VARIATIONS OF HODGE STRUCTURE

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Notation and Conventions

For a complex manifold X, we let $\mathcal{O}_X$ denote its sheaf of holomorphic functions. We let Ω_X^k denote the sheaf of holomorphic k-forms on X, and we let $\mathcal{A}_X^k$ denote the sheaf of smooth complex k-forms on X. We let TX denote the smooth tangent bundle of X, and let $T_X = T^{1,0}X$ denote the holomorphic tangent bundle.

1. Pure Hodge Structures

The purpose of a Hodge structure is to abstract away the properties of the cohomology groups of a compact Kähler manifold into a linear algebraic gadget.

Definition 1.1. An (***pure***) ***integral Hodge structure of weight*** k is a finitely generated abelian group V along with a decomposition of $V_{\mathbb{C}} := V \otimes_{\mathbb{Z}} \mathbb{C}$ as

$$V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{p,q}$$

satisfying the condition that $\overline{V^{p,q}} = V^{q,p}$. Such a decomposition is called the ***Hodge decomposition***. A ***morphism*** of Hodge structures is a map of abelian groups $\varphi : V \to W$ such that the complexified map $\varphi_{\mathbb{C}} : V_{\mathbb{C}} \to W_{\mathbb{C}}$ preserves the Hodge decomposition, i.e. $\varphi_{\mathbb{C}}(V^{p,q}) \subset W^{p,q}$.

We see that this is an abstraction of the k^{th} integral cohomology group of a Kähler manifold, i.e. for a Kähler manifold X, we have that $H^k(X, \mathbb{Z})$ is an integral Hodge structure of weight k, where the decomposition of $H^k(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^k(X, \mathbb{C})$ is given by the Hodge decomposition. Rational, real and complex Hodge structures are defined analogously. Note that given an integral Hodge structure $V_{\mathbb{Z}}$, we get a rational Hodge structure

$V_{\mathbb{Q}} = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$, and similarly for real and complex Hodge structures, so the level of generality is opposite to the inclusions $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

Definition 1.2. Let V be a Hodge structure of weight k. Then the ***Hodge filtration*** is the filtration $F^{\bullet}V$ on $V_{\mathbb{C}}$ defined by

$$F^p V_{\mathbb{C}} := \bigoplus_{r \geq p} V^{r,k-1}$$

The first thing to note is how the Hodge filtration behaves with the Hodge decomposition, which results from a few simply computations.

Proposition 1.3. *Let V be a Hodge structure of weight k.*

(1) $V_{\mathbb{C}} = F^p V_{\mathbb{C}} \oplus \overline{F^{k-p+1} V_{\mathbb{C}}}$.
(2) $V^{p,q} = F^p V_{\mathbb{C}} \cap \overline{F^q V_{\mathbb{C}}}$.

Proof.

(1) We have

$$\begin{aligned} F^p V_{\mathbb{C}} \oplus \overline{F^{k-p+1} V_{\mathbb{C}}} &= \left(\bigoplus_{r \geq p} V^{r,k-r} \right) \oplus \left(\bigoplus_{s \geq k-p+1} \overline{V^{s,k-s}} \right) \\ &= \left(\bigoplus_{r \geq p} V^{r,k-r} \right) \oplus \left(\bigoplus_{s \geq k-p+1} V^{k-s,s} \right) \\ &= \bigoplus_{p+q=k} V^{p,q} \\ &= V_{\mathbb{C}} \end{aligned}$$

an easy way to see the second to last equality is to keep an eye on the first term of the bidegree. For the first summand, we have that as r increases from p to k, the first term of the bidegree follows in suit. For the other term, as s ranges from $k-p+1$ to k, the first term of the bidegree decreases from $p-1$ to 0, so we have all the (p,q) accounted for.

(2) We have

$$\begin{aligned} F^p V_{\mathbb{C}} \cap \overline{F^q V_{\mathbb{C}}} &= \left(\bigoplus_{r \geq p} V^{r,k-r} \right) \cap \left(\bigoplus_{s \geq q} \overline{V^{s,k-s}} \right) \\ &= \left(\bigoplus_{r \geq p} V^{r,k-r} \right) \cap \left(\bigoplus_{s \geq q} V^{k-s,s} \right) \\ &= V^{p,q} \end{aligned}$$

Again, if we just keep an eye on the first term of the bidegree, a summand is in the intersection if and only if these terms coincide, which we see only happens when $r = p$ and $s = q$.

■

In the case of Kähler manifolds, recall that the cohomology groups $H^{p,q}(X,\mathbb{C})$ can be realized as the de Rham cohomology classes that admit representatives that are smooth closed (p,q)-forms. Applying this to the Hodge filtration on $H^k(X,\mathbb{C})$, we have that the subspaces $F^pH^k(X,\mathbb{C})$ are generated by the cohomology classes with representatives with "at least p many dz^i terms." In fact, more can be said.

Proposition 1.4. *Let $F^p\mathcal{A}^k_X$ denote the subspace of smooth k-forms spanned by the k-forms of type $(r,k-r)$ for $r \geq p$. Then*

$$F^pH^k(X,\mathbb{C}) = \frac{\ker(d : F^p\mathcal{A}^k_X \to F^p\mathcal{A}^{k+1}_X)}{\operatorname{Im}(d : F^p\mathcal{A}^{k-1}_X \to F^p\mathcal{A}^k_X)}$$

Proof. Let Z^p and B^p denote the kernel and image specified in the statement respectively, so we want to show that $F^pH^k(X,\mathbb{C}) = Z^p/B^p$. We have a map $Z \to H^k(X,\mathbb{C})$ taking a differential form to its cohomology class, and the image clearly contains $F^pH^k(X,\mathbb{C})$. Conversely, suppose we have that α is closed form whose cohomology class is contained in $F^pH^k(X,\mathbb{C})$. Let $\Delta = dd^* + d^*d$ be the Hodge Laplacian, and can write α uniquely as $\alpha = \beta + \Delta\gamma$, where β is harmonic. Then since Δ preserves the bidegree of forms, we get that β and γ are both in $F^p\mathcal{A}^k_X$. Then since both β and γ are d-closed, we have

$$0 = d\Delta\gamma = d(dd^*\gamma + d^*d\gamma) = dd^*d\gamma = 0$$

By orthogonality of the image of d^* and the kernel of d (since they are adjoint operators), we have that $d^*d\gamma = 0$. Therefore, we can write $\alpha = \beta + dd^*\gamma$, so α defines the same cohomology class as β. Therefore, using the identification of $H^{p,q}(X,\mathbb{C})$ with harmonic (p,q) forms, we get that $[\alpha] = [\beta] \in F^pH^k(X,\mathbb{C})$.

We now want to show that the kernel of the map specified above is B^p. We note that B^p is clearly contained in the kernel, so we only need to verify the the opposite inclusion. We do this by decreasing induction on p, so the base case is when $p = k$. When $p = k$, we have that $F^pH^k(X,\mathbb{C}) = H^{k,0}(X,\mathbb{C})$. Then suppose we have a closed form α with $[\alpha] \in H^{k,0}(X,\mathbb{C})$, so α can be assumed to be type $(k,0)$. Therefore, we have that $\overline{\partial}\alpha = 0$. Therefore, we have that $d\alpha = \overline{\partial}\alpha = 0$, so if α is d-exact, we may apply $\partial\overline{\partial}$-lemma to write $\alpha = \partial\overline{\partial}\xi$. But since forms cannot have a negative term in the bidegree, we conclude that $\alpha = 0$, which is what we wanted since $F^k\mathcal{A}^{k-1}_X = 0$.

We now assume we have proven the fact for $p+1$ to prove it for p. Suppose we have $\alpha \in F^p\mathcal{A}^k_X$ such that $d\alpha = 0$. We want to show that $\alpha = d\xi$ for some $\xi \in F^p\mathcal{A}^{k-1}_X$. Then we can write $\alpha = \Delta\beta$, since the harmonic representative of $[\alpha]$ is 0. Then let q such that $p + q = k$. We then get that $\alpha^{p,q} = \Delta\beta^{p,q} = 2\Delta_{\overline{\partial}}\beta^{p,q}$, where $\Delta_{\overline{\partial}}$ is the $\overline{\partial}$-Laplacian $\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ and we are using the Kähler identity $2\Delta_{\overline{\partial}} = \Delta$. Therefore, we have that $\alpha^{p,q} = 2\Delta_{\overline{\partial}}\beta^{p,q}$ is $\overline{\partial}$-closed. Again using orthogonality of the kernel of $\overline{\partial}$ with the image of $\overline{\partial}^*$, we find that $\overline{\partial}^*\overline{\partial}\beta^{p,q} = 0$, so $\alpha^{p,q} = 2\overline{\partial}\overline{\partial}^*\beta^{p,q}$. Then we have that $\alpha - 2d\overline{\partial}^*\beta^{p,q}$ is exact. Furthermore, we have that it is contained in $F^{p+1}\mathcal{A}^k_X$, since we have zeroed out the only component of α contained in $F^p\mathcal{A}^k_X$ by subtracting a form of type $(p+1, q-1)$. Applying the induction hypothesis to this form, we find that $\alpha - 2d\bar{\partial}^* \beta^{p,q} = d\gamma$ for some $\gamma \in F^{p+1}\mathcal{A}^{k-1}$, so we get $\alpha = 2d\bar{\partial}^* \beta^{p,q} + d\gamma$, which shows the desired result. ■

Given two Kähler manifolds X and Y and a holomorphic map $\varphi : X \to Y$, we get a pullback map on cohomology $\varphi^* : H^k(Y, \mathbb{Z}) \to H^k(X, \mathbb{Z})$ which when complexified, corresponds to the pullback of complex differential forms. Then since the pullback of differential forms under a holomorphic map preserves the type, we see that the pullback map on the k^{th} cohomology groups is a morphism of Hodge structures of weight k.

