

SHEAF COHOMOLOGY, LINE BUNDLES, AND DIVISORS

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1. ČECH COHOMOLOGY

For the most part, we will consider “sufficiently nice” topological spaces. For the most part, think of a space X as a (smooth, complex) manifold, an analytic space, or a quasi-compact separated scheme if you’re feeling adventurous.

Definition 1.1. Let X be a space and \mathcal{F} a sheaf of abelian groups over X . Let $\mathcal{U} = \{U_i\}_{i \in \mathbb{N}}$ be a countable open cover of X that is locally finite, i.e. for any $x \in X$, only finitely many U_i contain x . For $I = \{i_1, \dots, i_k\}$, let

$$U_I := \bigcap_{i \in I} U_i$$

Then define the *Čech cochain groups* of \mathcal{F} for the cover \mathcal{U} by

$$C^k(\mathcal{U}, \mathcal{F}) := \prod_{|I|=k+1} \mathcal{F}(U_I)$$

an element of $C^k(\mathcal{U}, \mathcal{F})$ is called a *Čech cochain*. For a k -cochain σ , and $I = \{i_0, \dots, i_k\}$, we denote the component of σ over U_I as σ_I or σ_{i_0, \dots, i_k} .

The Čech cochain groups are equipped with a differential $d : C^k(\mathcal{U}, \mathcal{F}) \rightarrow C^{k+1}(\mathcal{U}, \mathcal{F})$ where for $\sigma \in C^k(\mathcal{U}, \mathcal{F})$, the i_0, \dots, i_{k+1} component of $d\sigma$ is given by

$$(d\sigma)_{i_0, \dots, i_{k+1}} = \sum_{j=1}^{k+1} (-1)^j \sigma_{i_0, \dots, \widehat{i_j}, \dots, i_{k+1}}|_{U_0 \cap \dots \cap U_{k+1}}$$

where $\widehat{i_j}$ denotes that i_j is missing. We have that $d^2 = 0$ for a similar reason that $d^2 = 0$ for singular cohomology, you get repeats of terms with opposite signs due to the omitted

index. We denote the kernel of $d : C^i(\mathcal{U}, \mathcal{F}) \rightarrow C^{i+1}(\mathcal{U}, \mathcal{F})$ as $Z^i(\mathcal{U}, \mathcal{F})$, and we say that the elements are i -cocycles. We denote the image of $d : C^{i-1}(\mathcal{U}, \mathcal{F}) \rightarrow C^i(\mathcal{U}, \mathcal{F})$ as $B^i(\mathcal{U}, \mathcal{F})$, and we call the elements i -coboundaries.

Definition 1.2. The *Čech cohomology groups* of \mathcal{F} with respect to the cover \mathcal{U} , denoted $\check{H}^i(\mathcal{U}, \mathcal{F})$ is the cohomology of the Čech complex

$$0 \xrightarrow{d} C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{d} \dots$$

i.e. we have

$$\check{H}^i(\mathcal{U}, \mathcal{F}) := \frac{Z^i(\mathcal{U}, \mathcal{F})}{B^i(\mathcal{U}, \mathcal{F})}$$

Definition 1.3. Given an open cover $\mathcal{U} = \{U_i\}$, a *refinement* of \mathcal{U} is another open cover $\mathcal{V} = \{V_j\}$ such that every V_j is contained in some U_i . If \mathcal{V} is a refinement of \mathcal{U} , we write $\mathcal{V} < \mathcal{U}$.

If $\mathcal{V} < \mathcal{U}$, then we know we can find a map $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ such that $V_i \subset U_{\varphi(i)}$. Consequently, we can restrict sections over U_i to sections over $V_{\varphi(i)}$, so this induces a chain map $\rho_\varphi : C^k(\mathcal{U}, \mathcal{F}) \rightarrow C^k(\mathcal{V}, \mathcal{F})$ where

$$(\rho_\varphi(\sigma))_{i_0, \dots, i_k} = \sigma_{\varphi(i_0), \dots, \varphi(i_k)}|_{U_{i_0} \cap \dots \cap U_{i_k}}$$

This map commutes with the differentials for $C^\bullet(\mathcal{U}, \mathcal{F})$ and $C^\bullet(\mathcal{V}, \mathcal{F})$, so it descends to homomorphisms $\check{H}^i(\mathcal{U}, \mathcal{F}) \rightarrow \check{H}^i(\mathcal{V}, \mathcal{F})$. It can be shown that a different choice of chain map ρ_ψ for $\psi : \mathbb{N} \rightarrow \mathbb{N}$ is chain homotopic to ρ_φ , so the induced maps on cohomology are independent of our choice of φ .

Definition 1.4. The *Čech cohomology groups* of a sheaf \mathcal{F} over X is the limit over refinements

$$\check{H}^i(X, \mathcal{F}) := \lim_{\mathcal{V} < \mathcal{U}} \check{H}^i(\mathcal{V}, \mathcal{F})$$

i.e. the quotient disjoint union $\coprod_{\mathcal{U}} \check{H}^i(\mathcal{U}, \mathcal{F})$ over all refinements, where we identify $\sigma \in \check{H}^i(\mathcal{U}, \mathcal{F})$ and $\tau \in \check{H}^i(\mathcal{V}, \mathcal{F})$ if \mathcal{V} refines \mathcal{U} and $\rho(\sigma) = \tau$ under the map induced on cohomology by the refinement.

This definition of the Čech cohomology groups is essentially useless for computation. Its true power comes from the following theorem, due to Leray.

Theorem 1.5 (Leray). Suppose $\mathcal{U} = \{U_i\}$ is an open cover of X that is *acyclic* with respect to the sheaf \mathcal{F} , i.e. for $|I| > 1$ and any i ,

$$\check{H}^i(U_I, \mathcal{F}) = 0$$

Then

$$\check{H}^i(\mathcal{U}, \mathcal{F}) = \check{H}^i(X, \mathcal{F})$$

The intuition to keep in mind for an acyclic cover is the notion of a good cover in differential geometry. On a smooth manifold M , there exists a covering of M by open sets $\{U_i\}$ such that any nonempty intersections are contractible, which is done by taking geodesic balls around each point in M . Since the homotopical information on each U_i is trivial, the only nontrivial topological information in the cohomology of M comes from how the sets

are glued together to form M . As with exact sequences of chain complexes, short exact sequences of sheaves give long exact sequences in sheaf cohomology.

Theorem 1.6. *Let*

$$0 \longrightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \longrightarrow 0$$

be an exact sequence of sheaves over X . Then this induces a long exact sequence in cohomology:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \check{H}^0(X, \mathcal{E}) & \xrightarrow{\alpha^*} & \check{H}^0(X, \mathcal{F}) & \xrightarrow{\beta^*} & \check{H}^0(X, \mathcal{G}) \\ & & & & \searrow \delta & & \\ & & \check{H}^1(X, \mathcal{E}) & \xleftarrow{\alpha^*} & \check{H}^1(X, \mathcal{F}) & \xrightarrow{\beta^*} & \check{H}^1(X, \mathcal{G}) \\ & & & & \searrow \delta & & \\ & & \leftarrow & \dots & & \dots & \end{array}$$

Proof. We first define the maps $\alpha^* : \check{H}^i(X, \mathcal{E}) \rightarrow \check{H}^i(X, \mathcal{F})$ and $\beta^* : \check{H}^i(X, \mathcal{F}) \rightarrow \check{H}^i(X, \mathcal{G})$, and will then define the connecting homomorphism $\delta : \check{H}^i(X, \mathcal{E}) \rightarrow \check{H}^{i+1}(X, \mathcal{G})$. Given an open cover $\mathcal{U} = \{U_i\}$ of X , the sheaf morphism α gives for each open set U_i a homomorphism $\alpha(U_i) : \mathcal{E}(U_i) \rightarrow \mathcal{F}(U_i)$, which induces a chain map $C^\bullet(\mathcal{U}, \mathcal{F}) \rightarrow C^\bullet(\mathcal{U}, \mathcal{G})$. Since the maps $\alpha(U_i)$ commute with restriction maps, this chain map commutes with the differentials, so it descends to a map on cohomology $\check{H}^\bullet(\mathcal{U}, \mathcal{F}) \rightarrow \check{H}^\bullet(\mathcal{U}, \mathcal{G})$, which, after taking the limit over refinements or choosing \mathcal{U} to be simultaneously acyclic for \mathcal{E} and \mathcal{F} , gives us the induced map $\alpha^* : \check{H}^\bullet(X, \mathcal{F}) \rightarrow \check{H}^\bullet(X, \mathcal{G})$. The map β^* is defined similarly.

The construction of the connecting homomorphism mirrors the construction for singular (or de Rham) cohomology. We represent an element of $\check{H}^i(X, \mathcal{G})$ with a cocycle $\sigma \in Z^i(\mathcal{U}, \mathcal{G})$ with respect to some open cover \mathcal{U} . By surjectivity of β , by potentially passing to a refinement, we can write $\sigma = \beta(\tau)$ for some $\tau \in C^i(\mathcal{U}, \mathcal{F})$. Then since the induced map on chains commutes with the differentials, we have that $d\beta(\tau) = \beta(d\tau) = 0$, since τ is a cocycle. Therefore, $d\tau$ is in the kernel of β^* , so we can write $d\tau = \alpha(\eta)$ for some $\eta \in C^{i+1}(\mathcal{U}, \mathcal{F})$. We then define $\delta(\sigma)$ to be the class of η in the limit. We note that this is independent of our choice of τ , since any other choice of preimage of σ differs by an element of the form $\alpha(e)$ for some cocycle e by exactness. Then since α commutes with the differentials, we get $d(\tau + \alpha(e)) = d\tau + d\alpha(e) = d\tau + \alpha(de) = d\tau$. ■

We make one observation about Čech cohomology

Proposition 1.7. *The 0^{th} Čech cohomology group is isomorphic to the space of global sections, i.e.*

$$\check{H}^0(\mathcal{U}, \mathcal{F}) \cong \Gamma(X, \mathcal{F})$$

Proof. The 0^{th} cohomology is just the kernel of the map $d : C^0(\mathcal{U}, \mathcal{F}) \rightarrow C^1(\mathcal{U}, \mathcal{F})$. For a 0-cochain σ , we have that

$$(d\sigma)_{ij} = \sigma_j|_{U_i \cap U_j} - \sigma_i|_{U_i \cap U_j}$$

We then claim that the map $\Gamma(X, \mathcal{F}) \rightarrow \ker d$ sending a section σ to the cocycle $\tilde{\sigma}$ defined by

$$\tilde{\sigma}_i = \sigma|_{U_i}$$

is bijective. It is surjective, since any 0-cocycle contained in the kernel is a collection of local sections that agrees on intersections, which is exactly a global section of \mathcal{F} . In addition, it is injective, since if a section restricts to 0 on every open set, it is the zero section. ■

2. ČECH COHOMOLOGY ON \mathbb{CP}^n

We now compute Čech cohomology for various sheaves over \mathbb{CP}^n . The main objects of interest are the line bundles $\mathcal{O}(k)$. We have that \mathbb{CP}^n admits a nice cover $\mathcal{U} = \{U_i\}$ where U_i is the open set where the coordinate z_i does not vanish. We will soon show that this covering is acyclic for the structure sheaf $\mathcal{O}_{\mathbb{CP}^n}$, and since all line bundles are trivial on the open sets in this cover, this means that the covering will be acyclic for any line bundle $\mathcal{O}(k)$. So it suffices to compute Čech cohomology with respect to this cover. In particular, we note that for any line bundle $\mathcal{O}(k) \rightarrow \mathbb{CP}^n$, the transition functions ψ_{ij} for $\mathcal{O}(k)$ are $\psi_{ij}(\ell) = (z_j/z_i)^k$. We already know one cohomology group:

Theorem 2.1. *The 0th cohomology group $\check{H}^0(\mathbb{CP}^n, \mathcal{O}(k))$ is isomorphic to the space $\mathbb{C}[x_0, \dots, x_n]_k$ of homogeneous degree k polynomials in the variables x_0, \dots, x_n .*

We compute the rest of the cohomology now, which we do in stages.

