Thesis

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CHAPTER 1

Preliminaries

1. Clifford Algebras

Definition 1.1. Let V be a real finite dimensional vector space with a nondegenerate symmetric bilinear form $b : V \times V \to \mathbb{R}$. Then the ***Clifford Algebra*** of V is the data of a unital associative $\mathbb{R}$-algebra $\mathrm{Cliff}(V, b)$ and a linear map $i : V \to \mathrm{Cliff}(V, b)$ satisfying the following universal property: Given any linear map $\varphi : V \to A$ of V into any unital associative $\mathbb{R}$-algebra A satisfying $\varphi(v)^2 = b(v, v)$, there exists a unique map $\tilde{\varphi} : \mathrm{Cliff}(V, b) \to A$ such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[d, "i"'] \arrow[dr, "\varphi"] & \\ \mathrm{Cliff}(V,b) \arrow[r, "\tilde{\varphi}"'] & A \end{tikzcd}$$

Explicitly, we can construct $\mathrm{Cliff}(V, b)$ as a quotient of the tensor algebra

$$\mathcal{T}(V) = \bigoplus_{n \in \mathbb{Z}^{\geq 0}} V^{\otimes n}$$

by the left ideal generated by the elements $v \otimes v - b(v, v)$, and the map $i : V \to \mathcal{T}(V)/(v \otimes v - b(v, v))$ is given by the inclusion $V \hookrightarrow \mathcal{T}(V)$ followed by the quotient map. From this definition, it follows that the map i is injective, and we can identify V with it's image $i(V)$ as a subspace of $\mathrm{Cliff}(V, b)$. Using this identification, we can use the polarization identity to derive the useful relation

$$vw + wv = 2b(v, w)$$

In particular, if $b(v, w) = 0$, this implies that $vw = -wv$.

After fixing a basis for V, the bilinear form b will be given by a symmetric invertible matrix B, which is conjugate to a diagonal matrix where all the diagonal entries are either 1 or -1. If B has p positive entries and q negative entries, we say b has signature (p, q). Given a bilinear form b with signature (p, q), we then see that V admits a basis $\{e_i\}$ satisfying

(1) For $1 \leq i \leq p$, we have $b(e_i, e_i) = 1$
(2) For $p + 1 \leq j \leq p + q$, we have $b(e_j, e_j) = -1$
(3) For $i \neq j$, we have $b(e_i, e_j) = 0$

Any such basis then determines an isomorphism $(V, b) \to \mathbb{R}^{p|q}$, where $\mathbb{R}^{p|q}$ denotes $\mathbb{R}^{p+q}$ with the bilinear form given by the matrix

$$\begin{pmatrix} \mathrm{id}_{\mathbb{R}^p} & 0 \\ 0 & -\mathrm{id}_{\mathbb{R}^q} \end{pmatrix}$$

In addition, we get a basis for $\mathrm{Cliff}(V, b)$, given by

$$\left\{ e_{i_1} e_{i_2} \cdots e_{i_k} \ : \ 0 \leq k \leq n,\ 1 \leq i_j \leq n \right\}$$

where we interpret the product of 0 basis vectors to be the unit element 1. which then implies the dimension of $\mathrm{Cliff}(V, b)$ as a vector space is $2^{\dim V}$. This basis also determines an isomorphism $\mathrm{Cliff}(V, b) \to \mathrm{Cliff}_{p,q}(\mathbb{R})$, where $\mathrm{Cliff}_{p,q}(\mathbb{R})$ is the Clifford algebra for $\mathbb{R}^{p|q}$. Finally, this basis allows us to see that $\mathrm{Cliff}(V, b)$ has a $\mathbb{Z}/2\mathbb{Z}$ grading

$$\mathrm{Cliff}(V, b) = \mathrm{Cliff}^0(V, b) \oplus \mathrm{Cliff}^1(V, b)$$

Where $\mathrm{Cliff}^0(V,b)$ is the $\mathbb{R}$-span of all even products of basis vectors, and $\mathrm{Cliff}^1(V,b)$ is the $\mathbb{R}$-span of all odd products of basis vectors. In particular, $\mathrm{Cliff}^0(V,b)$ forms a subalgebra, and is called the ***even subalgebra***, and we say that the elements of $\mathrm{Cliff}^0(V,b)$ are even. We then call $\mathrm{Cliff}^1(V,b)$ the odd subspace, and say that its elements are odd. Elements that are contained in the odd or even subspace are called ***homogeneous***, and given a homogeneous element $a \in \mathrm{Cliff}(V,b)$, we define it's ***parity*** $|a|$ as

$$|a| = \begin{cases} 0 & a \in \mathrm{Cliff}^0(V,b) \\ 1 & a \in \mathrm{Cliff}^1(V,b) \end{cases}$$

There is an extremely nice relationship between a Clifford algebra and it's even subalgebra.

