

DIFFERENTIAL GEOMETRY LECTURE SERIES

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These are notes for the spring 2019 differential geometry lecture series for the math club. The plan is for the course to give an introduction to math club members to the basics and terminology of smooth manifolds. A good reference if you want to read more is John Lee's *Introduction to Smooth Manifolds*, and these notes will follow that text somewhat closely. In terms of prerequisites, it would be good to have a background in point set topology, multivariable calculus, and linear algebra.

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0. PRELIMINARIES

These are some preliminary details regarding multivariable calculus. Don't worry if you're not 100% familiar with them, but we will build upon these concepts when we study manifolds. Since the goal of differential geometry is to use calculus to study nonlinear spaces via linear approximation, it's good to have a foundation in calculus in the standard setting.

Recall that given a differentiable function $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$, where $F = (F^1, \dots, F^m)$ for functions $F^i : \mathbb{R}^n \rightarrow \mathbb{R}$, its derivative is given by its **Jacobian matrix** DF , which is the matrix

$$DF = \begin{pmatrix} \frac{\partial F^1}{\partial x^1} & \cdots & \frac{\partial F^1}{\partial x^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial x^1} & \cdots & \frac{\partial F^m}{\partial x^n} \end{pmatrix}$$

We denote the Jacobian evaluated at a point p to be DF_p . Given maps

$$\mathbb{R}^k \xrightarrow{F} \mathbb{R}^n \xrightarrow{G} \mathbb{R}^m$$

We have that $D(G \circ F)_p = DG_{F(p)} \circ DF_p$. This is the **multivariable chain rule**. Concisely, "the differential of the composition is the composition of the differentials."

Definition 0.1. A map $F : U \rightarrow V$ where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ is **smooth** if all of its component functions are infinitely differentiable, i.e. the partial derivatives of all order exist and are continuous.

Note that smoothness is a local condition – to know a function is smooth at a point, it suffices to check in a small neighborhood. A function is smooth if and only if it is smooth at every point.

Definition 0.2. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open subsets. A map $F : U \rightarrow V$ is called a **diffeomorphism** if F is smooth, bijective, and admits a smooth inverse $F^{-1} : V \rightarrow U$.

Note that a smooth bijection is not sufficient for a map to be a diffeomorphism. The map $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^3$ is a smooth bijection, but its inverse $x \mapsto x^{1/3}$ is not smooth at 0, so f is not a diffeomorphism. Since smoothness is a local property, we also have the notion of a local diffeomorphism.

Definition 0.3. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$. A smooth function $F : U \rightarrow V$ is a **local diffeomorphism** if for every $p \in U$, there exists an open neighborhood $U_p \subset U$ such that $F|_{U_p}$ is a diffeomorphism onto its image.

It is clear that every diffeomorphism is a local diffeomorphism, but the converse is not true, since local diffeomorphisms can fail to be bijective. An example of such a map is the polar transformation $p : \mathbb{R}^2 - \{0\} \rightarrow \mathbb{R}^2 - \{0\}$ given by $p(r, \theta) = (r \cos \theta, r \sin \theta)$. An important theorem tells us that the Jacobian matrix captures all the local information we need.

Theorem 0.4 (The Inverse Function Theorem). Let $F : U \rightarrow V$ be smooth. Then if DF_p is an isomorphism, F is a local diffeomorphism at p .

Finally, there's a widespread convention in differential geometry regarding summations, called the **Einstein summation convention**. We can compactly represent the summation $\sum_i v^i e_i$ as just $v^i e_i$. The rule is that if you see an upper index and a lower index, there is an implicit summation. For example, matrix multiplication in this notation is given by $(AB)_j^i = A_k^i B_j^k$. Where the upper index denotes the row, and the lower index denotes the column, and the implicit summation is over the dummy index k . This convention becomes useful when there are a lot of indices floating around, which is a common occurrence in differential geometry.

1. INTRODUCTION TO MANIFOLDS

Definition 1.1. A *topological manifold* is a Hausdorff space X such there exists a countable open cover $\{U_\alpha\}$ of X , along with homeomorphisms $\varphi_\alpha : U_\alpha \rightarrow V_\alpha$, where V_α is an open subset of \mathbb{R}^n . Given a chart (U, φ) , we can write φ in terms of its component functions

$$\varphi(p) = (x^1(p), \dots, x^n(p))$$

The functions x^i are called *local coordinates* on U .

In this way, we see that a manifold is a topological space that is locally topologically indistinguishable from Euclidean space. Both modifiers are important here – there could be global and geometric properties that differ from \mathbb{R}^n .

Example 1.2.

