A QUICK SURVEY OF HODGE THEORY

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Contents

1. Linear Algebra

Definition 1.1. Let V be a finite dimensional $\mathbb{R}$-vector space. An ***almost complex structure*** on V is a linear map $J : V \to V$ satisfying $J^2 = \mathrm{id}_V$. An ***almost complex vector space*** is tuple (V, J), where V is a finite dimensional $\mathbb{R}-$vector space equipped with an almost complex structure J.

The data of an almost complex structure J is equivalent to giving V the structure of a complex vector space, where we define $(a + bi) \cdot v = ab + bJv$. Because of this, we may call J a ***complex structure***. We use the name *almost* complex structure to emphasize the differences between the analogous constructions in the nonlinear world of manifolds. Note that since an almost complex structure is equivalent to a complex structure, this immediately implies that V is even dimensional.

One way to think about an almost complex structure is through the geometric interpretation of complex multiplication. If we regard $\mathbb{C}$ as a 2-dimensional vector space over $\mathbb{R}$, multiplication by i corresponds to a rotation by $\pi/2$ in the counterclockwise direction, and multiplication by $-i$ corresponds to a rotation by $\pi/2$ in the clockwise direction. From this we see that a choice of a square root of -1 comes with a choice of clockwise or counterclockwise. This implies that every complex vector space is canonically oriented as a real vector space. Given a $\mathbb{C}$-vector space V with $\mathbb{C}$-basis $\{z_1, \ldots, z_n\}$, we have that the ordered basis $(z_1, iz_1, \ldots z_n, iz_n)$ defines a positively oriented basis for V over $\mathbb{R}$.

Definition 1.2. Let V by any $\mathbb{R}$-vector space. The ***complexification*** of V is the complex vector space $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$. The complexification $V_{\mathbb{C}}$ is naturally a complex vector space, where for $\lambda \in \mathbb{C}$, the action of λ on a homogeneous element $\mu \otimes v$ is given by

$$\lambda \cdot (\mu \otimes v) = \lambda\mu \otimes v$$

The complexification is a different way of obtaining a complex vector space from a real one. The complexification $V_{\mathbb{C}}$ is a vector space of twice the dimension of V as a vector space over $\mathbb{R}$. In addition, there is a natural inclusion $V \hookrightarrow V_{\mathbb{C}}$ given by $v \mapsto 1 \otimes v$. Complexification is an instance of *extension of scalars* – every element of $V_{\mathbb{C}}$ is of the form $av + biw$ where $a, b \in \mathbb{R}$, and $v, w \in V$. Therefore, we will denote the element $(a + bi) \otimes v \in V_{\mathbb{C}}$ by $av + biv$, and we have a direct sum decomposition $V_{\mathbb{C}} = V \oplus iV$. Given a linear map $T : V \to W$ of $\mathbb{R}$-vector spaces, we can extend T to a complexified map $T_{\mathbb{C}} : V_{\mathbb{C}} \to W_{\mathbb{C}}$, where $T_{\mathbb{C}}(av + biw) = aTv + biTv$. In other words, $T_{\mathbb{C}} = \mathrm{id}_{\mathbb{C}} \otimes T$. In this way, we see that complexification defines a covariant functor $\mathsf{Vect}_{\mathbb{R}} \to \mathsf{Vect}_{\mathbb{C}}$ from the category of $\mathbb{R}$-vector spaces to the category of $\mathbb{C}$-vector spaces.

A natural question to ask is how an almost complex structure interacts with the process of complexification. Let (V, J) be an almost complex vector space. Then the complexified map $J_{\mathbb{C}} : V_{\mathbb{C}} \to V_{\mathbb{C}}$ squares to -1, and admits eigenvalues $\pm i$. For example, consider $V = \mathbb{C}$. We then can then make the natural identification of $\mathbb{C}$ with the almost complex vector space $(\mathbb{R}^2, J)$, where J is given by the matrix

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

using the ordered $\mathbb{R}$ basis $(1, i)$. Then $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}^2$, and $J_{\mathbb{C}}$ is given by the matrix

$$J_{\mathbb{C}} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

in the ordered $\mathbb{R}$ basis $(1, i, 1, i)$, where the first pair elements are elements of one copy of $\mathbb{C}^2$, and the second pair of elements are elements of a separate copy of $\mathbb{C}^2$, giving an $\mathbb{R}$ basis for $\mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$. The matrix $J_{\mathbb{C}}$ clearly has eigenvalues $\pm i$, with the i-eigenspace being spanned by the vectors

$$\begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{i}{2} \\ 0 \\ -\frac{i}{2} \\ 0 \end{pmatrix}$$

and the $-i$ eigenspace being the span of

$$\begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{i}{2} \\ 0 \\ -\frac{i}{2} \end{pmatrix}$$

The case for general V is similar, and amounts to fixing an $\mathbb{R}$ basis of the form $\{v_1, Jv_1, \ldots v_n, Jv_n\}$ for (V, J). The decomposition of $V_{\mathbb{C}}$ into the $\pm i$-eigenspaces of $J_{\mathbb{C}}$ gives a direct sum decomposition

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$$

where $V^{1,0}$ denotes the i-eigenspace, and $V^{0,1}$ denotes the $-i$-eigenspace. Note that in the bases we chose above for $\mathbb{C}^2$, complex conjugation is given by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and determines an isomorphism of complex vector spaces $V^{1,0} \cong \overline{V^{0,1}}$ and vice versa. In addition, complex conjugation picks out a distinguished subspace of $V_{\mathbb{C}}$ isomorphic to V – the 1-eigenspace. Elements of this eigenspace are called ***real*** (much like how the $\mathbb{R}^n$ is left invariant under complex conjugation in $\mathbb{C}^n$).

