CHERN-WEIL THEORY

JEFFREY JIANG

Contents

1. Lie Algebra Valued Differential Forms

Definition 1.1. let $E \to M$ be a vector bundle. An E***-valued differential*** k***-form*** is a section of $\Lambda^k T^*M \otimes E$. The space of E-valued k-forms is denoted $\Omega^k_M(E)$. In the case that E is a trivial bundle $M \times V$, we abbreviated this to V-valued k-forms, and denote this space by $\Omega^k_M(V)$. In a local trivialization of E, an element $\omega \in \Omega^k_M(E)$ can be thought of as a vector of k-forms.

One thing to keep in mid is that, unless E is a trivial bundle, the vector of k-forms in a local trivialization does not transform tensorially – the transformation law depends both on the coordinates on M as well was the transition functions of the bundle E. That being said, it is still extremely useful for local computations.

The situation we will be interested in is when E is a trivial bundle $P \times \mathfrak{g}$, where $\mathfrak{g}$ is some Lie algebra. We will briefly discuss some of the common notation and operations on Lie algebra valued forms. Many of these operations are analogous to operations on real valued forms, but can differ in subtle ways. Fix a basis $\{\xi_i\}$ for $\mathfrak{g}$. Then any Lie algebra valued k-form $\Theta \in \Omega^k_P(\mathfrak{g})$ can be written uniquely as

$$\Theta = \Theta^i \otimes \xi_i$$

for real valued k-forms $\Theta^i \in \Omega^k_P$. For brevity, we usually omit the tensor symbol, and abbreviate this to $\Theta^i\xi_i$. Given a k-form $\Theta \in \Omega^k_P(\mathfrak{g})$ and an ℓ-form $\Omega \in \Omega^\ell_P(\mathfrak{g})$, we define their wedge product $\Theta \wedge \Omega \in \Omega^{k+\ell}(\mathfrak{g} \otimes \mathfrak{g})$to be

$$\Theta \wedge \Omega := (\Theta^i \wedge \Omega^j)\xi_i \otimes \xi_j$$

Using the usual wedge product on differential forms. Note that this wedge product does not have a normalizing factor, which will explain various factors of 2 and 1/2 when doing computations with Lie algebra valued forms. In addition, we note that, unlike for real valued forms, we do not necessarily have $\Theta \wedge \Theta = 0$. We also note that this definition for the wedge product works for general vector bundle valued forms as well, but when we have a Lie algebra, we have a Lie bracket $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, which allows us to define

$$[\Theta \wedge \Omega] := (\Theta^i \wedge \Omega^j)[\xi_i, \xi_j]$$

Finally, we can extend the usual exterior derivative $d : \Omega^k_P \to \Omega^{k+1}_P$ to $\Omega^k_P(\mathfrak{g})$ in a natural way, namely

$$d\Theta := d\Theta^i \xi_i$$

2. Connections on Principal Bundles

Definition 2.1. Let M be a smooth manifold and G a Lie group. A ***principal G-bundle*** is a smooth manifold P equipped with a smooth right G action and a map $\pi : P \to M$ such that

(1) π is a submersion
(2) The action of G on P preserves the fibers of π
(3) The action of G on P is free and transitive on each fiber
(4) For each $p \in M$, there exists a neighborhood U containing p and a G-equivariant diffeomorphism $\varphi : \pi^{-1}(U) \to U \times G$ such that the we get the commutative diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr] \arrow[dr, "\pi"'] & & U \times G \arrow[dl] \\ & U & \end{tikzcd}$$

where the map $U \times G$ is projection onto the first factor, and the G action on $U \times G$ is right multiplication on the second factor.

Example 2.2. Let $E \to M$ be a rank k vector bundle. From E we can construct the ***frame bundle*** $\mathcal{B}(E)$, which as a set is

$$\mathcal{B}(E) = \Big\{ (p, b) \ : \ p \in M, \ b : \mathbb{R}^k \to E_p \text{ is an isomorphism.} \Big\}$$

This comes with a natural projection $\mathcal{B}(E) \to M$ by forgetting the second factor, and local coordinates for M and local trivializations for E endow $\mathcal{B}(E)$ the structure of a smooth manifold such that the projection map is a submersion. There is a natural right action of $\mathrm{GL}_k\mathbb{R}$ on $\mathcal{B}(E)$ given by precomposition, and it is clear that this action is both free and transitive on any given fiber $\mathcal{B}(E)_p$. In addition, local trivializations of E induce local trivializations for $\mathcal{B}(E)$, which are $\mathrm{GL}_k\mathbb{R}$-equivariant. Therefore, we have that $\mathcal{B}(E)$ is a principal $\mathrm{GL}_k\mathbb{R}$-bundle over M.

