DIFFERENTIAL GEOMETRY LECTURE SERIES

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These are notes for the spring 2019 differential geometry lecture series for the math club. The plan is for the course to give an introduction to math club members to the basics and terminology of smooth manifolds. A good reference if you want to read more is John Lee's *Introduction to Smooth Manifolds*. In terms of prerequisites, it would be good to have a background in point set topology, multivariable calculus, and linear algebra.

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1. Introduction to Manifolds

Definition 1.1. A ***topological manifold*** is a Hausdorff space X such there exists a countable open cover $\{U_\alpha\}$ of X, along with homeomorphisms $\varphi_\alpha : U_\alpha \to V_\alpha$, where V_α is an open subset of $\mathbb{R}^n$.

In this way, we see that a manifold is a topological space that is locally topologically indistinguishable from Euclidean space. Both modifiers are important here – there could be global and geometric properties that differ from $\mathbb{R}^n$.

Example 1.2.

(1) $\mathbb{R}^n$ is a topological manifold – it admits a global chart $(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})$.
(2) The 2-sphere $S^2 = \left\{(x,y,z) \in \mathbb{R}^3 \ : \ x^2+y^2+z^2 = 1\right\}$ is a topological manifold. If we let N and S denote the North and South poles repsectively, then we can construct charts using ***stereographic projection***. Stereographic projection from the North pole is a map $\varphi_N : S^2 - \{N\} \to \mathbb{R}^n$, where given a point $p \in S^2 - \{N\}$, we take the line cotaining both N and p, which intersects the $z = 0$ plane at one point q. We then define $\varphi_N(p) = q$. Stereographic projection φ_S from the South pole is defined in an analogous manner. An explicit formula for φ_N is
$$\varphi_N(x,y,z) = \left(\frac{x}{1-z}, \frac{x}{1-z}\right)$$

Exercise 1.3.

(1) S^2 does not admit a global chart. Why?
(2) Give an inverse map $\mathbb{R}^n \to \mathrm{S}^n - N$ for stereographic projection from the North Pole
(3) Generalize stereographic projection to the n-sphere $S^n = \left\{p \in \mathbb{R}^{n+1} \ : \ \|p\| = 1\right\}$