibers of a vector bundle carry the extra structure of a vector space, and the fibers of a principal bundle have the extra structure of a G-torsor for a Lie group G.

Definition 0.4. *A* ***vector bundle of rank*** *k is a fiber bundle $E \to M$ such that each fiber E_x has the structure of a k-dimensional vector space (usually over $\mathbb{R}$ or $\mathbb{C}$). A* ***vector bundle homomorphism*** *is a bundle homomorphism that restricts to a linear map on each fiber.*

Vector bundles form a familiar family of fiber bundles, as tangent bundles, cotangent bundles, and their associated tensor bundles are all vector bundles.

Definition 0.5. *Let G be a Lie group. a (left)* ***G-torsor*** *is a smooth manifold M equipped with a smooth (left) G-action that is free in transitive i.e.*

1. *For any $g \in G$ and $p \in M$, if $g \cdot p = p$, then $g = e$.*

2. For any $p \in M$, the orbit $G \cdot p$ is all of M

combined, these conditions give that for any fixed point p, there exists a unique group element $g \in G$ that takes p to any other point of M. In other words, the map $g \mapsto g \cdot p$ is a diffeomorphism.

Example 0.1. *Let V be a vector space. A* ***basis*** *(also called a* ***frame****) of V is a linear isomorphism $\mathbb{R}^n \to V$, and the set of frames of V is denoted $\mathcal{B}(V)$. There is a natural right action of $GL_n\mathbb{R}$ on $\mathcal{B}(V)$ by precomposition, i.e. for $g \in GL_n\mathbb{R}$ and $b \in \mathcal{B}(V)$, we have $b \cdot g = b \circ g$. This action is free and transitive, which allows us to define a topology and smooth structure on $\mathcal{B}(V)$, giving it the structure of a right $GL_n\mathbb{R}$-torsor.*

G-torsors can be thought of as Lie groups without a fixed choice of identity element. Indeed, if we fix a point p of a G-torsor M, this determines a unique Lie group structure on M where p is the identity element via the diffeomorphism $g \mapsto g \cdot p$. For example, there is no distinguished basis for a vector space V, which is reflected by the fact that $\mathcal{B}(V)$ is a torsor, rather than a group. This then allows us to define a principal G-bundle.

Definition 0.6. *A* ***principal G-bundle*** *is a fiber bundle $\pi : P \to M$ with a smooth* right *G-action such that every fiber $\pi^{-1}(x)$ is a G-torsor, and every point $p \in M$ in the base has a local trivialization $\pi^{-1}(U) \to U \times G$ that is G-equivariant, where the G-action on $U \times G$ is right multiplication on the second factor. The group G is often called the* ***structure group****.*

Example 0.2 (*Bundles of frames*). *Let M be a smooth manifold, and $p \in M$. Let $\mathcal{B}_p$ denote the $GL_n\mathbb{R}$-torsor of frames of the tangent space T_pM, and let $\mathcal{B}(M)$ be the set*

$$\mathcal{B}(M) = \coprod_{p \in M} \mathcal{B}_p$$

This set comes with a natural map $\pi : \mathcal{B}(M) \to M$ that maps a frame $b \in \mathcal{B}_p$ to p, so the fiber $\pi^{-1}(p)$ over any point is $\mathcal{B}_p$. Local coordinates x^i on an open set $U \subset M$ induce bijections $\pi^{-1}(U) \to U \times GL_n\mathbb{R}$, where we map $b \in \mathcal{B}_p$ to (p, A_b), where $A_b \in GL_n\mathbb{R}$ is the matrix where the i^{th} column are the components of $b(e_i)$ with respect to the coordinate vectors ∂_i. This then induces a topology and smooth structure on $\mathcal{B}(M)$ such that the projection map $\pi : \mathcal{B}(M) \to M$ is smooth and the maps $\pi^{-1}(U) \to U \times GL_n\mathbb{R}$ are smooth local trivializations.

