REPRESENTATION THEORY AND TOPOLOGICAL QUANTUM FIELD THEORIES

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1. Bordism and TQFTs

Definition 1.1. Let X and Y be n-dimensional closed manifolds (i.e. compact and without boundary). A ***bordism*** from X to Y is an $n+1$ dimensional manifold M such that the boundary is diffeomorphic to the disjoint union $X \coprod Y$. The ***Bordism category*** Bord_n is the category where the objects are closed n-dimensional manifolds, and the morphisms are bordisms between them.

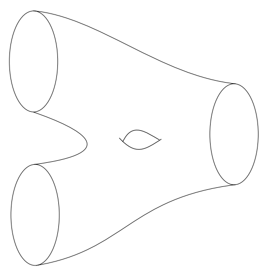

Figure 1. An example of a bordism $S^1 \coprod S^1 \to S^1$.

We make a distinction between the *incoming* and *outgoing* manifolds. In the bordism drawn above, we think of the two circles as the incoming manifolds, and single circle as the outgoing manifold. One way to think of a bordism is the time evolution of the incoming manifold to the outgoing one (think of the figure above extending outwards from the two circles and growing towards the single circle over some fixed amount of time).

The category of bordisms comes with a natural product – the disjoint union $\coprod$, where given two n-manifolds X and Y, we obtain a new n-manifold $X \coprod Y$. This operation is *symmetric*, i.e. there is a natural isomorphism $X \coprod Y \to Y \coprod X$, and gives the set of objects the structure of a commutative monoid. We say that $(\mathsf{Bord}_n, \coprod)$ is a ***symmetric monoidal category***. Another important example of a symmetric monoidal category is the category $\mathsf{Vect}_{\mathbb{F}}$ of finite dimensional $\mathbb{F}$-vector spaces, where the operation is the tensor product $\otimes$.

Definition 1.2. An $n+1$-dimensional ***topological quantum field theory (TQFT)*** is a symmetric monoidal functor $Z : (\mathsf{Bord}_n, \coprod) \to (\mathsf{Vect}_{\mathbb{C}}, \otimes)$, i.e. a functor $\mathsf{Bord}_n \to \mathsf{Vect}_{\mathbb{C}}$ satisfying:

(1) $Z(\varnothing) = \mathbb{C}$
(2) $Z(X \coprod Y) = Z(X) \otimes Z(Y)$

Note that the conditions on Z essentially state that it is a homomorphism of symmetric monoidal categories, hence the name. Note that the empty set is vacuously a manifold of *any* dimension. While this sounds silly at first, it ends up being very important, as it is the unit element under the disjoint union. In addition, we can interpret a closed manifold M as a bordism $\varnothing \to \varnothing$. Applying

Z to M, we get a linear map $\mathbb{C} \to \mathbb{C}$, which is just multiplication by a complex number λ. The number is called the ***partition function*** of M.

While the definition might seem somewhat abstract, it is very grounded in physical motivation. The incoming manifold of a bordism can be thought of as a space or system at an initial time, and the outgoing manifold can be thought of as the end state after undergoing the time evolution specified by the bordism. The functor Z then assigns to the initial and final states a state space. The fact that the state space of a disjoint union is the tensor product of the state spaces also matches the physical model.

2. Dijkgraaf-Witten Theory With Finite Gauge Group

We now define a specific TQFT, which is a toy model developed by Dijkgraaf and Witten. Fix a finite group G. For a fixed manifold M, let $\mathsf{Bun}_G(M)$ denote the category of principal G-bundles over M. Any morphism $P \to Q$ of principal G-bundles is an isomorphism, so this category is a ***groupoid***. For any given groupoid $\mathcal{G}$, we let $\pi_0(\mathcal{G})$ denote the set of isomorphism classes of objects in $\mathcal{G}$. Given a basepoint $x \in M$, we let $\mathsf{Bun}_G(M, x)$ denote the category of pointed G-bundles over M, which are pairs (P, p) where $P \to M$ is a principal G-bundle, and $p \in P$ is an element of the fiber P_x over x.

