MODERN MODULI THEORY EXERCISES

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Exercise 1.2. Show that the restriction $\mathrm{Fun}(\mathsf{Sch}^{\mathrm{op}}, \mathsf{Set}) \to \mathrm{Fun}(\mathsf{Ring}, \mathsf{Set})$ is not fully faithful.

Proof. Consider the functor H^1, which maps $X \mapsto H^1(X, \mathcal{O}_X)$ and the functor F that maps every scheme to $\{0\}$ and every morphism to the identity map $\{0\} \to \{0\}$. On affine schemes, H^1 and F agree. However, nonisomorphic schemes (e.g. any two nonisomorphic nonaffine schemes) get mapped to isomorphic objects in Set, so the functor cannot be fully faithful. ∎

Exercise 1.3.

(1) For schemes T and X, the assignment $U \subset T \mapsto \mathrm{Map}(U, X)$ is a sheaf of sets over T.
(2) Use this to show that the functor $\mathrm{Sch} \to \mathrm{Fun}(\mathsf{Ring}, \mathsf{Set})$ is fully faithful.

Proof.

(1) The assignment clearly forms a presheaf with the natural restriction maps given by restricting a morphism to a subset. To verify that the assignment defines a sheaf, we need to verify two more conditions. Let $\{U_i\}$ be an open covering of an open set U. Then let $f, g \in \mathrm{Map}(U, X)$ such that $f|_{U_i} = g|_{U_i}$ for all i. Then f and g agree on the points of U. In addition, f and g induce the same map on stalks, so they induce the same map of sheaves. Therefore, f and g define the same morphism. For the other condition, suppose we have $f_i \in \mathrm{Map}(U_i, X)$ such that f_i and f_j agree on $U_i \cap U_j$. Then since functions and sheaf morphisms glue, the f_i determine a morphism $f \in \mathrm{Map}(U, X)$ with $f|_{U_i} = f_i$.
(2) To show that the functor is fully faithful, we want to show that for schemes X, T, the map

$$\mathrm{Map}(T, X) \to \mathrm{Map}_{\mathrm{Fun}(\mathsf{Ring},\mathsf{Set})}(\mathrm{Map}(\cdot, T), \mathrm{Map}(\cdot, X))$$

is bijective. In the case that $T = \mathrm{Spec}(A)$ is affine, the standard proof of the Yoneda lemma works, where given a natural transformation $\eta : \mathrm{Map}(\cdot, T) \to \mathrm{Map}(\cdot, X))$, it is induced by postcomposition with the map $\eta_T(\mathrm{id}_T) : T \to X$, where we note that T is a valid input the the functors since it is affine. We then want to verify bijectivity when T is not necessarily affine. For injectivity, let $\{U_i\}$ be an affine open cover of T, and let $f, g : T \to X$ be two morphisms that induce the same natural transformation $\mathrm{Map}(\cdot, T) \to \mathrm{Map}(\cdot, X)$. Then let $f_i, g_i : U_i \to X$ be the restrictions of f and g to the U_i. Since the U_i are affine, we know that the natural transformations $\mathrm{Map}(\cdot, U_i) \to \mathrm{Map}(\cdot, X)$ induced by the f_i and g_i are the same, so we know $f_i = g_i$ on U_i. Therefore, $f = g$ since $U \mapsto \mathrm{Map}(U, X)$ is a sheaf. For surjectivity, let $\eta : \mathrm{Map}(\cdot, T) \to \mathrm{Map}(\cdot, X)$ be a natural transformation. We have natural inclusions $\mathrm{Map}(\cdot, U_i) \hookrightarrow \mathrm{Map}(\cdot, T)$ given by composition with the inclusions $U_i \hookrightarrow T$, and η restricts to natural transformations $\eta_i : \mathrm{Map}(\cdot, U_i) \to \mathrm{Map}(\cdot, X)$, which are determined by morphisms $f_i : U_i \to X$. By restricting the natural transformation to intersections $U_i \cap U_j$, we find that f_i and f_j agree on $U_i \cap U_j$. Therefore, the f_i glue to a morphism $T \to X$, which much necessarily induce η. Therefore, the map

$$\mathrm{Map}(T, X) \to \mathrm{Map}_{\mathsf{Fun}(\mathsf{Ring},\mathsf{Set})}(\mathrm{Map}(\cdot, T), \mathrm{Map}(\cdot, X))$$

is bijective in the non-affine case as well.