THEOREM 1.2. *The even subalgebra* $\mathrm{Cliff}^0_{p,q}(\mathbb{R})$ *is isomorphic to both* $\mathrm{Cliff}_{p-1,q}$ *and* $\mathrm{Cliff}_{p,q-1}$ *as ungraded algebras (as long as $p-1>0$ and $q-1>0$.)*

PROOF. Fix a basis $\left\{e_1^+, \dots e_p^+, e_1^- \dots e_q^-\right\}$ for $\mathbb{R}^{p|q}$, where $(e_i^+)^2 = 1$ and $(e_i^-)^2 = -1$. Then a quick computation shows that

$$\begin{aligned}(e_i^+e_j^+)^2 &= -(e_i^+)^2(e_j^+)^2 = -1 \\ (e_i^-e_j^-)^2 &= -(e_i^-)^2(e_j^-)^2 = -1 \\ (e_i^+e_j^-)^2 &= -(e_i^+)^2(e_i^-)^2 = 1 \\ (e_i^-e_j^+)^2 &= -(e_i^-)^2(e_j^+)^2 = 1\end{aligned}$$

Assume $q \neq 0$. Then a generating set for $\mathrm{Cliff}^0_{p,q}(\mathbb{R})$ is

$$\left\{e_1^-e_j^+ \;:\; 1 \le j \le p\right\} \cup \left\{e_1^-e_k^- \;:\; 2 \le k \le q\right\}$$

All the elements in the first set square to 1, and all the elements in the second set square to -1. We then get an isomorphism $\mathrm{Cliff}^0_{p,q}(\mathbb{R}) \to \mathrm{Cliff}_{p,q-1}$ via the mappings

$$\begin{aligned}e_1^-e_j^+ &\mapsto e_j^+ \\ e_1^-e_k^- &\mapsto e_{k-1}^-\end{aligned}$$

In the case $p \neq 0$, we can give a similar isomorphism, with all the plus signs swapped with the minus signs. ■

Given two $\mathbb{R}$-algebras A and B, we can form their tensor product $A \otimes B$, which has $A \otimes B$ as the underlying vector space, and the multiplication is defined as

$$(a \otimes b)(c \otimes d) = ac \otimes bd$$

In the case that both A and B are $\mathbb{Z}/2\mathbb{Z}$ graded algebras, we have an alternate version of the tensor product, where the underlying vector space is also $A \otimes B$, but the multiplication is given by

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|}(ac \otimes bd)$$

We see that in the multiplication, we are formally commuting the elements of b and c, and we want to introduce a sign whenever elements are moved past each other. This is the ***Koszul sign rule***.

One remarkable fact is that Clifford algebras are closed under the graded tensor product, i.e. the graded tensor products of two Clifford algebras is another Clifford algebra. For the remainder of this section, we will let $\otimes$ denote the graded tensor product.

THEOREM 1.3. $\mathrm{Cliff}_{p+t,q+s}(\mathbb{R}) \cong \mathrm{Cliff}_{p,q}(\mathbb{R}) \otimes \mathrm{Cliff}_{t,s}(\mathbb{R})$

PROOF. To give a map $\varphi : \mathrm{Cliff}_{p+t,q+s}(\mathbb{R}) \to \mathrm{Cliff}_{p,q}(\mathbb{R}) \otimes \mathrm{Cliff}_{t,s}(\mathbb{R})$, it is sufficient to specify it's action on $\mathbb{R}^{p+t|q+s}$, and checking that the Clifford relations hold. Let $\left\{b_1^+, \dots, b_{p+t}^+, b_1^-, \dots b_{q+s}^-\right\}$ denote the standard orthogonal basis for $\mathbb{R}^{p+t|q+s}$ where $(b_i^+)^2 = 1$ and $(b_i^-)^2 = -1$. We then define the bases $\{e_i^\pm\}$ and

$\{f_i^{\pm}\}$ analogously for $\mathbb{R}^{p|q}$ and $\mathbb{R}^{t|s}$ respectively. Then define φ by

$$\varphi(b_i^+) = \begin{cases} e_i^+ \otimes 1 & 1 \leq i \leq p \\ 1 \otimes f_i^+ & p+1 \leq i \leq p+t \end{cases}$$

$$\varphi(b_i^-) = \begin{cases} e_i^- \otimes 1 & 1 \leq i \leq q \\ 1 \otimes f_i^- & q+1 \leq i \leq q+s \end{cases}$$

