

# **Clifford Algebras and Spin Geometry**

**Senior Honors Thesis: supervised by Professor Daniel Freed**

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*The mathematician does not study pure mathematics because it is useful; he studies it because he delights in it and he delights in it because it is beautiful.*

– **Henri Poincaré**

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## Introduction



*Every morning in the early part of October 1843, on my coming down to breakfast, your brother William Edwin and yourself used to ask me: "Well, Papa, can you multiply triples?" Where to I was always obliged to reply, with a sad shake of the head, "No, I can only add and subtract them."*

– **William Rowan Hamilton**

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On October 16, 1843, William Rowan Hamilton famously carved the equations

$$i^2 = j^2 = k^2 = ijk = -1$$

into Broom Bridge, marking his discovery of the quaternions with an act of mathematical vandalism. It was well known at the time that the algebraic operations on the complex numbers  $\mathbb{C}$  encoded a great deal of the geometry of the plane  $\mathbb{R}^2$  – addition corresponded to vector translation, and the multiplication of complex numbers corresponded to scalings and rotations. After fruitlessly attempting to construct an analogous algebraic structure on  $\mathbb{R}^3$ , Hamilton came to the realization that he needed an additional dimension to encode the rotations of  $\mathbb{R}^3$ , thereby constructing the quaternions  $\mathbb{H}$ .

Later in the 19<sup>th</sup> century, William Clifford further generalized the algebras  $\mathbb{C}$  and  $\mathbb{H}$  to higher dimensions when he constructed his geometric algebras, which now bear his name. Clifford's general construction created an  $\mathbb{R}$ -algebra that encoded the geometry of a vector space  $V$  equipped with a nondegenerate quadratic form  $Q : V \rightarrow \mathbb{R}$ . Clifford algebras would prove to play a central part in the development of geometry, topology, and physics in the 20<sup>th</sup> century.

Mathematicians quickly found uses for Clifford algebras. In the early 20<sup>th</sup> century, mathematicians were puzzled by an anomaly regarding linear representations of the special orthogonal groups  $SO_n$ . When studying the representations of the Lie algebras  $\mathfrak{so}_n$ , they realized that some of the Lie algebra representations failed to exponentiate to linear representations of the group  $SO_n$ , and instead gave rise to projective representations. The root of the problem was that the special orthogonal groups are not simply connected – for  $n > 2$ , the fundamental group  $\pi_1(SO_n)$  is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$ , indicating the existence of a simply connected double cover.<sup>1</sup> These double covers are the sources of the missing representations. At this point, Clifford algebras came to the rescue, and these double covers were realized as subgroups of the multiplicative groups of Clifford algebras. These groups were dubbed the Spin groups, and the mysterious missing representations were named the Spin representations.

In 1928, Paul Dirac was developing a relativistic theory of the electron when he realized he needed a first order differential operator that squared to the Laplacian. No first order operator

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<sup>1</sup>This can be illustrated in 3 dimensions with the well-known belt trick, in which a 360 degree rotation of a belt buckle leaves the belt twisted

with scalar coefficients was able to satisfy this, but Dirac realized the the problem could be solved with the use of matrices, and constructed his gamma matrices. Remarkably, the relations

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} I$$

satisfied by Dirac's gamma matrices are exactly the defining relations for the Clifford algebra for Minkowski spacetime  $\mathbb{R}^{1,3}$ . Using these matrices, Dirac derived his famous equation

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

where  $\psi$  was an element of the Spin representation of the group  $\text{Spin}_{1,3}$ . Spin groups and Clifford algebras would prove to be vital elements in the development of quantum field theory.

Not to be outdone by the physicists, mathematicians also made extensive use of Clifford algebras and Spin groups to produce astounding results in geometry and topology. Through the work of many prominent mathematicians such as Michael Atiyah, Isadore Singer, and Raoul Bott, Clifford algebras and Spin groups saw remarkable applications in fields such as  $K$ -theory, homotopy theory, and gauge theory, producing numerous seminal theorems like the Atiyah-Singer index theorem and the Bott periodicity theorem. In particular, Atiyah and Singer generalized Dirac's differential operator to manifolds equipped with a Spin structure – the Dirac operator. The methods and tools developed would prove to be fundamental to the rapid advancement in the fields of geometry and topology at the time, and remain important to this day.

