# STABLE VECTOR BUNDLES ON RIEMANN SURFACES

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## 1. The Harder-Narasimhan Filtration

**Definition 1.1.** Let $M$ be a compact Riemann surface and $E \to M$ a holomorphic vector bundle. The complex structure of $M$ induces a natural orientation on $M$, giving an isomorphism $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$. The ***degree*** of $E$, denoted $\deg(E)$, is the image of the first Chern class $c_1(E)$ under this isomorphism. The ***slope*** of $E$, denoted $\mu(E)$, is the ratio

$$\mu(E) = \frac{\deg(E)}{\operatorname{rank}(E)}$$

*Remark.* Note that for a holomorphic vector bundle $E \to M$ of rank $k$, we have

$$\deg(E) = \deg(\Lambda^k E)$$

One important property of degree is that it places heavy limitations on the existence of holomorphic sections.

**Proposition 1.2.** *Let $L \to M$ be a holomorphic line bundle over a compact Riemann surface $M$ with $\deg(L) < 0$. Then $H^0(M, L) = 0$.*

*Proof.* Since $L$ is a negative line bundle, we have that $L^*$ is a positive line bundle. Kodaira vanishing gives us that $H^1(M, L^* \otimes K_M) = 0$, where $K_M := \Omega^{1,0}_M$ is the canonical bundle. Serre duality then tells us that $H^0(M, L) = 0$. ∎

A different, lower tech proof follows almost immediately from an alternate characterization of degree.

**Proposition 1.3.** *Let $L \to M$ be a holomorphic line bundle over a compact Riemann surface $M$ of degree $n$. Then for any nonzero meromorphic section $\sigma$ of $L$, the number of zeroes minus the number of poles, counted with multiplicity, is equal to $n$.*

**Corollary 1.4.** *Let $L_1, L_2 \to M$ be holomorphic line bundles over a compact Riemann surface $M$. If there exists a nonzero holomorphic bundle homomorphism $\varphi : L_1 \to L_2$, then $\deg(L_1) \leq \deg(L_2)$, with equality if and only if $\varphi$ is an isomorphism.*

*Proof.* We use the bundle isomorphism $\mathsf{Hom}(L_1, L_2) \cong L_1^* \otimes L_2$. Since degree is additive on tensor products, we have $\deg(L_1^* \otimes L_2) = \deg(L_2) - \deg(L_2)$. Therefore, if $\deg(L_1)$ is less than $\deg(L_2)$, we know that $\mathsf{Hom}(L_1, L_2)$ is of negative degree, so it admits no nonzero global sections.

For the other part, we note that if the degrees of $L_1$ and $L_2$ are equal, then $\mathsf{Hom}(L_1, L_2)$ is a degree 0 bundle. We then note that if $\mathsf{Hom}(L_1, L_2)$ admits a nonzero holomorphic section, it must necessarily have no zeroes, which implies that it is isomorphic to the trivial line bundle $\mathcal{O}_M$, which implies that any global section of $\mathsf{Hom}(L_1, L_2)$ is an isomorphism. ∎

**Corollary 1.5.** *Let $E, F \to M$ be holomorphic vector bundles of the same rank over a compact Riemann surface $M$. Then If there exists a nonzero holomorphic bundle homomorphism $E \to F$, we must have* $\deg(E) \leq \deg(F)$.

*Proof.* Apply the above corollary to the induced map $\Lambda^k E \to \Lambda^k F$, noting that the degree of $E$ and $F$ are equal to the degrees of their respective determinant line bundles. ∎

**Proposition 1.6.** *Let $E \to M$ be a holomorphic vector bundle over a compact Riemann surface $M$. Then there exists an integer $q > 0$ such that for any line bundle $L$ with $\deg(L) > q$, the bundle $\mathsf{Hom}(L, E)$ has no nonzero holomorphic sections.*