This map is injective on generators, so if we show that this satisfies the Clifford relations, then the map given by extending the map to all of $\mathrm{Cliff}_{p+t,q+s}(\mathbb{R})$ will be an isomorphism by dimension reasons. Showing the Clifford relations amounts to showing

(1) $\varphi(b_i^+)^2 = 1$
(2) $\varphi(b_i^-)^2 = -1$
(3) The images of any pair of distinct basis vectors anticommute.

The first two are relations are clear from how we defined φ. To show that the images of distinct basis vectors anticommute, there are serveral cases to consider. Given b_i^+ and b_j^+ where $1 \leq i, j \leq p$, they anticommute, because e_i^+ and e_j^+ anticommute. In the case where $1 \leq i \leq p$ and $p+1 \leq j \leq p+t$, we compute

$$\begin{aligned} \varphi(b_i^+)\varphi(b_j^+) + \varphi(b_j^+)\varphi(b_i+) &= (e_i^+ \otimes 1)(1 \otimes f_j^+) + (1 \otimes f_j^+)(e_i^+ \otimes 1) \\ &= e_i^+ \otimes f_j^+ - e_i^+ \otimes f_j^+ \end{aligned}$$

where we use the Koszul sign rule for the second term, noting that f_j+ and e_i^+ are both odd. The proof that the images of the b_i^- anti commute with each other, as well as the proof that the images of the b_i^+ and b_i^- anticommute are exactly the same. ■

Because of this theorem, once we compute a few of the lower dimensional Clifford algebras, we will have enough data to fully classify all Clifford algebras over $\mathbb{R}$. In the case $p = q = 0$, we let $\mathrm{Cliff}_{0,0}(\mathbb{R}) = \mathbb{R}$.

Example 1.4 (***Some low dimensional examples***).

(1) As ungraded algebras, the Clifford algebra $\mathrm{Cliff}_{0,1}(\mathbb{R})$ is isomorphic to $\mathbb{C}$, where the isomorphism is given by $e_1 \mapsto i$.
(2) As ungraded algebras, $\mathrm{Cliff}_{0,2}(\mathbb{R})$ is isomorphic to the quaternions $\mathbb{H}$, where the isomorphism is given by $e_1 \mapsto i$ and $e_2 \mapsto j$.
(3) As a graded algebra, $\mathrm{Cliff}_{1,1}(\mathbb{R})$ is isomorphic to $\mathrm{End}(\mathbb{R}^{1|1})$. The isomorphism is given by

$$e_1^+ \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad e_1^- \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(4) As ungraded algebras $\mathrm{Cliff}_{1,0}(\mathbb{R})$ is isomorphic to the product algebra $\mathbb{R} \times \mathbb{R}$, where $e_1 \mapsto (1, -1)$.
(5) As ungraded algebras, $\mathrm{Cliff}_{2,0}(\mathbb{R})$ is isomorphic to $M_2\mathbb{R}$. The isomorphism is given by

$$e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad e_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

To classify all Clifford algebras as ungraded algebras, it suffices to know the following table:

$M_8\mathbb{C}$	$M_8\mathbb{H}$	$M_8\mathbb{H} \times M_8\mathbb{H}$	$M_{16}\mathbb{H}$	$M_{32}\mathbb{C}$	$M_{64}\mathbb{R}$	$M_{64}\mathbb{R} \times M_{64}\mathbb{R}$	$M_{128}\mathbb{R}$
$M_4\mathbb{H}$	$M_4\mathbb{H} \times M_4\mathbb{H}$	$M_8\mathbb{H}$	$M_{16}\mathbb{C}$	$M_{32}\mathbb{R}$	$M_{32}\mathbb{R} \times M_{32}\mathbb{R}$	$M_{64}\mathbb{R}$	$M_{64}\mathbb{C}$
$M_2\mathbb{H} \times M_2\mathbb{H}$	$M_4\mathbb{H}$	$M_8\mathbb{C}$	$M_{16}\mathbb{R}$	$M_{16}\mathbb{R} \times M_{16}\mathbb{R}$	$M_{32}\mathbb{R}$	$M_{32}\mathbb{C}$	$M_{32}H$
$M_2\mathbb{H}$	$M_4\mathbb{C}$	$M_8\mathbb{R}$	$M_8\mathbb{R} \times M_8\mathbb{R}$	$M_{16}\mathbb{R}$	$M_{16}\mathbb{C}$	$M_{16}\mathbb{H}$	$M_{16}\mathbb{H} \times M_{16}\mathbb{H}$
$M_2\mathbb{C}$	$M_4\mathbb{R}$	$M_4\mathbb{R} \times M_4\mathbb{R}$	$M_8\mathbb{R}$	$M_8\mathbb{C}$	$M_8\mathbb{H}$	$M_8\mathbb{H} \times M_8\mathbb{H}$	$M_{16}\mathbb{H}$
$M_2\mathbb{R}$	$M_2\mathbb{R} \times M_2\mathbb{R}$	$M_4\mathbb{R}$	$M_4\mathbb{C}$	$M_4\mathbb{H}$	$M_4\mathbb{H} \times M_4\mathbb{H}$	$M_8\mathbb{H}$	$M_{16}\mathbb{C}$
$\mathbb{R} \times \mathbb{R}$	$M_2\mathbb{R}$	$M_2\mathbb{C}$	$M_2\mathbb{H}$	$M_2\mathbb{H} \times M_2\mathbb{H}$	$M_4\mathbb{H}$	$M_8\mathbb{C}$	$M_{16}\mathbb{R}$
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \times \mathbb{H}$	$M_2\mathbb{H}$	$M_4\mathbb{C}$	$M_8\mathbb{R}$	$M_8\mathbb{R} \times M_8\mathbb{R}$

To read the table, the bottom right entry is $\mathrm{Cliff}_{0,0} \cong \mathbb{R}$, and moving to the right increments the signature from (p, q) to $(p, q+1)$, and moving up increments the signature (p, q) to $(p+1, q)$. Any other Clifford algebra can be obtained from an algebra on this table by tensoring with $M_{16}\mathbb{R}$, since incremeting the signature by 8 (by adding to either p or q) results in tensoring with $M_{16}\mathbb{R}$.

Definition 1.5. A (left) ***Clifford module*** for the Clifford algebra $\mathrm{Cliff}_{p,q}(\mathbb{R})$ is a module for $\mathrm{Cliff}_{p,q}(\mathbb{R})$ in the usual sense i.e. a real vector space V equipped with an algebra action $\bullet : \mathrm{Cliff}_{p,q} \times V \to V$ satisfying

(1) Every element of $\mathrm{Cliff}_{p,q}(\mathbb{R})$ acts linearly on V.
(2) $(AB) \cdot v = A \cdot (B \cdot v)$ for all $v \in V$.
(3) $(A + B) \cdot v = A \cdot v + B \cdot V$ for all $v \in V$.

Equivalently, it is the data of a real vector space V and a homomorphism $\mathrm{Cliff}_{p,q}(\mathbb{R}) \to \mathrm{End}(V)$.

Definition 1.6. A Clifford module is ***irreducible*** if there exist no proper nontrivial submodules.

From the classification of Clifford algebras, all the Clifford algebras are either matrix algebras $M_n\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, or products $M_n\mathbb{F} \times M_n\mathbb{F}$ of two copies of the same matrix algebra. This is sufficient to conclude that Clifford algebras are semisimple, so all Clifford modules will be direct sums of irreducible modules. Therefore, classifying all Clifford modules reduces to classifying the irreducible Clifford modules.

Theorem 1.7. *Let $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Then any nontrivial irreducible module for $M_n\mathbb{F}$ is isomorphic to $\mathbb{F}^n$ with the standard action.*

Proof. We first note that $M_n\mathbb{F}$ acts transitively on $\mathbb{F}^n$, which implies that it is irreducible. We then must show that $\mathbb{F}^n$ is, up to isomorphism, the only irreducible $M_n\mathbb{F}$ module. The matrix algebra $M_n\mathbb{F}$ admits an increasing chain of left ideals

$$0 = I_0 \subset I_1 \subset \ldots \subset I_n = M_n\mathbb{F}$$

where I_k is the set of matrices where only the first k columns are nonzero. These ideals have the property that the quotient I_k/I_{k-1} is isomorphic to $\mathbb{F}^n$ as a left $M_n\mathbb{F}$ module. Then let M be some nontrivial irreducible $M_n\mathbb{F}$ module, and fix $m \in M$. Then the orbit $M_n\mathbb{F} \cdot m$ of m under the algebra action is a nonzero submodule, so it must be all of M. Then the map $\varphi : M_n\mathbb{F} \to M$ given by $A \mapsto A \cdot m$ is a surjective map of left $M_n\mathbb{F}$ modules. Then there must exist some smallest k such that $\varphi(I_k)$ is nonzero, and by construction, $\varphi|_{I_k}$ factors through the quotient I_k/I_{k-1}, which is isomorphic to $\mathbb{F}^n$ with the standard action. Then since $\mathbb{F}^n$ is irreducible, this gives us a nontrivial map between irreducible modules, which is an isomorphism by Shur's Lemma. ∎

