ABELIAN YANG-MILLS

JEFFREY JIANG

Contents

1. The General Case

In the case where the structure group is $U(1)$, the Yang-Mills equations for a connection A on principal $U(1)$ bundle $\pi : P \to M$ over a Riemannian manifold M reduce to

$$dF_A = 0$$
$$d^*F_A = 0$$

Where $F_A \in \Omega^2_M$ is the curvature form of A. The Yang-Mills equations are equivalent to $\Delta F_A = 0$, where Δ is the Laplacian on M. By Hodge theory, the cohomology class of F_A has a unique harmonic minimizer Θ, and Yang-Mills connections are the connections A satisfying $F_A = \Theta$. For a connection A and a 1-form $\eta \in \Omega^1_M$, we have the identity

$$F_{A+\eta} = F_A + d\eta$$

from which we can conclude that the space $\mathscr{A}_{\mathrm{YM}}(P)$ of Yang-Mills connections over M are a torsor over the vector space Z^1_M of closed 1-forms on M.

The gauge group in this situation is the group $\mathscr{G}(P) := \mathrm{Map}(M, U(1))$, which follows from the fact that $U(1)$ is abelian. The right action of $\mathscr{G}(P)$ on $\mathscr{A}_{\mathrm{YM}}(P)$ is given by the mapping

$$A \cdot f = A + \pi^* f^* \theta$$

where $\theta \in \Omega^1_{U(1)}$ is the Maurer-Cartan form. The Yang-Mills equations are invariant under the action of $\mathscr{G}(P)$, so we are interested in the space $\mathscr{A}_{\mathrm{YM}}(P)/\mathscr{G}(P)$ of Yang-Mills connections up to gauge equivalence. The gauge group also acts on Z^1_M, where $\eta \cdot f = \eta + f^*\theta$. Therefore, upon fixing a reference connection $A_0 \in \mathscr{A}_{\mathrm{YM}}(P)$ to identify $\mathscr{A}_{\mathrm{YM}}(P)$ with Z^1_M, we may instead compute the quotient $Z^1_M/\mathscr{G}(P)$. To do so we first quotient by the identity component $\mathscr{G}_0(P)$, and the quotient by the component group $\pi_0\mathscr{G}(P) = \mathscr{G}(P)/\mathscr{G}_0(P)$ The components of $\mathscr{G}(P)$ are given by homotopy classes of maps $M \to U(1)$, so $\mathscr{G}_0(P)$ is the space of nullhomotopic maps. Any such map $f : M \to S^1$ lifts to a map $\tilde{f}$ such that $e^{\tilde{f}} = f$, and since θ pulls back to dx along the exponential $\mathbb{R} \to U(1)$, we have that the action of f on a closed form η is just $\eta + d\tilde{f}$. In particular, since any function $h : M \to \mathbb{R}$ descends to a nullhomotopic map $e^h : M \to S^1$, this tells us that $Z^1_M/\mathscr{G}_0(P) = H^1(M, \mathbb{R})$. To quotient this space by $\pi_0\mathscr{G}(P)$, we note that $U(1)$ is a $K(\mathbb{Z}, 1)$, so homotopy classes of maps $M \to U(1)$ are classified by $H^1(X, \mathbb{Z})$. Therefore, upon quotienting by $\pi_0\mathscr{G}(P)$, we get

$$\mathscr{A}_{\mathrm{YM}}(P)/\mathscr{G}(P) \cong Z^1_M/\mathscr{G}(P) \cong H^1(X, \mathbb{R})/H^1(X, \mathbb{Z})$$

which is a torus $\mathbb{T}^{b_1(M)}$, where $b_1(M)$ is the first Betti number. It is important to note that the isomorphism $\mathscr{A}_{\mathrm{YM}}(P)/\mathscr{G}(P) \cong H^1(X, \mathbb{R})/H^1(X, \mathbb{Z})$ is only as topological spaces, as we will see later that the Yang-Mills moduli space is a torsor over $H^1(X, \mathbb{R})/H^1(X, \mathbb{Z})$.

