THE LAPLACE-DE RHAM OPERATOR ON A RIEMANNIAN MANIFOLD

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In $\mathbb{R}^2$, we know about the standard Laplace operator on $C^\infty(\mathbb{R}^2)$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

In a more general setting, let (M, g) be a Riemannian manifold. We can define an analogous operator

$$\Delta = \operatorname{div}(\operatorname{grad} f)$$

In local coordinates (x^i), we have that for $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$

$$\operatorname{grad} f = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}$$

$$\operatorname{div} X = \frac{1}{\sqrt{\det g_{ij}}} \frac{\partial}{\partial x^i} \left((X^i \sqrt{\det g_{ij}} \right)$$

Where g_{ij} is the symmetric matrix given by $g_{ij} = \langle \partial_i, \partial_j \rangle$ and g^{ij} is the inverse of g_{ij}. This gives the coordinate formula for

$$\Delta f = \frac{1}{\sqrt{\det g_{ij}}} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{\det g_{ij}} \frac{\partial f}{\partial x^j} \right)$$

Which using the standard metric $g_{ij} = \delta_{ij}$ on $\mathbb{R}^2$ recovers the standard Laplacian. However, we want to generalize Δ to arbitrary differential forms, which requires us to construct a bit of machinery.

To do this, we first note that the metric g determines an inner product on each tangent space T_pM where $\langle v, w \rangle = g_p(v, w)$. From this, we can construct an inner product on the alternating tensors $\Lambda^k(T_pM)$, which will give us a smoothly varying inner product on $\Omega^K(M)$. To do this, we will use the fact that g determines a bundle isomorphism $TM \to T^*M$ via the mapping $(x, v) \mapsto (x, \langle v, \cdot \rangle)$.

Proposition 1.1. *For a Riemannian manifold (M, g), there is a unique inner product on each $\Lambda^k(T_pM)$ characterized by the formula*

$$\langle \omega^1 \wedge \ldots \wedge w^k, \eta^1 \wedge \ldots \wedge \eta^k = \det \left(\langle (\omega^i)^\sharp, (\eta^j)^\sharp \right) \rangle$$

Where $\sharp$ is the index raising operator $\omega_i dx^i \mapsto g^{ij} \omega_j \frac{\partial}{\partial x^i}$.

Proof. We define the inner product locally in terms of an orthonomal frame E_i, and show that it is independent of the choice of frame. Let ε^i denote the coframe to E_i. We first claim that the set of ε^I where I is a strictly increasing multi-index of length k form an orthonormal basis. To see this, we compute

$$\langle \varepsilon^I, \varepsilon^J \rangle = \det \left(E_{i_k}, E_{j_\ell} \right)$$

We note that this is 1 if and only if $I = J$, since then the matrix we are taking the determinant of is $\mathrm{id}_{\mathbb{R}^k}$, otherwise, I contains some i_k not in J, which implies the k^{th} row of the matrix is 0, so the determinant is 0. This then defines an inner product by extending linearly to arbitrary k-forms.

To show that this is independent of our choice of frame, let B_i be another orthonormal frame with coframe β^i. Then we know that $B_i = A_i^j E_j$ with smooth functions A_j^i forming an orthogonal matrix every point. We then compute

$$\begin{aligned}\langle \beta^I, \beta^J \rangle &= \det\langle B_{i_k}, B_{j_\ell} \rangle \\ &= \det\langle A^j_{i_k} E_j, A^p_{j_\ell} E_p \rangle\end{aligned}$$

Noting that $A^j_{i_k} E_j$ is just the i_k^{th} column of the matrix A, we have that this is equal to $\det\langle A_{i_k}, A_{j_\ell} \rangle$. Again, if $I = J$, this is just the identity matrix, but if $I \neq J$, there will be a row of zeroes in the matrix $\langle A_{i_k}, A_{j_\ell}$, so the determinant will be 0. This shows that $\langle \cdot, \cdot \rangle$ is uniquely characterized. ∎

Remark. One observation is that the Riemannian volume form dV_g is the unique n-form on M with norm 1.

We can then use this inner product to produce an important operator. Recall that given a function $f \in C^\infty(M)$, we can define the integral of f over M by integrating the n-form $f dV_g$, which is a bundle homomorphism $\Omega^0(M) \to \Omega^n(M)$. We can generalize this to arbitrary k forms.