An important construction is the ***associated bundle construction***, where given a principal G-bundle $P \to M$ and a space F with a left G action, we can construct a fiber bundle $P \times_G F \to M$ with model fiber F.

Definition 2.3. Let $P \to M$ be a principal G-bundle, and F a smooth manifold with a left G-action. The ***associated fiber bundle*** $P \times_G F \to M$ is the set

$$P \times_G F := P \times F / (p, f) \sim (p \cdot g, g^{-1} \cdot f)$$

Local trivializations for P induce local trivializations on $P \times_G F$, which allows us to equip the set $P \times_G F$ with a topology and smooth structure such that the induced map $P \times_G F \to M$ is a fiber bundle over M with model fiber F.

Example 2.4. For a Lie group G, we have the adjoint action $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$. Then given a principal G-bundle $P \to M$, we can construct the ***adjoint bundle*** $\mathfrak{g}_P \to M := P \times_G \mathfrak{g}$, which is a bundle with model fiber $\mathfrak{g}$.

Example 2.5. Let $E \to M$ be a rank k vector bundle, and $\pi : \mathcal{B}(E) \to M$ its associated $\mathrm{GL}_k\mathbb{R}$ bundle of frames. There is a natural action of $\mathrm{GL}_k\mathbb{R}$ on $\mathbb{R}^k$, which allows us to constructed the associated bundle $\mathcal{B}(E) \times_{\mathrm{GL}_k\mathbb{R}} \mathbb{R}^k$. We claim that this bundle is isomorphic to the original bundle E.

To show this, we construct an explicit isomorphism. Let $[b, v]$ denote the equivalence class of (b, v) in $\mathcal{B}(E) \times_{\mathrm{GL}_k\mathbb{R}} \mathbb{R}^k$. Define the map $\varphi : \mathcal{B}(E) \times_{\mathrm{GL}_k\mathbb{R}} \to E$ by $\varphi[b, v] = b(v)$. We first need to verify that this map is well defined. Suppose we have another representative (a, w) for the equivalence class $[b, v]$. Then we know there exists some $g \in \mathrm{GL}_k\mathbb{R}$ such that $(a, w) = (b \circ g, g^{-1}(v))$. Then we would have $\varphi[a, w] = (b \circ g)(g^{-1}(v)) = b(v)$. Therefore φ is well defined. In addition, it is clear that this map has an inverse. For any $e \in E$, define the map $\psi : E \to \mathcal{B}(E) \times_{\mathrm{GL}_k\mathbb{R}} \mathbb{R}^k$ by $\psi(e) = [b, e']$ where b is a basis for the fiber containing e, and e' is the coordinate representation of e with respect to the basis b.

The example with frame bundles gives us a good idea as to how to think of principal bundles and associated bundles. A principal bundle can be thought of as coordinates systems for an associated bundle, and the relation we used to define the associated bundle can be thought of as the transformation law of the associated bundle.

We now move on to connections on principal bundles. Let $P \to M$ be a principal G-bundle. Since π is a submersion, it is constant rank, so $\ker d\pi$ defines a subbundle V called the ***vertical distribution***. This gives us the exact sequence of vector bundles

$$0 \longrightarrow V \longrightarrow TP \longrightarrow \pi^* TM \longrightarrow 0$$

Definition 2.6. A ***connection*** on P is a distribution $H \subset TP$ such that

(1) $V \oplus H = TP$
(2) $H_{p\cdot g} = d(R_g)_p H_p$, where $R_g : P \to P$ is the action of g