The torus $H^1(X, \mathbb{R})/H^1(X, \mathbb{Z})$ can also be realized ***Jacobian*** $\mathrm{Jac}(M)$ of M, which parameterizes flat $U(1)$ bundles over M up to gauge equivalence. The space $\mathscr{A}_{\mathrm{Flat}}(M)$ of flat $U(1)$ bundles up to gauge equivalence over M is well known to be the space of unitary representations $\rho : \pi_1(M) \to U(1)$. Since $U(1)$ is abelian, this factors through the abelianization, which, upon doing so, gives the same identification of $\mathrm{Jac}(M)$ as a torus of dimension $b_1(M)$.

To realize $\mathscr{A}_{\mathrm{YM}}(P)$ as a torsor over $\mathrm{Jac}(M)$, we pass though the correspondence

$$\{U(1)\text{-bundles } P \to M \text{ with connection}\} \leftrightarrow \{\text{Line bundles } L \to M \text{ with unitary connection}\}$$

The correspondence is obtained in one direction by taking associated bundles with the defining representation of $U(1)$, and the other direction comes from taking the unitary frame bundle with respect to some Hermitian fiber metric, along with the Chern connection. Then fix any principal $U(1)$ bundle $P \to M$, and let $L \to M$ be its associated line bundle. Taking the tensor product with a trivial bundle clearly results in an isomorphic line bundle, and for a Yang-Mills connection A on L, it is also clear that tensoring A with a flat connection yields another Yang-Mills connection, giving us an action $\mathscr{A}_{\mathrm{Flat}}(M)$ on $\mathscr{A}_{\mathrm{YM}}(P)$. Furthermore, tensoring by gauge equivalent flat connections clearly results in gauge equivalent connections, so this action factors through to an action of $\mathrm{Jac}(M)$ on $\mathscr{A}_{\mathrm{YM}}(P)/\mathscr{G}(P)$. To show that this gives $\mathscr{A}_{\mathrm{YM}}(P)$ the structure of a $\mathrm{Jac}(M)$-torsor, it suffices to show that for topologically isomorphic line bundles $L_1, L_2 \to M$ equipped with Yang-Mills connections A_1 and A_2, the product bundle $L_1 \otimes L_2^* \to M$ equipped with connection $A_1 \otimes A_2^*$ is a flat bundle. This follows immediately from the fact that $A_2 = A_1 + \eta$ for some closed form η, so $A_1 \otimes A_2^* = d + \eta$, which is a flat connection on the trivial bundle.

The case of principal $\mathbb{T}^n$ bundles is only a minor extension of the $U(1)$ case. We have a similar correspondence between principal $\mathbb{T}^n$-bundles and rank n vector bundles $E \to M$ along with the data of a direct sum decomposition $E = L_1 \oplus \cdots \oplus L_n$ of E into Hermitian line bundles. Identifying the Lie algebra of $\mathbb{T}^n$ with $\mathbb{R}^n$, we see that the curvature form F_A of a Yang-Mills connection A on E can be viewed as a vector 2-forms with each component being the curvature form for a connection on a direct summand L_i. The condition that a connection A on E is a Yang-Mills connection is then seen to be equivalent to each component of F_A being a harmonic 2-form on M. Therefore, the data of a Yang-Mills connection on a principal $\mathbb{T}^n$-bundle $P \to M$ is equivalent to the data of Yang-Mills connections A_i for each of the line bundles L_i, noting that the direct sum of Yang-Mills connections is also a Yang-Mills connection.