2. Polarized Hodge Structures

The Kähler class $[\omega] \in H^2(X, \mathbb{R})$ defines a Lefschetz operator $L : H^k(X, \mathbb{R}) \to H^{k+2}(X, \mathbb{R})$ where $L[\alpha] = [\omega \wedge \alpha]$. This gives rise to the ***Lefschetz decomposition*** of $H^k(X, \mathbb{R})$ as

$$H^k(X, \mathbb{R}) = \bigoplus_{r \geq 0} L^r H^{k-2r}(X, \mathbb{R})_{\text{Prim}}$$

where the primitive cohomology $H^i(X, \mathbb{R})_{\text{Prim}} \subset H^i(X, \mathbb{R})$ denotes the subspace primitive cohomology classes, which are the classes annhilated by L^{n-i+1}. This tells us that we can understand the cohomology of X in terms of L and the primitive pieces. Complexifying gives us an analogous decomposition on the complex cohomology, and the Lefschetz decomposition on $H^k(X, \mathbb{C})$ is compatible with the Hodge decomposition in the sense that

$$H^k(X, \mathbb{C})_{\text{Prim}} = \bigoplus_{p,q} (H^{p,q}(X, \mathbb{C}) \cap H^k(X, \mathbb{C})_{\text{Prim}})$$

So we can further break down the primitive pieces of the cohomology in terms of the Hodge decomposition. The Kähler class also gives a bilinear pairing Q_k on the cohomology group $H^k(X, \mathbb{R})$ by

$$Q_k([\alpha], [\beta]) = \int_X \omega^{n-k} \wedge \alpha \wedge \beta$$

The form Q_k is symmetric when k is even and alternating if k is odd. This then induces a Hermitian H_k form on $H^k(X, \mathbb{C})$ given by

$$H_k([\alpha], [\beta]) = i^k Q([\alpha], [\overline{\beta}])$$

In addition, the complexified Lefschetz decomposition is orthogonal with respect to H_k, and the restriction of H_k to the primitive component $L^r H^{k-2r}(X, \mathbb{C})_{\text{Prim}}$ is the same as the restriction of the form $(-1)^r H_{k-2r}$, where we are using the fact that L^r is injective on $H^{k-2r}(X, \mathbb{C})$ to identify $L^r H^{k-2r}(X, \mathbb{C})_{\text{Prim}}$ and $H^{k-2r}(X, \mathbb{C})_{\text{Prim}}$. Furthermore, the Hodge decomposition is orthogonal with respect to the H_k, and the restriction of $(-1)^{k(k-1)/2} i^{p-q-k} H_k$ to $H^{p,q}(X, \mathbb{C})_{\text{Prim}}$ is positive definite. Since $p - q$ always has the same parity as k, this amounts to saying that the restriction of H_k to $H^{p,q}(X, \mathbb{C})_{\text{Prim}}$ alternates between being positive and negative definite as we increment p.

Suppose the manifold X is not only Kähler, but is a complex submanifold of some $\mathbb{CP}^N$. The restriction of the Fubini-Study metric on $\mathbb{CP}^N$ gives a Kähler metric ω on X whose cohomology class $[\omega]$ is integral, i.e. $[\omega] \in H^2(X, \mathbb{Z})$, which corresponds to the curvature form of the Chern connection on a positive Hermitian line bundle on X. Kodaira's embedding theorem tells us that the opposite is also true – given an integral Kähler class $[\omega]$, we can find a positive Hermitian holomorphic line bundle $L \to X$ where the curvature of the Chern connection is $[\omega]$ such that L defines an embedding $X \hookrightarrow \mathbb{CP}^N$ where $\mathcal{O}_{\mathbb{CP}^N}(1)|_X \cong L$ and $[\omega]$ is represented by the restriction of the Fubini-Study metric. Then the operator L corresponds to intersecting with the hyperplane class, giving a geometric interpretation of the H_k as a notion of intersection forms on cohomology. Finally, $[\omega]$ being an integral class implies that the above discussion restricts to the integral cohomology, giving us bilinear forms Q_k on $H^k(X, \mathbb{Z})$ and primitive integral classes $H^k(X, \mathbb{Z})_{\text{Prim}}$ Putting everything together, this motivates the following:

Definition 2.1. A ***polarized integral Hodge structure*** of weight k is a pure integral Hodge structure V of weight k along with a bilinear form Q on V such that Q is alternating if k is odd, and symmetric if k is even, such that the Hermitian form H

$$H(\alpha, \beta) = i^k Q(\alpha, \bar{\beta})$$

on $V_{\mathbb{C}}$ satisfies:

(1) The Hodge decomposition is orthogonal with respect to H.
(2) the form $(-1)^{k(k-1)/2} i^{p-q-k} H$ is positive definite on $V^{p,q}$.

Taking the above discussion into account, this abstracts the primitive integral cohomology of a smooth projective algebraic variety. The abelian group V represents $H^k(X, \mathbb{Z})_{\text{Prim}}$, the $V^{p,q}$ in the Hodge decomposition represent $H^{p,q}(X, \mathbb{C})_{\text{Prim}}$, and Q represents the intersection pairing Q_k constructed above.

3. Flat Bundles and Local Systems

Definition 3.1. Let X be a complex manifold. A ***local system*** on X is a sheaf H of abelian groups on X that is isomorphic to the constant sheaf $\underline{A}$ for some abelian group A.

In most cases, we will discuss local systems where the abelian group A is a vector space over $\mathbb{R}$ or $\mathbb{C}$. Given a local system H of $\mathbb{C}$-vector spaces, we can obtain a sheaf $\mathcal{H}$ of free $\mathcal{O}_X$-modules over X by taking the tensor product

$$\mathcal{H} := H \otimes_{\underline{\mathbb{C}}} \mathcal{O}_X$$

Where we regard $\mathcal{O}_X$ as a sheaf of $\underline{\mathbb{C}}$-algebras via the inclusion $\underline{\mathbb{C}} \hookrightarrow \mathcal{O}_X$. The same discussion holds if if H is a local system of abelian groups or $\mathbb{R}$ vector spaces by tensoring over $\underline{\mathbb{Z}}$ and $\underline{\mathbb{R}}$ respectively. We note that the sheaf $\mathcal{H}$ is isomorphic to $\mathcal{O}_X^n$, which is the sheaf of holomorphic sections of the trivial rank n vector bundle $X \times \mathbb{C}^n$. However, we also have the data of our original sheaf H, which we can identify as a subsheaf of $\mathcal{H}$ via the mapping $h \mapsto h \otimes 1$. Our goal will be to identify H as a subsheaf of distinguished sections of the trivial bundle $X \times \mathbb{C}^n$.

Definition 3.2. Let $\mathcal{E}$ be a sheaf of $\mathcal{O}_X$-modules. A ***connection*** on $\mathcal{E}$ is a sheaf morphism $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X$ such that

(1) ∇ is $\underline{\mathbb{C}}$-linear.
(2) For an open set U, a function $f \in \mathcal{O}_X(U)$, and a section $s \in \mathcal{E}(U)$, we have that ∇ satisfies the Leibniz rule

$$\nabla(fs) = s \otimes df + f\nabla(s)$$

In the case that $\mathcal{E}$ is the sheaf of holomorphic sections of a holomorphic vector bundle $E \to X$, the connection ∇ is corresponds to the notion of a *holomorphic* connection on E. By replacing $\mathcal{O}_X$ with the sheaf of smooth complex valued functions and Ω^1_X with $\mathcal{A}^1_X$, the same definition yields the more traditional definition of a connection on a smooth complex vector bundle.

Definition 3.3. Let $\mathcal{E}$ be a sheaf of $\mathcal{O}_X$-modules equipped with a connection ∇. A section $s \in \mathcal{E}(U)$ is ***flat*** if $\nabla(s) = 0$.

Given a sheaf $\mathcal{E}$ with connection ∇, the flat sections of $\mathcal{E}$ form a subsheaf. In the case that $\mathcal{E}$ comes from a vector bundle $\pi : E \to X$, the flat sections correspond to the equivalent interpretation of the connection ∇ as a ***horizontal distribution*** on the total space E, i.e. a vector subbundle $D \subset TE$ such that $D \oplus \ker d\pi = TE$. From this perspective, the flat sections of ∇ correspond to sections of E that lie inside D. Another thing to note is that by a dimension count, we have that the rank of D as a vector bundle over E is the same as a the dimension of the base space X, which can be seen by noting that for any point $e \in E$, the fiber D_e projects isomorphically onto the tangent space $T_{\pi(e)}X$ under $d\pi_e$ since it is complentary to the kernel.

