HYPERCOHOMOLOGY AND SPECTRAL SEQUENCES

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Contents

Conventions and Notation

We will use cohomological indexing – for a complex $(A^\bullet, d)$, the differential increases the degree. We fix an abelian category $\mathcal{A}$ with enough injectives – for our purposes it is harmless to assume that $\mathcal{A}$ is one of Ab, Mod_R, or a category of sheaves over a space (e.g. $\mathsf{Sh}_{\mathsf{Ab}}(X)$ or $\mathrm{QCoh}(X)$), because of this, we will be relatively cavalier about using elements when doing homological algebra.

1. Derived Functors and Hypercohomology

Definition 1.1. A ***double complex*** is a collection of objects $K^{p,q} \in \mathcal{A}$ with $p, q \in \mathbb{Z}$ equipped with differentials

$$D_1 : K^{p,q} \to K^{p+1,q}$$
$$D_2 : K^{p,q} \to K^{p,q+1}$$

such that $D_i^2 = 0$ and $D_1 \circ D_2 = D_2 \circ D_1$.

There is a sign convention with double complexes – some prefer to replace the condition that the differentials commute with the condition that they *anticommute*, i.e. $D_1 \circ D_2 = -D_2 \circ D_1$.

The way to visualize a double complex is to arrange the objects $K^{p,q}$ in a grid, with $K^{p,q}$ in the position with coordinate (p,q).

$$\begin{tikzcd}
K^{0,3} \arrow[r, "D_1"] & K^{1,3} \arrow[r, "D_1"] & K^{2,3} \arrow[r, "D_1"] & K^{3,3} \\
K^{0,2} \arrow[r, "D_1"] \arrow[u, "D_2"] & K^{1,2} \arrow[r, "D_1"] \arrow[u, "D_2"] & K^{2,2} \arrow[r, "D_1"] \arrow[u, "D_2"] & K^{3,2} \arrow[u, "D_2"] \\
K^{0,1} \arrow[r, "D_1"] \arrow[u, "D_2"] & K^{1,1} \arrow[r, "D_1"] \arrow[u, "D_2"] & K^{2,1} \arrow[r, "D_1"] \arrow[u, "D_2"] & K^{3,1} \arrow[u, "D_2"] \\
K^{0,0} \arrow[r, "D_1"] \arrow[u, "D_2"] & K^{1,0} \arrow[r, "D_1"] \arrow[u, "D_2"] & K^{2,0} \arrow[r, "D_1"] \arrow[u, "D_2"] & K^{3,0} \arrow[u, "D_2"]
\end{tikzcd}$$

Because of this, we refer to D_1 as the ***horizontal differential*** and D_2 as the ***vertical differential***. One thing to note is that the defintion of a double complex has no finiteness or positivity assumption – we can have a nonzero objects $K^{p,q}$ for arbitrarily large p and q, and we may also have nonzero $K^{p,q}$ for $p,q < 0$.

Example 1.2. Given a complex manifold X, we have the sheaves $\mathcal{A}_X^{p,q}$ of smooth (p,q)-forms. These come equipped with differentials $\partial : \mathcal{A}_X^{p,q} \to \mathcal{A}_X^{p+1,q}$ and $\overline{\partial} : \mathcal{A}_X^{p,q} \to \mathcal{A}_X^{p,q+1}$. The operators ∂ and $\overline{\partial}$ anticommute, so with our sign convention, we need to introduce the sign $(-1)^p$ to $\overline{\partial}$ to get the double complex of sheaves $(\mathcal{A}_X^{p,q}, \partial, (-1)^p\overline{\partial})$.

In the case of some finiteness conditions, we can produce a chain complex from a double complex.

Definition 1.3. Let $(K^{p,q}, D_1, D_2)$ be a double complex such that $K^{p,q} = 0$ when $p > p_0$ or $q > q_0$, for some fixed $p_0, q_0 \in \mathbb{Z}$ (i.e. all the objects become zero once we move far enough down and to the left). Then the ***total complex*** (also called the ***simple complex***) associated to the double complex is the complex $(K^\bullet, D)$ where

$$K^n := \bigoplus_{p+q=n} K^{p,q}$$

and the differential D is given by

$$D := \bigoplus_{p+q=n} (D_1 \oplus (-1)^p D_2)$$

For convenience, we will call a double complex that satisfies the conditions above to be ***bounded below***. Given a complex $(A^\bullet, d)$, we will also say that it is bounded below if there are only finitely many terms in negative degree. By shifting degrees, we may as well assume that a bounded below double complex begins indexing at $(0,0)$, and that a bounded below complex begins at index 0.