Proposition 2.1. *There is a bijective correspondence*

$$\mathsf{Hom}(\pi_1(M, x), G) \longleftrightarrow \mathsf{Bun}_G(M, x)$$

Proof. Given a pointed bundle $(P, p) \to (M, x)$, we obtain a map $\varphi : \pi_1(M, x) \to G$ as follows: For a homotopy class of a loop $\sigma : I \to M$ based at x, we can lift σ to a path $\tilde{\sigma}$ on P starting at p. Then the endpoint $\tilde{\sigma}(1)$ is an element of the fiber P_x, so it can be uniquely written as $p \cdot g$ for some $g \in G$. Then defining $\varphi(\sigma) = g$ gives the desired homomorphism (though one should check that this construction is well defined on homotopy classes) called the ***holonomy*** of the bundle. This map is also called the ***monodromy***. In the other direction, given a homomorphism $\varphi : \pi_1(M, x) \to G$, we want to construct a pointed bundle $(P, p) \to (M, x)$ with holonomy φ. The manifold M admits a universal cover $(\widetilde{M}, \tilde{x})$, which is a principal $\pi_1(M, x)$-bundle over M. The homomorphism φ induces a left action of $\pi_1(M, x)$ on G, so we construct the associated bundle

$$P = \widetilde{M} \times_{\pi_1(M,x)} G$$

where we choose the basepoint of P to be $p = [\tilde{x}, e]$. It is easy to verify that the holonomy of this bundle is φ, and it is also easy to verify that the holonomy of this bundle is indeed φ, giving us the desired bijection. ∎

The group G acts on the category $\mathsf{Bun}_G(M, x)$ by permuting basepoints. Given a pointed bundle $(P, p) \to (M, x)$ and a group element $g \in G$, the action of g on (P, p) is the bundle $(P, p \cdot g)$. In addition, g acts on maps $(P, p) \to (Q, q)$ by precomposition. Taking the quotient by this group action, we obtain the groupoid $\mathsf{Bun}_G(M)$, since quotienting by the group action amounts to forgetting the basepoint. It is also useful to see the group action from the perspective of homomorphisms $\varphi : \pi_1(M, x) \to G$, which we can determine by studying how the holonomy of a bundle changes when we permute its basepoint. Let φ be the holonomy of a bundle $(P, p) \to (M, x)$, and let $\gamma : I \to M$ be a loop based at x. Then let $\widetilde{\gamma}$ denote its lift to a loop based at p. By the uniqueness of path lifting, the lift of γ to the a loop based at $p \cdot g$ is the loop $\widetilde{\gamma}_g(t) = \widetilde{\gamma(t)} \cdot g$. We the compute

$$\begin{aligned}
\widetilde{\gamma}_g(1) &= \widetilde{\gamma}(1) \cdot g \\
&= p \cdot \varphi[\gamma] \cdot g \\
&= p \cdot g \cdot g^{-1} \cdot \varphi[\gamma] \cdot g
\end{aligned}$$

So the holonomy φ transforms under the group action by conjugation, i.e. the holonomy φ_g of $(P, p \cdot g) \to (M, x)$ is $g^{-1} \cdot \varphi \cdot g$.

We now have the necessary tools to define the TQFT. Given a closed n-manifold M, define $Z(M) = \mathbb{C}_G(M)$, where $\mathbb{C}_G(M)$ denotes the vector space generated by complex valued functions on $\pi_0(\mathsf{Bun}_G(M))$. Then given a bordism X between manifolds M, and N, we need to produce a map $Z(X) : \mathbb{C}_G(M) \to \mathbb{C}_G(N)$. We do so as follows. We have inclusions $M \hookrightarrow X$ and $N \hookrightarrow X$ of M and N into ∂X, giving us the diagram

$$\begin{tikzcd} & X & \\ M \arrow[ur] & & N \arrow[ul] \end{tikzcd}$$

the inclusion maps induce maps $\mathsf{Bun}_G(X) \to \mathsf{Bun}_G(M)$ and $\mathsf{Bun}_G(X) \to \mathsf{Bun}(Y)$ by pullback (also called restriction), where we map a bundle $P \to X$ to the pullback bundle $P|_M$ and $P|_N$ respectively, where we identify M and N with the images under the inclusion into X, giving the diagram

$$\begin{tikzcd} & \mathsf{Bun}_G(X) \arrow[dl] \arrow[dr] & \\ \mathsf{Bun}_G(M) & & \mathsf{Bun}_G(N) \end{tikzcd}$$

