DIFFERENTIAL GEOMETRY LECTURE SERIES

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These are notes for the spring 2019 differential geometry lecture series for the math club. The plan is for the course to give an introduction to math club members to the basics and terminology of smooth manifolds. A good reference for additional reading is John Lee's *Introduction to Smooth Manifolds*, and these notes will follow that text somewhat closely. In terms of prerequisites, it would be good to have a background in point set topology, multivariable calculus, and linear algebra.

Contents

0. Preliminaries

These are some preliminary details regarding multivariable calculus. Don't worry if you're not 100% familiar with them, but we will build upon these concepts when we study manifolds. Since the goal of differential geometry is to use calculus to study nonlinear spaces via linear approximation, it's good to have a foundation in calculus in the standard setting.

Recall that given a differentiable function $F : \mathbb{R}^n \to \mathbb{R}^m$, where $F = (F^1, \ldots, F^m)$ for functions $F^i : \mathbb{R}^n \to \mathbb{R}$, its derivative is given by it's ***Jacobian matrix*** DF, which is the matrix

$$DF = \begin{pmatrix} \frac{\partial F^1}{\partial x^1} & \cdots & \frac{\partial F^1}{\partial x^n} \\ \vdots & \vdots & \vdots \\ \frac{\partial F^m}{\partial x^1} & \cdots & \frac{\partial F^m}{\partial x^n} \end{pmatrix}$$

We denote the Jacobian evaluated at a point p to be DF_p. Given maps

$$\mathbb{R}^k \xrightarrow{F} \mathbb{R}^n \xrightarrow{G} \mathbb{R}^m$$

We have that $D(G \circ F)_p = DG_{F(p)} \circ DF_p$. This is the ***multivariable chain rule***. Concisely, "the differential of the composition is the composition of the differentials."

Definition 0.1. A map $F : U \to V$ where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ is ***smooth*** if all of its component functions are infinitely differentiable, i.e. the partial derivatives of all ordersexist and are continuous.

Note that smoothness is a local condition – to know a function is smooth at a point, it suffices to check in a small neighborhood. A function is smooth if and only if it is smooth at every point.

Definition 0.2. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open subsets. A map $F : U \to V$ is called a ***diffeomorphism*** if F is smooth, bijective, and admits a smooth inverse $F^{-1} : V \to U$.

Note that a smooth bijection is not sufficient for a map to be a diffeomorphism. The map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is a smooth bijection, but it's inverse $x \mapsto x^{1/3}$ is not smooth at 0, so f is not a diffeomorphism. Since smoothness is a local property, we also have the notion of a local diffeomorphism.

Definition 0.3. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$. A smooth function $F : U \to V$ is a ***local diffeomorphism*** if for every $p \in U$, there exists an open neighborhood $U_p \subset U$ such that $F|_{U_p}$ is a diffeomorphism onto its image.

It is clear that every diffeomorphism is a local diffeomorphism, but the converse is not true, since local diffeomorphisms can fail to be bijective. An example of such a map is the polar transformation $p : \mathbb{R}^2 - \{0\} \to \mathbb{R}^2 - \{0\}$ given by $p(r, \theta) = (r\cos\theta, r\sin\theta)$. An important theorem tells us that the Jacobian matrix captures all the local information we need.

Theorem 0.4 (***The Inverse Function Theorem***)**.** *Let $F : U \to V$ be smooth. Then F is a local diffeomorphism at p if and only if dF_p is an isomorphism.*

Finally, there's a widespread convention in differential geometry regarding summations, called the ***Einstein summation convention***. We can compactly represent the summation $\sum_i v^i e_i$ as just $v^i e_i$. The rule is that if you see an upper index and a lower index, there is an implicit summation. For example, matrix multiplication in this notation is given by $(AB)^i_j = A^i_k B^k_j$. Where the upper index denotes the row, and the lower index denotes the column, and the implicit summation is over the dummy index k. This convention becomes useful when there are a lot of indices floating around, which is a common occurence in differential geometry.

1. Introduction to Manifolds

Definition 1.1. A ***topological manifold*** is a Hausdorff space X such there exists a countable open cover $\{U_\alpha\}$ of X, along with homeomorphisms $\varphi_\alpha : U_\alpha \to V_\alpha$, where V_α is an open subset of $\mathbb{R}^n$. Given a chart (U, φ), we can write φ in terms of its component functions

$$\varphi(p) = (x^1(p), \dots, x^n(p))$$

The functions x^i are called ***local coordinates*** on U.

In this way, we see that a manifold is a topological space that is locally topologically indistinguishable from Euclidean space. Both modifiers are important here – there could be global and geometric properties that differ from $\mathbb{R}^n$.

Example 1.2.

(1) $\mathbb{R}^n$ is a topological manifold – it admits a global chart $(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})$.
(2) The 2-sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ is a topological manifold. If we let N and S denote the North and South poles repsectively, then we can construct charts using ***stereographic projection***. Stereographic projection from the North pole is a map $\varphi_N : S^2 - \{N\} \to \mathbb{R}^n$, where given a point $p \in S^2 - \{N\}$, we take the line cotaining both N and p, which intersects the $z = 0$ plane at one point q. We then define $\varphi_N(p) = q$. Stereographic projection φ_S from the South pole is defined in an analogous manner. An explicit formula for φ_N is

$$\varphi_N(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

Exercise 1.3.

