SYMPLECTIC GEOMETRY AND KHÄLER MANIFOLDS: AN INTRODUCTION

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Linear Algebra

As typical when studying smooth manifolds, we first look at the linear algebra that will be non-linearized. To discuss Khäler manifolds (and consequently, Khäler vector spaces), we will need to assemble several mutually compatible structures – an inner product, a symplectic form, and a complex structure.

Definition 0.1. An ***inner product space*** is a vector space V equipped with a bilinear map $g : V \times V \to V$ that is symmetric and positive definite. In the case that V is a complex vector space, we replace the bilinear condition with the ***sesquilinear*** condition – conjugate linear in the first term, and linear in the second term. In this case, g is often referred to as a ***hermitian inner product***.

You're probably familiar with these, so we won't delve into them.

Definition 0.2. A ***symplectic vector space*** is a vector space V equipped with a nondegenerate skew symmetric bilinear form $\omega : V \times V \to V$.

Nondegeneracy here means that if for some v, $\omega(v, w) = 0$ for all $w \in V$, then $v = 0$. In other words, the form ω gives an isomorphism $V \to V^*$ via the mapping

$$v \mapsto \omega(v, \cdot)$$

We note that the positive definite condition on an inner product g implies nondegeneracy, so that also defines an analogous isomorphism. An equivalent formulation of nondegeneracy for ω is that the wedge product $\omega \wedge \ldots \wedge \omega$ with itself $n/2$ times gives a nonzero volume form for V.

Example 0.3. $\mathbb{R}^{2n}$ with coordinates (x_i, y_i) comes with a symplectic structure given by

$$\tilde{\omega} = \sum_i dx^i \wedge dy^i$$

which can be expressed in coordinates by

$$\tilde{\omega}(v, w) = v^T \Omega\, w$$

where

$$\Omega = \begin{pmatrix} 0 & \mathrm{id}_{\mathbb{R}^n} \\ -\mathrm{id}_{\mathbb{R}^n} & 0 \end{pmatrix}$$

In some sense, this is the *only* symplectic vector space, which should be made clear very soon.

Theorem 0.4. *Every symplectic vector space (V, ω) admits a* ***symplectic basis*** *$\{e_i, f_i\}$ where*

$$\omega(e_i, e_j) = 0 \qquad \omega(e_i, f_i) = \delta_{ij}$$

Corollary 0.5. *Every symplectic vector space is* ***symplectomorphic*** *to $(\mathbb{R}^{2n}, \tilde{\omega})$. That is, there exists a linear isomorphism $\varphi : (V, \omega) \to (\mathbb{R}^{2n}, \tilde{\omega})$ where $\varphi^*\tilde{\omega} = \omega$.*

Proof. Let $\{f_i, g_i\}$ be a symplectic basis for V on $\mathbb{R}^n$ and consider the map $\varphi : V \to \mathbb{R}^{2n}$ given by mapping $f_i \mapsto e_i$ and $g_i \mapsto e_{n+i}$ ■

Definition 0.6. A ***complex structure*** on a vector space V is an automorphism $J : V \to V$ such that $J^2 = -\mathrm{id}_V$

Given a complex structure J and an $\mathbb{R}$-vector space V, we can make V a $\mathbb{C}$-vector space by defining the action of $i \in \mathbb{C}$ by $i \cdot v = Jv$. We can then extend this to arbitrary complex numbers $\alpha + \beta i$ by $(\alpha + \beta i) \cdot v = \alpha v + \beta J v$.

If you know some linear algebra, you might know that there is another way to turn a $\mathbb{R}$-vector space into a complex one.

Definition 0.7. Given an $\mathbb{R}$-vector space V, define its complexification $V_{\mathbb{C}}$ as $V \otimes_{\mathbb{R}} \mathbb{C}$.

Note that adding a complex structure to a vector space is *not* the same as complexifying it. If we find some $J \in GL(V)$ such that $J^2 = -\operatorname{id}_V$, we haven't changed the dimension of V, but complexifying V doubles its dimension over $\mathbb{R}$. Despite this, the concepts are quite similar. If we have a vector space V with complex structure J, then if we complexify V, then J extends to a map $V_{\mathbb{C}} \to V_{\mathbb{C}}$, where $J(v + iw) = Jv + iJw$. This gives a decomposition

$$V_{\mathbb{C}} = V^+ \oplus V^-$$

where $V^+ = \{v + iJv : v \in V\}$ and $V^- = \{v - iJv : v \in V\}$, and we get an isomorphism of $\mathbb{C}$-vector spaces $V \to V^{\pm}$ where $v \mapsto v \pm iJv$, using the fact that $J^2 v = -v$.

Now that we have defined all of these, we would like to define when these structures are compatible, and see what results

Definition 0.8. A complex structure J is ***compatible*** with a symplectic form ω if $\omega(Jv, Jw) = \omega(v, w)$.

An analogous definition is used for an inner product g.

Theorem 0.9. *Given a vector space V with complex structure J, given a J-compatible inner product g, we obtain a symplectic form ω where $\omega(v, w) = g(Jv, w)$. Likewise, given a J-compatible symplectic form ω, we obtain an inner product g where $g(v, w) = \omega(Jv, w)$. Symplectic forms/inner products obtained in this manner are said to be* ***compatible*** *with the other.*

Definition 0.10. A vector space V is ***Kähler*** if it has compatible structures J, g, ω.