Then given a function $f \in \mathbb{C}_G(M)$ on principal bundles over M, we need to produce a function $Z(X)(f) \in \mathbb{C}_G(N)$. To do so, we need to define one more object. Let $Q \to N$ be a principal bundle over N. We then construct a groupoid $\mathcal{G}_Q$, where the objects are pairs (P, φ), where $P \to X$ is a principal bundle, and φ is an isomorphism $P|_N \to Q$, and the morphisms from $(P, \varphi) \to (P', \varphi')$ are bundle morphisms $\psi : P \to P'$ such that the diagram

$$\begin{tikzcd} P|_N \arrow[rr, "\psi"] \arrow[dr, "\varphi"'] & & P'|_N \arrow[dl, "\varphi'"] \\ & Q & \end{tikzcd}$$

commutes, where we pullback the map ψ using the inclusion map $N \hookrightarrow X$. We refer to $\mathcal{G}_Q$ as the ***groupoid preimage*** of Q. We now construct the linear map $Z(X) : \mathbb{C}_G(M) \to \mathbb{C}_G(N)$. Let $f \in \mathbb{C}_G(M)$ by defining $Z(X)(f) \in \mathbb{C}_G(N)$ by

$$Z(X)(f)(Q) = \sum_{(P,\varphi) \in \pi_0(\mathcal{G}_Q)} \frac{f(P|_M)}{|\operatorname{Aut}(P, \varphi)|}$$

While a bit opaque, you can think of this as a sort of categorified Fourier transform. Given a function over M, we pull it back to a function over X, convolve with some sort of kernel (where we sum instead of integrate because everything is finite), and then push down to N. Such push-pull constructions arise often as a sort of generalized Fourier transform.

3. Dijkgraaf-Witten Theory in Two Dimensions

We now restrict our attention to the 2-dimensional version of the TQFT Z we constructed above. First, we recall some of the details of the classification of $2d$-TQFTs, which are in bijection with ***commutative Frobenius algebras***. It is well known that the only connected closed 1-manifold is the circle S^1, up to diffeomorphism. Therefore, all the objects in the category Bord_n (again, up to diffeomorphism) are just disjoint unions of a finite number of circles. Using the fact that $Z(M \coprod N) = Z(M) \otimes Z(N)$, this means we can interpret the morphisms in Bord_n as operations on $Z(S^1)$, which will form the underlying vector space of our commutative Frobenius algebra. To classify the algebra, it suffice to know the following list of bordisms and the operations they induce on $Z(S^1)$

(1) The ***pair of pants*** P is a bordism $S^1 \coprod S^1 \to S^1$. Applying the functor Z, we get a ***multiplication*** $Z(P) : Z(S^1) \otimes Z(S^1) \to Z(S^1)$.

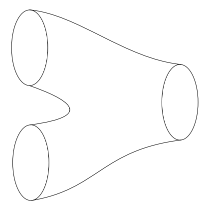

FIGURE 2. A pair of pants

(2) We can flip the pair of pants around to get a bordism $S^1 \to S^1 \coprod S^1$, which then induces a map $Z(S^1) \to Z(S^1) \otimes Z(S^1)$ called ***comultiplication***.

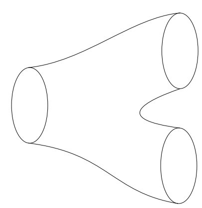

FIGURE 3. The pair of pants, flipped around

(3) The ***cap*** C is a bordism $\varnothing \to S^1$. Applying the functor Z, we get a linear map $\mathbb{C} \to Z(S^1)$. The image of 1 is the identity element of $Z(S^1)$ under the multiplication defined by the pair of pants.

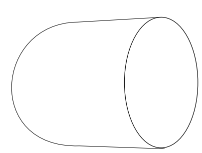

FIGURE 4. A cap

(4) By flipping the cap around, we get a bordism $S^1 \to \varnothing$, which determines a map $Z(S^1) \to \mathbb{C}$ called the ***trace***.

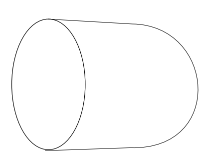

FIGURE 5. A cocap

The operations induced by the bordisms above are essentially the defining ingredients for a Frobenius algebra (along with some relations they satisfy, which we will not prove).