An almost complex structure J on V induces a dual map $J^* : V^* \to V^*$. By functoriality of taking dual spaces, $(J^*)^2 = -\operatorname{id}_{V^*}$, giving V^* the structure of an almost complex vector space via J^*. Explicitly, given a linear functional $\alpha \in V^*$, we have that the action of J^* on α is given by

$$(J^*\alpha)(v) = \alpha(Jv)$$

for every $v \in V$. We then get an analogous decomposition of the complexified dual space$V^*_{\mathbb{C}} = \mathsf{Hom}_{\mathbb{R}}(V, \mathbb{C})$ as

$$V^*_{\mathbb{C}} = (V^*)^{1,0} \oplus (V^*)^{0,1}$$

into the $\pm i$ eigenspaces of J^*, and have very natural interpretations in terms of V. The subspace $(V^*)^{1,0}$ consists of the α such that $\alpha(Jv) = i\alpha(v)$. We have a natural pairing $(V^*)^{1,0} \otimes V^{1,0} \to \mathbb{C}$ given by

$$\langle \alpha, v \rangle = \alpha(v)$$

which is nondegenerate, establishing the two vector spaces as dual to each other (as complex vector spaces). A similar statement holds for $V^{0,1}$ and $(V^*)^{0,1}$, putting them in duality as well. Another perspective to take is that the condition that $\alpha(Jv) = i\alpha(v)$ is equivalent to α being *complex linear* with respect to the complex structure J on V and i on $\mathbb{C}$, giving us an isomorphism $(V^*)^{1,0} \cong \mathsf{Hom}_{\mathbb{C}}(V, \mathbb{C})$. Similarly, we have that $(V^*)^{0,1} \cong \mathsf{Hom}_{\mathbb{C}}(V, \overline{\mathbb{C}})$.

We the consider the effects of complexification on another linear algebraic construction – the exterior algebra. Given a finite dimensional $\mathbb{R}$-vector space V, we have that the exterior algebra $\Lambda^\bullet V$ has a natural $\mathbb{Z}$-grading

$$\Lambda^\bullet V = \bigoplus_{k=0}^{n} \Lambda^k V$$

Upon complexification, we have a canonical isomorphism $(\Lambda^\bullet V)_{\mathbb{C}} \cong \Lambda^\bullet V_{\mathbb{C}}$, where the right hand side is the exterior algebra of $V_{\mathbb{C}}$ as a *complex* vector space (i.e. $\Lambda^0 V_{\mathbb{C}} \cong \mathbb{C}$, not $\mathbb{R}$), and $\Lambda^\bullet V$ is canonically embedded in $\Lambda^\bullet V_{\mathbb{C}}$ as the subspace invariant under complex conjugation. Further assume that we have an almost complex structure $J : V \to V$. This gives a direct sum decomposition $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$, which then induces a decomposition

$$\Lambda^k V_{\mathbb{C}} \cong \bigoplus_{p+q=k} \Lambda^p V^{1,0} \otimes_{\mathbb{C}} \Lambda^q V^{0,1}$$

giving the complexified exterior algebra a bigrading via the subspaces

$$\Lambda^{p,q}V := \Lambda^p V^{1,0} \otimes_{\mathbb{C}} \Lambda^q V^{0,1}$$

elements of $\Lambda^{p,q}$ are said to have ***bidegree*** (p,q). We now make several simple observations regarding the bigrading of $\Lambda^\bullet V_{\mathbb{C}}$, which stem from our observations regarding the decomposition $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$.

(1) From the fact that $\overline{V^{1,0}} = V^{0,1}$, we get that $\overline{\Lambda^{p,q}V} = \Lambda^{q,p}V$.
(2) Given $\omega \in \Lambda^{p,q}V$ and $\eta \in \Lambda^{s,t}V$, we have that $\omega \wedge \eta \in \Lambda^{p+s,q+t}V$.

2. Complex Manifolds

We now transfer our linear algebraic knowledge to complex manifolds. Let $U \subset \mathbb{C}$ be an open subset, and $z = x + iy$ the usual coordinate function on U. Considered as a subset of $\mathbb{R}^2$, for any point $p \in U$, the tangent space T_pU is spanned by the coordinate vectors $\partial_x|_p$ and $\partial_y|_p$. If we then complexify, the $\partial_x|_p$ and $\partial_y|_p$ also form a basis for $(T_pU)_{\mathbb{C}}$ as a *complex* vector space. Consequently, we can also form a basis for $(T_pU)_{\mathbb{C}}$ with the vectors

$$\left.\frac{\partial}{\partial z}\right|_p := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$
$$\left.\frac{\partial}{\partial \overline{z}}\right|_p := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

which we abbreviate as $\partial_z|_p$ and $\partial\overline{z}|_p$ respectively. Doing this for all p, we obtain vector fields ∂_z and $\partial_{\overline{z}}$ on all of U, where a function $f \in C^\infty(U, \mathbb{C})$ is holomorphic if and only if $\partial_{\overline{z}} f = 0$. Functions f annihilated by ∂_z are ***antiholomorphic***, i.e. $\overline{f}$ is holomorphic. These vector fields give a framing for the complexified tangent bundle $TU_{\mathbb{C}}$, i.e. an isomorphism $TU_{\mathbb{C}} \to U \times \mathbb{C}^2$.

We get a similar description of the complexified cotangent bundle $T^*U_{\mathbb{C}}$, which is framed by the covector fields

$$dz = \frac{1}{2}(dx + idy)$$
$$d\overline{z} = \frac{1}{2}(dx - idy)$$

which are easily verified to the duals of ∂_z and $\partial_{\overline{z}}$ respectively. For an open subset $U \subset \mathbb{C}^n$ with complex coordinates $z^i = x^i + iy^i$, we have analogous vector fields $\partial/\partial z^i$ and $\partial/\partial\overline{z}^i$ and covector fields dz^i and $d\overline{z}^i$.

Definition 2.1. An ***almost complex manifold*** is the data of a smooth manifold X and a smooth section $J \in \Gamma_X(\mathrm{End}(TX))$ such that $J^2 = -\mathrm{id}_{TX}$.

Definition 2.2. A ***complex manifold*** is a smooth manifold X such that there is an open cover $\{(U_\alpha, \varphi_\alpha)\}$ of X by charts U_α and diffeomorphisms $\varphi : U_\alpha \to V_\alpha$ to open sets $V_\alpha \subset \mathbb{R}^{2n} \cong \mathbb{C}^n$ such that the transition maps $\varphi_\alpha \circ \varphi_\beta^{-1}$ are holomorphic.

Definition 2.3. Let X be a complex manifold. A complex valued function $f \in C^\infty(X, \mathbb{C})$ is ***holomorphic*** if for any holomorphic chart (φ, U), we have that $f \circ \varphi^{-1}$ is holomorphic. The ring of holomorphic functions over X is denoted $\mathscr{O}_X$.