Equivalently, it is a choice of splitting of the exact sequence specified above. Another fruitful way to view connections is in terms of differential forms. For a Lie group G with Lie algebra $\mathfrak{g}$, we have an exponential map $\exp : \mathfrak{g} \to G$, where given a left invariant vector field $X \in \mathfrak{g}$, we have $\exp(X) = c_X(1)$, where $c_X : (-\varepsilon, \varepsilon) \to G$ is the integral curve of X with $c(0) = e$. Then if we have a principal G-bundle $P \to M$ and a point $p \in P$, we get a curve γ_X defined by

$$\gamma_X(t) = p \cdot \exp(tX)$$

Since the action of G preserves the fiber P_p, the tangent vector

$$\dot{\gamma}_X = \frac{d}{dt}\bigg|_{t=0} \gamma_X(t)$$

is contained in the vertical space V_p. In addition, since the action is free, we have that $\dot{\gamma}_X = 0$ if and only if $X = 0$. Therefore, we get an injective linear map given by $X \mapsto \dot{\gamma}_X$. By a dimension count, this gives an isomorphism $\mathfrak{g} \to V_p$. Doing this over all points $p \in P$, we get a vertical vector field $\widehat{X}$ on P, and gives us an isomorphism $\underline{\mathfrak{g}} \to V$, where $\underline{\mathfrak{g}}$ denotes the trivial bundle $P \times \mathfrak{g}$. Suppose we are given a connection $H \subset TP$. Then the decomposition $T_pP = V_p \oplus H_p$ gives us a projection map $T_pP \to V_p$ with kernel H_p. Using the identification $V_p \cong \mathfrak{g}$, this gives us an element of $\mathfrak{g}$. Doing this over all points, we see that H is equivalent to the data of a $\mathfrak{g}$-valued 1-form $\Theta \in \Omega^1_P(\mathfrak{g})$, called the ***connection* 1*-form***. The first thing we want to observe is how Θ transforms with respect to the G-action on P. To do this, we first observe how the vector fields we defined above transform.

Proposition 2.7. *Let $X \in \mathfrak{g}$ and let $\widehat{X} \in \mathfrak{X}(P)$ be the vector field induced by X. For $g \in G$, let R_g denote the diffeomorphism given by the right action of g. Then*

$$(R_g)_*(\widehat{X}) = \widehat{\operatorname{Ad}_{g^{-1}} X}$$

Proof. We compute

$$\begin{aligned}
(R_g)_*(\widehat{X})_p &= (R_g)_* \left(\frac{d}{dt}\bigg|_{t=0} p \cdot \exp(tX) \right) \\
&= \frac{d}{dt}\bigg|_{t=0} p \cdot (\exp(tX) g) \\
&= \frac{d}{dt}\bigg|_{t=0} pg \cdot (g^{-1} \exp(tX) g) \\
&= \widehat{\operatorname{Ad}_{g^{-1}} X}
\end{aligned}$$

■

This allows us to compute how a connection 1-form transforms.

Proposition 2.8. *A connection 1-form $\Theta \in \Omega^1_P(\mathfrak{g})$ satisfies*

$$R_g^* \Theta = \operatorname{Ad}_{g^{-1}} \Theta$$

Where $\operatorname{Ad}_{g^{-1}} \Theta$ is 1-form defined by

$$(\operatorname{Ad}_{g^{-1}} \Theta)(v) = \operatorname{Ad}_{g^{-1}}(\Theta(v))$$

Proof. For any p, we get a decomposition $T_pP = V_p \oplus H_p$, so any tangent vector $v \in T_pP$ can be uniquely decomposed as $v = \widehat{X} + h$ where $X \in \mathfrak{g}$ and $h \in H_p$. We compute

$$\begin{aligned}
(R_g^* \Theta)_p(v) &= (R_g^* \Theta)_p(\widehat{X} + h) \\
&= \Theta_{p \cdot g}((R_g)_*(\widehat{X}) + (R_g)_*(h)) \\
&= \Theta_{p \cdot g}(\widehat{\operatorname{Ad}_{g^{-1}} X}) \\
&= \widehat{\operatorname{Ad}_{g^{-1}} X} \\
&= \operatorname{Ad}_{g^{-1}} \Theta_p(\widehat{X} + h)
\end{aligned}$$

where we use the transformation law for $\widehat{X}$, the fact that H_p is the kernel of Θ_p, and the fact that H is G-invariant. ■

The identification of the vertical bundle with the trivial bundle $\underline{\mathfrak{g}}$ gives us a nice characterization of differential forms valued in an associated bundle of P.

Theorem 2.9. *Let $P \to M$ be a principal G bundle, and $E = P \times_G W$ an associated vector bundle induced by a representation $\rho : G \to \mathrm{GL}(W)$. Then there is a bijective correspondence*

$$\Omega^k(E) \leftrightarrow \left\{ \alpha \in \Omega^k_P(W) \ : \ R_g^* \alpha = \rho(g)^{-1}\alpha,\ \forall \xi \in \mathfrak{g}, \iota_\xi \alpha = 0 \right\}$$

This comes from a general principle that appropriately G-equivariant objects on P should descend to objects on M.