(1) S^2 does not admit a global chart. Why?
(2) Give an inverse map $\mathbb{R}^n \to \mathbb{S}^n - \{N\}$ for stereographic projection from the North Pole
(3) Generalize stereographic projection to the n-sphere $S^n = \{p \in \mathbb{R}^{n+1} : \|p\| = 1\}$

One of the main appeals of Euclidean space is that we have a lot of tools at our disposal, like linear algebra and calculus. We would like to generalize these notions to manifolds. One of the wonderful properties of manifolds is that they locally look like $\mathbb{R}^n$, so we can use the charts to translate concepts we know in $\mathbb{R}^n$ to concepts on the manifold. We know what it means for a function on $\mathbb{R}^n$ to be smooth, how do we translate this to a manifold X? Given a map $F : X \to \mathbb{R}^m$, what does it mean for F to be smooth? Our first guess is to use our charts. Given a map $F : X \to \mathbb{R}^m$, we want to say that it is smooth if for every $p \in X$, there exists a chart (U_p, φ_p) such that the composition $F \circ \varphi_p^{-1} : U_p \to \mathbb{R}^m$ is smooth. The intuition here is correct, there there are some technicalities that need to be addressed. Namely, suppose $p \in X$ lies in the domain of two charts (U_1, φ_1), and (U_2, φ_2). Then what if $F \circ \varphi_1^{-1}$ was smooth, but $F \circ \varphi_2^{-1}$ wasn't? This suggests that we need a compatibility condition for our charts if we want a notion of smoothness. We say two charts (U_1, φ_1) and (U_2, φ_2) are ***smoothly compatible*** if both $\varphi_1 \circ \varphi_2^{-1}$ and $\varphi_2 \circ \varphi_1^{-1}$ are smooth maps.

Definition 1.4. A collection $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ of smoothly compatible charts such that the U_α cover X is called an ***atlas***. An atlas is ***maximal*** if it is not property contained in any other atlas.

Definition 1.5. A ***smooth manifold*** is a topological manifold X equipped with a maximal atlast $\mathcal{A}$.

Now if we have a smooth manifold X, we have a notion of smooth maps $X \to \mathbb{R}^n$

Definition 1.6. A map $F : X \to \mathbb{R}^k$ is ***smooth*** if for every chart $(U_\alpha, \varphi_\alpha)$, we have that $F \circ \varphi_\alpha^{-1}$ is smooth in the Euclidean sense. We call $F \circ \varphi_\alpha^{-1}$ a local ***coordinate representation*** of the map F.

We also know what if means for maps between manifolds to be smooth.

Definition 1.7. A map $F : X \to Y$ of smooth manifolds is ***smooth*** if for every p, there exists charts (U, φ) and (V, ψ) of p and $F(p)$ respectively, such that (after appropriate shrinking of U and V), we have that $\psi \circ F \circ \varphi^{-1}$ is smooth in the Euclidean sense.

For convenience, we will make the a distinction between the words map and function, which is also common in the literature. A *map* will denote an arbitrary mapping $X \to Y$, where X and Y can denote any kind of set. A *function* on a space X is a mapping $X \to \mathbb{R}$. We let $C^\infty(X)$ denote the vector space of functions of X, which forms a commutative ring under pointwise multiplication and addition.

This is nice, but we've excluded a huge class of spaces. For example, the unit interval isn't a manifold, since there exist no charts about the endpoints satisfying our definition (why?). To include the spaces, we need another definition. Let $\mathbb{H}^n$ denote ***Euclidean half space***, where

$$\mathbb{H}^n = \left\{(x^1, \dots x^n) \in \mathbb{R}^n \ : \ x^n \geq 0\right\}$$

Definition 1.8. A ***manifold with boundary*** is a Hausdorff space X such that there exists a countable cover by charts $(U_\alpha, \varphi_\alpha)$ where the φ_α are homeomorphisms to open sets in $\mathbb{H}^n$. Given a point $p \in X$, we say that p is in the ***boundary*** of X if there exists a chart (U, φ) containing p such that $\varphi(p) = (x^1, \dots, x^n)$, and $x^n = 0$. We say that p is in the ***interior*** of X if no such chart exists. We denote the boundary and interior of X as ∂X and $\operatorname{Int} X$ respectively.

Under this definition, $\mathbb{H}^n$ is a smooth manifold with boundary, with a global chart given by $(\mathbb{H}^n, \mathrm{id}_{\mathbb{H}^n})$, and $\partial\mathbb{H}^n = \left\{(x^1, \dots x^n) \in \mathbb{H}^n \ : \ x^n = 0\right\}$

Exercise 1.9. Let X be a smooth n-manifold with boundary

(1) Show that ∂X is well defined. Namely, show that if we have a point $p \in X$ such that there exists a chart (U, φ) where $\varphi_1(p) \in \partial\mathbb{H}^n$, show that any other chart (U', φ') for p must also have $\varphi'(p) \in \partial\mathbb{H}$. (Hint: Use the inverse function theorem to show there exists no diffeomorphism from an open set in $\mathbb{R}^n$ to an open set in $\mathbb{R}^{n-1}$.)
(2) Prove that ∂X is a smooth $(n-1)$-manifold without boundary.