Unlike the linear case, the notion of an almost complex manifold and a complex manifold are *not* equivalent. Let (X, J) be an almost complex manifold. For a point $x \in X$, we have that J_x is an almost complex structure on the tangent space T_xX, which gives a decomposition of the complexified tangent space

$$T_xX_{\mathbb{C}} = T_x^{1,0}X \oplus T_x^{0,1}X$$

into the $\pm i$-eigenspaces of complexified J_x. Doing this over all points x, we get a decomposition of the complexified tangent bundle

$$TX_{\mathbb{C}} = T^{1,0}X \oplus T^{0,1}X$$

into the $\pm i$-eigenbundles of J.

Suppose X is a complex manifold. Then every tangent space T_xX inherits an almost complex structure coming from multiplication by i in $\mathbb{C}^n$, giving an almost complex structure J on X. To show this explicitly, let $z^i = x^i + iy^i$ be local holomorphic coordinates for X. Then the coordinate functions $(x^1, y^1, \ldots x^n, y^n)$ form a coordinate system for X as a *real* manifold, and the action of J in these coordinates is simply given by

$$\begin{aligned} \frac{\partial}{\partial x^i} &\mapsto \frac{\partial}{\partial y^i} \\ \frac{\partial}{\partial y^i} &\mapsto -\frac{\partial}{\partial x^i} \end{aligned}$$

and the subbundle $T^{1,0}X \subset TX_{\mathbb{C}}$ is spanned by the $\partial/\partial z^i$, and $T^{0,1}X$ is spanned by the $\partial/\partial \bar{z}^i$. The composition

$$TX \hookrightarrow TX_{\mathbb{C}} \longrightarrow T^{1,0}X$$

consisting of the inclusion followed by projecting then defines an isomorphism of the real tangent bundle with $T^{1,0}X$, which is called the ***holomorphic tangent bundle***. Explicitly, this map is given in local coordinates by

$$\begin{aligned} \frac{\partial}{\partial x^i} &= \frac{\partial}{\partial z^i} + \frac{\partial}{\partial \bar{z}^i} \mapsto \frac{\partial}{\partial z^i} \\ \frac{\partial}{\partial y^i} &= i\frac{\partial}{\partial z^i} - i\frac{\partial}{\partial \bar{z}^i} \mapsto i\frac{\partial}{\partial z^i} \end{aligned}$$

and the $\partial/\partial z^i$ along with the $i\partial/\partial z^i$ form a local frame for the holomorphic tangent bundle $T^{1,0}X$.

Given an almost complex manifold (X, J), a natural question to ask is whether or not X admits the structure of a complex manifold where J is multiplication by i. The answer to this question is a celebrated theorem of Newlander and Nirenberg

Theorem 2.4 (***Newlander-Nirenberg***). *An almost complex manifold (X, J) is a complex manifold (where J is multiplication by i) if and only if the subbundle $T^{1,0}X \subset TX_{\mathbb{C}}$ is integrable in the sense of Frobenius, i.e.*

$$[T^{1,0}X, T^{1,0}X] \subset T^{1,0}X$$

via complex conjugation, this is equivalent to $T^{0,1}X$ being integrable.

Intuitively, the existence of local *holomorphic* coordinates amounts to finding an integral submanifold for $T^{1,0}X$, since if X was complex, $T^{1,0}X$ would arise as the holomorphic tangent bundle of X.

3. Cohomology of Complex Manifolds.

Let X be a complex manifold, and let J denote the canonical almost complex structure on X. Applying our linear algebraic knowledge pointwise, we get a decomposition of the complexified cotangent bundle

$$T^*X_{\mathbb{C}} = (T^*X)^{1,0} \oplus (T^*X)^{0,1}$$

which then gives a decomposition of the space $\Omega^k_X(\mathbb{C})$ of complex valued differential k-forms as

$$\Omega^k_X(\mathbb{C}) = \bigoplus_{p+q=k} \Omega^{p,q}_X(\mathbb{C})$$

where $\Omega^{p,q}_X(\mathbb{C})$ denotes the space of (p,q)-forms, i.e. sections of $\Lambda^{p,q}T^*X$, and the direct sum is taken as modules over $C^\infty(X, \mathbb{C})$. Out of all the $\Lambda^{p,q}T^*X$, there is a distinguished bundle.

Definition 3.1. Let X be a complex manifold of complex dimension $\dim_{\mathbb{C}} X = n$ (i.e. $\dim X = 2n$). The ***canonical bundle*** of X, denote K_X, is the complex vector bundle

$$K_X = \Lambda^{n,0}T^*X$$

Let d denote the de Rham differential $\Omega^k_X \to \Omega^{k+1}_X$ after complexifying it to obtain a map $\Omega^k_X(\mathbb{C}) \to \Omega^{k+1}_X(\mathbb{C})$. In local holomorphic coordinates $\{z^i\}$, we have that the dz^i form a basis for the space of smooth $(1,0)$-forms over the ring $C^\infty(M, \mathbb{C})$, and the $d\overline{z}^i$ form a basis for the space of smooth $(0,1)$-forms. Therefore, given $f \in C^\infty(X, \mathbb{C})$, we can uniquely write $df \in \Omega^1_X(\mathbb{C})$ as

$$df = \frac{\partial f}{\partial z^i} dz^i + \frac{\partial f}{\partial \overline{z}^i} d\overline{z}^i$$

where we use Einstein summation notation. Given an arbitrary (p,q) form α, it has the local coordinate expression

$$\alpha = \sum_{p,q} \alpha_{p,q} dz^{i_1} \wedge \ldots \wedge dz^{i_p} \wedge d\overline{z}^{i_1} \ldots \wedge d\overline{z}^{i_q}$$

for smooth $\alpha_{p,q} \in C^\infty(X, \mathbb{C})$. Then we have that

$$d\alpha = \left(\sum_{p,q} \frac{\partial \alpha_{p,q}}{\partial z^i} dz^i \wedge dz^{i_1} \wedge \ldots \wedge dz^{i_p} \wedge d\overline{z}^{i_1} \ldots \wedge d\overline{z}^{i_q} \right)$$
$$+ \left(\sum_{p,q} \frac{\partial \alpha_{p,q}}{\partial \overline{z}^i} d\overline{z}^i \wedge dz^{i_1} \wedge \ldots \wedge dz^{i_p} \wedge d\overline{z}^{i_1} \ldots \wedge d\overline{z}^{i_q} \right)$$