- (1) \mathbb{R}^n is a topological manifold – it admits a global chart $(\mathbb{R}^n, \text{id}_{\mathbb{R}^n})$.
- (2) The 2-sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ is a topological manifold. If we let N and S denote the North and South poles respectively, then we can construct charts using *stereographic projection*. Stereographic projection from the North pole is a map $\varphi_N : S^2 - \{N\} \rightarrow \mathbb{R}^2$, where given a point $p \in S^2 - \{N\}$, we take the line containing both N and p , which intersects the $z = 0$ plane at one point q . We then define $\varphi_N(p) = q$. Stereographic projection φ_S from the South pole is defined in an analogous manner. An explicit formula for φ_N is

$$\varphi_N(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right)$$

Exercise 1.3.

- (1) S^2 does not admit a global chart. Why?
- (2) Give an inverse map $\mathbb{R}^2 \rightarrow S^2 - \{N\}$ for stereographic projection from the North Pole
- (3) Generalize stereographic projection to the n -sphere $S^n = \{p \in \mathbb{R}^{n+1} : \|p\| = 1\}$

One of the main appeals of Euclidean space is that we have a lot of tools at our disposal, like linear algebra and calculus. We would like to generalize these notions to manifolds. One of the wonderful properties of manifolds is that they locally look like \mathbb{R}^n , so we can use the charts to translate concepts we know in \mathbb{R}^n to concepts on the manifold. We know what it means for a function on \mathbb{R}^n to be smooth, how do we translate this to a manifold X ? Given a map $F : X \rightarrow \mathbb{R}^m$, what does it mean for F to be smooth? Our first guess is to use our charts. Given a map $F : X \rightarrow \mathbb{R}^m$, we want to say that it is smooth if for every $p \in X$, there exists a chart (U_p, φ_p) such that the composition $F \circ \varphi_p^{-1} : U_p \rightarrow \mathbb{R}^m$ is smooth. The intuition here is correct, there are some technicalities that need to be addressed. Namely, suppose $p \in X$ lies in the domain of two charts (U_1, φ_1) , and (U_2, φ_2) . Then what if $F \circ \varphi_1^{-1}$ was smooth, but $F \circ \varphi_2^{-1}$ wasn't? This suggests that we need a compatibility condition for our charts if we want a notion of smoothness. We say two charts (U_1, φ_1) and (U_2, φ_2) are *smoothly compatible* if both $\varphi_1 \circ \varphi_2^{-1}$ and $\varphi_2 \circ \varphi_1^{-1}$ are smooth maps.

Definition 1.4. A collection $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ of smoothly compatible charts such that the U_α cover X is called an *atlas*. An atlas is *maximal* if it is not properly contained in any other atlas.

Definition 1.5. A *smooth manifold* is a topological manifold X equipped with a maximal atlas \mathcal{A} .

Now if we have a smooth manifold X , we have a notion of smooth maps $X \rightarrow \mathbb{R}^n$

Definition 1.6. A map $F : X \rightarrow \mathbb{R}^k$ is *smooth* if for every chart $(U_\alpha, \varphi_\alpha)$, we have that $F \circ \varphi_\alpha^{-1}$ is smooth in the Euclidean sense. We call $F \circ \varphi_\alpha^{-1}$ a *local coordinate representation* of the map F .

We also know what it means for maps between manifolds to be smooth.

Definition 1.7. A map $F : X \rightarrow Y$ of smooth manifolds is *smooth* if for every p , there exists charts (U, φ) and (V, ψ) of p and $F(p)$ respectively, such that (after appropriate shrinking of U and V), we have that $\psi \circ F \circ \varphi^{-1}$ is smooth in the Euclidean sense.

For convenience, we will make the a distinction between the words map and function, which is also common in the literature. A *map* will denote an arbitrary mapping $X \rightarrow Y$, where X and Y can denote any kind of set. A *function* on a space X is a mapping $X \rightarrow \mathbb{R}$. We let $C^\infty(X)$ denote the vector space of functions of X , which forms a commutative ring under pointwise multiplication and addition.

This is nice, but we've excluded a huge class of spaces. For example, the unit interval isn't a manifold, since there exist no charts about the endpoints satisfying our definition (why?). To include the spaces, we need another definition. Let \mathbb{H}^n denote *Euclidean half space*, where

$$\mathbb{H}^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n : x^n \geq 0\}$$

Definition 1.8. A *manifold with boundary* is a Hausdorff space X such that there exists a countable cover by charts $(U_\alpha, \varphi_\alpha)$ where the φ_α are homeomorphisms to open sets in \mathbb{H}^n . Given a point $p \in X$, we say that p is in the *boundary* of X if there exists a chart (U, φ) containing p such that $\varphi(p) = (x^1, \dots, x^n)$, and $x^n = 0$. We say that p is in the *interior* of X if no such chart exists. We denote the boundary and interior of X as ∂X and $\text{Int } X$ respectively.