2. Tangent Spaces and the Tangent Bundle

Recall from multivariable calculus that the derivative is the best linear approximation to a map. Because of this, we need to introduce some sort of linear structure on our smooth manifold in order to make sense of derivatives of maps between manifolds. In the case where our manifold M is embedded in $\mathbb{R}^n$, this is easy, since we can take a chart (U, φ), and treat is as a map $\mathbb{R}^k \to \mathbb{R}^n$. Then the image of derivative $d\varphi_p^{-1} : \mathbb{R}^k \to \mathbb{R}^n$ is the best linear approximation to our manifold M at the point $p \in M \subset \mathbb{R}^n$. This is nice, but is not an intrinsic definition, since it relies on our manifold lying in some ambient Euclidean space. While this is always possible thanks to the Whitney Embedding theorem, there's a more natural way to talk about tangent vectors to an abstract manifold, that need not be embedded in $\mathbb{R}^n$.

We'll provide some motivation in the Euclidean case first. We naturally think of tangent vectors at a point $p \in \mathbb{R}^n$ as another copy of $\mathbb{R}^n$ based at p. However, there is a different way to think about tangent vectors. Let e_i denote the standard basis for $\mathbb{R}^n$, and let $v \in \mathbb{R}^n$. Then there exists a linear differential operator D_v on $\mathbb{C}^\infty(\mathbb{R}^n)$ that takes a function $f : \mathbb{R}^n \to \mathbb{R}$, and takes its directional derivative in the direction of v. If we write $v = v^i e_i$, then

$$D_v|_p = v^i \frac{\partial}{\partial x^i}\bigg|_p$$

We also note that D_v satisfies to product rule (also referred to as the Leibniz rule), i.e. for $f, g \in C^\infty(\mathbb{R}^n)$, we have that

$$D_v|_p(fg) = f(p)D_v g + g(p)D_v f$$

Definition 2.1. A linear map $D : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ satisfying the product rule at $p \in \mathbb{R}^n$, i.e.

$$D(fg) = f(p)Dg + g(p)Df$$

is called a ***derivation at p***

Theorem 2.2. *There is a bijection*

$$\{\textit{Derivations at } p\} \leftrightarrow \{\textit{tangent vectors based at } p\}$$

given by the mapping $v \mapsto D_v$

This theorem tells us that thinking about derivations is equivalent to thinking of tangent vectors based at a point p. In addition, this definition gives more structure to our original idea of tangent vectors, since derivations act on smooth functions, rather than just being some arrow. This definition also generalizes nicely to abstract manifolds, since the only real tools we have as of now are notions of smoothness.

Definition 2.3. Let M be a smooth manifold and $p \in M$. Then the ***tangent space*** at p, denoted T_pM, is the vector space of derivations at p.

Then given a smooth map of manifolds $F : M \to N$, we want the derivative of F at p to be a linear map $dF_p : T_pM \to T_{f(p)}N$. Given some derivation $v \in T_pM$, how should $dF_p(v)$ act on some function $g \in C^\infty(N)$? We don't have too much at our disposal, so the only reasonable choice here is to have

$$dF_p(v)g = v(g \circ F)$$

The derivative is also referred to as the ***pushforward***, and some choose to denote it as F_*.

Now that we have an abstract definition of tangent spaces, we'd like to see how these objects behave in local coordinates. Let M be a smooth n-manifold, and let (U, φ) be a chart for p with $\varphi = (x^1, \dots x^n)$. Then the ***coordinate vectors*** at p for the chart (U, φ) are the derivations $\partial/\partial x^i|_p$. The action of the coordinate vectors is given by

$$\frac{\partial}{\partial x^i} f = \left.\frac{\partial(f \circ \varphi^{-1})}{\partial x^i}\right|_{\varphi(p)}$$

So the coordinate vectors are literally the partial derivative operations on the coordinate representation of a function, justifying the notation. We often abbreviate them as just ∂_i when the choice of coordinates are clear.

Theorem 2.4. *The coordinate vectors $\partial/\partial x^i|_p$ form a basis for T_pM.*

The following exercise should help you build intuition about the derivative, since it will resemble something you might already know. In addition, it's a good exercise to make sure you understand the definition of vectors as derivations and the derivative.

Exercise 2.5. Let M and N be smooth manifolds, and let $F : M \to N$ be a smooth map. Fix $p \in M$ and charts (U, φ) and (V, ψ) for p and $F(p)$ respectively. Show that dF_p is given in local coordinates by the Jacobian matrix of $\psi \circ F \circ \varphi^{-1}$. (Hint: To do this, observe the action of $dF_p(\partial/\partial x^i|_p)$ on an arbitrary smooth function $f \in C^\infty(N)$).

