CHAPTER

1

Principal Bundles

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Why Fiber Bundles?

Suppose we want to study some space M, which for our purposes is a smooth manifold. One way to study M is to study functions $M \to F$ for some fixed target space F, e.g. a manifold , $\mathbb{R}$, or a vector space. This is a perfectly good method of studying M, but is sometimes not enough. More often then not, we want to study "function" from M into a vector space that varies over the base manifold M. For example, a vector field $X \in \mathfrak{X}(M)$ is not really a map $M \to \mathbb{R}^n$, it is an assignment to each $p \in M$ a vector in T_pM. In this way, we are led to the study of a smoothly parameterized family of vector spaces – the tangent bundle TM. This leads us to define a fiber bundle.

Definition 0.1. *Let M and F be a smooth manifolds. A **fiber bundle** over M with model fiber F is the data of a smooth manifold E with a smooth surjective map $\pi : E \to M$ such that for each $p \in M$, there exists an open set U and a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ such that*

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "p_1"] \\
& U &
\end{tikzcd}$$

*where $p_1 : U \times F \to U$ is projection onto the first factor. The map $\varphi : \pi^{-1} \to U \times F$ is called a **local trivialization**. The space E is called the **total space**, while the manifold M is called the **base space**. We often omit naming the map π and denote the fiber $\pi^{-1}(x)$ by E_x.*

This definition captures the notion of a family of manifolds diffeomorphic to F that are smoothly parameterized by the base space M. We also have a notion of a morphism between bundles.

Definition 0.2. *Let $\pi_E E \to M$ and $\pi_F : F \to M$ be fiber bundles with model fiber X. A* ***bundle homomorphism*** *is the data of a smooth map $\varphi : E \to F$ such that the diagram*

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "\pi_E"'] & & F \arrow[dl, "\pi_F"] \\ & M & \end{tikzcd}$$

commutes.

Our original motivation for thinking about fiber bundles was for a generalized notion of a function. To this end, we specify a special class of maps associated to a fiber bundle.

Definition 0.3. *Let $\pi : E \to M$ be a a fiber bundle with model fiber F. A* ***local section*** *of $\pi : E \to M$ is a smooth map $\sigma : U \to E$ such that $\pi \circ \sigma = \mathrm{id}_U$ for some open set $U \subset M$. If $U = M$, we call σ a* ***global section****. Equivalently, it is a smooth assignment to each $p \in U$ a point in the fiber E_p. We denote the space of sections over an open set U as $\Gamma_U(E)$.*

A section of $E \to M$ can be thought of as our desired generalization of a function. A map $M \to F$ is the same data as a section of the trivial bundle $M \times F \to M$. However, not every fiber bundle is trivial – there can be a nontrivial "twisting." An example of this is the Möbius band. Can you see why this bundle over S^1 is not isomorphic to the trivial bundle $S^1 \times [0, 1]$?

We are especially interested in two special classes of fiber bundles that carry additional structure – the fibers of a vector bundle carry the extra structure of a vector space, and the fibers of a principal bundle have the extra structure of a G-torsor for a Lie group G.

Definition 0.4. *A* ***vector bundle of rank*** *k is a fiber bundle $E \to M$ such that each fiber E_x has the structure of a k-dimensional vector space (usually over $\mathbb{R}$ or $\mathbb{C}$). A* ***vector bundle homomorphism*** *is a bundle homomorphism that restricts to a linear map on each fiber.*

Vector bundles form a familiar family of fiber bundles, as tangent bundles, cotangent bundles, and their associated tensor bundles are all vector bundles.

Definition 0.5. *Let G be a Lie group. a (left)* ***G-torsor*** *is a smooth manifold M equipped with a smooth (left) G-action that is free in transitive i.e.*

1. *For any $g \in G$ and $p \in M$, if $g \cdot p = p$, then $g = e$.*

2. For any $p \in M$, the orbit $G \cdot p$ is all of M

combined, these conditions give that for any fixed point p, there exists a unique group element $g \in G$ that takes p to any other point of M. In other words, the map $g \mapsto g \cdot p$ is a diffeomorphism.

Example 0.1. *Let V be a vector space. A* ***basis*** *(also called a* ***frame****) of V is a linear isomorphism $\mathbb{R}^n \to V$, and the set of frames of V is denoted mathcal$B(V)$. There is a natural right action of $GL_n\mathbb{R}$ on $\mathcal{B}(V)$ by precomposition, i.e. for $g \in GL_n\mathbb{R}$ and $b \in \mathcal{B}(V)$, we have $b \cdot g = b \circ g$. This action is free and transitive, which allows us to define a topology and smooth structure on $\mathcal{B}(V)$, giving it the structure of a right $GL_n\mathbb{R}$-torsor.*

G-torsors can be thought of as Lie groups without a fixed choice of identity element. Indeed, if we fix a point p of a G-torsor M, this determines a unique Lie group structure on M where p is the identity element. This allows us to define a principal G-bundle. For example, there is no distinguished basis for a vector space V, which is reflected by the fact that $\mathcal{B}(V)$ is a torsor, rather than a group.

