Thesis

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CHAPTER 1

Clifford Algebras and Spin Groups

1. Clifford Algebras

Definition 1.1. Let V be a real finite dimensional vector space with a nondegenerate symmetric bilinear form $b : V \times V \to \mathbb{R}$. Then the ***Clifford Algebra*** of V is the data of a unital associative $\mathbb{R}$-algebra $\mathrm{Cliff}(V, b)$ and a linear map $i : V \to \mathrm{Cliff}(V, b)$ satisfying the following universal property: Given any linear map $\varphi : V \to A$ of V into any unital associative $\mathbb{R}$-algebra A satisfying $\varphi(v)^2 = b(v, v)$, there exists a unique map $\tilde{\varphi} : \mathrm{Cliff}(V, b) \to A$ such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[d, "i"'] \arrow[dr, "\varphi"] & \\ \mathrm{Cliff}(V,b) \arrow[r, "\tilde{\varphi}"'] & A \end{tikzcd}$$

Explicitly, we can construct $\mathrm{Cliff}(V, b)$ as a quotient of the tensor algebra

$$\mathcal{T}(V) = \bigoplus_{n \in \mathbb{Z}^{\geq 0}} V^{\otimes n}$$

where we quotien tby the left ideal generated by elements of the form $v \otimes v - b(v, v)$, and the linear map $i : V \to \mathcal{T}(V)/(v \otimes v - b(v, v))$ is given by the inclusion $V \hookrightarrow \mathcal{T}(V)$ followed by the quotient map. We identify V with its image $i(V)$ as a subspace of $\mathrm{Cliff}(V, b)$.

Definition 1.2. Define the bilinear form $b : \mathbb{R}^{p+q} \times \mathbb{R}^{p+q} \to \mathbb{R}$ by

$$b(v, w) = \sum_{i=1}^{p} v^i w^i - \sum_{i=p+1}^{p+1} v^i w^i$$

where the v^i and w^i are the components of v and w with respect to the standard basis on $\mathbb{R}^{p+q}$. We denote the vector space $\mathbb{R}^{p+q}$ equipped with this bilinear form as $\mathbb{R}^{p,q}$.

Let V be a vector space equipped with a nondegenerate bilinear form b and fix a basis for V. Then the bilinear form b is given by a symmetric invertible matrix B, which is conjugate to a diagonal matrix where all the diagonal entries are either 1 or -1. If after conjugation B has p positive entries and q negative entries, we say b has signature (p, q). Any bilinear form b of signature (p, q) admits a basis $\{e_i\}$ satisfying

(1) For $1 \leq i \leq p$, we have $b(e_i, e_i) = 1$
(2) For $p + 1 \leq j \leq p + q$, we have $b(e_j, e_j) = -1$
(3) For $i \neq j$, we have $b(e_i, e_j) = 0$

Any such basis then determines an isomorphism $(V, b) \to \mathbb{R}^{p,q}$. In addition, we get a basis for $\mathrm{Cliff}(V, b)$, given by

$$\left\{ e_{i_1} e_{i_2} \cdots e_{i_k} \;:\; 0 \leq k \leq n,\ 1 \leq i_j \leq n \right\}$$

where we interpret the product of 0 basis vectors to be the unit element 1. This then implies the dimension of $\mathrm{Cliff}(V, b)$ as a vector space is $2^{\dim V}$. This basis also determines an isomorphism $\mathrm{Cliff}(V, b) \to \mathrm{Cliff}_{p,q}(\mathbb{R})$, where $\mathrm{Cliff}_{p,q}(\mathbb{R})$ is the Clifford algebra for $\mathbb{R}^{p,q}$. Given $v, w \in V$, we write v and w in these bases as $v^i e_i$ and $w^i e_i$ (using Einstein summation convention), and derive the useful relation

$$\begin{aligned} vw + wv &= v^i w^j e_i e_j + v^i w^j e_j e_i \\ &= v^i w^i (e_i^2 + e_i^2) \\ &= 2b(v, w) \end{aligned}$$

where we use the fact that the e_i are orthgonal to deduce that $e_i e_j = -e_j e_i$ if $i \neq j$.

Definition 1.3. [1] Let V be a vector space. A $\mathbb{Z}/2\mathbb{Z}$ ***grading*** on V is a decomposition $V = V^0 \oplus V^1$. Equivalently, it is the data of a linear map $\varepsilon : V \to V$ such that ε acts by identity on a subspace V^0 of V and negative identity on a complementary subspace V^1, which gives us the desired direct sum decomposition of V as the ± 1 eigenspaces of ε. The map ε is called the ***grading operator***. Elements of V^0 are said to be ***even*** and elements of V^1 are said to be ***odd***. Elements of the even and odd subspaces are said to be ***homogeneous***. Given an homogeneous element $v \in V$, defines it's ***parity***, denoted $|v|$ by

$$|v| = \begin{cases} 0 & v \in V^0 \\ 1 & v \in V^1 \end{cases}$$

Definition 1.4. A $\mathbb{Z}/2\mathbb{Z}$ ***graded algebra*** A over $\mathbb{R}$ (often called a superalgebra) is an $\mathbb{R}$-algebra A equipped with a grading $A = A^0 \oplus A^1$ such that the multiplication respects the grading, i.e. given elements $a, b \in A$, Their product ab is an element of $A^{|a|+|b|}$ where the addition is done mod 2.

Example 1.5.

(1) Any $\mathbb{R}$-algebra A can be made into a graded algebra where we let $A^0 = A$ and $A^1 = 0$.
(2) The exterior algebra $\Lambda^\bullet V$ of a vector space V is a $\mathbb{Z}/2\mathbb{Z}$ graded algebra (in fact, it is $\mathbb{Z}$ graded), where the grading is $\Lambda^\bullet V = \Lambda^0 V \oplus \Lambda^1 V$ where $\Lambda^0 V$ is the subspace spanned by even products of vectors and $\Lambda^1 V$ is the subspace spanned by odd products of vectors.

For the most part, the algebras we work with will be $\mathbb{Z}/2\mathbb{Z}$ graded, so the term "graded" may be used in lieu of "$\mathbb{Z}/2\mathbb{Z}$ graded." In the case of ambiguity, we will specify the grading.
The Clifford algebra $\mathrm{Cliff}(V, b)$ is naturally a $\mathbb{Z}/2\mathbb{Z}$ graded algebra. Fix a basis $\{e_i\}$ for $\mathrm{Cliff}(V, b)$. We then define a grading

$$\mathrm{Cliff}(V, b) = \mathrm{Cliff}^0(V, b) \oplus \mathrm{Cliff}^1(V, b)$$

Where $\mathrm{Cliff}^0(V, b)$ is the $\mathbb{R}$-span of all even products of basis vectors, and $\mathrm{Cliff}^1(V, b)$ is the $\mathbb{R}$-span of all odd products of basis vectors. In particular, $\mathrm{Cliff}^0(V, b)$ forms a subalgebra, and is called the ***even subalgebra***. There is an extremely nice relationship between a Clifford algebra and it's even subalgebra.

Theorem 1.6. *The even subalgebra* $\mathrm{Cliff}^0_{p,q}(\mathbb{R})$ *is isomorphic to both* $\mathrm{Cliff}_{q,p-1}$ *and* $\mathrm{Cliff}_{p,q-1}$ *as ungraded algebras (as long as* $p - 1 > 0$ *or* $q - 1 > 0$.*)*

Proof. Fix a basis $\left\{e_1^+, \dots e_p^+, e_1^- \dots e_q^-\right\}$ for $\mathbb{R}^{p,q}$, where $(e_i^+)^2 = 1$ and $(e_i^-)^2 = -1$. Then a quick computation shows that

$$\begin{aligned} (e_i^+ e_j^+)^2 &= -(e_i^+)^2 (e_j^+)^2 = -1 \\ (e_i^- e_j^-)^2 &= -(e_i^-)^2 (e_j^-)^2 = -1 \\ (e_i^+ e_j^-)^2 &= -(e_i^+)^2 (e_i^-)^2 = 1 \\ (e_i^- e_j^+)^2 &= -(e_i^-)^2 (e_j^+)^2 = 1 \end{aligned}$$

Assume $q \neq 0$. Then a generating set for $\mathrm{Cliff}^0_{p,q}(\mathbb{R})$ is

$$\left\{e_1^- e_j^+ \ : \ 1 \leq j \leq p\right\} \cup \left\{e_1^- e_k^- \ : \ 2 \leq k \leq q\right\}$$

[1] It is common in the literature to refer to $\mathbb{Z}/2\mathbb{Z}$ graded vector spaces as super vector spaces. The "super" prefix often refers to a $\mathbb{Z}/2\mathbb{Z}$ grading on the relevant object.

All the elements in the first set square to 1, and all the elements in the second set square to -1. We then get an isomorphism $\mathrm{Cliff}^0_{p,q}(\mathbb{R}) \to \mathrm{Cliff}_{p,q-1}$ via the mappings

$$e_1^- e_j^+ \mapsto e_j^+$$
$$e_1^- e_k^- \mapsto e_{k-1}^-$$

In the case where $p \neq 0$, we have that an equally good generating set for $\mathrm{Cliff}^0_{p,q}(\mathbb{R})$ is

$$\left\{e_1^+ e_j^+ \ : \ 2 \leq j \leq p\right\} \cup \left\{e_1^+ e_i^- \ : \ 1 \leq i \leq q\right\}$$

Where the elements in the first set square to -1 and the elements of the second set square to 1. Then the mappings

$$e_1^+ e_j^+ \mapsto e_{j-1}^-$$
$$e_1^+ e_j^- \mapsto e_j^+$$

gives the isomorphism $\mathrm{Cliff}^0_{p,q}(\mathbb{R}) \to \mathrm{Cliff}_{q,p-1}$. ■

Given two $\mathbb{R}$-algebras A and B, we can form their tensor product $A \otimes B$, which has $A \otimes B$ as the underlying vector space, and the multiplication is defined as

$$(a \otimes b)(c \otimes d) = ac \otimes bd$$

In the case that both A and B are $\mathbb{Z}/2\mathbb{Z}$ graded algebras, we have an alternate version of the tensor product, where the underlying vector space is also $A \otimes B$, but the multiplication is given by

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|}(ac \otimes bd)$$

We see that in the multiplication, we are formally commuting the elements of b and c, and we want to introduce a sign whenever elements are moved past each other. This is the ***Koszul sign rule***. Another concept that needs a slight modification in the graded case is the opposite algebra. In the normal case, given an $\mathbb{R}$-algebra A, the ***opposite algebra*** is the algebra A^{op} with the same underlying vector space, but the multiplication in A^{op} is given by $a * b = ba$, where ba is the multiplication in A. In doing so, we are formally commuting a and b, so in the graded situation, we invoke the Koszul sign rule when defining the opposite algebra, and define the multiplication in A^{op} to be $a * b = (-1)^{|a||b|}ba$.