Definition 2.10. let $P \to M$ be a principal G bundle with connection Θ. The ***curvature form*** of the connection Θ is a 2-form $\Omega \in \Omega^2_P(\mathfrak{g})$ defined by

$$\Omega = d\Theta + \frac{1}{2}[\Theta \wedge \Theta]$$

where in a local trivialization, $d\Theta$ is a matrix of 2-forms given by $(d\Theta)^i_j = d(\Theta)^i_j$, and

$$[\Theta \wedge \Theta](X, Y) = [\Theta(X), \Theta(Y)] - [\Theta(Y), \Theta(X)] = 2[\Theta(X), \Theta(Y)]$$

As with the connection 1-form, we first observe how the curvature form Ω transforms with respect to the G-action on P.

Proposition 2.11.

(1) $R_g^* \Omega = \mathrm{Ad}_{g^{-1}} \Omega$
(2) $\iota_\xi(\Omega) = 0$ *for all* $\xi \in \mathfrak{g}$.

Before proving the proposition, we state an extremely helpful formula.

Lemma 2.12 (***Cartan's magic formula***)**.** *Let ξ be a vector field, and let ω be a k-form. Let $\mathcal{L}_\xi$ denote the Lie derivative along ξ. Then*

$$\mathcal{L}_\xi \omega = d(\iota_\xi(\omega)) + \iota_\xi(d\omega)$$

Proof of proposition.

(1) We compute

$$\begin{aligned} R_g^*\Omega &= R_g^* d\Theta + \frac{1}{2} R_g^*[\Theta \wedge \Theta] \\ &= d(R_g^*\Theta) + \frac{1}{2}[R_g^*\Theta \wedge R_g^*\Theta] \\ &= \mathrm{Ad}_{g^{-1}} d\Theta + \frac{1}{2}[\mathrm{Ad}_{g^{-1}} \Theta \wedge \mathrm{Ad}_{g^{-1}} \Theta] \\ &= \mathrm{Ad}_{g^{-1}} \Omega \end{aligned}$$

(2) Let $\xi \in \mathfrak{g}$. Then

$$\iota_\xi(\Omega) = \iota_\xi(d\Theta) + \frac{1}{2}\iota_\xi[\Theta \wedge \Theta]$$

Using Cartan's magic formula, we have that

$$\mathcal{L}_\xi \Theta = d(\iota_\xi(\Theta)) + \iota_\xi(d\Theta) = \iota_\xi(\Theta)$$

Since $\iota_\xi(\Theta) = \xi$, which is constant, which then implies that $d\iota_\xi(\Theta) = 0$. We then compute

$$\begin{aligned}\mathcal{L}_\xi \Theta &= \frac{d}{dt}\bigg|_{t=0} R^*_{\exp(t\xi)}\Theta \\ &= \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_{\exp(t\xi)^{-1}}\Theta \\ &= [-\xi, \Theta]\end{aligned}$$

We also compute

$$\frac{1}{2}\iota_\xi[\Theta\wedge\Theta] = \frac{1}{2}[\iota_\xi(\Theta)\wedge\Theta] = [\xi,\Theta]$$

Therefore, $\iota_\xi(\Omega) = 0$.

■

As a result, the curvature form Ω descends to the base manifold, giving us a 2-form valued in the adjoint bundle $\mathfrak{g}_P$.

Theorem 2.13 (***The Bianchi Identity***)**.**

$$d\Omega + [\Theta\wedge\Omega] = 0$$

Proof. Fix a basis $\{\xi_i\}$ for $\mathfrak{g}$, and let $\Theta = \Theta^i\xi_i$. We then compute

$$\begin{aligned}d\Omega &= d^2\Theta + \frac{1}{2}d[\Theta\wedge\Theta] \\ &= \frac{1}{2}d(\Theta^i\wedge\Theta^j)[\xi_i,\xi_j] \\ &= \frac{1}{2}(d\Theta^i\wedge\Theta^j - \Theta^i\wedge d\Theta^j)[\xi_i,\xi_j] \\ &= \frac{1}{2}(d\Theta^i\wedge\Theta^j - d\Theta^j\wedge\Theta^i)[\xi_i,\xi_j] \\ &= (d\Theta^i\wedge\Theta^j - d\Theta^j\wedge\Theta^i)[\xi_i,\xi_j]\end{aligned}$$

