THE LAPLACE-DE RAHM OPERATOR ON A RIEMANNIAN MANIFOLD

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In $\mathbb{R}^2$, we know about the standard Laplace operator on $C^\infty(\mathbb{R}^2)$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

In a more general setting, let (M, g) be a Riemannian manifold. We can define an analogous operator

$$\Delta = \operatorname{div}(\operatorname{grad} f)$$

In local coordinates (x^i), we have that for $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$

$$\operatorname{grad} f = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}$$

$$\operatorname{div} X = \frac{1}{\sqrt{\det g_{ij}}} \frac{\partial}{\partial x^i} \left((X^i \sqrt{\det g_{ij}} \right)$$

Where g_{ij} is the symmetric matrix given by $g_{ij} = \langle \partial_i, \partial_j \rangle$ and g^{ij} is the inverse of g_{ij}. This gives the coordinate formula for

$$\Delta f = \frac{1}{\sqrt{g_{ij}}} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{\det g_{ij}} \frac{\partial f}{\partial x^j} \right)$$

Which using the standard metric $g_{ij} = \delta_{ij}$ on $\mathbb{R}^2$ recovers the standard Laplacian. However, we want to generalize Δ to arbitrary differential forms, which requires us to construct a bit of machinery.

To do this, we first note that the metric g determines an inner product on each tangent space T_pM where $\langle v, w \rangle = g_p(v, w)$. From this, we can construct an inner product on the alternating tensors $\Lambda^k(T_pM)$, which will give us a smoothly varying inner product on $\Omega^K(M)$. To do this, we will use the fact that g determines a bundle isomorphism $TM \to T^*M$ via the mapping $(x, v) \mapsto (x, \langle v, \cdot \rangle)$.

Proposition 1.1. *For a Riemannian manifold (M, g), there is a unique inner product on each $\Lambda^k(T_pM)$ characterized by the formula*

$$\langle \omega^1 \wedge \ldots \wedge w^k, \eta^1 \wedge \ldots \wedge \eta^k = \det \left(\langle (\omega^i)^\sharp, (\eta^j)^\sharp \right) \rangle$$

Where $\sharp$ is the index raising operator $\omega_i dx^i \mapsto g^{ij} \omega_j \frac{\partial}{\partial x^i}$.