Theorem 2.2. *For any $i > n$, we have $\check{H}^i(\mathcal{U}, \mathcal{O}(k)) = 0$.*

Proof. The cover of \mathbb{CP}^i by the U_i has cardinality $n + 1$. Therefore, $C^{n+1}(\mathcal{U}, \mathcal{O}(k)) = 0$. ■

To compute the rest of the cohomology groups, we first prove a lemma characterizing local sections of $\mathcal{O}(k)$.

Lemma 2.3. *Let $\pi : \mathbb{C}^{n-1} - \{0\} \rightarrow \mathbb{CP}^n$ be the usual projection sending $z \in \mathbb{C}^{n-1}$ to $\text{span}\{z\}$. Then the space of sections $\mathcal{O}(k)(U)$ is isomorphic to the space of homogeneous of degree k holomorphic functions $f : \pi^{-1}(U) \rightarrow \mathbb{C}$ i.e.*

$$f(tz_0, \dots, tz_n) = t^k f(z_0, \dots, z_n)$$

Proof. Let $\sigma \in \mathcal{O}(k)(U)$ be a section. Then the set $\{U \cap U_i\}$ is an open cover of U , so σ is determined by its restrictions $\sigma_i := \sigma|_{U \cap U_i}$. Since the bundle $\mathcal{O}(d)$ is trivial over the U_i , the local sections σ_i can be identified with holomorphic functions $U \cap U_i \rightarrow \mathbb{C}$ with the compatibility condition

$$\sigma_i([z_0 : \dots : z_n]) = \left(\frac{z_j}{z_i}\right)^k \sigma_j([z_0 : \dots : z_n])$$

We then give maps in both directions. Given a section $\sigma \in \mathcal{O}(k)(U)$, define the function f_σ by

$$f_\sigma(z_0, \dots, z_n) = z_i^k \sigma_i(\pi(z_0, \dots, z_n))$$

We must verify that this is well-defined, i.e. it is independent of our choice of i . We compute

$$\begin{aligned} f_\sigma(z_0, \dots, z_n) &= z_i^k \sigma_i(\pi(z_0, \dots, z_n)) \\ &= \left(\frac{z_j}{z_i}\right)^k z_i^k \sigma_j(\pi(z_0, \dots, z_n)) \\ &= z_j^k \sigma_j(\pi(z_0, \dots, z_n)) \end{aligned}$$

So this determines a well defined function on $\pi^{-1}(U)$. In addition, it is visibly homogeneous of degree k , since the σ_i are constant on lines and z_i^k is homogeneous of degree k . To show this is an isomorphism, we provide an inverse. Given a homogeneous function f of degree k on $\pi^{-1}(U)$, define the section σ_f locally by

$$(\sigma_f)_i([z_0 : \dots : z_n]) = \frac{f(z_0, \dots, z_n)}{z_i^k}$$

then to show that this defines a section, we must show that they agree on intersections using the transition functions. We compute

$$\left(\frac{z_i}{z_j}\right)^k (\sigma_f)_i|_{U \cap U_i \cap U_j}([z_0 : \dots : z_n]) = \left(\frac{z_i}{z_j}\right)^k \frac{f(z_0, \dots, z_n)}{z_i^k} = \frac{f(z_0, \dots, z_n)}{z_j^k} = (\sigma_f)_j$$

The two mappings provided are visibly inverses, since one is essentially multiplication by z_j^k and the other is essentially division by z_j^k . ■

Over intersections of the distinguished open sets U_i , the sections have a particularly nice form. Under the projection $\pi : \mathbb{C}^{n+1} - \{0\} \rightarrow \mathbb{CP}^n$, the preimage of U_I for $I = \{i_0, \dots, i_d\}$ is just \mathbb{C}^{n+1} minus the coordinate axes $z_{i_j} = 0$. By taking power series, a holomorphic function on $\pi^{-1}(U_I)$ is given by Laurent series where the z_{i_j} can appear in negative degree. Being homogeneous of degree d implies that all the terms in the series expansion must be homogeneous of degree k , where the degree of $(z_k)^a / (z_{i_j})^b$ is $a - b$. Consequently, all such holomorphic functions must be polynomials in $\mathbb{C}[z_0, \dots, z_n, z_{i_0}^{-1}, \dots, z_{i_d}^{-1}]$ of degree k .

3. ČECH COHOMOLOGY AND LINE BUNDLES

Let $\mathcal{U} = \{U_i\}$ be a covering of X , and \mathcal{F} a sheaf of abelian groups over X . Then with respect to this cover, a Čech 2-cocycle $\sigma \in Z^2(\mathcal{U}, \mathcal{F})$ is defined by the equation

$$0 = (d\sigma)_{ijk} = \sigma_{jk}|_{U_i \cap U_j \cap U_k} - \sigma_{ik}|_{U_i \cap U_j \cap U_k} + \sigma_{ij}|_{U_i \cap U_j \cap U_k}$$

Written multiplicatively (and omitting the restriction), this becomes

$$1 = \sigma_{jk} \sigma_{ik}^{-1} \sigma_{ij}$$

which, since the group is abelian, is equivalent to

$$\sigma_{ik} = \sigma_{ij} \sigma_{jk}$$

which looks exactly like a cocycle condition for transition functions of a line bundle. Recall that given a holomorphic line bundle $\pi : L \rightarrow X$, we have local trivializations – we can find a cover $\mathcal{U} = \{U_i\}$ with maps $\varphi_i : \pi^{-1}(U_i) \rightarrow U_i \times \mathbb{C}$ such that

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\varphi_i} & U_i \times \mathbb{C} \\ & \searrow & \swarrow \\ & U_i & \end{array}$$

commutes, where the maps to U_i are the projections. Therefore, if we consider the map $\varphi_i \circ \varphi_k^{-1} : U_i \cap U_k \times \mathbb{C} \rightarrow U_i \cap U_k \times \mathbb{C}$, we have that $\varphi(x, \lambda) = (x, \psi_{ij}(x)(\lambda))$, where the functions $\psi_{ij} : U_i \cap U_j \rightarrow \text{GL}_1 \mathbb{C}$ are holomorphic. The ψ_{ij} are called the **transition functions** of the line bundle L .

Proposition 3.1. *The transition functions ψ_{ij} satisfy the following conditions*

- (1) $\psi_{ij}\psi_{ji} = 1$ (i.e. the constant function $x \mapsto 1$)
- (2) $\psi_{ij}\psi_{jk} = \psi_{ik}$.

The second condition is often called the **cocycle condition**, in reference to the identity we derived above for the defining property of a Čech cocycle.

Proof. Consider the map $\varphi_i \circ \varphi_j^{-1} \circ \varphi_j \circ \varphi_i^{-1} = \text{id}$. We compute under the action on a general element (x, λ)

$$(x, \lambda) \xrightarrow{\varphi_j \circ \varphi_i^{-1}} (x, \psi_{ji}(x)(\lambda)) \xrightarrow{\varphi_i \circ \varphi_j^{-1}} (x, \psi_{ij}(x)\psi_{ji}(x)(\lambda))$$

Therefore, we have that $\psi_{ij}(x)\psi_{ji}(x) = 1$ for all x , showing the first property. For the second property, we do the same thing. Consider the function $\varphi_i \circ \varphi_j^{-1} \circ \varphi_j \circ \varphi_k = \varphi_i \circ \varphi_k^{-1}$. Then for (x, λ) , we compute the action of this function to be

$$(x, \lambda) \xrightarrow{\varphi_j \circ \varphi_k^{-1}} (x, \psi_{jk}(x)(\lambda)) \xrightarrow{\varphi_i \circ \varphi_j^{-1}} (x, \psi_{ij}(x)\psi_{jk}(x)(\lambda))$$

So we get that $\psi_{ij}(x)\psi_{jk}(x) = \psi_{ik}(x)$ for all x . ■

Under the Čech differential, the image of a Čech 0-cochain σ is given by

$$(d\sigma)_{ij} = \sigma_j - \sigma_i$$

written multiplicatively, this becomes

$$(d\sigma)_{ij} = \sigma_j \sigma_i^{-1}$$

In the same spirit, we translate this to a statement regarding transition functions of a line bundle.

Proposition 3.2. *Let $\pi L \rightarrow X$ be a holomorphic line bundle where the transition functions ψ_{ij} with respect to a cover $\{U_i\}$ satisfy the **coboundary condition**, i.e. there exist holomorphic functions $\sigma_i : U_i \rightarrow \text{GL}_1 \mathbb{C}$ such that*

$$\psi_{ij} = \sigma_j \sigma_i^{-1}$$

Then L is a trivial line bundle.

Proof. It suffices to provide a nonvanishing section $X \rightarrow L$. A section $s : X \rightarrow L$ is equivalent to functions $s_i : U_i \rightarrow \mathbb{C}$ with the compatibility condition

$$s_i = \psi_{ij}s_j$$

define the s_i by $s_i = \sigma_i^{-1}$. Then they satisfy the compatibility condition, since

$$\psi_{ij}\sigma_j = \sigma_j\sigma_i^{-1}\sigma_j^{-1} = \sigma_i^{-1} = s_i$$

then since the σ_i are functions to $\mathrm{GL}_1\mathbb{C} = \mathbb{C}^\times$, they glue to a global nonvanishing section, so L is isomorphic to the trivial line bundle $X \times \mathbb{C}$. ■

Recall that isomorphism classes of line bundles over X form a group under tensor product, where the inverse of a line bundle L is the dual bundle L^* . Given line bundles $L, L' \rightarrow X$ and an open cover $\mathcal{U} = \{U_i\}$ of X in which both L and L' are trivialized over the U_i (for instance, a good cover of X), let ψ_{ij} be the transition functions for L and let φ_{ij} be the transition functions for L' . Then the transition functions for $L \otimes L'$ are $\psi_{ij}\varphi_{ij}$, and the transition functions for L^* are given by φ_{ij}^{-1} .

