Multilinear Algebra

General theme – Do the linear algebra, then transfer the concepts to manifolds by doing it pointwise on tangent spaces.

Recall: Given a vector space V, the dual space V^* is $V^* = \{\text{linear maps } V \to \mathbb{R}\}$

Given a basis $\{e_i\}$ for V, $\{e^i\}$ is a basis for V^* where $e^i(e_j) = \delta_{ij}$

Def: Let V_i, W be vector spaces. A map $F: V_1 \times \dots \times V_k \to W$ is multilinear if

$$F(v_1 \dots \lambda v_i + \mu v_i' \dots v_k) \qquad \lambda, \mu \in \mathbb{R}$$

$$= \lambda F(v_1 \dots v_i \dots v_k) + \mu F(v_1 \dots v_i' \dots v_k)$$

Def: Let $V_1 \dots V_k$ be vector spaces. The tensor product of $V_1 \dots V_k$ is a vector space $V_1 \otimes \dots \otimes V_k$ equipped with a multilinear map $p: V_1 \times \dots \times V_k \to V_1 \otimes \dots \otimes V_k$ such that for any multilinear map $\varphi: V_1 \times \dots \times V_k \to W$, there exists a unique $\tilde{\varphi}: V_1 \otimes \dots \otimes V_k \to W$ s.t.

$$\begin{tikzcd} V_1 \times \dots \times V_k \arrow[d, "p"'] \arrow[dr, "\varphi"] & \\ V_1 \otimes \dots \otimes V_k \arrow[r, "\tilde{\varphi}"'] & W \end{tikzcd}$$ commutes.

In other words, linear maps out of the tensor product are <u>equivalent data</u> to multilinear maps out of $V_1 \times \dots \times V_k$

This property uniquely characterizes the tensor product up to unique isomorphism, but we haven't guaranteed it exists.

<u>Thm:</u> The tensor product exists.

<u>Proof</u> For vector spaces $V_1 \dots V_k$, let $F(V_1 \dots V_k)$ be the free vector space on $V_1 \dots V_k$, i.e. formal linear combinations of elements of $V_1 \times \dots \times V_k$. Then

Let R be the subspace spanned by elts of the form

$$(v_1 \dots \lambda v_i \dots v_k) - \lambda(v_1 \dots \dots v_k)$$
$$(v_1 \dots v_i + v_i' \dots v_k) - (v_1 \dots v_i \dots v_k) - (v_1 \dots v_i' \dots v_k)$$

Then $F(V_1 \dots V_k)/R$ w/ the quotient map satisfies the universal property of the tensor product. We denote the image of $(v_1 \ldots v_k)$ under the quotient map by $v_1 \otimes \ldots \otimes v_k$

Note - not all elements are of this form e.g. $e_1 \otimes e_2 + e_3 \otimes e_4$

For vector spaces V, W with bases $\{e_i\}$ and $\{f_j\}$, a basis for $V \otimes W$ is

$\{e_i \otimes f_j\}$. In particular,

$$\dim(V \otimes W) = (\dim V)(\dim W).$$

Thm

There is a canonical isomorphism

$$V_1^* \otimes \ldots \otimes V_k^* \longleftrightarrow \{\text{Multilinear maps } V_1 \times \ldots \times V_k \to \mathbb{R}\}$$

The action of $v^1 \otimes \ldots \otimes v^k$ is

$$v^1 \otimes \ldots v^k(w_1, \ldots w_k) = \prod_i v^i(w_i)$$

Def: Let V be a vector space. A covariant k-tensor is an element of $\underbrace{V^* \otimes \ldots \otimes V^*}_{k \text{ times}}$

a contravariant k-tensor is an element of $\underbrace{V \otimes \ldots \otimes V}_{k \text{ times}}$. A mixed tensor of type (k, l)

is an element of $\underbrace{V^* \otimes \cdots \otimes V^*}_{k\text{-times}} \otimes \underbrace{V \otimes \cdots \otimes V}_{l\text{-times}}$.

The spaces of covariant, contravariant, and mixed tensors are denoted $T^k(V)$, $T_k(V)$, $T^k_l(V)$ respectively.