*Proof.* Since $M$ is projective, we can find an ample line bundle $H \to M$, which is necessarily of positive degree. Then by Serre Vanishing, we know that for a sufficiently large $m$, the bundle $H^m \otimes E^* \otimes K_M$ has no higher cohomology. Serre Duality then lets us conclude that $H^0(M, H^{-m} \otimes E) = 0$. Using the isomorphism $H^{-m} \otimes E \cong \mathsf{Hom}(H^m, E)$, this tells us that there exist no nonzero holomorphic bundle homomorphisms $H^m \to E$. Riemann-Roch then tells us that for any line bundle $L$, we have

$$h^0(M, H^{-m} \otimes L) - h^0(M, H^m \otimes L^* \otimes K_M) = \deg(H^{-m} \otimes L) + 1 - g$$

where $K_M$ is the canonical bundle of $M$. Since degree is additive on tensor products, this tell us that

$$h^0(M, H^{-m} \otimes L) \geq -m \deg(H) + \deg(L) + 1 - g$$

Therefore, if $\deg(L)$ is sufficiently large, $H^0(M, H^{-m} \otimes L)$ is nonzero. Then using the fact that $H^{-m} \otimes L \cong \mathsf{Hom}(H^m, L)$, this tells us that there exists a nonzero holomorphic bundle homomorphism $H^m \to L$. Therefore, when $L$ is of sufficiently high degree, $\mathsf{Hom}(L, E)$ admits no global sections, since otherwise, this would imply the existence of a nonzero map $H^m \to E$, which is impossible since $H^0(M, \mathsf{Hom}(H^m, E)) = 0$. ∎

The previous proposition allows us to give a uniform bound on the slopes of holomorphic subbundles of a holomorphic vector bundle.

**Proposition 1.7.** *Let $E \to M$ be a holomorphic vector bundle over a compact Riemann surface $M$. Then there exists a nonnegative integer $m \geq 0$ such that*

$$\mu(F) \leq m$$

*for all proper nontrivial holomorphic subbundles $F \subset E$.*

*Proof.* Consider the bundles $\Lambda^k E$. The previous proposition gives nonnegative integers $m(k)$ such that any holomorphic line bundle $L \to M$ with degree greater than $m(k)$ admits no nonzero holomorphic bundle homomorphisms to $\Lambda^k E$. Then let $F \subset E$ be a proper nontrivial holomorphic subbundle of rank $k$. The inclusion $F \hookrightarrow E$ induces an inclusion $\Lambda^k F \hookrightarrow \Lambda^k E$, which tells us that $\deg(F) = \deg(\Lambda^k F) \leq m(k)$. Therefore, any holomorphic subbundle $F \subset E$ of rank $k$ must satisfy $\mu(F) \leq m(k)/k$. Taking the maximum among the $m(k)/k$ over all $k$ less than the rank of $E$ then gives the desired slope bound. ∎

We now collect a few facts about slopes.

**Lemma 1.8.** *Let*

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$$

*be an exact sequence of holomorphic vector bundles over a compact Riemann surface M. Then*

$$\mu(F) = \frac{\deg(E) + \deg(G)}{\operatorname{rank}(E) + \operatorname{rank}(G)}$$

*Proof.* Since the degree is purely topological, we may forget the holomorphic structures, and treat the exact sequence as an exact sequence of smooth vector bundles. Any such exact sequence splits in the smooth category, giving us a direct sum decomposition $F = E \oplus G$ as smooth vector bundles. Then since the first Chern class is additive on direct sums, we get that $\deg(F) = \deg(E) + \deg(G)$. Rank is also clearly additive on exact sequences, and putting these two together gives the desired formula for the slope of $F$. ■

**Corollary 1.9.** *Let*

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$$

*be an exact sequence of holomorphic vector bundles over a compact Riemann surface M. Then if $\mu(E) \geq \mu(F)$, we have $\mu(F) \geq \mu(G)$. Likewise, if $\mu(E) \leq \mu(F)$, then $\mu(F) \leq \mu(G)$.*

**Definition 1.10.** A holomorphic vector bundle $E \to M$ over a compact Riemann surface is ***stable*** if for all proper nontrivial vector subbundles $F \subset E$, we have the strict inequality

$$\mu(F) < \mu(E)$$

We say $E$ is ***semistable*** if instead we have the inequality

$$\mu(F) \leq \mu(E)$$

To iteratively construct the Harder-Narasimhan filtration for a holomorphic vector bundle, we will need to find some sort of maximal subbundles.