which shows that $d\alpha$ is a $(p+1, q+1)$ form, i.e. d is of bidegree $(1,1)$. This expression for $d\alpha$ allows us to decompose d as a sum $d = \partial + \overline{\partial}$, where

$$\partial \alpha = \sum_{p,q} \frac{\partial \alpha_{p,q}}{\partial z^i} dz^i \wedge dz^{i_1} \wedge \ldots \wedge dz^{i_p} \wedge d\overline{z}^{i_1} \ldots \wedge d\overline{z}^{i_q}$$

$$\overline{\partial} \alpha = \sum_{p,q} \frac{\partial \alpha_{p,q}}{\partial \overline{z}^i} d\overline{z}^i \wedge dz^{i_1} \wedge \ldots \wedge dz^{i_p} \wedge d\overline{z}^{i_1} \ldots \wedge d\overline{z}^{i_q}$$

from the definition, it is clear that ∂ is of bidegree $(1,0)$ and $\overline{\partial}$ is of bidegree $(0,1)$.

Proposition 3.2.

(1) $\partial^2 = 0$
(2) $\overline{\partial}^2 = 0$
(3) $\partial\overline{\partial} + \overline{\partial}\partial = 0$

Proof. We know that $d = \partial + \overline{\partial}$ satisfies $d^2 = 0$. Therefore, we have that

$$d^2 = (\partial + \overline{\partial})^2 = \partial^2 + \partial\overline{\partial} + \overline{\partial}\partial + \overline{\partial}^2$$

we then note that the bidegrees of all the components. We have:

(1) ∂^2 has bidegree $(2,0)$.
(2) $\partial\overline{\partial} + \overline{\partial}\partial$ has bidegree $(1,1)$
(3) $\overline{\partial}^2$ has bidegree $(0,2)$

in order for d^2 to vanish, all the terms of different bidegrees must vanish, proving all three parts of the proposition. ■

The operators ∂ and $\overline{\partial}$ satisfy many of the same identities as d. Another helpful identity to note is that

$$\partial \alpha = \overline{\overline{\partial}\overline{\alpha}}$$

Definition 3.3. Let X be a complex manifold. A ***holomorphic vector bundle*** over X is a complex vector bundle $\pi : E \to X$ such that the total space E is a complex manifold and the projection π is a holomorphic map.

Example 3.4. For a complex manifold X, the complex vector bundles $\Lambda^{q,0}T^*X$ are all holomorphic vector bundles, which is easily seen using local trivializations defined by local holomorphic coordinates.

Note again that a holomorphic vector bundle is *not* a complex vector bundle (for example, a complex vector bundle can be odd dimensional if the base space is odd dimensional).

Instead, it is a complex bundle with extra structure. Let $\sigma : X \to E$ be a *smooth* section. In a local holomorphic trivialization of E, we can write σ as

$$\sigma = (\sigma^1, \ldots, \sigma^k)$$

for smooth complex-valued functions σ^i, where k is the rank of E as a complex vector bundle. In this trivialization, we can apply the operator $\overline{\partial}$ component-wise to get

$$\overline{\partial}\sigma = (\overline{\partial}\sigma^1, \ldots, \overline{\partial}\sigma^k)$$

and this local definition glues together to give a well defined operator $\overline{\partial}_E : \Omega_X^{0,0}(E) \to \Omega_X^{0,1}(E)$, where $\Omega_X^{p,q}(E)$ denotes the space of sections $\Gamma_X(T^{p,q}X \otimes_{\mathbb{C}} E)$. We then note that the holomorphic sections $X \to E$ are exactly those annihilated by $\overline{\partial}_E$. Since $\overline{\partial}_E$ is defined locally in terms of $\overline{\partial}$, we immediately see that $\overline{\partial}_E^2 = 0$, giving us the ***Dolbeault complex*** of E.

$$\Omega_X^{0,0}(E) \xrightarrow{\overline{\partial}_E} \Omega_X^{0,1}(E) \xrightarrow{\overline{\partial}_E} \cdots\cdots \xrightarrow{\overline{\partial}_E} \Omega_X^{0,n}(E) \xrightarrow{\overline{\partial}_E} 0$$

The ***Dolbeault cohomology groups*** of E are the cohomology groups of the Dolbeault complex, and are denoted $H^p(X, E)$. In the case $E = \Lambda^{0,q}T^*X$, we have that $\overline{\partial}_E = \overline{\partial}$, and we denote these cohomology groups by

$$H^{p,q}(X) = H^p(X, \Lambda^{0,q}T^*X)$$

4. Kähler Manifolds

Kähler manifolds will form a special class of manifolds where Hodge theory will prove especially useful. Before diving into the world of Kähler manifolds, we do some prerequisite linear algebra, which is closely tied to Hermitian geometry.

Definition 4.1. Let V be a complex vector space. A ***Hermitian form*** on V is a bilinear map $h : \overline{V} \otimes_{\mathbb{C}} V \to \mathbb{C}$ satisfying $h(v, w) = \overline{h(w, v)}$.

Another way to define a Hermitian form is as a ***sesquilinear*** form $h : V \otimes_{\mathbb{C}} V \to \mathbb{C}$, i.e. conjugate linear in the first term, and linear in the second term, along with the Hermitian symmetry condition $h(v, w) = \overline{h(v, w)}$. In a basis for V, a Hermitian form is given by a Hermitian matrix H, i.e. $H^\dagger = H$.

Given a Hermitian form h on a complex vector space V, we can decompose h into its real and imaginary parts. Let $g = \mathfrak{Re}(h)$ and $\omega = \mathfrak{Im}(h)$, so we have $h = g + i\omega$.

Proposition 4.2. *Let $h = g + i\omega$ be a hermitian form. Then we have*

(1) g is symmetric
(2) ω is skew-symmetric

Proof.