One remarkable fact is that Clifford algebras are closed under the graded tensor product, i.e. the graded tensor products of two Clifford algebras is another Clifford algebra. Likewise, the graded opposite algebra of a Clifford algebra is again a Clifford algebra. For the remainder of this section, we will let $\otimes$ denote the graded tensor product, and the superscript op will denote the graded opposite algebra.

Theorem 1.7. $\mathrm{Cliff}_{p+t,q+s}(\mathbb{R}) \cong \mathrm{Cliff}_{p,q}(\mathbb{R}) \otimes \mathrm{Cliff}_{t,s}(\mathbb{R})$

Proof. To give a map $\varphi : \mathrm{Cliff}_{p+t,q+s}(\mathbb{R}) \to \mathrm{Cliff}_{p,q}(\mathbb{R}) \otimes \mathrm{Cliff}_{t,s}(\mathbb{R})$, it is sufficient to specify its action on $\mathbb{R}^{p+t,q+s}$, to check that the Clifford relations hold. Let $\left\{b_1^+, \dots, b_{p+t}^+, b_1^-, \dots b_{q+s}^-\right\}$ denote the standard orthogonal basis for $\mathbb{R}^{p+t|q+s}$ where $(b_i^+)^2 = 1$ and $(b_i^-)^2 = -1$. We then define the bases $\{e_i^\pm\}$ and $\{f_i^\pm\}$ analogously for $\mathbb{R}^{p,q}$ and $\mathbb{R}^{t,s}$ respectively. Then define φ by

$$\varphi(b_i^+) = \begin{cases} e_i^+ \otimes 1 & 1 \leq i \leq p \\ 1 \otimes f_i^+ & p+1 \leq i \leq p+t \end{cases}$$
$$\varphi(b_i^-) = \begin{cases} e_i^- \otimes 1 & 1 \leq i \leq q \\ 1 \otimes f_i^- & q+1 \leq i \leq q+s \end{cases}$$

This map is injective on generators, so if we show that this satisfies the Clifford relations, then the map given by extending the map to all of $\mathrm{Cliff}_{p+t,q+s}(\mathbb{R})$ will be an isomorphism by dimension reasons. Showing the Clifford relations amounts to showing

(1) $\varphi(b_i^+)^2 = 1$
(2) $\varphi(b_i^-)^2 = -1$

(3) The images of any pair of distinct basis vectors anticommute.

The first two are relations are clear from how we defined φ. To show that the images of distinct basis vectors anticommute, there are serveral cases to consider. Given b_i^+ and b_j^+ where $1 \leq i, j \leq p$, they anticommute, because e_i^+ and e_j^+ anticommute. In the case where $1 \leq i \leq p$ and $p+1 \leq j \leq p+t$, we compute

$$\begin{aligned}\varphi(b_i^+)\varphi(b_j^+) + \varphi(b_j^+)\varphi(b_i+) &= (e_i^+ \otimes 1)(1 \otimes f_j^+) + (1 \otimes f_j^+)(e_i^+ \otimes 1) \\ &= e_i^+ \otimes f_j^+ - e_i^+ \otimes f_j^+\end{aligned}$$

where we use the Koszul sign rule for the second term, noting that f_j+ and e_i^+ are both odd. The proof that the images of the b_i^- anti commute with each other, as well as the proof that the images of the b_i^+ and b_i^- anticommute are exactly the same. ■

Theorem 1.8. *The graded opposite algebra* $\mathrm{Cliff}_{p,q}^{op}$ *is isomorphic to* $\mathrm{Cliff}_{q,p}$.

Proof. Fix an orthgonal basis $\{e_i^{\pm}\}$ for $\mathbb{R}^{p,q}$, where $(e_i^{\pm})^2 = \pm 1$. We note that since the $e_i^{\pm}$ are odd elements, they square to $e_i^{\mp}$ in the opposite algebra. Indeed, the mapping $e_i^{\pm} \to e_i^{\mp}$ defines the isomorphism $\mathrm{Cliff}_{p,q}^{\mathrm{op}} \to \mathrm{Cliff}_{q,p}$. ■

Because of these theorems, once we compute a few of the lower dimensional Clifford algebras, we will have enough data to fully classify all Clifford algebras over $\mathbb{R}$.

Example 1.9 (***Some low dimensional examples***).

(1) The Clifford algebra $\mathrm{Cliff}_{0,0}(\mathbb{R})$ is isomorphic to $\mathbb{R}$.
(2) As ungraded algebras, the Clifford algebra $\mathrm{Cliff}_{0,1}(\mathbb{R})$ is isomorphic to $\mathbb{C}$, where the isomorphism is given by $e_1 \mapsto i$.
(3) As ungraded algebras, $\mathrm{Cliff}_{0,2}(\mathbb{R})$ is isomorphic to the quaternions $\mathbb{H}$, where the isomorphism is given by $e_1 \mapsto i$ and $e_2 \mapsto j$.
(4) As graded algebras, $\mathrm{Cliff}_{1,1}(\mathbb{R})$ is isomorphic to $\mathrm{End}(\mathbb{R}^{1|1})$. The isomorphism is given by

$$e_1^+ \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad e_1^- \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(5) As ungraded algebras $\mathrm{Cliff}_{1,0}(\mathbb{R})$ is isomorphic to the product algebra $\mathbb{R} \times \mathbb{R}$, where $e_1 \mapsto (1, -1)$.
(6) As ungraded algebras, $\mathrm{Cliff}_{2,0}(\mathbb{R})$ is isomorphic to the algebra $M_2\mathbb{R}$ of 2×2 matrices with coefficients in $\mathbb{R}$. The isomorphism is given by

$$e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad e_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

To classify all Clifford algebras as ungraded algebras, it suffices to know the following table:

$M_8\mathbb{C}$	$M_8\mathbb{H}$	$M_8\mathbb{H} \times M_8\mathbb{H}$	$M_{16}\mathbb{H}$	$M_{32}\mathbb{C}$	$M_{64}\mathbb{R}$	$M_{64}\mathbb{R} \times M_{64}\mathbb{R}$	$M_{128}\mathbb{R}$
$M_4\mathbb{H}$	$M_4\mathbb{H} \times M_4\mathbb{H}$	$M_8\mathbb{H}$	$M_{16}\mathbb{C}$	$M_{32}\mathbb{R}$	$M_{32}\mathbb{R} \times M_{32}\mathbb{R}$	$M_{64}\mathbb{R}$	$M_{64}\mathbb{C}$
$M_2\mathbb{H} \times M_2\mathbb{H}$	$M_4\mathbb{H}$	$M_8\mathbb{C}$	$M_{16}\mathbb{R}$	$M_{16}\mathbb{R} \times M_{16}\mathbb{R}$	$M_{32}\mathbb{R}$	$M_{32}\mathbb{C}$	$M_{32}\mathbb{H}$
$M_2\mathbb{H}$	$M_4\mathbb{C}$	$M_8\mathbb{R}$	$M_8\mathbb{R} \times M_8\mathbb{R}$	$M_{16}\mathbb{R}$	$M_{16}\mathbb{C}$	$M_{16}\mathbb{H}$	$M_{16}\mathbb{H} \times M_{16}\mathbb{H}$
$M_2\mathbb{C}$	$M_4\mathbb{R}$	$M_4\mathbb{R} \times M_4\mathbb{R}$	$M_8\mathbb{R}$	$M_8\mathbb{C}$	$M_8\mathbb{H}$	$M_8\mathbb{H} \times M_8\mathbb{H}$	$M_{16}\mathbb{H}$
$M_2\mathbb{R}$	$M_2\mathbb{R} \times M_2\mathbb{R}$	$M_4\mathbb{R}$	$M_4\mathbb{C}$	$M_4\mathbb{H}$	$M_4\mathbb{H} \times M_4\mathbb{H}$	$M_8\mathbb{H}$	$M_{16}\mathbb{C}$
$\mathbb{R} \times \mathbb{R}$	$M_2\mathbb{R}$	$M_2\mathbb{C}$	$M_2\mathbb{H}$	$M_2\mathbb{H} \times M_2\mathbb{H}$	$M_4\mathbb{H}$	$M_8\mathbb{C}$	$M_{16}\mathbb{R}$
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \times \mathbb{H}$	$M_2\mathbb{H}$	$M_4\mathbb{C}$	$M_8\mathbb{R}$	$M_8\mathbb{R} \times M_8\mathbb{R}$

To read the table, the bottom left entry is $\mathrm{Cliff}_{0,0} \cong \mathbb{R}$, and moving to the right increments the signature from (p, q) to $(p, q+1)$, and moving up increments the signature (p, q) to $(p+1, q)$. Any other Clifford algebra can be obtained from an algebra on this table by tensoring with $M_{16}\mathbb{R}$, since incremeting the signature by 8 (by adding to either p or q) results in tensoring with $M_{16}\mathbb{R}$.

