HYPERCOHOMOLOGY AND SPECTRAL SEQUENCES

JEFFREY JIANG

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Conventions and Notation

We will use cohomological indexing – for a complex $(A^\bullet, d)$, the differential increases the degree. We fix an abelian category $\mathcal{A}$ with enough injectives – for our purposes it is harmless to assume that $\mathcal{A}$ is one of Ab, Mod_R, or a category of sheaves over a space (e.g. $\mathsf{Sh}_{\mathsf{Ab}}(X)$ or $\mathrm{QCoh}(X)$), because of this, we will be relatively cavalier about using elements when doing homological algebra.

1. Derived Functors and Hypercohomology

Definition 1.1. A ***double complex*** is a collection of objects $K^{p,q} \in \mathcal{A}$ with $p, q \in \mathbb{Z}$ equipped with differentials

$$D_1 : K^{p,q} \to K^{p+1,q}$$
$$D_2 : K^{p,q} \to K^{p,q+1}$$

such that $D_i^2 = 0$ and $D_1 \circ D_2 = D_2 \circ D_1$.

There is a sign convention with double complexes – some prefer to replace the condition that the differetials commute with the condition that they *anticommute*, i.e. $D_1 \circ D_2 = -D_2 \circ D_1$.

The way to visualize a double complex is to arrange the objects $K^{p,q}$ in a grid, with $K^{p,q}$ in the position with coordinate (p,q).

$$\begin{array}{ccccccc}
K^{0,3} & \xrightarrow{D_1} & K^{1,3} & \xrightarrow{D_1} & K^{2,3} & \xrightarrow{D_1} & K^{3,3} \\
{\scriptstyle D_2}\big\uparrow & & {\scriptstyle D_2}\big\uparrow & & {\scriptstyle D_2}\big\uparrow & & {\scriptstyle D_2}\big\uparrow \\
K^{0,2} & \xrightarrow{D_1} & K^{1,2} & \xrightarrow{D_1} & K^{2,2} & \xrightarrow{D_1} & K^{3,2} \\
{\scriptstyle D_2}\big\uparrow & & {\scriptstyle D_2}\big\uparrow & & {\scriptstyle D_2}\big\uparrow & & {\scriptstyle D_2}\big\uparrow \\
K^{0,1} & \xrightarrow{D_1} & K^{1,1} & \xrightarrow{D_1} & K^{2,1} & \xrightarrow{D_1} & K^{3,1} \\
{\scriptstyle D_2}\big\uparrow & & {\scriptstyle D_2}\big\uparrow & & {\scriptstyle D_2}\big\uparrow & & {\scriptstyle D_2}\big\uparrow \\
K^{0,0} & \xrightarrow{D_1} & K^{1,0} & \xrightarrow{D_1} & K^{2,0} & \xrightarrow{D_1} & K^{3,0}
\end{array}$$

Because of this, we refer to D_1 as the ***horizontal differential*** and D_2 as the ***vertical differential***. One thing to note is that the defintion of a double complex has no finiteness or positivity assumption – we can have a nonzero objects $K^{p,q}$ for arbitrarily large p and q, and we may also have nonzero $K^{p,q}$ for $p,q<0$.

Example 1.2. Given a complex manifold X, we have the sheaves $\mathcal{A}_X^{p,q}$ of smooth (p,q)-forms. These come equipped with differentials $\partial : \mathcal{A}_X^{p,q} \to \mathcal{A}_X^{p+1,q}$ and $\overline{\partial} : \mathcal{A}_X^{p,q} \to \mathcal{A}_X^{p,q+1}$. The operators ∂ and $\overline{\partial}$ anticommute, so with our sign convention, we need to introduce the sign $(-1)^p$ to $\overline{\partial}$ to get the double complex of sheaves $(\mathcal{A}_X^{p,q}, \partial, (-1)^p\overline{\partial})$.

In the case of some finiteness conditions, we can produce a chain complex from a double complex.

Definition 1.3. Let $(K^{p,q}, D_1, D_2)$ be a double complex such that $K^{p,q} = 0$ when $p > p_0$ or $q > q_0$, for some fixed $p_0, q_0 \in \mathbb{Z}$ (i.e. all the objects become zero once we move far enough down and to the left). Then the ***total complex*** (also called the ***simple complex***) associated to the double complex is the complex $(K^\bullet, D)$ where

$$K^n := \bigoplus_{p+q=n} K^{p,q}$$

and the differential D is given by

$$D := \bigoplus_{p+q=n} (D_1 \oplus (-1)^p D_2)$$

For convenience, we will call a double complex that satisfies the conditions above to be ***bounded below***. Given a complex $(A^\bullet, d)$, we will also say that it is bounded below if there are only finitely many terms in negative degree. By shifting degrees, we may as well assume that a bounded below double complex begins indexing at $(0,0)$, and that a bounded below complex begins at index 0.