(1) Since h is Hermitian, we have that $h(v, w) = \overline{h(w, v)}$, which have the same real part. Therefore, $g(v, w) = g(w, v)$, so g is symmetric
(2) We again use the fact that $h(v, w) = \overline{h(w, v)}$. Since conjugation reverses the sign on the imaginary component, we immediately get that $\omega(v, w) = -\omega(w, v)$.

∎

Remark. The decomposition of h into its real and imaginary components is one way of seeing the useful identity

$$U_n = O_{2n} \cap Sp_{2n} \cap GL_n\mathbb{C}$$

where we regard all of the groups as subgroups of $GL_{2n}\mathbb{R}$. Any matrix preserving the standard Hermitian form on $\mathbb{C}^n$ must necessarily preserve both the real and imaginary components, so it must preserve the standard Euclidean inner product on $\mathbb{R}^{2n} \cong \mathbb{C}^n$ and the standard symplectic form on $\mathbb{R}^{2n} \cong \mathbb{C}$.

Given a complex vector space V equipped with a hermitian metric h, let J denote the natural almost complex structure induced by multiplication by i. Writing $h = g + i\omega$, we see that ω and g are intimately related, i.e.

(1) $g(v, Iw) = \omega(v, w)$
(2) $\omega(v, Iw) = g(v, w)$

both of which come from the fact that h is compatible with J in the sense that $h(Jv, Jw) = h(v, w)$. Finally, we note that ω can be naturally identified with a real element of $(V^*)^{1,1}$.

Definition 4.3. Let h be a Hermitian form, and $\omega = \mathfrak{Im}(h)$. Then ω is called the ***Kähler form*** of h.

We now move back to the world of manifolds. Let (X, J) be an almost complex manifold. The almost complex structure gives each tangent space T_xX the structure of a complex vector space, so it makes sense to discuss a Hermitian form h_x on T_xX.

Definition 4.4. A ***Kähler manifold*** is an almost complex manifold (X, J) with a smoothly varying Hermitian metric (i.e. a positive definite Hermitian form) h where J is integrable and the Kähler form ω obtained from h is closed.

From the decomposition $h = g + i\omega$, we see that a Kähler manifold is a melting pot of several compatible structures, namely the Riemannian metric g, the almost complex structure J, and the symplectic form ω.

From the basic definition, we can already make some surprisingly deep statements about Kähler manifolds.

Proposition 4.5. *Let X be a Kähler manifold with Hermitian form $h = g + i\omega$. Then the volume form of X with respect to the Riemannian metric g is $\omega^n/n!$, where $n = \dim_{\mathbb{C}} X$.*

Proof. It suffices to verify this pointwise for a fixed point $x \in X$. The Riemannian volume form dV_g is uniquely characterized by the fact that $(dV_g)_x(e_1, \dots, e_{2n}) = 1$ for an oriented orthonormal basis $\{e_1, \dots e_{2n}\}$ for T_xX. Arrange local holomorphic coordinates $\{z^i\}$ about x such that at x, the Hermitian form h_x is given by the identity matrix, i.e.

$$h_x\left(\left.\frac{\partial}{\partial z^i}\right|_x, \left.\frac{\partial}{\partial z^j}\right|_x\right) = \delta_{ij}$$

Let ∂_i denote $\partial/\partial z^i|_x$. We then have that $\{\partial_1, i\partial_1, \dots \partial_n, i\partial_n\}$ is an oriented orthonormal $\mathbb{R}$-basis for T_xX with respect to g, so it suffices to evaluate ω_x^n on this basis, and verify that it evaluates to $n!$. To do this, we use a lemma.

Lemma 4.6. *In local holomorphic coordinates $\{z^i\}$, the Kähler form ω is given by*

$$\omega = \frac{i}{2}\sum_i H^i_j dz^i \wedge d\overline{z}^i$$

where $H^i_j = h(\partial_i, \partial_j)$.

The proof is an easy consequence of ω being of type $(1,1)$ and the fact that it is the imaginary component of h.

In the coordinates we specified above, we have that H_x is the identity matrix, so the lemma gives us

$$\omega_x = \frac{i}{2}\sum_i dz^i \wedge d\overline{z}^i$$

we note that when we wedge ω_x with itself n times, we constantly pick up duplicates of the dz^i and $d\overline{z}^i$, so only one term survives, leaving us with

$$\omega^n_x = \frac{i}{2}\left(dz^1 \wedge d\overline{z}^i \wedge \cdots \wedge dz^i \wedge d\overline{z}^i\right) = dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n = \frac{dV_g}{n!}$$

where $z^i = x^i + iy^i$. We use the convention that $v \wedge w = \mathrm{Alt}(v \otimes w)$ without the normalizing factor, which explains the extra term of $n!$. ■

Proposition 4.7. *Let X be a Kähler manifold. Then all the even degree de Rham cohomology groups $H^{2n}_{dR}(X, \mathbb{R})$ are nontrivial.*

Proof. The fact that $\omega^n = dV_g/n!$ implies that $\int_M \omega^n = \int_M dV_g = \mathrm{Vol}(M) > 0$, so by Stokes' theorem, ω^n cannot be exact. From this, we can also conclude that ω^k for any k cannot be exact, for if $\omega^k = d\alpha$, then we would have that

$$\omega^n = \omega^{n-k} \wedge d\alpha = d(\omega^{n-k} \wedge \alpha)$$

Therefore, since ω is a degree 2 form, we must have that all the even dimensional cohomology groups are nontrivial ■

This shows that the Kähler condition is quite rigid.

5. Laplacians and Harmonic Forms

We start in the most general case, that of a compact Riemannian manifold (X, g). For the remainder of these notes, all manifolds will be compact and boundaryless, unless denoted otherwise.

Definition 5.1. Let (X, g) be a Riemannian manifold. Then the vector bundles $\Lambda^k T^*X$ inherit natural fiber metrics $(\cdot, \cdot)$, which are defined by the set

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} \,:\, 1 \leq i_1 < i_2 < \cdots < i_k\}$$

being an orthonormal basis for $\Lambda^k T^*_x X$ where $\{e_i\}$ is an orthonormal basis for $T^*_x X$. This then introduces an L^2 ***inner product*** $(\cdot, \cdot)_{L^2}$ on the space Ω^k_X of differential k-forms, where we define

$$(\alpha, \beta)_{L^2} = \int_X (\alpha, \beta)\, dV_g$$

where (α, β) denotes the smooth function $x \mapsto (\alpha_x, \beta_x)$.