Proposition 1.4. *The total complex of a bounded below double complex actually forms a complex.*

Proof. It suffices to verify this on the component $K^{p,q} \subset K^n$. The differential D^2 on this component maps to

$$K^{p+q+2} = K^{p+2} \oplus K^{p+1,q+1} \oplus K^{p,q+2}$$

and is given component wise by the map $(D_1^2, (-1)^{p+1}(D_2 \circ D_1) + (-1)^p(D_1 \circ D_2), D_2^2)$. Then since $D_i^2 = 0$ and the differentials commute, we have that this is the zero map. ■

To make use of double complexes, we first prove a useful fact.

Proposition 1.5. *Let $(I^{p,q}, D_1, D_2)$ be a double complex with total complex $(I^\bullet, D)$, and let $(M^\bullet, d)$ be another complex in $\mathcal{A}$. Then if there exist injective maps $i^p : M^p \to I^{p,0}$ such that $(I^{p,\bullet}, D_2)$ is a resolution of M^p (i.e. the complex $M^p \to I^{p,\bullet}$ is exact for all p), then the morphism $M^\bullet \to I^\bullet$ by the i^p induces an isomorphism from the cohomology of $(M^\bullet, d)$ to the cohomology of the total complex $(I^\bullet, D)$.*

Pictorially, the setup is given by the diagram

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \\
I^{0,2} \arrow[u] \arrow[r, "D_1"] & I^{1,2} \arrow[u] \arrow[r, "D_1"] & I^{2,2} \arrow[u] \arrow[r, "D_1"] & I^{3,2} \arrow[u] \arrow[r] & \cdots \\
I^{0,0} \arrow[u, "D_2"] \arrow[r, "D_1"] & I^{1,0} \arrow[u, "D_2"] \arrow[r, "D_1"] & I^{2,0} \arrow[u, "D_2"] \arrow[r, "D_1"] & I^{3,0} \arrow[u, "D_2"] \arrow[r] & \cdots \\
M^0 \arrow[u, "i^0"] \arrow[r, "d"] & M^1 \arrow[u, "i^1"] \arrow[r, "d"] & M^2 \arrow[u, "i^2"] \arrow[r, "d"] & M^3 \arrow[u, "i^3"] \arrow[r] & \cdots
\end{tikzcd}
$$

Where the columns are all exact.

Proof of proposition. We prove this for abelian groups. The proof for R-modules is near identical, and the proof can also be adapted for an abelian category of sheaves by looking at the induced maps on stalks.

Let $\alpha \in I^k := \bigoplus_{p+q=k} I^{p,q}$ such that $D\alpha = 0$, and let $\alpha_{p,q} \in I^{p,q}$ denote the component of α lying in $I^{p,q}$, so $\alpha = \sum_{p,q} \alpha_{p,q}$. Then since $D|_{I^{p,q}}$ is given by $D = D_1 + (-1)^p D_2$, we have that $D_2\alpha_{0,k} = 0$, since no cancellation can occur, since you cannot obtain an element in $I^{0,k+1}$ by appplying D_1. Furthermore, we know that for $q < k$ we have $D_1\alpha_{p,q} + (-1)^p\alpha_{p+1,q-1} = 0$. By assumption, the complex $(I^{0,\bullet}, D_2)$ is exact, since it is a resolution of M^0, so we have that $\alpha_{0,k} = D_2\beta$ for some $\beta \in I^{0,k-1}$. We then consider the element $\alpha' := \alpha - D\beta$, which defines the same element as α in $H^k(I^\bullet)$. We note that α' has the property that $\alpha'_{0,k} = 0$. Because of this, the fact that $D\alpha' = 0$ implies that $D_2\alpha'_{1,k-1} = 0$, since the only way to get cancellation is by applying D_1 to $\alpha_{0,k}$, which we just noted was 0. Repeating this argument, we obtain an element $\tilde{\alpha} \in I^{k,0}$ that defines the same element in $H^k(I^\bullet)$ as our original element α. We then note that $D\tilde{\alpha} = 0$ implies that $D_i\tilde{\alpha} = 0$, again since there is no way to obtain cancellation. Then since $I^{k\bullet}$ is a resolution of M^k via i^k, the fact that $D_2\tilde{\alpha} = 0$ implies by exactness that $\tilde{\alpha} = i^k(\beta)$ for some $\beta \in M^k$. The fact that