■

Exercise 1.5. For any category $\mathcal{C}$ and an object $T \in \mathcal{C}$, the ***slice category*** $\mathcal{C}_{/T}$ is the category where the objects are morphisms $X \to T$ in $\mathcal{C}$, and the morphisms $(X \to T) \to (Y \to T)$ are morphisms $X \to Y$ in $\mathcal{C}$ such that

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & T & \end{tikzcd}$$

is a commutative diagram. Show that the canonical functor $\mathcal{P}(\mathcal{C}_{/T}) \to \mathcal{P}(\mathcal{C})_{/h_T}$ is an equivalence.

Proof. Let $A : \mathcal{P}(\mathcal{C}_{/T}) \to \mathcal{P}(\mathcal{C})_{/h_T}$ denote the canonical functor. Explicitly, given a functor $F \in \mathcal{P}(\mathcal{C}_{/T}) = \mathrm{Fun}(\mathcal{C}_{/T}^{\mathrm{op}}, \mathsf{Set})$, $A(F)$ is the functor $\mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ given on objects by

$$[A(F)](X) = \coprod_{p \in \mathrm{Map}(X,T)} F\left(X \xrightarrow{p} T\right)$$

Given a morphism $f : X \to Y$, we let

$$[A(F)](f) = \coprod_{\substack{p \in \mathrm{Map}(X,T) \\ q \text{ compatible}}} F\left(\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & T & \end{tikzcd}\right)$$

where $q \in \mathrm{Map}(Y, T)$ is compatible with p if the diagram commutes. The functor $A(F)$ comes equipped with a canonical natural transformation η to h_T corresponding to forgetting everything except for the morphism $p : X \to T$. To show that A is an equivalence, we provide an inverse functor $B : \mathcal{P}(\mathcal{C}_{/h_T}) \to \mathcal{P}(\mathcal{C}_{/T})$. Given a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ with a natural transformation η to h_T, define the functor $B(F) \in \mathcal{P}(\mathcal{C}_{/T})$ on objects by

$$B(F)(X \xrightarrow{p} T) = \eta_X^{-1}(p) \subset F(X)$$

On morphisms, we let

$$B(F)\left(\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & T & \end{tikzcd}\right) = F(f)|_{\eta_X^{-1}(p)}$$

Where we use the fact that η is a natural transformation and the fact that the diagram commutes to deduce that the image of $\eta_X^{-1}(p)$ under $F(f)$ is contained in $\eta_Y^{-1}(q)$.

We then show that A and B are inverses of each other. We first consider the functor $A \circ B : \mathcal{P}(\mathcal{C})_{/h_T} \to \mathcal{P}(\mathcal{C})_{/h_T}$. Given a functor $F \in \mathcal{P}(\mathcal{C})_{/h_T}$ equipped with a natural transformation η to h_T, we have that

$$[(A \circ B)(F)](X) = \coprod_{p \in \mathrm{Map}(X,T)} \eta_X^{-1}(p) = F(X)$$

So $A \circ B$ is equivalent to the identity functor. In the other direction, given a functor $F \in \mathcal{P}(\mathcal{C}_{/T})$, we have that

$$[(B \circ A)(F)](X \xrightarrow{p} T) = \eta_X^{-1}(p)$$

where $\eta_X : F(X) \to \mathrm{Map}(X, T)$ is the map that takes an element of

$$F(X \xrightarrow{p} T) \subset \coprod_{q \in \mathrm{Map}(X,T)} F(X \xrightarrow{q} T)$$

and maps it to p, so $\eta_X^{-1}(p)$ is exactly $F(X \xrightarrow{p} T)$, so $B \circ A$ is also equivalent to the identity functor. ■

Exercise 2.2. Show that if $X \to Y$ is an étale morphism of smooth varieties over $\mathbb{C}$, the every point of X has a neighborhood in the analytic topology which is a homeomorphism onto its image.