This construction works for vector bundles as well. For a rank k vector bundle $E \to M$, there is a principal $GL_n\mathbb{R}$-bundle $\mathcal{B}(E) \to M$, where the fiber over a point p is the $GL_k\mathbb{R}$-torsor of frames for the fiber E_x.

Heuristically, principal bundles can be thought of bundles of symmetries of some object, and in many cases, this object is a vector bundle, as we will soon see.

Groups Control the Geometry

Given any geometric object M (e.g. a vector space, manifold, fiber bundle), an extremely important question to as is – "what are the symmetries of M?" More concretely, we would like to know the *automorphisms* of M. Any quantity intrinsic to the object M must be invariant under these automorphisms, sine we have no "preferred frame of reference" for the object M. In this way, we see that any geometric properties of M are defined by the symmetry group G, and we say the group G "controls" the geometry of M. For example, given a finite dimensional vector space V, the group of symmetries is the general linear group $GL(V)$ of invertible linear transformations. If we further fix an inner product $\langle \cdot, \cdot \rangle$ on V, we have a smaller class of symmetries – the orthogonal group $O(V) \subset GL(V)$ of linear automorphisms preserving the inner product. Additional structure (e.g. orientation, symplectic form, complex structure) allows us to pick out subgroups of automorphisms, which gives a more restrictive class of symmetries of our vector space, and these subgroups define the geometry of the space.

This proves to be a very fruitful philosophy for approaching geometry. Smooth manifolds are locally modeled on vector spaces, and given a structure on the linear world of vector spaces, we often get an analogous structure in the nonlinear world of manifolds. For example, a Riemannian metric g on a manifold M is the nonlinear analogue of an inner product on a vector space. Because of this, we would expect the orthogonal group O_n to play an important role in the geometry of a Riemannaian manifold. The interaction of the group with the geometry comes in play through the language of principal bundles. Before we mentioned that principal bundles can be thought of as bundles of symmetries, which we make precise with the construction of a ***associated bundle***.

Definition 0.7. *Let $P \to M$ be a principal G-bundle, and F a smooth manifold with a smooth left G-action. The **associated bundle** is the bundle*

$$P \times_G F = P \times F/(p, f) \sim (p \cdot g, g^{-1} \cdot f)$$

This is a fiber bundle over M with model fiber F, and it is a good exercise to see why this is true by explicitly constructing the local trivializations in terms of local trivializations of $\mathcal{B}(M)$.

The construction is a bit obtuse, so we make a few observations to make sense of why we want to care about associated bundles.

Example 0.3 (***The tangent bundle***)**.** *Let M be a smooth manifold. From M, we have a principal GL_n-bundle $\mathcal{B}(M) \to M$ of frames for TM. In addition, $GL_n\mathbb{R}$ admits a natural left action on $\mathbb{R}^n$ via matrix multiplication. The associated bundle $E = \mathcal{B}(M) \times_{GL_n\mathbb{R}} \mathbb{R}^n$ is isomorphic to the tangent bundle TM. To see this, we construct maps in both directions.*

Let $[b, v]$ denote an equivalence class in E, where $b \in \mathcal{B}_p$ is a frame for T_pM and $v \in \mathbb{R}^n$. Then define $\varphi : E \to TM$ by $\varphi[b, v] = b(v)$. This is well defined, since

$$\varphi[b \cdot g, g^{-1} \cdot v] = b \circ g(g^{-1}(v)) = b(v)$$

In the other direction, let $(p, v) \in TM$, i.e. $p \in M$ and $v \in T_pM$. Fix a basis $b : \mathbb{R}^n \to T_pM$, and define $\psi : TM \to E$ by $\psi(p, v) = [b, b^{-1}(v)]$. It's an easy exercise to check that these two maps compose to identity in both directions.

The above construction works in the setting of vector bundles as well. Given a rank k vector bundle $E \to M$, we can construct the principal $GL_k\mathbb{R}$-bundle $\mathcal{B}(E)$, and the associated bundle $\mathcal{B}(E) \times_{GL_k\mathbb{R}} \mathbb{R}^k$ is isomorphic to the bundle E, which tells us that the processes of taking frame bundles and constructing associated bundles are inverses to each other – no information is lost in either direction, so we can work in whichever setting proves most convenient.