Theorem 3.3. *Let X be a complex manifold, and \mathcal{O}_X its sheaf of holomorphic functions. Then let \mathcal{O}_X^\times be the sheaf of invertible functions, which is a sheaf of abelian groups under multiplication. Then we have a group isomorphism*

$$\check{H}^1(X, \mathcal{O}_X^\times) \cong \mathrm{Pic}(X)$$

Proof. Fix a good cover $\mathcal{U} = \{U_i\}$ for X . Since all the sets and their nonempty intersections are contractible, we have that $\check{H}^i(U_i, \mathcal{F}) = 0$ for all $i > 0$ where \mathcal{F} is the sheaf of sections of any line bundle. Since all the U_i are contractible, we also have that any line bundle over U_i is trivial, so it admits transition functions ψ_{ij} with respect to this cover. As shown above, the functions ψ_{ij} exactly define a Čech 1-cocycle, and any Čech coboundary defines a trivial bundle. In addition, we have that the transition functions of a tensor product are exactly the products of the transition functions. Putting everything together, this tells us that the mapping $L \mapsto \{\psi_{ij}\}$ sending a line bundle to the cocycles determined by its transition functions is a bijective group homomorphism. ■

4. DIVISORS

Given a complex manifold X , a complex hypersurface $Y \subset X$ (also called an analytic hypersurface) is locally cut out a single holomorphic function, i.e. there is an open cover $\{U_i\}$ of X such that $Y \cap U_i$ is the vanishing locus of a single holomorphic function g_i . We say that the functions g_i are **defining functions** for Y . The hypersurface Y is said to be **irreducible** if at any $p \in Y$, the defining function g for Y in a neighborhood about p is irreducible in the local ring $\mathcal{O}_{X,p}$. If the defining function for a hypersurfaces is not irreducible, then it can be written as a product of irreducible functions g_i in $\mathcal{O}_{X,p}$, in which case we can write Y as the union of the irreducible hypersurfaces determined by the g_i .

On the complex line \mathbb{C} , the irreducible hypersurfaces are just points $p \in \mathbb{C}$, which have defining functions $f(z) = (z - p)$. Recall that all meromorphic functions on \mathbb{C} are of the form $p(z)/q(z)$ for polynomials $p(z), q(z) \in \mathbb{C}[z]$. A great deal of the study of meromorphic functions on \mathbb{C} revolves around studying their zeroes and poles, which is something that divisors will encapsulate for higher dimensional complex manifolds.

Definition 4.1. Let $Y \subset X$ be an irreducible hypersurface. Let $f \in \mathcal{K}(X)$ be a meromorphic function, and fix a point $y \in Y$. Then Y is locally cut out by an irreducible holomorphic function $g \in \mathcal{O}_{X,y}$. The **order** of f at y , denoted $\text{ord}_{Y,y}(f)$ is the smallest integer n such that there exists an invertible holomorphic function $h \in \mathcal{O}_{X,y}$ such that $f = g^n h$.

We note that this definition is independent of our choice of defining function, since any two irreducible defining functions for Y differ by a unit in $\mathcal{O}_{X,y}$. In addition, if Y is irreducible, it is also independent of our choice of point $y \in Y$, since if a holomorphic function g is irreducible in $\mathcal{O}_{X,y}$, then it is also irreducible in $\mathcal{O}_{X,y'}$ for any point y' sufficiently close. Therefore, we can simply write $\text{ord}_Y(f) := \text{ord}_{Y,y}(f)$ for any point $y \in Y$, provided that Y is an irreducible hypersurface. The intuition behind the definition is that the order of f at y should be the degree to which the function vanishes at the point y , or the degree of the pole at y .

Example 4.2. Let $X = \mathbb{C}$, and $Y = \{p\}$, so Y has locally (in fact globally) defining function $g(z) = (z - p)$. Then let f be the meromorphic function

$$f(z) = \frac{(z - p)^k}{q(z)}$$

where $q(z)$ is a polynomial that does not vanish at p . Then the order of f at p is, as you would expect, k . The role of h is played by the locally invertible function $1/q(z)$, which is a unit in the local ring $\mathcal{O}_{\mathbb{C},p}$. If instead we have

$$f(z) = \frac{q(z)}{(z - p)^k}$$

for a polynomial $q(z)$ that does not vanish at p , then the order of f at y is $-k$, where the role of h is played by $q(z)$.

A simple, but important observation is that order is additive on products.

Proposition 4.3. Let $Y \subset X$ be an irreducible hypersurface with locally defining function g at $y \in Y$. Let $f_1, f_2 \in \mathcal{K}(X)$ be meromorphic functions. Then

$$\text{ord}_Y(f_1 f_2) = \text{ord}_Y(f_1) + \text{ord}_Y(f_2)$$

Some more key observations are that for a nonvanishing holomorphic function f , we have that $\text{ord}_Y(f) = 0$ for any irreducible hypersurface $Y \subset X$, and that $\text{ord}_Y(f) = 0 \text{ord}_Y(1/f)$.

Proof. In the local ring $\mathcal{O}_{X,y}$, we have that

$$f_1 = g^{\text{ord}_Y(f_1)} h_1 \quad f_2 = g^{\text{ord}_Y(f_2)} h_2$$

Therefore, we have that

$$f_1 f_2 = g^{\text{ord}_Y(f_1) + \text{ord}_Y(f_2)} h_1 h_2$$

The function $h_1 h_2$ does not vanish at y , and we have that $\text{ord}_Y(f_1) + \text{ord}_Y(f_2)$ must be the smallest integer for $f_1 f_2$, since $\text{ord}_Y(f_1)$ and $\text{ord}_Y(f_2)$ are minimal for f_1 and f_2 respectively. ■

Definition 4.4. A *divisor* D on X is a formal integral combination

$$D = \sum a_i [Y_i]$$

where Y_i is an irreducible hypersurface for all i . The group of divisors on X is denoted $\text{Div}(X)$.

The intuition for a divisor is that it should be interpreted as a prescription for zeroes and poles of a meromorphic function. For example, in \mathbb{C} , you should think of the divisor

$$D = p_1 + 4p_2 - 4p_3 - 6p_4$$

as corresponding to the meromorphic function

$$f(z) = \frac{(z - p_1)(z - p_2)^4}{(z - p_3)^4(z - p_4)^6}$$

Of course, this isn't exactly true or precise. For example, we could multiply the numerator or denominator by nonvanishing holomorphic functions and then obtain a function with the same poles and zeroes as f .

Definition 4.5. Let $f \in \mathcal{K}(X)$ be a meromorphic function. Then the *zero divisor* of f is the divisor

$$(f)_0 = \sum_{\text{ord}_Y(f) > 0} \text{ord}_Y(f) [Y]$$

where the sum is taken over all irreducible hypersurfaces Y with $\text{ord}_Y(f) > 0$. Likewise, the *pole divisor* of f is defined to be the divisor

$$(f)_\infty = \sum_{\text{ord}_Y(f) < 0} \text{ord}_Y(f) [Y]$$

The *associated divisor* of f is the divisor

$$(f) = (f)_0 - (f)_\infty = \sum_Y \text{ord}_Y(f) [Y]$$

This makes precise what we mean by divisors being prescriptions for poles and zeroes, but the correspondence between meromorphic functions and divisors is far from bijective, as we noted above. Our observation was that if any two meromorphic functions differed by a nonvanishing holomorphic function, then they will define the same divisor. This turns out to be the only redundancy.

Theorem 4.6. Let X be a complex manifold. Then let \mathcal{K}_X^\times denote the sheaf of nonzero meromorphic functions, and $\mathcal{O}_X^\times \subset \mathcal{K}_X^\times$ the sheaf of nonvanishing holomorphic functions, which are both sheaves of abelian groups under multiplication. Then we have an isomorphism

$$\Gamma(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times) \rightarrow \text{Div}(X)$$

Proof. A global section σ of the quotient sheaf $\mathcal{K}_X^\times / \mathcal{O}_X^\times$ is a collection of pairs (U_i, f_i) of meromorphic functions $f_i \in \mathcal{K}_X^\times(U_i)$ such that on the intersections $U_i \cap U_j$, their quotient f_i/f_j is an element of \mathcal{O}_X^\times , i.e. a nonvanishing holomorphic function over $U_i \cap U_j$. Therefore, over the intersection, we have that

$$\text{ord}_Y(f_i) - \text{ord}_Y(f_j) = \text{ord}_Y(f_i/f_j) = 0$$

For any irreducible hypersurface Y . Therefore, we have that $\text{ord}_Y(f_i) = \text{ord}_Y(f_j)$. Therefore, the order $\text{ord}_Y(\sigma)$ of σ is well-defined for any irreducible hypersurface $Y \subset X$, so we get a well-defined map

$$\begin{aligned} \Gamma(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times) &\rightarrow \text{Div}(X) \\ \sigma &\mapsto (\sigma) := \sum_Y \text{ord}_Y(\sigma)[Y] \end{aligned}$$

To show that this mapping is bijective, we provide an inverse map. Let $D = \sum_i a_i[Y_i] \in \text{Div}(X)$. Then we can find an open cover $\{U_i\}$ of X where in each open set U_j , any irreducible hypersurface Y_i intersecting U_j nontrivially is the vanishing locus of a holomorphic function g_{ij} , which is unique up to multiplication by an element in $\mathcal{O}_X^\times(U_j)$. Then we claim that the functions $f_j := \prod_i g_{ij}$ glue to a global section of $\mathcal{K}_X^\times / \mathcal{O}_X^\times$, i.e. on any $U_j \cap U_k$, we have that f_j / f_k is a nonvanishing holomorphic function. However, on any intersection $U_j \cap U_k$, the functions g_{ij} and g_{ik} define the same irreducible hypersurface Y_i , so they differ by multiplication by a nonvanishing holomorphic function. Therefore, the f_j glue to a global section.

It is easily verified that these two constructions are inverses, so the maps are bijective. Furthermore, the map is a group homomorphism since order is additive on products of meromorphic functions, so it is a group isomorphism. ■

Remark. For complex manifolds, we have that global sections of the quotient sheaf and divisors are isomorphic, but in the algebro-geometric setting, this is not true without some smoothness assumptions. In algebraic geometry, elements of $\Gamma(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times)$ are called **Weil divisors**, while the elements of what we called $\text{Div}(X)$ are called **Cartier divisors**.