In this thesis we aim to explore and develop some of the theory of Clifford algebras, Spin and Pin groups, and Spin/Pin structures on manifolds. In particular, we plan to construct and classify Clifford algebras and their modules, and transport these constructions to the nonlinear world of manifolds. We will develop some of the general theory for Spin and Pin structures on manifolds, and will give several examples.



## CHAPTER 1

# Clifford Algebras and Spin Groups



*No one fully understands spinors. Their algebra is formally understood, but their geometrical significance is mysterious. In some sense they describe the “square root” of geometry and, just as understanding the concept of  $\sqrt{-1}$  took centuries, the same might be true of spinors.*

– *Sir Michael Atiyah*



### 1. Clifford Algebras

**DEFINITION 1.1.** Let  $V$  be a real finite dimensional vector space with a nondegenerate symmetric bilinear form  $b : V \times V \rightarrow \mathbb{R}$ . Then the **Clifford Algebra** of  $V$  is the data of a unital associative  $\mathbb{R}$ -algebra  $\text{Cliff}(V, b)$  and a linear map  $i : V \rightarrow \text{Cliff}(V, b)$  satisfying the following universal property: Given any linear map  $\varphi : V \rightarrow A$  of  $V$  into any unital associative  $\mathbb{R}$ -algebra  $A$  satisfying  $\varphi(v)^2 = b(v, v)$ , there exists a unique algebra homomorphism  $\tilde{\varphi} : \text{Cliff}(V, b) \rightarrow A$  such that the following diagram commutes:

$$\begin{array}{ccc} V & & \\ \downarrow i & \searrow \varphi & \\ \text{Cliff}(V, b) & \xrightarrow{\tilde{\varphi}} & A \end{array}$$



This universal property uniquely characterizes the Clifford algebra  $\text{Cliff}(V, b)$  up to unique isomorphism.

**THEOREM 1.2.** *The Clifford algebra is unique up to unique isomorphism, i.e. given another unital associative algebra  $C$  equipped with a linear map  $j : V \rightarrow C$  satisfying the universal property, there exists a unique algebra isomorphism  $\varphi : \text{Cliff}(V, b) \rightarrow C$ .*

**PROOF.** Given such an algebra  $C$  with a linear map  $j : V \rightarrow C$ , the map  $j$  satisfies the Clifford relation  $j(v)^2 = b(v, v)$ , so it induces a unique algebra homomorphism  $\varphi : \text{Cliff}(V, b) \rightarrow C$ . The linear map  $i : V \rightarrow \text{Cliff}(V, b)$  also satisfies the Clifford relation, so it induces an algebra homomorphism  $\psi : C \rightarrow \text{Cliff}(V, b)$ . We claim that these maps are inverses. To do so, we show the compositions  $\varphi \circ \psi$  and  $\psi \circ \varphi$  are identity. Using the universal property of the Clifford algebra once more, the map  $i$  induces a unique map  $\text{Cliff}(V, b) \rightarrow \text{Cliff}(V, b)$  such that

$$\begin{array}{ccc} V & & \\ \downarrow i & \searrow i & \\ \text{Cliff}(V, b) & \longrightarrow & \text{Cliff}(V, b) \end{array}$$

commutes. The identity map makes this diagram commute, so by uniqueness, this is the induced map. The map  $\psi \circ \varphi$  also makes this diagram commute, so it must be identity by uniqueness. An identical proof shows that  $\varphi \circ \psi$  is the identity map  $\text{id}_C$ . ■

Explicitly,  $\text{Cliff}(V, b)$  is realized as a quotient of the tensor algebra

$$\mathcal{T}(V) = \bigoplus_{n \in \mathbb{Z}^{\geq 0}} V^{\otimes n}$$

where we quotient by the two-sided ideal generated by elements of the form  $v \otimes v - b(v, v)$ , and the linear map  $i : V \rightarrow \mathcal{T}(V)/(v \otimes v - b(v, v))$  is given by the inclusion  $V \hookrightarrow \mathcal{T}(V)$  followed by the quotient map. We identify  $V$  with its image  $i(V)$  as a subspace of  $\text{Cliff}(V, b)$ .