$D_1\tilde{\alpha} = 0$ implies that $d\beta = 0$ by commutativity of the square

$$\begin{array}{ccc} I^{k,0} & \xrightarrow{D_1} & I^{k+1,0} \\ i^k \uparrow & & \uparrow i^{k+1} \\ M^k & \xrightarrow[d]{} & M^{k+1} \end{array}$$

Therefore, we have that in the induced map on cohomology, the cohomology class $[\alpha] = [\tilde{\alpha}] \in H^k(I^\bullet)$ is the image of the class $[\beta] \in H^k(M^\bullet)$, so the map $H^k(M^\bullet) \to H^k(I^\bullet)$ is surjective.

For injectivity, suppose we have that a cohomology class $[\alpha] \in H^k(M^\bullet)$ maps to 0, so $i^k(\alpha) = D\beta$ for some $\beta \in I^{k-1}$. We note that $(D\beta)_{0,k} = 0$ since i^k is a map $M^k \to I^{k,0}$. Furthermore, we note that $(D\beta)_{0,k} = D_2\beta_{0,k-1}$. Therefore, we can find a $\gamma \in I^{0,k-2}$ such that $\beta_{0,k-1} = D_2\gamma$. Then setting $\beta' = \beta - D\gamma$, we have that $D\beta' = D\beta = i^k(\alpha)$ but β' has the property that $\beta'_{0,k-1} = 0$. By repeating this argument, we may assume that $i^k(\alpha) = D\beta$, where $\beta \in I^{k-1,0}$. Then since $D_2\beta \in I^{k-1,1}$ and there is no possibility of cancellation, we have that $D_2\beta = 0$ and $D_1\beta = \alpha$. In the case $k = 1$, we can immediately write $\alpha = D\beta$ with $\beta \in I^{0,0}$, and the same reasoning as above lets us conclude that $D_2\beta = 0$ and $D_1\beta = \alpha$. By exactness, $D_2\beta = 0$ implies that $\beta = i^{k-1}(\gamma)$ for some $\gamma \in M^{k-1}$. Then commutativity of the square

$$\begin{array}{ccc} I^{k-1,0} & \xrightarrow{D_1} & I^{k,0} \\ i^{k-1} \uparrow & & \uparrow i^{k} \\ M^{k-1} & \xrightarrow[d]{} & M^{k} \end{array}$$

this implies that $d\gamma = \alpha$, so $[\alpha] = 0$. Therefore, the induced map on cohomology is injective. ■

We will use this to prove the next proposition.

Proposition 1.6. *let $(M^\bullet, d)$ be a bounded below complex in $\mathcal{A}$. Then there exists another bounded below complex $(I^\bullet, D)$ in $\mathcal{A}$ and a quasi-isomorphism $\varphi : M^\bullet \to I^\bullet$ such that*

(1) *Each of the I^k is an injective object in $\mathcal{A}$.*
(2) *Each $\varphi^k : M^k \to I^k$ is injective.*

Proof. Our goal will be to construct a double complex of injective objects satisfying the conditions of the previous proposition, i.e. a double complex $(I^{p,q}, D_1, D_2)$ that is a level-wise resolution of $M^\bullet$. The desired complex will then be the total complex of the double complex $I^{p,q}$. To do this, it suffices to produce a chain map $i^\bullet : M^\bullet \to I^{1,\bullet}$ such that each i^k is an injection into an injective object $I^{1,k}$. Once we do this, we can iteratively construct the double complex by repeating the argument with $M^\bullet = \operatorname{coker} i^\bullet$ to obtain a chain map $\operatorname{coker} i^\bullet \to I^{2,\bullet}$, and then defining the map $I^{1,\bullet} \to I^{2,\bullet}$ to be the composition of the quotient map $I^{1,\bullet} \to \operatorname{coker} i^\bullet$ with the map $\operatorname{coker} i^\bullet \to I^{2,\bullet}$.

