SPIN GEOMETRY CONFERENCE COURSE

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Week 1

Exercise 1.1. Prove $SL_n(\mathbb{R})$ and $O(n)$ are manifolds

Exercise 1.2. What is the "shape" of $SL_2(\mathbb{R})$?

Exercise 1.3. Prove that

$$O(2) = \left\{ \begin{pmatrix} \cos\theta - \sin\theta \\ \sin\theta \cos\theta \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} \cos\theta \sin\theta \\ \sin\theta - \cos\theta \end{pmatrix} \right\}$$

The first set consists of rotations and the second set consists of reflections. Which rotations commute? Which reflections commute? Do reflections commute with reflections?

Exercise 1.4. Investigate $O(3)$. What is it's "shape?"

Week 2

Exercise 2.1. What is the derivative of det?

Exercise 2.2. Explore the exponetial map $\mathfrak{sl}_2(\mathbb{R}) \to SL_2(\mathbb{R})$

Exercise 2.3. Prove that every element of $O(n)$ can be written as the composition of at most n reflections about hyperplanes in $\mathbb{R}^n$.

Proof. We do this by induction. For $n = 1$, this is obvious, since $O(1) \cong \pm 1$. The assuming that this holds for dimension $n - 1$, Let $A \in O(n)$, and let $v \in \mathbb{R}$. We want to construct a hyperplane reflection R such that $RAv = v$, which is obtained by taking R to be the hyperplane reflection about the bisector of v and Av. More explicitly, take R to be the hyperplane reflection about the vector

$$\frac{Av - v}{\|Av - v\|}$$

which is given by the equation

$$Rw = w - 2\frac{\langle Av - v, v\rangle}{\langle Av - v, Av - v\rangle}(Av - v)$$

Computing its action on v, we get

$$\begin{aligned} Rv &= v - 2\frac{\langle Av - v, v\rangle}{\langle Av - v, Av - v\rangle}(Av - v) \\ &= v - \frac{2\langle Av, v\rangle - 2\langle v, v\rangle}{2\langle v, v\rangle - 2\langle Av, v\rangle}(Av - v) \\ &= v + Av - v \\ &= Av \end{aligned}$$

Then since R is its own inverse (being a reflection), we have that $RAv = v$, so RAv fixes v and its orthogonal complement. ∎

TODO insert motivation of $A_n^{\pm}$

Definition 2.4. Define $A_n^{\pm}$ to be the unital algebra generated by $\mathbb{R}^n$ such that $\xi^2 = \pm 1$. and $\xi\eta =? \eta\xi$. Determine the sign of $\eta\xi$. Explore these algebras. Find $A\pm_1$, $A_2^{\pm}$ .... What are they isomorphic to? Can you identify $O(n)$ as a subgroup?

Week 3

Exercise 3.1. Classify the algebras A_n^+ (we messed these up week 2).

Exercise 3.2. Prove that
$$\{e_{i_1}e_{i_2}\dots e_{i+k} \mid 1 \leq i_1 < i_2 < \dots < i_k \leq n\}$$
is a basis for $A_n^\pm$.

Exercise 3.3. Modify the isomorphisms found for A_n^- by choosing $\mathbb{Z}/2\mathbb{Z}$ gradings for the domains and codomains such that the isomorphisms are now isomorphisms as superalgebras.

Exercise 3.4. Construct a tensor product for super vector spaces and superalgebras.

Exercise 3.5. Explore the "shape" of the group
$$G = \langle v \mid \|v\| = 1\rangle \subset (A_n^-)^\times$$
and the nature of the surjection $G \twoheadrightarrow O(n)$. What is the kernel of this map?

Week 4

Exercise 4.1. Define $\varphi : A_n^\pm \to A_n^\pm$ by $\varphi(v) = -v$ and $\varphi(vw) = (wv)$, and extending linearly to sums. Does $\varphi(x) \cdot x$ define a norm on $A_n^\pm$?

Exercise 4.2. Let $(A, \|\|)$ be a normed $\mathbb{R}$-algebra such that $\|ab\| \leq \|a\| \|b\|$ for all $a, b \in A$. Show that the multiplicative units form an open subset.

Exercise 4.3. An algebra A is called a ***matrix algebra*** if there exists an isomorphism $A \cong \mathrm{End}(V)$ for some vector space V. Which $A_n^\pm$ are matrix algebras?

Exercise 4.4. Let V be a vector space and $b : V \times V \to V$ a bilinear form. We want to construct the Clifford algebra $\mathrm{Cliff}(V, b)$ as the "best" associative unital $\mathbb{R}$-algebra generated by V subject to the relation $v_1v2 + v_2v_1 = 2b(v_1, b_2)1_A$, where 1_A denotes the multiplicative unit in A.