Local coordinates provide an excellent tool for giving concrete descriptions of functions and objects of a manifold, but working in coordinates has a few nuances. If we define some quantity (a function, tensor, etc.) using local coordinates, we need to ensure that our definition still works under a change of coordinates. Otherwise, the quantity we attempted to define is not well-defined at all! Because of this, it's in our best interest to understand how tangent vectors behave under coordinate transformation.

Let (U, φ) and (V, ψ) be two charts about a point $p \in M$, with $\varphi = (x^1, \dots x^n)$ and $\psi = (y^1, \dots, y^n)$. We want to express $\partial/\partial x^i|_p$ in terms of the $\partial/\partial y^i|_p$. To do this, we want to compute the image of $\partial/\partial x^i$ under the derivative $d(\psi \circ \varphi^{-1})_p$. The computation yields

$$\left.\frac{\partial}{\partial x^i}\right|_p = \frac{\partial y^j}{\partial x^i}(\psi(p)) \left.\frac{\partial}{\partial y^j}\right|_p$$

which looks just like the chain rule from multivariable calculus. Likewise, for an arbitrary vector $v = v^i \partial/\partial x^i$, it's componenets $\tilde{v}^i$ in the coordinate system given by the y^i is

$$\tilde{v}^j = v^i \frac{\partial y^j}{\partial x^i}(\psi(p))$$

It's important to note that each point $p \in M$ gets its own tangent space – there's no natural identification between tangent spaces, and no way to transport information between then (at least not yet!). However, we can construct a new object using the technology we have now – the tangent bundle.

Definition 2.6. Let M be a smooth manifold. Then the ***tangent bundle*** of M is the set

$$TM = \coprod_{p \in M} T_pM = \{(p, v) \, : \, p \in M, v \in T_pM\}$$

The tangent bundle comes equipped with a natural map $\pi : TM \to M$ given by $\pi(p, v) = p$ called the ***projection***.

We only said set in the definition, but there's more structure here. As you might expect, TM is a smooth manifold.

Theorem 2.7. *The tangent bundle TM is a smooth manifold, and has a natural smooth structure such that the projection $\pi : TM \to M$ is smooth.*

Proof. For a point $p \in M$, fix a chart (U, φ), where $\varphi = (x^1, \dots, x^n)$. We then use this to construct what will be chart for $\pi^{-1}(U)$, and we will use these charts to define the topology. We know that for each point in U, the vectors ∂_i form a basis for the tangent space at this point, so each pair $(q, v) \in \pi^{-1}(U)$, we can write it uniquely in coordinates as $(x^1(q), \dots x^n(q), v^1, \dots v^n)$, where the v^i are the components of v in terms of the ∂_i. This gives a bijection $\Phi_p : \pi^{-1}(U) \to U \times \mathbb{R}^n$. Doing this for all $p \in M$ gives a cover of TM, and these sets satisfies the conditions for a basis, so we can take the topology generated by these sets. In addition, the transition maps $\Phi_p \circ \Phi_q$ are smooth, so they determine a smooth structure on TM, making a smooth manifold of dimension $2n$. ∎

In addition, given a smooth map $F : M \to N$, this induces a map $dF : TM \to TN$, where in local coordinates, $dF(p, v) = (F(p), dF_p(v))$.

Exercise 2.8. Show that the assignments $M \mapsto TM$ and $F \mapsto dF$ determines a covariant functor from the category of smooth manifolds to itself, called the ***Tangent Functor***. Can you recognize this statement in terms a familiar statement in calculus?

Note that the charts we constructed for TM are of a particular form, namely they are isomorphisms to $U \times \mathbb{R}^n$, so TM locally looks like a product manifold, but globally there may be some twisting. In addition, each fiber $\pi^{-1}(p)$ has the structure of a real vector space. These are defining features of a geometric object called a ***vector bundle***, which are central objects in differential geometry.

3. A Short Introduction to Vector Bundles

Definition 3.1. Let M be a smooth manifold. A ***vector bundle*** over M is another smooth manifold E equipped with a smooth map $\pi : E \to M$ satisfying the following properties.

(1) π is surjective
(2) Each fiber $\pi^{-1}(p)$ has the structure of a real vector space.
(3) For every $p \in M$, there exists a neighborhood U of p and a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that we get the commutative diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^k \arrow[dl, "p"] \\ & U & \end{tikzcd}$$

where p denotes the standard projection $U \times \mathbb{R}^k \to U$. The map Φ is called a ***local trivialization***.

The dimension k of each fiber $\pi^{-1}(p)$ is called the ***rank*** of the vector bundle. We may just call E the vector bundle, and the fiber $\pi^{-1}(p)$ as E_p where the map π is implicit.

Example 3.2.

(1) Given any manifold M and a vector space V, the product manifold $M \times V$ equppied with the projection $M \times V \to M$ is a vector bundle, called the ***trivial bundle***.
(2) The tangent bundle TM of a smooth n-manifold M is a vector bundle of rank n.