We construct the complex $I^{1,\bullet}$ iteratively. To start, we pick an arbitrary injection $i^0 : M^0 \hookrightarrow I^{0,0}$ from M^0 to an injective object $I^{0,0}$. We have an injective map $M^0 \to I^{0,0} \oplus M^1$

given by $(i^0, -d)$. Let $j : M^1 \to I^{0,0} \oplus M^1$ and $k : I^{0,0} \to I^{0,0} \oplus M^1$ be the inclusions, and let $\pi : I^{0,0} \oplus M^1 \to \operatorname{coker}(I^0, -d)$ be the quotient map. Then fix an injection $\eta : \operatorname{coker}(i^0, -d) \to I^{1,0}$ for an arbitrary injective object $I^{1,0}$. Then define $i^1 : M^1 \to I^{1,0}$ by $i^1 := \eta \circ \pi \circ j$, and define $D_1 : I^{0,0} \to I^{1,0}$ by $D_1 := \eta \circ \pi \circ k$. We claim that the map i^1 is injective. By the definition of the cokernel, an element $\alpha \in M^1$ maps to 0 if it is contained in the image of $(i^0, -d) : M^0 \to I^{0,0} \oplus M^1$. However, since i^0 is injective, the only element with a 0 in the first component is $(0, 0)$. Therefore, since j and η are injective, $i^1(\alpha) = 0$ if and only if $\alpha = 0$. We then claim that $D_1 \circ i^0 = i^1 \circ d$. We note that for $\alpha \in M^0$ we have

$$i^1(d\alpha) = (\eta \circ \pi)(0, d\alpha)$$
$$D_1(i^0(\alpha)) = (\eta \circ \pi)(i^0(\alpha), 0)$$

So if suffices to show that $\pi(0, d\alpha) = \pi(i^0(\alpha), 0)$, but we have that $(i^0(\alpha), 0) - (0, d_\alpha) = (i^0(\alpha), -d\alpha) \in \operatorname{coker}(i^0, -d)$, so this is true. In summary, what we have constructed so far can be summarized by the diagram

$$\begin{array}{ccccc} I^{0,0} & \xrightarrow{D_1} & I^{1,0} & & \\ {\scriptstyle i^0}\uparrow & & {\scriptstyle i^1}\uparrow & & \\ M^0 & \xrightarrow{d} & M^1 & \xrightarrow{d} & M^2 \end{array}$$

To continue this, let $\overline{i^1} : M^1 \to \operatorname{coker} D_1$ denote the composition of i^1 with the quotient map. This gives us a map $(\overline{i^1}, -d) : M^1 \to \operatorname{coker} D_1 \oplus M^2$. As we did before, fix an injection $\eta : \operatorname{coker}(\overline{i^1}, -d) \to I^{2,0}$, and define the maps i^2 and D_1 analogously to the previous construction. The map i^2 is injective and $D_1 \circ i^1 = i^2 \circ d$ by near identical reasoning to before. What's left to show is that $D_1^2 = 0$, but this comes immediately from the fact that $\eta \circ \pi$ vanishes on the image of $(\overline{i^1}, -d)$ by construction. Continuing this process then constructs the desired complex $I^{\bullet,0}$ of injectives. ■

A consequence of this proposition is that for the purposes of cohomology, we can always assume we are computing the cohomology of a complex of injective objects. In more modern language, this says that any bounded below complex $M^\bullet$ is isomorphic to a complex of injectives in the derived category $D(\mathcal{A})$.

The goal of the preceding discussion is to extend the classical notion of the right derived functors of a left exact functor $F : \mathcal{A} \to \mathcal{B}$. The classical examples (e.g. Ext and Tor) are defined on objects of $\mathcal{A}$. However, the preceding discussion allows us to extend the functors R^iF to bounded below complexes $M^\bullet$ in $\mathcal{A}$. Echoing the classical definition, we would like to define $R^iF(M^\bullet)$ on a bounded below complex $M^\bullet$ to be the cohomology groups $H^i(F(I^\bullet))$, where $I^\bullet$ is a complex of injective objects admitting a quasi-isomorphism $i^\bullet : M^\bullet \to I^\bullet$, with each i^k injective (which always exists by the previous discussion). Indeed, this is the correct definition, but there is work to be done to show that it is well defined. We fix once and for all a left exact functor F. We first collect some lemmas.

Lemma 1.7. *Let $i^\bullet : M^\bullet \to I^\bullet$ be a quasi-isomorphism of a bounded below complex $M^\bullet$ into a complex $I^\bullet$ of injective objects with each i^k injective. Then for another complex of injective objects*

$J^\bullet$ with a quasi-isomorphism $j^\bullet : M^\bullet \to J^\bullet$ (where each j^k need not be injective), there exists a chain map $\varphi^\bullet I^\bullet \to J^\bullet$, well defined up to homotopy, such that $\varphi^\bullet \circ j^\bullet$.

