

THE FRÖLICHER SPECTRAL SEQUENCE AND THE $\partial\bar{\partial}$ LEMMA

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For any complex manifold X , we have the Frölicher spectral sequence, which computes the de Rham cohomology of a complex manifold in terms of the ∂ and $\bar{\partial}$ cohomology. On the E_0 page, it is given by the Dolbeault cohomology of the bundles $\Omega^{p,0}$, i.e. the item in the (p, q) position is the bundle $\Omega^{p,q}$ of (p, q) forms, and the differential on E_0 is just $\bar{\partial}$. For example, a small section of the E_0 page would be:

$$\begin{array}{c|cccc}
 & \Omega^{0,2} & \Omega^{1,2} & \Omega^{2,2} & \Omega^{3,2} \\
 2 & \uparrow \bar{\partial} & \uparrow \bar{\partial} & \uparrow \bar{\partial} & \uparrow \bar{\partial} \\
 & \Omega^{0,1} & \Omega^{1,1} & \Omega^{2,1} & \Omega^{3,1} \\
 1 & \uparrow \bar{\partial} & \uparrow \bar{\partial} & \uparrow \bar{\partial} & \uparrow \bar{\partial} \\
 & \Omega^{0,0} & \Omega^{1,0} & \Omega^{2,0} & \Omega^{3,0} \\
 0 & & & & \\
 \hline
 & 0 & 1 & 2 & 3
 \end{array}$$

Then in the E_1 page, we have $E_1^{p,q}$ is the cohomology in the (p, q) slot in the E_0 page, which is just $H^{p,q}(X)$ (abbreviated to $H^{p,q}$) by Hodge theory in the compact case. The differentials going from left to right are the operators ∂ , which descends to cohomology because $\partial\bar{\partial} = -\bar{\partial}\partial$. A small section of the E_1 page would be :

$$\begin{array}{c|cccc}
 & H^{0,2} & \xrightarrow{\partial} & H^{1,2} & \xrightarrow{\partial} & H^{2,2} & \xrightarrow{\partial} & H^{3,2} \\
 2 & & & & & & & \\
 & H^{0,1} & \xrightarrow{\partial} & H^{1,1} & \xrightarrow{\partial} & H^{2,1} & \xrightarrow{\partial} & H^{3,1} \\
 1 & & & & & & & \\
 & H^{0,0} & \xrightarrow{\partial} & H^{1,0} & \xrightarrow{\partial} & H^{2,0} & \xrightarrow{\partial} & H^{3,0} \\
 0 & & & & & & & \\
 \hline
 & 0 & 1 & 2 & 3
 \end{array}$$

For a general compact complex manifold X , this continues to the E_2 page, where the differential “rotates,” and the (p, q) slot is the cohomology in the (p, q) slot of the E_1 page. The big theorem we want to prove is:

Theorem 1.1. *For a compact Kähler manifold X , the Frölicher spectral sequence degenerates at the E_1 page, i.e. all the differentials are 0.*

In other words, we can terminate our spectral sequence computations at E_1 . Going to the E_1 page is easy, since all the computations are done with the operators ∂ and $\bar{\partial}$. In practice, continuing on to further pages is difficult. The entire spectral sequence story seems very difficult (and it is), but in the compact Kähler story, it reduces to a simple lemma.

Theorem 1.2 (The $\partial\bar{\partial}$ lemma). *Let X be a Kähler manifold, and η a complex k -form that is ∂ and $\bar{\partial}$ -closed. Then if η is d , ∂ , or $\bar{\partial}$ -exact, there exists a form ξ such that $\eta = \partial\bar{\partial}\xi$.*

The proof of this lemma requires the following results from Hodge Theory:

Theorem 1.3 (Comparison of the Laplacians). *Let X be a compact Kähler manifold. Then*

$$\Delta = 2\Delta_{\partial} = 2\Delta_{\bar{\partial}}$$

where Δ , Δ_{∂} , and $\Delta_{\bar{\partial}}$ are the Laplacians

$$\Delta = dd^* + d^*d$$

$$\Delta_{\partial} = \partial\partial^* + \partial^*\partial$$

$$\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$$

The proof of this theorem requires certain commutation relations to hold, called the **Kähler identities**. This identity is not true in general for an arbitrary compact complex manifold. The other result we need is

Theorem 1.4 (The Hodge Decomposition). *Any complex valued form $\alpha \in \Omega^{p,q}$ can be written as*

$$\alpha = \beta + \Delta\gamma$$

where β is harmonic, i.e. $\Delta\beta = 0$.

This theorem is true for general compact complex manifolds, not just Kähler manifolds.

Proof of the $\partial\bar{\partial}$ lemma. The proof for all three cases is much the same, so we do just the one where $\eta = \bar{\partial}\alpha$ is $\bar{\partial}$ -exact. By the Hodge decomposition, we write $\alpha = \beta + \Delta\gamma$, with β harmonic. Since $\Delta = 2\Delta_{\bar{\partial}}$, and β is $\Delta_{\bar{\partial}}$ -harmonic if and only if $\bar{\partial}\beta = \bar{\partial}^*\beta = 0$, we have that $\bar{\partial}\beta = 0$. We then compute

$$\begin{aligned} \eta &= \bar{\partial}\alpha \\ &= \bar{\partial}(\beta + \Delta\gamma) \\ &= \bar{\partial}\beta + 2\bar{\partial}(\Delta_{\partial}\gamma) \\ &= 0 + 2\bar{\partial}(\partial\partial^*\gamma + \partial^*\partial\gamma) \\ &= 2\bar{\partial}\partial\partial^*\gamma - 2\partial^*\bar{\partial}\partial\gamma \\ &= -2\partial\bar{\partial}\partial^*\gamma - 2\partial^*\bar{\partial}\partial\gamma \end{aligned}$$

Then since η is ∂ -closed, we have that $\partial^*\bar{\partial}\partial\gamma$ must also be ∂ -closed. By orthogonality of the image of ∂^* with the kernel of ∂ , we have that $\partial^*\bar{\partial}\partial\gamma = 0$, so $\eta = -2\partial\bar{\partial}\partial^*\gamma = 2\bar{\partial}\partial\partial^*\gamma$, so letting $\xi = \partial^*\gamma$, we are done. \blacksquare

We now use this to prove Theorem 1.1.

Proof of 1.1. We want to show that all the differentials on the E_1 page are 0, i.e. for a cohomology class $[\alpha] \in H^{p,q}$, $[\partial\alpha] = 0$. Since $[\alpha]$ is a Dolbeault cohomology class, we know that α is $\bar{\partial}$ -closed. Therefore, $\partial\alpha$ is both $\bar{\partial}$ and ∂ closed, since ∂ and $\bar{\partial}$ anticommute. Then by the $\partial\bar{\partial}$ lemma, we have that $\partial\alpha = \partial\bar{\partial}\eta$ for some η . Therefore, using the fact that ∂ and $\bar{\partial}$ commute one final time, we find that $\partial\alpha$ is $\bar{\partial}$ -exact, i.e. $[\partial\alpha] = 0$ ■

The idea of the Frölicher spectral sequence is that we can compute the cohomology of $d = \partial + \bar{\partial}$ in terms of the cohomology of ∂ and $\bar{\partial}$ themselves. However, for general complex manifolds, this is hard, since the differentials and cohomology beyond the E_1 page become much more complicated than the ones on the E_0 and E_1 pages, which are all objects we are familiar with, and are relatively easy to compute with. What Theorem 1.1 tells us is that in the case that X is compact Kähler, this is enough.