Proposition 1.2. *For every $k \in \{0, \dots n\}$, there exists a unique bundle homomorphism*

$$\star : \Omega^k(M) \to \Omega^{n-k}(M)$$

*called the **Hodge star operator** such that for any $\omega, \eta \in \Omega^k(M)$, we have that $\omega \wedge \star\eta = \langle \omega, \eta \rangle dV_g$ where dV_g is the Riemannian volume form. The $n - k$-form $\star\omega$ is often referred to as the **Hodge dual** to ω*

Proof. We first prove uniqueness. Let ε^i be the coframe to an orthonomal basis E_i. Then for and increasing index set I of length k, we have that $\star$ must satisify

$$\varepsilon \wedge \star\varepsilon^I = dV_g$$

Therefore, we must have that $\star\varepsilon^I = \pm\varepsilon^J$, where $I \cup J = \{1, \dots n\}$ and J is an increasing index and the sign are chosen such that when we permute I and J to be in increasing order, the sign chosen for $\star\varepsilon^I$ cancel the ones that come from the permutation, since otherwise, $\varepsilon \wedge \star\varepsilon^I = 0$. This uniquely characterizes $\star$ on a basis, so it uniquely extends linearly to $\Omega^k(M)$. ∎

One observation we make is that $\star\star\varepsilon^I = (-1)^{k(n-k)}\varepsilon^I$,which can be verified by shuffling the wedge products and carefully tracking signs. This extends to all k-forms, so $\star\star\omega = (-1)^{k(n-k)}\omega$. Another observation is that this determines a bundle *isomorphism* $\Omega^k(M) \to \Omega^{n-k}(M)$, since it maps an orthonormal basis to an orthonormal basis.

Example 1.3. In $\mathbb{R}^n$ with the standard coordinates x^i and the standard metric tensor $g_{ij} = \delta_{ij}$, we have that the dx^i form an global orthonormal frame for $\mathbb{R}^n$. Given any dx^i, we have that

$$\star dx^i = (-1)^{i-1} dx^1 \wedge \dots \wedge \hat{dx^i} \wedge \dots \wedge dx^n$$

Where $\hat{dx^i}$ indicates that dx^i is missing from the wedge product. The sign comes from the fact that

$$dx^i \wedge dx^1 \wedge \dots \wedge \hat{dx^i} \wedge \dots \wedge dx^n = (-1)^{i-1} dx^1 \wedge \dots \wedge dx^n$$

Example 1.4. For $\mathbb{R}^5$, consider $\star\star dx^1 \wedge dx^3$. We first compute

$$\begin{aligned}\star dx^1 \wedge dx^3 &= -dx^2 \wedge dx^4 \wedge dx^5 \\ \star\star dx^1 \wedge dx^3 &= \star - dx^2 \wedge dx^4 \wedge dx^5 = dx^1 \wedge d^3\end{aligned}$$

Finally, we can use the Hodge star to define yet another operator

Definition 1.5. Let (M, g) be a compact oriented Riemannian manifold. Then the ***codifferential*** δ (also denoted in the literature by d^*) is a map

$$\begin{aligned}\delta &: \Omega^k(M) \to \Omega^{k-1}(M) \\ \delta\omega &= (-1)^{n(k+1)+1} \star d \star \omega\end{aligned}$$

Where δ is defined on $\Omega^0(M) = C^\infty(M)$ by $\delta f = 0$ for all smooth functions f.

Proposition 1.6. *The codifferential δ on a Riemannian manifold (M, g) without boundary satisfies the following properties:*

(1) $\delta^2 = 0$

(2) For $\omega, \eta \in \Omega^k(M)$, let

$$(\omega, \eta) = \int_M \langle \omega, \eta \rangle \, dV_g$$

Then for $\omega \in \Omega^k(M)$ and $\eta \in \Omega^{k-1}(M)$, we have that

$$(\delta\omega, \eta) = (\omega, d\eta)$$

*where d is the exterior derivative. In this way, we see that δ is the **adjoint** of d with respect to the inner product, justifying the name **codifferential**.*

Proof. (1) We have that

$$\begin{aligned}\delta^2 &= (-1)^{n(k+1)+1} \delta \star d \star \\ &= (-1)^{n(k+1)+1} (-1)^{nk+1} \star d \star \star d \star\end{aligned}$$

We note that $\star\star = (-1)^{k(n-k)} \operatorname{id}_{\Omega^k(M)}$, so this simplifies to

$$(-1)^p \star dd\star = 0$$

Since $d^2 = 0$.

(2) We first verify that $(\cdot, \cdot)$ determines an inner product. We note that it is symmetric since $\langle \cdot, \cdot \rangle$ is symmetric, and it is also bilinear since integration is linear and $\langle \cdot, \cdot \rangle$ is as well. All that remains is to show that it is positive definite. We note that it is positive since $\langle \omega, \omega \rangle$ is positive for all ω, so

$$\int_M \langle \omega, \omega \rangle dV_g > 0$$

. In addition, we have that $\langle \omega, \omega \rangle = 0$ if and only if $\omega = 0$, and $\int_M f dV_g = 0$ if and only if $f = 0$. Therefore, $(\cdot, \cdot)$ is positive definite, so it defines an inner product on $\Omega^k(M)$.