Proof. It suffices to check on affine opens, and we may further assume that $X \to Y$ is standard étale. Then the condition that the determinant of $[\partial_j f_i]$ is a unit implies that the morphism induces an isomorphism on cotangent spaces, which implies that the map is a local diffeomorphism. Therefore, in the analytic topology, we may find sufficeintly small open sets such that the map is a diffeomorphism onto its image. ■

Exercise 2.3. Use the adjunction between f^* and f_* to show that $\mathrm{Qcoh}_{/S}$ is a fibered category over Sch/S and an arrow $(X, E) \to (Y, F)$ is cartesian if and only if the homomorphism $F \to f_*(E)$ induces an isomorphism $f^*(F) \cong E$.

Proof. Let $p : \mathrm{Qcoh}_{/S} \to \mathrm{Sch}_{/S}$ denote the functor mapping a pair (X, E) to X. To show that $\mathrm{Qcoh}_{/S}$ is a fibered category over $\mathrm{Sch}_{/S}$, we want to show that given a morphism of S-schemes $f : X \to Y$ and a quasi-coherent sheaf F over Y that there exists a p-cartesian morphism $\phi : (X, E) \to (Y, F)$ with E quasi-coherent over X and $p(\phi) = f$.

Suppose we have a cartesian arrow $(X, E) \to (Y, F)$ given by a morphism $f : X \to Y$ of schemes and a sheaf morphism $\varphi_f : F \to f_*E$. Then for any other pair $(Z, G) \in \mathrm{Qcoh}_{/S}$ we know that that a scheme morphism $g : Z \to X$ and an arrow $(Z, G) \to (Y, F)$ given by scheme morphism h and sheaf map φ_h satisfying the compatibility condition $h = f \circ g$ is equivalent to an arrow $(Z, G) \to (X, E)$ which is given by g and a sheaf morphism φ_g such that $\varphi_h : F \to h_*G = f_*g_*G$ is equal to the composition of φ_f and the map $f_*E \to f_*g_*G$ induced by φ_g. Using the adjunction between f^* and f^*, we get an isomorphism $F \to f_*f^*F$ by applying the adjunction to the identity map $f^*F \to f^*F$. We claim that the arrow $(X, f^*F) \to (Y, F)$ given by f and this isomorphism satisfies the conditions of a cartesian arrow. Let g and (h, φ_h) be as above. We want to show that this induces a unique arrow

$(g, \varphi_g) : (Z, G) \to (X, f^*F)$. We know by the adjunction that a map $f^*F \to g_*G$ is equivalent to a map $F \to f_*g_*G$, so we can and must take φ_g to be the map corresponding to φ_h.

Now suppose we are given a cartesian arrow $(f, \varphi_f) : (X, E) \to (Y, F)$. We claim that the map $f^*F \to E$ induced by φ_f is an isomorphism. It suffices to provide an inverse arrow. Since $(X, f^*F) \to (Y, F)$ is also cartesian, we have that (f, φ_f) induces an arrow $(X, E) \to (X, f^*F)$. By standard arguments, this must be the inverse to (f, φ_f), since the composition in both directions must be equal to the identity by the universal property of being cartesian. ∎

Exercise 3.1. If $\mathcal{F} = \mathrm{Qcoh}_{/S} \to \mathrm{Sch}_{/S}$ and D is a diagram consisting of two schemes and two non-identity arrows $f, g : X \to Y$, then show That the category of cartesian sections of $\mathcal{F}$ over D is equivalent to the category of quasi-coherent sheaves E over Y along with an isomorphism $f^*(E) \cong g^*(E)$.

Proof. We first unpack some of the definitions. A cartesian section $\sigma \in \Gamma^{\mathrm{cart}}_{\mathrm{Sch}_{/S}}(D, \mathrm{Qcoh}_{/S})$ is a diagram in $\mathrm{Qcoh}_{/S}$ of the form

$$\begin{tikzcd} (X, F) \arrow[r, bend left, "{(f,\varphi_f)}"] \arrow[r, bend right, "{(g,\varphi_g)}"'] & (Y, E) \end{tikzcd}$$

where the arrows (f, φ_f) and (g, φ_g) are cartesian. By the previous problem, the arrows being cartesian implies that the maps $\psi_f : f^*E \to F$ and $\psi_g : g^*E \to F$ by applying the adjunction to φ_f and φ_g are isomorphisms. Given two cartesian sections σ_1 and σ_2 where

