DIFFERENTIAL GEOMETRY LECTURE SERIES

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These are notes for the spring 2019 differential geometry lecture series for the math club. The plan is for the course to give an introduction to math club members to the basics and terminology of smooth manifolds. A good reference if you want to read more is John Lee's *Introduction to Smooth Manifolds*. In terms of prerequisites, it would be good to have a background in point set topology, multivariable calculus, and linear algebra.

Contents

1. Introduction to Manifolds

Definition 1.1. A ***topological manifold*** is a Hausdorff space X such there exists a countable open cover $\{U_\alpha\}$ of X, along with homeomorphisms $\varphi_\alpha : U_\alpha \to V_\alpha$, where V_α is an open subset of $\mathbb{R}^n$. Given a chart (U, φ), we can write φ in terms of its component functions

$$\varphi(p) = (x^1(p), \ldots, x^n(p))$$

The functions x^i are called ***local coordinates*** on U.

In this way, we see that a manifold is a topological space that is locally topologically indistinguishable from Euclidean space. Both modifiers are important here – there could be global and geometric properties that differ from $\mathbb{R}^n$.

Example 1.2.

(1) $\mathbb{R}^n$ is a topological manifold – it admits a global chart $(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})$.

(2) The 2-sphere $S^2 = \left\{(x,y,z) \in \mathbb{R}^3 \,:\, x^2+y^2+z^2 = 1\right\}$ is a topological manifold. If we let N and S denote the North and South poles repsectively, then we can construct charts using ***stereographic projection***. Stereographic projection from the North pole is a map $\varphi_N : S^2 - \{N\} \to \mathbb{R}^n$, where given a point $p \in S^2 - \{N\}$, we take the line cotaining both N and p, which intersects the $z = 0$ plane at one point q. We then define $\varphi_N(p) = q$. Stereographic projection φ_S from the South pole is defined in an analogous manner. An explicit formula for φ_N is

$$\varphi_N(x,y,z) = \left(\frac{x}{1-z}, \frac{x}{1-z}\right)$$

Exercise 1.3.

(1) S^2 does not admit a global chart. Why?

(2) Give an inverse map $\mathbb{R}^n \to \mathrm{S}^n - N$ for stereographic projection from the North Pole

(3) Generalize stereographic projection to the n-sphere $S^n = \left\{p \in \mathbb{R}^{n+1} \,:\, \|p\| = 1\right\}$

One of the main appeals of Euclidean space is that we have a lot of tools at our disposal, like linear algebra and calculus. We would like to generalize these notions to manifolds. One of the wonderful properties of manifolds is that they locally look like $\mathbb{R}^n$, so we can use the charts to translate concepts we know in $\mathbb{R}^n$ to concepts on the manifold.

Definition 1.4. A map $F : U \to V$ where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ is ***smooth*** if all of its component functions are infinitely differentiable, i.e. the partial derivatives of all orders exist and are continuous.

Note that smoothness is a local condition – to know a function is smooth at a point, it suffices to check in a small neighborhood. Now that we know what it means for a function on $\mathbb{R}^n$ to be smooth, how do we translate this to a manifold X? Given a map $F : X \to \mathbb{R}^m$, what does it mean for F to be smooth? Our first guess is to use our charts. Given