Theorem 1.8. *Any nontrivial irreducible module for $M_n\mathbb{F} \times M_n\mathbb{F}$ is isomorphic to either $\mathbb{F}^n$ where the left factor acts in the usual way, and the right factor acts by 0, or $\mathbb{F}^n$ where the left factor acts by 0 and the right factor acts in the usual way.*

Proof. Let R denote $\mathbb{F}^n$ where the right factor acts nontrivially, and let L denote $\mathbb{F}^n$ where the left factor acts nontrivially. Both L and R are irreducible since $M_n\mathbb{F} \times M_n\mathbb{F}$ acts transitively on them. To show that they are the only irreducible modules up to isomorphism, we use a similar techinique as above. Let I_k denote the chain of increasing ideals in $M_n\mathbb{F}$, as we used above. Then $M_n\mathbb{F} \times M_n\mathbb{F}$ admits a chain of increasing left ideals J_k

$$0 = J_0 \subset I_1 \times \{0\} \subset \ldots \subset M_n\mathbb{F} \times \{0\} \subset M_n\mathbb{F} \times I_1 \subset \ldots \subset M_n\mathbb{F} \times M_n\mathbb{F} = J_{2n}$$

We note that for $1 \leq k \leq n$, we have that J_k/J_{k-1} is isomorphic to L, and for $n+1 \leq k \leq 2n$, we have that J_k/J_{k-1} is isomorphic to R. Then given a nontrivial irreducible module M and a nonzero element m, we get a surjective map $\varphi : M_n\mathbb{F} \times M_n\mathbb{F}$ where $A \mapsto A \cdot m$. Like before, there exists some smallest k such that $\varphi(J_k)$ is nonzero, which then factors through to an isomorphism $J_k/J_{k-1} \to M$, so M is either isomorphic to R or L. ∎

This then gives a full classification of the irreducible ungraded Clifford modules.

2. The Spin and Pin Groups

The group of invertible elements in $\mathrm{Cliff}_{p,q}(\mathbb{R})$, denoted $\mathrm{Cliff}^{\times}_{p,q}(\mathbb{R})$ contains a group $\mathrm{Pin}_{p,q}$, which a double cover of the group $O_{p,q}$ of matrices preserving the standard bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{p|q}$. Inside of $\mathrm{Pin}_{p,q}$, there exists a subgroup $\mathrm{Spin}_{p,q} \subset \mathrm{Pin}_{p,q}$, which double covers the group $SO_{p,q}$, which consists of the subgroup of $O_{p,q}$ with determinant equal to 1.

Definition 2.1. The ***Pin group*** $\mathrm{Pin}_{p,q}$ is the subgroup of $\mathrm{Cliff}^{\times}_{p,q}(\mathbb{R})$ generated by the set

$$\left\{v \in \mathbb{R}^{p,q} \ : \ v^2 = \pm 1\right\}$$

The ***Spin group*** $\mathrm{Spin}_{p,q}$ is the subgroup of $\mathrm{Pin}_{p,q}$ generated by even products of basis vectors. In other words, $\mathrm{Spin}_{p,q} = \mathrm{Pin}_{p,q} \cap \mathrm{Cliff}^0_{p,q}(\mathbb{R})$.

To show that the Pin and Spin groups cover $O_{p,q}$ and $\mathrm{Spin}_{p,q}$, we make a short digression. Given a vector $v \in \mathbb{R}^{p,q}$, we can define a reflection map $R_v : \mathbb{R}^{p,q} \to \mathbb{R}^{p,q}$ given by $R_v(w) = w - 2\langle v, w\rangle v$, which will reflect across the hyperplane $v^{\perp}$.