**THEOREM 1.3.** *The Clifford algebra is functorial in the following way: Given vector spaces  $V$  and  $V'$  equipped with nondegenerate symmetric bilinear forms  $b$  and  $b'$  respectively and a linear map  $T : V \rightarrow V'$  such that  $b(v, w) = b'(Tv, Tw)$  for all  $v, w \in V$ , there exists a unique algebra homomorphism  $T_* : \text{Cliff}(V, b) \rightarrow \text{Cliff}(V', b')$  such that*

$$\begin{array}{ccc} V & \xrightarrow{T} & V' \\ i \downarrow & & \downarrow i' \\ \text{Cliff}(V, b) & \xrightarrow{T_*} & \text{Cliff}(V', b') \end{array}$$

commutes, where  $i$  and  $i'$  are the inclusions of  $V$  and  $V'$  into their respective Clifford algebras.

**PROOF.** We use the realization of  $\text{Cliff}(V, b)$  as the quotient

$$\text{Cliff}(V, b) = \mathcal{T}(V)/(v \otimes v - b(v, v))$$

The map  $T$  induces a map  $\tilde{T} : \mathcal{T}(V) \rightarrow \mathcal{T}(V')$  where the action on homogeneous elements is given by

$$\tilde{T}(v_1 \otimes \cdots \otimes v_k) = Tv_1 \otimes \cdots \otimes Tv_k$$

The fact that  $b(v, w) = b'(Tv, Tw)$  implies that  $\tilde{T}$  maps the ideal  $(v \otimes v - b(v, v))$  to the ideal  $(v' \otimes v' - b'(v', b'))$ , so the map  $\tilde{T}$  descends to the quotients, giving a map  $T_* : \text{Cliff}(V, b) \rightarrow \text{Cliff}(V', b')$ . The fact that the diagram commutes follows from the fact that the maps  $i$  and  $i'$  are the inclusions of  $V$  and  $V'$  into their respective tensor algebras, followed by the quotient maps into  $\text{Cliff}(V, b)$  and  $\text{Cliff}(V', b')$  respectively. ■

**DEFINITION 1.4.** Define the bilinear form  $b : \mathbb{R}^{p+q} \times \mathbb{R}^{p+q} \rightarrow \mathbb{R}$  by

$$b(v, w) = \sum_{i=1}^p v^i w^i - \sum_{i=p+1}^{p+q} v^i w^i$$

where the  $v^i$  and  $w^i$  are the components of  $v$  and  $w$  with respect to the standard basis on  $\mathbb{R}^{p+q}$ . We denote the vector space  $\mathbb{R}^{p+q}$  equipped with this bilinear form as  $\mathbb{R}^{p,q}$ . ❖

Let  $V$  be a vector space equipped with a nondegenerate bilinear form  $b$  and fix a basis for  $V$ . Then the bilinear form  $b$  is given by a symmetric invertible matrix  $B$ , which is conjugate to a diagonal matrix where all the diagonal entries are either 1 or  $-1$ . If after conjugation,  $B$  has  $p$  positive entries and  $q$  negative entries, we say  $b$  has signature  $(p, q)$ . Any bilinear form  $b$  of signature  $(p, q)$  admits a basis  $\{e_i\}$  satisfying

- (1) For  $1 \leq i \leq p$ , we have  $b(e_i, e_i) = 1$
- (2) For  $p+1 \leq j \leq p+q$ , we have  $b(e_j, e_j) = -1$
- (3) For  $i \neq j$ , we have  $b(e_i, e_j) = 0$

Any such basis then determines an isomorphism  $(V, b) \rightarrow \mathbb{R}^{p,q}$ , and we call such a basis an **orthogonal basis** for  $(V, b)$ . In addition, we get a basis for  $\text{Cliff}(V, b)$  given by

$$\{e_{i_1}e_{i_2}\cdots e_{i_k} : 0 \leq k \leq n, 1 \leq i_j \leq n\}$$

where we interpret the product of 0 basis vectors to be the unit element 1. This then implies the dimension of  $\text{Cliff}(V, b)$  as a vector space is  $2^{\dim V}$ . This basis also determines an isomorphism  $\text{Cliff}(V, b) \rightarrow \text{Cliff}_{p,q}(\mathbb{R})$ , where  $\text{Cliff}_{p,q}(\mathbb{R})$  is the Clifford algebra for  $\mathbb{R}^{p,q}$ . Given  $v, w \in V$ , we write  $v$  and  $w$  in these bases as  $v^i e_i$  and  $w^j e_j$  (using Einstein summation convention), and derive the useful relation

$$\begin{aligned} vw + wv &= v^i w^j e_i e_j + v^i w^j e_j e_i \\ &= v^i w^i (e_i^2 + e_i^2) \\ &= 2b(v, w) \end{aligned}$$

where we use the fact that the  $e_i$  are orthogonal to deduce that  $e_i e_j = -e_j e_i$  if  $i \neq j$ .

