Diff Geo Lecture 4

The Plan

- Wrap up differential forms
  - The exterior derivative
  - Integration + Stokes' Thm
- Some fun with Lie groups
  - Definitions
  - Lie group actions "Orbit stabilizer for Lie groups"

The Exterior Derivative

Let $\omega \in \Omega^k(M)$ $\quad \omega = \underset{I}{\overset{\text{increasing } I}{\sum}} \omega_I \, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$

The <u>exterior derivative</u> of ω, denoted $d\omega$ is a $k+1$ form given by

$$d\omega = \sum_I \frac{\partial \omega_I}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

<u>Example</u>

Consider the case in $\mathbb{R}^3$

$$C^\infty(\mathbb{R}^3) = \Omega^0(\mathbb{R}^3) \xrightarrow{d} \Omega^1(\mathbb{R}^3)$$

$$f \longmapsto \frac{\partial f}{\partial x^i} dx^i \approx \text{grad } f$$

$$\Omega^1(\mathbb{R}^3) \xrightarrow{\quad d \quad} \Omega^2(\mathbb{R}^3)$$

$$f_1 dx^1 + f_2 dx^2 + f_3 dx^3 \longmapsto \left(\frac{\partial f_1}{\partial x^1} dx^1 \wedge dx^1 + \frac{\partial f_1}{\partial x_2} dx^2 \wedge dx^1 + \frac{\partial f_1}{\partial x^3} dx^3 \wedge dx^1 \right)$$

$$+ \left(\text{Similar terms for } f_2 dx^2 \text{ and } dx^3 \right)$$

$$= \left(\frac{\partial f_2}{\partial x^1} - \frac{\partial f_1}{\partial x^2} \right) dx^1 \wedge dx^2 + \left(\frac{\partial f_3}{\partial x^2} - \frac{\partial f_2}{\partial x^3} \right) dx^2 \wedge dx^3 + \left(\frac{\partial f_1}{\partial x^3} - \frac{\partial f_3}{\partial x^1} \right) dx^1 \wedge dx^3$$

* Curl

$$\Omega^2(\mathbb{R}^3) \xrightarrow{d} \Omega^3(\mathbb{R}^3)$$

$$\begin{aligned} &f_1\, dx^1 \wedge dx^2 \\ &+ f_2\, dx^2 \wedge dx^3 \\ &+ f_3\, dx^1 \wedge dx^3 \end{aligned} \longmapsto \left(\frac{\partial f_1}{\partial x^1} + \frac{\partial f_2}{\partial x^2} + \frac{\partial f_3}{\partial x^3} \right) dx^1 \wedge dx^2 \wedge dx^3$$

??

div.

One thing to note, $d \circ d = 0$

(curl grad $= 0$, div curl $= 0$)

If $d\eta = 0$, we say η is closed

if $\omega = d\eta$ for some η, ω is exact

$d \circ d = 0 \Rightarrow$ exact forms are closed.

We can then take the quotient

$\{\text{closed}\} / \{\text{exact}\}$ giving us the

de Rham cohomology groups $H^k_{dR}(M)$

Integration

Rmk: We'll assume our manifolds are oriented.

In the Euclidean case, we know how to integrate. For $U \subset \mathbb{R}^n$ open,

$$\omega = f\, dx^1 \wedge \dots \wedge dx^n,$$

$$\int_U \omega = \int_U f\, dx^1 \dots dx^n$$ in the sense we know

Recall that pullback by any map $\varphi: V \to U$ gives us a factor of $\det D\varphi$, giving us the coordinate change formula.

Indeed, $\int_V \varphi^* \omega = \int_U \omega$ if φ is a diffeomorphism.

On manifolds, suppose the support of ω is contained in a single coordinate chart (U, φ). Then we can define $\int_M \omega = \int_{\varphi(U)} (\varphi^{-1})^* \omega$

In the general case, we'll need <u>a partition of unity</u>

<u>Thm</u> (Generalized Stokes')

Let M be a smooth manifold with boundary, and let $\omega \in \Omega^{n-1}(M)$. Then

$$\int_{\partial M} \omega = \int_M d\omega$$

In particular, if $\partial M = \emptyset$, the integral of any exact form is 0

<u>Lie groups</u>

<u>Def</u>: A Lie group is a group G that is a smooth manifold such that the maps

$$G \times G \to G$$
$$(g,h) \mapsto gh$$

$$G \longrightarrow G$$
$$g \longmapsto g^{-1}$$

are smooth.