We now evaluate our TQFT on the circle. From before, we saw that pointed principal G-bundles are in bijection with group homomorphisms $\pi_1(S^1, x) \to G$. Since $\pi_1(S^1, x) \cong \mathbb{Z}$, this is entirely determined by the image of 1, so we get that the objects of $\mathsf{Bun}_G(S^1, x)$ is in bijection with the elements of G. Since we are considering functions from isomorphisms classes of principal bundles *without basepoints*, we have that the objects of $\mathsf{Bun}_G(S^1)$ are in bijection with *conjugacy classes* of G. Therefore, the vector space $Z(S^1)$ assigned to the circle by our TQFT is the complex vector space $\mathbb{C}[G]^G$ of class functions on G, which is the underlying vector space of our commutative Frobenius algebra. We then determine the linear maps determined by the bordisms specified above.

We first do the cap C , which determines a linear functional $Z(C) : Z(S^1) \to \mathbb{C}$. Let $f \in Z(S^1)$. To compute this map, we make two observations.

(1) Topologically, C is homeomorphic to the 2-disk D^2, which is contractible. Therefor, the only principal G-bundle over C is the trivial bundle $C \times G$.
(2) The only principal G-bundle over the empty set is the empty set, which is vacuously a principal G-bundle.

We the see that the groupoid preimage of $\varnothing$ is just the groupoid $\mathsf{Bun}_G(C)$, which consists of a single object representing $C \times G$, with $|G|$ automorphisms corresponding to multiplication by the group elements. Restriction of this trivial bundle to the boundary S^1 is then the trivial bundle $S^1 \times G$, which corresponds to the map $\mathbb{Z} \to G$ where $1 \mapsto e$. Therefore, evaluating the function f on the trivial bundle is equivalent to evaluating its corresponding class function(which we also denote f) on the identity element e. Normalizing by the order of the group then gives us

$$Z(C)(f) = \frac{f(e)}{|G|}$$

We now compute the map determined by the cocap, which is a map $\mathbb{C} \to \mathbb{C}[G]^G$. Fix a number $\alpha \in \mathbb{C}$, and a principal bundle Q over S^1. Again, the cocap is homeomorphic to D^2, which has a single principal bundle up to isomorphism with $|G|$ automorphisms. Then the groupoid preimage $\mathcal{G}_Q$ is the empty groupoid unless Q is isomorphic to the trivial bundle $S^1 \times G$. Therefore, we have that the map associated to the cocap is given by $\alpha \mapsto f_\alpha$ where

$$f_\alpha(Q) = \begin{cases} \frac{\alpha}{|G|} & Q \cong S^1 \times G \\ 0 & \text{otherwise} \end{cases}$$

which corresponds to the class function that is $\alpha/|G|$ on e and 0 on all other conjugacy classes.

We now determine the multiplication induced by the pair of pants M. Topologically, the pair of pants is homeomorphic to a a twice punctured disk. Using the identification of principal bundles with their holonomy, this picture proves to be quite useful. The fundamental group $\pi_1(M, x)$ is isomorphic to the free group on 2 elements, so a principal bundle over M is equivalent data to a choice of two group elements of G, up to simultaneous conjugation. These correspond to the holonomies by traveling once around each of the inner circles, and the holonomy traveling around the outer circle is the product of the holonomies around the two inner circles We then compute the linear map determined by M, thought of as a bordism $S^1 \coprod S^1 \to S^1$. In our picture, the outer boundary of this disk corresponds to the single S^1 boundary component, and the two inner circles correspond to the $S^1 \coprod S^1$ boundary component. Let Q be a principal bundle over S^1, with holonomy k. Then the groupoid preimage $\mathcal{G}_Q$ corresponds to pairs of elements (g, h) such

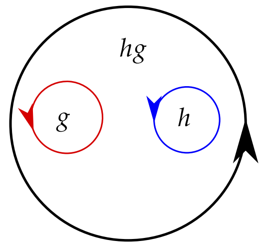

FIGURE 6. Holonomies around the pair of pants.

that $gh = k$, up to simultaneous conjugation. Then given functions $f_1, f_2 \in Z(S^1)$, evaluating these functions on this bundle corresponds to evaluating the corresponding class functions of f_1 on g and f_2 on h, and taking their product. Putting it all together, we get that the multiplication map defined the the pair of pants $Z(M) : \mathbb{C}[G]^G \otimes \mathbb{C}[G]^G \to \mathbb{C}[G]$ maps $f_1 \otimes f_2 \mapsto f_1 * f_2$, where