2. Clifford Modules

Definition 2.1. A (left) ***Clifford module*** for the Clifford algebra $\mathrm{Cliff}_{p,q}(\mathbb{R})$ is a module for $\mathrm{Cliff}_{p,q}(\mathbb{R})$ in the usual sense i.e. a real vector space V equipped with an algebra action $\bullet : \mathrm{Cliff}_{p,q} \times V \to V$ satisfying

(1) Every element of $\mathrm{Cliff}_{p,q}(\mathbb{R})$ acts linearly on V.
(2) $(AB) \cdot v = A \cdot (B \cdot v)$ for all $v \in V$.
(3) $(A + B) \cdot v = A \cdot v + B \cdot V$ for all $v \in V$.

Equivalently, it is the data of a real vector space V and a homomorphism $\mathrm{Cliff}_{p,q}(\mathbb{R}) \to \mathrm{End}(V)$.

Definition 2.2. A Clifford module is ***irreducible*** if there exist no proper nontrivial submodules.

From the classification of Clifford algebras, all the Clifford algebras are either matrix algebras $M_n\mathbb{F}$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, or products $M_n\mathbb{F} \times M_n\mathbb{F}$ of two copies of the same matrix algebra. This is sufficient to conclude that Clifford algebras are semisimple, so all Clifford modules will be direct sums of irreducible modules. Therefore, classifying all Clifford modules reduces to classifying the irreducible Clifford modules.

Theorem 2.3. *Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. Then any nontrivial irreducible module for $M_n\mathbb{F}$ is isomorphic to $\mathbb{F}^n$ with the standard action.*

Proof. We first note that $M_n\mathbb{F}$ acts transitively on $\mathbb{F}^n$, which implies that it is irreducible. We then must show that $\mathbb{F}^n$ is, up to isomorphism, the only irreducible $M_n\mathbb{F}$ module. The matrix algebra $M_n\mathbb{F}$ admits an increasing chain of left ideals

$$0 = I_0 \subset I_1 \subset \ldots \subset I_n = M_n\mathbb{F}$$

where I_k is the set of matrices where only the first k columns are nonzero. These ideals have the property that the quotient I_k/I_{k-1} is isomorphic to $\mathbb{F}^n$ as a left $M_n\mathbb{F}$ module. Then let M be some nontrivial irreducible $M_n\mathbb{F}$ module, and fix $m \in M$. Then the orbit $M_n\mathbb{F} \cdot m$ of m under the algebra action is a nonzero submodule, so it must be all of M. Then the map $\varphi : M_n\mathbb{F} \to M$ given by $A \mapsto A \cdot m$ is a surjective map of left $M_n\mathbb{F}$ modules. Then there must exist some smallest k such that $\varphi(I_k)$ is nonzero, and by construction, $\varphi|_{I_k}$ factors through the quotient I_k/I_{k-1}, which is isomorphic to $\mathbb{F}^n$ with the standard action. Then since $\mathbb{F}^n$ is irreducible, this gives us a nontrivial map between irreducible modules, which is an isomorphism by Schur's Lemma. ■

Theorem 2.4. *Any nontrivial irreducible module for $M_n\mathbb{F} \times M_n\mathbb{F}$ is isomorphic to either $\mathbb{F}^n$ where the left factor acts in the usual way, and the right factor acts by 0, or $\mathbb{F}^n$ where the left factor acts by 0 and the right factor acts in the usual way.*

Proof. Let R denote $\mathbb{F}^n$ where the right factor acts nontrivially, and let L denote $\mathbb{F}^n$ where the left factor acts nontrivially. Both L and R are irreducible since $M_n\mathbb{F} \times M_n\mathbb{F}$ acts transitively on them. To show that they are the only irreducible modules up to isomorphism, we use a similar techinique as above. Let I_k denote the chain of increasing ideals in $M_n\mathbb{F}$, as we used above. Then $M_n\mathbb{F} \times M_n\mathbb{F}$ admits a chain of increasing left ideals J_k

$$0 = J_0 \subset I_1 \times \{0\} \subset \ldots \subset M_n\mathbb{F} \times \{0\} \subset M_n\mathbb{F} \times I_1 \subset \ldots \subset M_n\mathbb{F} \times M_n\mathbb{F} = J_{2n}$$

We note that for $1 \leq k \leq n$, we have that J_k/J_{k-1} is isomorphic to L, and for $n+1 \leq k \leq 2n$, we have that J_k/J_{k-1} is isomorphic to R. Then given a nontrivial irreducible module M and a nonzero element m, we get a surjective map $\varphi : M_n\mathbb{F} \times M_n\mathbb{F}$ where $A \mapsto A \cdot m$. Like before, there exists some smallest k such that $\varphi(J_k)$ is nonzero, which then factors through to an isomorphism $J_k/J_{k-1} \to M$, so M is either isomorphic to R or L. ■

This then gives a full classification of the irreducible ungraded Clifford modules.

3. Complex Clifford Algebras

Much of the discussion regarding Clifford algebras can be reconstructed using complex vector spaces. However, there is an important distinction to be made. Over $\mathbb{C}$, the notion of signature no longer makes sense when discussing bilinear forms. Given a bilinear form $B : V \times V \to \mathbb{C}$ and a vector $v \in V$ with $B(v, v) = 1$, we have that $B(iv, iv) = -1$. Therefore, the complex Clifford algebras generated by V with any nondegenerate bilinear form are entirely determined by their dimension. In the case of $\mathbb{C}^n$ with the standard bilinear form

$$\langle v, w \rangle = \sum_i v^i w^i$$

we denote the Clifford algebra by $\mathrm{Cliff}_n(\mathbb{C})$. In the complex case, the classification is much simpler, and is determined by the parity of the dimension.

In the even case of $\mathbb{C}^{2n}$, we first prove a lemma.

LEMMA 3.1. *There exists a basis $\{e_1, \ldots e_n, f_1, \ldots, f_n\}$ for $\mathbb{C}^{2n}$ satisfying*

(1) $\langle e_i, e_i \rangle = \langle f_i, f_i \rangle = 0$
(2) $\langle e_i, f_j \rangle = \delta_{ij}$

where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$.

PROOF. Let $\{a_i\}$ denote the first n standard basis vectors for $\mathbb{C}^{2n}$, and let $\{b_i\}$ denote the last n standard basis vectors. Then setting $e_i = a_i + ib_i$ and $f_i = a_i - ib_i$, we get a basis $\{e_1, \ldots e_n, f_1, \ldots, f_n\}$ for $\mathbb{C}^{2n}$. We then compute

$$\begin{aligned}
\langle e_i, e_i \rangle &= \langle a_i + ib_i, a_i + ib_i \rangle \\
&= \langle a_i, a_i \rangle + 2\langle a_i, ib_i \rangle + \langle ib_i, ib_i \rangle \\
&= 1 + 0 + -1 \\
&= 0 \\
\langle f_i, f_i \rangle &= \langle a_i - ib_i, a_i - ib_i \rangle \\
&= \langle a_i, a_i \rangle - 2\rangle a_i, ib_i \rangle + \langle ib_i, ib_i \rangle \\
&= 0 \\
\langle e_i, f_j \rangle &= \langle a_i + ib_i, a_j - ib_j \rangle \\
&= \langle a_i, a_j \rangle - \langle a_i, ib_j \rangle + \langle a_j, ib_i \rangle - \langle ib_i, ib_j \rangle \\
&= \delta_{ij} + 0 + 0 + \delta ij \\
&= 2\delta_{ij}
\end{aligned}$$

So normalizing the e_i and f_i by dividing by $\sqrt{2}$ gives the desired basis. ∎

This basis gives a direct sum decomposition $\mathbb{C}^{2n} = W \oplus W'$ where W is the span of the e_i and W' is the span of the f_i. We then claim that $\mathrm{Cliff}_{2n}(\mathbb{C})$ is isomorphic to the endomorphism algebra $\mathrm{End}(\Lambda^\bullet W)$. To give a map $\mathrm{Cliff}_{2n}(\mathbb{C}) \to \mathrm{End}(\Lambda^\bullet W)$, we need to specify two maps $\varphi : W \to \mathrm{End}(\Lambda^\bullet W)$ and $\varphi' : W' \to \mathrm{End}(\Lambda^\bullet W)$ such that for all $w, p \in W$ and $w', p' \in W'$ we have

(1) $\varphi(w) \circ \varphi(p) + \varphi(p) \circ \varphi(w) = 0$
(2) $\varphi'(w') \circ \varphi(p') + \varphi(p') \circ \varphi(w') = 0$
(3) $\varphi(w) \circ \varphi'(w') + \varphi'(w') \circ \varphi(w) = 2\langle w, w' \rangle$

where we use the fact that $\langle \cdot, \cdot \rangle$ vanishes on W and W'. For notational convenience, we will denote $\varphi(w)$ as φ_w, and will do the same with φ'. Define $\varphi_w : \Lambda^\bullet W \to \Lambda^\bullet W$ by $\varphi_w(\alpha) = w \wedge \alpha$ and $\varphi'(w')$ by

$$\varphi'_{w'}(v_1 \wedge \cdots \wedge v_k) = 2 \sum_i (-1)^{i-1} \langle w', v_i \rangle v_1 \wedge \cdots \widehat{v_i} \cdots \wedge v_k$$

which will satisfy these relations, and these two maps define the desired isomorphism $\mathrm{Cliff}_{2n}(\mathbb{C}) \to \mathrm{End}(\Lambda^\bullet W)$.

In the odd dimensional case of $\mathbb{C}^{2n+1}$, we can decompose $\mathbb{C}^{2n+1}$ as $\mathbb{C}^{2n+1} = W \oplus W' \oplus U$, where W and W' are the same as in the decomposition in the even case, and U is the orthogonal complement to $W \oplus W'$. We then define the maps from W and W' into $\mathrm{End}(\Lambda^\bullet W)$, and then define a map $U \to \mathrm{End}(\Lambda^\bullet W)$ where a unit vector in U acts by identity on the odd elements of $\Lambda^\bullet W$ and by negative identity on the even elements. This then defines a map $\varphi : \mathrm{Cliff}_{2n+1}(\mathbb{C}) \to \mathrm{End}(\Lambda^\bullet W)$. Repeating this with $\mathrm{End}(\Lambda^\bullet W')$ gives another map $\psi : \mathrm{Cliff}_{2n+1}(\mathbb{C}) \to \mathrm{End}(\Lambda^\bullet W')$. Then the product map $\varphi \times \psi : \mathrm{Cliff}_{2n+1}(\mathbb{C}) \to \mathrm{End}(\Lambda^\bullet W) \times \mathrm{End}(\Lambda^\bullet W')$

is the desired isomorphism.