This construction also shows why we might care about associated bundles – the quotient by relation we specified exactly gives the correct transformation law for tangent vectors. You may have heard the joke that physicists define a vector as "something that transforms like a vector." A more precise joke here would perhaps be "a tangent vector is something that transforms like a tangent vector." What do we mean by this? Physically, the only "real" quantities are those invariant under a change of reference frame. In terms our example, we see that a tangent vector $v \in T_pM$ is *not* just an n-tuple of numbers, it's the collection of all coordinate representations of v with respect to any basis of T_pM, which is exactly what the associated bundle construction captures. We can see this another way. Fix an element $b \in \mathcal{B}_p \subset \mathcal{B}(M)$. This then determines an isomorphism of $\mathbb{R}^n$ to the fiber of the associated bundle $\mathcal{B}(M) \times_{GL_n\mathbb{R}} \mathbb{R}^n$ by $v \mapsto [b, v]$, this exactly what happens when we fix a basis for T_pM! In this way, we see that fixing the first component in the equivalence class of an element of an associated bundle is essentially a choice of basis or reference frame, and doing the computations with the equivalence classes themselves is in essense, working in a coordinate-free manner by working in all coordinate systems simultaneously.

Another important property of the associated bundle construction is that we can interpret sections of any associated bundle in terms of maps on the principal bundle.

Proposition 0.1. *Let $P \to M$ be a principal G-bundle, and F a manifold with a left G-action, and let $E = P \times_G F$ denote the corresponding associated bundle. Then there is a bijective correspondence*

$$\Gamma_M(E) \longleftrightarrow \{G\text{-equivariant maps } P \to F\}$$

Proof. We provide maps in both directions. First, let $\psi : M \to E$ be a section. We then define the corresponding G-equivariant map $\widetilde{\psi} : P \to F$. We know that for any $x \in M$, we have that $\psi(x) = [p, f]$ for $p \in P_x$. Then define $\widetilde{\psi}(p)$ to be the second factor of the unique representative of $[p, f]$ with p as a first factor. By the definition of the associated bundle, this map is G-equivariant.

In the other direction, let $\widetilde{\varphi} : P \to F$ be a G-equivariant map. We then construct its corresponding section $\varphi : M \to E$. For $x \in M$, let $\varphi(x) = [p, \tilde{\varphi}(p)]$ for any $p \in P_x$. The definition of the associated bundle ensures that this is well defined, and it is easy to see that the two maps we defined are inverses to each other. ■

This is a specific example of a very general phenomenon – we can either work on the base manifold M, or we can work G-equivariantly on the principal bundle $P \to M$ and use the G-equivariance to descend to the base manifold. The idea of working G-equivariantly captures the idea in all coordinate systems at once.

We now address how fixing additional structures on our manifold M changes this picture. The best example here is that of a Riemannian metric g on M, which gives us a notion of an ***orthonormal frame*** of the tangent space T_pM, which is a linear *isometry* $b : (\mathbb{R}^n, \langle \cdot, \cdot \rangle) \to (T_pM, g_p)$ where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$, and g_p is the Riemannian metric evaluated at the point p. We then construct the ***orthonormal frame bundle*** $\mathcal{B}_O(M)$ in the same way we constructed the frame bundle $\mathcal{B}(M)$, which is a principal O_n-bundle, which reflects that we now have a more restrictive view on symmmetry – our automorphisms now take values in O_n instead of $GL_n\mathbb{R}$.