We're mostly concerned with covariant tensors for our discussion - a key property they have is that they pullback (which makes the name a bit awkward -- a relic of older times).

~~Given~~ Given a linear map $A: V \to W$, this induces a map $A^*: T^k(W) \to T^k(V)$ where $A^*\omega(v_1 \ldots v_k) = \omega(Av_1 \ldots Av_k)$.

Back to manifolds.

Let M be a smooth manifold, $p \in M$. Then the cotangent space, denoted T_p^*M is the dual space to T_pM. The cotangent bundle is $T^*M = \coprod_{p \in M} T_p^*M$ equipped with the natural projection.

Def: A <u>covector field</u> ω is a global section of T^*M. The space of covector fields is denoted $\Omega^1(M)$. Like with vector fields, we denote its value at p by ω_p

Def: Let $f \in C^\infty(M)$ then the <u>differential</u> df is a covector field s.t. $df_p(v) = vf$

Notation is the same as the derivative... in the case of $f \in C^\infty(M)$, the concepts are essentially the same.

In coordinates x^i, the component functions w.r.t the dual basis ξ^i to ∂_i are determined by $df(\partial_i) = \partial_i f$

$\Rightarrow$ in coordinates $df = \partial_i f \, \xi^i$

In particular, we compute

$$dx^j = \partial_i x^j \xi^i = \delta_{ij} \xi^i = \xi^j$$

So the dx^i are the dual basis to the ∂_i

How do they transform?

Let $\{x^i\}$ $\{y^i\}$ be coordinate functions and $\omega \in \Omega^1(M)$ where $\omega = \omega_i dx^i$

Then $$\omega_i = \omega\left(\frac{\partial}{\partial x^i}\right) = \omega\left(\frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}\right)$$

$$= \frac{\partial y^j}{\partial x^i}\tilde{\omega}_j$$

where $\tilde{\omega}_j$ are the component functions of ω in the y^i coordinate system.

Covector fields are the natural objects to integrate along curves. This is because they pullback.

<u>Def</u>: Let $F: M \to N$, and $\omega \in \Omega^1(N)$. Then the <u>pullback</u> of ω is a covector field $F^*\omega \in \Omega^1(M)$ where

$$(F^*\omega)_p = dF_p^*\omega_{F(p)}$$

You've been computing pullbacks for a long time under a different name — u-substitution.

Example

$F: \mathbb{R}^2 \to \mathbb{R}^2 \quad F(s,t) = (st, e^t)$

$\omega = x\,dy - y\,dx$

$$F^*\omega = st\,d(e^t) - e^t d(st)$$
$$= st(e^t dt) - e^t(s\,dt + t\,ds)$$
$$= (ste^t - s)\,dt - te^t ds$$

Def: A smooth curve is a smooth map $\gamma: [0,1] \to M$.

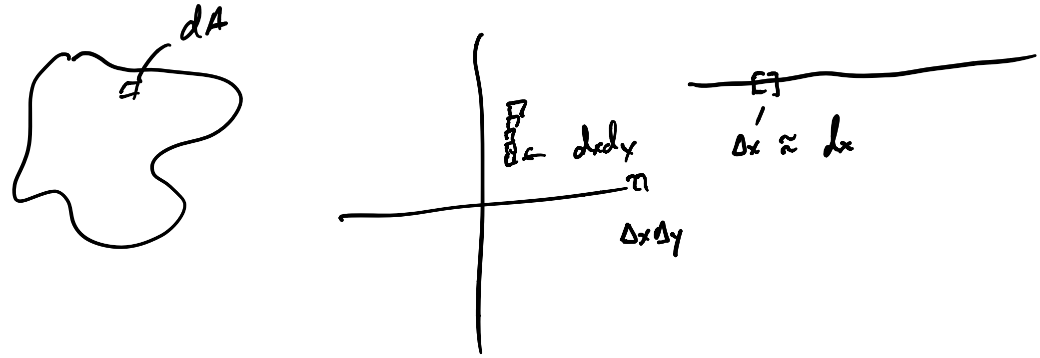
We know how to integrate along $[0,1]$, any covector field $\omega \in \Omega^1([0,1])$ is globally

$f\,dt$ for f a smooth function $[0,1] \to \mathbb{R}$

$$\int_{[0,1]} f\,dt = \int_0^1 f\,dt$$

Differential Forms

To integrate, we need a notion of volume



Recall that for a matrix A, $\det A$ is the signed volume of the parallelopiped spanned by its columns. In addition, if the columns are linearly dependent, $\det A = 0$. The determinant will be our model for our signed area functions.

