THE RISING SEA: CATEGORIES AND SHEAVES

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These are some notes + exercises I've compiled working through the first 2 chapters of Ravi Vikhil's *The Rising Sea*, with the main purpose being to gain some familiarity and comfort with categories and sheaves.

1. Category Theory

Definition 1.1. A ***category*** $\mathscr{C}$ is a collection[1] of ***objects***, denoted $\mathsf{Ob}(\mathscr{C})$ and a collection[2] ***morphisms*** $\mathsf{Hom}(A, B)$[3] for every pair of objects $A, B \in \mathsf{Ob}(\mathscr{C})$ satisfying the following axioms:

(1) Given morphisms $f : A \to B$ and $g : B \to C$, there is a unique map $g \circ f : A \to C$ that makes the following diagram commute

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

(2) For every object $A \in \mathsf{Ob}(\mathscr{C})$, there exists an ***identity morphism*** $\mathrm{id}_A \in \mathsf{Hom}(A, A)$ such that for any morphisms $f : A \to B$ and $g : C \to A$, we have that $\mathrm{id}_A \circ f = f$ and $g \circ \mathrm{id}_A = g$

A morphism $f : A \to B$ is an ***isomorphism*** is there exists a morphism $g : B \to A$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$. We then call g the ***inverse*** to f. Isomorphisms $A \to A$ are called ***automorphisms*** of A.

Example 1.2. The category of sets, often denoted Set has sets as its objects, and maps of sets as its morphisms.

Example 1.3. Vector spaces over a field $\mathbb{F}$ also form a category, denoted $\mathsf{Vec}_{\mathbb{F}}$, where the objects are $\mathbb{F}$-vector spaces, and the objects are $\mathbb{F}$-linear maps.

Exercise 1.4. Let A be an object of a category $\mathscr{C}$. Show that the automorphisms of $\mathsf{Hom}(A, A)$ form a group, called the ***Automorphism group*** of A. Show that two isomorphisc objects in $\mathscr{C}$ have isomorphic automorphism groups.

Proof. Verifying that $\mathrm{Aut}(A)$ is a group is mainly an exercise in definition chasing. Associativity comes from the category axioms, the identity element is the identity morphism, and inverses exist by the definition of the automorphism group.

For the second part, let $A, B \in \mathsf{Ob}(\mathscr{C})$ be isomorphic, with $f : A \to B$ an isomorphism. We then define a map $f^* : \mathrm{Aut}(B) \to \mathrm{Aut}(A)$ where $f^*\sigma = f^{-1} \circ \sigma \circ f$. This is clearly a group homomorphism, and is an isomorphism with inverse given by $(f^{-1})^*$. ■

Now that we have categories, the natural thing to study are maps of categories.

Definition 1.5. A (covariant)***functor*** $\mathcal{F}$ from a category $\mathscr{C}$ to another category $\mathscr{D}$ assigns each object $A \in \mathsf{Ob}(\mathscr{C})$ an object $\mathcal{F}(A) \in \mathsf{Ob}(\mathscr{D})$ and to each map $f : A \to B$ in $\mathscr{C}$ a map $\mathcal{F}(f) : \mathcal{F}(A) \to \mathcal{F}(B)$ such that $\mathcal{F}$ respects composition, i.e.

$$\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$$

a ***contravariant functor*** can be defined similarly, except it reverses the arrows, i.e. $\mathcal{F}(f)$ is now a map $B \to A$, rather than $A \to B$.

Example 1.6. For mathematical objects that are just sets with extra structure (e.g. vector spaces, groups, rings, etc), the ***forgetful functor*** is the functor that takes each object to its underlying set, and each map to itself (thought of as a map of sets). Categories that admit a forgetful functor into Set are called ***concrete categories***.

[1] Loosely speaking; there's some set-theoretic issues here, but it's not that important for us

[2] Again, ignoring set-theoretic problems

[3] Vikhil uses Mor, but we'll use the more standard notation of Hom

Example 1.7 (***The functor of points***)**.** For a category $\mathscr{C}$, fix an object $A \in \mathsf{Ob}(\mathscr{C})$. We use this to define the functor $\mathcal{F}_A : \mathscr{C} \to \mathsf{Set}$, where for $B \in \mathsf{Ob}(\mathscr{C})$, we let $\mathcal{F}(B) = \mathsf{Hom}(A, B)$ and for $f : B \to C$, we let $\mathcal{F}_A(f) : \mathsf{Hom}(A, B) \to \mathsf{Hom}(A, C)$ be the map where $\mathcal{F}_A(f)(g) = f \circ g$.

Like functions, we can compose functors $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ and $\mathcal{G} : \mathscr{B} \to \mathscr{C}$ to obtain $\mathcal{G} \circ \mathcal{F} : \mathscr{A} \to \mathscr{C}$, where for $A \in \mathsf{Ob}(\mathscr{A})$, we have that $\mathcal{G} \circ \mathcal{F}(A) = \mathcal{G}(\mathcal{F}(A))$, and the same thing for morphisms in $\mathscr{A}$. Also like functions, we have notions of injectivity and surjectivity.

