THE FRÖLICHER SPECTRAL SEQUENCE AND THE $\partial\overline{\partial}$ LEMMA

JEFFREY JIANG

For any complex manifold X, we have the Frölicher spectral sequence, which computes the de Rham cohomology of a complex manifold in terms of the ∂ and $\overline{\partial}$ cohomology. On the E_0 page, it is given by the Dolbeault cohomology of the holomorphic vector bundles $\Omega^{p,0}$, i.e. the item in the (p,q) position is the space $\mathcal{A}^{p,q}$ of smooth (p,q) forms, and the differential on E_0 is just $\overline{\partial}$. For example, a small section of the E_0 page would be:

$$
\begin{array}{c|cccc}
2 & \mathcal{A}^{0,2} & \mathcal{A}^{1,2} & \mathcal{A}^{2,2} & \mathcal{A}^{3,2} \\
 & \overline{\partial}\uparrow & \overline{\partial}\uparrow & \overline{\partial}\uparrow & \overline{\partial}\uparrow \\
1 & \mathcal{A}^{0,1} & \mathcal{A}^{1,1} & \mathcal{A}^{2,1} & \mathcal{A}^{3,1} \\
 & \overline{\partial}\uparrow & \overline{\partial}\uparrow & \overline{\partial}\uparrow & \overline{\partial}\uparrow \\
0 & \mathcal{A}^{0,0} & \mathcal{A}^{1,0} & \mathcal{A}^{2,0} & \mathcal{A}^{3,0} \\
\hline
 & 0 & 1 & 2 & 3
\end{array}
$$

Then in the E_1 page, we have $E_1^{p,q}$ is the cohomology in the (p,q) slot in the E_0 page, which is just $H^{p,q}(X)$ (abbreviated to $H^{p,q}$) by Hodge theory in the compact case. The differentials going from left to right are the operators ∂, which descends to cohomology because $\partial\overline{\partial} = -\overline{\partial}\partial$. A small section of the E_1 page would be :

$$
\begin{array}{c|l}
2 & H^{0,2} \xrightarrow{\partial} H^{1,2} \xrightarrow{\partial} H^{2,2} \xrightarrow{\partial} H^{3,2} \\
 & \\
1 & H^{0,1} \xrightarrow{\partial} H^{1,1} \xrightarrow{\partial} H^{2,1} \xrightarrow{\partial} H^{3,1} \\
 & \\
0 & H^{0,0} \xrightarrow{\partial} H^{1,0} \xrightarrow{\partial} H^{2,0} \xrightarrow{\partial} H^{3,0} \\
\hline
 & \quad 0 \qquad\quad 1 \qquad\quad 2 \qquad\quad 3
\end{array}
$$

For a general compact complex manifold X, this continues to the E_2 page, where the differential "rotates," and the (p,q) slot is the cohomology in the (p,q) slot of the E_1 page. The big theorem we want to prove is:

Theorem 1.1. *For a compact Kähler manifold X, the Frölicher spectral sequence degenerates at the E_1 page, i.e. all the differentials are* 0*.*

In other words, we can terminate our spectral sequence computations at E_1. Going to the E_1 page is easy, since all the computations are done with the operators ∂ and $\overline{\partial}$. In practice, continuing on to further pages is difficult. The entire spectral sequence story seems very difficult (and it is), but in the compact Kähler story, it reduces to a simple lemma.

