THE PONTRYAGIN-THOM THEOREM

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Framed Bordism

Let M be a smooth manifold of dimension n, and $Y \subset M$ a submanifold of codimension q, i.e. $\dim Y = n - q$. Then the tangent bundle TY is naturally a subbundle of the ambient tangent bundle $TM|_Y$ over Y. The normal bundle of Y in M, denoted NY, is the vector bundle over Y with each fiber being the quotient space T_pM/T_pY. This gives us the short exact sequence

$$0 \longrightarrow TY \longrightarrow TM|_Y \longrightarrow NY \longrightarrow 0$$

You can think of NY as the vector bundle of normal vectors pointing outward from Y.

Definition 1.1. A ***framing*** of a codimension q submanifold $Y \subset M$ is an isomorphism $NY \to R^q \times Y$. A ***framed submanifold*** of a manifold M is the data of a submanifold $Y \subset M$ and a framing $NY \to R^q \times Y$. We note that the isomorphism $NY \to R^q \times Y$ is equivalent data to a global frame (basis of sections) for NY, hence the name.

For example, if we consider $S^1 \subset \mathbb{R}^2$, we have that the tangent space $T_pS^1 \cong p^\perp$. Therefore, an example of a framing for S^1 is the assignment $p \mapsto p$, since p spans the orthogonal complement of T_pS^1, giving a global frame for NS^1.

Definition 1.2. Let X_0 and X_1 be framed codimension q submanifolds of M. Then X_1 and X_2 are ***framed bordant*** if there exists a framed submanifold $Y \subset [0,1] \times M$ such that

(1) $Y \cap \{i\} \times M = \{i\} \times X_i$ for $i = 0, 1$.
(2) The framings at $Y \cap \{i\}$ agree with the framings of X_i.

We denote the framed cobordism group of codimension q as $\Omega^{\mathrm{fr}}_{m-q}(M)$.

There's a little subtlety by what we mean by "the framings agree." By this, we note that the boundary of $[0,1] \times M$ contains two copies of M at the boundary, $\partial([0,1] \times M) = \{0\} \times M \coprod \{1\} \times M$, and the inclusion $M \hookrightarrow \{i\} \times M$ gives us an isomorphism of vector bundles $TM \cong T(\{i\} \times M) \subset T([0,1] \times M)$. Likewise, we get a isomorphism of normal bundles $NX_i \cong N(\{i\} \times Y)$, where the normal bundle of $\{i\} \times Y$ is taken inside of $\{i\} \times M$. Hence a framing of $\{i\} \times X_i$ is equivalent to a framing of X_i, and we ask for the framings to agree with the framing for Y. A trivial, but easy to visualize example is to take $S^1 \subset \mathbb{R}^2$, with the framing we gave earlier. Then the cylinder $[0,1] \times S^1 \subset [0,1] \times \mathbb{R}^2$ is a framed bordism $S^1 \to S^1$, where the framing is given by our previous framing at each $\{t\} \times S^1$.

In particular, there is one easy way to obtain a huge family of framed submanifolds. Let M be an m dimensional manifold, and N an n dimensional manifold with $n \leq m$. Then by Sards theorem, for any map $f : M \to N$, the set of critical points of f forms a set of measure 0. In other words, for almost any point $p \in N$, we have that $df_x : T_xM \to T_pN$ is a surjective map, giving us that $f^{-1}(p)$ is a submanifold of M of codimension n. The tangent space $T_xf^{-1}(p)$ is naturally identified with the kernel of df_x, so df_x factors through the quotient $T_xM/T_xf^{-1}(p)$, giving an isomorphism $T_xM/T_xf^{-1}(p) \to T_pN$. Therefore, fixing a basis for T_pN gives us an isomorphism $T_xM/T_xf^{-1}(p) \to \mathbb{R}^n \times f^{-1}(p)$, which is the definition of a framing for $f^{-1}(p)$.

The Pontryagin-Thom Theorem

We now consider the special case where we have M mapping into a sphere S^q. Let $[M, S^q]$ denote the set of all homotopy classes of maps $M \to S^q$. Fix a framed submanifold $Y \subset M$. Then it admits a ***tubular***

neighborhood, which is the data of an open set $U \subset M$ containing Y equipped with a submersion $U \to Y$, and a diffeomorphism $NY \to U$ such that

$$\begin{tikzcd} NY \arrow[rr] \arrow[dr] & & U \arrow[dl] \\ & Y & \end{tikzcd}$$

commutes. You can think of U as a "fattened up" version of Y, and the submersion $U \to Y$ squishes U onto Y. Then the framing for Y gives us an isomorphism $NY \to \mathbb{R}^q \times Y$, and then taking the inverse diffeomorphism $U \to NY$, we obtain a map $U \to NY \to \mathbb{R}^q \times Y \to \mathbb{R}^q$, where the last map is projection. We denote the map as $h : U \to \mathbb{R}^q$. Intuitively, the map $h(p)$ is the distance from p to Y. Then by fixing a cutoff function ρ, we define the ***Pontryagin-Thom collapse map*** of Y, $f_Y : M \to S^q$ by

$$f_Y(p) = \begin{cases} \frac{h(p)}{\rho(|h(p)|)} & p \in U \\ \infty & p \in M - U \end{cases}$$

Where we identify S^q as the one point compactification $\mathbb{R}^q \cup \{\infty\}$ via stereographic projection. Intuitively, the map collapses Y to a point, (namely 0), and smoothly stretches out U onto the sphere before collapsing the rest of M onto the point at infinity.

Theorem 3.1. *Let $\varphi : [M, S^q] \to \Omega^{fr}_{n-q}$ be the map taking $[f]$ to the preimage $f^{-1}(p)$ of a regular value, and let $\psi : \Omega^{fr}_{n-q} \to [M, S^q]$ be the map taking the bordism class $[Y]$ to the homotopy class of the collapse map f_Y. Then φ and ψ are inverse maps.*

Proof. There's a lot of details to check here, and not much time, so we'll list what needs to be shown, and the tools used to show them

(1) Show φ is independent of choice of p, this is seen by noting that we can compose with a rotation to move any point of S^q onto our choice of p.
(2) Show φ is well defined on homotopy classes. This is shown by taking a homotopy F from f_0 to f_1 and (after some fiddling with transversality) taking the framed submanifold $F^{-1}(p) \subset [0,1] \times M$, which will be a bordism from $f_0^{-1}(p)$ to $f_1^{-1}(p)$.
(3) Show that ψ is independent of cutoff function and tubular neighborhood. This amounts to retracting one neighborhood onto another, and showing a similar cutoff function would produce a homotopic map
(4) Show that ψ is well defined on boridsm classes. Given bordant submanifolds Y_0 and Y_1, let $X \subset [0,1]$ be the bordism between them. Then the collapse map f_X is the homotopy from f_{Y_0} to f_{Y_1}.
(5) Show that $\varphi \circ \psi$ is the identity map. This is easily seen, since the collapse map for a submanifold has regular value 0, and preimage Y by construction
(6) Show that $\psi \circ \varphi$ is identity. This one is a bit tricky. The key fact here is that given a map f_0, $Y = f_0^{-1}(p)$, then we have $df_0|_Y = df_Y|_Y$, which is enough to deduce they are homotopic with some work.

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The theorem gives us a remarkable dictionary from bordism to algebraic topology. In the case that $M = S^n$, we have that the maps are actually group isomorphisms $\pi_n(S^q) \cong \Omega^{\text{fr}}_{n-q}(S^n)$. Going a bit further, if we had a bit more machinery we could establish an isomorphism from the ring of stable homotopy groups of the sphere, and the framed bordism ring.