Proposition 1.4. *The total complex of a bounded below double complex actually forms a complex.*

Proof. It suffices to verify this on the component $K^{p,q} \subset K^n$. The differential D^2 on this component maps to

$$K^{p+q+2} = K^{p+2} \oplus K^{p+1,q+1} \oplus K^{p,q+2}$$

and is given component wise by the map $(D_1^2, (-1)^{p+1}(D_2 \circ D_1) + (-1)^p(D_1 \circ D_2), D_2^2)$. Then since $D_i^2 = 0$ and the differentials commute, we have that this is the zero map. ∎

To make use of double complexes, we first prove a useful fact.

Proposition 1.5. *Let $(I^{p,q}, D_1, D_2)$ be a double complex with total complex $(I^\bullet, D)$, and let $(M^\bullet, d)$ be another complex in $\mathcal{A}$. Then if there exist injective maps $i^p : M^p \to I^{p,0}$ such that $(I^{p,\bullet}, D_2)$ is a resolution of M^p (i.e. the complex $M^p \to I^{p,\bullet}$ is exact for all p), then the morphism $M^\bullet \to I^\bullet$ by the i^p induces an isomorphism from the cohomology of $(M^\bullet, d)$ to the cohomology of the total complex $(I^\bullet, D)$.*

Pictorially, the setup is given by the diagram

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \\
I^{0,2} \arrow[u] \arrow[r, "D_1"] & I^{1,2} \arrow[u] \arrow[r, "D_1"] & I^{2,2} \arrow[u] \arrow[r, "D_1"] & I^{3,2} \arrow[u] \arrow[r] & \cdots \\
I^{0,0} \arrow[u, "D_2"] \arrow[r, "D_1"] & I^{1,0} \arrow[u, "D_2"] \arrow[r, "D_1"] & I^{2,0} \arrow[u, "D_2"] \arrow[r, "D_1"] & I^{3,0} \arrow[u, "D_2"] \arrow[r] & \cdots \\
M^0 \arrow[u, "i^0"] \arrow[r, "d"] & M^1 \arrow[u, "i^1"] \arrow[r, "d"] & M^2 \arrow[u, "i^2"] \arrow[r, "d"] & M^3 \arrow[u, "i^3"] \arrow[r] & \cdots
\end{tikzcd}$$

Where the columns are all exact.

Proof of proposition. We prove this for abelian groups. The proof for R-modules is near identical, and the proof can also be adapted for an abelian category of sheaves by looking at the induced maps on stalks.

Let $\alpha \in I^k := \bigoplus_{p+q=k} I^{p,q}$ such that $D\alpha = 0$, and let $\alpha_{p,q} \in I^{p,q}$ denote the component of α lying in $I^{p,q}$, so $\alpha = \sum_{p,q} \alpha_{p,q}$. Then since $D|_{I^{p,q}}$ is given by $D = D_1 + (-1)^p D_2$, we have that $D_2\alpha_{0,k} = 0$, since no cancellation can occur, since you cannot obtain an element in $I^{0,k+1}$ by appplying D_1. Furthermore, we know that for $q < k$ we have $D_1\alpha_{p,q} + (-1)^p\alpha_{p+1,q-1} = 0$. By assumption, the complex $(I^{0,\bullet}, D_2)$ is exact, since it is a resolution of M^0, so we have that $\alpha_{0,k} = D_2\beta$ for some $\beta \in I^{0,k-1}$. We then consider the element $\alpha' := \alpha - D\beta$, which defines the same element as α in $H^k(I^\bullet)$. We note that α' has the property that $\alpha'_{0,k} = 0$. Because of this, the fact that $D\alpha' = 0$ implies that $D_2\alpha'_{1,k-1} = 0$, since the only way to get cancellation is by applying D_1 to $\alpha_{0,k}$, which we just noted was 0. Repeating this argument, we obtain an element $\tilde{\alpha} \in I^{k,0}$ that defines the same element in $H^k(I^\bullet)$ as our original element α. We then note that $D\tilde{\alpha} = 0$ implies that $D_i\tilde{\alpha} = 0$, again since there is no way to obtain cancellation. Then since $I^{k\bullet}$ is a resolution of M^k via i^k, the fact that $D_2\tilde{\alpha} = 0$ implies by exactness that $\tilde{\alpha} = i^k(\beta)$ for some $\beta \in M^k$. The fact that