Proof. Let d_M, d_I, and d_J denote the differentials for $M^\bullet$, $I^\bullet$, and $J^\bullet$ respectively. Since $i^0 : M^0 \to I^0$ is an injective map, and J^0 is an injective object, the map $j^0 : M^0 \to J^0$ extends to a map $\varphi^0 : I^0 \to J^0$ such that $j^0 = \varphi^0 \circ i^0$. We then define $\varphi^1 : I^1 \to J^1$. To do this, we first claim that the kernel of $d_I + i^1 : I^0 \oplus M^1 \to I^1$ is equal to the image of $(i^0, -d_M) : M^0 \to I^0 \oplus M^1$. The inclusion $\operatorname{Im}(i^0, -d_M) \subset \ker(d_I + i^1)$ follows immediately from $i^\bullet$ being a chain map. For the opposite inclusion, let $(x, m) \in \ker(d_I + i^1)$, so $i^1(m) = -d_I x$. Therefore, $i^1(m)$ is 0 in $H^1(I^\bullet)$. Then since $i^\bullet$ is a quasi-isomorphism, the induced map $H^1(M^\bullet) \to H^1(I^\bullet)$ is injective, so we know that $m = d_M m_0$ for some $m_0 \in M^0$. Then since $H^0(M^\bullet) \to H^0(I^\bullet)$ is surjective, and there exist no terms in negative degree, we have that $x = i^0(n)$ for some $n \in M^0$. Therefore, what remains is to show that $n = m_0$, which again follows immediately from the fact that there is no term in negative degree. Then if we consider the map $(d_J \circ \varphi^0) + j^1 : I^0 \oplus M^1 \to J^1$, we have that $d_J \circ \varphi^0 \circ i^0 = d_J \circ j^0$, so we have that $(d_J \circ \varphi^0) + j^1$ vanishes on $\operatorname{Im}(i^0, -d_M)$ since $j^\bullet$ is a chain map. Then since $d_I + i^1$ also vanishes on $\operatorname{Im}(i^0, -d_M)$, we have that $(d_J \circ \varphi^0) + j^1$ factors through the inclusion $\operatorname{Im}(d_I + i^1) \hookrightarrow I^1$. Then since J^1 is injective, this extends to a map $\varphi^1 : I^1 \to J^1$. We construct the rest of the maps φ^k in the same fashion, and omit the proof of the uniqueness of $\varphi^\bullet$ up to homotopy out of laziness. ■

With this, we can prove that our tentative definition of $R^i F(M^\bullet)$ is reasonable.

Proposition 1.8. *The definition of $R^i F(M^\bullet)$ is independent (up to canonical isomorphism) from the choice of injective quasi-isomorphism $i^\bullet : M^\bullet \to I^\bullet$. In other words, given another injective quasi-isomorphism $j^\bullet : M^\bullet \to J^\bullet$, there exists a canonical isomorphism $H^i(F(I^\bullet)) \to H^i(F(J^\bullet))$.*

Proof. By the preceding lemma, we get chain maps $\varphi^\bullet : I^\bullet \to J^\bullet$ and $\psi^\bullet : J^\bullet \to I^\bullet$ such that $j^\bullet = \varphi^\bullet \circ i^\bullet$ and $i^\bullet = \psi^\bullet \circ i^\bullet$. We have that $\mathrm{id}_{I^\bullet} - (\psi^\bullet \circ \varphi^\bullet)$ is zero on $M^\bullet$, regarded as asubcomplex of $I^\bullet$ and similarly for $\mathrm{id}_{J^\bullet} - (\varphi^\bullet \circ \psi^\bullet)$, which show that φ and ψ induce isomorphisms on cohomology. In fact, it can be shown that the compositions are chain homotopic to the identity, so applying the functor F gives a chain homotopy from $F(I^\bullet) \to F(J^\bullet)$, giving us the desired isomorphism $H^i(F(I^\bullet)) \to H^i(F(J^\bullet))$. ■

This definition is perfectly fine, but we want to weaken the assumptions for the quasi-isomorphism $i^\bullet : M^\bullet \to I^\bullet$ used to define $R^i F(M^\bullet)$. The first thing is to remove the requirement that the $i^k : M^k \to I^k$ be injective. To prove this, we first introduce some new defintions.