Definition 0.8. *Let M be a smooth manifold, and $\mathcal{B}(M)$ its $GL_n\mathbb{R}$-bundle of frames. Let G be a Lie group, and $\rho : G \to GL_n\mathbb{R}$ be a homomorphism. Then a* ***reduction of structure group*** *to G is the data of a principal G-bundle $Q \to M$ equipped with a G-equivariant map $Q \to \mathcal{B}(M)$, where G acts on the left of $\mathcal{B}(M)$ by $g \cdot b = b \circ \rho(g)^{-1}$.*

In the case we illustrated above with a Riemannian metric g, the homomorphism ρ is just the inclusion $O_n \hookrightarrow GL_n\mathbb{R}$, the bundle Q is $\mathcal{B}_O(M)$, and the O_n-equivariant map is the inclusion $\mathcal{B}_O(M) \hookrightarrow \mathcal{B}(M)$. However, the homomorphism ρ does *not* need to be injective, which makes the name a bit a misnomer – the group does not need to be a subgroup of $GL_n\mathbb{R}$. For example, when working with the Spin group Spin_n, the map ρ is often the double cover $\mathrm{Spin}_n \to SO_n$.

Our construction of the tangent bundle as an associated bundle can also be generalized to other tensor bundles built out of the tangent bundle (e.g. $\mathcal{T}^k_\ell(M)$, T^*M, $\Lambda^k(T^*M)$, etc), by using the induced representations of the stucture group G on tensor products, the dual representation, and the induced representations on exterior powers respectively. From this, we see that the associated bundle construction is the bridge between the theory of principal bundles and vector bundles with representation theory, and reductions of structure group allows us to import our linear model geometries (e.g. an inner product space, a vector space with a complex structure I, an oriented vector space, etc.) to the nonlinear world of manifolds by asking for a reduction of structure group to the appropriate group. For example, an orientation is equivalent to a reduction of structure group to $GL_n^+\mathbb{R} = \{A \in GL_n\mathbb{R} \ : \ \det A > 0\}$, and an almost complex structure on a $2n$-dimensional manifold is equivalent to a reduction of structure group from $GL_{2n}\mathbb{R}$ to $GL_n\mathbb{C}$.

Connections

Another vital piece of the theory of principal bundles is the theory of connections. Unfortunately, this discussion will require some additional background. This section will assome some knowledge of algebraic topology (on the level over covering spaces), and a decent amount of comfort with differential geometry.

We first motivate why we would care about connections. Suppose our group G is finite. Then a principal G-bundle $P \to M$ is the same as a (possibly disconnected) $|G|$-fold cover of M equipped with a free and transitive G action. For example, double covers of M are exactly principal $\mathbb{Z}/2\mathbb{Z}$-bundles over M. An extremely important property of covering spaces is the ***path lifting property***. Given a cover $P \to M$, a point $x \in M$, and a lift $p \in P_x$ of x, let $\gamma : I \to M$ be a path with $\gamma(0) = x$. Then there exists a unique path $\widetilde{\gamma} : I \to P$ such that $\widetilde{\gamma}(0) = p$, and $\pi \circ \widetilde{\gamma} = \gamma$. We call the path $\widetilde{\gamma}$ a ***lift*** of γ. This makes use of the fact that there is a neighborhood about p that is mapped diffeomorphically onto a neighborhood of x.

We would also like a path lifting property for a general principal bundle $P \to M$, but our hopes are immediately dashed by a simple dimension count – the dimension of P is $\dim G + \dim M$, so if G is not discrete, we have no hope of lifting a path, since there will be no local diffeomorphisms we will be able to use to lift a path. To accomplish our goal of path lifting, we need additional data.

Definition 0.9. *Let M a smooth manifold. A **rank k distribution** H over M is a vector subbundle $H \subset TM$, such that the fiber H_x over any $x \in M$ is a k-dimensional subspace of T_xM.*

A distribution can be thought of as a higher dimensional analogue of a vector field, where instead of tangent vectors, we are asking for tangent k-planes. A principal bundle comes equipped with a natural distribution.

Definition 0.10. *Let $\pi : E \to M$ be a fiber bundle with model fiber F. Then the subbundle $\ker \pi_*$ is a distribution of rank $\dim F$ called the **vertical distribution**.*

Since a fiber bundle is locally modeled on a product $U \times F$, we can think of it locally as several copies of F sitting over out manifold M. The vertical distribution consists of all the directions in the fiber. If you move in any of the directions of the vertical distribution, you stay in the same fiber you started with.