Examples

- Any finite group (w/ discrete topology) is a Lie group. (In fact, any countable discrete group)

- $(\mathbb{R}^n, +)$ is a Lie group

- $\mathbb{T} = \{ z \in \mathbb{C} \mid |z| = 1 \}$ is a Lie group diffeomorphic to S^2

- $GL_n\mathbb{R} = \{ A : \mathbb{R}^n \to \mathbb{R}^n \mid A \text{ is invertible} \}$ is an open submanifold of $M_n\mathbb{R} \cong \mathbb{R}^{n^2}$

  Similarly, $GL_n\mathbb{C}$ and $GL_n\mathbb{H}$

- $SL_n\mathbb{R} = \{ A \in GL_n\mathbb{R} \mid \det A = 1 \}$
  Similarly, $SL_n\mathbb{C}$, "$SL_n\mathbb{H}$"

- $O_n = \{ A \in GL_n\mathbb{R} \mid \langle Av, Aw \rangle = \langle v, w \rangle \}$
  $= \{ A \in GL_n\mathbb{R} \mid AA^T = I \}$

- $SO_n = SL_n\mathbb{R} \cap O_n$

The dream of a group is to act on something. This is often how they are all defined e.g. $D_n \circlearrowright$ regular n-gon

$S_n \circlearrowright \{1, \ldots, n\}$

The dream of a <u>Lie group</u> is to act on a manifold

<u>Def</u>: Let G be a Lie group and M a manifold. Then a <u>Lie group action</u> of G on M $(G \circlearrowright M)$ is a smooth map Θ $G \times M \to M$

$$\Theta(h, \Theta(g,m)) = \Theta(hg, m)$$

We often denote $\Theta(g, -) = \Theta_g : M \to M$

Equivalently, it is the data of a smooth group homomorphism $\Theta : G \to \mathrm{Diff}(M)$

$$g \mapsto \Theta_g$$

Examples

- The trivial action $g \mapsto Id_M$
- $\mathbb{Z}^n \curvearrowright \mathbb{R}^n$ by integer translations
  (What is the quotient group?)
- $SL_n\mathbb{R} \curvearrowright \mathbb{R}^n$ by volume preserving transformations.
- SO_n acts transitively on $S^2 \subset \mathbb{R}^3$

A space with a transitive G-action is called a homogeneous G-space.

homogeneous $\approx$ looks the same at all points.

Given a Lie group action $G \curvearrowright M$ and a point $p \in M$, the isotropy subgroup (alternatively, stabilizer subgroup) is

$$G_p = \{ g \in G \mid g \cdot p = p \}$$

Thm (Orbit-Stabilizer for Lie Groups)

Let $G \curvearrowright M$ be a transitive Lie group action. Then M is diffeomorphic to the coset space G/G_p for any $p \in M$ via the map

$$g \cdot G_p \longmapsto g \cdot p.$$

(Motto: Any transitive action is an action on cosets)

Example.

$$SO_3 \curvearrowright S^2 \subset \mathbb{R}^3$$

What is the isotropy subgroup of the North pole $N = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$?

$$H = \left\{ \begin{pmatrix} 0 & \multicolumn{2}{c}{(A)} \\ 0 & & \\ 1 & 0 & 0 \end{pmatrix} \;\middle|\; A \in SO_2 \right\}$$

So $S^3 \cong SO_3 / H \cong SO_2$

Given a group action $G \curvearrowright M$, we can take the quotient $M/G = M/\sim$ where $m \sim g.m$. Under sufficient conditions this is still a manifold. With discrete groups this is covering space theory.

<u>Example</u> (The Hopf Fibration)

Consider $S^3 \subset \mathbb{C}$

$$S^3 = \{ (z_1, z_2) \mid |z_1|^2 + |z_2|^2 \}$$

S^1 acts on S^3 by multiplication

$$s \cdot (z_1, z_2) = (sz_1, sz_2)$$

This is a free action, so the orbits are circles. The quotient S^3/S^1 is $\mathbb{C}P^1 \cong S^2$. The quotient projection $S^3 \longrightarrow S^2$ is the <u>Hopf Fibration</u>

You can do the same process with the quaternions $\mathbb{H}$.

$$Sp_1 = \{q \mid |q| = 1\} \cong S^3$$

$$Sp_1 \curvearrowright S^7 \subset \mathbb{H}^2$$

$S^7/Sp_1 \cong \mathbb{HP}^1 \cong S^4$, giving another

Hopf fibration $S^7 \to S^4$