$$\sigma_1 = \left(\begin{tikzcd} (X, F_1) \arrow[r, bend left, "{(f_1,\varphi_{f_1})}"] \arrow[r, bend right, "{(g_1,\varphi_{g_1})}"'] & (Y, E_1) \end{tikzcd} \right) \qquad \sigma_2 = \left(\begin{tikzcd} (X, F_2) \arrow[r, bend left, "{(f_2,\varphi_{f_2})}"] \arrow[r, bend right, "{(g_2,\varphi_{g_2})}"'] & (Y, E_2) \end{tikzcd} \right)$$

a morphism $\sigma_1 \to \sigma_2$ is a natural transformation η, which is given by the data of two arrows $(\mathrm{id}_X, \eta_X) : (X, F_1) \to (X, F_2)$ and $(\mathrm{id}_Y, \eta_Y) : (Y, E_1) \to (Y, E_2)$ along with the condition that

$$(f_2, \varphi_{f_2}) \circ (\mathrm{id}_X, \eta_X) = (\mathrm{id}_Y, \eta_Y) \circ (f_1, \varphi_{f_1})$$

Then let $\mathcal{C}$ denote the category where the objects are pairs (E, ρ) where $E \in \mathrm{Qcoh}_{/S}(Y)$ and $\rho : f^*E \to g^*E$ is an isomorphism, and the morphisms $(E_1, \rho_1) \to (E_2, \rho_2)$ are sheaf morphisms $\eta : E_1 \to E_2$ such that the following square commutes:

$$\begin{tikzcd} f^*E_1 \arrow[r, "\rho_1"] \arrow[d, "f^*\eta"'] & g^*E_1 \arrow[d, "g^*\eta"] \\ f^*E_2 \arrow[r, "\rho_2"'] & g^*E_2 \end{tikzcd}$$

We want to show that there is an equivalence of categories $\Gamma^{\mathrm{cart}}_{\mathrm{Sch}_{/S}}(D, \mathrm{Qcoh}_{/S}) \to \mathcal{C}$. Consider the functor given on objects by mapping a cartesian section

$$\begin{tikzcd} (X,F) \arrow[r, bend left, "{(f,\varphi_f)}"] \arrow[r, bend right, "{(g,\varphi_g)}"'] & (Y,E) \end{tikzcd}$$

to the pair $(E, \psi_g^{-1} \circ \psi_f)$, where ψ_g, ψ_f are the maps given by applying the adjunction to φ_g and φ_f respectively. On morphisms, we take a natural transformation η to the sheaf morphism η_Y, noting that the following square commutes:

$$\begin{array}{ccc} f^*E_1 & \xrightarrow{\psi_{g_1}^{-1}\circ\psi_{f_1}} & g^*E_1 \\ {\scriptstyle f^*\eta_Y}\downarrow & & \downarrow{\scriptstyle g^*\eta_Y} \\ f^*E_2 & \xrightarrow[\psi_{g_2}^{-1}\circ\psi_{f_2}]{} & g^*E_2 \end{array}$$

To show that this functor is an equivalence, we show it is fully faithful and essentially surjective. The fact that the functor is essentially surjective is relatively clear, is we can realize any isomorphism $f^*E \to g^*E$ as going through some intermediate sheaf F (for example, take $F = f^*E$). Showing that the functor is fully faithful amounts to showing that a natural transformation η

$$\left(\begin{tikzcd} (X,F_1) \arrow[r, bend left, "{(f_1,\varphi_{f_1})}"] \arrow[r, bend right, "{(g_1,\varphi_{g_1})}"'] & (Y,E_1) \end{tikzcd}\right) \xrightarrow{\eta} \left(\begin{tikzcd} (X,F_2) \arrow[r, bend left, "{(f_2,\varphi_{f_2})}"] \arrow[r, bend right, "{(g_2,\varphi_{g_2})}"'] & (Y,E_2) \end{tikzcd}\right)$$

is equivalent to a sheaf morphism $\alpha : E_1 \to E_2$. This is clear, since the condition that η be a natural transformation is that

$$(f_2, \varphi_{f_2}) \circ (\mathrm{id}_X, \eta_X) = (\mathrm{id}_Y, \eta_Y) \circ (f_1, \varphi_{f_1})$$

which implies that we can recover η_X from η_Y. Therefore, declaring $\eta_Y = \alpha$ specifies η uniquely, so the functor is fully faithful, so it is an equivalence. ■