Definition 0.6. *A* ***principal*** ***G-bundle*** *is a fiber bundle $\pi : P \to M$ with a smooth* right *G-action such that every fiber $\pi^{-1}(x)$ is a G-torsor, and every point $p \in M$ in the base has a local trivialization $\pi^{-1}(U) \to U \times G$ that is G-equivariant, where the G-action on $U \times G$ is right multiplication on the second factor. The group G is often called the* ***structure group****.*

Example 0.2 (***Bundles of frames***)**.** *Let M be a smooth manifold, and $p \in M$. Let $\mathcal{B}_p$ denote the $GL_n\mathbb{R}$-torsor of frames of the tangent space T_pM, and let $\mathcal{B}(M)$ be the set*

$$\mathcal{B}(M) = \coprod_{p \in M} \mathcal{B}_p$$

This set comes with a natural map $\pi : \mathcal{B}(M) \to M$ that maps a frame $b \in \mathcal{B}_p$ to p, so the fiber $\pi^{-1}(p)$ over any point is $\mathcal{B}_p$. Local coordinates x^i on an open set $U \subset M$ induce bijections $\pi^{-1}(U) \to U \times GL_n\mathbb{R}$, where we map $b \in \mathcal{B}_p$ to (p, A_b), where $A_b \in GL_n\mathbb{R}$ is the matrix where the i^{th} column are the components of $b(e_i)$ with respect to the coordinate vectors ∂_i. This then induces a topology and smooth structure on $\mathcal{B}(M)$ such that the projection map $\pi : \mathcal{B}(M) \to M$ is smooth and the maps $\pi^{-1}(U) \to U \times GL_n\mathbb{R}$ are smooth local trivializations.

This construction works for vector bundles as well. For a rank k vector bundle $E \to M$, there is a principal $GL_n\mathbb{R}$-bundle $\mathcal{B}(E) \to M$, where the fiber over a point p is the $GL_k\mathbb{R}$-torsor of frames for the fiber E_x.

Heuristically, principal bundles can be thought of bundles of symmetries of some object, and in many cases, this object is a vector bundle, as we will soon see.

Groups Control the Geometry

Given any geometric object M (e.g. a vector space, manifold, fiber bundle), an extremely important question to as is – "what are the symmetries of M?" More concretely, we would like to know the *automorphisms* of M. Any quantity intrinsic to the object M must be invariant under these automorphisms, sine we have no "preferred frame of reference" for the object M. In this way, we see that any geometric properties of M are defined by the symmetry group G, and we say the group G "controls" the geometry of M. For example, if we let V be any finite dimensional vector space, the group of symmetries is the general linear group $GL(V)$ of invertible linear transformations. If we further fix an inner product $\langle \cdot, \cdot \rangle$ on V, we have a smaller class of symmetries – the orthogonal group $O(V) \subset GL(V)$ of linear automorphisms preserving the inner product. Additional structure (e.g. orientation, symplectic form, complex structure) allows us to pick out subgroups of automorphisms, which gives a more restrictive class of symmetries for our vector spaces,

This proves to be a very fruitful philosophy for approaching geometry. Smooth manifolds are locally modeled on vector spaces, and given a structure on the linear world of vector spaces, we often get an analogous structure in the nonlinear world of manifolds. For example, a Riemannian metric g on a manifold M is the nonlinear analogue of an inner product on a vector space. Because of this, we would expect the orthogonal group O_n to play an important role in the geometry of a Riemannaian manifold. The interaction of the group with the geometry comes in play through the language of principal bundles. Before we mentioned that principal bundles can be thought of as bundles of symmetries, which we make precise with the construction of a ***associated bundle***.

Definition 0.7. *Let $P \to M$ be a principal G-bundle, and F a smooth manifold with a smooth left G-action. The* ***associated bundle*** *is the bundle*

$$P \times_G F = P \times F/(p, f) \sim (p \cdot g, g^{-1} \cdot f)$$

This is a fiber bundle over M with model fiber F, and it is a good exercise to see why this is true by explicitly constructing the local trivializations in terms of local trivializations of $\mathcal{B}(M)$.

The construction is a bit obtuse, so we make a few observations to make sense of why we want to care about associated bundles.

Example 0.3 (***The tangent bundle***)**.** *Let M be a smooth manifold. From M, we have a principal GL_n-bundle $\mathcal{B}(M) \to M$ of frames for TM. In addition, $GL_n\mathbb{R}$ admits a natural left action on $\mathbb{R}^n$ via matrix multiplication. The associated bundle $E = \mathcal{B}(M) \times_{GL_n\mathbb{R}} \mathbb{R}^n$ is isomorphic to the tangent bundle TM. To see this, we construct maps in both directions.*

Let $[b, v]$ denote an equivalence class in E, where $b \in \mathcal{B}_p$ is a frame for T_pM and $v \in \mathbb{R}^n$. Then define $\varphi : E \to TM$ by $\varphi[b, v] = b(v)$. This is well defined, since

$$\varphi[b \cdot g, g^{-1} \cdot v] = b \circ g(g^{-1}(v)) = b(v)$$

In the other direction, let $(p, v) \in TM$, i.e. $p \in M$ and $v \in T_pM$. Fix a basis $b : \mathbb{R}^n \to T_pM$, and define $\psi : TM \to E$ by $\psi(p, v) = [b, b^{-1}(v)]$. It's an easy exercise to check that these two maps compose to identity in both directions.