4. The Pin and Spin Groups

The group of invertible elements in $\mathrm{Cliff}_{p,q}(\mathbb{R})$, denoted $\mathrm{Cliff}^\times_{p,q}(\mathbb{R})$ contains a group $\mathrm{Pin}_{p,q}$, which a double cover of the group $O_{p,q}$ of matrices preserving the standard bilinear form $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^{p,q}$. Inside of $\mathrm{Pin}_{p,q}$, there exists a subgroup $\mathrm{Spin}_{p,q} \subset \mathrm{Pin}_{p,q}$, which double covers the group $SO_{p,q}$, which consists of the subgroup of $O_{p,q}$ where all the elements have determinant equal to 1.

Definition 4.1. The ***Pin group*** $\mathrm{Pin}_{p,q}$ is the subgroup of $\mathrm{Cliff}^\times_{p,q}(\mathbb{R})$ generated by the set

$$\left\{v \in \mathbb{R}^{p,q}\ :\ v^2 = \pm 1\right\}$$

The ***Spin group*** $\mathrm{Spin}_{p,q}$ is the subgroup of $\mathrm{Pin}_{p,q}$ generated by even products of basis vectors. In other words, $\mathrm{Spin}_{p,q} = \mathrm{Pin}_{p,q} \cap \mathrm{Cliff}^0_{p,q}(\mathbb{R})$. In the case that the bilinear form is definite, we let $\mathrm{Pin}^+_n = \mathrm{Pin}_{n,0}$ and $\mathrm{Pin}^-_n = \mathrm{Pin}_{0,n}$.

Theorem 4.2. $\mathrm{Spin}_{0,n} \cong \mathrm{Spin}_{n,0}$.

Proof. Recall that we have an isomorphism $\mathrm{Cliff}^{\mathrm{op}}_{p,q} \to \mathrm{Cliff}_{q,p}$ where $e^\pm_i \mapsto e^\mp_i$. In addition, the even subalgebra $\mathrm{Cliff}^0_{q,p}$ is isomorphic to the (ungraded) opposite algebra of the even subalgebra $\mathrm{Cliff}^0_{p,q}$. Therefore, the Spin group $\mathrm{Spin}_{q,p} \subset \mathrm{Cliff}_{q,p}$ is isomorphic to the opposite group $\mathrm{Spin}^{\mathrm{op}}_{p,q}$. We then know that every group is isomorphic to its opposite group via the map $g \mapsto g^{-1}$, giving us the desired isomorphism. ∎

In particular, this implies that the Spin groups in definite signatures are isomorphic, so we will henceforth denote them as Spin_n. To show that the Pin and Spin groups cover $O_{p,q}$ and $\mathrm{Spin}_{p,q}$, we make a short digression. Given a vector $v \in \mathbb{R}^{p,q}$, we can define a reflection map $R_v : \mathbb{R}^{p,q} \to \mathbb{R}^{p,q}$ given by $R_v(w) = w - 2\langle v, w\rangle v$, which will reflect across the hyperplane $v^\perp$.

Theorem 4.3 (***Cartan-Dieudonné***). *Any orthogonal transformation $A \in O_{p,q}$ can be written as the composition of at most $p+q$ hyperplane reflections, where we interpret the identity map as the composition of 0 reflections.*

Proof. We prove this by induction on $n = p + q$. The case $n = 1$ is trivial, since $O_1 = \{\pm 1\}$. Then given $A \in O_{p,q}$, fix some nonzero $v \in \mathbb{R}^{p,q}$. Then define $R : \mathbb{R}^{p,q} \to \mathbb{R}^{p,q}$ by

$$R(w) = w - 2\frac{\langle Av - v, w\rangle}{\langle Av - v, Av - v\rangle}(Av - v)$$

Then R will be a reflection about the hyperplane orthogonal to $Av - v$, and will interchange v and Av. Therefore, RA is an orthogonal transformation fixing v. Since RA is orthogonal, it will also fix the orthogonal complement $v^\perp$, so it will restrict to an orthgonal transformation on $v^\perp$. We then see that $v^\perp$ is 1 dimension lower thant $\mathbb{R}^{p,q}$, and restricting the bilinear form to $v^\perp$, we know by the inductive hypotehsis that $RA|_{v^\perp}$ can be written as at most $n-1$ hyperplane reflections in $v^\perp$. Since RA fixes v, we can extend all of these transformations to a hyperplane reflection on all of $\mathbb{R}^{p,q}$, by taking the span of each hyperplane wiht v, giving us that RA is a composition of at most $n-1$ reflections. Finally, composing RA with R gives us that A can be written as a composition of at most n hyperplane reflections. ∎

The Cartan-Dieudonné theorem will be the central piece for showing that the Pin and Spin groups cover the orthogonal groups.

Theorem 4.4. *There exist 2-to-1 group homomorphisms $\mathrm{Pin}_{p,q} \to O_{p,q}$ and $\mathrm{Spin}_{p,q} \to SO_{p,q}$, i.e. there exist short exact sequences of groups*

$$0 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Pin}_{p,q} \longrightarrow O_{p,q} \longrightarrow 0$$

$$0 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}_{p,q} \longrightarrow SO_{p.q} \longrightarrow 0$$

Proof. We first consider the case of $\mathrm{Pin}_{p,q}$. To do this, we need to construct a group action where $\mathrm{Pin}_{p,q}$ acts on $\mathbb{R}^{p,q}$ by orthogonal transformations. There exists an involution $T : \mathrm{Cliff}_{p,q}(\mathbb{R}) \to \mathrm{Cliff}_{p,q}(\mathbb{R})$, where given the standard orthogonal basis $\{e_1, \ldots, e_{p+1}\}$, we define

$$T(e_{i_1} \cdots e_{i_k}) = e_{i_k} \cdots e_{i_1}$$

and extending linearly to the remainder of $\mathrm{Cliff}_{p,q}(\mathbb{R})$. Given $a \in \mathrm{Cliff}_{p,q}(\mathbb{R})$, we denote $T(a)$ by a^T. We note that for a vector $v \in \mathbb{R}^{p,q}$, identifying $\mathbb{R}^{p,q}$ as a subspace of $\mathrm{Cliff}_{p,q}(\mathbb{R})$, satisfying $\langle v, v\rangle = \pm 1$, we have that $v^T = v$ and $v^{-1} = \pm v$. Then given $g \in \mathrm{Pin}_{p,q}$, and $v \in \mathbb{R}^{p,q}$, we claim that the left action

$$g \cdot v = -gvg^{-1}$$

defines the group action we desire. To show this, we must show that this indeed maps $\mathbb{R}^{p,q}$ back into itself, and that the group elements act by orthogonal trasnformations. It suffices to check this on the generating set of elements v with $\langle v, v\rangle = \pm 1$. First assume that $\langle v, v\rangle = 1$. Then given $w \in \mathbb{R}^{p,q}$, we compute

$$\begin{aligned} -vwv^{-1} &= -vwv \\ &= (wv - 2\langle v, w\rangle)v \\ &= w - 2\langle v, w\rangle v \end{aligned}$$

Which is hyperplane reflection about the orthogonal complement of v. In the case that $\langle v, v\rangle = -1$, we compute

$$\begin{aligned} -vwv^{-1} &= -vw(-v) \\ &= (2\langle -v, w\rangle + wv)(-v) \\ &= w - 2\langle -v, w\rangle \end{aligned}$$

which is hyperplane reflection about the orthogonal complement of $-v^{\perp}$, which is the same as the orthogonal complement of $v^{\perp}$. Therefore, $\mathrm{Pin}_{p,q}$ acts by orthogonal transformations, giving us a homomorphism $\mathrm{Pin}_{p,q} \to O_{p,q}$. This map is surjective by the Cartan-Dieudonné theorem, and it can be verified that the kernel is $\{\pm 1\}$. ■

Most of the constructions carry over to the complex Clifford algebras, giving us the complex Pin and Spin groups, denoted $\mathrm{Pin}_n\mathbb{C}$ and $\mathrm{Spin}_n\mathbb{C}$, which double cover the complex orthogonal groups $O_n\mathbb{C}$ and $SO_n\mathbb{C}$ respectively.

Two simple examples of spin groups occur in dimensions 2 and 3. Since $SO_2 \cong \mathbb{T}$, where

$$\mathbb{T} = \{z \in \mathbb{C} \ : \ |z| = 1\}$$

we have that $\mathrm{Spin}_2 \cong \mathbb{T}$, where the covering map is given by $z \mapsto z^2$. In the case of SO_3, we consider the unit quaternions, which form a Lie group isomorphic to the group SU_2. Then given $q \in SU_2$, we define the map $\varphi_q : \mathbb{R}^3 \to \mathbb{R}^3$ where $\varphi_q(v) = qv\bar{q}$, where $\bar{q}$ is the quaternionic conjugate of q (e.g. $\overline{a + bi + cj + dk} = a - bi - cj - dk$), and $v = v^i e_i$ is idenitified with $v^1 i + v^2 j + v^3 k \in \mathbb{H}$. The mapping $q \mapsto \varphi_q$ then gives a double cover $SU_2 \to SO_3$. In particular, SU_2 is diffeomorphic to the sphere S^3, so the double covering realizes SO_3 as the quotient of S^3 by the antipodal map, giving us that $SO_3 \cong \mathbb{RP}^3$.