Definition 1.8. A covariant functor $\mathcal{F} : \mathscr{C} \to \mathscr{D}$ is ***faithful*** if the induced map $\mathsf{Hom}_{\mathscr{C}}(A, B) \to \mathsf{Hom}_{\mathscr{D}}(\mathcal{F}(A), \mathcal{F}(B))$ is injective and ***full*** if it surjective.

Example 1.9. The forgetful functor $\mathsf{Vec}_k \to \mathsf{Set}$ is not full, since there are more maps of sets than there are maps of vector spaces, since an element of $\mathsf{Hom}_{\mathsf{Set}}(V, W)$ need not be linear. However, the "inclusion" functor $i : \mathsf{Ab} \to \mathsf{Grp}$ from abelian groups into groups is full. Given two abelian groups G and H, we clearly have that $\mathsf{Hom}_{\mathsf{Ab}}(G, H) = \mathsf{Hom}_{\mathsf{Grp}}(G, H)$, so we have that i is a faithful functor as well.

Functors in a lot of ways act just like functions, but there's some more things we can do with them. For example, it some sense, we can have maps between functors.

Definition 1.10. Given covariant functors $\mathcal{F}, \mathcal{G} : \mathscr{A} \to \mathscr{B}$, a ***natural transformation*** $\mathcal{F} \to \mathcal{G}$ assigns each object $A \in \mathsf{Ob}(\mathscr{A})$ a morphism $m_A : \mathcal{F}(A) \to \mathcal{G}(A)$, such that for every morphism $f : A \to B$ in $\mathscr{A}$, the following diagram commutes

$$\begin{tikzcd} \mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "m_A"'] & \mathcal{F}(B) \arrow[d, "m_B"] \\ \mathcal{G}(A) \arrow[r, "\mathcal{G}(f)"'] & \mathcal{G}(B) \end{tikzcd}$$

a ***natural isomorphism*** is when all the given morphisms m_A are isomorphisms.

Example 1.11. A trivial example of a natural transformation between $\mathcal{F} \to \mathcal{G}$ assigns the zero map $z : \mathcal{F}(A) \to \mathcal{G}(A)$ for every object A. Such a zero map only exists in some categories, such as Vec_k and Grp.

This gives a rigorous definition for the words "canonical" and "natural" that get thrown around a bit too often. The prototypical example for this idea is the natural isomorphism from a finite dimensional vector space to its double dual.

Exercise 1.12. Let $\mathcal{D} : \mathsf{Vec}_k \to \mathsf{Vec}_k$ be the functor that maps each k-vector space V to its dual space V^* of linear functions $V \to k$ and maps a linear map $A : V \to W$ to its pullback (sometimes called the transpose) $A^* : W^* \to V^*$. Show that the double dual functor $\mathcal{D} \circ \mathcal{D}$ is naturally isomorphic to the identity functor id.

Proof. For a vector v in a vector space V, define the map $\xi_v : V^* \to k$ by $\xi_v(\omega) = \omega(v)$. The let $\Xi_V : V \to V^{**}$ be the map that sends $v \mapsto \xi_v$. We claim that the morphisms Ξ_V define a natural isomorphism from $\mathcal{D} \circ \mathcal{D}$ to id. We first show that Ξ_V defines an isomorphism. Since V and V^{**} are the same dimension, it suffices to check that Ξ_V has trivial kernel. Suppose $v \mapsto \xi_v = 0$. Then $\omega(v) = 0$ for all $\omega \in V^*$, which is only true when $v = 0$. Therefore, Ξ_V is an isomorphism for every V. Showing that the Ξ_V define a natural isomorphism now amounts to showing that the following diagram commutes

$$\begin{tikzcd} V \arrow[r, "A"] \arrow[d, "\Xi_V"'] & W \arrow[d, "\Xi_W"] \\ V^{**} \arrow[r, "A^{**}"'] & W^{**} \end{tikzcd}$$

which amounts to showing that $\Xi_W \circ A = A^{**} \circ \Xi_V$. Let $v \in V$ and $\omega \in W^*$. Then we compute

$$\begin{aligned} (\Xi_W \circ A)(v)(\omega) &= \Xi_W(Av)(\omega) \\ &= \xi_{Av}(\omega) \\ &= \omega(Av) \end{aligned}$$

We also compute

$$\begin{aligned}(A^{**} \circ \Xi_V)(v)(\omega) &= A^{**}\xi_v(\omega) \\ &= \xi_v(A^*\omega) \\ &= A^*\omega(v) \\ &= \omega(Av)\end{aligned}$$

So we have given a natural isomorphism $\mathcal{D} \circ \mathcal{D} \to \mathrm{id}$ ■