In the special case that we are working with a principal bundle, an important fact is that the vertical distribution has a canonical identification with the product bundle $M \times \mathfrak{g}$ of M with the Lie algebra $\mathfrak{g}$ of G.

Proposition 0.2. *Let $P \to M$ be a principal G-bundle with vertical distribution V. Then every fiber V_p of the vertical distribution has a canonical identification with $\mathfrak{g}$.*

Proof. Using the identification of $\mathfrak{g}$ with left-invariant vector fields on G, there is an ***exponential map*** $\exp : \mathfrak{g} \to G$ where for an element $X \in \mathfrak{g}$, we define $\exp(X) = \gamma_X(1)$, where γ_X is the integral curve of the vector field X with $\gamma_X(0) = e$. The exponential map maps a small line segment $\{tX \ : \ t \in (-\varepsilon, \varepsilon)\}$ to a curve $\gamma : (-\varepsilon, \varepsilon) \to G$, with $\gamma(0) = e$. Fix a point $p \in P$. Then any Lie algebra element $X \in \mathfrak{g}$ determines a tangent vector $\widehat{X} \in T_pP$ by the formula

$$\widehat{X} = \frac{d}{dt}\bigg|_{t=0} p \cdot \exp(tX)$$

Since the group action preserves the fibers, $\widehat{X} \in V_p$, and it is a useful exercise to check that the mapping $X \mapsto \widehat{X}$ is an isomorphism $\mathfrak{g} \to V_p$. ■

There is an analogous identification for a vector bundle.

Proposition 0.3. *Let $\pi : E \to M$ be a vector bundle with vertical distribution V. Then for $e \in E$, there is a canonical identification of V_e with $E_{\pi(e)}$, where the mapping is given by*

$$v \mapsto \frac{d}{dt}\bigg|_{t=0} e + tv$$

Definition 0.11. *Let $\pi : E \to M$ be a fiber bundle with vertical distribution V. Then a* ***connection*** *is a choice of* ***horizontal distribution****, i.e. a distribution $H \subset TE$ such that for every $p \in P$, we have that $V_p \oplus H_p = T_pE$.*

A connection on E gives us distinguished "manifold directions" which are complementary to the "fiber directions" defined by the vertical distribution V. By the first isomorphism theorem, the differential $\pi_*|_{H_p} : H_p \to T_{\pi(p)}M$ is an isomorphism, which is the essential fact we will use to define path lifting on a fiber bundle.

Specializing to principal bundles $P \to M$, one useful thing to note is that the choice of complementary subspace to V_p is equivalent data to the choice of a projection map $T_pP \to V_p$, with H_p as the kernel. Using the identification of V_p with $\mathfrak{g}$, this is equivalent to a surjective map $T_pP \to \mathfrak{g}$. Doing this fiberwise, we see that a connection H is equivalent data to a $\mathfrak{g}$ valued 1-form $\Theta \in \Omega^1_P(\mathfrak{g})$, i.e. a section of $\Lambda^1(P) \otimes \mathfrak{g}$, where we recover H as the kernel of Θ. In addition, since the projection map is identity on V, it is clear that for $X \in \mathfrak{g}$, we have that $\Theta(\widehat{X}) = X$. The form Θ is called the ***connection form***. An analogous construction can be made for a vector bundle $\pi : E \to M$ as well, giving a connection form $\omega \in \Omega^1_E(\pi^* E)$ valued in the pullback bundle $\pi^* E$.

We now return to the issue of path lifting. Given a fiber bundle $\pi : E \to M$ equipped with a connction $H \subset TE$, we remarked earlier that $\pi_*|_{H_p} : H_p \to T_{\pi(p)}M$ is an isomorphism. This means that for any tangent vector $v \in T_{\pi(p)}$, there is a *unique* vector $\tilde{v} \in H_p \subset T_pE$ such that $\pi_*(\tilde{v}) = v$. We call $\tilde{v}$ the ***horizontal lift*** of the vector v. We now have all the ingredients in place for path lifting.