Proposition 3.4. *Let H be a local system of $\mathbb{C}$-vector spaces, and $\mathcal{H} = H \otimes_{\underline{\mathbb{C}}} \mathcal{O}_X$ the sheaf of holomorphic sections of the associated vector bundle. Then there exists a connection ∇ on $\mathcal{H}$ such that H is the sheaf of flat sections of ∇.*

Proof. Over an open set $U \subset X$, we have that $\mathcal{H}(U)$ is isomorphic to $\mathcal{O}^n_X(U)$. Furthermore, we may choose the isomorphism $\mathcal{O}^n_X(U) \to \mathcal{H}(U)$ such that the standard basis vector e_i maps to an element h_i of H, thought of as a subsheaf of $\mathcal{H}$. Then given a local section s, it can be written in this trivialization as $s = \sum_i f_i h_i$ for some smooth functions $f_i \in \mathcal{O}_X(U)$. Define ∇ by the formula

$$\nabla(s) = \sum_i h_i \otimes df_i$$

Furthermore, we claim that this independent of our choice of trivialization. Given another trivialization $\{h'_i\}$, we have that h_i and h'_i are related by some matrix $A \in \mathrm{GL}_n\mathbb{C}$, (i.e. $h_i = A^j_i h'_j$, using Einstein summation convention) since H is locally constant. Therefore, in the local trivialization $\{h'_i\}$, the section s can be represented as $s = \sum_i f_i A^j_i h'_j$. Therefore, if we use our definition of ∇ in this trivialization, we obtain

$$\nabla(s) = \sum_{i,j} h'_j \otimes d(f_i A^j_i) = \sum_{i,j} h'_j \otimes A^j_i df_i = \sum_{i,j} A^j_i h'_j \otimes df_i = \sum_i h_i \otimes df_i$$

where we use the fact that the matrix A is locally constant on U and $\mathbb{C}$-linearity of d. Therefore, our local definition of ∇ is well-defined, and glues to a connection on all of $\mathcal{H}$. Furthermore, we see that a section s is flat if and only if it is locally of the form $s = \sum_i \lambda_i h_i$ for scalars $\lambda_i \in \underline{\mathbb{C}}(U)$. Then by identifying H with the subsheaf $H \otimes_{\underline{\mathbb{C}}} 1 \subset \mathcal{H}$, we immediately see that the flat sections are exactly the sections of H. ∎

The connection ∇ defined in the proof has several special properties that are not shared by all connections. One thing to observe is that the $(0,1)$ part of ∇ is exactly $\overline{\partial}$ in a trivialization of $\mathcal{H}$ mapping the standard basis of $\mathcal{O}^n_X$ to sections of $H \subset \mathcal{H}$. This completely characterizes the holomorphic structure on the associated smooth complex vector bundle.

To identify the second property, we need to define a notion of curvature.

Let $\mathcal{E}$ be a sheaf of $\mathcal{O}_X$-modules equipped with a connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X$. This induces a map $\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^k_X \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^{k+1}_X$, which we also call ∇, by the formula

$$\nabla(s \otimes \omega) = \nabla(s) \wedge \omega + s \otimes d\omega$$

Where $\nabla(s) \wedge \omega$ is obtained by writing $\nabla(s)$ locally as a sum of the form $\sum_i s_i \otimes \omega_i$ for $s_i \in \mathcal{E}(U)$ and $\omega_i \in \Omega^1_X(U)$ and wedging the ω_i with ω.

Proposition 3.5. *The extended connection $\nabla : \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^k_X \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^{k+1}_X$ satisfies a graded Leibniz rule, i.e. for a function $f \in \mathcal{O}_X(U)$ and a local section $\theta \in \mathcal{E}(U) \otimes \Omega^k_X(U)$, we have*

$$\nabla(f\theta) = (-1)^k \theta \wedge df + f \nabla\theta$$

Proof. By linearity, it suffices to check this when θ is of the form $s \otimes \omega$ for a local section $s \in \mathcal{E}(U)$ and $\omega \in \Omega^1_X(U)$. We then compute

$$\begin{aligned}
\nabla(f\theta) &= \nabla(f(s \otimes \omega)) \\
&= \nabla(fs \otimes \omega) \\
&= \nabla(fs) \wedge \omega + fs \otimes d\omega \\
&= (s \otimes df + f\nabla(s)) \wedge \omega + fs \otimes d\omega \\
&= s \otimes (df \wedge \omega) + f\nabla(s) \wedge \omega + fs \otimes d\omega \\
&= s \otimes (df \wedge \omega) + f\nabla(s \otimes \omega) \\
&= (-1)^k s \otimes (\omega \wedge df) + f\nabla(s \otimes \omega) \\
&= (-1)^k \theta \wedge df + f\nabla(\theta)
\end{aligned}$$

■

Definition 3.6. The ***curvature*** of a connection ∇ is the map $\Omega : \mathcal{E} \to \mathcal{E} \otimes \Omega^2_X$ given by $\Omega := \nabla \circ \nabla$.

Proposition 3.7. *The curvature Ω of any connection ∇ is $\mathcal{O}_X$-linear.*

Proof. Fix a local section s and a holomorphic function f. Then we compute

$$\begin{aligned}
\Omega(fs) &= \nabla(s \otimes df + f\nabla(s)) \\
&= \nabla(s \otimes df) + \nabla(f\nabla(s)) \\
&= (\nabla(s) \wedge df + s \otimes d^2 f) - \nabla(s) \wedge df + f\nabla(\nabla(s)) \\
&= f\Omega(s)
\end{aligned}$$

■

If a connection has curvature equal to zero, it is said to be ***flat***.

From the perspective of horizontal distributions, the curvature tensor Ω has an interpretation as the ***Frobenius tensor*** of the horizontal distribution D, which measures the failure of D to be ***involutive***, i.e. the failure of vector fields on E lying in D to be closed under the Lie bracket. By Frobenius' theorem, this is equivalent to D being ***integrable***, so Ω measures the obstruction to the existence of integral submanifolds of D inside of the total space E. From this perspective, flatness of a connection is exactly integrability of the horizontal distribution.

Proposition 3.8. *The connection ∇ on the vector bundle $\mathcal{H} \otimes_{\underline{\mathbb{C}}} \mathcal{O}_X$ is flat.*

Proof. Fix a local trivialization of $\mathcal{H}$ such that a local basis of sections is given by a set $\{h_i\}$ with $h_i \in H(U)$. The local basis of sections determines an isomorphism $\mathcal{O}_X^n(U) \to \mathcal{H}(U)$, and identifies ∇ with the operator that applies the de Rham differential d to each component of $\mathcal{O}_X^n(U)$. Then since $d^2 = 0$, we have that $\Omega = \nabla \circ \nabla = 0$. ■

This suggests the following.

Theorem 3.9. *There is an bijective correspondence*

$$\{\text{Local systems of } \mathbb{C}\text{-vector spaces}\} \longleftrightarrow \{\text{Holomorphic vector bundles with flat connection}\}$$

Proof. In one direction, we can associate to a local system H the vector bundle $\mathcal{H}$ with the connection we defined above. In the other direction, given a holomorphic vector bundle $\pi : E \to X$ with flat connection ∇, we claim that the sheaf H of flat sections of E forms a local system. Once we do so, it's clear that the mappings specified above are inverses to each other, giving us the desired correspondence.

Since the bundle $\pi : E \to X$ has a flat connection ∇, we have that the horizontal distribution D is integrable, so there exists a integral submanifolds of E whose tangent spaces correspond to the fibers of $D \to E$. Fix a basepoint $x \in X$, and consider the zero element $0 \in E_x$ the the fiber lying over x. Then let $Y \subset E$ denote an integral submanifold of D containing 0. Since D is complementary to $\ker d\pi$ and $T_0 Y = D_0$, we have that $\pi|_Y : Y \to X$ is a local diffeomorphism at 0, so there exists connected open neighborhoods $U \subset Y$ and $V \subset X$ such that $\pi|_U$ is a diffeomorphism $U \to V$. Furthermore, we can find a neighborhood U' of Y diffeomorphic to $Y \times (-\varepsilon, \varepsilon)$ such that the slice $Y \times \{t\}$ projects diffeomorphically onto V. Then U' is the union of the images of flat sections $V \to E$, since it is fibered by integral submanifolds of the horizontal distribution. Then since U' is an open neighborhood of the zero section $V \to E$, taking fiberwise linear combinations of elements of U' spans the fibers of E lying over V. Since linear combinations of flat sections are flat, we have that the fibers of E lying over V are generated by flat sections. Therefore, we can identify the flat sections of E over U' with elements of the fiber E_0, since a section σ whose value at x is a vector $v \in E_0$ is uniquely extended to a flat section in a neighborhood of x by existence and uniqueness of solutions to the differential equation $\nabla\sigma = 0$. Putting everything together, we find that the flat sections over U' can be identified with constant functions $U \to E_x$, which tells us that the sheaf of flat sections can be identified with the constant sheaf $\underline{E_x}$, so they form a local system. ■

4. Derived Pushforwards of Sheaves

To discuss the Gauss-Manin connection, we need to make a short digression into homological algebra and derived pushforwards of sheaves. For a topological space X, let $\mathrm{Sh}(X)$ denote the abelian category of sheaves of R-modules over X, where $R = \mathbb{Z}$, $\mathbb{R}$, or $\mathbb{C}$.

Definition 4.1. Let $\pi : X \to Y$ be a continuous map. The ***pushforward*** (also known as the ***direct image***) along π is a functor $\pi_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$, where given a sheaf $\mathcal{F}$ over X, the sheaf $\pi_*\mathcal{F}$ is defined by the mapping

$$\pi_*\mathcal{F}(U) := \mathcal{F}(\pi^{-1}(U))$$

Given a sheaf morphism $\varphi : \mathcal{F} \to \mathcal{G}$, the sheaf morphism $\pi_*\varphi : \pi_*\mathcal{F} \to \pi_*\mathcal{G}$ is given by

$$\pi_*\varphi(U) := \varphi(\pi^{-1}(U))$$

Definition 4.2. Given a continuous map $\pi : X \to Y$ and the ***pullback*** along π is a functor $\pi^* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ where for a sheaf $\mathcal{F} \in \mathrm{Sh}(Y)$, the sheaf $\pi^*\mathcal{F}$ is the sheafification of the presheaf

$$U \mapsto \lim_{V \supset \pi(U)} \mathcal{F}(V)$$

Given a morphism $\varphi : \mathcal{F} \to \mathcal{G}$, the sheaf morphism $\pi^*\varphi : \pi^*\mathcal{F} \to \pi^*\mathcal{G}$ is given by the sheaf morphism induced by the map of presheaves

$$\lim_{V \supset \pi(U)} \varphi(V) : \lim_{V \supset \pi(U)} \mathcal{F}(V) \to \lim_{V \supset \pi(U)} \mathcal{G}(V)$$

It's clear from the definition of the pullback that for a point $x \in X$, the stalk $(\pi^*\mathcal{F})_x$ is equal to the stalk $\mathcal{F}_{\pi(x)}$. Furthermore, given a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves over Y, we have that the morphism $\pi^*\varphi$ is given on stalks by the same maps induced by φ. Then since injectivity and surjectivity for sheaf maps can be checked on the level of stalks, this immediately yields the result:

Proposition 4.3. *The functor π^* is exact, i.e. for an exact sequence of sheaves over Y,*

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

the sequence of sheaves over X given by

$$0 \longrightarrow \pi^*\mathcal{E} \longrightarrow \pi^*\mathcal{F} \longrightarrow \pi^*\mathcal{G} \longrightarrow 0$$

is exact.