Exercise 3.2. Show that if the indexing category $\mathcal{I}$ has a terminal object $*$, then restricting any diagram $D : \mathcal{I} \to \mathcal{C}$ to $*$ defines an equivalence $\Gamma^{\mathrm{cart}}_{\mathcal{C}}(D, \mathcal{F}) \cong \mathcal{F}(D(*))$.

Proof. Explicitly, the functor restricting $*$ is given by mapping a cartesian section σ to $\sigma(*)$ and mapping a natural transformation $\eta : \sigma_1 \to \sigma_2$ to map $\eta_* : \sigma_1(*) \to \sigma_2(*)$. This is fully faithful by essentially the same argument in the previous exercise, so all that remains is to show that the functor is essentially surjective. This amounts to showing that given any object $\xi \in \mathcal{F}$ with $p(\xi) = D(*)$, there exists a cartesian section σ with $\sigma(*) = \xi$. If we fix a cleavage for F, we can construct a cartesian section where an object $A \in \mathcal{I}$ maps to source of the unique cartesian arrow lifting the map $D(A) \to D(*)$, which gives us that the functor is essentially surjective. ■

Exercise 3.3. Let $p : \mathcal{F} \to \mathcal{C}$ be a fibered category, and fix a cleavage for $\mathcal{F}$. Show that for any cover $\mathcal{U} = \{U_\alpha \to U\}$, the category $\mathrm{Desc}_{\mathcal{F}}(\mathcal{U})$ is equivalent to the category $\mathcal{D}$ of pairs of collections $(\{\xi_\alpha\}, \{\phi_{\alpha\beta}\})$, of objects $\xi_\alpha \in \mathcal{F}(U_\alpha)$ and isomorphisms $\phi_{\alpha\beta} : \mathrm{pr}_1^*(\xi_\beta) \to \mathrm{pr}_0^*(\xi_\alpha)$ such that for any α, β, γ, we have

$$\mathrm{pr}_{02}^*(\phi_{\alpha\gamma}) = \mathrm{pr}_{01}^*(\phi_{\alpha\beta}) \circ \mathrm{pr}_{12}^*(\phi_{\beta\gamma}) : \mathrm{pr}_2^*(\xi_\gamma) \to \mathrm{pr}_0^*(\xi_\alpha)$$

Proof. We first note that a morphism in $\mathcal{D}$ between $(\{\xi_\alpha\}, \{\phi_{\alpha\beta}\})$ and $(\{\zeta_\alpha\}, \{\psi_{\alpha\beta}\})$ consists of a collection of morphisms $\{\rho_\alpha : \xi_\alpha \to \zeta_\alpha\}$ such that for any α, β, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{pr}_1^*(\xi_\beta \arrow[r, "\phi_{\alpha\beta}"] \arrow[d, "\mathrm{pr}_1^*\rho_\beta"'] & \mathrm{pr}_0^*(\xi_\alpha) \arrow[d, "\mathrm{pr}_0^*\rho_\alpha"] \\
\mathrm{pr}_1^*\zeta_\beta \arrow[r, "\psi_{\alpha\beta}"'] & \mathrm{pr}_0^*\zeta_\alpha
\end{tikzcd}$$

We then define the functor $\mathrm{Desc}_{\mathcal{F}}(\mathcal{U}) \to \mathcal{D}$ Let $D_{\mathcal{U}} : \mathcal{I}_I \to \mathcal{C}$ denote the diagram in $\mathcal{C}$, and let $\sigma : \mathcal{I}_I \to \mathcal{F}$ be a cartesian section. Then we map σ to the pair $(\{\sigma(J)\}_{J\in\mathcal{I}}, \{\sigma(\varphi)\}_{\varphi\in K})$ where $K \subset \mathrm{Mor}(\mathcal{I})$ is the set of arrows between the first two "levels" of the descent diagram. We map a natural transformation $\eta : \sigma_1 \to \sigma_2$ to the collection $\{\eta_J : \sigma_1(J) \to \sigma_2(J)\}$. We first note that this functor is essentially surjective, since given an object $(\{\xi_\alpha\}, \{\phi_{\alpha\beta}\}) \in \mathcal{D}$, we can make a cartesian section σ, where the objects on the first "level" are given by the ξ_α, and we fill in the rest of the objects in the diagram using the cleavage. The morphisms are then induced using the isomorphisms from the cleavage and the isomorphism $\phi_{\alpha\beta}$, and the fact that the choice doesn't matter for the third "level" amounts to the cocycle condition.