$D_1\tilde{\alpha} = 0$ implies that $d\beta = 0$ by commutativity of the square

$$\begin{array}{ccc} I^{k,0} & \xrightarrow{D_1} & I^{k+1,0} \\ {\scriptstyle i^k}\uparrow & & \uparrow{\scriptstyle i^{k+1}} \\ M^k & \xrightarrow[d]{} & M^{k+1} \end{array}$$

Therefore, we have that in the induced map on cohomology, the cohomology class $[\alpha] = [\tilde{\alpha}] \in H^k(I^\bullet)$ is the image of the class $[\beta] \in H^k(M^\bullet)$, so the map $H^k(M^\bullet) \to H^k(I^\bullet)$ is surjective.

For injectivity, suppose we have that a cohomology class $[\alpha] \in H^k(M^\bullet)$ maps to 0, so $i^k(\alpha) = D\beta$ for some $\beta \in I^{k-1}$. We note that $(D\beta)_{0,k} = 0$ since i^k is a map $M^k \to I^{k,0}$. Furthermore, we note that $(D\beta)_{0,k} = D_2\beta_{0,k-1}$. Therefore, we can find a $\gamma \in I^{0,k-2}$ such that $\beta_{0,k-1} = D_2\gamma$. Then setting $\beta' = \beta - D\gamma$, we have that $D\beta' = D\beta = i^k(\alpha)$ but β' has the property that $\beta'_{0,k-1} = 0$. By repeating this argument, we may assume that $i^k(\alpha) = D\beta$, where $\beta \in I^{k-1,0}$. Then since $D_2\beta \in I^{k-1,1}$ and there is no possibility of cancellation, we have that $D_2\beta = 0$ and $D_1\beta = \alpha$. In the case $k = 1$, we can immediately write $\alpha = D\beta$ with $\beta \in I^{0,0}$, and the same reasoning as above lets us conclude that $D_2\beta = 0$ and $D_1\beta = \alpha$. By exactness, $D_2\beta = 0$ implies that $\beta = i^{k-1}(\gamma)$ for some $\gamma \in M^{k-1}$. Then commutativity of the square

$$\begin{array}{ccc} I^{k-1,0} & \xrightarrow{D_1} & I^{k,0} \\ {\scriptstyle i^{k-1}}\uparrow & & \uparrow{\scriptstyle i^{k}} \\ M^{k-1} & \xrightarrow[d]{} & M^{k} \end{array}$$

this implies that $d\gamma = \alpha$, so $[\alpha] = 0$. Therefore, the induced map on cohomology is injective. ∎

We will use this to prove the next proposition.

Proposition 1.6. *let $(M^\bullet, d)$ be a bounded below complex in $\mathcal{A}$. Then there exists another bounded below complex $(I^\bullet, D)$ in $\mathcal{A}$ and a quasi-isomorphism $\varphi : M^\bullet \to I^\bullet$ such that*

(1) *Each of the I^k is an injective object in $\mathcal{A}$.*
(2) *Each $\varphi^k : M^k \to I^k$ is injective.*

Proof. Our goal will be to construct a double complex of injective objects satisfying the conditions of the previous proposition, i.e. a double complex $(I^{p,q}, D_1, D_2)$ that is a level-wise resolution of $M^\bullet$. The desired complex will then be the total complex of the double complex $I^{p,q}$. To do this, it suffices to produce a chain map $i^\bullet : M^\bullet \to I^{1,\bullet}$ such that each i^k is an injection into an injective object $I^{1,k}$. Once we do this, we can iteratively construct the double complex by repeating the argument with $M^\bullet = \operatorname{coker} i^\bullet$ to obtain a chain map $\operatorname{coker} i^\bullet \to I^{2,\bullet}$, and then defining the map $I^{1,\bullet} \to I^{2,\bullet}$ to be the composition of the quotient map $I^{1,\bullet} \to \operatorname{coker} i^\bullet$ with the map $\operatorname{coker} i^\bullet \to I^{2,\bullet}$.