Def: A covariant tensor ω is alternating if it changes sign whenever two arguments are exchanged. The space of alternating k-tensors is denoted $\Lambda^k V^*$

Alternating tensors come with a product, called the wedge product, denoted $\wedge$

We'll omit the gory details, treat the properties as axioms.

Properties of the wedge product

1) $\wedge$ is bilinear, i.e. $\omega \wedge (\eta + \xi) = \omega\wedge\eta + \omega\wedge\xi$
2) $\wedge$ is associative, i.e. $\omega \wedge (\eta \wedge \xi) = (\omega\wedge\eta)\wedge\xi$
3) For $\omega \in \Lambda^k V^*$ $\eta \in \Lambda^l V^*$, $\omega \wedge \eta \in \Lambda^{k+l} V^*$
4) $\omega \in \Lambda^k V^*$ $\eta \in \Lambda^l V^*$ $\omega\wedge\eta = (-1)^{kl} \eta \wedge \omega$

Under $\wedge$, the vector space $\Lambda^\bullet V^* = \bigoplus_k \Lambda^k V^*$ forms an $\mathbb{R}$-algebra called the <u>exterior algebra</u> of V^*

Given covectors $\omega^1 \dots \omega^k \in V^*$, the alternating tensor
$$\omega^1 \wedge \dots \wedge \omega^k (v_1 \dots v_k) = \det \omega^i(v_j)$$

<u>Fact</u>: Let $\{e_i\}$ be a basis for V, $\{e^i\}$ its dual basis. Then $\{ e_{i_1} \wedge \dots \wedge e_{i_k} : 1 \leq i_1 < i_2 < \dots < i_k \leq n \}$ is a basis for $\Lambda^k V^*$

Back to manifolds.

<u>Rmk</u>: There's a lot of constructions for making new vector spaces from old, e.g direct sum, tensor product, Λ^k, Hom. We can do this

fiberwise on vector bundles – when does this result in another vector bundle? The answer is when we have a <u>smooth functor</u>, but we don't need to worry about it. Just take it as a fact that Λ^k is a smooth functor, so we can construct the vector bundles

$$\Lambda^k T^*M = \coprod_{p\in M} \Lambda^k T_p^* M$$

<u>Def</u> A <u>differential k-form</u> is a smooth global section of $\Lambda^k T^*M$.

We denote the space of k-forms as $\Omega^k(M)$. (We consider 0-forms as smooth functions. $\Omega^0(M) = C^\infty(M)$)

For a k-form ω, an l-form η, the wedge product $\omega\wedge\eta$ is the $k+l$ form where $(\omega\wedge\eta)_p = \omega_p \wedge \eta_p$.

In local coordinates, any k-form ω can be written as $\omega = \sum_I \omega_I dx^{i_1} \wedge \dots \wedge dx^{i_k}$

summing over all increasing index sequences $I = i_1 \dots i_k$

Like 1-forms (covector fields), differential k-forms pullback, so we should know how to compute them.

Let $F: M \to N$ be a smooth map.

Then $F^*: \Omega^k(N) \to \Omega^k(M)$ satisfies:

1) F^* is linear
2) $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$
3) Summing over all increasing index sets

$$I = 1 \leq i_1 < i_2 < \dots < i_k \leq n$$

$$F^*\left(\sum_I \omega_I \, dy^{i_1} \wedge \dots \wedge dy^{i_k}\right)$$

$$= \sum_I (\omega_I \circ F) \, d(y^{i_1} \circ F) \wedge \dots \wedge d(y^{i_k} \circ F)$$

In particular, for a top degree form

$$\omega = u \, dy^1 \wedge \dots \wedge dy^n$$

$$F^*\omega = (u \circ F)(\operatorname{Det} DF) \, dx^1 \wedge \dots \wedge dx^n$$

The change of variables formula from multivariable analysis!