The key fact we will need regarding π^* and π_* is the fact that they are adjoint functors.

Proposition 4.4. *Let $\pi : X \to Y$ be a continuous map, and let $\mathcal{F} \in \mathrm{Sh}(X)$ and $\mathcal{G} \in \mathrm{Sh}(Y)$. Then there is a bijection*

$$\mathrm{Hom}(\pi^*\mathcal{G}, \mathcal{F}) \longleftrightarrow \mathrm{Hom}(\mathcal{G}, \pi_*\mathcal{F})$$

that is natural in $\mathcal{F}$ and $\mathcal{G}$, i.e. functorial with respect to maps into and out of $\mathcal{F}$ and $\mathcal{G}$.

Proof. We provide maps in both directions. For one direction, let $\varphi \in \mathrm{Hom}(\pi^*\mathcal{G}, \mathcal{F})$. For an open set $U \subset X$, this gives us a map $\varphi(U) : \pi^*\mathcal{G}(U) \to \mathcal{F}(U)$. By definition, we have that $\pi^*\mathcal{G}(U) = \lim_{V \supset \pi(U)} \mathcal{G}(V)$, so the map $\varphi(U)$ is equivalent to the data of maps $\phi_V : \mathcal{G}(V) \to \mathcal{F}(U)$ for all open sets $V \supset \pi(U)$ such that if $W \supset V$, then ϕ_W is equal to the composition of ϕ_V with the restriction map $\mathcal{G}(W) \to \mathcal{G}(V)$. From this data, we want to produce a sheaf morphism $\mathcal{G} \to \pi_*\mathcal{F}$. To do this, for each open set $U \subset Y$, we want to produce a map $\mathcal{G}(U) \to \mathcal{F}(\pi^{-1}(U))$. We note that $\pi(\pi^{-1}(U)) = U$, so if we consider the map $\varphi(\pi^{-1}(U)) : \lim_{V \supset U} \mathcal{G}(V) \to \mathcal{F}(\pi^{-1}(U))$, we may take the map $\phi_U : \mathcal{G}(U) \to \mathcal{F}(\pi^{-1}(U))$. Doing this for all open sets gives us the desired sheaf morphism $\mathcal{G} \to \pi_*\mathcal{F}$.

In the other direction, suppose we are given a sheaf map $\psi \in \mathrm{Hom}(\mathcal{G}, \pi_*\mathcal{F})$. Then for an open set $U \subset Y$ we have a map $\psi(U) : \mathcal{G}(U) \to \mathcal{F}(\pi^{-1}(U))$. From this we want to produce a sheaf map $\pi^*\mathcal{G} \to \mathcal{F}$. Let $W \subset X$ be an open set. We then must give a map $\pi^*\mathcal{G}(W) \to \mathcal{F}(W)$. By definition, we have that $\pi^*\mathcal{G}(W) = \lim_{V \supset \pi(W)} \mathcal{G}(V)$. So we need to give the data of maps $\xi_V : \mathcal{G}(V) \to \mathcal{F}(W)$ for every $V \supset \pi(W)$ that are compatible with the restriction maps of $\mathcal{G}$ in the same sense as we specified above with the ϕ_V. We observe that for $V \supset \pi(W)$, we have that $\pi^{-1}(V) \supset \pi^{-1}(\pi(W)) \supset W$, so we have a restriction map $\mathcal{F}(\pi^{-1}(V)) \to \mathcal{F}(W)$. We can then define the maps ξ_V to the composition of $\psi(V) : \mathcal{G}(V) \to \mathcal{F}(\pi^{-1}(V))$ with the restriction $\mathcal{F}(\pi^{-1}(V)) \to \mathcal{F}(W)$. The fact that these maps are compatible follows from ψ being a sheaf morphism.

From here, it is a simple but tedious verification to show that the two constructions provided above are inverse to each other and are natural in $\mathcal{F}$ and $\mathcal{G}$, giving us the desired adjunction. ∎

With the preliminaries out of the way, we can discuss what we need for the Gauss-Manin connection. The first observation to make is that for a continuous map $\pi : X \to Y$, the pushforward functor π_* is left-exact. This follows immediately from the definition and the fact that injectivity of a sheaf morphism can be checked on sections. However, π_* is not right exact in general – as with most things involving sheaves, the issue arises from the existence of surjective sheaf morphisms that are not surjective on sections.

Definition 4.5. Let $\pi : X \to Y$ be a continuous map, and let $\mathcal{F} \in \mathrm{Sh}(X)$. The ***derived pushforward sheaves*** of $\mathcal{F}$ (also referred to as ***higher direct image sheaves***) are the right derived functors $R^i\pi_*\mathcal{F}$.

Explicitly, the derived pushforward sheaves of $\mathcal{F}$ can be computed by taking an injective resolution of $\mathcal{F}$

$$0 \longrightarrow \mathcal{F} \xrightarrow{d^0} \mathcal{I}^1 \xrightarrow{d^1} \mathcal{I}^2 \xrightarrow{d^2} \cdots$$

Applying π_* to this resolution gives a complex of sheaves over Y

$$0 \longrightarrow \pi_*\mathcal{F} \xrightarrow{\pi_*d^0} \pi_*\mathcal{I}^1 \xrightarrow{\pi_*d^1} \pi_*\mathcal{I}^2 \xrightarrow{\pi_*d^2} \cdots$$

and we have that the sheaf $R^i\pi_*\mathcal{F}$ is the i^{th} cohomology sheaf in this sequence, i.e.

$$R^i\pi_*\mathcal{F} := \frac{\ker \pi_* d^i}{\operatorname{Im} \pi_* d^{i-1}}$$

where we let π_*d^{-1} denote the zero map $0 \to \pi_*\mathcal{F}$.

To give a more usable characterization of the derived pushforward sheaves, we first prove a lemma.

Lemma 4.6. *Let $\pi : X \to Y$ be a continuous map, and $\mathcal{I} \in \mathrm{Sh}(X)$ an injective sheaf. Then $\pi_*\mathcal{I}$ is an injective sheaf.*

Proof. We want to show that given a sheaf morphism $\varphi : \mathcal{F} \to \pi_*\mathcal{I}$ and an injection $\mathcal{F} \to \mathcal{G}$, there exists a sheaf map $\widetilde{\varphi} : \mathcal{G} \to \pi_*\mathcal{I}$ such that the follwing diagram commutes:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d, hook] & \pi_*\mathcal{I} \\ \mathcal{G} \arrow[ur, dashed, "\widetilde{\varphi}"'] & \end{tikzcd}$$

Using the adjunction of π^* with π_*, this is equivalent to finding a map $\widehat{\varphi} : \pi^*\mathcal{G} \to \mathcal{I}$ such that the following diagram commutes :

$$\begin{tikzcd} \pi^*\mathcal{F} \arrow[r] \arrow[d, hook] & \mathcal{I} \\ \pi^*\mathcal{G} \arrow[ur, dashed, "\widehat{\varphi}"'] & \end{tikzcd}$$

where the map $\pi^*\mathcal{F} \to \mathcal{I}$ is the composition of $\pi^*\varphi : \pi^*\mathcal{F} \to \pi^*\pi_*\mathcal{I}$ with the natural map $\pi^*\pi_*\mathcal{I} \to \mathcal{I}$ obtained by taking the image of $\mathrm{id}_{\pi_*\mathcal{I}}$ under the map $\mathrm{Hom}(\pi_*\mathcal{I}, \pi_*\mathcal{I}) \to \mathrm{Hom}(\pi^*\pi_*\mathcal{I}, \mathcal{I})$ given by the adjunction. In addition, we use the fact that π^* is exact to conclude that the map $\pi^*\mathcal{F} \to \pi^*\mathcal{G}$ is injective. We then note that since $\mathcal{I}$ is injective, such a $\widehat{\varphi}$ exists. ■

The lemma then lets us find the nice characterization we desire.