(3) Consider $\mathbb{RP}^n$, the space of lines in $\mathbb{R}^{n+1}$. A point in $\mathbb{RP}^n$ is a 1-dimensional subspace of $\mathbb{R}^{n+1}$, so we can construct the vector bundle E where the fiber over a point $\ell \in \mathbb{RP}^n$ is the the 1-dimensional subspace ℓ. This is called the ***tautological bundle*** over $\mathbb{RP}^n$, and exists over more general spaces called ***Grassmannians***.

Exercise 3.3. Let M and N be smooth manifolds, $\pi : E \to N$ a vector bundle, and $F : M \to N$ a smooth map. Let

$$F^*E = \left\{ (p, v) \; : \; p \in M, \, v \in \pi^{-1}(f(p)) \right\}$$

Show that F^*E is a vector bundle over M, called the ***pullback bundle***.

Definition 3.4. Let $E \to M$ be a vector bundle. Then a ***local section*** of E is a right inverse $\sigma : U \to E$ for some open subset $U \subset M$, i.e. $\pi \circ \sigma = \mathrm{id}_U$. In the case that $U = M$, we call σ a ***global section***.

For a vector bundle $E \to M$, there is a canonical global section called the ***zero section***, which is the map $p \mapsto (p, 0)$. where 0 denotes the 0 vector in E_p. A lot of the concepts in differential geometry can be expressed in terms of vector bundles and their sections, especially when it comes to vector and tensor fields on manifolds.

Exercise 3.5. Prove that the tangent bundle TS^2 is not isomorphic to a trivial bundle, i.e. there exists no diffeomorphism $TS^2 \to S^2 \times \mathbb{R}^2$ such that the diagram

$$\begin{tikzcd} TS^2 \arrow[rr] \arrow[dr] & & S^2 \times \mathbb{R}^2 \arrow[dl] \\ & S^2 & \end{tikzcd}$$

commutes. (Hint : Use the Hairy Ball Theorem, which states that there does not exist an everywhere nonzero vector field on S^2. What would triviality of TS^2 imply about vector fields on S^2?)

4. Multilinear Algebra

Tensors fields will play an important role in thr study of manifolds, so it's important to see the linear model first. In this way. we can see how to transport knowledge of the linear case to the nonlinear case of manifolds. As it turns out, tensors are the proper way the think about integration on a manifold, which is one of their many uses.

Recall that the ***dual space*** V^* of a vector space V is the vector space of linear maps $V \to \mathbb{R}$. Dual spaces will be the building blocks of multilinear functions The central idea of tensors is *multilinearity*. Let $V_1 \dots V_n$ be vector spaces. Then a function $F : V_1 \times \cdots \times V_n \to \mathbb{R}$ is ***multilinear*** if it is linear in each term with the other terms fixed, i.e.

$$F(v_1, \dots \lambda v_i + \mu v_i', \dots, v_n) = \lambda F(v_1, \dots v_i, \dots v_n) + \mu F(v_1 \dots, v_i', \dots, v_n)$$

Definition 4.1. Let $V_1 \cdots V_k$ be finite dimensional vector spaces. Then the ***tensor product*** of $V_1 \cdots V_k$ is a vector space, denoted $V_1 \otimes \cdots \otimes V_k$ equipped with a multilinear map $V_1 \times \cdots \times V_k \to V \otimes \cdots \otimes V_k$ satisfying the follwing ***universal property*** : Given a multilinear map $\varphi : V_1 \times \cdots \times V_k \to W$ for any vector space W, there exists a unique linear map $\tilde{\varphi} : V_1 \otimes \cdots \otimes V_k$ such that the following diagram

$$\begin{tikzcd}
V_1 \times \cdots \times V_k \arrow[d] \arrow[dr, "\varphi"] & \\
V_1 \otimes \cdots \otimes V_k \arrow[r, "\tilde{\varphi}"'] & W
\end{tikzcd}$$

commutes.

We've given the universal property, but we haven't guaranteed that such a vector space exists.

Theorem 4.2. *Let $V_1, \dots V_k$ be finite dimensional vector spaces, and let $F(V_1, \dots, V_k)$ denote the **free vector space** on $V_1, \dots V_k$, i.e. formal linear combinations of k-tuples $(v_1, \dots, v_k)$ with $v_i \in V_i$. Then let R denote the subspace spanned by elements of the form*

$$(v_1, \dots \lambda v_i, \dots v_k) - \lambda(v_1, \dots v_i, \dots v_k)$$
$$(v_1, \dots, v_i + v_i', \dots v_k) - (v_1, \dots v_i, \dots v_k) - (v_1, \dots v_i', \dots v_k)$$

Let $v_1 \otimes \cdots \otimes v_k$ denote the image of $(v_1, \dots v_k)$ under the quotient projection. Then the vector space $F(V_1, \dots V_k)/R$, equipped with the map $V_1 \times \cdots V_k \to F(V_1, \dots V_k)/R$ given by $(v_1, \dots, v_k) \mapsto v_1 \otimes \cdots \otimes v_k$ satisfies the universal property of the tensor product.