**DEFINITION 1.5.**<sup>1</sup> Let  $V$  be a vector space. A  $\mathbb{Z}/2\mathbb{Z}$  **grading** on  $V$  is a direct sum decomposition  $V = V^0 \oplus V^1$ . Elements of  $V^0$  are said to be **even** and elements of  $V^1$  are said to be **odd**. Elements of the even and odd subspaces are said to be **homogeneous**. Given an homogeneous element  $v \in V$ , define its **parity**, denoted  $|v|$  by

$$|v| = \begin{cases} 0 & v \in V^0 \\ 1 & v \in V^1 \end{cases}$$

Equivalently, it is the data of a linear map  $\varepsilon : V \rightarrow V$  such that  $\varepsilon$  acts by identity on a subspace  $V^0$  of  $V$  and negative identity on a complementary subspace  $V^1$ , which gives the direct sum decomposition of  $V$  as the  $\pm 1$  eigenspaces of  $\varepsilon$ . The map  $\varepsilon$  is called the **grading operator**. ❖

**DEFINITION 1.6.** A  $\mathbb{Z}/2\mathbb{Z}$  **graded algebra**  $A$  over  $\mathbb{R}$  (often called a superalgebra) is an  $\mathbb{R}$ -algebra  $A$  equipped with a grading  $A = A^0 \oplus A^1$  such that the multiplication respects the grading, i.e. given homogeneous elements  $a, b \in A$ , their product  $ab$  is an element of  $A^{|a|+|b|}$  where the addition is done mod 2. ❖

**EXAMPLE 1.7.**

- (1) Any  $\mathbb{R}$ -algebra  $A$  can be made into a graded algebra where we let  $A^0 = A$  and  $A^1 = 0$ .
- (2) The exterior algebra  $\Lambda^\bullet V$  of a vector space  $V$  is a  $\mathbb{Z}/2\mathbb{Z}$  graded algebra (in fact, it has a  $\mathbb{Z}$  grading as well), where the grading is  $\Lambda^\bullet V = \Lambda^{\text{even}} V \oplus \Lambda^{\text{odd}} V$  where  $\Lambda^{\text{even}} V$  is the subspace spanned by even products of vectors and  $\Lambda^{\text{odd}} V$  is the subspace spanned by odd products of vectors.
- (3) Let  $V = V^0 \oplus V^1$  be a  $\mathbb{Z}/2\mathbb{Z}$  graded vector space. Then the algebra of endomorphisms  $\text{End } V$  can be endowed with the structure of a graded algebra, where the even subspace consists of linear maps preserving the decomposition  $V^0 \oplus V^1$ , and the odd subspace consists of linear maps  $T$  reversing the decomposition, i.e.  $T(V^i) = V^{i+1 \bmod 2}$ . In a ordered basis where the first elements are all even and the last elements are all odd, the even elements of  $\text{End } V$  are block diagonal, while the odd elements are block off-diagonal. ◀

For the most part, the algebras we work with will be  $\mathbb{Z}/2\mathbb{Z}$  graded, so the term “graded” may be used in lieu of “ $\mathbb{Z}/2\mathbb{Z}$  graded.” In the case of ambiguity, we will specify the grading.

<sup>1</sup>It is common in the literature to refer to  $\mathbb{Z}/2\mathbb{Z}$  graded vector spaces as super vector spaces. The “super” prefix often refers to a  $\mathbb{Z}/2\mathbb{Z}$  grading on the relevant object.