Proposition 4.7. *For a continuous map $\pi : X \to Y$ and a sheaf $\mathcal{F} \in \mathrm{Sh}(X)$, the derived pushforward sheaves $R^i\pi_*\mathcal{F}$ are the sheafifications of the presheaves defined by*

$$U \mapsto H^i(\pi^{-1}(U), \mathcal{F}|_{\pi^{-1}(U)})$$

Proof. Consider the complex of sheaves over Y obtained by applying π_* to an injective resolution of $\mathcal{F}$.

$$0 \longrightarrow \pi_*\mathcal{F} \xrightarrow{\pi_*d^0} \pi_*\mathcal{I}^1 \xrightarrow{\pi_*d^1} \pi_*\mathcal{I}^2 \xrightarrow{\pi_*d^2} \cdots$$

By the previous lemma, this is an injective resolution of the sheaf $\pi_*\mathcal{F}$. For an open set $U \subset Y$, we have that the sections of $R^i\pi_*\mathcal{F}$ over U are

$$(R^i\pi_*\mathcal{F})(U) := \left(\frac{\ker \pi_*d^i}{\mathrm{Im}\, \pi_*d^{i-1}}\right)(U)$$

By definition, the quotient sheaf $\ker \pi_*d^i \,/\, \mathrm{Im}\, \pi_*d^{i-1}$ is the sheafification of the presheaf

$$U \mapsto \frac{(\ker \pi_*d^i)(U)}{(\mathrm{Im}\, \pi_*d^{i-1})(U)}$$

We then note that since the complex of sheaves is an injective resolution for the sheaf $\pi_*\mathcal{F}$, the R-module given by $(\ker \pi_*d^i)(U)\,/\,(\mathrm{Im}\, \pi_*d^{i-1})(U)$ is the i^{th} right derived functor for the functor $\Gamma(U, -)$ that takes a sheaf over Y to its sections over U. We then note that this is exactly the i^{th} sheaf cohomology group $H^i(U, \pi_*\mathcal{F}|_U)$, which is the same as $H^i(\pi^{-1}(U), \mathcal{F}|_{\pi^{-1}(U)})$. ■

In this way we see that the derived pushforwards behave like the sheaf cohomology groups for a sheaf relative to the map $\pi : X \to Y$. Indeed, if Y is a point, then the functor π_* is just taking global sections, and the derived pushforwards $R^i\pi_*\mathcal{F}$ become the constant sheaves associated to the sheaf cohomology groups $H^i(X, \mathcal{F})$.

5. The Gauss-Manin Connection

To study variations of Hodge structure, we first need a notion of what it means to vary the complex structure on a smooth manifold X.

Definition 5.1. A ***family of complex manifolds*** is a proper holomorphic submersion $\pi : \mathcal{X} \to B$.

If we fix a basepoint $b \in B$, then the fiber $X_b := \pi^{-1}(b)$ is a complex submanifold of $\mathcal{X}$. The idea is that fibers near X_b should be deformations of X_b. This is made precise by a theorem due to Ehresmann.

Theorem 5.2 (***Ehresmann***)**.** *Let $\pi : \mathcal{X} \to B$ be a smooth proper submersion, where B is contractible. Let $b \in B$ be a basepooint, and X_b the fiber over b. Then there exists a difeomorphism $\mathcal{X} \to X_b \times B$ such that the following diagram commutes:*

$$\begin{tikzcd} \mathcal{X} \arrow[rr] \arrow[dr, "\pi"'] & & X_b \times B \arrow[dl] \\ & B & \end{tikzcd}$$

where the map $X_b \times B \to B$ is projection onto the second factor.

In particular, if π is a family of complex manifolds, this statement can be refined further.

Theorem 5.3. *Let $\pi : \mathcal{X} \to B$ be a proper holomorphic submersion where B is contractible. Fix a basepoint $b \in B$. Then there exists a smooth map $T_b : \mathcal{X} \to X_b$ such that the map $(T_b, \pi) : \mathcal{X} \to X_b \times B$ is a trivialization of $\mathcal{X}$ and the fibers of T_b are complex submanifolds of $\mathcal{X}$.*

Note that the map T_b need not be (and often is not) holomorphic. However, this tells us that in some sense, the complex structure is varying holomorphically. To make this more precise, we note that by forgetting the complex structure, we can view X_b as a smooth manifold. From this perspective, a complex structure on X_b can be interpreted as a diffeomorphism $X \to X_b$ where X is a complex manifold – by declaring this map to be an isomorphism of complex manfolds, this uniquely determines a complex structure on the smooth manifold X_b. Using this perspective, the map $T_b : \mathcal{X} \to X_b$ can be interpreted as a family of diffeomorphisms $X_p \to X_b$ parameterized by the points $p \in B$ via restriction to the fibers of π. If we pick a point x in some fiber X_p, it lies in some fiber of the map $T_b|_{X_p}$, which is a complex manifold diffeomorphic to B. Moving in the "B-direction" takes us though different fibers of π (in other words, different complex structures on X_b), and since the fibers of T_b are complex manifolds, this movement is "holomorphic." Since all the points in the fiber $T_b|_{X_p}$ map to the same point as x under T_b, this can be seen as viewing the same point $T_b(x)$ of the underlying smooth manifold X_b using the different complex structures holomorphically parameterized by B.

Given a family of complex manifolds $\pi : \mathcal{X} \to B$ manifolds over a contractible base B such that the fiber X_b over a fixed basepoint $b \in B$ is isomorphic to X as a complex manifold, we get an exact sequence of holomorphic vector bundles over X

$$0 \longrightarrow T_X \longrightarrow T_{\mathcal{X}}|_X \longrightarrow (\pi^* T_B)|_X \longrightarrow 0$$

Since B is contractible, the bundle T_B is isomorphic to the trivial bundle $B \times T_{B,b}$, so the bundle $(\pi^* T_B)|_X$ is isomorphic to the trivial $X \times T_{B,b}$. The long exact sequence on sheaf cohomology induced by the short exact sequence then gives us a boundary map $\rho : H^0(X, X \times T_{B,b}) \to H^1(X, T_X)$. The cohomology group $H^0(X, X \times T_{B,b})$ is equal to the space of global sections of $X \times T_{B,b}$, which is simply the space of holomorphic maps $X \to T_{B,b}$. Then since X is compact, any such map is constant, so we may interpret ρ as a map $T_{B,b} \to H^1(X, T_X)$. This map is called the ***Kodaira-Spencer map***. Though we won't get into it here, the Kodaira-Spencer map classifies the infinitesimal deformation of complex structure in the family $\pi : \mathcal{X} \to B$.

We now make use of our detour into derived pushforwards. Let $\pi : \mathcal{X} \to B$ be a family of complex manifolds over a contractible base B with basepoint b. Letting $\underline{R}$ denote the constant sheaf over $\mathcal{X}$ with stalk R where $R = \mathbb{Z}, \mathbb{R}$, or $\mathbb{C}$, we get the derived pushforward sheaves $R^i \pi_* \underline{R} \in \mathrm{Sh}(B)$. Over a neighborhood U of b, we get that

$$(R^i \pi_* \underline{R})(U) = H^i(\pi^{-1}(U), \underline{R}|_{\pi^{-1}(U)})$$

Since $\underline{R}|_{\pi^{-1}(U)}$ is the same as the constant sheaf over $\pi^{-1}(U)$, these are the same as the singular cohomology groups $H^i(\pi^{-1}(U), R)$, which are topological invariants. Then using Ehresmann's theorem, we know that $\mathcal{X} \to B$ can be trivialized to $X_b \times B \to B$, so $\pi^{-1}(U) \cong X_b \times U$. By restricting our attention to contractible neighborhoods U of b (which we can do since B is locally contractible), we have that $X_b \times U$ is homotopy equivalent to X_b. Putting this all together, we find that the sheaves $R^i \pi_* \underline{R}$ are local systems with stalks isomorphic to the singular cohomology groups $H^i(X_b, R)$.

Definition 5.4. Let $\pi : \mathcal{X} \to B$ be a family of complex manifolds, and let $b \in B$ be a basepoint. Let $\mathcal{H}^i$ denote the vector bundle obtained from the local system $R^i \pi_* \underline{R}$. Then the ***Gauss-Manin connection*** on $\mathcal{H}^i$ is the flat connection ∇ induced by the local system.

A section of $\mathcal{H}^i$ can be thought of as a family of cohomology classes $\alpha_t \in H^i(X_b, R)$ parameterized by B. From this perspective, the flat sections are exactly the ones that define the same cohomology class as α_b, where we use the isomorphism $X_t \to X_b$ induced by the map $T_b : \mathcal{X} \to X_b$ obtained by trivializing $\pi : \mathcal{X} \to B$ to identify the cohomology groups $H^i(X_t, R) \cong H^i(X_b, R)$.

6. Variations of Hodge Structure

Fix a compact Kähler manifold X and a family of deformations $\pi : \mathcal{X} \to B$ over a contractible base B, i.e. a family where the fiber over a distinguished basepoint $b \in B$ is X. We then have the following result.

Theorem 6.1. *For a sufficiently small neighborhood of the basepoint $b \in B$, all the fibers are Kähler manifolds.*

Because of this, up to restricting B to a smaller neighborhood, we can assume that all the fibers are Kähler manifolds diffeomorphic to X. In particular, this tell us that all the Betti numbers $b^i(X_t)$ are the same for all $t \in B$. However, more is true.

Theorem 6.2. *For a sufficiently small neighborhood of the basepoint b, the Hodge numbers $h^{p,q}(X_t) := \dim H^{p,q}(X_t, \mathbb{C})$ are equal to $h^{p,q}(X)$.*

Though we won't prove it here, the result can be deduced from upper semicontinuity of the dimensions of the kernels of a smoothly varying elliptic differential operator.

Since B is contractible, we get isomorphisms $H^k(\mathcal{X}, \mathbb{C}) \cong H^k(X, \mathbb{C})$ and $H^k(\mathcal{X}, \mathbb{C}) \cong H^k(X_t, \mathbb{C})$ given by the restriction of differential forms, so we get a canonical identification $H^k(X_t, \mathbb{C}) \cong H^k(X, \mathbb{C})$. for all $t \in B$. The fact that the Hodge numbers are constant might suggest that the Kähler structure is not varying, but this isn't true. One way to see this is through the period maps.