The reason we are about tensor products, is that it gives us the tools to describe multilinear maps.

Theorem 4.3. *Let $L(V_1, \dots, V_k)$ denote the space of multilinear maps $V_1 \times \cdots \times V_k \to \mathbb{R}$. Then there is a canonical isomorphism $V_1^* \otimes \cdots \otimes V_k^* \cong L(V_1 \dots, V_k)$*

Proof. We define $\varphi : V_1 \otimes \cdots \otimes V_k \to L(V_1, \dots, V_k)$ by specifying it's action on the spanning set of elements of the form $v^1 \otimes \cdots \otimes v^k$ and extending linearly to linear combinations.

$$\varphi(v^1 \otimes \cdots \otimes v^k)(w_1, \dots w_k) = \prod_i v^i(w_i)$$

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Exercise 4.4. Prove the map φ is an isomorphism.

Definition 4.5. A ***covariant*** k-tensor on a vector space V is an element of $\underbrace{V^* \otimes \dots \otimes V^*}_{k \text{ times}}$. A ***contravariant*** k tensor is an element of $\underbrace{V \otimes \dots \otimes V}_{k \text{ times}}$. A (k, ℓ)-tensor is an element of $\underbrace{V^* \otimes \dots \otimes V^*}_{k \text{ times}} \otimes \underbrace{V \otimes \dots \otimes V}_{\ell \text{ times}}$. We denote the vector space of covariant, contravariant, and mixed tensors as $T^k(V)$, $T_k(V)$, and $T^k_\ell(V)$ respectively.

For the most part, we will be focused on covaraint tensors, though contravariant and mixed tensors do show up as well in differential geometry. An important property of covariant tensors is that they *pull back* (which admittedly makes their name quite confusing – the name is a relic from older times). What we mean by that is that given vector spaces V and W, a linear map $\varphi : V \to W$ induces a map $\varphi^* : T^k(W) \to T^k(V)$, where a k-tensor ω is mapped to the tensor $\varphi^*\omega$ defined by

$$\varphi^*\omega(v_1, \dots, v_k) = \omega(\varphi(v_1), \dots \varphi(v_k))$$

5. Cotangent Spaces and Covectors

Now that we've defined tensors on vector spaces, we want to transport this concept fo the nonlinear case of manifolds.

Definition 5.1. Let M be a smooth manifold, and $p \in M$. Then the ***cotangent space*** at p is the dual space to T_pM, and is denoted T_p^*M. The ***cotangent bundle*** is the vector bundle $T^*M = \coprod_{p \in M} T_p^*M$.

Definition 5.2. A ***covector field*** is a global section of T^*M, i.e. for every $p \in M$, we assign it a covector $\omega_p \in T_p^*M$. We denote the vector space of covector fields as $\mathfrak{X}^*(M)$. Given a covector field ω, we often denote it's value at $p \in M$ as ω_p.

In particular, there's an easy way to obtain covector fields. Given a smooth function $f \in C^\infty(M)$, define the ***differential*** of f, denoted df, to be the covector field $df_p(v) = vf$, where $c \in T_pM$. We note that this is the same notation for the derivative, but in the case for a smooth real-valued function, the two definitions essentially coincide. In coordinates, we have that, from the definition, $df(\partial_i) = \partial_i f$, so the components of df in terms of the dual basis ξ^i to the ∂_i are the functions $\partial_i f$. In particular, we can compute the differentials for the coordinate functions x^i. We have that

$$dx^j = \partial_i x^j \xi^j = \delta_i^j \xi^i = \xi^j$$

Therefore, the differentials dx^i are the dual basis to the ∂_i. We can then write the coordinate formula for df in a more suggestive notation $df = \partial_i f dx^i$, which looks like a gradient.

As with tangent vectors, we would like to know how the covectors dx^i transform under a change of coordinates. Let $(y^1, \dots y^n)$ denote another set of coordinate functions, and let $\omega = \omega_i dx^i$ be some covector. Then using the transformation law for the tangent vectors $\partial/\partial x^i$, we compute

$$\omega_i = \omega\left(\frac{\partial}{\partial x^i}\right) = \omega\left(\frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}\right) = \frac{\partial y^j}{\partial x^i}\tilde{\omega}^j$$

where $\tilde{\omega}^j$ denotes the component functions of ω with respect to the basis dy^i.

As it turns out, covector fields are the natural things to integrate along curves, and will form the basis for the discussion on integration on manifolds. The main reason for this is that covector fields pull back, which means that they play nicely with coordinate charts.

Definition 5.3. Let $F : M \to N$ be a smooth map of manifolds, and $\omega \in \mathfrak{X}^*(N)$. Define the ***pullback*** of ω along F to be the covector field $F^*\omega \in \mathfrak{X}^*(M)$ defined by $(F^*\omega)_p = dF_p^*\omega_p$, i.e. the point-wise pullback

$$dF_p^*\omega_p(v) = \omega_{F(p)}(dF_p(v))$$

for $v \in T_pM$.

Note that this is defined for *any* smooth map $M \to N$, not just for diffeomorphisms like in the case of vector fields.