The Clifford algebra  $\text{Cliff}(V, b)$  is naturally a  $\mathbb{Z}/2\mathbb{Z}$  graded algebra. Fix a basis  $\{e_i\}$  for  $V$ . We then define the grading

$$\text{Cliff}(V, b) = \text{Cliff}^0(V, b) \oplus \text{Cliff}^1(V, b)$$

Where  $\text{Cliff}^0(V, b)$  is the  $\mathbb{R}$ -span of all even products of basis vectors, and  $\text{Cliff}^1(V, b)$  is the  $\mathbb{R}$ -span of all odd products of basis vectors. Since the product of even elements is again even, the subspace  $\text{Cliff}^0(V, b)$  forms a subalgebra, and is called the *even subalgebra*.

As  $\mathbb{Z}/2\mathbb{Z}$ -graded *vector spaces*,  $\text{Cliff}(V, b)$  is *naturally* isomorphic to the exterior algebra  $\Lambda^\bullet(V)$ . To construct the isomorphism, we first construct an algebra homomorphism  $\text{Cliff}(V, b) \rightarrow \text{End}(\Lambda^\bullet(V))$  where we map  $v \in V$  to  $\varepsilon(v) + 1/2\iota(v)$ , where

$$\begin{aligned} \varepsilon(v)(\omega) &= v \wedge \omega \\ \iota(v)(v_1 \wedge \cdots \wedge v_k) &= \sum_{i=1}^k (-1)^{k+1} \langle v, v_i \rangle v_1 \wedge \cdots \widehat{v_i} \cdots \wedge v_k \end{aligned}$$

where  $\widehat{v_i}$  indicates that the  $i^{\text{th}}$  term in the wedge product is missing. This map satisfies the Clifford relations, so it extends to a unique map  $\varphi : \text{Cliff}(V, b) \rightarrow \text{End}(\Lambda^\bullet(V))$ . Then the mapping  $v \mapsto \varphi(v)(1)$  is the desired isomorphism  $\text{Cliff}(V, b) \rightarrow \Lambda^\bullet(V)$ . In particular, if  $e_i$  is an orthogonal basis for  $V$ ,

$$\varphi(e_{i_1} e_{i_2} \cdots e_{i_k}) = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$$

Of course,  $\text{Cliff}(V, b)$  and  $\Lambda^\bullet(V)$  are noncanonically isomorphic, since they are the same dimension and the even and odd subspaces are the same dimensions, but the key point here is that the isomorphism is canonical. Since their map is not an algebra homomorphism, a useful thing to note is how multiplication in the Clifford algebra looks like in the exterior algebra under the isomorphism  $\varphi$ .

PROPOSITION 1.8. For  $v \in V$  and  $\eta \in \text{Cliff}(V, b)$

$$\varphi(v\eta) = v \wedge \eta + \iota(v)(\eta)$$

PROOF. Fix an orthogonal basis  $\{e_i^\pm\}$  where  $(e_i^+)^2 = 1$ ,  $(e_i^-)^2 = -1$ . It suffices to check that the identity holds when  $v$  is a basis vector and  $\eta$  is a product of basis vectors. We compute

$$e_i^\pm e_{i_1}^\pm \cdots e_{i_k}^\pm = \begin{cases} \pm e_{i_2} \cdots e_{i_k} & e_{i_1}^\pm = e_i^\pm \\ e_1 e_{i_1} \cdots e_{i_k} & e_{i_1}^\pm \neq e_i^\pm \end{cases}$$

In addition, we have that

$$e_i^\pm \wedge (e_{i_1}^\pm \wedge \cdots \wedge e_{i_k}^\pm) + \iota(e_i^\pm)(e_{i_1}^\pm \wedge \cdots \wedge e_{i_k}^\pm) = \begin{cases} 0 \pm e_{i_2}^\pm \wedge \cdots \wedge e_{i_k}^\pm & e_{i_1}^\pm = e_i^\pm \\ e_i^\pm \wedge e_{i_1}^\pm \wedge \cdots \wedge e_{i_k}^\pm, & e_{i_1}^\pm \neq e_i^\pm \end{cases}$$

Then using the fact that  $\varphi$  sends products of orthogonal vectors to wedges of orthogonal vectors, we see that the desired identity holds on a basis, completing the proof.  $\blacksquare$

There is an extremely nice relationship between a Clifford algebra and its even subalgebra.