Theorem 1.2 (***The*** $\partial\overline{\partial}$ ***lemma***)**.** *Let X be a Kähler manifold, and η a complex k-form that is ∂ and $\overline{\partial}$-closed. Then if η is d, ∂, or $\overline{\partial}$-exact, there exists a form ξ such that $\eta = \partial\overline{\partial}\xi$.*

The proof of this lemma requires the following results from Hodge Theory:

Theorem 1.3 (***Comparison of the Laplacians***)**.** *Let X be a compact Kähler manifold. Then*

$$\Delta = 2\Delta_{\partial} = 2\Delta_{\overline{\partial}}$$

where Δ, Δ_{∂}, and $\Delta_{\overline{\partial}}$ are the Laplacians

$$\begin{aligned} \Delta &= dd^* + d^*d \\ \Delta_{\partial} &= \partial\partial^* + \partial^*\partial \\ \Delta_{\overline{\partial}} &= \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} \end{aligned}$$

The proof of this theorem requires certain commutation relations to hold, called the ***Kähler identities***. This identity is not true in general for an arbitrary compact complex manifold. The other result we need is

Theorem 1.4 (***The Hodge Decomposition***)**.** *Any complex valued form $\alpha \in \Omega^{p,q}$ can be written as*

$$\alpha = \beta + \Delta\gamma$$

where β is harmonic, i.e. $\Delta\beta = 0$.

This theorem is true for general compact complex manifolds, not just Kähler manifolds.

Proof of the $\partial\overline{\partial}$ lemma. The proof for all three cases is much the same, so we do just the one where $\eta = \overline{\partial}\alpha$ is $\overline{\partial}$-exact. By the Hodge decomposition, we write $\alpha = \beta + \Delta\gamma$, with β harmonic. Since $\Delta = 2\Delta_{\overline{\partial}}$, and β is $\Delta_{\overline{\partial}}$-harmonic if and only if $\overline{\partial}\beta = \overline{\partial}^*\beta = 0$, we have that $\overline{\partial}\beta = 0$. We then compute

$$\begin{aligned} \eta &= \overline{\partial}\alpha \\ &= \overline{\partial}(\beta + \Delta\gamma) \\ &= \overline{\partial}\beta + 2\overline{\partial}(\Delta_{\partial}\gamma) \\ &= 0 + 2\overline{\partial}(\partial\partial^*\gamma + \partial^*\partial\gamma) \\ &= 2\overline{\partial}\partial\partial^*\gamma - 2\partial^*\overline{\partial}\partial\gamma \\ &= -2\partial\overline{\partial}\partial^*\gamma - 2\partial^*\overline{\partial}\partial\gamma \end{aligned}$$

Then since η is ∂-closed, we have that $\partial^*\overline{\partial}\partial\gamma$ must also be ∂-closed. By orthogonality of the image of ∂^* with the kernel of ∂, we have that $\partial^*\overline{\partial}\partial\gamma = 0$, so $\eta = -2\partial\overline{\partial}\partial^*\gamma = 2\overline{\partial}\partial\partial^*\gamma$, so letting $\xi = \partial^*\gamma$, we are done. ■

We now use this to prove Theorem 1.1.

Proof of 1.1. We want to show that all the differentials on the E_1 page are 0, i.e. for a cohomology class $[\alpha] \in H^{p,q}$, $[\partial\alpha] = 0$. Since $[\alpha]$ is a Dolbeault cohomology class, we know that α is $\bar{\partial}$-closed. Therefore, $\partial\alpha$ is both $\bar{\partial}$ and ∂ closed, since ∂ and $\bar{\partial}$ anticommute. Then by the $\partial\bar{\partial}$ lemma, we have that $\partial\alpha = \partial\bar{\partial}\eta$ for some η. Therefore, using the fact that ∂ and $\bar{\partial}$ anticommute one final time, we find that $\partial\alpha$ is $\bar{\partial}$-exact, i.e. $[\partial\alpha] = 0$ ■

The idea of the Frölicher spectral sequence is that we can compute the cohomology of $d = \partial + \bar{\partial}$ in terms of the cohomology of ∂ and $\bar{\partial}$ themselves. However, for general complex manifolds, this is hard, since the differentials and cohomology beyond the E_1 page become much more complicated than the ones on the E_0 and E_1 pages, which are all objects we are familiar with, and are relatively easy to compute with. What Theorem 1.1 tells us is that in the case that X is compact Kähler, this is enough.