Example 5.4 (Computing pullbacks)**.** Computing pullbacks of covector fields when the domain and codomain are Euclidean space is relatively easy, and agrees with the calculus idea of u-substitution. For example, let $F : \mathbb{R}^2 \to \mathbb{R}^2$ where $F(s,t) = (st, e^t)$ and $\omega = xdy - ydx$ (Exercise 11.8 in Lee). Then to compute the pullback $F^*\omega$, we substitute st for x and e^t for y, i.e.

$$\begin{aligned} F^*\omega &= std(e^t) - e^t d(st) \\ &= st(e^t dt) - e^t(tds + sdt) \\ &= (ste^t - se^t)dt - te^t ds \end{aligned}$$

Definition 5.5. Let M be a smooth manifold. Then a ***smooth curve*** is a smooth map $\gamma : [0,1] \to M$.

The key point here is that integration on the interval $[0,1]$ makes sense to us, so given a covector field ω, we want to define its integral in terms of what we already know. Let ω be a covector field on $[0,1]$, with global coordinate t. Then we know $\omega = f dt$ for some smooth function f, and we can define the integral to be our usual notion of integration

$$\int_{[0,1]} \omega = \int_0^1 f dt$$

Definition 5.6. Let $\gamma : [0,1] \to M$ be a smooth curve, and $\omega \in \mathfrak{X}^*(M)$. Define the ***integral*** of ω along γ to be

$$\int_\gamma \omega = \int_{[0,1]} \gamma^* \omega$$

Example 5.7 (Exercise 11-14 in Lee)**.** Let ω be the covector field on $\mathbb{R}^3$ defined by

$$\omega = -\frac{4z}{(x^2+1)} dx + \frac{2y}{y^2+1} dy + \frac{2x}{{}^2+1} dz$$

and let S denote the line segment from $(0,0,0)$ to $(1,1,1)$. To compute the line integral $\int_S \omega$, we parameterize S with $\gamma(t) = (t,t,t)$, and compute the pullback to be

$$\begin{aligned} \gamma^*\omega &= -\frac{4t}{(t^2+1)} dt + \frac{2t}{t^2+1} dt + \frac{2t}{{}^2+1} dt \\ &= \left(\frac{4t}{t^2+1} - \frac{4t}{(t^2+1)^2} \right) dt \end{aligned}$$

which gives us that

$$\int_S \omega = \int_0^1 \left(\frac{4t}{t^2+1} - \frac{4t}{(t^2+1)^2} \right) dt$$

Exercise 5.8. Define the covector field ω on $\mathbb{R}^2 - \{0\}$ by

$$\omega = \frac{x dy - y dx}{x^2 + y^2}$$

compute the integral of ω along the unit circle. (Note that the integral along the unit circle is the same as the integral along the unit circle missing a point, which can be parameterized with the curve $\gamma(t) = (\cos t, \sin t)$)

6. Differential Forms and Integration on Manifolds

The central idea behind integration is *volume*. If you think about your calculus classes (especially vector calculus), you recall that the things you integrate tend to be infinitesimal areas or volumes, and you might have thought of the formal symbols dA and $dx\, dy$ as symbols representing tiny squares or parallelogram, and you probably think of dt as an infinitesimal line segment. Therefore, it's in our best interest to understand a general framework for volume. You've probablty encountered the determinant of a matrix before, and you might have seen that it computes the volume of the parallelopiped spanned by the columns of the matrix. This will be the prototypical "volume function" we will work with. There are several properties of determinants that make them ideal "volume functions," and there are some properties we should keep in mind.

Firstly, the determinant is multilinear, much like how the volumes of parelleopipieds change when we add or scale vectors. Secondly, the determinant evaluates to 0 on any linearly dependent set of vectors, which should be intuitive, since we are missing a dimension for our parallelopiped. A consequence of this is that the determinant is *alternating* – it switches sign if we swap two columns of the matrix. This hints that we should really be thinking about *signed* volume. Therefore, the class of volume functions we're looking for should be multiliear alternating tensors on the vector space.

Definition 6.1. A tensor $\omega \in T^k(V)$ is ***alternating*** if it changs sign whenever two arguments are interchaged, i.e. for vectors $v_1, \ldots v_k$, we have that for all distinct i, j,

$$\omega(v_1, \ldots, v_i, \ldots v_j, \ldots v_k) = -\omega(v_1, \ldots v_j, \ldots v_i, \ldots v_k)$$

Another way to define this is using the sign of a permutation. Given an permutation $\sigma \in S_k$, the ***signature*** of σ is the number of transpositions needed to generated σ, and is denoted $\operatorname{sgn} \sigma$. Then let $(-1)^\sigma$ denoted $(-1)^{\operatorname{sgn} \sigma}$. An equivalent defintion for ω to be alternating is that

$$\omega(v_1, \ldots v_k) = (-1)^\sigma \omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

We denote the space of all alternating covariant k-tensors as $\Lambda^k(V^*)$.

