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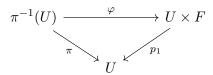
Principal Bundles

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Why Fiber Bundles?

Suppose we want to study some space M, which for our purposes is a smooth manifold. One way to study M is to study functions $M \to F$ for some fixed target space F, e.g. a manifold, \mathbb{R} , or a vector space. This is a perfectly good method of studying M, but is sometimes not enough. More often then not, we want to study "function" from M into a vector space that varies over the base manifold M. For example, a vector field $X \in \mathfrak{X}(M)$ is not really a map $M \to \mathbb{R}^n$, it is an assignment to each $p \in M$ a vector in T_pM . In this way, we are led to the study of a smoothly parameterized family of vector spaces – the tangent bundle TM. This leads us to define a fiber bundle.

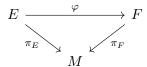
Definition 0.1. Let M and F be a smooth manifolds. A **fiber bundle** over M with model fiber F is the data of a smooth manifold E with a smooth surjective map $\pi: E \to M$ such that for each $p \in M$, there exists an open set U and a diffeomorphism $\varphi: \pi^{-1}(U) \to U \times F$ such that



where $p_1: U \times F \to U$ is projection onto the first factor. The map $\varphi: \pi^{-1} \to U \times F$ is called a **local trivialization**. The space E is called the **total space**, while the manifold M is called the **base space**. We often omit naming the map π and denote the fiber $\pi^{-1}(x)$ by E_x .

This definition captures the notion of a family of manifolds diffeomorphic to F that are smoothly parameterized by the base space M. We also have a notion of a morphism between bundles.

Definition 0.2. Let $\pi_E E \to M$ and $\pi_F : F \to M$ be fiber bundles with model fiber X. A bundle homomorphism is the data of a smooth map $\varphi : E \to F$ such that the diagram



commutes.

Our original motivation for thinking about fiber bundles was for a generalized notion of a function. To this end, we specify a special class of maps associated to a fiber bundle.

Definition 0.3. Let $\pi : E \to M$ be a a fiber bundle with model fiber F. A **local section** of $\pi : E \to M$ is a smooth map $\sigma : U \to E$ such that $\pi \circ \sigma = \mathrm{id}_U$ for some open set $U \subset M$. If U = M, we call σ a **global section**. Equivalently, it is a smooth assignment to each $p \in U$ a point in the fiber E_p . We denote the space of sections over an open set U as $\Gamma_U(E)$.

A section of $E \to M$ can be thought of as our desired generalization of a function. A map $M \to F$ is the same data as a section of the trivial bundle $M \times F \to M$. However, not every fiber bundle is trivial – there can be a nontrivial "twisting." An example of this is the Möbius band. Can you see why this bundle over S^1 is not isomorphic to the trivial bundle $S^1 \times [0,1]$?

We are especially interested in two special classes of fiber bundles that carry additional structure – the fibers of a vector bundle carry the extra structure of a vector space, and the fibers of a principal bundle have the extra structure of a G-torsor for a Lie group G.

Definition 0.4. A vector bundle of rank k is a fiber bundle $E \to M$ such that each fiber E_x has the structure of a k-dimensional vector space (usually over \mathbb{R} or \mathbb{C}). A vector bundle homomorphism is a bundle homomorphism that restricts to a linear map on each fiber.

Vector bundles form a familiar family of fiber bundles, as tangent bundles, cotangent bundles, and their associated tensor bundles are all vector bundles.