DIFFERENTIAL GEOMETRY LECTURE SERIES

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These are notes for the spring 2019 differential geometry lecture series for the math club. The plan is for the course to give an introduction to math club members to the basics and terminology of smooth manifolds. A good reference if you want to read more is John Lee's *Introduction to Smooth Manifolds*. In terms of prerequisites, it would be good to have a background in point set topology, multivariable calculus, and linear algebra.

Contents

0. Preliminaries

Definition 0.1. A map $F : U \to V$ where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ is ***smooth*** if all of its component functions are infinitely differentiable, i.e. the partial derivatives of all orders exist and are continuous.

Note that smoothness is a local condition – to know a function is smooth at a point, it suffices to check in a small neighborhood.

1. Introduction to Manifolds

Definition 1.1. A ***topological manifold*** is a Hausdorff space X such there exists a countable open cover $\{U_\alpha\}$ of X, along with homeomorphisms $\varphi_\alpha : U_\alpha \to V_\alpha$, where V_α is an open subset of $\mathbb{R}^n$. Given a chart (U, φ), we can write φ in terms of its component functions

$$\varphi(p) = (x^1(p), \dots, x^n(p))$$

The functions x^i are called ***local coordinates*** on U.

In this way, we see that a manifold is a topological space that is locally topologically indistinguishable from Euclidean space. Both modifiers are important here – there could be global and geometric properties that differ from $\mathbb{R}^n$.

Example 1.2.

(1) $\mathbb{R}^n$ is a topological manifold – it admits a global chart $(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})$.
(2) The 2-sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ is a topological manifold. If we let N and S denote the North and South poles repsectively, then we can construct charts using ***stereographic projection***. Stereographic projection from the North pole is a map $\varphi_N : S^2 - \{N\} \to \mathbb{R}^n$, where given a point $p \in S^2 - \{N\}$, we take the line cotaining both N and p, which intersects the $z = 0$ plane at one point q. We then define $\varphi_N(p) = q$. Stereographic projection φ_S from the South pole is defined in an analogous manner. An explicit formula for φ_N is

$$\varphi_N(x, y, z) = \left(\frac{x}{1 - z}, \frac{x}{1 - z} \right)$$

Exercise 1.3.

(1) S^2 does not admit a global chart. Why?
(2) Give an inverse map $\mathbb{R}^n \to \mathbb{S}^n - N$ for stereographic projection from the North Pole
(3) Generalize stereographic projection to the n-sphere $S^n = \{p \in \mathbb{R}^{n+1} : \|p\| = 1\}$

One of the main appeals of Euclidean space is that we have a lot of tools at our disposal, like linear algebra and calculus. We would like to generalize these notions to manifolds. One of the wonderful properties of manifolds is that they locally look like $\mathbb{R}^n$, so we can use the charts to translate concepts we know in $\mathbb{R}^n$ to concepts on the manifold. We know what it means for a function on $\mathbb{R}^n$ to be smooth, how do we translate this to a manifold X? Given a map $F : X \to \mathbb{R}^m$, what does it mean for F to be smooth? Our first guess is to use our charts. Given a map $F : X \to \mathbb{R}^m$, we want to say that it is smooth if for every $p \in X$, there exists a chart (U_p, φ_p) such that the composition $F \circ \varphi_p^{-1} : U_p \to \mathbb{R}^m$ is smooth. The intuition here is correct, there there are some technicalities that need to be addressed. Namely, suppose $p \in X$ lies in the domain of two charts (U_1, φ_1), and (U_2, φ_2). Then what if $F \circ \varphi_1^{-1}$ was smooth, but $F \circ \varphi_2^{-1}$ wasn't? This suggests that we need a compatibility condition for our charts if we want a notion of smoothness. We say two charts (U_1, φ_1) and (U_2, φ_2) are ***smoothly compatible*** if both $\varphi_1 \circ \varphi_2^{-1}$ and $\varphi_2 \circ \varphi_1^{-1}$ are smooth maps.

Definition 1.4. A collection $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ of smoothly compatible charts such that the U_α cover X is called an ***atlas***. An atlas is ***maximal*** if it is not property contained in any other atlas.

Definition 1.5. A ***smooth manifold*** is a topological manifold X equipped with a maximal atlast $\mathcal{A}$.

Now if we have a smooth manifold X, we have a notion of smooth maps $X \to \mathbb{R}^n$

Definition 1.6. A map $F : X \to \mathbb{R}^k$ is ***smooth*** if for every chart $(U_\alpha, \varphi_\alpha)$, we have that $F \circ \varphi_\alpha^{-1}$ is smooth in the Euclidean sense. We call $F \circ \varphi_\alpha^{-1}$ a local ***coordinate representation*** of the map F.

We also know what if means for maps between manifolds to be smooth.

Definition 1.7. A map $F : X \to Y$ of smooth manifolds is ***smooth*** if for every p, there exists charts (U, φ) and (V, ψ) of p and $F(p)$ respectively, such that (after appropriate shrinking of U and V), we have that $\psi \circ F \circ \varphi^{-1}$ is smooth in the Euclidean sense.

For convenience, we will make the a distinction between the words map and function, which is also common in the literature. A *map* will denote an arbitrary mapping $X \to Y$, where X and Y can denote any kind of set. A *function* on a space X is a mapping $X \to \mathbb{R}$. We let $C^\infty(X)$ denote the vector space of functions of X, which forms a commutative ring under pointwise multiplication and addition.

This is nice, but we've excluded a huge class of spaces. For example, the unit interval isn't a manifold, since there exist no charts about the endpoints satisfying our definition (why?). To include the spaces, we need another definition. Let $\mathbb{H}^n$ denote ***Euclidean half space***, where

$$\mathbb{H}^n = \left\{(x^1, \dots x^n) \in \mathbb{R}^n \;:\; x^n \geq 0\right\}$$

Definition 1.8. A ***manifold with boundary*** is a Hausdorff space X such that there exists a countable cover by charts $(U_\alpha, \varphi_\alpha)$ where the φ_α are homeomorphisms to open sets in $\mathbb{H}^n$. Given a point $p \in X$, we say that p is in the ***boundary*** of X if there exists a chart (U, φ) containing p such that $\varphi(p) = (x^1, \dots, x^n)$, and $x^n = 0$. We say that p is in the ***interior*** of X if no such chart exists. We denote the boundary and interior of X as ∂X and $\operatorname{Int} X$ respectively.

Under this definition, $\mathbb{H}^n$ is a smooth manifold with boundary, with a global chart given by $(\mathbb{H}^n, \mathrm{id}_{\mathbb{H}^n})$, and $\partial\mathbb{H}^n = \left\{(x^1, \dots x^n) \in \mathbb{H}^n \;:\; x^n = 0\right\}$

Exercise 1.9. Let X be a smooth n-manifold with boundary

(1) Show that ∂X is well defined. Namely, show that if we have a point $p \in X$ such that there exists a chart (U, φ) where $\varphi_1(p) \in \partial\mathbb{H}^n$, show that any other chart (U', φ') for p must also have $\varphi'(p) \in \partial\mathbb{H}$. (Hint: Use the inverse function theorem to show there exists no diffeomorphism from an open set in $\mathbb{R}^n$ to an open set in $\mathbb{R}^{n-1}$)
(2) Prove that ∂X is a smooth $(n-1)$-manifold without boundary