Many examples of low dimensional Spin groups arise from investigating the relationship between a 4 dimensional complex vector space V and it's second exterior power $\Lambda^2 V$. Fix a volume form $\mu \in \Lambda^4 V^*$. This then induces a symmetric, nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ on $\Lambda^2 V$ by

$$\langle \alpha, \beta\rangle = \langle \alpha \wedge \beta, \mu\rangle$$

where $\langle \alpha \wedge \beta,\, u\rangle$ denotes the natural pairing of the vector space $\Lambda^4 V$ with its dual $\Lambda^4 V^*$. By fixing a basis $\{e_i\}$ for V where $\mu(e_1 \wedge e_2 \wedge e_3 \wedge e_4) = 1$, we see that the group of transformations $\mathrm{Aut}(V, \mu)$ perserving μ is isomorphic to the group $SL_4\mathbb{C}$. In addition, each map $T \in \mathrm{Aut}(V, \mu)$ induces a map $\Lambda^2 T : \Lambda^2 V \to \Lambda^2 V$, which is determined by the formula $\Lambda^2 T(v \wedge w) = Tv \wedge Tw$. For any $T \in \mathrm{Aut}(V, \mu)$, the induced map $\Lambda^2 T$ preserves the bilinear form on $\Lambda^2 V$, so the mapping $T \mapsto \Lambda^2 V$ determines a group homomorphism

$\mathrm{Aut}(V,\mu) \to \mathrm{Aut}(\Lambda^2 V, \langle\cdot,\cdot\rangle)$, where $\mathrm{Aut}(\Lambda^2 V, \langle\cdot,\cdot\rangle)$ denotes the group of linear automorphisms perserving the bilinear form. The kernel of this map is $\{\pm\,\mathrm{id}_V\}$, and fixing an orthogonal basis for $\langle\cdot,\cdot\rangle$ gives us that this map is a double cover $SL_4\mathbb{C} \to SO_6\mathbb{C}$, so $SL_4\mathbb{C}$ is isomorphic to the complex spin group $\mathrm{Spin}_4\mathbb{C}$

If we then fix a hermitian inner product $h : V \times V \to \mathbb{C}$, we can consider the automorphisms $\mathrm{Aut}(V,\mu,h)$ preserving h and μ, which is isomorphic to the group SU_4. The bilinear form h induces a hermitian inner product (which we also denote h) on $\Lambda^2 V$ defined by

$$h(v_1 \wedge v_2, v_3 \wedge v_4) = \det \begin{pmatrix} h(v_1,v_3) & h(v_1,v_4) \\ h(v_2,v_3) & h(v_2,v_4) \end{pmatrix}$$

Then if $T \in \mathrm{Aut}(V,\mu,h)$, $\Lambda^2 T$ preserves the bilinear form $\langle\cdot,\cdot\rangle$ induced by μ as well as the hermitian inner product induced by h. The group that preserves both of these structures is isomorphic to $SO_6\mathbb{C} \cap U_6$, which is $SO_6\mathbb{R}$. This gives us that $SU_4 \cong \mathrm{Spin}_6$.

In general, one can play the game of fixing additional structure on V (e.g. a real stucture, quaternionic structure, symplectic form) and look for the induced structure on $\Lambda^2 V$. This then gives a map from automorphisms of V preserving this additional structure to automorphisms of $\Lambda^2 V$ preserving the induced structure. Playing this game then determines several other low dimensional Spin groups.

$$\begin{array}{lll} \mathrm{Spin}_5\mathbb{C} \cong Sp_4\mathbb{C} & \mathrm{Spin}_4 \cong Sp(4) & \mathrm{Spin}_4\mathbb{C} \cong SL_2\mathbb{C} \times SL_2\mathbb{C} \\ \mathrm{Spin}_{1,3} \cong SL_2\mathbb{C} & \mathrm{Spin}_{1,2} \cong SL_2\mathbb{R} & \mathrm{Spin}_{1,5} \cong SL_2\mathbb{H} \end{array}$$

Where $Sp_4\mathbb{C}$ denotes the group of 4×4 matrices preserving a symplectic form, $Sp(4) = Sp_4\mathbb{C} \cap U_4$, and $SL_2\mathbb{H}$ denotes the automorphisms of a 2 dimensional quaternionic vector space with determinant 1 when regarded as 4×4 complex matrices. Any Clifford module immediately gives a representation of the Pin and Spin groups simply by restricting the action.

Definition 4.5. Given a Pin group $\mathrm{Pin}_{p,q}$, the ***Pinor representations*** are representations of $\mathrm{Pin}_{p,q}$ that arise from an irreducible Clifford module M (i.e. the action of $\mathrm{Pin}_{p,q}$ can be extended to an action of $\mathrm{Cliff}_{p,q}(\mathbb{R})$). The ***Spinor representations*** are defined analogously for the group $\mathrm{Spin}_{p,q}$.

From the classification of Clifford modules, we get a classification of all the Pinor representations. From the relationship between a Clifford algebra and its even subalgebra, we also get a complete classification of all the Spinor representations.

5. Projective Spin Representations

Using the isomorphisms of the Clifford algebras with matrix algebras or products of matrix algebras along with the identification of the even subalgebra with another Clifford algebra, we have a complete classification of the irreducible modules over the even subalgebras $\mathrm{Cliff}^0_n(\mathbb{R})$. Restricting to the Spin group $\mathrm{Spin}_n \subset \mathrm{Cliff}^0_n(\mathbb{R})$, this gives us the Spin representations. These Spin representations define projective representations of the group SO_n.

Proposition 5.1. *Let G be a group, and V a finite dimensional irreducible representation of G. Then every element of the center acts by scalars, i.e. for $g \in Z(G)$, there exists a scalar λ_g such that for all v in V, we have*

$$g \cdot v = \lambda_g v$$

Proof. Let $g \in Z(G)$. Then for every $h \in G$ and $v \in V$, we have that

$$h \cdot (g \cdot v) = g \cdot (h \cdot v)$$

Therefore, the action of g defines a G-equivariant map $V \to V$, which by Schur's lemma must necessarily be a scalar multiple of the identity. ∎

Given a Spin representation $\mathbb{S}$, we have that that the elements ± 1 act by scalars. Then since SO_n is the quotient of Spin_n by the subgroup $\{\pm 1\}$ of the center, we get a projective representation of SO_n on the projectivization $\mathbb{PS}$ in the following way: Given an element $A \in SO_n$, we have that A lifts to two elements $\{\pm\tilde{A}\} \subset \mathrm{Spin}_n$ which differ by -1. Since -1 acts by a scalar on $\mathbb{S}$, these elements determine the same action of $\mathbb{PS}$, giving us a well defined action of SO_n on $\mathbb{PS}$.

In some sense, the Spin representation $\mathbb{S}$ is not realized canonically, while the projective Spin representation $\mathbb{PS}$ is canonical. Given an isomorphism $\varphi : V \to W$, this induces a unique algebra isomorphism $\operatorname{End} V \to \operatorname{End} W$ where $A \in \operatorname{End} V$ is mapped to $\varphi \circ A \circ \varphi^{-1}$. However, the converse is not true.

PROPOSITION 5.2. *Let $\varphi : \operatorname{End} V \to \operatorname{End} W$ be an algebra isomorphism. Then there does not exist a unique isomorphism $\psi : V \to W$ inducing φ.*

To prove this, we need a lemma.

LEMMA 5.3. *The group of algebra automorphisms $\operatorname{Aut}(M_n\mathbb{F})$ is isomorphic to the projective general linear group $PGL_n\mathbb{F} = GL_n\mathbb{F}/Z(GL_n\mathbb{F})$.*

PROOF. Let $\alpha : M_n\mathbb{F} \to M_n\mathbb{F}$ be an algebra automorphism. We know that $M_n\mathbb{F}$ admits a single irreducible module M, which is $\mathbb{F}^n$ with the standard action. Then α defines another module M^α, which is the same underlying vector space as M, with the algebra action given by $M \cdot v = \alpha(M)v$, where the right hand side is the action of $\alpha(M)$ on the module M. Since α is an algebra automorphism, $M_n\mathbb{F}$ acts transitively on M^α, so it is also an irreducible module, which must be isomorphic to M. Therefore, there exists a module isomorphism $A : M \to M^\alpha$. Since M and M^α are the same underlying vector space, A is also a linear isomorphism $A : M \to M$, thought of as a vector space instead of a module. Then since A is a module homomorphism. we have that for any $M \in M_n\mathbb{F}$,

$$A \circ M = \alpha(M) \circ A \implies A \circ M \circ A^{-1} = \alpha(M)$$

so α is given by conjugation by $A \in GL(M)$. In a basis, this tells us that the map $GL_n\mathbb{F} \to M_n\mathbb{F}$ given by conjugation is surjective, and the kernel of this map is the center of $GL_n\mathbb{F}$, so by the first isomorphism theorem, we have that $\operatorname{Aut}(M_n\mathbb{F}) \cong PGL_n\mathbb{F}$. ■

PROOF OF PROPOSITION. Fix bases $\mathbb{F}^n \to V$ and $\mathbb{F}^n \to W$. This induces isomorphisms $M_n\mathbb{F} \to V$ and $M_n\mathbb{F} \to W$. Then in these bases, the algebra isomorphism φ is given by an automorphism $M_n\mathbb{F} \to M_n\mathbb{F}$, and the question now translates to asking whether this automorphism induces an isomorphism $\mathbb{F}^n \to \mathbb{F}^n$. From the lemma, we know this is false – an automorphism of $M_n\mathbb{F}$ only determines an element of $PGL_n\mathbb{F}$, so it is induced by an entire family of automorphisms differing by $Z(GL_n\mathbb{F})$. ■

However, an isomorphism $\varphi : \operatorname{End} V \to \operatorname{End} W$ does induce an isomorphism $\mathbb{P}V \to \mathbb{P}W$ of projective spaces. To see, this we make an identification between 1 dimensional subspaces of V with maximal left ideals of $\operatorname{End} V$.