$$(f_1 * f_2)(k) = \sum_{gh=k} \frac{f_1(g) f_2(h)}{|\operatorname{Aut}(g,h)|}$$

where $\operatorname{Aut}(g,h)$ denotes the automorphism group of (g,h) in $\mathcal{G}_Q$, which correspond to pairs (g,h) (up to simultaneous conjugation) whose product gh is equal to k. The stabilizer of (g,h) under the conjugation action $\ell \cdot (g,h) = (\ell g \ell^{-1}, \ell h \ell^{-1})$ is the intersection $C(g) \cap C(h)$ of the centralizers of g and h. The automorphism group of k the centralizer $C(k)$, so we get that

$$|\operatorname{Aut}(g,h)| = \frac{|C(g) \cap C(h)|}{|C(k)|}$$

which then gives us that the multiplication is

$$(f_1 * f_2)(k) = \sum_{gh=k} \frac{|C(k)|}{|C(g) \cap C(h)|} f_1(g) f_2(h)$$

There is a nice interpretation of this multiplication.

Proposition 3.1. *There is an algebra isomorphism of $Z(S^1)$ with the center of the group algebra $\mathbb{C}[G]$ given by the mapping*

$$f \mapsto \sum_{g \in G} f(g) g$$

To show this, we first use a lemma.

Lemma 3.2. *Let $C_1, \dots, C_n$ denote the conjugacy classes of G. Then the elements*

$$b_i = \sum_{g \in C_i} g$$

form a basis for $Z(\mathbb{C}[G])$.

Proof. Since the conjugacy classes are disjoint, it is clear that the b_i are linearly independent, so it suffices to show that they span. Let $\alpha \in Z(\mathbb{C}[G])$, which can be written as

$$\alpha = \sum_{g \in G} \alpha_g g$$

for $\alpha_g \in \mathbb{C}$. In addition, $\alpha \in Z(\mathbb{C}[G])$ is equivalent to $h\alpha = \alpha h$ for any $h \in G$, which is then equivalent to $h\alpha h^{-1} = \alpha$ Expanding this out, we get that this condition is equivalent to

$$\sum_{g \in G} \alpha_g h g h^{-1} = \sum_{g \in G} \alpha_g g$$

We can then substitute $u = hgh^{-1}$, which gives us that $g = h^{-1}uh$, and the summation on the left hand side becomes

$$\sum_{u \in G} \alpha_{h^{-1}uh} u$$

renaming u back to g, we have that the condition that $\alpha \in Z(\mathbb{C}[G])$ is given by the equation

$$\sum_{g} \alpha_{h^{-1}gh} g = \sum_{g} \alpha_g g$$

which is true if and only if we have the coefficients $\alpha_{h^{-1}gh}$ and α_g are equal. It is clear that the elements b_i we constructed satisfy the condition for their coefficients, so all the b_i lie in $Z(\mathbb{C}[G])$, in addition, they span, since if $\sum_{g \in G} a_g g \in Z(\mathbb{C}[G])$, the coefficients are determined by the conjugacy classes of the elements, so we can just take the appropriate linear combinations of the b_i. ∎

Proof of Proposition 3.1*.* We know that a basis $\mathbb{C}[G]^G$ is given by the indicator functions χ_i for the conjugacy classes C_i. Under the mapping $f \mapsto \sum_{g \in G} f(g)g$, we have that $\chi_i \mapsto b_i$, where b_i is the same as we defined above, so this is a linear isomorphism. We then check that this mapping is an algebra homomorphism (and therefore, an algebra isomorphism). It suffices to verify this on a basis, which we choose to be the indicator functions χ_i. Given indicators χ_i and χ_j for the conjugacy classes C_i and C_j. Under the mapping $\mathbb{C}[G]^G \to Z(\mathbb{C}[G])$, the χ_i are mapped to the b_i, and we compute

$$\begin{aligned} b_i b_j &= \left(\sum_{g \in C_i} g \right) \left(\sum_{h \in C_j} h \right) \\ &= \sum_{g} \lambda_g g \end{aligned}$$

where λ_g is the cardinality of the set

$$\Lambda_g = \left\{ (h, k) \in C_i \times C_j \ : \ hk = g \right\}$$

To determine the value of the λ_g, we note that the centralizer $C(g)$ of g acts on Λ_g by simultaneous conjugation, since for $\ell \in C(g)$ and $(h, k) \in \Lambda_g$, we have that

$$(\ell h \ell^{-1})(\ell k \ell^{-1}) = \ell hk \ell^{-1} = g$$

In addition, it is clear that the stabilizer subgroup of an element $(h, k) \in \Lambda_g$ is the intersection $C(h) \cap C(k) \subset C(g)$. ∎