Under this definition, \mathbb{H}^n is a smooth manifold with boundary, with a global chart given by $(\mathbb{H}^n, \text{id}_{\mathbb{H}^n})$, and $\partial \mathbb{H}^n = \{(x^1, \dots, x^n) \in \mathbb{H}^n : x^n = 0\}$

Exercise 1.9. Let X be a smooth n -manifold with boundary

- (1) Show that ∂X is well defined. Namely, show that if we have a point $p \in X$ such that there exists a chart (U, φ) where $\varphi_1(p) \in \partial \mathbb{H}^n$, show that any other chart (U', φ') for p must also have $\varphi'(p) \in \partial \mathbb{H}^n$. (Hint: Use the inverse function theorem to show there exists no diffeomorphism from an open set in \mathbb{R}^n to an open set in \mathbb{R}^{n-1} .)
- (2) Prove that ∂X is a smooth $(n-1)$ -manifold without boundary.

2. TANGENT SPACES AND THE TANGENT BUNDLE

Recall from multivariable calculus that the derivative is the best linear approximation to a map. Because of this, we need to introduce some sort of linear structure on our smooth manifold in order to make sense of derivatives of maps between manifolds.

In the case where our manifold M is embedded in \mathbb{R}^n , this is easy, since we can take a chart (U, φ) , and treat it as a map $\mathbb{R}^k \rightarrow \mathbb{R}^n$. Then the image of derivative $d\varphi_p^{-1} : \mathbb{R}^k \rightarrow \mathbb{R}^n$ is the best linear approximation to our manifold M at the point $p \in M \subset \mathbb{R}^n$. This is nice, but is not an intrinsic definition, since it relies on our manifold lying in some ambient Euclidean space. While this is always possible thanks to the Whitney Embedding theorem, there's a more natural way to talk about tangent vectors to an abstract manifold, that need not be embedded in \mathbb{R}^n .

We'll provide some motivation in the Euclidean case first. We naturally think of tangent vectors at a point $p \in \mathbb{R}^n$ as another copy of \mathbb{R}^n based at p . However, there is a different way to think about tangent vectors. Let e_i denote the standard basis for \mathbb{R}^n , and let $v \in \mathbb{R}^n$. Then there exists a linear differential operator D_v on $C^\infty(\mathbb{R}^n)$ that takes a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, and takes its directional derivative in the direction of v . If we write $v = v^i e_i$, then

$$D_v|_p = v^i \frac{\partial}{\partial x^i}|_p$$

We also note that D_v satisfies to product rule (also referred to as the Leibniz rule), i.e. for $f, g \in C^\infty(\mathbb{R}^n)$, we have that

$$D_v|_p(fg) = f(p)D_v g + g(p)D_v f$$

Definition 2.1. A linear map $D : C^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$ satisfying the product rule at $p \in \mathbb{R}^n$, i.e.

$$D(fg) = f(p)Dg + g(p)Df$$

is called a *derivation at p*

Theorem 2.2. *There is a bijection*

$$\{\text{Derivations at } p\} \leftrightarrow \{\text{tangent vectors based at } p\}$$

given by the mapping $v \mapsto D_v$

This theorem tells us that thinking about derivations is equivalent to thinking of tangent vectors based at a point p . In addition, this definition gives more structure to our original idea of tangent vectors, since derivations act on smooth functions, rather than just being some arrow. This definition also generalizes nicely to abstract manifolds, since the only real tools we have as of now are notions of smoothness.

Definition 2.3. Let M be a smooth manifold and $p \in M$. Then the **tangent space** at p , denoted $T_p M$, is the vector space of derivations at p .

Then given a smooth map of manifolds $F : M \rightarrow N$, we want the derivative of F at p to be a linear map $dF_p : T_p M \rightarrow T_{f(p)} N$. Given some derivation $v \in T_p M$, how should $dF_p(v)$ act on some function $g \in C^\infty(N)$? We don't have too much at our disposal, so the only reasonable choice here is to have

$$dF_p(v)g = v(g \circ F)$$

The derivative is also referred to as the **pushforward**, and some choose to denote it as F_* .

Now that we have an abstract definition of tangent spaces, we'd like to see how these objects behave in local coordinates. Let M be a smooth n -manifold, and let (U, φ) be a chart for p with $\varphi = (x^1, \dots, x^n)$. Then the **coordinate vectors** at p for the chart (U, φ) are the derivations $\partial/\partial x^i|_p$. The action of the coordinate vectors is given by

$$\frac{\partial}{\partial x^i} f = \frac{\partial(f \circ \varphi^{-1})}{\partial x^i} \Big|_{\varphi(p)}$$

So the coordinate vectors are literally the partial derivative operations on the coordinate representation of a function, justifying the notation. We often abbreviate them as just ∂_i when the choice of coordinates are clear.