One of the big punchlines for divisors is their correspondence with holomorphic line bundles over X , which we can make explicit using the isomorphism with $\Gamma(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times)$. In the proof we gave above, we noted that a global section of $\mathcal{K}_X^\times / \mathcal{O}_X^\times$ can be seen as a collection of meromorphic functions f_i on an open cover $\{U_i\}$, such that on intersections, the function f_i / f_j is a nonvanishing holomorphic function. In addition, it is easily seen that the functions satisfy the cocycle condition, so they can be used to define transition functions for a line bundle, which we call $\mathcal{O}(D)$. The mapping $D \mapsto \mathcal{O}(D)$ is clearly a group homomorphism, so it defines a natural map $\text{Div}(X) \rightarrow \text{Pic}(X)$. In particular, this shows that line bundles are closely related to the codimension 1 geometry of the complex manifold X .

5. SHEAF COHOMOLOGY

Over a complex manifold X , we have many different cohomology theories at our disposal:

- (1) The singular cohomology groups $H_{\text{sing}}^i(X, \mathbb{Z})$.
- (2) The de Rham cohomology groups $H_{\text{dR}}^i(X)$.
- (3) The Dolbeault cohomology groups of a holomorphic vector bundle $H_{\bar{\partial}}^i(X, E)$.
- (4) The Čech cohomology groups of a sheaf $\check{H}^i(X, \mathcal{F})$.

We want to compare these various cohomology theories. To do so, we show that many of these cohomology theories are computing the same thing : sheaf cohomology.

Remark. While we might explicitly work with sheaves of abelian groups, the following discussion is applicable to sheaves of \mathcal{O}_X modules, C^∞ modules, etc.

Definition 5.1. The *global sections functor* $\Gamma(X, \cdot)$ is a functor $\text{Ab}(X) \rightarrow \text{Ab}$ of the category of sheaves of abelian groups over X to the category of abelian groups, where given a sheaf of abelian groups \mathcal{F} ,

$$\Gamma(X, \mathcal{F}) := \mathcal{F}(X)$$

The functor is left-exact, i.e. given an exact sequence of sheaves

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}$$

we get an exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{E}) \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{G})$$

However, the functor is not right exact, which is due to the local definitions of injectivity and surjectivity. The sheaf axiom guarantees that being injective on stalks implies that a sheaf morphism is injective on sections, but it does not imply the same thing for surjectivity. As an example, let \mathcal{Z}^k be the sheaf of closed smooth k -forms, and let \mathcal{B}^k be the sheaf of exact k -forms. Then the inclusion $\mathcal{B}^k \hookrightarrow \mathcal{Z}^k$ is surjective, since every closed k -form is exact in a sufficiently small neighborhood. However, if $H_{dR}^k(X) \neq 0$, then $\Gamma(X, \mathcal{Z}^k) \rightarrow \Gamma(X, \mathcal{B}^k)$ is not surjective.

Definition 5.2. The *sheaf cohomology groups* $H^i(X, \mathcal{F})$ of a sheaf \mathcal{F} over X are the right derived functors of the global sections functor applied to \mathcal{F}

$$H^i(X, \mathcal{F}) := R^i\Gamma(X, \mathcal{F})$$

i.e, we take an injective resolution $\mathcal{I}^\bullet = \{\mathcal{I}^j\}$ of \mathcal{F}

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \dots$$

and apply $\Gamma(X, \cdot)$ term-wise to the sequence to get

$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{I}^0) \longrightarrow \Gamma(X, \mathcal{I}^1) \longrightarrow \dots$$

and then compute the cohomology of this sequence.

We note that in order for these right derived functors to be defined, there need to be *enough injectives*. We say that an abelian category \mathcal{A} has enough injectives if every object $A \in \text{Ob}(\mathcal{A})$ admits an injective map $A \hookrightarrow I$ into an injective object I , where an injective object is defined to be an object I where given any map $X \rightarrow I$ and an injection $X \hookrightarrow I$, there exists a map $Y \rightarrow Q$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xhookrightarrow{\quad} & Y \\ & \searrow & \swarrow \text{dashed} \\ & Q & \end{array}$$

Alternatively, the pullback map $\text{Hom}(Y, Q) \rightarrow \text{Hom}(X, Q)$ is surjective. We will take the following results on faith:

Theorem 5.3. *The categories \mathbf{Ab} , $\mathbf{Ab}(X)$, $\mathbf{Mod}_{\mathcal{O}_X}$, and \mathbf{Mod}_{C^∞} have enough injectives.*

Like with the definition of the Čech cohomology groups as limits, the definition in terms of injective resolutions is practically useless computationally, since injective sheaves are hard to write down and difficult to find in the wild. The name of the game here is to find a nicer class of resolutions we can take.

Definition 5.4. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a left-exact additive functor. An object $A \in \mathbf{Ob}(\mathcal{A})$ is *acyclic* for the functor F if $R^i F(A) = 0$ for all $i > 0$.

Proposition 5.5. *Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor, and $A \in \mathbf{Ob}(\mathcal{A})$. Then let $A \rightarrow M^\bullet$ be a resolution of A by F -acyclic objects. Then $R^i F(A)$ is isomorphic to the i^{th} cohomology of the complex $F(M^\bullet)$.*

Proof. Let $d^0 : M^0 \rightarrow M^1$. Then let $B = \text{coker}(A \rightarrow M^0)$. By exactness of

$$0 \longrightarrow A \longrightarrow M^0 \xrightarrow{d^0} M^1$$

we get a short exact sequence

$$0 \longrightarrow A \longrightarrow M^0 \longrightarrow B \longrightarrow 0$$

where the map $M^0 \rightarrow B$ is the map d^0 , which descends to B by exactness. We then take injective resolutions $A \rightarrow I^\bullet$, $M^0 \rightarrow J^\bullet$, and $B \rightarrow K^\bullet$. The maps $A \rightarrow M^0$ and $M^0 \rightarrow B$, which gives a short exact sequence of chain maps between resolutions by using the defining property of injective objects, giving us

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & M^0 & \longrightarrow & B \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & I^0 & \longrightarrow & J^0 & \longrightarrow & K^0 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & I^1 & \longrightarrow & J^1 & \longrightarrow & K^1 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \vdots & & \vdots & & \vdots \end{array}$$

which gives us a long exact sequence in cohomology. Noting that the cohomology of the respective sequences are just the right derived functors of A , M^0 , and B , we get the long

exact sequence

$$\begin{array}{ccccccc}
 0 & \longrightarrow & R^0F(A) & \longrightarrow & R^0F(M^0) & \longrightarrow & R^0F(B) \\
 & & & & \searrow & & \\
 & & R^1F(A) & \longrightarrow & R^1F(M^0) & \longrightarrow & R^1F(B) \\
 & & & & \searrow & & \\
 & & R^2F(A) & \longrightarrow & R^2F(M^0) & \longrightarrow & R^2F(B) \\
 & & & & \searrow & & \\
 & & & & \dots & & \dots
 \end{array}$$

Then using the fact that M^0 is F -acyclic, along with the fact that $R^0F = F$, we get that this long exact sequence is actually

$$\begin{array}{ccccccc}
 0 & \longrightarrow & F(A) & \longrightarrow & F(M^0) & \longrightarrow & F(B) \\
 & & & & \searrow & & \\
 & & R^1F(A) & \longrightarrow & 0 & \longrightarrow & R^1F(B) \\
 & & & & \searrow & & \\
 & & R^2F(A) & \longrightarrow & 0 & \longrightarrow & R^2F(B) \\
 & & & & \searrow & & \\
 & & & & \dots & & \dots
 \end{array}$$

which gives us isomorphisms $R^iF(B) \rightarrow R^{i+1}F(A)$ for $i > 0$, as well as an isomorphism $R^1F(A) \cong \text{coker } F(M^0) \rightarrow F(B)$. To compute the cokernel of that map, we note that B admits the resolution

$$0 \longrightarrow B \longrightarrow M^1 \longrightarrow \dots$$

where the map $B \rightarrow M^1$ is the map induced by d^0 , using exactness of the acyclic resolution. Then since F is left-exact, we get that

$$0 \longrightarrow F(B) \longrightarrow F(M^1) \longrightarrow \dots$$

is exact, giving us that $F(B)$ is isomorphic to the kernel of $F(M^1) \rightarrow F(M^2)$. We then note that the image of $F(M^0) \rightarrow F(M^1)$ is contained in $F(B)$, regarded as the kernel of $F(M^1) \rightarrow F(M^2)$. We then get

$$R^1F(A) \cong \frac{\ker(F(M^1) \rightarrow F(M^2))}{\text{Im}(F(M^1) \rightarrow F(M^2))} = H^1(F(M^\bullet))$$

Then to get the isomorphism for $R^2F(A)$, we note that since $R^1F(B) \cong R^2F(A)$, we can play the same game using the resolution of B to compute $R^1F(B)$ to be $H^2(F(M^\bullet))$, and then inductively repeat the process with the cokernel of $M^1 \rightarrow M^2$ to get $R^3F(A)$ and so on. ■

Something that is silly to observe, but useful.

Proposition 5.6. *Injective objects are F -acyclic.*

Proof. Let I be injective. Then take the injective resolution $0 \rightarrow I \rightarrow I \rightarrow 0$ where the map is the identity map. ■

Therefore, to compute right derived functors, it suffices to find acyclic resolutions. This gets us one step closer to finding nice resolutions for computing sheaf cohomology.

Definition 5.7. A sheaf \mathcal{F} over X is *flasque* (also called *flabby*) if the restriction maps are surjective.

Most sheaves in nature aren't flasque. However, flasque sheaves are useful for giving us acyclic resolutions. To do this, we'll need some lemmas.

Lemma 5.8. *Let*

$$0 \longrightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \longrightarrow 0$$

be a short exact sequence where \mathcal{E} is flasque. Then for any open set U , the sequence

$$0 \longrightarrow \mathcal{E}(U) \xrightarrow{\alpha(U)} \mathcal{F}(U) \xrightarrow{\beta(U)} \mathcal{G}(U) \longrightarrow 0$$

is exact.