Definition 5.2. The ***Hodge star*** operator, denoted $\star$ (though many choose to denote it $*$), is a bundle homomorphism $\star : \Lambda^k T^*X \to \Lambda^{n-k}T^*X$ defined by the property such that for any $\alpha, \beta \in \Lambda^k T^*X$

$$\alpha \wedge \star\beta = (\alpha, \beta)\, dV_g$$

this property uniquely characterizes $\star$.

Proposition 5.3. *$\star$ exists.*

Proof. The defining property uniquely characterizes $\star$, so it suffices to specify a local definition, which will then glue to a smooth bundle homomorphism. Let E_i be a local orthonormal frame for T^*X. Then we know that the set of k-fold wedge products of the E_i with increasing indices forms a basis for each $\Lambda^K T_x^*X$. In addition, we know that $E_1 \wedge \cdots \wedge E_n = dV_g$. Putting this all together, We must have that for $E_I = E_{i_1} \wedge \cdots E_{i_k}$, $\star E_I = E_J$, with $J = \{j_{i_1}, \dots, j_{i_{n-k}}\}$ where the tuple $(i_1, \dots i_k, j_1, \dots j_k)$ is an even permutation of $(1, \dots, n)$. We have then specified $\star$ on an orthonormal basis. ■

One thing to note is that $\star$ is an isomorphism, since it maps orthonormal frames to orthonormal frames. It also satisfies the useful identity

$$\star^2 \alpha = (-1)^{k(n-k)} \alpha$$

where $\alpha \in \Omega_X^k$.

Complexification allows us to obtain analogous operations on complex valued forms on an almost complex manifold (X, J), where we extend g and its induced fiber metrics on the $\Lambda^k T^*X$ to a Hermitian metric h on $TX_{\mathbb{C}}$ and fiber metrics on $\Lambda^K T^*X_{\mathbb{C}}$. Likewise, we complexify the Hodge star $\star$, which gives the identity

$$(\alpha, \beta)\, dV_g = \alpha \wedge \overline{\star\beta}$$

We then get an analogous L^2 metric on complex valued forms, defined by

$$(\alpha, \beta)_{L^2} = \int_X \alpha \wedge \overline{\star\beta}\, dV_g$$

The Hodge star operators allows us to define formal adjoints to d, ∂, and $\overline{\partial}$ with respect to the L^2 metrics.

Definition 5.4. Define the operators

(1) $d^* : \Omega_X^k \to \Omega_X^{k-1}$
(2) $\partial^* : \Omega_X^{p,q}(\mathbb{C}) \to \Omega_X^{p-1,q}(\mathbb{C})$
(3) $\overline{\partial}^* : \Omega_X^{p,q}(\mathbb{C}) \to \Omega_X^{p,q-1}(\mathbb{C})$

by the formulas

(1) $d^* = (-1)^k \star^{-1} d\star$
(2) $\partial^* = -\star$
$dbar\star$
(3) $\overline{\partial}^* = -\star$
$partial\star$

Proposition 5.5. *The operators d^*, ∂^*, and $\overline{\partial}^*$ are formally adjoints to their respective operators with respect to the appropriate L^2 metrics, i.e.*

$$\begin{aligned}(d\alpha, \beta)_{L^2} &= (\alpha, d^*\beta)_{L^2}\\ (\partial\alpha, \beta)_{L^2} &= (\alpha, \partial^*\beta)\\ (\overline{\partial}\alpha, \beta)_{L^2} &= (\alpha, \overline{\partial}^*\beta)\end{aligned}$$

Proof. We prove the case for d and $\overline{\partial}^*$. The proof for ∂^* is near identical to the one for $\overline{\partial}^*$. We know that the exterior derivative is an ***anti-derivation*** i.e. it satisfies

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k d\beta$$

where $k = |\alpha|$ denotes the degree of α. In particular, this implies that for a k-form α and a $k+1$-form β, we have that

$$d(\alpha \wedge \star\beta) = d\alpha \wedge \star\beta + (-1)^k \alpha \wedge d \star \beta$$

From Stokes' theorem, we have that

$$\int_X d(\alpha \wedge \star\beta) = \int_{\partial X} \alpha \wedge \star\beta = 0$$

since $\partial X = \varnothing$. Therefore, we have that

$$\int_X d\alpha \wedge \star\beta = \int_X (-1)^{k+1} \alpha \wedge d \star \beta$$

we further note that the left hand side is the definition of $(d\alpha, \beta)_{L^2}$. We also compute

$$\begin{aligned}(\alpha, d^*\beta)_{L^2} &= \int_X \alpha \wedge \star d^*\beta\\ &= \int_X \alpha \wedge ((-1)^{k+1} \star \star^{-1} d \star \beta)\\ &= \int_X (-1)^{k+1} \alpha \wedge d \star \beta\end{aligned}$$

proving the desired equality.

For $\overline{\partial}^*$, we have a similar proof. Let α be a complex k-form, and β a complex $k+1$-form. Then we have

$$(\overline{\partial}\alpha, \beta)_{L^2} = \int_X \alpha \wedge \overline{\star\beta}$$

again, we have that

$$\overline{\partial}(\alpha \wedge \overline{\star\beta}) = \overline{\partial}\alpha \wedge \overline{\star\beta} + (-1)^k \overline{\partial}(\overline{\star\beta})$$

the integral of the left hand side over all of X is 0 as a consequence of Stokes' theorem, which gives us

$$\int_X \overline{\partial}\alpha \wedge \overline{\star\beta} = \int_X (-1)^{k+1} \alpha \wedge \overline{\partial}(\overline{\star\beta}) = \int_X (-1)^{k+1} \alpha \wedge \overline{\partial \star \beta}$$

The left hand side is $(\overline{\partial}\alpha, \beta)_{L^2}$. We claim (as was the case with d^*), that the right hand side is equal to $(\alpha, \overline{\partial}^*\beta)$. We compute

$$\begin{aligned}(\alpha, \overline{\partial}^*\beta)_{L^2} &= \int_X \alpha \wedge \overline{\star\partial^*\beta} \\ &= \int_X \alpha \wedge \overline{\star(-\star\partial\star\beta)} \\ &= -\int_X \alpha \wedge \overline{\star^2\partial\star\beta} \\ &= (-1)^{k(n-k)+1}\int_X \alpha \wedge \overline{\partial\star\beta}\end{aligned}$$