Theorem 2.4. *The coordinate vectors $\partial/\partial x^i|_p$ form a basis for $T_p M$.*

The following exercise should help you build intuition about the derivative, since it will resemble something you might already know. In addition, it's a good exercise to make sure you understand the definition of vectors as derivations and the derivative.

Exercise 2.5. Let M and N be smooth manifolds, and let $F : M \rightarrow N$ be a smooth map. Fix $p \in M$ and charts (U, φ) and (V, ψ) for p and $F(p)$ respectively. Show that dF_p is given in local coordinates by the Jacobian matrix of $\psi \circ F \circ \varphi^{-1}$. (Hint: To do this, observe the action of $dF_p(\partial/\partial x^i|_p)$ on an arbitrary smooth function $f \in C^\infty(N)$).

Local coordinates provide an excellent tool for giving concrete descriptions of functions and objects of a manifold, but working in coordinates has a few nuances. If we define some quantity (a function, tensor, etc.) using local coordinates, we need to ensure that our definition still works under a change of coordinates. Otherwise, the quantity we attempted to define is not well-defined at all! Because of this, it's in our best interest to understand how tangent vectors behave under coordinate transformation.

Let (U, φ) and (V, ψ) be two charts about a point $p \in M$, with $\varphi = (x^1, \dots, x^n)$ and $\psi = (y^1, \dots, y^n)$. We want to express $\partial/\partial x^i|_p$ in terms of the $\partial/\partial y^j|_p$. To do this, we want to compute the image of $\partial/\partial x^i$ under the derivative $d(\psi \circ \varphi^{-1})_p$. The computation yields

$$\frac{\partial}{\partial x^i} \Big|_p = \frac{\partial y^j}{\partial x^i}(\psi(p)) \frac{\partial}{\partial y^j} \Big|_p$$

which looks just like the chain rule from multivariable calculus. Likewise, for an arbitrary vector $v = v^i \partial/\partial x^i$, its components \tilde{v}^j in the coordinate system given by the y^j is

$$\tilde{v}^j = v^i \frac{\partial y^j}{\partial x^i}(\psi(p))$$

It's important to note that each point $p \in M$ gets its own tangent space – there's no natural identification between tangent spaces, and no way to transport information between them (at least not yet!). However, we can construct a new object using the technology we have now – the tangent bundle.

Definition 2.6. Let M be a smooth manifold. Then the *tangent bundle* of M is the set

$$TM = \coprod_{p \in M} T_p M = \{(p, v) : p \in M, v \in T_p M\}$$

The tangent bundle comes equipped with a natural map $\pi : TM \rightarrow M$ given by $\pi(p, v) = p$ called the *projection*.

We only said set in the definition, but there's more structure here. As you might expect, TM is a smooth manifold.

Theorem 2.7. *The tangent bundle TM is a smooth manifold, and has a natural smooth structure such that the projection $\pi : TM \rightarrow M$ is smooth.*

Proof. For a point $p \in M$, fix a chart (U, φ) , where $\varphi = (x^1, \dots, x^n)$. We then use this to construct what will be chart for $\pi^{-1}(U)$, and we will use these charts to define the topology. We know that for each point in U , the vectors ∂_i form a basis for the tangent space at this point, so each pair $(q, v) \in \pi^{-1}(U)$, we can write it uniquely in coordinates as $(x^1(q), \dots, x^n(q), v^1, \dots, v^n)$, where the v^i are the components of v in terms of the ∂_i . This gives a bijection $\Phi_p : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$. Doing this for all $p \in M$ gives a cover of TM , and these sets satisfies the conditions for a basis, so we can take the topology generated by these sets. In addition, the transition maps $\Phi_p \circ \Phi_q$ are smooth, so they determine a smooth structure on TM , making a smooth manifold of dimension $2n$. ■

In addition, given a smooth map $F : M \rightarrow N$, this induces a map $dF : TM \rightarrow TN$, where in local coordinates, $dF(p, v) = (F(p), dF_p(v))$.

Exercise 2.8. Show that the assignments $M \mapsto TM$ and $F \mapsto dF$ determines a covariant functor from the category of smooth manifolds to itself, called the *Tangent Functor*. Can you recognize this statement in terms a familiar statement in calculus?

Note that the charts we constructed for TM are of a particular form, namely they are isomorphisms to $U \times \mathbb{R}^n$, so TM locally looks like a product manifold, but globally there may be some twisting. In addition, each fiber $\pi^{-1}(p)$ has the structure of a real vector space. These are defining features of a geometric object called a *vector bundle*, which are a central object in differential geometry.