We then show the functor is fully faithful. Let $\{\rho_\alpha : \xi_\alpha \to \zeta_\alpha\}$ be a morphism in $\mathcal{D}$. Then we can construct a natural transformation η, where the maps $\eta_\alpha : \xi_\alpha \to \zeta_\alpha$ is given by the ρ_α, which then uniquely determines the rest of the maps of the diagram by using the cleavage and the pullbacks of the maps in a similar fashion to 3.2 and 3.1. ∎

Exercise 3.4. We know that the composition of cartesian fibrations $\mathcal{F}' \to \mathcal{F} \to \mathcal{C}$ is again a cartesian fibration. Given a Grothendieck topology on $\mathcal{C}$, we can define a Grothendieck topology on $\mathcal{F}$ whose coverings consist of families of cartesian arrows $\{\xi_i \to \xi\}_{i\in I}$ such that $\{p(\xi_i) \to p(\xi)\}_{i\in I}$ is a covering in $\mathcal{C}$. Show that if $\mathcal{F}$ is a stack over $\mathcal{C}$ and $\mathcal{F}'$ is a stack over $\mathcal{F}$ with the induced topology, then $\mathcal{F}'$ is a stack over $\mathcal{C}$.

Proof. We want to show that for any covering $\mathcal{U} = \{U_\alpha \to U\}$ in $\mathcal{C}$, the restriction functor $\Gamma_{\mathcal{C}}^{\mathrm{cart}}(D_{\mathcal{U}}^+, \mathcal{F}) \to \Gamma_{\mathcal{C}}^{\mathrm{cart}}(D_{\mathcal{U}}, \mathcal{F})$ is an equivalence. Let $\mathcal{I}$ denote the indexing category for the descent diagram, and $\mathcal{I}_+$ the indexing category for the augmented descent diagram. Then let σ be a cartesian section of $\mathcal{F}' \to \mathcal{C}$. We note that σ is cartesian over $p \circ \sigma$. Then $p \circ \sigma$ is a cartesian section of $\mathcal{F} \to \mathcal{C}$, since any cartesian arrow of $\mathcal{F}' \to \mathcal{C}$ must project to a cartesian arrow of $\mathcal{F} \to \mathcal{C}$. Since $\mathcal{F} \to \mathcal{C}$ is a stack, this is equivalent to an augmented cartesian section $\sigma_+ : \mathcal{I}_+ \to \mathcal{F}$. Since $\mathcal{F}' \to \mathcal{F}$ is a cartesian fibration, we can lift the diagram σ_+ to a diagram $\widetilde{\sigma_+} : \mathcal{I}_+ \to \mathcal{F}'$, which is cartesian over $\mathcal{F}$, so since $\mathcal{F}' \to \mathcal{F}$ is a stack, this is equivalent to the original section σ. ∎

Exercise 3.5. Show that if $f : \mathcal{C} \to \mathcal{D}$ and $\mathcal{F}$ is a fibered category over $\mathcal{D}$, then $f^{-1}(\mathcal{F})$ is a fibered category over $\mathcal{C}$, where an arrow in $f^{-1}(\mathcal{F})$ is cartesian if and only if the corresponding arrow in $\mathcal{F}$ is cartesian.