The above construction works in the setting of vector bundles as well. Given a rank k vector bundle $E \to M$, we can construct the principal $GL_k\mathbb{R}$-bundle $\mathcal{B}(E)$, and the associated bundle $\mathcal{B}(E) \times_{GL_k\mathbb{R}} \mathbb{R}^k$ is isomorphic to the bundle E.

This construction shows why we might care about associated bundles – the quotient by relation we specified exactly gives the correct transformation law for tangent vectors. You may have heard the joke that physicists define a vector as "something that transforms like a vector." A more precise joke here would perhaps be "a tangent vector is something that transforms like a tangent vector." What do we mean by this? Physically, the only "real" quantities are those invariant under a change of reference frame. In terms our example, we see that a tangent vector $v \in T_pM$ is *not* just an n-tuple of numbers, it's the collection of all coordinate representations of v with respect to any basis of T_pM, which is exactly what the associated bundle construction captures. We can see this another way. Fix an element $b \in \mathcal{B}_p \subset \mathcal{B}(M)$. This then determines an isomorphism of $\mathbb{R}^n$ to the fiber of the associated bundle $\mathcal{B}(M) \times_{GL_n\mathbb{R}} \mathbb{R}^n$ by $v \mapsto [b, v]$, this exactly what happens when we fix a basis for T_pM! In this way, we see that fixing the first component in the equivalence class of an element of an associated bundle is essentially a choice of basis or reference frame, and doing the computations with the equivalence classes themselves is in essense, working in a coordinate-free manner.

We now address how fixing additional structures on our manifold M changes this picture. The best example here is that of a Riemannian metric g on M, which gives us a notion of an ***orthonormal frame*** of the tangent space T_pM, which is a linear *isometry* $b : (\mathbb{R}^n, \langle \cdot, \cdot \rangle) \to (T_pM, g_p)$ where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$, and g_p is the Riemannian metric evaluated at the point p. We then construct the ***orthonormal frame bundle*** $\mathcal{B}_O(M)$ in the same way we constructed the frame bundle $\mathcal{B}(M)$, which is a principal O_n-bundle, which reflects that we now have a more restrictive view on symmmetry – our automorphisms now take values in O_n instead of $GL_n\mathbb{R}$.

Definition 0.8. *Let M be a smooth manifold, and $\mathcal{B}(M)$ its $GL_n\mathbb{R}$-bundle of frames. Let G be a Lie group, and $\rho : G \to GL_n\mathbb{R}$ be a homomorphism. Then a **reduction of structure group** to G is the data of a principal G-bundle $Q \to M$ equipped with a G-equivariant map $Q \to \mathcal{B}(M)$, where G acts on the left of $\mathcal{B}(M)$ by $g \cdot b = b \circ \rho(g)^{-1}$.*

In the case we illustrated above with a Riemannian metric g, the homomorphism ρ is just the inclusion $O_n \hookrightarrow GL_n\mathbb{R}$, the bundle Q is $\mathcal{B}_O(M)$, and the O_n-equivariant map is the inclusion $\mathcal{B}_O(M) \hookrightarrow \mathcal{B}(M)$. However, the homomorphism ρ does *not* need to be injective, which makes the name a bit a misnomer – the group does not need to be a subgroup of $GL_n\mathbb{R}$. For example, when working with the Spin group Spin_n, the map ρ is often the double cover $\mathrm{Spin}_n \to SO_n$.

Our construction of the tangent bundle as an associated bundle can also be generalized to other tensor bundles built out of the tangent bundle (e.g. $\mathcal{T}^k_\ell(M)$, T^*M, $\Lambda^k(T^*M)$, etc), by using the induced representations of the stucture group G on tensor products, the dual representation, and the induced representations on exterior powers respectively. From this, we see that the associated bundle construction is the bridge between the theory of principal bundles and vector bundles with representation theory, and reductions of structure group allows us to import our linear model geometries (e.g. an inner product space, a vector space with a complex structure I, an oriented vector space, etc.) to the nonlinear world of manifolds by asking for a reduction of structure group to the appropriate group. For example, an orientation is equivalent to a reduction of structure group to $GL_n^+\mathbb{R} = \{A \in GL_n\mathbb{R} \ : \ \det A > 0\}$, and an almost complex structure on a $2n$-dimensional manifold is equivalent to a reduction of structure group from $GL_{2n}\mathbb{R}$ to $GL_n\mathbb{C}$.

References

Most of the content of this article came from many discussions with Professor Dan Freed. If you want to learn more, I heavily recommend finding some time to talk to him. A good reference for the theory of principal bundles can be found in Kobayashi and Nomizu's books *Foundations of Differential Geometry.*