DIFFERENTIAL GEOMETRY LECTURE 2

JEFFREY JIANG

1. Vector Fields

Recall that given a smooth n dimensional manifold M, we get the ***tangent bundle*** TM, which is a $2n$ dimensional smooth manifold, equipped with a map $\pi : TM \to M$, where the fiber $\pi^{-1}(p)$ over a point $p \in M$ is the tangent space T_pM. Also recall that a ***section*** of a π is a map $\sigma : U \to TM$ such that $\pi \circ \sigma = \mathrm{id}_U$. If $U = M$, we call σ a global section.

Definition 1.1. A ***vector field*** on a smooth manifold is a global section $X : M \to TM$. The set of vector fields on M is denoted $\mathfrak{X}(M)$.

If you unwrap the definitions, we see that a section is exactly the data we want – for every point $p \in M$, we are assigning to it a vector in T_pM. We will often denote the vector $X(p)$ at p by X_p.

If we fix a chart U with coordinates x^i, we get the ***coordinate vector fields*** $\partial/\partial x^i$, where

$$\left(\frac{\partial}{\partial x^i}\right)_p = \frac{\partial}{\partial x^i}\bigg|_p$$

Then given an arbitrary smooth vector field X, we have that X_p is a linear combination of the coordinate vector fields. Smoothness of X then tells us that we can write X locally as

$$X = X^i \frac{\partial}{\partial x^i}$$

for smooth functions $X^i : U \to \mathbb{R}$. We also note that $\mathfrak{X}(M)$ admits more structure than that of a set.

Proposition 1.2.

(1) Let $X, Y \in \mathfrak{X}(M)$. Then $fX + gY$ defined a smooth vector field, for $f, g \in C^\infty(M)$.
(2) $\mathfrak{X}(M)$ is a $C^\infty(M)$ module.

In more sheafy language, this tells us that the sheaf of vector fields on M is a sheaf of modules over the sheaf of smooth functions. In fact, we can say even more about the algebraic structure of vector fields. Given a chart U with coordinates x^i, we know we can write any vector field as $X^i\partial_i$. This tells us that the ∂_i form a basis for the local vector fields $\mathfrak{X}(U)$ as a $C^\infty(U)$ module, i.e. locally, the vector fields form a free module over $C^\infty(U)$. Note that this might not hold globally though.

Definition 1.3. A ***local frame*** for M is an collection of smooth vector fields E_i defined on an open set $U \subset M$ such that for each $p \in U$, we have that the $E_i|_p$ form a basis for T_pM. If $U = M$, we say that the E_i form a ***global frame***.

We've already seen a local frame, the coordinate vector fields ∂_i.

Recall that a vector $v \in T_pM$ acts on functions $f \in C^\infty(M)$ – it takes as an input a smooth function, and produces a real number. We see then that vector fields act on smooth functions as well, where we define the action pointwise to produce a new function. Explicity, given $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, we have

$$(Xf)(p) = X_pf$$

In this way, we see that a vector field X determines a linear map $C^\infty(M) \to C^\infty(M)$. In fact, it defines a ***derivation***, i.e.

$$X(fg) = fXg + gXf$$

since each vector X_p is a derivation at p.

Proposition 1.4. *Vector fields on M are in bijection with derivations $D : C^\infty(M) \to C^\infty(M)$, where the mapping is given by $X \mapsto D_X$ where $D_X f = Xf$.*

Given a vector field $X \in \mathfrak{X}(M)$ and a smooth map $F : M \to N$, we can apply the differential dF_p pointwise to X, but the result may not be well defined. For example, if F is not injective, there will exists at least two points p, q such that $F(p) = F(q) = y$. Then if we want to use F and X to define a vector field on Y, we have a conundrum – what vector should we assign to y? Should it be $dF_p(X_p)$ or $dF_q(X_q)$? Therefore in order for X to push forward to a vector field on Y, we must impose the condition on each fiber $F^{-1}(y)$ that for all $p \in F^{-1}(y)$, we have $dF_p(X_p)$ is the same.

Definition 1.5. Given smooth manifolds M and N, a smooth map $F : M \to N$, and vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. We say that X and Y are ***F-related*** if for all $q \in N$ and all $p \in F^{-1}(q)$, we have $dF_p(X_p) = Y_q$.

Given an arbitrary vector field X and a smooth map $F : M \to N$, it's not true in general that an F-related vector field exists in $\mathfrak{X}(N)$, however, in the case that F is a diffeomorphism, a unique F-related vector field exists, called the pushforward $F_* X$. In order for a vector field to be F-related to X, we see that it must be define by

$$(F_* X)_p = dF_{F^{-1}(p)}(X_{F^{-1}(p)})$$

which is well defined since F is invertible.