Proof. We first show that arrows in $f^{-1}(\mathcal{F})$ are cartesian if and only if the corresponding arrows in $\mathcal{F}$ are cartesian. Suppose we have a cartesian arrow $(X, \xi) \to (Y, \eta)$ in $f^{-1}(\mathcal{F})$. Then $(Z, \alpha) \to (X, \xi)$ is equivalent data to a morphism $Z \to X$ in $\mathcal{C}$ and a morphism $\alpha \to \eta$ in $\mathcal{D}$ such that $\alpha \to \eta$ covers the map $f(Z) \to f(X)$. But this is equivalent to the arrow $\xi \to \eta$ covering $f(X) \to f(Y)$ being cartesian. To show that $f^{-1}(\mathcal{F})$ is a fibered category, given a morphism $X \to Y$ in $\mathcal{C}$ and an object (Y, η) lying over Y, we can pick any cartesian lift of the morphism $f(X \to Y)$ in $\mathcal{F}$, which gives us the desired cartesian lift in $f^{-1}(\mathcal{F})$ ∎

Exercise 3.13. Let $\mathcal{F}$ be a fibered category over $\mathrm{Sch}_{/S}$ such that for any set of schemes $\{U_i\}$, the canonical morphism $\mathcal{F}(\coprod_i U_i) \to \prod_i \mathcal{F}(U_i)$ is an equivalence of categories. Show that if $\mathcal{F}$ satisfies descent with respect to the covering $\{U_i \to U\}$ if and only if it satisfies descent with respect to the covering $\{U' := \coprod_i U_i \to U\}$. Use this to show that a fibered category over $\mathrm{Sch}_{/S}$ is a stack if and only if $\mathcal{F}$ maps disjoint unions of schemes to products of categories, and $\mathcal{F}$ satisfies descent to all coverings $U' \to U$.

Proof. For the first part, let $D_{U'}$ denote the descent diagram for the covering U', and let D_U denote the descent diagram for the covering by all the U_i. Then let $\sigma \in \Gamma^{\mathrm{cart}}_{\mathrm{Sch}_{/S}}(D_{U'}, \mathcal{F})$ be a cartesian section. Then upon fixing a cleavage, we know that σ is equivalent to an object ξ over $\coprod_i U_i$ and an isomorphism $\phi : \mathrm{pr}_1^* \xi \to \mathrm{pr}_0^* \xi$. Then under canonical functor $\mathcal{F}(\coprod_i U_i) \to \prod_i \mathcal{F}(U_i)$, we have that ξ is mapped to the tuple (ξ_i) where ξ_i is the source of the cartesian lift of the inclusion $U_i \hookrightarrow U$ determined by a cleavage. Similarly, the isomorphism ϕ becomes a tuple (ϕ_{ij}) of maps $\phi_{ij} : \mathrm{pr}_1^* \xi_j \to \mathrm{pr}_0^* \xi_i$, which satisfy the cocycle condition because ϕ did. This is then equivalent to a cartesian section of the diagram D_U. Therefore, if $\mathcal{F}$ satisfies descent with respect to either cover, by going through the equivalence, we get that it satisfies descent with respect to the other cover, since we have that the category of cartesian sections over either augmented descent diagram is equivalent to $\mathcal{F}(U)$.

For the second part, let $p : \mathcal{F} \to \mathrm{Sch}_{/S}$ be any fibered category. ∎

Exercise 3.14. Use Corollary 3.14 to show that a fibered category over $\mathrm{Sch}_{/S}$ is a stack for the étale topology if and only if it is a stack for the smooth topology.

Proof. Let $\mathcal{F}$ be a fibered category over $\mathrm{Sch}_{/S}$. If $\mathcal{F}$ is a stack for the smooth topology, then it is certainly a stack for the étale topology, since every étale map is smooth. For the other direction, suppose we have a smooth covering $U_0 \to U$. By factoring the morphism $U_0 \to U$ locally into an étale map followed by the standard smooth morphism and taking disjoint unions, we get $V \to U_0 \to U$, where the first map is étale. This gives us an étale cover $V \to U$, and by assumption, $\mathcal{F}$ satisfies descent along $T \times_U V$ for any T since $\mathcal{F}$ is a stack for the étale topology. Therefore, by Corollary 3.14, we have that $\mathcal{F}$ satisfies descent along the covering $U_0 \to U$. ∎

Exercise 3.15. Show that a fibered category over $\mathrm{Sch}_{/S}$ is a stack for the étale topology if and only if it is a stack for the Zariski topology and satisfies descent along étale standard smooth morphisms.