2. The Case of Riemann Surfaces

Let Σ be a Riemann surface of genus $g \geq 1$, and fix a Riemannian metric on Σ such that the Riemannian volume form ω satisfies

$$\int_\Sigma \omega = 1$$

In this case, the first Chern class $c_1(P)$ of a principal $U(1)$-bundle is an integer, using the identification $H^2(\Sigma, \mathbb{Z}) \cong \mathbb{Z}$ using the orientation induced by the complex structure. We abuse notation and let $c_1(P)$ denote the integer under this correspondence, and let $[c_1(P)]$ denote the cohomology class. Fix once and for all a principal $U(1)$-bundle $Q \to \Sigma$ with $c_1(Q) = 1$ and a Yang-Mills connection A_0, which will serve as our reference bundle with connection. The standard Hodge theory argument shows that the curvature F_{A_0} must be $2\pi i\omega$. Let $p : \widetilde{\Sigma} \to \Sigma$ be the universal cover of Σ, and consider the pullback bundle

$$\begin{tikzcd} p^*Q \arrow[r] \arrow[d] & Q \arrow[d] \\ \widetilde{\Sigma} \arrow[r, "p"'] & \Sigma \end{tikzcd}$$

Since the genus of Σ is greater than 0, we know $\widetilde{\Sigma}$ is contractible, so p^*Q is a trivial bundle $\widetilde{\Sigma} \times U(1)$, and the pullback connection still has curvature $2\pi i\omega$. We have a covering map $\widetilde{\Sigma} \times \mathbb{R} \to \widetilde{\Sigma} \times U(1)$ given by exponentiation in the second factor, which gives us

$$\begin{tikzcd} \widetilde{\Sigma} \times \mathbb{R} \arrow[d] & \\ \widetilde{\Sigma} \times U(1) \arrow[r] \arrow[d] & Q \arrow[d] \\ \widetilde{\Sigma} \arrow[r, "p"'] & \Sigma \end{tikzcd}$$

Then since $\widetilde{\Sigma} \times U(1)$ is a trivial bundle, the composite map $\widetilde{\Sigma} \times \mathbb{R} \to \Sigma$ is a principal bundle. Denote the structure group of this bundle by $\Gamma_\mathbb{R}$. We then determine the structure of $\Gamma_\mathbb{R}$. Since the action of $\pi_1(M)$ on $\widetilde{\Sigma}$ commutes with the $\mathbb{R}$ action on $\widetilde{\Sigma} \times \mathbb{R}$, it follows that $\Gamma_\mathbb{R}$ is a central extension of $\pi_1(M)$ by $\mathbb{R}$ so it fits into the short exact sequence of groups

$$1 \longrightarrow \mathbb{R} \longrightarrow \Gamma_\mathbb{R} \longrightarrow \pi_1(M) \longrightarrow 1$$

Finally, the connection on $\widetilde{\Sigma} \times U(1)$ lifts to a connection on $\widetilde{\Sigma} \times \mathbb{R}$, which follows from the fact that we can lift horizontal distributions along covering spaces. By a slight abuse of notation, we also refer to this connection as A_0. Once more, the curvature of the connection on $\widetilde{\Sigma} \times \mathbb{R}$ remains equal to $2\pi i\omega$. To determine the group $\Gamma_\mathbb{R}$ we identify Σ as a quotient of the $2g$-gon with edges labeled by $a_1, b_1 \ldots, a_g, b_g$ and their inverses. Then it suffices to compute the holonomy of the connection A_0 about the boundary path $\prod_i [a_i, b_i]$, since the holonomy about this path determines the $U(1)$ action on $\widetilde{\Sigma} \times U(1)$, which determines the action on $\widetilde{\Sigma} \times \mathbb{R}$ by lifting along the covering map. By pushing the path $\prod_i [a_i, b_i]$ into the interior of the $2g$-gon, we obtain a closed loops that bounds a disk in Σ, so the holonomy about the boundary of the disk is given by the integral of the curvature form $2\pi i\omega$. Taking the limit as we push this path out to the boundary, we find that the holonomy is computed by the integral

$$\int_\Sigma 2\pi i\omega = 2\pi c_1(Q) = 2\pi$$

which tells us the holonomy traverses the fiber once. Putting everything together, we get that $\Gamma_\mathbb{R}$ is the central extension of $\pi_1(M)$ obtained by adjoining a central element J that generates a subgroup isomorphic to $\mathbb{R}$, along with the relation $\prod_i [a_i, b_i] = J$.

