CHAPTER

1

Principal Bundles

Jeffrey Jiang
jeffjiang@utexas.edu

Why Fiber Bundles?

Suppose we want to study some space M, which for our purposes is a smooth manifold. One way to study M is to study functions $M \to F$ for some fixed target space F, e.g. a manifold , $\mathbb{R}$, or a vector space. This is a perfectly good method of studying M, but is sometimes not enough. More often then not, we want to study "function" from M into a vector space that varies over the base manifold M. For example, a vector field $X \in \mathfrak{X}(M)$ is not really a map $M \to \mathbb{R}^n$, it is an assignment to each $p \in M$ a vector in T_pM. In this way, we are led to the study of a smoothly parameterized family of vector spaces – the tangent bundle TM. This leads us to define a fiber bundle.

Definition 0.1. *Let M and F be a smooth manifolds. A* ***fiber bundle*** *over M with model fiber F is the data of a smooth manifold E with a smooth surjective map $\pi : E \to M$ such that for each $p \in M$, there exists an open set U and a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ such that*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "p_1"] \\ & U & \end{tikzcd}$$

where $p_1 : U \times F \to U$ is projection onto the first factor.