Proof. Let $\mathcal{F} \to \mathrm{Sch}_{/S}$ be a fibered category. If $\mathcal{F}$ is a stack for the étale topology, then it certainly is a stack for the Zariski topology, since every Zariski cover is an étale cover (since open immersions are étale). It also clear satisfies descent along standard étale maps. For the other direction, suppose $\mathcal{F}$ is a stack for the Zariski topology and satisfies descent along standard étale morphisms, and let $U \to X$ be an étale covering. By passing to a Zariski covering $Z \to X$, this factors as $U \to Z \to X$, where $U \to Z$ is standard étale. Then since $\mathcal{F}$ satisfies descent along $U \to Z$ and is a stack for the Zariski topology, we have that $\mathcal{F}$ satisfies descent for the covering $U \to X$. Therefore, we get that $\mathcal{F}$ is a stack for the étale topology. ■

Exercise 5.3. Show that any morphism between principal G-bundles is an isomorphism

Proof. Suppose we have a morphism $f : T \times G \to T \times G$ of trivial G-bundles. We know that this morphism is equivalent to a section of the trivial bundle $T \times G \to T$, i.e. a morphism $\varphi_f : T \to G$. Then the inverse morphism $f^{-1} : T \times G \to T \times G$ is given by the morphism corresponding to the section φ_f^{-1}, where $\varphi_f^{-1}(t) := \varphi_f(t)^{-1}$. Then suppose we have a morphism $f : P \to Q$ of G-bundles. Then since $P \to T$ is smooth, it admits sections étale locally on T, so $P|_U \cong U \times T$ for an étale cover $U \to G$. By refining U, we may assume that $Q|_U$ is also trivial. Then pulling back f along $U \to T$ gives a morphism of trivial G-bundles over U, which is an isomorphism. Therefore, f is an isomorphism. ■

Exercise 6.3. Show that when $X_\bullet = (G \rightrightarrows \mathrm{pt})$ is a group scheme over S, then the category of $X_\bullet$ spaces over T is equivalent to the usual category whose objects are algebraic spaces with a G-action relative to T, and whose morphisms are G-equivariant morphisms.

Proof. Let $\pi_\bullet : P_\bullet \to (X_\bullet)_T$ be a $X_\bullet$-space, and let $s, t : P_1 \to P_0$ denote the source and target maps for $P_\bullet$. Then we have the cartesian square

$$\begin{tikzcd} P_1 \arrow[r, "s"] \arrow[d, "\pi_1"'] & P_0 \arrow[d, "\pi_0"] \\ G \times T \arrow[r] & \mathrm{pt} \times T \end{tikzcd}$$

Where the map $G \times T \to \mathrm{pt} \times T$ is given by the terminal map $G \times T \to \mathrm{pt}$ and the projection to T. We first note that π_0 gives P_0 the structure of a T-space. Furthermore, the square being cartesian allows us to identify P_1 as $P_1 = P_0 \times_T G$. Then the action map $t : P_1 \to P_0$ gives P_0 the structure of a G-space over T. This satisfies the usual axioms of a group action due to the fact that $P_\bullet$ is a groupoid.

Then let $P_\bullet$ and $P'_\bullet$ be two $X_\bullet$-spaces over T and let $F_\bullet : P_\bullet \to P'_\bullet$ be a morphism lying over the identity map $T \to T$. We note that the "action" maps for both P_0 and P'_0 are just the terminal maps to pt, so we may identify P_1 and P'_1 with the products

$$P_1 = P_0 \times G$$
$$P'_1 = P'_0 \times G$$

Furthermore, using these identifications, we know that F_1 is given on U-points by the mapping $F_1(p, g) = (F_0(p), g)$. Then the fact that $F_\bullet$ is a map of $X_\bullet$-spaces tells us that $t'(F_1(p, g)) = F_0(t(p, g))$. Putting these together, we find that

$$t'(F_0(p), g) = F_0(t(p, g))$$

Then since t' and t are the action maps for the G-actions on P_0 and P_0' respectively, this is exactly the statement that F_0 is G-equivariant. Therefore, the functor mapping $P_\bullet$ to the G-space P_0 and sending a map $F_\bullet$ to the G-equivariant map F_0 defines an equivalence of categories between $X_\bullet$-spaces over T and G-spaces over T. ■