Proof. By left-exactness of taking sections, it suffices to show that $\beta(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is surjective. Let $\sigma \in \mathcal{G}(U)$. Since β is a surjective sheaf morphism, for any $x \in U$, the induced map on stalks $\beta_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$ is surjective, which implies that there exists a sufficiently small neighborhood $V_x \subset U$ of x where $\beta(V_x) : \mathcal{F}(V_x) \rightarrow \mathcal{G}(V_x)$ is surjective, so we can find $\tau_x \in \mathcal{F}(V_x)$ such that $\beta(V_x)(\tau_x) = \sigma|_{V_x}$. Then let $y \in U$ such that $V_x \cap V_y \neq \emptyset$, and let $\tau_y \in \mathcal{F}(V_y)$ such that $\beta(V_y)(\tau_y) = \sigma|_{V_y}$. Then we know that $\tau_x|_{V_x \cap V_y} - \tau_y|_{V_x \cap V_y} \in \ker \beta(V_x \cap V_y)$, so it is the image of an element $k \in \mathcal{E}(V_x \cap V_y)$. Since \mathcal{E} is flasque, we know that we can lift k to an element $\chi_{x,y} \in \mathcal{E}(V_x)$. Then the element $\tau'_x = \tau_x - \alpha(V_x)(\chi_{x,y})$ still maps to $\sigma|_{V_x}$ under $\beta(V_x)$ by exactness, and restricts to τ_y on $V_x \cap V_y$. Therefore, τ'_x and τ_y glue to a section over $V_x \cup V_y$ that maps to $\sigma|_{V_x \cup V_y}$. We then find a maximal pair (W, τ) such that $\tau \in \mathcal{F}(W)$ and $\beta(W)(\tau) = \sigma|_W$. Then we must necessarily have $W = U$, since otherwise, we can find another open subset V of U and a section ϕ over V mapping to $\sigma|_V$, and extend τ to a section over $W \cup V$, since if $W \cap V \neq \emptyset$, we can use our above argument, and otherwise, no work needs to be done. Either way, finding such a U and ϕ contradicts maximality of (W, τ) . ■

Lemma 5.9. *Let*

$$0 \longrightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \longrightarrow 0$$

be a short exact sequence of sheaves where \mathcal{E} and \mathcal{F} are flasque. Then \mathcal{G} is flasque.

Proof. Let $V \subset U$. Then given $\sigma \in \mathcal{G}(V)$, we want to show that there exists $\tilde{\sigma} \in \mathcal{G}(U)$ that restricts to σ . Since \mathcal{E} is flasque, we have that

$$0 \longrightarrow \mathcal{E}(V) \xrightarrow{\alpha(V)} \mathcal{F}(V) \xrightarrow{\beta(V)} \mathcal{G}(V) \longrightarrow 0$$

is exact, so we can find a section $\tau \in \mathcal{F}(V)$ with $\beta_V(\tau) = \sigma$. Then since \mathcal{F} is flasque, this lifts to an element $\tilde{\tau} \in \mathcal{F}(U)$. Then taking $\tilde{\sigma} = \beta(U)(\tilde{\tau})$ gives us the desired section, since the properties of a sheaf morphism implies that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\beta(U)} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \xrightarrow{\beta(V)} & \mathcal{G}(V) \end{array}$$

■

Proposition 5.10. *Flasque sheaves are acyclic for the global sections functor $\Gamma(X, \cdot)$.*

Proof. Let \mathcal{F} be a flasque sheaf. We first embed \mathcal{F} into an injective flasque sheaf \mathcal{I} . Since \mathbf{Ab} has enough injectives, we can embed each stalk \mathcal{F}_x into an injective group I_x . Then define the sheaf \mathcal{I} by

$$\mathcal{I}(U) = \prod_{x \in U} I_x$$

and the restriction maps are the projection maps $\prod_{x \in U} I_x \rightarrow \prod_{x \in V} I_x$. Since these maps are surjective, \mathcal{I} is flasque. In addition, it is injective by construction. Then \mathcal{F} embeds into \mathcal{I} by composing the inclusions $\mathcal{F}(U) \hookrightarrow \prod_{x \in U} \mathcal{F}_x \hookrightarrow \prod_{x \in U} I_x$. Then let \mathcal{G} be the cokernel of $\mathcal{F} \hookrightarrow \mathcal{I}$, giving us the exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{G} \longrightarrow 0$$

taking resolutions of \mathcal{F} , \mathcal{I} and \mathcal{G} gives us a long exact sequence in cohomology

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(X, \mathcal{F}) & \longrightarrow & H^0(X, \mathcal{I}) & \longrightarrow & H^0(X, \mathcal{G}) \\ & & & & \swarrow & & \\ & & H^1(X, \mathcal{F}) & \longrightarrow & H^1(X, \mathcal{I}) & \longrightarrow & H^1(X, \mathcal{G}) \\ & & & & \swarrow & & \\ & & H^2(X, \mathcal{F}) & \longrightarrow & H^2(X, \mathcal{I}) & \longrightarrow & H^2(X, \mathcal{G}) \\ & & & & \swarrow & & \\ & & & & \dots & & \dots \end{array}$$

The since H^0 is just $\Gamma(X, \cdot)$ and \mathcal{I} is injective, this becomes

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{F}(X) & \longrightarrow & \mathcal{I}(X) & \longrightarrow & \mathcal{G}(X) \\
 & & & & & \searrow & \\
 & & H^1(X, \mathcal{F}) & \longrightarrow & 0 & \longrightarrow & H^1(X, \mathcal{G}) \\
 & & & & & \searrow & \\
 & & H^2(X, \mathcal{F}) & \longrightarrow & 0 & \longrightarrow & H^2(X, \mathcal{G}) \\
 & & & & & \searrow & \\
 & & & & \dots & & \dots
 \end{array}$$

Since \mathcal{F} is flasque, we have that

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{I}(X) \longrightarrow \mathcal{G}(X) \longrightarrow 0$$

is exact, so this implies that $H^1(X, \mathcal{F}) = 0$. In addition, for $i > 0$, we get isomorphisms $H^i(X, \mathcal{G}) \rightarrow H^{i+1}(X, \mathcal{F})$. To get that $H^2(X, \mathcal{F}) = 0$, we note that \mathcal{G} is flasque, so we can repeat the argument to get that $H^1(X, \mathcal{G}) = H^2(X, \mathcal{F})$, and then continue for the other cohomology groups. ■

As a consequence, it suffices to find resolutions by flasque sheaves to compute sheaf cohomology. The good news here is that every sheaf admits a canonical acyclic resolution, the **Godement resolution**. For a sheaf \mathcal{F} , let \mathcal{F}_{God} be the sheaf $U \mapsto \prod_{x \in U} \mathcal{F}_x$, which is clearly flasque. Then we construct a flasque resolution for \mathcal{F} as follows: embed $\mathcal{F} \hookrightarrow \mathcal{F}_{\text{God}}$, and let \mathcal{G}^1 be the cokernel. Then let the next sheaf in the sequence be $\mathcal{G}_{\text{God}}^1$, where the map $\mathcal{F}_{\text{God}} \rightarrow \mathcal{G}_{\text{God}}^1$ is the quotient map $\mathcal{F}_{\text{God}} \rightarrow \mathcal{G}^1$ composed in the the inclusion $\mathcal{G}^1 \hookrightarrow \mathcal{G}_{\text{God}}^1$. Then take \mathcal{G}^2 to be the cokernel of $\mathcal{F}_{\text{God}} \rightarrow \mathcal{G}_{\text{God}}^1$, and continue with $\mathcal{G}_{\text{God}}^2$ and so on. Pictorially, the Godement construction is constructed as follows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}_{\text{God}} & \longrightarrow & \mathcal{G}^1 \longrightarrow 0 \\
 & & & & \searrow & & \downarrow \\
 & & & & & & \mathcal{G}_{\text{God}}^1 \longrightarrow \mathcal{G}^2 \longrightarrow 0 \\
 & & & & & \searrow & \downarrow \\
 & & & & & & \mathcal{G}_{\text{God}}^2 \longrightarrow \mathcal{G}^3 \longrightarrow 0 \\
 & & & & & & \downarrow \\
 & & & & & & \dots
 \end{array}$$

Once more, this resolution is computationally useless, but it serves the purpose of showing that every sheaf admits a resolution by flasque sheaves. With this in hand, we can finally show that we can compute sheaf cohomology with a nice class of sheaves.

Definition 5.11. Let \mathcal{A} be a sheaf of rings over X such that \mathcal{A} admits *partitions of unity*, i.e. for any open cover $\{U_i\}$ of X , there exist global sections $f_i \in \mathcal{A}(X)$ such that $\sum_i f_i = 1$

and f_i is supported in U_i , and where over any particular open set, all but finitely many of the f_i are 0. Then a sheaf of \mathcal{A} -modules is a *fine sheaf*.

These are sheaves we care about, and appear in nature.

Example 5.12. Let M be a smooth manifold. Then the sheaf C^∞ of smooth functions admits partitions of unity. Therefore, Mod_{C^∞} consists of fine sheaves.

The punchline is that fine sheaves are acyclic, which gives us a source of reasonable sheaves with which to build resolutions.

Theorem 5.13. *Fine sheaves are acyclic with respect to $\Gamma(X, \cdot)$.*

Proof. Let \mathcal{F} be a fine sheaf – a sheaf of modules over a sheaf of rings \mathcal{A} that admits partitions of unity. Then by taking the Godement resolution of \mathcal{F} , we get an injective resolution of flasque \mathcal{A} -modules.

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \xrightarrow{d^0} \mathcal{I}^1 \xrightarrow{d^1} \dots$$

Then since flasque sheaves are acyclic, we get that we can compute the sheaf cohomology of \mathcal{F} as

$$H^i(X, \mathcal{F}) = \frac{\ker d^{i+1}(X) : \mathcal{I}^i(X) \rightarrow \mathcal{I}^{i+1}(X)}{\text{Im } d^i(X) : \mathcal{I}^{i-1}(X) \rightarrow \mathcal{I}^i(X)}$$

Then let $\alpha \in \ker d^{i+1}(X)$, by exactness, we know that locally, in a sufficiently small open cover $\{U_j\}$, the map $d^i(U_j)$ is surjective, so we can find $\beta_j \in \mathcal{I}^{i-1}(U_j)$ such that $d^i(U_j)(\beta_j) = \alpha|_{U_j}$. Then $f_j \beta_j$ determines a global section that is equal to $f_j \beta_j$ on U_j and 0 elsewhere, and we get that $\sum_j f_j \beta_j$ maps to $\sum_j f_j \alpha|_{U_j} = \alpha$ under $d^i(X)$. Therefore, the sequence is exact on global sections for $i > 0$, so \mathcal{F} is acyclic. ■

This tells us that many of the sheaves we know about, like sheaves of smooth sections of a vector bundle are trivial from the perspective of sheaf cohomology.

6. COMPARISON OF COHOMOLOGY THEORIES

The fact that many of the sheaves we encounter naturally have trivial sheaf cohomology might come as a surprise, since we know we can extract topological data from these sheaves. The reason for this is that they provide good resolutions of other sheaves with nontrivial sheaf cohomology. U

Proposition 6.1 (Poincaré Lemma). *Every closed smooth k -form ω is locally exact, i.e. for a sufficiently small U , we have that $\omega|_U = d\eta$ for some $k-1$ -form η .*

Proposition 6.2 ($\bar{\partial}$ -Poincaré Lemma). *Every closed smooth (p, q) -form ω is locally $\bar{\partial}$ -exact*

Corollary 6.3. *The de Rham complex*

$$A^0(X) \xrightarrow{d} A^1(X) \xrightarrow{d} \dots$$

is an exact sequence of sheaves.