**Proposition 1.11.** *Let $E \to M$ be a holomorphic vector bundle over a compact Riemann surface. Then there exists a unique holomorphic subbundle $E_1 \subset E$ such that for every holomorphic subbundle $F \subset E$, we have*

(1) $\mu(F) \leq \mu(E_1)$
(2) $\operatorname{rank}(F) \leq \operatorname{rank}(E_1)$ *if* $\mu(F) = \mu(E_1)$.

*Furthermore, $E_1$ is semistable, and is called the* ***maximal semistable subbundle****.*

*Proof.* If $E$ is already semistable, then taking $E_1 = E$ works, so we assume that $E$ is not semistable. From the previous proposition, we know that the slopes $\mu(F)$ of all proper nontrivial subbundles is bounded by some integer $m$. We then claim that there exists a subbundle of maximal slope. To see this, we note that by definition, the slope of a proper subbundle $F \subset E$ of rank $k$ is given by

$$\mu(F) := \frac{\deg(F)}{k} \leq m$$

Since the degree of $F$ is a nonnegative integer, this tells us that $\mu(F)$ is contained in the finite set $\{0, 1/k, \ldots, mk/k\}$. Therefore, the set of all possible slopes is a finite set, so there exists bundles that attain this maximum. We then choose $E_1$ to be a bundle with maximal rank among those with maximal slope. Then $E_1$ satisfies the two specified conditions, and is semistable by the first condition.

We then want to show uniqueness of $E_1$. Let $F$ be another subbundle satisfying the specified properties. If $F \subset E_1$, the the second property guarantees that $F = E_1$, so we assume that $F \not\subset E_1$. Then consider the holomorphic bundle $E/E_1$, and let $\varphi : F \to E/E_1$ be the restriction of the quotient map to $F$. By assumption, $\varphi$ is a nonzero holomorphic map. This gives us the exact sequence of coherent sheaves

$$0 \longrightarrow \ker\varphi \longrightarrow F \xrightarrow{\varphi} \varphi_* F \longrightarrow 0$$

We note that we must pass to sheaves, since the kernel and image of $\varphi$ may fail to be vector bundles. However, we may return to the land of vector bundles via the following observation : the stalks of the structure sheaf $\mathcal{O}_M$ are all principal ideal domains, so the stalks of $\ker\varphi$ and $\varphi_* F$ have direct sum decompositions into a torsion submodule and a free module, the free components define vector bundles $A \supset \ker\varphi$ and $B \supset \varphi_* F$, giving us the exact sequence of vector bundles

$$0 \longrightarrow A \longrightarrow F \xrightarrow{\varphi} B \longrightarrow 0$$

More explicitly, $A$ and $B$ may be realized as the smallest holomorphic subbundles containing $\ker\varphi$ and $\operatorname{Im}\varphi$ respectively. To see this, let $k$ be the rank of $\varphi_* F$ as an $\mathcal{O}_M$-module, and let $\{s_i\}$ be a local holomorphic frame for $F$. Generically, we have that the set $\{\varphi(s_i)\}$ spans a $k$-dimensional subbundle of $E/E_1$, except when all the sections of $\Lambda^k(E/E_1)$ of the form

$$\varphi(s_{i_1}) \wedge \cdots \wedge \varphi(s_{i_k})$$

simultaneously vanish, i.e. there exists no linearly independent subset of $\{\varphi(s_i)\}$ of cardinality $k$. Then in a sufficiently small neighborhood of a point where all these sections vanish, we may fix local holomorphic coordinates and a holomorphic trivialization of $\Lambda^k(E/E_1)$ such that the sections $\varphi(s_{i_1}) \wedge \cdots \wedge \varphi(s_{i_k})$ are identified with holomorphic functions $U \to \mathbb{C}$, where $U$ is an open subset in $\mathbb{C}$. Then we can factor out all of the zeroes of these sections to obtain a set of nonvanishing sections of $\Lambda^k(E/E_1)$, which then locally determine a rank $k$ subbundle of $E/E_1$. Gluing these local definitions together then gives us the bundle $B$. We construct $A$ from $\ker\varphi$ via a similar procedure. In total, we get the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & F & \longrightarrow & F/A & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\widetilde{\varphi}} & & \\
0 & \longleftarrow & (E/E_1)/B & \longleftarrow & E/E_1 & \longleftarrow & B & \longleftarrow & 0
\end{array}$$