PROPOSITION 5.4. *There is a bijection*

$$\{\textit{Maximal left ideals of } \operatorname{End} V\} \longleftrightarrow \mathbb{P}V$$

PROOF. Given a line $L \in \mathbb{P}V$, the ***annihilator*** of L is the set

$$\operatorname{Ann}(L) = \{M \in \operatorname{End} V \ : \ M(L) = 0\}$$

In fact, $\operatorname{Ann}(L)$ is a left ideal in $\operatorname{End} V$, since given $A \in \operatorname{End} V$ and $M \in \operatorname{Ann}(L)$, L lies in the kernel of $A \circ M$. We claim that $\operatorname{Ann}(L)$ is maximal. Suppose $\operatorname{Ann}(L) \subset I$ is properly contained in a left ideal I. Fix an ordered basis for V in which the first basis element is a nonzero element of L, then elements of $\operatorname{Ann}(L)$ are represented by matrices with all zeroes in the first column. Then since $\operatorname{Ann}(L)$ is properly contained in I, there exists some $M \in I$ such that $M \notin \operatorname{Ann}(L)$, which implies that as a matrix, the first column of M is nonzero. Then pick a matrix $A \in \operatorname{Ann}(L)$ in which the nonzero columns complete the first column into a basis for $\mathbb{R}^n$. Then $A + M$ is an invertible element of $\operatorname{End} V$, so I must be all of $\operatorname{End} V$. Therefore $\operatorname{Ann}(L)$ is maximal. To show that the mapping $L \mapsto \operatorname{Ann}(L)$ is a bijection, we produce an inverse. Let $I \subset \operatorname{End} V$ be a maximal ideal. Then we claim that the subspace

$$\mathbb{V}(I) = \bigcap_{M \in I} \ker M$$

is a 1 dimensional subspace of V. We note that $\mathbb{V}(I)$ cannot be trivial, since this would imply that I would contain an invertible element, contradicting that it is a proper ideal. We also see that it cannot be higher than 2 dimensional, since otherwise, I woud be contained in the annihilator of a proper nontrivial subspace of $\mathbb{V}(I)$, contradicting maximality of I. We then claim that these two mappings are inverses. We certainly have that $\mathrm{Ann}(\mathbb{V}(I)) \supset I$, so by maximality, this must be I. In addition it is clear that $\mathbb{V}(\mathrm{Ann}(L)) = L$ by the definition of $\mathbb{V}(I)$ and the annihilator. Therefore, these mappings are inverses. ■

Therefore, given an algebra isomorphism $\varphi : \mathrm{End}\, V \to \mathrm{End}\, W$, this induces a map $\mathbb{P}V \to \mathbb{P}W$ since the image of a maximal ideal under an isomorphism is a maximal ideal. In addition, the induced map is a bijection, since it has an inverse given by the induced map of φ^{-1}. In addition, the group of units $GL(V)$ acts on $\mathbb{P}V$ by right multiplication – given a maximal ideal I and $A \in GL(V)$, the ideal $I \cdot A$ is also a maximal left ideal.

This gives us a canonical realization of the Spin representations. In the case that the even subalgebra is isomorphic to a matrix algebra $M_n\mathbb{F}$, the projective Spin representation is restriction of the action of $\mathrm{Cliff}_n^0(\mathbb{R})$ on maximal left ideals of $\mathrm{Cliff}_n^0(\mathbb{F})$. In the case that the even subalgebra is isomorphic to a product $M_n\mathbb{F} \times M_n$, the irreducible modules identify the subalgebras L and R isomorphic to $M_n\mathbb{F} \times \{0\}$ and $\{0\} \times M - n\mathbb{F}$ by finding the maximal subalgebra that acts nontrivially. Looking at the maximal left ideals of these subalgebras then identifies the two projective Spin representations. In addition, since -1 acts trivially on left ideals, these projective Spin represention descend to the quotient $\mathrm{Spin}_n / \{\pm 1\} \cong SO_n$, giving us the projective representations of SO_n.

CHAPTER 2

Spin Structures on Manifolds

1. Fiber Bundles

Definition 1.1. Let M and F be smooth manifolds. Then a ***fiber bundle*** over M with model fiber F is a the data of a smooth manifold E with a smooth map $\pi : E \to M$ such that for every point $p \in M$, there is a neighborhood $U \subset M$ containing p such that there exists an diffeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ such that the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "p_U"] \\ & U & \end{tikzcd}$$

Where P_U denotes projection onto the first factor. The map φ is called a ***local trivialization*** of the fiber bundle $\pi : E \to M$.

Given a fiber bundle $\pi : E \to M$ we often denote the fiber $\pi^{-1}(p)$ by E_p.

Definition 1.2. let $\pi : E \to M$ and $p : L \to M$ be fiber bundles with model fiber F over M. A ***bundle homomorphism*** is a smooth map $\varphi : E \to L$ such that the following diagram commutes

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & L \arrow[dl, "p"] \\ & M & \end{tikzcd}$$

An important property of fiber bundles is that they pull back.

Definition 1.3. Let $\pi : E \to M$ be a fiber bundle with model fiber F and let $f : X \to M$ be a smooth map. Then the ***pullback*** of E by f is the data of a smooth manifold

$$f^*E = \left\{ (x, p) \ : \ x \in X, p \in \pi^{-1}(f(x)) \right\}$$

along with the projection $f^*E \to M$ given by $(x, p) \mapsto x$, giving $f^*E \to X$ the structure of a fiber bundle over X with model fiber F.

Definition 1.4. Let $\pi : E \to M$ be a fiber bundle. A ***local section*** of $\pi : E \to M$ is a smooth map $\sigma : U \to E$ of an open set $U \subset M$ such that $\pi \circ \sigma = \mathrm{id}_U$. If $U = M$, the section is called a ***global section***. Equivalently, it is the smooth assignment of an element in E_p to each point $p \in U$. We denote the set of sections of $\pi : E \to M$ over a set U as $\Gamma_U(E)$.

A fiber bundle is a very general construction in which the fibers F do not necessarily admit extra structure. An important special case of a fiber bundle is a vector bundle, where the fibers are vector spaces.

Definition 1.5. Let M be a smooth manifold. A ***vector bundle*** over M is fiber bundle $\pi : E \to M$ with model fiber $\mathbb{R}^n$ (or $\mathbb{C}^n$) such that the local trivializations $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ (or $\mathbb{C}^n$) restrict to linear isomorphisms on the fibers, i.e. for all $p \in U$, the restriction $\varphi|_{\pi^{-1}(p)}$ is an isomorphism $\pi^{-1}(p) \to \{p\} \times \mathbb{R}^n$ (or $\mathbb{C}^n$). The dimension n of the fibers is called the ***rank*** of the vector bundle. A ***vector bundle homomorphism*** $\varphi : E \to L$ is a bundle homomorphism with the added stipulation that $\varphi|_{E_x}$ is a linear map.

Example 1.6.

(1) Given a smooth manifold M, the tangent bundle $TM = \coprod_{p \in M} T_pM$ is a vector bundle, where the rank is the dimension of M.

(2) The ***tautological bundle*** over $\mathbb{RP}^n$ is the vector bundle that assigns to each subspace $\ell \in \mathbb{RP}^n$ itself as it's fiber. An analgous construction defines the tautological bundle over the Grassmannian $\mathrm{Gr}_k(\mathbb{R}^n)$.

Definition 1.7. A ***Lie group*** is a smooth manifold G with a group structure such that the multiplication map $(g, h) \mapsto gh$ and the inversion map $g \mapsto g^{-1}$ are smooth.

Example 1.8.

(1) The group $GL_n\mathbb{R}$ of invertible $n \times n$ matrices is an open subset of $M_n\mathbb{R}$, and therefore a smooth n^2 dimensional manifold.
(2) The group $SL_n\mathbb{R}$ of $n \times n$ matrices with determinant 1 is a closed submanifold of $GL_{\mathbb{R}}$.
(3) The orthogonal groups O_n and special orthogonal groups SO_n are Lie groups.
(4) The unitary groups U_n and special unitary groups SU_n are Lie groups.

Another important class of fiber bundles are principal bundles, in which the fibers have the structure of G-torsors.

Definition 1.9. Let G be a Lie group, and M a smooth manifold. A ***Principal G-bundle*** over M is the data of

(1) A smooth manifold P with a map $\pi : P \to M$.
(2) A smooth right G-action on P that is free and transitive on the fibers of π.
(3) For every point $p \in M$, a neighborhood $U \subset M$ containing p and a G-equivariant diffeomorphism $\varphi : \pi^{-1}(U) \to U \times G$ (where the right action on $U \times G$ is right multiplication on the second factor) such that we get the commutative diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "p_U"] \\ & U & \end{tikzcd}$$

where p_U denotes projection onto the first factor.

A ***principal bundle homomorphism*** is a bundle homomorphism $\varphi : P \to Q$ that is G-equivariant.

Example 1.10. Given a smooth manifold M and a point $p \in M$, a basis of the tangent space is a linear isomorphism $b : \mathbb{R}^n \to T_pM$. The group $GL_n\mathbb{R}$ acts on the set of bases $\mathcal{B}_p$ on the right by $b \cdot g = b \circ g$. Then the ***frame bundle*** of M, denoted $\mathcal{B}(M)$ is the disjoint union

$$\mathcal{B}(M) = \coprod_{p \in M} \mathcal{B}_p$$

where π is the projection map $(p, b) \mapsto p$. Then $\mathcal{B}(M)$ is a principal $GL_n\mathbb{R}$ bundle over M.

Example 1.11. Given a smooth manifold M with a Riemannian metric g, this induces an inner product on each tangent space. Then the set of orthonormal bases of T_pM is the set of all linear isometries $(\mathbb{R}^n, \langle \cdot, \cdot \rangle) \to (M, g_p)$ where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$ and g_p is the Riemannian metric evaluated at p. Then the disjoint union over all points p of orthonormal bases for the tangent spaces forms the ***orthonormal frame bundle*** $\mathcal{B}_O(M)$, which is a principal O_n bundle.