The constant sheaf $\underline{\mathbb{R}}$ of locally constant real-valued functions naturally lives as a subsheaf of $A^0(X)$, and we know that this is exactly the kernel of $d : A^0(X) \rightarrow A^1(X)$. This tells us that the inclusion $0 \rightarrow \underline{\mathbb{R}} \rightarrow A^\bullet(X)$ is a resolution of $\underline{\mathbb{R}}$, called the *de Rham resolution*. Furthermore, since all the $A^i(X)$ are sheaves of C^∞ -modules, they are fine, so the resolution is a resolution of $\underline{\mathbb{R}}$ by acyclic sheaves. Therefore, we get the isomorphisms

$$H^i(X, \underline{\mathbb{R}}) \cong H_{dR}^i(X)$$

A similar story holds for the sheaf cohomology of the sheaf of sections of a holomorphic vector bundle $E \rightarrow X$. The $\bar{\partial}$ -Poincaré lemma implies that the Dolbeault complex

$$\mathcal{A}^0(E) \xrightarrow{\bar{\partial}_E} \mathcal{A}^1(E) \xrightarrow{\bar{\partial}_E} \dots$$

of sheaves of smooth sections of $(\Lambda^i T^* X)_{\mathbb{C}} \otimes E$ is an exact sequence, since $\bar{\partial}_E$ is defined locally in terms of the operator $\bar{\partial}$ on X . Then since the kernel of $\bar{\partial}_E : \mathcal{A}^0(E) \rightarrow \mathcal{A}^1(E)$ is exactly the sheaf \mathcal{E} of holomorphic sections of E , we get that $0 \rightarrow \mathcal{E} \rightarrow \mathcal{A}^\bullet(E)$ is an acyclic resolution of \mathcal{E} , which gives us isomorphisms

$$H^i(X, \mathcal{E}) \cong H_{\bar{\partial}}^i(X, E)$$

7. THE FIRST CHERN CLASS

On any complex manifold X , we have the following exact sequence of sheaves:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \longrightarrow 0$$

where \exp is the sheaf morphism where a function $f \in \mathcal{O}_X(U)$ is mapped to the function e^f . This gives us the long exact sequence in sheaf cohomology

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(X, \mathbb{Z}) & \longrightarrow & H^0(X, \mathcal{O}_X) & \longrightarrow & H^0(X, \mathcal{O}_X^\times) \\ & & & & & \swarrow & \\ & & H^1(X, \mathbb{Z}) & \longrightarrow & H^1(X, \mathcal{O}_X) & \longrightarrow & H^1(X, \mathcal{O}_X^\times) \\ & & & & \searrow c_1 & & \\ & & H^2(X, \mathbb{Z}) & \longrightarrow & \dots & & \end{array}$$

We are particularly interested in the boundary map $H^1(X, \mathcal{O}_X^\times) \rightarrow H^2(X, \mathbb{Z})$, which we called c_1 . Recall that we have an isomorphism $H^1(X, \mathcal{O}_X^\times) \cong \text{Pic}(X)$.

Definition 7.1. The *first Chern class* of a line bundle L is its image under the map c_1 , using the canonical identification $H^1(X, \mathcal{O}_X^\times) \cong \text{Pic}(X)$.

The first Chern class will prove to be a powerful invariant of line bundles, and our goal will be to understand these cohomology classes in terms of the geometry of the manifold X . We first note some properties of c_1 that can be immediately deduced from the definition and our knowledge of the group structure on $\text{Pic}(X)$.

Proposition 7.2.

$$(1) \quad c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$$

- (2) $c_1(L^*) = -c_1(L)$.
 (3) For a map $F : X \rightarrow Y$, we have $c_1(F^*L) = F^*c_1(L)$.

The first two observations are immediate, and the last one follows from functoriality of cohomology.

Using some Hodge theory, we can already learn something about \mathbb{CP}^n using the first Chern class

Theorem 7.3.

$$\text{Pic}(\mathbb{CP}^n) \cong \mathbb{Z}$$

Proof. Hodge theory gives us the Hodge decomposition

$$H^k(\mathbb{CP}^n, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X, \mathbb{C})$$

as well as isomorphisms

$$H^q(\mathbb{CP}^n, \Omega_{\mathbb{CP}^n}^p) \cong H^{p,q}(\mathbb{CP}^n, \mathbb{C})$$

We know the singular cohomology of \mathbb{CP}^n is given by

$$H^k(\mathbb{CP}^n, \mathbb{C}) \cong \begin{cases} \mathbb{C} & k \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

In particular, if $p + q$ is odd, then we know that $H^{p,q}(X, \mathbb{C}) = 0$. A special case is

$$0 = H^{1,0}(\mathbb{CP}^n, \mathbb{C}) \cong H^1(\mathbb{CP}^n, \Omega_{\mathbb{CP}^n}^0) = H^1(\mathbb{CP}^n, \mathcal{O}_{\mathbb{CP}^n})$$

Therefore, we have that in the long exact sequence induced by the exponential exact sequence, we have

$$0 \longrightarrow \text{Pic}(\mathbb{CP}^n) \xrightarrow{c_1} \mathbb{Z}$$

which tells us that c_1 is injective. Since the only subgroups of \mathbb{Z} are isomorphic to \mathbb{Z} , this gives $\text{Pic}(\mathbb{CP}^n) \cong \mathbb{Z}$. In addition, it tells us that c_1 is a complete isomorphism invariant. Two holomorphic line bundles over \mathbb{CP}^n are isomorphic if and only if they have the same first Chern class. ■

Given that the first Chern class is defined using sheaves of holomorphic functions, as well as our preliminary observation with \mathbb{CP}^n , one might first think that c_1 is an invariant of holomorphic line bundles, but the truth is that it is a slightly coarser invariant – it cannot distinguish between holomorphic line bundles and smooth complex line bundles. To see this, let C_c^∞ denote the sheaf of smooth \mathbb{C} -valued functions on X . We note that we also have the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow C_c^\infty \xrightarrow{\exp} (C_c^\infty)^\times \longrightarrow 0$$

which similarly induces a long exact sequence in cohomology, yielding a boundary map $\delta : H^1(X, (C_c^\infty)^\times) \rightarrow H^2(X, \mathbb{Z})$. Using functoriality of sheaf cohomology, the inclusions

$\mathcal{O}_X \hookrightarrow C^\infty$ and $\mathcal{O}_X^\times \hookrightarrow (C^\infty)^\times$ induce maps on cohomology, giving us the commutative diagram

$$\begin{array}{ccccc} H^1(X, C^\infty) & \longrightarrow & H^1(X, (C^\infty)^\times) & \xrightarrow{\delta} & H^2(X, \mathbb{Z}) \\ \uparrow & & \uparrow & & \parallel \\ H^1(X, \mathcal{O}_X) & \longrightarrow & H^1(X, \mathcal{O}_X^\times) & \xrightarrow{c_1} & H^2(X, \mathbb{Z}) \end{array}$$

with exact rows. Given a holomorphic line bundle L , we can forget its complex structure, regarding it as a complex line bundle, i.e. an element of $H^1(X, (C^\infty)^\times)$, and then take the image under δ . Commutativity of the diagram implies that this is the same thing as $c_1(L)$. Since C^∞ is a fine sheaf, we have that this diagram is actually

$$\begin{array}{ccccc} 0 & \longrightarrow & H^1(X, (C^\infty)^\times) & \xrightarrow{\delta} & H^2(X, \mathbb{Z}) \\ \uparrow & & \uparrow & & \parallel \\ H^1(X, \mathcal{O}_X) & \longrightarrow & H^1(X, \mathcal{O}_X^\times) & \xrightarrow{c_1} & H^2(X, \mathbb{Z}) \end{array}$$

which tells us that δ is injective. Therefore, on smooth complex line bundles, the first Chern class is a perfect invariant – two complex line bundles are smoothly isomorphic if and only if they have the same first Chern class. However, this also reveals to us that it cannot distinguish between different holomorphic structures on the same underlying complex line bundle. However, we can use the complex geometry as a tool to compute certain quantities of a holomorphic bundle, which will turn out to be topological invariants of the bundle, independent of the holomorphic structure. Our goal will be to realize the first Chern class of a line bundle as the cohomology class associated to the curvature of a connection. Eventually, we will add a third characterization to the mix by introducing the Atiyah class.

We first discuss the relationship of c_1 with the curvature of a connection on a complex line bundle. We will discuss connections on complex vector bundles in general, and will then specialize to the case of line bundles.

Definition 7.4. Let $E \rightarrow X$ be a complex vector bundle, and let $\mathcal{A}^k(E)$ denote the sheaf of smooth E -valued k -forms, i.e. the sheaf of sections of the bundle $\Lambda^k(T^*X)_\mathbb{C} \otimes E$, which is a sheaf of C^∞ -modules. A **connection** ∇ is a \mathbb{C} -linear map $\nabla : \mathcal{A}^0 \rightarrow \mathcal{A}^1$ where for a smooth function f and local section s , we have the **Leibniz rule**

$$\nabla(fs) = df \otimes s + f\nabla s$$

A connection can be naturally extended to an operator $\nabla : \mathcal{A}^k \rightarrow \mathcal{A}^{k+1}$, where for a k -form α and a local section s of E , we define

$$\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^k \alpha \wedge \nabla s$$

In a similar fashion, the connection ∇ on E induces connection on associated bundles (e.g. tensor powers, duals). In particular, it induces a connection on the bundle $\text{End}(E)$, which by abuse of notation we also denote ∇ . Given a section F of $\text{End}(E)$, we have that the action of ∇F on a section s is given by the formula

$$(\nabla F)(s) = \nabla(F(s)) - F(\nabla s)$$

Proposition 7.5. *Let ∇_1 and ∇_2 be two connections on a complex vector bundle $E \rightarrow X$. Then $\nabla_1 - \nabla_2$ is an element of $\mathcal{A}^1(\text{End } E)$.*

Proof. We want to show that the difference is C^∞ -linear. We compute

$$\begin{aligned} (\nabla_1 - \nabla_2)(fs) &= \nabla_1(fs) - \nabla_2(fs) \\ &= df \otimes s + f\nabla_1 s - (df \otimes s + f\nabla_2 s) \\ &= f(\nabla_1 - \nabla_2)(s) \end{aligned}$$

■

Proposition 7.6. *Let $a \in \mathcal{A}^1(\text{End } E)$ and let ∇ be a connection on E . Then $\nabla + a$ is a connection on E , where given a local section s , we define $\nabla + a$ by*

$$(\nabla + a)s = \nabla s + as$$

where given a tangent vector v , we have that a acts on s by $a(v)s$.

Proof. We clearly have that $\nabla + a$ is \mathbb{C} -linear. To check that it satisfies the Leibniz rule, we compute

$$\begin{aligned} (\nabla + a)(fs) &= \nabla(fs) + a(fs) \\ &= df \otimes s + f\nabla s + fas \\ &= df \otimes s + f(\nabla + a)(s) \end{aligned}$$

■

Therefore, given a complex vector bundle $E \rightarrow X$ equipped with connection, the connection is locally of the form $d + A$, where d is the usual complexified de Rham differential and A is a matrix of 1-forms.

