(1) How to identify a line in V with a subset of End V ?

- Ann(L) - prove its maximal

(A) - $\{$lines in $V\}$ $\longleftrightarrow$ $\{$maximal left ideals$\}$

(B) Projectors of rank 1

Tr 1, preserved und isomorphism?

Graded modules & more patterns in the table

Dirac operators (think in the graded sense) on S^1. What kind of twisting Can you impose on lifts to E'

Conjecture: Lines in V correspond to maximal left ideals of $\operatorname{End} V$

Need to show:

1. $\operatorname{Ann}(L) \subset \operatorname{End} V$ is maximal
2. $\mathbb{V}(I) = \{ v \in V \mid Av = 0 \ \forall A \in I \}$ is 1-dimensional

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1. How to show left ideal is maximal?

Suppose $\operatorname{Ann}(L) \subset I$ then I contains a unit.

If $\operatorname{Ann}(L) \subset I$, then exists $M \notin \operatorname{Ann}(L)$

Claim $\exists A \in \operatorname{Ann}(L)$ s.t. $A+M$ is invertible.

We can fix an ordered basis for V containing a nonzero elt of L as the first elt.

Then $\operatorname{Ann}(L) = \{ (0 \mid ?) \}$ in this basis. Then M will necessarily have a nonzero first col in this basis, and completing this column vector to a basis gives an invertible elt - so $\mathrm{Ann}(L)$ is maximal

(2) Let I be a maximal left ideal

WTS $\mathbb{V}(I)$ is 1-dimensional

Cannot be 0 dimensional, since this would imply I contains a unit.

Suppose $\mathbb{V}(I)$ is ≥ 2 dimensional.

$\Downarrow$

then no matrix in I vanishes on a smaller subspace of $\mathbb{V}(I)$

$\Rightarrow$ I is contained in the ideal of matrices vanishing on a proper subspace

So we have

$$L \xmapsto{\varphi} \operatorname{Ann}(L)$$

$$\{\text{Lines in } V\} \rightleftarrows \left\{\begin{matrix}\text{Maximal ideals of} \\ \operatorname{End} V\end{matrix}\right\}$$

$$\mathbb{V}(I) \xleftarrow{\psi} I$$

$\psi \circ \varphi$ is clearly identity.

$\varphi \circ \psi$ $\quad \operatorname{Ann}(\mathbb{V}(I)) = I$?

$I \subset \operatorname{Ann}(\mathbb{V}(I))$, so yes

So φ & ψ are inverses!

We have a right action

$$\{\text{Maximal left Ideals}\} \curvearrowleft GL(V)$$

$$I \cdot A = \{MA \mid M \in I\}$$

Clearly a group action if $I \cdot A$ is maximal given that I is maximal.

Since I is maximal, we can treat it as $\operatorname{Ann}(L)$ for $L \subset V$.

Then for $M \in Ann(L)$,

$M \cdot A \in Ann(A^{-1}(L))$ and this is invertible,

so A maps $Ann(L)$ bijectively to

$Ann(A^{-1}(L))$, so this is a

right action.

Every maximal left ideal is a codimension

n-subspace (Just check in a basis)

So we get a map

$$\{\text{Maximal left ideals}\} \to \{Gr_{n^2-n}(M_n \mathbb{F})\}$$

This should pick out a subspace diffeomorphic to $\mathbb{FP}'$

Dirac Operators on S^1

We have $\mathbb{R}$ with the standard inner product and $\mathbb{Z} \subset \mathbb{R}$.

This makes $\mathbb{R}/\mathbb{Z} \cong S^1$ a Riemannian manifold.

On $\mathbb{R}$, we have the Dirac operator

$D = e^1 \frac{\partial}{\partial x}$, which is a first order differential operator $C^\infty(\mathbb{R}, \mathrm{Cliff}_{0,1}) \to C^\infty(\mathbb{R}, \mathrm{Cliff}_{0,1})$

The irreducible clifford module M for $\mathrm{Cliff}_{0,1}$ is itself with the left action. Grading is

$$M = M^0 \oplus M^1 \quad \text{when} \quad \begin{aligned} M^0 &= \mathrm{span}\{1\} \\ M^1 &= \mathrm{span}\{e_1\} \end{aligned}$$

We know $\mathrm{Spin}_1 \cong \mathbb{Z}/2\mathbb{Z}$, so we only get two spin structures on S^1, namely the

1. trivial bundle $S^1 \times \mathrm{Spin}_1$ and the
2. double cover $S^1 \to S^1$