Definition 1.12. Let $\pi : P \to M$ be a principal G-bundle over M, and let F be a smooth manifold with a smooth left G action. Then the ***associated fiber bundle***, denoted $P \times_G F$, is the set

$$P \times_G F = P \times G / (p, g) \sim (p \cdot h, h^{-1}g)$$

Since the group action on P preserves the fibers, the projection $p_1 : P \times F \to P$ composed with the projection $\pi : P \to M$ descends to the quotient, givng us a projection map $P \times_G F \to M$.

The first thing to check is that $P \times_G F$ is a fiber bundle, justifying the name.

Proposition 1.13. *Let $\pi : P \to M$ be a principal G-bundle and F a manifold with a left G action. then the associated bundle $\Phi : P \times_G F \to M$ is a fiber bundle with model fiber F.*

Proof. We wish to provide local trivializations $\Phi^{-1}(U) \to U \times F$ for the associated bundle. Fix a local trivialization $\psi : \pi^{-1}(U) \to U \times G$. Then ψ is of the form $\psi(p) = (\pi(p), \tilde{\psi}(p))$ for some $\tilde{\psi} : \pi^{-1}(U) \to G$, and $\tilde{\psi}(p \cdot g) = \tilde{\psi}(p)g$. Define

$$\begin{aligned} \varphi : \Phi^{-1}(U) &\to U \times F \\ [p, f] &\mapsto (\pi(p), \tilde{\psi}(p) \cdot f) \end{aligned}$$

We first note that this is well defined on equivalence classes, since

$$[p \cdot g, g^{-1} \cdot f] \mapsto (\pi(p \cdot g), \tilde{\psi}(p \cdot g) \cdot g^{-1} \cdot f) = (\pi(p), \psi(p) \cdot f)$$

So φ is well defined. We also note that $\Phi(\varphi[p, f]) = \pi(p)$ by how Φ was defined, so φ is a local trivialization provided it is a homeomorphism. To show this, we construct an inverse. Define $\alpha : U \times F \to \Phi^{-1}(U)$ by $\alpha(u, f) = [\psi^{-1}(u, e), f]$, where e denotes the identity element of G. We then claim that α is the inverse. We compute

$$\begin{aligned} (\varphi \circ \alpha)(u, f) &= \varphi[\psi^{-1}(u, e), f] \\ &= (u, e \cdot f) \\ &= (u, f) \end{aligned}$$

In the other direction, we compute

$$\begin{aligned} (\alpha \circ \varphi)[p, f] &= \alpha(\pi(p), \tilde{\psi}(p)) \\ &= [\psi^{-1}(\pi(p), e), f] \\ &= [p, f] \end{aligned}$$

Therefore, φ is a local trivialization, giving us that $\Phi : P \times_G F \to M$ is a fiber bundle with model fiber F. ■

There is a correspondence between sections of an associated bundle and G-equivariant maps $P \to F$.

Proposition 1.14. *Let $\pi : P \to M$ be a principal G-bundle. Then there is a bijection*

$$\{G\text{-equivariant maps } P \to F\} \longleftrightarrow \Gamma_M(P \times_G F)$$

Proof. Since F has a left G-action, we first convert this to a right G-action by $f \cdot g = g^{-1} f$. Then what we mean by a G-equivariant map is a map $\varphi : P \to F$ such that

$$\varphi(p \cdot g) = g^{-1} \cdot \varphi(p)$$

We then wish to use G-equivariant map φ to produce a section $\tilde{\varphi} : M \to P \times_G F$ of the associated bundle. For a point $x \in M$, pick any $p \in \pi^{-1}(x)$ in the fiber. Then define

$$\tilde{\varphi}(x) = [p, \varphi(p)]$$

where $[p, \varphi(p)]$ denotes the equivalence class of $(p, \varphi(p))$ in $P \times_G F$. We first claim that this map is well defined, i.e. it is independent of our choice of point $p \in \pi^{-1}(x)$. We know that G acts freely and transitively on $\pi^{-1}(x)$, so all points in the fiber are of the form $p \cdot g$ for a unique $g \in G$. Then we have that for any $g \in G$

$$(p \cdot g, \varphi(p \cdot g)) = (p \cdot g, g^{-1} \cdot \varphi(p)) \sim (p, \varphi(p))$$

where we use the G-equivariance of φ and the definition of the equivalence relation on the associated bundle. In addition, this induced map is a section of $P \times_G F \to M$, since the image is represented by an element of $P \times F$ with an element of the fiber $\pi^{-1}(x)$ in the first factor.

Conversely, given a section $\sigma : M \to P \times_G F$, we wish to produce a G-equivariant map $P \to F$. Given such a section, and a point $x \in M$, we have that $\sigma(x) = [p, f]$ for some $p \in P$ and $f \in F$. Then define $\tilde{\sigma} : P \to F$ such that $\tilde{\sigma}(p) = f$, and $\tilde{\sigma}(p \cdot g) = g^{-1} \cdot f$. Since G acts freely and transitively on the fibers, this is well defined, and specifies the map on every point of P. In addition, $\tilde{\sigma}$ is G-equivariant by construction. Then the maps

$$\{G\text{-equivariant maps } P \to F\} \longleftrightarrow \Gamma_M(P \times_G F)$$

we provided are easily verified to be inverses to each other, giving us the correspondence. ■

In some sense, the geometry of the associated fiber bundle $P \times_G F$ is controlled by the group G, as G determines a disinguished group of symmetries of the fiber F.

EXAMPLE 1.15 (The tangent bundle as an associated bundle). Given a manifold M, we can take the frame bundle $\pi : \mathcal{B}(M) \to M$, which is a principal $GL_n\mathbb{R}$ bundle. The group $GL_n\mathbb{R}$ acts linearly on $\mathbb{R}^n$ in the standard way, giving us an associated vector bundle $\mathcal{B}(M) \times_{GL_n\mathbb{R}} \mathbb{R}^n$. We claim that this bundle is isomorphic to the tangent bundle TM, i.e. there exists an diffeomorphism $\varphi : \mathcal{B}(M) \times_{GL_n\mathbb{R}} \mathbb{R}^n \to TM$ that restricts to linear isomorphisms on the fibers and the diagram

$$\begin{tikzcd} \mathcal{B}(M) \times_{GL_n\mathbb{R}} \mathbb{R}^n \arrow[rr, "\varphi"] \arrow[dr] & & TM \arrow[dl] \\ & M & \end{tikzcd}$$

commutes, where the maps to M are the bundle projections. Recall that elements of $\mathcal{B}(M) \times_{GL_n\mathbb{R}} R^n$ are represented by pairs (b, v), where $b : \mathbb{R}^n \to T_{\pi(b)}M$ is a linear isomorphism, and v is a vector in $\mathbb{R}^n$. Then define φ by

$$\varphi[b, v] = (\pi(b), b(v))$$

This is well defined, since $\varphi[b \circ g, g^{-1}(v)] = (\pi(b \circ g), (b \circ g)(g^{-1}(v))) = (\pi(b), b(v))$. This is an isomorphism, where the inverse mapping maps $(p, v) \in TM$ to $(b, \tilde{v})$ where b is any basis of T_pM and $\tilde{v}$ is the coordinate representation of v in the basis b. In this example, we see that the associated bundle identifies the same vector under different coordinate transformations, which defines the symmetries of TM.

In general, given a rank n vector bundle $E \to M$, we can construct the frame bundle $\mathcal{B}(E)$ for E and recover E by taking the associated bundle $\mathcal{B}(E) \times_{GL_n\mathbb{R}} \mathbb{R}^n$, so the process of taking frames and constructing associated bundles are inverses.

DEFINITION 1.16. Let $\pi : P \to M$ be a principal G-bundle, and $\rho : H \to G$ a group homomorphism. The map ρ gives P a left H action where $h \cdot p = \rho(h) \cdot p$. A ***reduction of structure group*** is the data of a principal H bundle $\varphi : Q \to M$ and an H equivariant bundle homomorphism $F : Q \to P$.

The map $F : Q \to P$ induces a map $\tilde{F} : Q \times_H G \to P$, where we map the equivalence class $[q, g]$ to $F(q)g$. This is well defined on equivalences classes since

$$(q \cdot h, \rho(h)^{-1}g) \mapsto F(q \cdot h)\rho(h)^{-1}g = F(q)\rho(h)\rho(h)^{-1}g = F(q)g$$

EXAMPLE 1.17 (Reduction from $GL_n\mathbb{R}$ to O_n). Let M be a smooth manifold, and $\pi : \mathcal{B}(M) \to M$ it's bundle of frames. The inclusion map $\iota : O_n \hookrightarrow GL_n\mathbb{R}$ gives an action of O_n on $\mathcal{B}(M)$, where given $b \in \mathcal{B}(M)$ and $T \in O_n$, $b \cdot T = b \cdot \iota(T)$. We then take the quotient by this O_n action, giving us a quotient map $q :$ $\mathcal{B}(M)/O_n$. Since the inclusion is injective, the O_n action is free on $\mathcal{B}(M)$, so this gives $q : \mathcal{B}(M) \to \mathcal{B}(M)/O_n$ the structure of a O_n bundle. In addition, the action of O_n preserves the fibers of $\pi : \mathcal{B}(M) \to M$, so π descends to the quotient, so $\mathcal{B}(M)/O_n \to M$ is a fiber bundle with model fiber $GL_n\mathbb{R}/O_n$. Since $GL_n\mathbb{R}$ deformation retracts onto O_n via the Gram-Schmidit algorithm, $GL_n\mathbb{R}/O_n$ is contractible, so the fiber bundle $\mathcal{B}(M)/O_n \to M$ admits global sections. Then given a section $\sigma : M \to \mathcal{B}(M)/O_n$, this gives an O_n bundle over M via the pullback $\sigma^*\mathcal{B}(M)$. In addition, we get a O_n-equivariant map $\sigma^*\mathcal{B}(M) \to \mathcal{B}(M)$ given by $(p, b) \mapsto (p, \iota(b))$. The bubdle $\sigma^*\mathcal{B}(M)$ can be thought of as the bundle of orthonormal frames with respect to some Riemannian metric on M, and the fact that $\mathcal{B}(M)/O_n$ admits sections corresponds to the fact that every manifold admits a Riemannian metric.