where the last equality comes from skew-symmetry of the Lie bracket. On the other hand, we compute

$$\begin{aligned}[\Theta\wedge\Omega] &= \left[\Theta\wedge\left(d\Theta + \frac{1}{2}[\Theta\wedge\Theta]\right)\right] \\ &= [\Theta\wedge d\Theta] + \frac{1}{2}[[\Theta\wedge\Theta]\wedge\Theta] \\ &= (\Theta^i\wedge d\Theta^j)[\xi_i,\xi_j] \\ &= (d\Theta^j\wedge\Theta^i)[\xi_i,\xi_j] \\ &= -(d\Theta^i\wedge\Theta^j - d\Theta^j\wedge\Theta^i)[\xi_i,\xi_j]\end{aligned}$$

Where we use skew symmetry of the bracket for the last equality, and use the Jacobi identity to deduce the vanishing of $[[\Theta\wedge\Theta]\wedge\Theta]$. Putting these two computations together yields the Bianchi identity. ■

3. The Chern-Weil Homomorphism

The fact that a curvature form Ω descends to a $\mathfrak{g}_P$-valued 2-form on the base suggests that the curvature might have something to say about the topology of the base manifold M. As we'll see, this takes the form of characteristic classes.

Definition 3.1. Let G be a Lie group with Lie algebra $\mathfrak{g}$. An ***invariant polynomial of degree*** k is a function $p \in \mathrm{Sym}^k \mathfrak{g}^*$ such that

$$p(X_1, \ldots, X_k) = p(\mathrm{Ad}_g(X_1), \ldots \mathrm{Ad}_g(X_k))$$

We denote the space of degree k invariant polynomials by $I^k(G)$, and let $I(G) = \bigoplus_k I^k(G)$ denote the space of all invariant polynomials.

We'll discuss specific examples in detail later. For now, we move on with the general theory. But first, a short lemma

Lemma 3.2. *Let $p \in I^k(G)$. Then for any $\xi, X_1, \ldots X_k \in \mathfrak{g}$, we have the identity*

$$\sum_{i=1}^{k} p(X_1, \ldots [\xi, X_i], \ldots X_n) = 0$$

Proof. Since p is an invariant polynomial, we have the equation

$$p(\mathrm{Ad}_{\exp(t\xi)}(X_1), \ldots, \mathrm{Ad}_{\exp(t\xi)}(X_n)) = p(X_1, \ldots, X_n)$$

Differentiating with respect to t at $t = 0$, we get the desired identity, where we use the product rule, as well as the fact that

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp(t\xi)}(X) = [\xi, X]$$

∎

The algebra of invariant polynomials can be naturally identified with homogeneous polynomials on $\mathfrak{g}$, where given an invariant polynomial $p \in I^k(G)$, we get a function on $\mathfrak{g}$ via the function $X \mapsto p(X, \ldots, X)$. Upon fixing a basis for $\mathfrak{g}$, we can recover a polynomial. We can recover the polynomial p from this function via a formula analogous to the polarization identity for recovering a bilinear form from its quadratic form. In the case that G is a matrix group, the idea is to think of p as a polynomial in the entries of X.

Theorem 3.3 (***The Chern-Weil Homomorphism***)**.** *Let $P \to M$ be a principal G-bundle over M, and $\Omega \in \Omega^2_P(\mathfrak{g})$ the curvature form corresponding to some connection on P. Then we get a map $I(G) \to \Omega^\bullet_M$, where for $p \in I^k(G)$, we map*

$$p \mapsto \omega_p := p(\underbrace{\Omega \wedge \cdots \wedge \Omega}_{k \text{ times}})$$

There's a lot to unpack here.

Proof. We first make sense of the formula. We have that $\Omega \in \Omega^2_P(\mathfrak{g})$, so a priori, ω_p doesn't descend to M. Our hope is that invariance of p, along with the transformation laws of Ω means that this form will descend to the base. Punting that problem to later, we can feed $2k$ tangent vectors to $\Omega \wedge \cdots \wedge \Omega$, giving us k elements $X_1, \ldots, X_k \in \mathfrak{g}$, which we can then feed to p, giving us a real number. Therefore, the types check out.