Where $\widetilde{\varphi}$ is the natural map induced by $\varphi$. We then make the following observations:

(1) Since $F$ is semistable, we have that $\mu(A) \leq \mu(F)$. Then since the top row is exact. this tells us that $\mu(F) \leq \mu(F/A)$.
(2) We know that $F/A$ is the same rank as $B$, and by assumption, the map $\widetilde{\varphi}$ is nonzero, so $\deg(F/A) \leq \deg(B)$. Therefore, $\mu(F/A) \leq \mu(B)$.
(3) We note that since $E_1$ is of maximal slope, we in particular have that for any holomorphic subbundle $Q \subset E$ with $E_1 \subsetneq Q$, then $\mu(Q) \leq \mu(E_1)$. Then if we consider the exact sequence

$$0 \longrightarrow E_1 \longrightarrow Q \longrightarrow Q/E_1 \longrightarrow 0$$

this tells us that the slope of $\mu(Q/E_1) \leq \mu(Q)$. Since we may regard subbundles of $E/E_1$ as quotients $Q/E_1$, this tells us that for any holomorphic subbundle of $E/E_1$, the slope is bounded by $\mu(E_1)$. In our case, this implies that $\mu(B) \leq \mu(E_1)$.

We now put everything together. Our observations give us the chain of inequalities

$$\mu(F) \leq \mu(F/A) \leq \mu(B) \leq \mu(E_1)$$

We then observe that we have the exact sequence

$$0 \longrightarrow E_1 \longrightarrow \varphi^{-1}(B) \longrightarrow B \longrightarrow 0$$

By the first property of $E_1$, we have that $\mu(\varphi^{-1}(B)) \leq \mu(E_1)$. However, by assumption the rank of $\varphi^{-1}(B)$ is larger than the rank of $E_1$, so the second property guarantees that we have the strict inequality $\mu(\varphi^{-1}(B)) < \mu(E_1)$. The exact sequence then tells us that $\mu(\varphi^{-1}(B)) \geq \mu(B)$, so $\mu(B) < \mu(E_1)$. However, this tells us that our above chain of inequalities is actually

$$\mu(F) \leq \mu(F/A) \leq \mu(B) < \mu(E_1)$$

which in particular implies that $\mu(F) < \mu(E_1)$, contradicting maximality of $\mu(F)$. ∎

**Theorem 1.12** (***Harder-Narasimhan Filtration***)**.** *Any holomorphic vector bundle $E \to M$ over a compact Riemann surface $M$ admits a filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

*by holomorphic subbundles $E_i$ such that $E_i/E_{i-1}$ is semistable and*

$$\mu(E_1/E_0) > \mu(E_2/E_1) > \cdots > \mu(E_n/E_{n-1})$$

*Proof.* Let $E_1$ be the maximal semistable subbundle of $E$. Then let $E_2$ be the preimage of the maximal semistable subbundle of $E/E_1$ under the quotient map. We then iteratively construct the filtration by taking $E_i$ to be the preimage of the maximal semistable subbundle of $E/E_{i-1}$. By construction, the quotients $E_i/E_{i-1}$ are semistable. We then want to show the monotonicity of the slopes. We also note that since the slopes are strictly decreasing, this construction will eventually terminate. To show this, we use the exact sequence

$$0 \longrightarrow E_{i+1}/E_i \longrightarrow E/E_i \longrightarrow E/E_{i+1} \longrightarrow 0$$

By constuction, we have that $E_{i+1}/E_i$ is the maximal semistable bundle of $E/E_i$. Therefore, any subbundle of $E/E_{i+1}$ has a strictly smaller slope than $E_{i+1}/E_i$, in particular, this gives us that the slope of $E_{i+2}/E_{i+1}$ is less than the slope of $E_{i+1}/E_i$, which is what we wanted to show. ∎

## References

[1] S. Kobayashi. *Differential Geometry of Complex Vector Bundles*. 1987.