Definition 1.9. Let $(A^\bullet, d_A)$ and $(B^\bullet, d_B)$ be complexes in $\mathcal{A}$, and let $\varphi^\bullet : A^\bullet \to B^\bullet$ be a chain map. The ***cone*** of $\varphi^\bullet$, is the complex $(C^\bullet, d_C)$ where $C^k = A^k \oplus B^{k-1}$ and the differntial $d_C^k : C^k \to C^{k+1}$ is given by

$$d_C^k := \begin{pmatrix} d_A^k & (-1)^k \varphi^k \\ 0 & d_B^{k-1} \end{pmatrix}$$

Definition 1.10. Given a chain complex $A^\bullet$, the ***shift*** of A, denoted $A[n]^\bullet$ is the chain complex such that $A[n]^k = A^{k-n}$.

Given a chain map $\varphi^\bullet : A^\bullet \to B^\bullet$ with cone $C^\bullet$, we get a split short exact sequence of chain complexes

$$0 \longrightarrow B[1]^\bullet \longrightarrow C^\bullet \longrightarrow A^\bullet \longrightarrow 0$$

where the map $B[1]^k \to C^k$ is given by the inclusion $B[1]^k := B^{k-1} \hookrightarrow C^k := A^k \oplus B^{k-1}$ and the map $C^k \to A^k$ is given by projection onto A^k.

Lemma 1.11. *Let $\varphi^\bullet : A^\bullet \to B^\bullet$ be a quasi-isomorphism, and let $C^\bullet$ be the cone of $\varphi^\bullet$. Then $H^i(C^\bullet) = 0$*

Proof. We use the long exact sequence induced by the short exact sequence

$$0 \longrightarrow B[1]^\bullet \longrightarrow C^\bullet \longrightarrow A^\bullet \longrightarrow 0$$

A diagram chase reveals that the connecting homomorphism $H^k(A^\bullet) \to H^{k+1}(B[1]^\bullet) = H^k(B^\bullet)$ is the map induced by $\varphi^\bullet$, which is an isomorphism since $\varphi^\bullet$ is a quasi-isomorphism. Therefore, by exactness we can conclude that the cohomology groups of $C^\bullet$ vanish. ■

We will also use (but won't prove) a standard fact about injective objects

Proposition 1.12. *Any exact complex $I^\bullet$ of injective objects is homotopy equivalent to the trivial complex, i.e. There exists a chain homotopy $H^k : I^k \to I^{k-1}$ such that $H^{k+1} \circ d^k + d^{k-1} + \circ H^k = \mathrm{id}_{I^k}$.*

Corollary 1.13. *For an exact bounded below complex $I^\bullet$ of injective objects, the complex $F(I^\bullet)$ is exact.*

Proof. Appyling the functor F to to the chain homotopy $H^k : I^k \to I^{k-1}$ gives the desired chain homotopy for $F(I^\bullet)$ to the trivial complex in $\mathcal{B}$, so $F(I^\bullet)$ is exact. ■

Proposition 1.14. *Let $\alpha^\bullet : M^\bullet \to I^\bullet$ be any quasi-isomorphism, where $I^\bullet$ is a complex of injective objects. Then $\alpha^\bullet$ induces an isomorphsim $R^iF(M^\bullet) \cong H^i(F(I^\bullet))$.*

Proof. Let $i^\bullet : M^\bullet \to J^\bullet$ be a quasi-isomorphism, with each i^k injective, so $R^iF(M^\bullet) = H^i(F(J^\bullet))$. Then by Lemma 1.7, we have that there exists a chain map $\varphi^\bullet : J^\bullet \to I^\bullet$ such that $\alpha^\bullet = \varphi^\bullet \circ i^\bullet$. Since $\alpha^\bullet$ and $i^\bullet$ are quasi-isomorphisms, $\varphi^\bullet$ is necessarily a quasi-isomorphism. Then consider the cone $C^\bullet$ of $\varphi^\bullet$. By Lemma 1.11, we have that $C^\bullet$ has trivial cohomology, and consists of injective objects, since $I^\bullet$ and $J^\bullet$ are complexes of injective objects. Applying F to the short exact sequence gives us

$$0 \longrightarrow F(I[1]^\bullet) \longrightarrow F(C^\bullet) \longrightarrow F(J^\bullet) \longrightarrow 0$$

which remains exact, since the original short exact sequence was split. Then taking the long exact sequence in cohomology, we use the Corollary 1.13 to conclude that $F(C^\bullet)$ has no cohomology, so the map $H^k(F(J^\bullet)) \to H^k(F(I^\bullet)) = R^iF(M^\bullet)$ induced by $F(\varphi^\bullet)$ is an isomorphism. ■

As a consequence, we get a functoriality property for the functors R^iF.