We now check that the form $\omega_p \in \Omega_P^{2k}$ descends to M. This means we need to check two things :

(1) $R_g^* \omega_p = \mathrm{Ad}_{g^{-1}} \omega_p$
(2) $\iota_\xi \omega_p = 0$ for any $\xi \in \mathfrak{g}$.

For the first property, we compute

$$\begin{aligned} R_g^* \omega_p &= R_g^*(p(\Omega \wedge \cdots \wedge \Omega)) \\ &= p(R_g^* \Omega \wedge \cdots \wedge R_g^* \Omega) \\ &= p(\mathrm{Ad}_{g^{-1}} \Omega \wedge \cdots \mathrm{Ad}_{g^{-1}} \Omega) \\ &= p(\Omega \wedge \cdots \wedge \Omega) \end{aligned}$$

For the second property, let $\xi \in \mathfrak{g}$. Then

$$\begin{aligned} (\iota_\xi \omega_p) &= \iota_\xi(p(\Omega \wedge \cdots \wedge \Omega)) \\ &= \sum_i p(\Omega \wedge \cdots \wedge \iota_\xi \Omega \wedge \cdots \wedge \Omega) \\ &= 0 \end{aligned}$$

where we use the fact that $\iota_\xi \Omega = 0$. Therefore, we have that ω_p descends to the base, giving us a form in Ω_M^{2k}. ■

Theorem 3.4. *The Chern-Weil homomorphism maps an invariant polynomial $p \in I(G)$ to a closed form in Ω_M^{2k}, so it descends to a map $I(G) \to H^{2k}(M, \mathbb{R})$.*

Proof. Let Θ denote the connection 1-form. We want to show that ω_p is closed. We compute, using the Leibniz rule for exterior differentiation,

$$\begin{aligned} d\omega_p &= d(p(\Omega \wedge \cdots \wedge \Omega)) \\ &= \sum_i p(\Omega \wedge \cdots \wedge d\Omega \wedge \cdots \wedge \Omega) \\ &= -\sum_i p(\Omega \wedge \cdots \wedge [\Theta \wedge \Omega] \cdots \wedge \Omega) \\ &= 0 \end{aligned}$$

where we use the Bianchi identity, along with the fact that $\Theta(\xi) = \xi$. ■

4. The Case of $\mathrm{GL}_n\mathbb{C}$: Chern Classes

When we restrict our attention to $\mathrm{GL}_n\mathbb{C}$, and the principal bundle $P \to M$ is the $\mathrm{GL}_n\mathbb{C}$-bundle of frames for a complex vector bundle, the fruit of our labor will be the Chern classes of the vector bundle. To do this, we first need to understand the invariant polynomials on $\mathfrak{gl}_n\mathbb{C}$. In this case, the adjoint action of $\mathrm{GL}_n\mathbb{C}$ on $\mathfrak{gl}_n\mathbb{C}$ is conjugation. We also have a basis for $\mathfrak{gl}_n\mathbb{C}$ given by the elementary matrices E_{ij} that are zero everywhere except for a 1 in the (i, j) entry. The invariant polynomials on $\mathfrak{gl}_n\mathbb{C}$ will come from a familiar friend : the determinant. We know that the det is multiplicative, so we clearly have that

$\det(AXA^{-1}) = \det(X)$ for any $X \in \mathfrak{gl}_n\mathbb{C}$ and any $A \in \mathrm{GL}_n\mathbb{C}$, so det is an invariant polynomial. The determinant will be the only tool we will need to construct all the invariant polynomials.

Definition 4.1. The ***elementary invariant polynomials*** $f_0, \dots f_n \in I(\mathrm{GL}_n)$ are the polynomials defined by the formula

$$\det(tI + X) = \sum_{i=0}^{n} f_i(X) t^{n-i}$$

Where t is an indeterminate variable, and $X = X^i_j$ is an $n \times n$ matrix of indeterminate variables.