Definition 6.3. Let $b^{p,k} := \dim F^p H^k(X, \mathbb{C})$ and let $G = \mathrm{Gr}(b^{p,k}, H^k(X, \mathbb{C}))$. denote the Grasmannian of complex $b^{p,k}$-dimensional subspaces of $H^k(X, \mathbb{C})$. The ***period maps*** of the family $\pi : \mathcal{X} \to B$ are the maps

$$\begin{aligned} \mathcal{P}^{p,k} : B &\to G \\ t &\mapsto F^p H^k(X_t, \mathbb{C}) \subset H^k(X, \mathbb{C}) \end{aligned}$$

where we use the identification $H^k(X_t, \mathbb{C}) \cong H^k(X, \mathbb{C})$.

There are two key facts about the period maps that we will need, but won't prove.

Proposition 6.4.

(1) *The $\mathcal{P}^{p,k}$ are holomorphic.*

(2) *The image of differential of $\mathcal{P}^{p,k}$ at $t \in B$*

$$d\mathcal{P}^{p,k}_t : T_{B,t} \to T_{G, F^p H^k(X_t,\mathbb{C})} = \mathrm{Hom}(F^p H^k(X_t, \mathbb{C}), H^k(X, \mathbb{C})/F^p H^k(X_t, \mathbb{C}))$$

is contained in $\mathrm{Hom}(F^p H^k(X_t, \mathbb{C}), F^{p-1} H^k(X_t, \mathbb{C})/F^p H^k(X_t, \mathbb{C}))$.

Altogether, we may package the period maps $\mathcal{P}^{p,k}$ into a single map $\mathcal{P}^k$ from B into a product of Grassmannians, whose p^{th} component is $\mathcal{P}^{p,k}$. The first fact tells us that the Hodge filtration varies holomorphically in a family. It is less obvious as to how one should interpret the second fact – we will need it to discuss Griffiths transversality.

We now put everything together to discuss variations of Hodge structure. Let $\mathcal{H}^k$ denote the vector bundles over B obtained from the local systems $R^i\pi_*\underline{\mathbb{C}}$, equipped with the Gauss-Manin connection ∇. The Grassmannians $G^{p,k} := \mathrm{Gr}(b^{p,k}, H^k(X, \mathbb{C}))$ have natural holomorphic vector bundles $\mathcal{S}^{p,k} \to G$ called the ***tautological bundles*** where the fiber over a subspace $W \subset H^k(X, \mathbb{C})$ is W.

Definition 6.5. The ***Hodge bundles*** of the family of deformations $\pi : \mathcal{X} \to B$ are the holomorphic vector bundles $F^p\mathcal{H}^k := (\mathcal{P}^{p,k})^*\mathcal{S}^{p,k}$.

As the name suggests, the fiber of $F^p\mathcal{H}^k$ over $t \in B$ is the fiber of $\mathcal{S}^{p,k}$ over $\mathcal{P}^{p,k}(t) = F^pH^k(X_t,\mathbb{C})$. The Hodge bundles give a decreasing filtration of $\mathcal{H}^k$

$$\mathcal{H}^k = F^0\mathcal{H}^k \supset \cdots \supset F^k\mathcal{H}^k \supset F^{k+1}\mathcal{H}^k = 0$$

where over each $t \in B$, taking the fibers over t gives the Hodge filtration on X_t. Furthermore, the quotient bundles $\mathcal{H}^{p,q} := F^p\mathcal{H}^k/F^{p+1}\mathcal{H}$ (where $p+q=k$) have the property that the fibers over T are given by $F^pH^k(X_t,\mathbb{C})/F^{p+1}H^k(X_t,\mathbb{C}) = H^{p,q}(X_t,\mathbb{C})$, which tells us that the associated graded vector bundle of this filtration gives us the Hodge decompositions over each fiber. Furthermore, the Hodge subbundles satisfy a condition called ***Griffiths transversality***.

Theorem 6.6 (***Griffiths Transversality***)**.** *The Hodge bundles satisfy*

$$\nabla F^p\mathcal{H}^k \subset F^{p-1}\mathcal{H}^k \otimes \Omega^1_B$$

Proof. Since the flat sections of $\mathcal{H}^k$ are a local system with constant stalk $H^k(X,\mathbb{C})$, we can trivialize the bundle $\mathcal{H}^k$ to obtain an isomorphism of holomorphic vector bundles $\mathcal{H}^k \cong B \times H^k(X,\mathbb{C})$ which identifies the Gauss-Manin connection ∇ with the ordinary derivative on the trivial bundle. Putting things together, ∇ can be interpreted as follows: a section $\sigma : B \to \mathcal{H}^k$ maps a point $t \in B$ to a cohomology class $\sigma(t) \in H^k(X_t,\mathbb{C})$, which is identified with a cohomology class α_t in $H^k(X,\mathbb{C})$. Differentiating σ in the direction $v \in T_{B,t}$ amounts to taking the infinitesimal change in the cohomology class identified with α_t as we vary the Hodge filtration by considering the fibers of $\mathcal{X} \to B$ in the direction v, which is encoded by the image of v under the differential of the period map $d\mathcal{P}^k_t(v)$, which is an element of $\bigoplus_p \mathrm{Hom}(F^pH^k(X_t,\mathbb{C}), H^k(X,\mathbb{C})/F^p(X_t,\mathbb{C}))$. If the class $\sigma(t)$ lies in $F^pH^k(X_t,\mathbb{C})$, then applying the p^{th} component of $d\mathcal{P}^k_t(v)$ to $\sigma(t)$ gives us $\nabla_v\sigma(t)$. Griffiths transversality then follows immediately from the second fact in Proposition 5.4. ∎

Griffiths transversality tells us that the restrictions of ∇ to the Hodge bundles $F^p\mathcal{H}^k$ descend to the quotients $\mathcal{H}^{p,q}$ to maps $\theta^{p,q} : \mathcal{H}^{p,q} \to \mathcal{H}^{p-1,q+1} \otimes \Omega^1_B$.

Proposition 6.7. *The maps $\theta^{p,q} : \mathcal{H}^{p,q} \to \mathcal{H}^{p-1,q+1} \otimes \Omega^1_B$ are holomorphic bundle homomorphisms.*

Proof. We want to show that the $\theta^{p,q}$ are $\mathcal{O}_B$-linear. Let f be a holomorphic function and σ a section of $\mathcal{H}^{p,q} \subset \mathcal{H}^k$, which we can represent with a section of $F^p\mathcal{H}^k$, which we also call σ. Since ∇ is a connection, we have that

$$\nabla(f\sigma) = f\nabla(\sigma) + \sigma \otimes df$$

We then note that $\sigma \otimes df$ is a section of $F^p\mathcal{H}^k \otimes \Omega^1_B$, and that $\nabla(\sigma)$ is a section of $F^{p-1}\mathcal{H}^k$ by Griffiths transversality, so $f\nabla(\sigma)$ and $\nabla(f\sigma)$ define the same element of $\mathcal{H}^{p-1,q+1} \otimes \Omega^1_B$. Therefore, $\theta^{p,q}$ is $\mathcal{O}_B$-linear. ∎

Working fiberwise over $t \in B$, we have that $\theta^{p,q}$ restricted to the fiber over t is a map

$$\mathcal{H}^{p,q}_t = H^{p,q}(X,\mathbb{C}) \to (\mathcal{H}^{p-1,q+1} \otimes \Omega^1_B)_t = H^{p-1,q+1}(X_t,\mathbb{C}) \otimes T^*_{B,t}$$

Given a section σ of $\mathcal{H}^{p,q}$ whose value at t is the cohomology class $[\sigma(t)] \in H^{p,q}(X_t,\mathbb{C})$ applying $\theta^{p,q}$ to σ and evaluating the result at a tangent vector $v \in T_{B,b}$ gives us a cohomology class in $H^{p-1,q+1}(X_t,\mathbb{C})$, which can be interpreted as the infinitesimal change in

$[\sigma(t)]$ as we vary the fibers of $\mathcal{X} \to B$ in the direction v. The maps $\theta^{p,q}$ can explicitly be realized as the differentials of the period map. For this reason, the collection of maps $\{\theta^{p,q}\}$ evaluated at t is called the ***infinitesimal variation of Hodge structure*** at t. Another useful characterization of the $\theta^{p,q}$ comes from second fundamental forms with respect to ∇ of the short exact sequences

$$0 \longrightarrow F^p\mathcal{H} \longrightarrow \mathcal{H} \longrightarrow \mathcal{H}/F^p\mathcal{H} \longrightarrow 0$$

The second fundamental form gives a map $F^p\mathcal{H} \to \mathcal{H}/F^p\mathcal{H} \otimes \mathcal{A}_B^{1,0}$, which is obtained by composing ∇ with the map $\mathcal{H} \otimes \mathcal{A}_B^{1,0} \to \mathcal{H}/F^p\mathcal{H} \otimes \mathcal{A}_B^{1,0}$ induced by the quotient map. By Griffiths transversality, the image of $F^p\mathcal{H}$ under the second fundamental form is contained in $F^{p-1}\mathcal{H}/F^p\mathcal{H} = \mathcal{H}^{p-1,q}$ and $F^{p+1}\mathcal{H}$ is mapped to $F^p\mathcal{H}$, so the second fundamental form descends to the quotient $F^p\mathcal{H}/F^{p+1} = \mathcal{H}^{p,q}$ to give the map $\theta^{p,q}$. A nontrivial computation shows that the maps $\theta^{p,q}$ are given by composing interior multiplication with wedgeing with the Kodaira-Spencer map.

The above discussion motivates the definition of an abstract variation of Hodge structure.