We note that since X is a complex manifold, n is even, so this is equal to

$$(-1)^{k^2+1}\int_X \alpha \wedge \overline{\partial\star\beta} = (-1)^{k+1}\int_X \alpha \wedge \overline{\partial\star\beta} = (\overline{\partial}\alpha, \beta)_{L^2}$$

where we use the fact that k and k^2 have the same parity. ■

A similar story holds for the $\overline{\partial}$ operator on a holomorphic vector bundle $E \to X$. Suppose $\dim_{\mathbb{C}} X = n$, so $\dim X = 2n$, and let $E \to X$ be a holomorphic vector bundle equipped with a Hermitian metric. Let ω denote the Kähler form of X. Then $\omega^n \in \Omega_X^{n,n}$ determines a trivialization $\Lambda^{n,n}T^*X \to X \times \mathbb{C}$, which introduces a pairing

$$\begin{aligned}\Lambda^{0,q}T^*X \otimes \Lambda^{n,n-q}T^*X &\to \mathbb{C} \\ \alpha \otimes \beta &\mapsto \lambda\end{aligned}$$

where $\lambda \in \mathbb{C}$ is the unique number where $\alpha \wedge \beta = \lambda\omega^n$. This pairing is nondegenerate, which establishes the duality

$$\Lambda^{0,q}T^*X \cong (\Lambda^{n,n-q}T^*X)^*$$

tensoring with the holomorphic vector bundle E, we get the duality

$$\Lambda^{0,q}T^*X \otimes E \cong \Lambda^{n,n-q}T^*X \otimes E^*$$

Furthermore, the Hermitian metrics on both E and X gives an antilinear isomorphism

$$\Lambda^{0,q}T^*X \otimes E \to (\Lambda^{0,q}T^*X \otimes E)^*$$

putting these together, we get an antilinear isomorphism

$$\Lambda^{0,q}T^*X \otimes E \to (\Lambda^{0,q}T^*X \otimes E)^* \to \Lambda^{n,n-q}T^*X$$

which we denote $\star_E$. As the notation suggestions, $\star_E$ is the analogue of the Hodge star operator. Let $\langle\cdot,\cdot\rangle$ denote the Hermitian metric on E. Then given E-valued forms $\alpha, \beta \in \Omega_X^{0,q}(E)$, we get a smooth function $X \to \mathbb{C}$ where $x \mapsto \langle\alpha_x, \beta_x\rangle$. The Hermitian metric on X induces a Riemannian metric, which then gives a volume form $dV_g \in \Omega_X^{2n}$. This gives an L_2 metric $(\cdot,\cdot)_{L^2}$ on $\Omega_X^{0,q}(E)$ defined by

$$(\alpha, \beta)_{L^2} = \int_X \langle\alpha, \beta\rangle \, dV_g$$

as before, this inner product is related to $\star_E$ via the identity

$$(\alpha, \beta)_{L^2} = \int_X \alpha \wedge \star_E\beta$$

We first clarify the notation $\alpha \wedge \star_E \beta$. Given $\beta \in \Omega_X^{0,q}(E)$, we have that $\star_E \beta \in \Omega_X^{n,n-q}(E^*)$ Wedging with α gives us a form in $\Omega_X^{n,n}(E \otimes E^*)$, and contracting E with E^* gives us an element of $\Omega_X^{n,n}$, which we can integrate over X.

We then use $\star_E$ to define an adjoint to $\overline{\partial}_E$.

Definition 5.6. Define $\overline{\partial}_E^* : \Omega_X^{0,q}(E) \to \Omega_X^{0,q-1}(E)$ by

$$(-1)^q \star_E^{-1} \overline{\partial}_{K_X \otimes E^*} \star_E$$

Proposition 5.7. *$\overline{\partial}_E^*$ is the formal adjoint of $\overline{\partial}_E$ with respect to $(\cdot,\cdot)_{L^2}$.*

The proof is very similar to the ones we performed above.

With these adjoint operators, we construct Laplacian operators.

Definition 5.8. Define the Laplacians Δ, Δ_∂, $\Delta_{\overline{\partial}}$, and Δ_E by

$$\begin{aligned}
\Delta &= dd^* + d^*d \\
\Delta_\partial &= \partial\partial^* + \partial^*\partial \\
\Delta_{\overline{\partial}} &= \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} \\
\Delta_E &= \overline{\partial}_E\overline{\partial}_E^* + \overline{\partial}_E^*\overline{\partial}_E
\end{aligned}$$

Forms that are annihilated by their respective Laplacian operator are said to be ***harmonic***. The subspace of harmonic forms is denoted

(1) $\mathcal{H}^k(X)$ for Δ.
(2) $\mathcal{H}^{q,0}(X)$ for Δ_∂.
(3) $\mathcal{H}^{0,q}(X)$ for $\Delta_{\overline{\partial}}$.
(4) $\mathcal{H}^{0,q}(X,E)$ for Δ_E.

We first prove some helpful properties of these Laplacian operators

Proposition 5.9.

$$(\alpha, \Delta\beta)_{L^2} = (d\alpha, d\beta)_{L^2} + (d^*\alpha, d^*\beta)_{L^2}$$

and the analogous statement for all the other Laplacians.

Proof. We compute

$$\begin{aligned}
(\alpha, \Delta\beta)_{L^2} &= (\alpha, dd^*\beta + d^*d\beta)_{L^2} \\
&= (\alpha, dd^*\beta)_{L^2} + (\alpha, d^*d\beta)_{L^2} \\
&= (d^*\alpha, d^*\beta) + (d\alpha, d\beta)
\end{aligned}$$

where the last equality comes from the fact that d and d^* are adjoints. The proof is the same for all the other Laplacians. ■

Proposition 5.10. *A k-form α is harmonic if and only if $d\alpha = d^*\alpha = 0$, and the analogous statements for ∂, $\overline{\partial}$, and $\overline{\partial}_E$.*

Proof. This is an immediate corollary of the previous proposition, along with the fact that $(\cdot,\cdot)_{L^2}$ is positive definite. ■

Corollary 5.11. *The Laplacians are formally self-adjoint, i.e.*

$$(\Delta\alpha, \beta)_{L^2} = (\alpha, \Delta\beta)_{L^2}$$

and a similar statement for the other Laplacians.