Definition 7.7. Let $E \rightarrow X$ be a complex vector bundle equipped with a connection ∇ . The *curvature* of ∇ is defined to be the map $F_\nabla := \nabla^2 : \mathcal{A}^0(E) \rightarrow \mathcal{A}^2(E)$.

One might expect the curvature to be a second order differential operator, but a small miracle happens.

Proposition 7.8. *The curvature transformation F_∇ is C^∞ -linear, i.e. it defines a global section of $\mathcal{A}^2(\text{End } E)$.*

Proof. We compute for a function f and local section s

$$\begin{aligned} F_\nabla(fs) &= \nabla(\nabla(fs)) \\ &= \nabla(df \otimes s + f\nabla s) \\ &= d^2f \otimes s - df \wedge \nabla s + \nabla(f\nabla s) \\ &= -df \wedge \nabla s + df \wedge \nabla s + f\nabla^2 s \\ &= f\nabla^2 s \\ &= fF_\nabla(s) \end{aligned}$$

■

Theorem 7.9 (The Bianchi Identity). *Let F_∇ be the curvature form of a connection on a vector bundle $E \rightarrow X$. Then*

$$\nabla(F_\nabla) = 0$$

Proof. Let s be a section of E . By the definition of the induced connection on $\text{End}(E)$, we compute

$$\begin{aligned} (\nabla(F_\nabla))(s) &= \nabla(F_\nabla(s)) - F_\nabla(\nabla s) \\ &= \nabla^3 s - \nabla^3 s \\ &= 0 \end{aligned}$$

■

Finally we make one more observation

Proposition 7.10. *Let ∇ be a connection on $E \rightarrow X$, and $\nabla' = \nabla + A$. Then*

$$F_{\nabla'} = F_\nabla + dA + A \wedge A$$

Proof. We compute locally for a section s ,

$$\begin{aligned} F_{\nabla'} &= (d + A)((d + A)(s)) \\ &= (d + A)(ds + A(s)) \\ &= d^2 s + A(ds) + d(A(s)) + (A \wedge A)(s) \\ &= dA(s) + (A \wedge A)(s) \end{aligned}$$

■

In the case of a line bundle $L \rightarrow X$, much of this discussion simplifies. First of all we have that the bundle $\text{End}(L)$ is a trivial bundle, so we have that every connection is locally of the form $d + A$ for a 1-form A . This allows several small miracles to occur.

Proposition 7.11. *The curvature F_∇ of a connection on a complex line bundle $L \rightarrow M$ is a globally defined 2-form in $H^2(X, \mathbb{C})$.*

Proof. We have that locally, $\nabla = d + A$, so we have locally

$$F_\nabla = dA + A \wedge A = dA$$

We then need to show that these local definitions glue. In two trivializing neighborhoods U_i and U_j with transition functions ψ_{ij} and ψ_{ji} , let A_i and A_j denote the connection 1-forms over each trivialization. Let s be a local section over $U_i \cap U_j$ that is nonzero and constant when viewed as a section of the trivial bundle $U_i \cap U_j \times \mathbb{C}$. Therefore, we have that then the local representations s_i and s_j are related by $\psi_{ij}s_j = s_i$. In addition, we have that over U_i , $\nabla s = ds_i + A_i s_i = A_i s_i$, since s_i is a constant function, and we have a similar picture over U_j . We then compute

$$A_i \psi_{ij} s_j = \nabla(\psi_{ij} s_j) = d\psi_{ij} s_j + \psi_{ij} \nabla s_j = d\psi_{ij} s_j + \psi_{ij} A_j s_j$$

Therefore, we have

$$A_i \psi_{ij} = d\psi_{ij} + \psi_{ij} A_j$$

Multiplying both sides by ψ_{ij}^{-1} , moving terms around, we find

$$A_j = A_i - \psi_{ij}^{-1} d\psi_{ij}$$

We then verify that the local descriptions of F_∇ agree on the intersection. We compute

$$\begin{aligned}
 F_\nabla &= dA_i \\
 &= \psi_{ij}^{-1} d\psi_{ij} \wedge \psi_{ij}^{-1} d\psi_{ij} + dA_i \\
 &= \psi_{ij}^{-2} d\psi_{ij} \wedge d\psi_{ij} + dA_i \\
 &= -d\psi_{ij}^{-1} \wedge d\psi_{ij} + dA_i \\
 &= d(A_i - \psi_{ij}^{-1} d\psi_{ij}) \\
 &= dA_j
 \end{aligned}$$

Where we use the fact that $d(\psi_{ij}^{-1}) = \psi_{ij}^{-2} d\psi_{ij}$. Therefore, the local descriptions of F_∇ glue to a global 2-form. ■

In addition, since we have locally that F_∇ is given by dA_i , we have that F_∇ is closed, so it defines a cohomology class in $H^2(X, \mathbb{C})$. This gives us two ways of obtaining a cohomology class from a line bundle. The first is by taking the first Chern class from the exponential exact sequence, and the other is by taking the curvature form of a connection.

Theorem 7.12. *Let $L \rightarrow X$ be a complex line bundle, and let ∇ be any connection on L with curvature form F_∇ . Then*

$$\left[\frac{i}{2\pi} F_\nabla \right] = c_1(L)$$

Proof. We first give an explicit formula for the Čech cocycle defining $c_1(L)$. Fix a good cover $\mathcal{U} = \{U_i\}$ for X , and let the ψ_{ij} be the transition functions for L with respect to this covering. The ψ_{ij} determine a Čech cocycle in $H^1(X, (C_\mathbb{C}^\infty)^\times)$. By surjectivity of \exp , by passing to a finer cover we may assume that we can find a branch of the logarithm for each ψ_{ij} , so $\log \psi_{ij}$ is well defined, and defines a Čech cocycle in $H^1(X, (C^\infty)^\times)$. Then since the ψ_{ij} satisfy the cocycle condition

$$\psi_{ij}\psi_{jk}\psi_{ik}^{-1} = 1$$

we have that

$$\log \psi_{ij} + \log \psi_{jk} - \log \psi_{ik} \in 2\pi i\mathbb{Z}$$

So we have that

$$\frac{1}{2\pi i} \log \psi_{ij} + \log \psi_{jk} - \log \psi_{ik} \in \mathbb{Z}$$

is the cocycle representing $c_1(L)$. In the natural map $H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{C})$, we have that the cocycle has the same formula (up to a sign depending on convention).

Then let ∇ be any connection on L , with local connection forms A_i . We have that F_∇ is a closed 2-form, with local description $F_\nabla = dA_i$. Let $\mathcal{A}^k(X)$ denote the sheaf of smooth k -forms over X , and let $\mathcal{Z}^k(X)$ denote the sheaf of closed smooth k -forms over X . Then we have the exact sequences of sheaves

$$0 \longrightarrow \mathcal{Z}^1(X) \longrightarrow \mathcal{A}^1(X) \xrightarrow{d} \mathcal{Z}^2(X) \longrightarrow 0$$

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{A}^0(X) \xrightarrow{d} \mathcal{Z}^1(X) \longrightarrow 0$$

which give rise to boundary homomorphisms $\delta_1 : H^0(X, \mathcal{Z}^2(X)) \rightarrow H^1(X, \mathcal{Z}^2(X))$ and $\delta_2 : H^1(X, \mathcal{Z}^1(X)) \rightarrow H^2(X, \mathbb{C})$ respectively. The fact that the local definitions $F_\delta = dA_i$ glue to a global form is exactly the statement that the dA_i form a Čech cocycle in $H^0(X, \mathcal{Z}^2(X))$, i.e. $dA_i - dA_j = 0$ for all i, j with $U_i \cap U_j \neq \emptyset$. Following the definition of the boundary homomorphism, this implies that it is the image of the cocycle determined by the A_i under d , and then applying the Čech differential we get that $\delta_1(F_\nabla)$ is the cocycle $\{A_j - A_i\}$. Then recall from our earlier calculation that $A_j = A_i - \psi_{ij}^{-1} d\psi_{ij}$, we get that

$$A_j - A_i = -\psi_{ij}^{-1} d\psi_{ij}$$

which is exactly $-d \log \psi_{ij}$. Then tracing through the definition of the boundary homomorphism δ_2 , we have that $\delta_2(A_j - A_i)$ is the preimage of the Čech differential applied to the cocycle $\log \psi_{ij}$, which is exactly $-(\log \psi_{ij} + \log \psi_{jk} - \log \psi_{ik})$. Putting everything together, we find that

$$c_1(L) = \frac{1}{2\pi i} \log \psi_{ij} + \log \psi_{jk} - \log \psi_{ik} = \left[\frac{i}{2\pi} F_\nabla \right]$$

■

8. THE CHERN CONNECTION

Definition 8.1. Let $E \rightarrow X$ be a complex vector bundle. A **Hermitian structure** on E is the data of a smoothly varying hermitian form $H(x)$ on each fiber E_x . If E is a complex vector bundle equipped with a Hermitian structure, we call it a **Hermitian vector bundle**.

Hermitian structures always exist due to convexity of the space of Hermitian forms. Therefore, we can define Hermitian forms locally in neighborhoods where the bundle E is trivial, and then use a partition of unity to glue them together. Like in the case of Riemannian geometry, we can ask for a connection to be compatible with a Hermitian structure.

Definition 8.2. Let $E \rightarrow X$ be a Hermitian vector bundle. A connection ∇ on E is a **Hermitian connection** if for sections s_1, s_2 , we have

$$dh(s_1, s_2) = h(\nabla s_1, s_2) + h(s_1, \nabla s_2)$$

In the complex world, we can also impose a holomorphic structure on vector bundles, and we can also ask for a compatibility condition with the holomorphic structure.

Definition 8.3. Let $E \rightarrow X$ be a holomorphic structure. A connection ∇ on E is said to be **compatible with the holomorphic structure** if the $(0, 1)$ component $\nabla^{0,1}$ is equal to $\bar{\partial}_E$.

This leads us to a natural question : given a holomorphic vector bundle $E \rightarrow X$ equipped with a Hermitian structure h , can we find a connection ∇ that is compatible both with the Hermitian structure and the holomorphic structure?

Theorem 8.4 (The Chern connection). Let $E \rightarrow X$ be a Hermitian holomorphic vector bundle. There exists a unique Hermitian connection ∇ , called the **Chern connection** that is compatible with the holomorphic structure.