We note that by how we've define the $\star$ operator, the inner product is given by the equivalent definition for $\xi, \alpha \in \Omega^k(M)$

$$(\xi, \alpha) = \int_M \xi \wedge \star\alpha$$

Therefore, we wish to prove the equivalent statement that for $\omega \in \Omega^k(M)$ and $\eta \in \Omega^{k-1}(M)$

$$\int_M \delta\omega \wedge \star\eta = \int_M \omega \wedge \star d\eta$$

Then using symmetry of the inner product, this is equivalent to the statement

$$\int_M \eta \wedge \star\delta\omega = \int_M d\eta \wedge \star\omega$$

We then compute

$$\begin{aligned}d\eta \wedge \star\omega - \eta \wedge \star\delta\omega &= d\eta \wedge \star\omega - (-1)^{n(k+1)+1} \eta \wedge \star \star d \star \omega \\ &= d\eta \wedge \star\omega - (-1)^{n(k+1)+1} (-1)^{(n-k+1)(n-(n-k+1))} \eta \wedge d \star \omega \\ &= d\eta \wedge \star\omega + (-1)^{-k^2+1} \eta \wedge d \star \omega \\ &= d\eta \wedge \star\omega + (-1)^{k-1} \eta \wedge d \star \omega \\ &= d(\eta \wedge \star\omega)\end{aligned}$$

Where we use the fact that $-k^2 + 1$ has the opposite parity of k, and that d is an antiderivation on $\Omega(M)$ Therefore, we have by Stokes' Theorem

$$\int_M d\eta \wedge \star\omega - \eta \wedge \star d\omega = \int_M d(\eta \wedge \star\omega) = \int_{\partial M} \eta \wedge \star\omega = 0$$

Which gives us that

$$(\delta\omega, \eta) = (\omega, d\eta)$$

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Finally, we have the necessary tools to define the fabled ***Laplace-de Rham Operator*** (Also known as the ***Laplace-Beltrami Operator***.

Definition 1.7. On a oriented compact Riemannian manifold (M, g), define the ***Laplace-de Rham Operator***, denoted Δ, as the family of maps $\Omega^k(M) \to \Omega^k(M)$ such that

$$\Delta = \delta d + d\delta$$

A k-form ω satsifying $\Delta\omega = 0$ is called ***harmonic***.

Proposition 1.8. *The Laplace-de Rahm operator agrees with (up to sign) to the Laplacian on $C^\infty(M)$*

Proof. We have that for $f \in C^\infty(M)$ (letting Δ denote the Laplace-de Rham operator)

$$\Delta f = d\delta f + \delta df = \delta df$$

Since we defined δ to be 0 on $\Omega^0(M) = C^\infty(M)$. We then have that $\delta df = - \star d \star df$. We first compute $\star df$, which is the $n-1$-form satisfying

$$df \wedge \star df = \langle df, df \rangle \, dV_g$$

. Noting that $df^\sharp = \operatorname{grad} f$, we compute

$$\begin{aligned}
\langle \operatorname{grad} f, \operatorname{grad} f \rangle \, dV_g &= \frac{\partial f}{\partial x^i} dx^i \left(g^{jk} \frac{\partial f}{\partial j} \frac{\partial}{\partial x^k} \right) \, dV_g \\
&= g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} \, dV_g \\
&= \sqrt{\det g} \left(g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} dx^1 \right) \wedge \ldots \wedge dx^n
\end{aligned}$$

Therefore, we conclude that

$$\star df = (-1)^{i-1} \sqrt{\det g} \left(g^{ij} \frac{\partial f}{\partial x^j} \right) dx^1 \wedge \ldots \hat{dx^i} \wedge \ldots \wedge dx^n$$

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We then compute $d \star df$ to be

$$\begin{aligned}
d \star df &= (-1)^{i-1} \frac{\partial}{\partial x^k} \left(g^{ij} \sqrt{\det g} \frac{\partial f}{\partial x^j} \right) dx^k \wedge dx^1 \wedge \ldots \wedge \hat{dx^i} \wedge \ldots \wedge dx^n \\
&= \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{\det g} \frac{\partial f}{\partial x^j} \right) dx^1 \wedge \ldots \wedge dx^n \\
&= \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{\det g} \frac{\partial f}{\partial x^j} \right) dV_g
\end{aligned}$$

We then have that $\star d \star df$ must satsify $d \star f \wedge \star d \star df = \langle d \star f, d \star f \rangle \, dV_g$. We then compute

$$\begin{aligned}
&\left\langle \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{\det g} \frac{\partial f}{\partial x^j} \right) dV_g, \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^k} \left(g^{k\ell} \sqrt{\det g} \frac{\partial f}{\partial x^\ell} \right) \, dV_g \right\rangle dV_g \\
&= \frac{1}{\det g} \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{\det g} \frac{\partial f}{\partial x^j} \right) \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^k} \left(g^{k\ell} \sqrt{\det g} \frac{\partial f}{\partial x^k} \right) \, dV_g
\end{aligned}$$

Where we use the fact that $\langle dV_g, dV_g \rangle = 1$. Therefore, we conclude that

$$\delta df = - \star d \star f = - \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{\det g} \frac{\partial f}{\partial x^j} \right)$$

Due to our choice of sign convention, this is unfortunately the negative of the the original Laplacian we had. Some sign conventions define $\Delta f = - \operatorname{div}(\operatorname{grad} f)$ for this reason.