Corollary 1.15. *Let $\alpha^\bullet : M^\bullet \to N^\bullet$ be a quasi-isomorphism of bounded below complexes. Then for any choice of injective quasi-isomorphisms $i^\bullet : M^\bullet \to I^\bullet$ and $j^\bullet : N^\bullet \to J^\bullet$ into complexes $I^\bullet$ and $J^\bullet$ of injective objects, we get a canonical map*

$$R^iF(M^\bullet) = H^k(F(I^\bullet)) \to H^k(J^\bullet) = R^iF(N^\bullet)$$

Proof. Apply the previous proposition to the quasi-isomorphism $j^\bullet \circ \alpha^\bullet$. ■

We close with some functoriality properties, but omit the proofs out of laziness.

Proposition 1.16. *Let $\varphi^\bullet : A^\bullet \to B^\bullet$ be a chain map, and let $i^\bullet : A^\bullet \to I^\bullet$ and $j^\bullet : B^\bullet \to J^\bullet$ be quasi-isomorphisms to injective complexes with $i^\bullet$ injective. Then there exists a chain map $\psi^\bullet : I^\bullet \to J^\bullet$, unique up to homotopy, such that the following diagram commutes:*

$$\begin{tikzcd} A^\bullet \arrow[r, "\varphi^\bullet"] \arrow[d, "i^\bullet"'] & B^\bullet \arrow[d, "j^\bullet"] \\ I^\bullet \arrow[r, "\psi^\bullet"'] & J^\bullet \end{tikzcd}$$

Corollary 1.17. *Let $\varphi^\bullet : A^\bullet \to B^\bullet$ be any chain map. Then for injective quasi-isomorphisms $i^\bullet : A^\bullet \to I^\bullet$ and $j^\bullet : B^\bullet \to J^\bullet$ into complexes of injectives, we get a canonical map*

$$R^iF(A^\bullet) = H^i(F(I^\bullet)) \to H^i(F(J^\bullet)) = R^iF(B^\bullet)$$

Using this, we can show that we can replace the injective complexes with F-acyclic objects, i.e. complexes $N^\bullet$ where $F(N^i) = 0$.

Proposition 1.18. *Let $\varphi^\bullet : M^\bullet \to N^\bullet$ be a quasi-isomorphism from $M^\bullet$ to a complex of $N^\bullet$ of F-acyclic objects. Then $\varphi^\bullet$ induces an isomorphism $R^iF(M^\bullet) \to H^i(F(N^\bullet))$.*

In summary, we have extended the notion of R^iF to bounded below complexes – we can recover the original notion by considering an object A in $\mathcal{A}$ as a complex with a single term in degree 0, and zeroes elsewhere. In this case, the new definition coincides with the original, since we may regard an injective resolution $0 \to A \to I^\bullet$ of A as an injective quasi-isomorphism

$$\begin{tikzcd} A \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ I^1 \arrow[r] & I^2 \arrow[r] & I^3 \arrow[r] & \cdots \end{tikzcd}$$

2. The Holomorphic de Rham Theorem

We now apply our foray into derived functors to sheaves. Let $\mathcal{A}$ denote some abelian category of sheaves over a space X (e.g. $\mathrm{Sh}_{\mathsf{Ab}}(X)$, $\mathrm{Mod}_{\mathcal{O}_X}$, etc.). Let Γ denote the global sections functor for $\mathcal{A}$.

Definition 2.1. For a bounded below complex $\mathcal{F}^\bullet$ of sheaves, the ***hypercohomology*** of $\mathcal{F}^\bullet$, denoted $\mathbb{H}^k(X, \mathcal{F}^\bullet)$, is

$$\mathbb{H}^k(X, \mathcal{F}^\bullet) := R^k\Gamma(\mathcal{F}^\bullet)$$

An resolution $0 \to \mathcal{F} \to \mathcal{G}^\bullet$ is the same thing as a quasi-isomorphism from $\mathcal{F}$, regarded as a complex with the complex $\mathcal{G}^\bullet$, giving us that $H^k(X, \mathcal{F}) \cong \mathbb{H}^k(X, \mathcal{G}^\bullet)$.