Let $\pi : E \to M$ be a fiber bundle equipped with connection H, and let $\gamma : I \to M$ be a smooth path with $\gamma(0) = x$. For each time $t \in I$, we have a velocity vector $\gamma'(t) \in T_{\gamma(t)}M$. This determines a vector field V on the fibers along the path γ, where the value of V at any point $p \in P_{\gamma t}$ is the unique horizontal lift of $\gamma'(t)$ to p. By the theory of differential equations, we can integrate these vector fields to a flow on the fibers along the path, so fixing a point $p \in P_x$ determines a unique integral curve $\widetilde{\gamma} : I \to P$ starting at p, which satisfies $\pi \circ \widetilde{\gamma} = \gamma$, solving the path lifting problem. We can then use the lifted paths to define ***parallel transport*** along a curve. Given a curve $\gamma I \to M$, we get a family of maps $\tau_t : E_{\gamma(0)} \to E_\gamma(t)$, where $\tau_t(f) = \gamma_f(t)$, where γ_f is the unique lift of γ to a path starting at $f \in E_{\gamma(0)}$. These maps are isomorphisms, since they have an inverses given by the parallel transport maps induced by the path γ traversed backwards. The parallel transport maps give us a way to transport data between fibers along a curve, so a connection "connects" nearby fibers, justifying the name. Our usage of horizontal lifts of velocity vectors shows us why the connection was the missing data for path lifting. Since our group was not discrete, we did not have a distinguished "manifold" direction to lift the path, which was exactly the problem that the horizontal distribution solved. In this way, we see that a recipe for path lifting is equivalent data for a connection, since any recipe for path lifting needs to specify a distinguished "manifold" direction to connect nearby fibers.

We now restrict our attention to principal bundles. An extremely important fact is that a connection on a principal G-bundle $P \to M$ induces a connection on *any* associated bundle $P \times_G F$. There are several ways to see this. One way is through the horizontal distribution. Let $H \subset TP$ be the horizontal distribution on P. Then $T(P \times F)$ is canonically isomorphic to $TP \oplus TF$, so we get a distribution $H \times \{0\} \subset TP \oplus TF$. This is invariant under the G action on $P \times F$, so it descends to the quotient $P \times_G F$, giving us a horizontal distribution on the associated bundle. Another way to see this is that the connection on P gives us path lifting on any associated bundle. Fix a basepoint $x \in M$, and a basepoint (p, f) in the fiber of $P \times_G F$ over x. Then given a path $\gamma : I \to M$ based at x, we get a unique lift $\widetilde{\gamma} : I \to P$ to a curve in p. We then get a path $\widehat{\gamma} : I \to P \times_G F$ defined by $\widehat{\gamma}(t) = [\widetilde{\gamma}(t), f]$. This gives a useful intepretation of curves on fiber bundles with horizontal velocity vectors. Thinking of the principal bundle component as a sort of "coordinate system," These curves are constant in the second component, and the connection on P allows us to pick a distinguished path of frames. In the case of the tangent bundle, we can say even more. Given a path $\gamma : I \to M$ in M, the mapping $t \mapsto (\gamma(t), \gamma'(t))$ defines a curve in TM. Thinking of TM as the associated bundle $\mathcal{B}(M) \times_{GL_n\mathbb{R}} \mathbb{R}^n$, what does it mean for this curve in TM to be of the form $[\widetilde{\gamma}(t), v]$ for a fixed $v \in \mathbb{R}^n$? We can think of v as the *acceleration*. Therefore, asking for the curve in TM to be horizontal is like asking for the curve γ on M to be *constant accleration*, which is one definition of a geodesic.

References

Most of the content of this article came from many discussions with Professor Dan Freed. If you want to learn more, I heavily recommend finding some time to talk to him. A good reference for the theory of principal bundles can be found in Kobayashi and Nomizu's books *Foundations of Differential Geometry*.