Example 4.2 ($n = 2$)**.** Let X be the matrix

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, b, c, d are indeterminate variables. Then we have that

$$\begin{aligned} \det(tI + X) &= \det \begin{pmatrix} t+a & b \\ c & t+d \end{pmatrix} \\ &= (t+a)(t+d) - bc \\ &= t^2 + (a+d)t + ad - bc \end{aligned}$$

This give us

$$\begin{aligned} f_0(X) &= 1 \\ f_1(X) &= a + d \\ f_2(X) &= ad - bc \end{aligned}$$

All the polynomials are visibly invariant, since f_0 is vacuously invariant, f_1 is the trace, and f_2 is the determinant

Example 4.3 ($n = 3$)**.** Let X be the indeterminate matrix

$$X = \begin{pmatrix} a & b & c \\ p & q & r \\ x & y & z \end{pmatrix}$$

We then laboriously compute

$$\det \begin{pmatrix} a+t & b & c \\ p & q+t & r \\ x & y & z+t \end{pmatrix}$$

$$= t^3 + (a+q+z)t^2 + (aq + az + qz - ry - bp + cx)t + aqz - ary - pbz + bxr + cpy - cxq$$

Which gives us

$$\begin{aligned} f_0(X) &= 1 \\ f_1(X) &= a + q + z \\ f_2(X) &= (aq + az + qz - ry - bp + cx) \\ f_3(X) &= aqz - ary - pbz + bxr + cpy - cxq \end{aligned}$$

While things look very complicated, these polynomials are just the usual culprits. f_0 and f_1 are analogous to the case of $\mathfrak{gl}_2$. The polynomial f_2 is slightly tricky : it is the polynomial $1/2(\mathrm{tr}(X)^2 - \mathrm{tr}(X^2))$. The polynomial f_3 is just det.

We then make two key observations

(1) If $p \in I(\mathrm{GL}_n\mathbb{C})$ and $X \in \mathfrak{gl}_n\mathbb{C}$ is diagonalizable, conjugation invariance then implies that p is just a polynomial in the eigenvalues of X.

(2) The polynomials f_k suspiciously look like the elementary symmetric polynomials

$$\sigma_k = \sum_{1 \le i_1 < i_2 \ldots < i_k \le n} t_{i_1} \cdots t_{i_k}$$

To make use of the first observation, we need the following theorem.

Theorem 4.4. *The set of diagonalizable matrices in* $\mathrm{GL}_n\mathbb{C}$ *is dense.*

Proof. We note that any matrix with distinct eigenvalues is diagonalizable. Given any non-diagonalizable matrix A, we can always find an arbitrarily small perturbation $A + \varepsilon B$ that has distinct eigenvalues. ∎

Theorem 4.5. *let T be the indeterminate diagonal matrix*

$$T = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}$$

Then the mapping $I(\mathrm{GL}_n\mathbb{C}) \to \mathbb{C}[t_1, \ldots t_n]^{S_n}$

$$P \mapsto \widetilde{P}(t_1, \ldots t_n) := P(T)$$

is an isomorphism.

Proof. The map is clearly an algebra homomorphism, so to check surjectivity, it suffices to check that the image contains the elementary symmetric polynomials, since any symmetric polynomial is a polynomial $f(\sigma_1, \ldots \sigma_n)$ in the elementary symmetric polynomials σ_k. For an indeterminate variable λ, we have that

$$\det(\lambda I + X) \mapsto \det(\lambda I + T) = \prod_{i=1}^{n} (\lambda + t_i) = \sum_{k=0}^{n} \sigma_k(t_1, \ldots, t_k)\lambda^{n-k}$$

We also know that

$$\det(\lambda I + T) = \sum_{k=0}^{n} f_k(X)\lambda^{n-k}$$

Then since the map is a homomorphism, we can compare the terms of degree λ^j to conclude that $f_k(X) \mapsto \sigma_k(t_1, \ldots t_n)$. Consequently, the map is surjective, since given any symmetric polynomial $p(\sigma_1, \dots \sigma_k)$, we have that it is the image of the invariant polynomial $p(f_1, \dots f_k)$.

For injectivity, we have that if an invariant polynomial p is in the kernel, then it must vanish on all diagonal matrices. Since p is continuous as a function of the entries of the matrix, and the set of diagonalizable matrices is dense, we must have that $p = 0$. ∎

This gives a complete description of the invariant polynomials for $\mathrm{GL}_n\mathbb{C}$.

Definition 4.6. Let $P \to M$ be a principal $\mathrm{GL}_k\mathbb{C}$ bundle with curvature form Ω. Then the j^{th} ***Chern class*** is the cohomology class

$$\left[\frac{i}{2\pi} f_k(\Omega)\right]$$

The preceding discussion gives us something immediate

Theorem 4.7. *The Chern classes generate the image of the Chern-Weil homomorphism : every cohomology class in the image can be written as a polynomial in Chern classes.*