THE RISING SEA: CATEGORIES AND SHEAVES

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These are some notes + exercises I've compiled working through the first 2 chapters of Ravi Vikhil's *The Rising Sea*, with the main purpose being to gain some familiarity and comfort with categories and sheaves.

1. Category Theory

Definition 1.1. A ***category*** $\mathscr{C}$ is a collection[1] of ***objects***, denoted $\mathsf{Ob}(\mathscr{C})$ and a collection[2] ***morphisms*** $\mathsf{Hom}(A, B)$[3] for every pair of objects $A, B \in \mathsf{Ob}(\mathscr{C})$ satisfying the following axioms:

(1) Given morphisms $f : A \to B$ and $g : B \to C$, there is a unique map $g \circ f : A \to C$ that makes the following diagram commute

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

(2) For every object $A \in \mathsf{Ob}(\mathscr{C})$, there exists an ***identity morphism*** $\mathrm{id}_A \in \mathsf{Hom}(A, A)$ such that for any morphisms $f : A \to B$ and $g : C \to A$, we have that $\mathrm{id}_A \circ f = f$ and $g \circ \mathrm{id}_A = g$

A morphism $f : A \to B$ is an ***isomorphism*** is there exists a morphism $g : B \to A$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$. We then call g the ***inverse*** to f. Isomorphisms $A \to A$ are called ***automorphisms*** of A.

Example 1.2. The category of sets, often denoted Set has sets as its objects, and maps of sets as its morphisms.

Example 1.3. Vector spaces over a field $\mathbb{F}$ also form a category, denoted $\mathsf{Vec}_{\mathbb{F}}$, where the objects are $\mathbb{F}$-vector spaces, and the objects are $\mathbb{F}$-linear maps.

Exercise 1.4. Let A be an object of a category $\mathscr{C}$. Show that the automorphisms of $\mathsf{Hom}(A, A)$ form a group, called the ***Automorphism group*** of A.

Proof. ■

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[1] Loosely speaking; there's some set-theoretic issues here, but it's not that important for us

[2] Again, ignoring set-theoretic problems

[3] Vikhil uses Mor, but we'll use the more standard notation of Hom