We construct the complex $I^{1,\bullet}$ iteratively. To start, we pick an arbitrary injection $i^0 : M^0 \hookrightarrow I^{0,0}$ from M^0 to an injective object $I^{0,0}$. We have an injective map $M^0 \to I^{0,0} \oplus M^1$

given by $(i^0, -d)$. Let $j : M^1 \to I^{0,0} \oplus M^1$ and $k : I^{0,0} \to I^{0,0} \oplus M^1$ be the inclusions, and let $\pi : I^{0,0} \oplus M^1 \to \operatorname{coker}(I^0, -d)$ be the quotient map. Then fix an injection $\eta : \operatorname{coker}(i^0, -d) \to I^{1,0}$ for an arbitrary injective object $I^{1,0}$. Then define $i^1 : M^1 \to I^{1,0}$ by $i^1 := \eta \circ \pi \circ j$, and define $D_1 : I^{0,0} \to I^{1,0}$ by $D_1 := \eta \circ \pi \circ k$. We claim that the map i^1 is injective. By the definition of the cokernel, an element $\alpha \in M^1$ maps to 0 if it is contained in the image of $(i^0, -d) : M^0 \to I^{0,0} \oplus M^1$. However, since i^0 is injective, the only element with a 0 in the first component is $(0, 0)$. Therefore, since j and η are injective, $i^1(\alpha) = 0$ if and only if $\alpha = 0$. We then claim that $D_1 \circ i^0 = i^1 \circ d$. We note that for $\alpha \in M^0$ we have

$$i^1(d\alpha) = (\eta \circ \pi)(0, d\alpha)$$
$$D_1(i^0(\alpha)) = (\eta \circ \pi)(i^0(\alpha), 0)$$

So if suffices to show that $\pi(0, d\alpha) = \pi(i^0(\alpha), 0)$, but we have that $(i^0(\alpha), 0) - (0, d_\alpha) = (i^0(\alpha), -d\alpha) \in \operatorname{coker}(i^0, -d)$, so this is true. In summary, what we have constructed so far can be summarized by the diagram

$$\begin{tikzcd}
I^{0,0} \arrow[r, "D_1"] & I^{1,0} & \\
M^0 \arrow[u, "i^0"] \arrow[r, "d"] & M^1 \arrow[u, "i^1"] \arrow[r, "d"] & M^2
\end{tikzcd}$$

To continue this, let $\overline{i^1} : M^1 \to \operatorname{coker} D_1$ denote the composition of i^1 with the quotient map. This gives us a map $(\overline{i^1}, -d) : M^1 \to \operatorname{coker} D_1 \oplus M^2$. As we did before, fix an injection $\eta : \operatorname{coker}(\overline{i^1}, -d) \to I^{2,0}$, and define the maps i^2 and D_1 analogously to the previous construction. The map i^2 is injective and $D_1 \circ i^1 = i^2 \circ d$ by near identical reasoning to before. What's left to show is that $D_1^2 = 0$, but this comes immediately from the fact that $\eta \circ \pi$ vanishes on the image of $(\overline{i^1}, -d)$ by construction. Continuing this process then constructs the desired complex $I^{\bullet,0}$ of injectives. ■

A consequence of this proposition is that for the purposes of cohomology, we can always assume we are computing the cohomology of a complex of injective objects. In more modern language, this says that any bounded below complex $M^\bullet$ is isomorphic to a complex of injectives in the derived category $D(\mathcal{A})$.

The goal of the preceding discussion is to extend the classical notion of the right derived functors of a left exact functor $F : \mathcal{A} \to \mathcal{B}$. The classical examples (e.g. Ext and Tor) are defined on objects of $\mathcal{A}$. However, the preceding discussion allows us to extend the functors R^iF to bounded below complexes $M^\bullet$ in $\mathcal{A}$. Echoing the classical definition, we would like to define $R^iF(M^\bullet)$ on a bounded below complex $M^\bullet$ to be the cohomology groups $H^i(F(I^\bullet))$, where $I^\bullet$ is a complex of injective objects admitting a quasi-isomorphism $i^\bullet : M^\bullet \to I^\bullet$, with each i^k injective (which always exists by the previous discussion). Indeed, this is the correct definition, but there is work to be done to show that it is well defined. We first collect some lemmas.

Lemma 1.7. *Let $i^\bullet : M^\bullet \to I^\bullet$ be a quasi-isomorphism of a bounded below complex $M^\bullet$ into a complex $I^\bullet$ of injective objects with each i^k injective. Then for another complex of injective objects*

$J^\bullet$ with a quasi-isomorphism $j^\bullet : M^\bullet \to J^\bullet$ satisfying the same conditions as $i^\bullet$, there exists a chain map $\varphi^\bullet I^\bullet \to J^\bullet$, well defined up to homotopy, such that $\varphi^\bullet \circ j^\bullet$.

References

[1] C. Voisin. *Hodge Theory and Complex Algebraic Geometry, I*. 2002.