The purpose of this construction is to realize the bijective correspondence:

$$\{U(1)\text{-bundles } P \to \Sigma \text{ with Yang-Mills connection}\} / \mathscr{G}(P) \longleftrightarrow \mathsf{Hom}(\Gamma_\mathbb{R}, U(1))$$

One direction is clear– given a homomorphism $\rho : \Gamma_\mathbb{R} \to U(1)$, we can form the associated bundle $(\widetilde{\Sigma} \times \mathbb{R}) \times_{\Gamma_\mathbb{R}} \times U(1)$ with connection $\dot{\rho}(A_0)$. The fact that $\dot{\rho}(A_0)$ is a Yang-Mills connection follows from the observation

$$\dot{\rho}(d \star F_{A_0}) = d \star \dot{\rho}(F_A)$$

where $\dot{\rho} : \mathbb{R} \to \mathbb{R}$ is the derivative of ρ at the identity after making the identifications of $\mathrm{Lie}(\Gamma_\mathbb{R}) \cong \mathbb{R}$ and $\mathfrak{u}(1) \cong \mathbb{R}$.

For the other direction, let $P \to \Sigma$ be a $U(1)$-bundle with Yang-Mills connection A. By passing to the line bundle perspective, we have a line bundle $L \to \Sigma$ with Yang-Mills connection, and the original reference bundle Q corresponds to another line bundle $L_0 \to \Sigma$ with Yang-Mills connection A_0. The usual Hodge theory gives us that $F_A = 2\pi i c_1(L)\omega$, so up to gauge equivalence, we may write $L = L_0^{\otimes c_1(L)}$, equipped with the connection $A_0^{\otimes c_1(L)} \otimes \Theta$, where Θ is a flat connection on the trivial line bundle. The flat bundle with connection Θ furnishes us with a homomorphism $\varphi : \pi_1(M) \to U(1)$. The rest of the argument follows from the following observation : A group homomorphism $\rho : \Gamma_\mathbb{R} \to U(1)$ necessarily factors through $\pi_1(M)$, since $U(1)$ being abelian implies that

$$\rho\left(\prod_i [a_i, b_i]\right) = 1$$

So any such homomorphism must necessarily map the central generator J to the identity. However, we note that a map $\Gamma_\mathbb{R} \to U(1)$ is still more data than a map $\pi_1(\Sigma) \to U(1)$, since we are also provided the information of the differential $\dot{\rho} : \mathbb{R} \to \mathbb{R}$. The fact that $\rho(J) = 1$ implies that $\dot{\rho}(1)$ must be integral, since $e^{\dot{\rho}(1)} = \rho(J)$. Therefore, we construct the group homomorphism $\rho : \Gamma_\mathbb{R} \to U(1)$ maps all of $\mathbb{R}$ to 1, and agrees with $\varphi : \pi_1(\Sigma) \to U(1)$ on the generators $a_1, b_1, \ldots a_g, b_g$, but satisfies $\dot{\rho}(1) = c_1(L)$. In this way, we see that the central element J determines the topological type of the bundle, while the map $\pi_1(M) \to U(1)$ determines the connection up to gauge equivalence.

In the general case, recall that for a fixed bundle P, we fixed a reference connection A_0 on P, which allowed us to identify $\mathscr{A}_{\mathrm{YM}}(P)$ with the space of closed forms. In the special case of a Riemann surface, fixing a connection on the bundle Q provided a reference connection on *all* principal $U(1)$-bundles, since $U(1)$-bundles over Σ are classified by integers via the first Chern class, so they can all be written as tensor powers of L and L^*.