One wonderful fact about the alternating tensors is that the form an algebra under a product called the ***wedge product***, and is denoted $\wedge$. There's a few gory details to work out, which I will defer to Lee. We're mainly concerened with their properties, which you can take as axioms. If you want to read more into the construction of the wedge product, chapter 14 in Lee should have everything you need. We'll first lay out the rules for the wedge product.

(1) The wedge product $\wedge$ is bilinear, i.e.

$$\omega \wedge (\eta + \xi) = \omega \wedge \eta + \omega \wedge \xi$$

(2) The wedge product is associative, i.e. $\omega \wedge (\eta \wedge \xi) = (\omega \wedge \eta) \wedge \xi)$.
(3) For $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^\ell(V^*)$, we have that $\omega \wedge \eta \in \Lambda^{k+\ell}(V^*)$
(4) The wedge product is anticommutative, (sometimew called supercommutative if you are familiar with superalgebras) i.e. for $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^\ell(V^*)$, we have that $\omega \wedge \eta = (-1)^{k\ell} \eta \wedge \omega$. In particular, note that for $k = \ell = 1$, this implies that $\omega \wedge \omega = 0$.

Under this product, the vector space $\bigoplus_k \Lambda^k(V)$ forms an algebra over $\mathbb{R}$, called the ***exterior algebra***. In particular, this algebra is $\mathbb{Z}$-graded (if you know what that means).

The next thing we need to know is that given a k-form ω and an ℓ-form η, how does $\omega \wedge \eta$ act? We start in the simple case. Let ω and η both be 1-forms, i.e. covectors. Then

$$(\omega \wedge \eta)(v_1, v_2) = \omega(v_1)\eta(v_2) - \omega(v_2)\eta(v_1) = \det \begin{pmatrix} \omega(v_1) & \omega(v_2) \\ \eta(v_1) & \eta(v_2) \end{pmatrix}$$

In particular, given collection $\{\omega^i\}$ of covectors, their k-fold wedge product acts on k-vectors by

$$\omega^1 \wedge \ldots \wedge \omega^k(v_1, \ldots, v_k) = \det \omega^i(v_j)$$

If we fix a basis $\{e_i\}$ for V and let $\{\varepsilon^i\}$ denote the dual basis, then you can take for a fact that the set

$$\left\{\varepsilon^{i_1} \wedge \ldots \wedge \varepsilon^{i_k} \ : \ 1 \leq i_1 < i_2 \ldots < i_k \leq \dim V\right\}$$

is a basis for $\Lambda^k(V)$, which implies in particular that the dimension is $\binom{\dim V}{k}$.

Exercise 6.2. Show that if a set $\omega^1 \ldots \omega^k$ of covectors is linearly dependent, then $\omega^1 \wedge \ldots \wedge \omega^k = 0$.

We now move to the world of manifolds. First, we make a quick remark regarding vector bundles. We have a lot of tools for constructing new vector spaces from old (e.g. direct sum, tensor product, exterior algebra), and most of these constructions carry over to vector bundles, where we do the linear algebraic operations fiberwise. For example, for vector bundles $E \to M$ and $F \to M$, we can form the ***Whitney sum*** $E \oplus F$ by taking the fiberwise direct sum, i.e. $(E \oplus F)_p = E_p \oplus F_p$. There's some care that the fiberwise operations occur smoothly, which is made rigorous with the notion of a ***smooth functor***. For more details, you can check out Exercise 10-8 in Lee. For now, just assume that taking the exterior powers $\Lambda^k(V)$ forms a smooth functor, so we can in particular do this to the cotangent bundle of a manifold M, giving us vector bundles $\Lambda^k(T^*M) = \coprod_{p \in M} \Lambda^k(T_p^*M)$ with the obvious projections.

Definition 6.3. A section of $\Lambda^k(T^*M)$ is a ***differential k-form***. The space of k-forms is denoted $\Omega^k(M)$. The wedge product $\omega \wedge \eta$ for for $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$ is the $k + \ell$ form where $(\omega \wedge \eta)_p = \omega_p \wedge \eta_p$. Note that for local coordinates (x^i), any k-form can be written locally in the form $\omega = \sum_I \omega_I dx^{i_1} \wedge \ldots \wedge dx^{i_k}$, where the sums is over all increasing sequences $I = (i_1, \ldots i_k)$ with $1 \leq i_1 < \ldots < i_k \leq \dim M$.

Like how covector fields (1-forms) were integrated over curves, k-forms are objects meant to be integrated over k-dimensional submanifolds. Like with $1-$forms, we mainly do this via pullback, so it's important to know how to do the computations.

Proposition 6.4. *Let $F : M \to N$ be a smooth map.*

(1) $F^* : \Omega^k(N) \to \Omega^k(M)$ is linear

(2) F^* commutes with the wedge product, i.e.

$$F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$$

(3) Summing over all increasing sequences of numbers $1 \leq i_1 < \ldots i_k \leq \dim M$

$$F^* \sum_I \omega_I dy^{i_1} \wedge \ldots \wedge dy^{i_k} = \sum_I (\omega_I \circ F) d(y^{i_1} \circ F) \wedge \ldots \wedge d(y^{i_k} \circ F)$$