We will use the following properties of the Laplacian operators, but will not prove them.

Proposition 5.12.

(1) *The Laplacian operators defined above are elliptic.*
(2) *For an elliptic differential operator $D : E \to F$ between vector bundles equipped with fiber metrics of the same rank over a compact manifold X, the kernel of D is finite dimensional, the image of D is closed and of finite codimension, and there is an orthogonal decomposition*

$$\Gamma_X(E) = \ker D \oplus \operatorname{Im} D^*$$

where $D^* : F \to E$ is the adjoint of D with respect to the L^2 metrics on E and F

6. The Hodge Decomposition

The Hodge decomposition is an immediate corollary of the above proposition and the fact that the Laplacian is self-adjoint.

Corollary 6.1 (*The Hodge decomposition*). *For a compact Riemannian manifold (X, g), there is an orthogonal decomposition*

$$\Omega_X^k = \mathcal{H}^k(X) \oplus \operatorname{Im} \Delta$$

Corollary 6.2. *The canonical map $\mathcal{H}^k(X) \to H^k_{dR}(X)$ mapping a harmonic form α to its de Rham cohomology class is an isomorphism*

Proof. Let $\beta \in \Omega_X^k$ be a closed form. By the Hodge decomposition, we can write

$$\begin{aligned} \beta &= \alpha + \Delta\eta \\ &= \alpha + dd^*\eta + d^*d\eta \end{aligned}$$

with $\alpha \in \mathcal{H}^k(X)$ and some $\eta \in \Omega_X^k$. Since α is harmonic, we have that $d\alpha = 0$, and we also have that $ddd^*\eta = 0$ since $d^2 = 0$, so for β to be closed, we must have $dd^*d\eta = 0$ as well. Therefore, $[\beta] = [\alpha]$, so the map $\alpha \mapsto [\alpha]$ is surjective. For injectivity, let $\alpha \in \mathcal{H}^k(X)$ with $[\alpha] = 0$, i.e. α is exact. Since α is harmonic, we have that $d\alpha = d^*\alpha = 0$. Since the kernel of d^* is orthogonal to the image of d (which is a general fact about adjoint linear transformations), we conclude that $\alpha = 0$, so the map is injective, and therefore an isomorphism. ∎

As expected, we have a similar situation for the Δ_E-harmonic forms and the sheaf cohomology $H^q(X, E)$ for a holomorphic vector bundle E, which is naturally identified with the Dolbeault cohomology groups $H^{0,q}(X, E)$. A near identical proof gives us that the map $\mathcal{H}^{0,q}(X, E) \to H^q(X, E)$ is an isomorphism.

The Hodge decomposition gives us easy proofs of various duality theorems.

Theorem 6.3 (*Poincaré Duality*). *For a compact manifold X, the pairing $H^k_{dR}(X) \otimes H^{n-k}_{dR}(X) \to \mathbb{R}$ defined by*

$$\langle [\alpha], [\beta] \rangle = \int_X \alpha \wedge \beta$$

is nondegenerate, giving an isomorphism $H^k_{dR}(X) \cong (H^{n-k}_{dR}(X))^*$.

Proof. Fix a Riemannian metric g for X, which allows us to construct the Laplacian Δ and the spaces $\mathcal{H}^k(X)$, which are isomorphic to the de Rham cohomology groups $H^k_{dR}(X)$. We use a lemma, which is a result from a simple (but somewhat tedious) computation.

Lemma 6.4. *The Hodge star* $\star$ *commutes with* Δ.

In particular, the lemma implies that if $\alpha \neq 0$ is harmonic, then $\star\alpha$ is also a nonzero harmonic form. Therefore, it suffices to show that $\langle [\alpha], [\star\alpha] \rangle \neq 0$ for $\alpha \neq 0$. But

$$\langle [\alpha], [\star\alpha] \rangle = \int_X \alpha \wedge \star\alpha = (\alpha, \alpha)_{L^2} \neq 0$$

■

A little extra work gets the holomorphic vector bundle analogue, which is ***Serre duality***. Given a holomorphic vector bundle $E \to X$, there is a natural pairing

$$H^q(X, E) \otimes H^{n-q}(X, E^* \otimes K_X) \to H^q(X, K_X) \to \mathbb{C}$$

where the first map contracts the E and E^* factors, and the second map is given by integration over X. Once we make the proper identification of this pairing with Dolbeault cohomology, we can use the same technique we used for proving Poincaré duality to prove that this pairing is nondegenerate. In Dolbeault cohomology, let $\alpha \in H^{0,q}(X, E)$ and $\beta \in H^{0,n-q}(X, E^* \otimes K_X)$. Then we can take the exterior product $\alpha \wedge \beta \in H^{0,n}(X, E \otimes E^* \otimes K_X)$, which is obtained by class of the product of two representatives. Contracting the E and E^* factors gives us an element of $H^{0,n}(X, K_X)$, which is canonically identified with $H^{2n}(X, \mathbb{C}) = H^{n,n}(X)$. Integration then gives the pairing.

Theorem 6.5 (***Serre duality***)**.** *The pairing*

$$H^q(X, E) \otimes H^{n-q}(X, E^* \otimes K_X) \to \mathbb{C}$$

defined above is nondegenerate.

Proof. The proof is similar to the one of Poincaré duality. From the discussion above, we prove the pairing with Dolbeault cohomology. Fix Hermitian metrics on E and X, giving us the operator Δ_E and the spaces $\mathcal{H}^{0,q}(X, E)$ of harmonic forms, which are isomorphic to the cohomology groups $H^{0,q}(X, E) = H^q(X, E)$. As before, we use (without proof) that $\star_E$ commutes with Δ_E to deduce that for α harmonic, $\star_E\alpha$ is also harmonic. Therefore, it again suffices to show that for α harmonic and nonzero, the pairing $\langle [\alpha], [\star\alpha] \rangle$ is nonzero. However, the pairing is just $(\alpha, \star\alpha)_{L^2}$, which is nonzero since α is. ■