We now restrict to the case where X is a complex manifold of complex dimension n. Let $\mathcal{O}_X$ denote the sheaf of holomorphic functions on X, and let $\Omega_X^{p,0}$ denote the sheaves of holomorphic p-forms on X. The de Rham differential $d : \mathcal{A}_X^k \to \mathcal{A}_X^{k+1}$ on the sheaves

$\mathcal{A}^k_X$ of smooth k-forms splits as $d = \partial + \overline{\partial}$, where $\partial : \mathcal{A}^{p,q}_X \to \mathcal{A}^{p+1}$ and $\overline{\partial} : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}_X$ that anticommute. Furthermore, the operator ∂ preserves holomorphic forms, i.e. it maps $\Omega^{p,0}_X \subset \mathcal{A}^{p,0}_X$ into $\Omega^{p+1,0}_X \subset \mathcal{A}^{p+1,0}$ since the holomorphic p-forms are exactly those annihilated by $\overline{\partial}$. This gives us the ***holomorphic de Rham complex***:

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\partial} \Omega^{1,0} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega^{n,0} \longrightarrow 0$$

There is an inclusion $i : \underline{\mathbb{C}} \hookrightarrow \mathcal{O}_X$ of the locally constant functions into the sheaf of holomorphic functions.

Proposition 2.2 (***The holomorphic Poincaré Lemma***)**.** *The map $i : \underline{\mathbb{C}} \to \Omega^{\bullet,0}$ is a resolution of the constant sheaf $\underline{\mathbb{C}}$, i.e. the complex*

$$0 \longrightarrow \underline{\mathbb{C}} \xrightarrow{i} \mathcal{O}_X \xrightarrow{\partial} \Omega^{1,0} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega^{n,0} \longrightarrow 0$$

is exact

Proof. Consider the double complex $(\mathcal{A}^{p,q}_X, \partial, (-1)^p\overline{\partial})$. We have inclusions $\Omega^{\bullet,0}_X \hookrightarrow \mathcal{A}^{\bullet,q}_X$, giving us

$$\begin{tikzcd}
\mathcal{A}^{0,n} \arrow[r, "\partial"] & \mathcal{A}^{1,n} \arrow[r, "\partial"] & \cdots \arrow[r, "\partial"] & \mathcal{A}^{n,n} \\
\vdots \arrow[u, "\overline{\partial}"] & \vdots \arrow[u, "-\overline{\partial}"] & \vdots \arrow[u, "(-1)^p\overline{\partial}"] & \vdots \arrow[u, "(-1)^n\overline{\partial}"] \\
\mathcal{A}^{0,1} \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"] & \mathcal{A}^{1,1} \arrow[u, "-\overline{\partial}"] \arrow[r, "\partial"] & \cdots \arrow[u, "(-1)^p\overline{\partial}"] \arrow[r, "\partial"] & \mathcal{A}^{n,1} \arrow[u, "(-1)^n\overline{\partial}"] \\
C^\infty_X(\mathbb{C}) \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"] & \mathcal{A}^{1,0} \arrow[u, "-\overline{\partial}"] \arrow[r, "\partial"] & \cdots \arrow[u, "(-1)^p\overline{\partial}"] \arrow[r, "\partial"] & \mathcal{A}^{n,0} \arrow[u, "(-1)^n\overline{\partial}"] \\
\mathcal{O}_X \arrow[u, hook] \arrow[r, "\partial"] & \Omega^{1,0}_X \arrow[u, hook] \arrow[r, "\partial"] & \cdots \arrow[u, hook] \arrow[r, "\partial"] & \Omega^{n,0} \arrow[u, hook]
\end{tikzcd}$$

The columns are exact by the $\overline{\partial}$-Poincaré lemma, so by Proposition 1.5, the chain map from the holomorphic de Rham complex to the total complex of $(\mathcal{A}^{p,q}, \partial, (-1)^p\overline{\partial})$ is a quasi-isomorphism. We then note that the total complex is nothing but the smooth de Rham complex $(\mathcal{A}^\bullet, d)$, which is exact by the smooth Poincaré lemma, and that the map to the total complex induced by the inclusions $\Omega^{p,0} \hookrightarrow \mathcal{A}^{p,0}$ is just the inclusion $\Omega^{p,0} \hookrightarrow \mathcal{A}^k$. Furthermore, the 0^{th} cohomology sheaf of the smooth de Rham complex is the constant sheaf $\underline{\mathbb{C}}$. Therefore, we have that the holomorphic de Rham complex is a resolution of $\underline{\mathbb{C}}$. ∎

Corollary 2.3 (***The Holomorphic de Rham Theorem***)**.** *There is an isomorphism*

$$H^k(X, \mathbb{C}) \cong \mathbb{H}^k(X, \Omega^{\bullet,0}_X)$$

References

[1] C. Voisin. *Hodge Theory and Complex Algebraic Geometry, I*. 2002.