The spin representation is the sign rep $\{\pm 1\} \curvearrowright \mathbb{C}$

Since Spin_1 is 1-dimensional, there is a unique connection on both bundles, which is just the 0 form

① The associated spinor bundle is

$$(S^1 \times \mathrm{Spin}_1) \times_{\mathrm{Spin}_1} \mathbb{C}$$

$$\|$$

 -1

$$(S^1 \times \mathrm{Spin}_1) \times M \big/ (p, \pm 1, v) \sim (p, \mp 1, -v)$$

 $+1$

$p \longmapsto [p, 1, e_1]$ is a global section

② The associated spinor bundle to the nontrivial bundle $\Pi \to S^1$

$$\Pi \times_{\mathrm{Spin}_1} M = \Pi \times M \big/ (e^{i\theta}, z) \sim (e^{i(\theta+\pi)}, -z)$$

① How does D descend to the spinor bundle

$$(S^1 \times Spin_1) \times M$$

$\mathbb{R}$ covers S^1 via $\theta \overset{\varphi}{\longmapsto} e^{i\theta}$

Given $f \in C^\infty(S^1, M)$, this uniquely determines a 2π-periodic function $\tilde{f} : \mathbb{R} \to M$.

Then define D_1 by

$$\widetilde{D_1 f} = D\tilde{f}$$

$$D_1 = e^{i}\frac{\partial}{\partial \theta}$$

② How does D descend to the spinor bundle $S = \Pi \times_{Spin_1} M$?

Let $\varphi : S^1 \to S$ be a section.

$$\begin{tikzcd}
\mathbb{R} \arrow[r, dashed, "?"] \arrow[d] & \mathbb{R} \times M \arrow[d] \\
S^1 \arrow[r, "\varphi"'] \arrow[ur, dashed] & \Pi \times_{Spin_1} M
\end{tikzcd}$$

$$\Pi \times_{Spin_1} M = \Pi \times M \Big/ (e^{i\theta}, z) \overset{?}{} (e^{i(\theta+\pi)}, -z)$$

A section σ of $\mathbb{T} \times_{spin_1} M$ should be equivalent data to a $spin_1$ equivariant map

$\tilde{\sigma} : \mathbb{T} \longrightarrow M$. How?

define $\sigma(m) = [\pi^{-1}(m), \tilde{\sigma}(\pi^{-1}(m))]$

By how $\mathbb{T} \times_{spin_1} M$ is defined and the equivariance of σ, this is well-defined on fibers.

<u>Ex.</u> Consider the $Spin_1$ equivariant map

$$\tilde{\sigma} : \mathbb{T} \longrightarrow M$$

$$\tilde{\sigma}(e^{i\theta}) = \cos\theta + e^{i}\sin\theta = e^{i\theta}$$

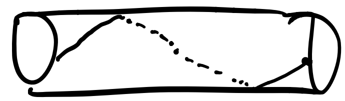
The corresponding section of $\mathbb{T} \times_{spin_1} M$ is

$$\sigma(e^{i\theta}) = [e^{i\theta/2}, e^{i\theta/2}] = [e^{i(\theta/2 + \pi)}, e^{i(\frac{\theta}{2} + \pi)}]$$

$\mathbb{T} \times_{spin_1} M$ should look like a open solid torus withe a twist

The section going along the center is the 0 section

What is σ?



A section of $\mathbb{T} \times_{\text{spin}} M$ is equivalent to a 2π periodic $f: \mathbb{R} \to M$ with $f(\theta + \pi) = -f(\theta)$

We can then do Df and push it down

What does D look like after we trivialize?

trivializing just untwists the torus.

$$\begin{tikzcd} \mathbb{T} \times_{\text{spin}} M \arrow[r, "\varphi"] \arrow[dr] & S^1 \times M \arrow[d] \\ & S^1 \end{tikzcd}$$

$$\underset{\substack{\downarrow \\ e^{i2\theta}}}{\varphi[e^{i\theta}, z]} = \left(\underset{\substack{\downarrow \\ e^{i2\theta}}}{e^{i2\theta}}, e^{i\theta} z \right)$$ is welldefined!

$$[e^{i\theta}, z] \longmapsto (e^{i2\theta}, e^{i\theta}z)$$

$$\wr$$

$$[e^{i(\theta+\pi)}, -z] \mapsto (e^{i2\theta}, -e^{i\theta}z) \quad \checkmark$$

How does D look after this trivialization?