Definition 6.8. A ***variation of Hodge structure*** of weight k over a complex manifold B is the data of a local system V of abelian groups over B, and a decreasing filtration of the vector bundle $\mathcal{V} := V \otimes_{\underline{\mathbb{C}}} \mathcal{O}_B$ by holomorphic subbundles $F^p\mathcal{V}$ such that

(1) $\mathcal{V} = F^p\mathcal{V} \oplus \overline{F^{k-p+1}\mathcal{V}}$ as smooth vector bundles, where conjugation is taken from the real structure defined by $V \otimes_{\underline{\mathbb{R}}} \mathbb{R}$.
(2) The subbundles $F^p\mathcal{V}$ satisfy the ***Griffiths transversality*** condition

$$\nabla(F^p\mathcal{V}) \subset \mathcal{F}^{p-1} \otimes_{\mathcal{O}_B} \Omega_B^1$$

where ∇ is the Gauss-Manin connection on $\mathcal{V}$ obtained from the local system V.

The prototypical example of a variation of Hodge structure of weight k is given by the local system $R^k\pi_*\underline{\mathbb{Z}}$ obtained from a family $\pi : \mathcal{X} \to B$. A variation of Hodge structure can also be defined in terms of the Hodge decomposition instead of the Hodge filtration, i.e. by specifying a direct sum decomposition of $\mathcal{H}$ by subbundles $\mathcal{H}^{p,q}$ giving a Hodge decomposition of each fiber. One can go from one viewpoint to the other in one direction by taking quotients $F^p\mathcal{V}/F^{p+1}\mathcal{V}$, and in the other direction by taking direct sums of the $\mathcal{H}^{p,q}$.

For polarized Hodge structures the prototypical example will be a family of smooth projective varieties $\pi : \mathcal{X} \to B$. The fibers of π come equipped with integral Kähler classes, giving us a section of the local system $R^2\pi_*\underline{\mathbb{Z}}$, which we use to define the Lefschetz operator fiberwise, giving us a bilinear form Q on each of the stalks of the local system $R^k\pi_*\underline{\mathbb{Z}}$ and a sub Hodge structure on each fiber given by the primitive component of $H^k(X_t, \mathbb{Z})$, which forms a local system $V_{\mathrm{Prim}} \subset R^k\pi_*\underline{\mathbb{Z}}$. Complexifying, we get a Hermitian fiber metric on the vector bundle $R^k\pi_*\underline{\mathbb{Z}} \otimes_{\underline{\mathbb{C}}} \mathcal{O}_B$, and the restrictions of Q to V_{Prim} and H to $\mathcal{H}_{\mathrm{Prim}} := V_{\mathrm{Prim}} \otimes_{\mathbb{C}} \mathcal{O}_B$ give the stalks of V_{Prim} the structure of polarized Hodge structures.

Furthermore, the compatibility of the Hodge decomposition with the primitive components tells us that the local system V_{Prim} defines a variation of Hodge structure. Putting things together, we get

Definition 6.9. A ***polarized variation of Hodge structure*** of weight k over B is a variation of Hodge structure V of weight k, along with a bilinear form Q on each stalk of V such that

(1) Q is symmetric if k is even and alternating if k is odd.
(2) The Hodge decomposition on each fiber of the associated vector bundle is orthogonal with respect to the Hermitian fiber metric H induced by Q.
(3) The hermitian form H is flat with respect to the Gauss-Manin connection, i.e. $\nabla H = 0$.
(4) The form $(-1)^{k(k-1)/2} i^{p-q-k} H$ is positive definite on each fiber of $V^{p,q}$.

The Hermitian fiber metric associated to a polarized variation of Hodge structure is often called the ***Hodge metric***.

7. Higgs Bundles and Complex Variations of Hodge Structure

Polarized complex variations of Hodge structure are in correspondence with objects called ***Higgs bundles***, which is the shadow of a the ***nonabelian Hodge theorem***[3], which provides an equivalence between the category of local systems with a certain subcategory of the category of Higgs bundles. For this section, we will assume that we are working with vector bundles over a smooth complex projective variety X.

Definition 7.1. A ***Higgs bundle*** is a holomorphic vector bundle $E \to X$ equipped with a holomorphic $\mathrm{End}(E)$ valued 1-form $\theta \in \Omega^1_X(\mathrm{End}(E))$ satisfying $\theta \wedge \theta = 0$. The form θ is called the ***Higgs field***.

Given a complex variation of Hodge structure, we want to construct a Higgs bundle. Suppose we have a complex variation of Hodge structure of weight k, which is given by a holomorphic vector bundle $E \to X$ along with a filtration of E by holomorphic subbundles $F^p E$ and a flat connection ∇ on E satisfying Griffiths transversality. We let $E_{\mathrm{Hodge}} = \bigoplus_p F^p E / F^{p+1} E$ denote the associated graded bundle coming from the filtration, and let $E^{p,q}$ (where $p + q = k$) denote the summands. We have that E is isomorphic to E_{Hodge} as smooth bundles, but not necessarily as holomorphic bundles. The infinitesimal variations of Hodge structures give us holomorphic maps $\theta^{p,q} : E^{p,q} \to E^{p-1,q} \otimes \Omega^1_X$, and taking their sum gives a holomorphic map $\theta : E \to E \otimes \Omega^1_X$, i.e. an element $\theta \in \Omega^1_X(\mathrm{End}(E))$. Furthermore, since θ is induced by the Gauss-Manin connection on E, which is flat, we get that $\theta^2 = 0$, so θ defines a Higgs field. Putting things together, a complex variation of Hodge structure E gives us a Higgs bundle $(E_{\mathrm{Hodge}}, \theta)$.

Suppose further that $E \to X$ comes from a polarized variation of Hodge structure, so we have a Hodge metric on E_{Hodge} that is flat with respect to the Gauss-Manin connection ∇, and whose restrictions to the $E^{p,q}$ are definite. By changing the signs of the Hodge metric on the $E^{p,q}$ to make it positive definite, we obtain a Hermitian metric on E_{Hodge}, which for simplicity we will also refer to as the Hodge metric – from now on "Hodge metric" will always refer to the positive definite form unless specified otherwise. The Hodge metric on

E_{Hodge} gives us a Chern connection D, and the isomorphism of E and E_{Hodge} as smooth bundles allows us to transport ∇ to a flat connection on E_{Hodge}. Furthermore, we can use the Hodge metric to define the splitting of the filtration of E by the bundles F^pE to obtain our isomorphism of E with E_{Hodge} as a smooth bundles. The difference $\nabla - D$ is an element of $\mathcal{A}^1_B(\text{End}(E_{\text{Hodge}}))$, and it turns out to be a familiar friend.

Proposition 7.2. *Let $\theta \in \Omega^1_X(\text{End}(E_{\text{Hodge}}))$ be the map obtained by summing the infinitesimal variations of Hodge structure. Then*

$$\nabla - D = \theta + \theta^*$$

Where θ^* is the Hermitian adjoint of θ with respect to the Hodge metric.

The proof of this amounts to understanding the second fundamental form of a holomorphic subbundle.

Lemma 7.3. *Let $E \to X$ be a Hermitian holomorphic vector bundle, $S \subset E$ a holomorphic subbundle, and $Q := E/S$ the quotient bundle. Equip S with the Hermitian metric by restricting the Hermitian metric on E, and equip Q with the Hermitian metric obtained by identifying the underlying smooth bundle with $S^\perp$. Let ∇_E,∇_S, and ∇_Q denote the respective Chern connections, and $\eta \in \mathcal{A}^{1,0}_X(\text{Hom}(S, Q))$ the second fundamental form. Then under the identification of the underlying smooth bundle of E with $S \oplus Q$, we have that*

$$\nabla_E - (\nabla_S + \nabla_Q) = \eta + \eta^*$$

where η^* is the Hermitian adjoint of η, and $\nabla_S + \nabla_Q$ is the direct sum connection on $S \oplus Q$.

Proof. It suffices to verify this locally. Fix an orthonormal frame $e_1, \dots, e_n$ for E such that $e_1, \dots, e_s$ forms an orthonormal frame for S, so $e_{s+1}, \dots e_n$ is a local orthonormal frame for $S^\perp$. Then by the definition of the second fundamental form we have that the connection matrix α_E for ∇_E in this frame is given in block form by

$$\begin{pmatrix} \alpha_S & \eta^* \\ \eta & \alpha_Q \end{pmatrix}$$

where α_S and α_Q are the connection matrices for ∇_S and ∇_Q respectively. Since the connection matrix for $\nabla_S + \nabla_Q$ is block diagonal with α_S and α_Q on the diagonal, this shows the desired result. ∎

Proof of proposition. Since ∇ is the Gauss-Manin connection coming from a local system, we know that the $(0, 1)$ part defines the holomorphic structure on E. Furthermore, since the Hodge metric is flat with respect to ∇, we get that ∇ is the Chern connection on E with respect to the holomorphic structure obtained from ∇. We have an canonical identification of $E^{k,0}$ with the bundle F^kE as holomorphic bundles, where k is the weight of the variation of Hodge structure. Then we can identify $(F^kE)^\perp$ with the quotient E/F^kE and $F^{k-1}E/F^k = E^{k-1,1}$ with a subbundle of $F^kE^\perp$ as smooth bundles. Iterating this construction with $E^{k-1,1}$ and $F^{k-2}E$ and so on, we get an orthogonal splitting identifying F^pE with $\bigoplus_{r \le p} E^{p,q}$, identifying the underlying smooth bundles of E and the Hodge bundle E_{Hodge}. Then using the fact that the Chern connection on E_{Hodge} is the sum of the Chern connections on the $E^{p,q}$, an iterative application of the previous lemma gives the desired result. ∎

We then want to identify the Higgs bundles that arise from variations of Hodge structure. The set of isomorphism classes of Higgs bundles admits a $\mathbb{C}^\times$-action by scaling the Higgs field, i.e. the action of $t \in \mathbb{C}^\times$ on a Higgs bundle (E, θ) is $(E, t\theta)$. Our first observation relates the fixed points of this $\mathbb{C}^\times$ action with what Simpson calls a ***system of Hodge bundles***, which is slightly weaker notion than a variation of Hodge structure.