Theorem 2.2 (***Cartan-Dieudonné***). *Any orthogonal transformation $A \in O_{p,q}$ can be written as the composition of at most $p+q$ hyperplane reflections, where we interpret the identity map as the composition of* 0 *reflections.*

Proof. We prove this by induction on $n = p+q$. The case $n = 1$ is trivial, since $O_1 = \{\pm 1\}$. Then given $A \in O_{p,q}$, fix some nonzero $v \in \mathbb{R}^{p|q}$. Then define $R : \mathbb{R}^{p|q} \to \mathbb{R}^{p|q}$ by

$$R(w) = w - 2\frac{\langle Av - v, w\rangle}{\langle Av - v, Av - v\rangle}(Av - v)$$

Then R will be a reflection about the hyperplane orthogonal to $Av - v$, and will interchange v and Av. Therefore, RA is an orthogonal transformation fixing v. Since RA is orthogonal, it will also fix the orthogonal complement $v^{\perp}$, so it will restrict to an orthgonal transformation on $v^{\perp}$. We then see that $v^{\perp}$ is 1 dimension lower thant $\mathbb{R}^{p|q}$, and restricting the bilinear form to $v^{\perp}$, we know by the inductive hypotehsis that $RA|_{v^{\perp}}$ can be written as at most $n-1$ hyperplane reflections in $v^{\perp}$. Since RA fixes v, we can extend all of these transformations to a hyperplane reflection on all of $\mathbb{R}^{p|q}$, by taking the span of each hyperplane wiht v, giving us that RA is a composition of at most $n-1$ reflections. Finally, composing RA with R gives us that A can be written as a composition of at most n hyperplane reflections. ∎

The Cartan-Dieudonné theorem will be the central piece for showing that the Pin and Spin groups cover the orthogonal groups.

Theorem 2.3. *There exist* 2*-to-*1 *group homomorphisms* $\mathrm{Pin}_{p,q} \to O_{p,q}$ *and* $\mathrm{Spin}_{p,q} \to SO_{p,q}$*, i.e. there exist short exact sequences*

$$0 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Pin}_{p,q} \longrightarrow O_{p,q} \longrightarrow 0$$

$$0 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}_{p,q} \longrightarrow SO_{p.q} \longrightarrow 0$$

Proof. We first consider the case of $\mathrm{Pin}_{p,q}$. To do this, we need to construct a group action where $\mathrm{Pin}_{p,q}$ acts on $\mathbb{R}^{p|q}$ by orthogonal transformations. There exists an involution $T : \mathrm{Cliff}_{p,q}(\mathbb{R}) \to \mathrm{Cliff}_{p,q}(\mathbb{R})$, where given the standard orthogonal basis $\left\{e_1, \ldots, e_{p+1}\right\}$, we define

$$T(e_{i_1} \cdots e_{i_k}) = e_{i_k} \cdots e_{i_1}$$

and extending linearly to the remainder of $\mathrm{Cliff}_{p,q}(\mathbb{R})$. Given $a \in \mathrm{Cliff}_{p,q}(\mathbb{R})$, we denote $T(a)$ by a^T. We note that for a vector $v \in \mathbb{R}^{p|q}$, identifying $\mathbb{R}^{p|q}$ as a subspace of $\mathrm{Cliff}_{p,q}(\mathbb{R})$, satisfying $\langle v, v\rangle = \pm 1$, we have that $v^T = v$ and $v^{-1} = \pm v$. Then given $g \in \mathrm{Pin}_{p,q}$, and $v \in \mathbb{R}^{p|q}$, we claim that the left action

$$g \cdot v = -gvg^{-1}$$

defines the group action we desire. To show this, we must show that this indeed maps $\mathbb{R}^{p|q}$ back into itself, and that the group elements act by orthogonal trasnformations. It suffices to check this on the generating set of elements v with $\langle v, v\rangle = \pm 1$. First assume that $\langle v, v\rangle = 1$. Then given $w \in \mathbb{R}^{p|q}$, we compute

$$\begin{aligned} -vwv^{-1} &= -vwv \\ &= (wv - 2\langle v, w\rangle)v \\ &= w - 2\langle v, w\rangle v \end{aligned}$$

Which is hyperplane reflection about the orthogonal complement of v. In the case that $\langle v, v\rangle = -1$, we compute

$$\begin{aligned}
-vwv^{-1} &= -vw(-v) \\
&= (2\langle -v, w\rangle + wv)(-v) \\
&= w - 2\langle -v, w\rangle
\end{aligned}$$

which is hyperplane reflection about the orthogonal complement of $-v^{\perp}$, which is the same as the orthogonal complement of $v^{\perp}$. Therefore, $\mathrm{Pin}_{p,q}$ acts by orthogonal transformations, giving us a homomorphism $\mathrm{Pin}_{p,q} \to O_{p,q}$. ■