The Exterior Derivative

Let $\omega \in \Omega^k(M)$ $\quad \omega = \sum_I \omega_I \, dx^{i_1} \wedge \dots \wedge dx^{i_k}$

increasing I

The <u>exterior derivative</u> of ω, denoted $d\omega$ is a $k+1$ form given by

$$d\omega = \sum_I \frac{\partial \omega_I}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

Example

Consider the case in $\mathbb{R}^3$

$$C^\infty(\mathbb{R}^3) = \Omega^0(\mathbb{R}^3) \xrightarrow{d} \Omega^1(\mathbb{R}^3)$$

$$f \longmapsto \frac{\partial f}{\partial x^i} dx^i \approx \text{grad } f$$

$$\Omega^1(\mathbb{R}^3) \xrightarrow{\quad d \quad} \Omega^2(\mathbb{R}^3)$$

$$f_1 dx^1 + f_2 dx^2 + f_3 dx^3 \longmapsto \left(\frac{\partial f_1}{\partial x^1} dx^1 \wedge dx^1 + \frac{\partial f_1}{\partial x_2} dx^2 \wedge dx^1 + \frac{\partial f_1}{\partial x^3} dx^3 \wedge dx^1 \right)$$

$+$ (Similar terms for $f_2 dx^2$ and dx^3)

$$= \left(\frac{\partial f_2}{\partial x^1} - \frac{\partial f_1}{\partial x^2} \right) dx^1 \wedge dx^2 + \left(\frac{\partial f_3}{\partial x^2} - \frac{\partial f_2}{\partial x^3} \right) dx^2 \wedge dx^3 + \left(\frac{\partial f_1}{\partial x^3} - \frac{\partial f_3}{\partial x^1} \right) dx^1 \wedge dx^3$$

* Curl

$$\Omega^2(\mathbb{R}^3) \xrightarrow{d} \Omega^3(\mathbb{R}^3)$$

$$\begin{aligned} &f_1\, dx^1 \wedge dx^2 \\ &+ f_2\, dx^2 \wedge dx^3 \\ &+ f_3\, dx^1 \wedge dx^3 \end{aligned} \longmapsto \left(\frac{\partial f_1}{\partial x^1} + \frac{\partial f_2}{\partial x^2} + \frac{\partial f_3}{\partial x^3} \right) dx^1 \wedge dx^2 \wedge dx^3$$

$\overset{??}{\text{div.}}$

One thing to note, $d \circ d = 0$

(curl grad $= 0$, div curl $= 0$)

If $d\eta = 0$, we say η is closed

if $\omega = d\eta$ for some η, ω is exact

$d \circ d = 0 \Rightarrow$ exact forms are closed.

We can then take the quotient

$\{\text{closed}\} / \{\text{exact}\}$ giving us the

de Rham cohomology groups $H^k_{dR}(M)$

Integration

<u>Rmk</u> : We'll assume our manifolds are oriented.

In the Euclidean case, we know how to integrate. For $U \subset \mathbb{R}^n$ open,

$$\omega = f\, dx^1 \wedge \cdots \wedge dx^n,$$

$$\int_U \omega = \int_U f\, dx^1 \cdots dx^n \quad \text{in the sense we know}$$

Recall that pullback by any map $\varphi: V \to U$ gives us a factor of $\det D\varphi$, giving us the coordinate change formula.

Indeed, $\displaystyle\int_V \varphi^* \omega = \int_U \omega$ if φ is a diffeomorphism.

On manifolds, suppose the support of ω is contained in a single coordinate chart (U, φ). Then we can define $\displaystyle\int_M \omega = \int_{\varphi(U)} (\varphi^{-1})^* \omega$

In the general case, we'll need a partition of unity

<u>Thm</u> (Generalized Stokes')

Let M be a smooth manifold with boundary, and let $\omega \in \Omega^{n-1}(M)$. Then

$$\int_{\partial M} \omega = \int_M d\omega$$

In particular, if $\partial M = \emptyset$, the integral of any exact form is 0