2. Dirac Operators

A physical motivation for Clifford algebras and the Spin group comes from physics. In the process of developing a relativistic equation for the electron, Paul Dirac saw the need for a first order differential operator D such that D squared to the Laplace operator.

$$\Delta = -\sum_i \frac{\partial^2}{\partial(x^i)^2}$$ [1]

[1]Our reasoning for choosing this sign convention for Δ is twofold – one reason is that the spectrum of Δ is positive with this choice of sign, and the second is that this definition coincides with a generalized Laplacian on a Riemannian manifold.

If D were the be first order, it would have to be written as

$$D = a^i \frac{\partial}{\partial x^i}$$

for some coefficients a^i. However, it is clear that choosing scalar coefficients for the a^i will not suffice. For example, in $\mathbb{R}^2$ with the standard coordinates x and y, we see that given any first order operator $D = a^1\partial_x + a^2\partial_y$ satisfies

$$D^2 = \left(a^1\frac{\partial}{\partial x} + a^2\frac{\partial}{\partial y}\right)^2 = (a^1)^2\frac{\partial^2}{\partial x^2} + a^1a^2\frac{\partial^2}{\partial x\partial y} + a^2a^1\frac{\partial^2}{\partial y\partial x} + (a^2)^2\frac{\partial^2}{\partial y^2}$$

From this equation, we see that the a^i must square to -1, and we must also have that $a^1a^2 + a^2a^1 = 0$ in order for the mixed partial terms to vanish. This is not possible if the a^i are scalars (in either $\mathbb{R}$ or $\mathbb{C}$). However, the required relations are exactly the relations between orthogonal basis vectors in the Clifford algebra $\mathrm{Cliff}_{0,n}(\mathbb{R})$!

Definition 2.1. Let $\{e_i\}$ be the standard basis for $\mathbb{R}^n$, $\{e^i\}$ its dual basis, and $x^1, \ldots, x^n$ the standard coordinates on $\mathbb{R}^n$. The ***Dirac operator*** on $\mathbb{R}^n$ is the first order differential operator

$$D = e^i \frac{\partial}{\partial x^i}$$

It is not clear from the definition what function space D should act on. The partial derivative operators make sense for any vector valued function, but multiplcation by e^i does not make sense an arbitrary vector space – it must be a Clifford module. Therefore, D is an operator $D : C^\infty(\mathbb{R}^n, M) \to C^\infty(\mathbb{R}^n, M)$ on smooth functions from $\mathbb{R}^n$ to a Clifford module M.

3. Spin Structures

Every manifold M admits a Riemannian metric, so there always exists a reduction of structure group from $GL_n\mathbb{R}$ to O_n. If M is orientable, we can reduce the structure group further to SO_n, which corresponds to a choice of orientation on M. A ***Spin structure*** on M is a further reduction of structure group to Spin_n, with respect to the double cover $\mathrm{Spin}_n \to SO_n$, i.e. the data of a principal Spin_n bundle $P \to M$, along with a Spin_n-equivariant map $P \to \mathcal{B}_{SO}(M)$, where $\mathcal{B}_{SO}(M)$ denotes the bundle of positively oriented orthonomal frames. Depending on the value n, there exist either one or two Spin representations for Spin_n. Let $\mathbb{S}$ denote the direct sum all of all the Spin representations for Spin_n, i.e. $\mathbb{S}$ is equal to the single Spin representation in the case that Spin_n only admits one representation, and $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ in the case that Spin_n admits two inequivalent Spin representations $\mathbb{S}^+$ and $\mathbb{S}^-$. Then given a Spin structure on M with a principal Spin_n bundle P, the ***spinor bundle*** over M is the associated vector bundle $P \times_{\mathrm{Spin}_n} \mathbb{S}$.

Example 3.1 (Spin structures on S^1). The group Spin_1 is equal to $\{\pm 1\}$, so a principal Spin_1-bundle over $\mathbb{S}^1$ is a double cover $\pi : P \to \mathbb{S}^1$ along with a Spin_1-equivariant map $P \to \mathcal{B}_{SO_1}(S^1)$. Since $SO_1 = 1$, $\mathcal{B}_{SO}(S^1)$ is the trivial bundle $S^1 \times \{1\}$. Therefore, specifying a Spin_1-equivariant map $P \to \mathbb{S}^1 \times \{1\}$ is no additional data, since we are forced to map all of the fiber $\pi^{-1}(x)$ to $(x, 1)$ for any $x \in S^1$. Consequently, all double covers give rise to a reduction of structure group to Spin_1. There are only two double covers of Spin_1. One of them is the disconnected double cover, which is the disjoint union $S^1 \coprod S^1$, which we denote as $\pi_1 : P_1 \to S^1$. The other is the connected double cover, which the circle covering itself via the map $z \mapsto z^2$, which we denote as $\pi_2 : P_2 \to S^1$. For convenience, we parameterize P_2 with angles $\theta \in [0, 4\pi)$, so the covering map is given by $\theta \mapsto e^{i\theta}$. The Spin representation is the sign representation on $\mathbb{R}$, where -1 acts by multiplication by -1, and the complexifying gives us an action on $\mathbb{C}$ where -1 acts by multiplication by -1, giving us two spinor bundles $P_1 \times_{\mathrm{Spin}_1} \mathbb{C}$ and $P_2 \times_{\mathrm{Spin}_1} \mathbb{C}$.

In the first case,the associated bundle is a trivial bundle. Using the identification with Spin_1-equivariant maps $P_1 \to \mathbb{C}$ with sections of the associated bundle, it suffices to find such a map to produce a global section of $\pi_1 : P_1 \to S^1$. Write P_1 as the disjoint union $S_1 \coprod S_1$, with the circles parameterized by angles $\theta, \varphi \in [0, 2\pi)$. The Spin_1 action is then given by $\theta \mapsto -\varphi$ and $\varphi \mapsto -\theta$. Then the mappings $\theta \mapsto e^{i\theta}$ and $\varphi \mapsto e^{-i\varphi}$ define a Spin_1-equivariant map $P_1 \to \mathbb{C}$, giving us a trivialization of the associated bundle $P_1 \times_{\mathrm{Spin}_1} \mathbb{C}$. In addition, we see that sections of $P_1 \times_{\mathrm{Spin}_1} \mathbb{C}$ are equivalent data to maps $S^1 \to \mathbb{C}$, since once we map one of the components of P_1 into $\mathbb{C}$, this entirely determines how we need to map the other component in order to remain Spin_1-equivariant. This further allows us to identify sections of the spinor bundle with 2π-periodic functions $\mathbb{R} \to \mathbb{C}$.

In the second case, the bundle is also trivial! We again contruct a trivialization for the associated bundle by providing a Spin_1-equivariant map $\sigma : P_2 \to \mathbb{C}$. Using the parameterization of P_2 with angles $\theta \in [0, 4\pi)$, the Spin_1 action on P_2 is given by $-1 \cdot \theta = \theta + 2\pi \mod 4\pi$. Then define σ by $\sigma(\theta) = e^{i\theta/2}$. This map is Spin_1-equivariant, so it produces a trivialization of the spinor bundle $P_2 \times_{\mathrm{Spin}_1} \mathbb{C}$. In addition, we see that sections of the spinor bundle correspond with 2π-antiperiodic functions, i.e. functions $\psi : \mathbb{R} \to \mathbb{C}$ such that $f(\theta) = -f(\theta + 2\pi)$, using the fact that we parameterized P_2 with angles from $[0, 4\pi)$. Another way to write a 2π-antiperiodic map ψ is as a product $\psi(\theta) = e^{i\theta/2} f(\theta)$ for a 2π-periodic function f, which is the representation of the section with respect to the trivialization σ defined above.

The two different spin structures produced two isomorphic vector bundles, but there is still a way to distinguish between the two – their Dirac operators. The Clifford algebra $\mathrm{Cliff}_{0,1}(\mathbb{R})$ is isomorphic to $\mathbb{C}$ as an $\mathbb{R}$-algebra via the mappings $1 \mapsto 1$, and $e_1 \mapsto i$, so the Dirac operator on $\mathbb{R}$ can also be written as $i\frac{d}{dt}$. We can then use our identifications of sections of the spinor bundles with functions $\mathbb{R} \to \mathbb{C}$ to investigate the Dirac operators on each bundle. In the case of the disconnected double cover P_1, we have the identifications of sections of $P_1 \times \mathrm{Spin}_1\mathbb{C}$ with 2π-periodic functions $\mathbb{R} \to \mathbb{C}$. Then given a section ψ, we can identify it as a function $\mathbb{R} \to \mathbb{C}$, and use the Dirac operator in $\mathbb{R}$, which will again be a 2π-periodic function, giving us another section, ginving us that the Dirac operator D_1 on sections of $P_1 \times_{\mathrm{Spin}_1} \mathbb{C}$ is just $i\frac{d}{d\theta}$. For the connected double cover, we identified sections of $P_2 \times_{\mathrm{Spin}_1} \mathbb{C}$ as products $\psi(\theta) = e^{i\theta/2} f(\theta)$ for a 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$. Then applying the Dirac operator from $\mathbb{R}$ to this function, we get

$$\begin{aligned} D\psi(\theta) &= i\frac{d}{d\theta}\left(e^{i\theta/2} f(\theta)\right) \\ &= e^{i\theta/2}\frac{\partial f}{\partial \theta} - \frac{1}{2}e^{i\theta/2} \end{aligned}$$

In the local trivialization $\sigma(\theta) = e^{i\theta/2}$, the operator D_2 operates on 2π-periodic functions, just like D_1, and is given by $D_2 = i\partial_\theta - \frac{1}{2}$. In particular, the first operator D_1 has integer spectrum, and the spectrum of D_2 is the spectrum of D_1 shifted by $\frac{1}{2}$, which allows us to distinguish to two Spin structures on S^1 by their Dirac operators.