Proof. The proof has a similar flavor to other proofs of existence and uniqueness of global objects. We use the compatibility conditions to determine local conditions that the connection must satisfy, and show that it uniquely specifies the connection. Therefore, the local definitions glue to a global one. In a local trivialization of E , we have that any connection ∇ is of the form $d + A$ for a matrix valued 1-form A . In addition, we have that the Hermitian structure gives a Hermitian form $H(x)$ at each point x . Let $\{E_i\}$ the local frame for E , where each E_i is just the constant section $x \mapsto e_i$. Then we get a coordinate representation of the Hermitian form H . $H_j^i(x) = h(E_i, E_j)$. We clearly have that $dE_i = 0$, since all the sections are constant, so $\nabla E_i = AE_i$. The condition that ∇ is Hermitian gives us

$$dH_j^i = dH(E_i, E_j) = h(AE_i, E_j) + h(E_i, AE_j)$$

Writing the matrix A in coordinates A_j^i , this gives us

$$dH_j^i = \overline{A_k^i} h(E_k, E_j) + h(E_i, E_k) A_j^k = \overline{A_k^i} H_j^k + H_k^i A_j^k$$

which tells us that we must have that

$$dH = A^\dagger H + HA$$

We then consider the restrictions imposed on ∇ by the condition that ∇ is compatible with the holomorphic structure. Decomposing $\nabla = \nabla^{1,0} + \nabla^{0,1}$, we want $\nabla^{0,1} = \bar{\partial}_E$. In our local picture, we have that $\nabla = \partial + \bar{\partial} + A$, and $\bar{\partial}_E$ is just $\bar{\partial}$. Therefore, we must have that $A^{0,1} = 0$, so $A = A^{1,0}$. Decomposing $d = \partial + \bar{\partial}$ and applying it to our Hermitian condition, we find that

$$\partial H + \bar{\partial} H = (A^{1,0})^\dagger H + HA^{1,0}$$

comparing the bidegrees then gives

$$\begin{aligned} \partial H &= HA^{1,0} \\ \bar{\partial} H &= (A^{1,0})^\dagger H \end{aligned}$$

Moving H to the other side of the equation and conjugating the equation gives

$$\partial \bar{H} (\bar{H}^{-1}) = (A^{1,0})^T = A^T$$

which gives a unique characterization of the matrix A , as well as a local definition, showing existence and uniqueness of the Chern connection. \blacksquare

Unsurprisingly, if we have a connection compatible with a Hermitian or holomorphic structure, then the corresponding curvature forms are also compatible.

Proposition 8.5.

- (1) Let $E \rightarrow X$ be a Hermitian vector bundle with Hermitian connection ∇ . Then F_∇ is locally given by a skew-Hermitian matrix C of 2-forms. Globally, we have

$$h(F_\nabla s_1, s_2) + h(s_1, F_\nabla s_2) = 0$$

- (2) Let $E \rightarrow X$ be a holomorphic vector bundle. Then F_∇ has no $(0,2)$ part, i.e. $F_\nabla = F_\nabla^{2,0} + F_\nabla^{1,1}$.

Proof.

- (1) Locally E is isomorphic to the trivial bundle with the standard Hermitian structure, so $\nabla = d + A$ for a skew-Hermitian matrix A . Then using the local formula

$$F_{\nabla} = dA + A \wedge A$$

we want to show that $F_{\nabla}^{\dagger} = -F_{\nabla}$. We compute

$$\begin{aligned} F_{\nabla}^{\dagger} &= d(A^{\dagger}) + (A \wedge A)^{\dagger} \\ &= d(A^{\dagger}) - A^{\dagger} \wedge A^{\dagger} \end{aligned}$$

The fact that $(A \wedge A)^{\dagger} = -A^{\dagger} \wedge A^{\dagger}$ is slightly tricky, and comes from the skew symmetry of the wedge product. Explicitly in components, we have

$$\begin{aligned} ((A \wedge A)^{\dagger})_j^i &= (\overline{A} \wedge \overline{A})_i^j \\ &= \overline{A}_k^j \wedge \overline{A}_i^k \\ &= -\overline{A}_k^i \wedge \overline{A}_j^k \\ &= -(A^{\dagger} \wedge A^{\dagger})_j^i \end{aligned}$$

We then complete the computation, using the fact that A is skew-Hermitian

$$\begin{aligned} F_{\nabla}^{\dagger} &= d(A^{\dagger}) - A^{\dagger} \wedge A^{\dagger} \\ &= -dA - A \wedge A \\ &= -F_{\nabla} \end{aligned}$$

- (2) Locally, we may write $\nabla = d + A$. Since ∇ is compatible with the holomorphic structure, we also have that $A = A^{1,0}$. We have that locally

$$F_{\nabla} = dA + A \wedge A$$

Splitting $d = \partial + \bar{\partial}$, we get

$$F_{\nabla} = (\partial + \bar{\partial})(A) + A \wedge A = \bar{\partial}A + \partial A + A \wedge A$$

and since A is type $(1,0)$, we get that $\bar{\partial}A$ is type $(1,1)$ and $\partial A + A \wedge A$ is type $(2,0)$. ■

Putting these together, we get a result on the type of the curvature of a Chern connection.

Proposition 8.6. *Let ∇ be the Chern connection for a Hermitian holomorphic vector bundle $E \rightarrow X$. Then F_{∇} is type $(1,1)$.*

Proof. Since ∇ is Hermitian, we have that $F_{\nabla}^{\dagger} = -F_{\nabla}$. Since ∇ is compatible with the holomorphic structure, we have that $F_{\nabla} = F_{\nabla}^{2,0} + F_{\nabla}^{1,1}$. Then consider

$$F_{\nabla}^{\dagger} = (F_{\nabla}^{2,0})^{\dagger} + (F_{\nabla}^{1,1})^{\dagger}$$

We have that $(F_{\nabla}^{2,0})^{\dagger}$ is type $(0,2)$ and $(F_{\nabla}^{1,1})^{\dagger}$ is type $(1,1)$. Therefore, in order to have $F_{\nabla}^{\dagger} = -F_{\nabla}$, we must have that $(F_{\nabla}^{1,1})^{\dagger} = -F_{\nabla}^{1,1}$ and $F_{\nabla}^{2,0} = 0$ by a simple type check. ■

This yields an important result regarding the curvature of Chern connection.

Theorem 8.7. *The curvature form F_∇ of a Chern connection on a holomorphic vector bundle $E \rightarrow X$ is $\bar{\partial}_E$ -closed, so it defines a Dolbeault cohomology class $[F_\nabla] \in H^1(X, \Omega_X \otimes \text{End}(E))$, where Ω_X denotes the bundle of holomorphic 1-forms.*

Proof. Applying the Bianchi Identity to the curvature form $F_\nabla = F_\nabla^{1,1}$, we find

$$0 = \nabla(F_\nabla)^{1,2} = \bar{\partial}_E F_\nabla$$

where we use the fact that the induced connection on $\text{End}(E)$ is also compatible with the holomorphic structure. ■

This gives some intuition behind the Chern connection. For an arbitrary connection on a holomorphic vector bundle E , we need not have that the curvature is $\bar{\partial}_E$ -closed. This shows that asking for a connection to be a Chern connection is a sort of integrability condition. In addition, it gives a generalization of the first Chern class for line bundles to holomorphic bundles of arbitrary rank. In the line bundle case, we did not have to worry about curvature being closed, but in the higher dimensional case, we have to restrict our attention to Chern connections.

Given a holomorphic vector bundle $E \rightarrow X$, we now get a recipe for producing Dolbeault cohomology classes in $H^1(X, \Omega_X \otimes \text{End}(E))$. In addition, We can place a Hermitian structure on E , and then compute the curvature of the Chern connection. Naturally, the next question is whether this cohomology class depends on the choice of Hermitian structure.

Theorem 8.8. *The cohomology class $[F_\nabla]$ of the curvature of a Chern connection on a holomorphic vector bundle E is independent on the choice of Hermitian structure.*

Proof. Let ∇ be the Chern connection with respect to some Hermitian structure on E . Then any other connection is of the form $\nabla + A$ with curvature $F_{\nabla+A} = F_\nabla + \nabla A + A \wedge A$. Then if $\nabla' = \nabla + A$ is the Chern connection with respect to a different Hermitian structure, then A must be type $(2,0)$. In addition, we know that both F_∇ and $F_{\nabla'} = F_\nabla + \nabla A + A \wedge A$ are both type $(1,1)$. In order for this to be true, we must have that $\nabla A + A \wedge A$ is of type $(1,1)$. Since $A \wedge A$ is of type $(2,0)$, we get

$$(\nabla A + A \wedge A)^{1,1} = (\nabla A)^{1,1} = \nabla A$$

Then since A is type $(1,0)$, the $(1,1)$ part of A is $\nabla^{0,1} A = \bar{\partial}_E A$ since ∇ is compatible with the holomorphic structure. Therefore, we have that $\nabla A + A \wedge A = \bar{\partial}_E A$, giving us

$$F_{\nabla'} = F_\nabla + \bar{\partial}_E A$$

so $[F_{\nabla'}] = [F_\nabla]$. ■

Definition 8.9. Let $E \rightarrow X$ be a holomorphic vector bundle of rank k , and $\mathcal{U} = \{U_i\}$ a good cover of X with local trivializations $\varphi_i : E|_{U_i} \rightarrow U_i \times \mathbb{C}^k$ and transition functions $\psi_{ij} : U_i \cap U_j \rightarrow \text{GL}_k \mathbb{C}$. The **Atiyah class** of E , denoted $A(E)$ is the element $A(E) \in H^1(X, \Omega_X \otimes \text{End } E)$ determined by the cocycle $\sigma_{ij} = \varphi_j^{-1} \circ (\psi_{ij}^{-1} d\psi_{ij}) \circ \varphi_i$, where Ω_X denotes the bundle of holomorphic 1-forms.

We first make sense of the formula for the cocycle. A component σ_{ij} of the Čech cocycle σ should be a local section $\sigma_{ij} : U_i \cap U_j \rightarrow \Omega_X \otimes \text{End } E$. What this means is that we should be able to feed σ_{ij} the following data:

- (1) A point $p \in U_i \cap U_j$
- (2) A tangent vector $v \in T_p X$
- (3) A vector $w \in E_p$ in the fiber

and we should expect another vector in E_p to be the output. We have that $\varphi(p, w)$ should be a pair $(p, w') \in U_i \cap U_j \times \mathbb{C}^k$. Then since $\psi_{ij} : U_i \cap U_j \rightarrow \text{GL}_k \mathbb{C}$, we have that $d(\psi_{ij})_p : T_p X \rightarrow M_n \mathbb{C}$. Therefore, we have that $d(\psi_{ij})_p(v)$ is a matrix, which we can then apply to w' . Then we can apply the matrix inverse $(\psi_{ij}(p))^{-1}$ to the result, leaving us with some other pair $(p, w'') \in U_i \cap U_j \times \mathbb{C}^k$, and then applying the inverse of the local trivialization φ_j to this gives us the desired output. The fact that this defines a cocycle essentially boils down to the fact that the ψ_{ij} satisfy the cocycle condition, along with using the chain rule for the differentials $d\psi_{ij}$.

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