Proposition 7.4. *Suppose a Higgs bundle (E, θ) admits the structure of a **system of Hodge bundles**, i.e. there exists some $k \in \mathbb{Z}^{\geq 0}$ such that E admits a direct sum decomposition $E = \bigoplus_{p+q=k} E^{p,q}$ such that θ maps $E^{p,q}$ to $E^{p-1,q+1} \otimes \Omega^1_X$. Then E is fixed under the $\mathbb{C}^\times$ action, i.e. $(E, \theta) \cong (E, t\theta)$ for all $t \in \mathbb{C}^\times$. Conversely, any Higgs bundle that is fixed by the $\mathbb{C}^\times$ action admits such a decomposition.*

Remark. The difference between a variation of Hodge structure and a system of Hodge bundles is somewhat subtle. Given a variation of Hodge structure, one obtains a system of Hodge bundles by taking the associated graded bundle and the infinitesimal variations of Hodge structure. In general, one cannot go the other direction, since we have forgotten the data of the Gauss-Manin connection and have only kept the "infinitesimal data."

Proof. First suppose (E, θ) admits the structure of a system of Hodge bundles. We want to produce isomorphisms $(E, \theta) \to (E, t\theta)$ for all $t \in \mathbb{C}^\times$. Fix such a t, and consider the map $\varphi_t : E \to E$ given by scaling $E^{p,q}$ by t^q. The map is a holomorphic bundle isomorphism $E \to E$, and we have that for $v \in E^{p,q}$

$$(t\theta \circ \varphi_t)(v) = t\theta(t^q v) = t^{q+1}\theta(v) = ((\varphi_t \otimes \mathrm{id}_{\Omega^1_X}) \circ \theta)(v)$$

where we use the fact that $\theta(v) \in E^{p-1,q+1}$.

For the other direction, suppose we have a Higgs bundle (E, θ) of rank k that is fixed by the $\mathbb{C}^\times$-action. In particular, for a $t \in \mathbb{C}^\times$ where t is not a root of unity, we have that there exists an isomorphism $\varphi_t : E \to E$ such that φ_t intertwines θ and $t\theta$ as above. Taking the characteristic polynomials of φ_t fiberwise on E, we get holmomorphic functions on B mapping $b \in B$ to the i^{th} coefficient of the characteristic polynomial of φ_t restricted to E_b. Since B is compact, these functions are constant, so the eigenvalues of φ_t are constant functions on B, so we can decompose $E = \bigoplus_\lambda E_\lambda$ into λ-eigenbundles of φ_t, so $E_\lambda := \ker(f - \lambda)^n$, where n is the multiplicity of the eigenvalue λ. By assumption, we have that $t\theta \circ \varphi_t = (\varphi_t \otimes \mathrm{id}_{\Omega^1_X}) \circ \theta$, so we get that $t^n\theta \circ (f - \lambda)^n = ((\varphi - t\lambda)^n \otimes \mathrm{id}_{\Omega^1_X}) \circ \theta$, which tells us that θ maps E_λ to $E_{t\lambda} \otimes \Omega^1_X$. Since t is not a root of unity, we may decompose the set of eigenvalues as a disjoint uniion of sets of the form $\{\lambda_i, t\lambda_i, t^2\lambda_i, \dots t^r\lambda_i\}$ for some subset of eigenvalues λ_i. Then we get a system of Hodge bundles by defining $E^{i,k-i}$ as the direct sum $\bigoplus_i E_{\lambda_i}$. ∎

This suggests that Higgs bundles are intimately related to complex variations of Hodge structure, and if we restrict ourselves to polarized complex variations of Hodge structure, we do in fact get a correspondence, which is a special case of the ***nonabelian Hodge theorem***. To state the nonabelian Hodge theorem, we use the fact that a local system V is equivalent to the data of a representation $\pi_1(X) \to \mathrm{GL}_k\mathbb{C}$. In one direction, you take the holonomy of the Gauss-Manin connection, which is independent of the homotopy class of a loop from flatness. In the other direction, you obtain a flat bundle from a represetation

$\pi_1(M) \to \mathrm{GL}_k\mathbb{C}$ by taking the associated bundle

$$\widetilde{X} \times_{\pi_1(M)} \mathbb{C}^n := (\widetilde{X} \times \mathbb{C}^n)/\pi_1(M)$$

with $\pi_1(M)$ acting diagonally. The second notion we'll need is that notion of stability of Higgs bundles.

Definition 7.5. The ***slope*** $\mu(E)$ of a Higgs bundle (E, θ) is $\mu(E) := \deg(E)/\mathrm{rank}(E)$. A Higgs bundle (E, θ) is :

(1) ***Stable*** if for every Higgs subbundle $S \subset E$, i.e. a holomorphic subbundle such that $\theta(S) \subset S \otimes \Omega^1_X$, we have $\mu(S) < \mu(E)$.
(2) ***Semistable*** if for every Higgs subbundle S, we have $\mu(S) \leq \mu(E)$.
(3) ***Polystable*** if E is a direct sum of stable Higgs subbundles of the same slope.

This allows us to state the nonabelian Hodge theorem.

Theorem 7.6 (***Nonabelian Hodge Theorem***)**.** *There is an equivalence of categories between the category of local systems arising from semisimple representations of $\pi_1(X)$ and polystable Higgs bundles with vanishing Chern classes.*

To apply this theorem, we must identify the systems of Hodge bundles that arise from complex variations of Hodge structure. This was done by Simpson[4], building upon work of Donaldson and Uhlenbeck-Yau on the existence of Hermitian-Yang-Mills connections on holomorphic vector bundles.

Theorem 7.7. *Every stable Higgs bundle (E, θ) admits a Hermitian-Yang-Mills metric. In particular, if the Chern classes of E vanish, then E admits a flat connection. If E comes from a system of Hodge bundles, (E, θ), then the flat connection satisfies Griffiths transversality, and the maps $\theta^{p,q}$ are induced by the flat connection.*

This theorem tells us that the construction taking a variation of Hodge structure to a system of Hodge bundles can be inverted in the case that the system of Hodge bundles forms a stable Higgs bundle.

Furthermore, we have the following results:

Proposition 7.8.

(1) *The system of Hodge bundles arising from an complex variation of Hodge structure with irreducible holonomy $\pi_1(M) \to \mathrm{GL}_k\mathbb{C}$ is a stable Higgs bundle.*
(2) *The representations $\pi_1(M) \to \mathrm{GL}_k\mathbb{C}$ arising from polarized complex variations of Hodge structure are semisimple, i.e. they decompose into direct sums of irreducible representations.*

Combinining the first part of the proposition with Theorem 7.7, we get that the stable systems of Hodge bundles are exactly in correspondence with variations of Hodge structure with irreducible holonomy. The second part of the proposition tells us that any complex variation of Hodge structure decomposes into a direct sum of irreducible ones, so the corresponding system of Hodge bundles decomposes into a direct sum of stable systems. Putting everything together, we get:

Proposition 7.9. *Polarized complex variations of Hodge structure are in bijection with semisimple representations of $\pi_1(X)$ whose corresponding Higgs bundles are fixed by the $\mathbb{C}^\times$ action.*

It's worth noting how this correspondence fits into the proof of the nonabelian Hodge theorem. The proof consists of proving that the categories of semisimple representations of $\pi_1(X)$ and polystable Higgs bundles with vanishing Chern classes are equivalent to the same category – the category of ***harmonic bundles***. The idea of harmonic bundles concerns the subtle interplay between a flat connection ∇ on a Hermitian vector bundle E and the Chern connection on E with respect to the holomorphic structure defined by the $(0,1)$ part of ∇.

Suppose we have a flat connection ∇ on a vector bundle E with Hermitian metric K. Let ∇' and ∇'' denote the $(1,0)$ and $(0,1)$ parts respectively, so $\nabla = \nabla' + \nabla''$. Let D be the Chern connection with respect to the holomorphic structure ∇'' and the Hermitian metric, and let δ'_K denote the $(1,0)$ part of D, so $D = \delta'_K + \nabla''$. Let $\theta_K = (\nabla' - \delta'_K)/2$. We then define the ***pseudocurvature*** to be the operator $G_K := (\overline{\partial} + \theta_K)^2$. The pseudocurvature vanishes precisely when $\theta_K \wedge \theta_K = 0$, i.e. θ_K defines a Higgs bundle structure on E.

Suppose instead we have a Higgs bundle (E, θ). Then we get a connection ∇ from the formula $\nabla := \partial + \overline{\partial} + \theta + \theta^*$. In general, the connection ∇ need not be flat. However, when it is, the construction above recovers the Higgs field θ.

Definition 7.10. A ***harmonic bundle*** is a holomorphic bundle E with flat connection ∇ such that there exists a Hermitian metric K such that $\theta_K := (\nabla' - \delta'_K)/2$ defines a Higgs bundle structure on E. Such a Hermitian metric is a ***harmonic metric***.

Therefore the idea of the proof of the nonabelian Hodge theorem involves proving the existence of Harmonic metrics on polystable Higgs bundles. The proof of this is in the spirit of, and draws upon the proofs of Narasimhan-Seshadri and Uhlenbeck-Yau. In the case of polarized complex variations of Hodge structure, we can reinterpret our observation as the statement that the Hodge metric is a harmonic metric.

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