THEOREM 1.9. The even subalgebra  $\text{Cliff}_{p,q}^0(\mathbb{R})$  is isomorphic to both  $\text{Cliff}_{q,p-1}$  and  $\text{Cliff}_{p,q-1}$  as ungraded algebras (as long as  $p-1 > 0$  or  $q-1 > 0$ ).

PROOF. Fix a basis  $\{e_1^+, \dots, e_p^+, e_1^-, \dots, e_q^-\}$  for  $\mathbb{R}^{p,q}$ , where  $(e_i^+)^2 = 1$  and  $(e_i^-)^2 = -1$ . We then compute

$$\begin{aligned}(e_i^+ e_j^+)^2 &= -(e_i^+)^2 (e_j^+)^2 = -1 \\ (e_i^- e_j^-)^2 &= -(e_i^-)^2 (e_j^-)^2 = -1 \\ (e_i^+ e_j^-)^2 &= -(e_i^+)^2 (e_j^-)^2 = 1 \\ (e_i^- e_j^+)^2 &= -(e_i^-)^2 (e_j^+)^2 = 1\end{aligned}$$

Assume  $q \neq 0$ . Then a generating set for  $\text{Cliff}_{p,q}^0(\mathbb{R})$  is

$$\{e_1^- e_j^+ : 1 \leq j \leq p\} \cup \{e_1^- e_k^- : 2 \leq k \leq q\}$$

All the elements in the first set square to 1, and all the elements in the second set square to  $-1$ . We then get an isomorphism  $\text{Cliff}_{p,q}^0(\mathbb{R}) \rightarrow \text{Cliff}_{p,q-1}$  via the mappings

$$\begin{aligned}e_1^- e_j^+ &\mapsto e_j^+ \\ e_1^- e_k^- &\mapsto e_{k-1}^-\end{aligned}$$

In the case where  $p \neq 0$ , we have that an equally good generating set for  $\text{Cliff}_{p,q}^0(\mathbb{R})$  is

$$\{e_1^+ e_j^+ : 2 \leq j \leq p\} \cup \{e_1^+ e_i^- : 1 \leq i \leq q\}$$

Where the elements in the first set square to  $-1$  and the elements of the second set square to 1. Then the mappings

$$\begin{aligned}e_1^+ e_j^+ &\mapsto e_{j-1}^- \\ e_1^+ e_i^- &\mapsto e_i^+\end{aligned}$$

gives the isomorphism  $\text{Cliff}_{p,q}^0(\mathbb{R}) \rightarrow \text{Cliff}_{q,p-1}$ . ■

Given two  $\mathbb{R}$ -algebras  $A$  and  $B$ , we can form their tensor product  $A \otimes B$ , which has  $A \otimes B$  as the underlying vector space, and the multiplication is defined as

$$(a \otimes b)(c \otimes d) = ac \otimes bd$$

In the case that both  $A$  and  $B$  are  $\mathbb{Z}/2\mathbb{Z}$  graded algebras, we have an alternate version of the tensor product, where the underlying vector space is also  $A \otimes B$ . The grading on the tensor product is the decomposition

$$A \otimes B = (A^0 \otimes B^0 \oplus A^1 \otimes A^1) \oplus (A^0 \otimes B^1 \oplus A^1 \otimes B^0)$$

and the multiplication of homogeneous elements is given by

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|}(ac \otimes bd)$$

We see that in the multiplication, we are formally commuting the elements of  $b$  and  $c$ , and we want to introduce a sign whenever elements are moved past each other. This is the **Koszul sign rule**. Another concept that needs a slight modification in the graded case is the opposite algebra. In the ungraded case, given an  $\mathbb{R}$ -algebra  $A$ , the **opposite algebra** is the algebra  $A^{\text{op}}$  with the same underlying vector space, but the multiplication in  $A^{\text{op}}$  is given by  $a * b = ba$ , where  $ba$  is the multiplication in  $A$ . In doing so, we are formally commuting  $a$  and  $b$ , so in the graded situation, we invoke the Koszul sign rule when defining multiplication in the opposite algebra, and define the multiplication of homogeneous elements in  $A^{\text{op}}$  to be  $a * b = (-1)^{|a||b|}ba$ .

One remarkable fact is that Clifford algebras are closed under the graded tensor product, i.e. the graded tensor products of two Clifford algebras is another Clifford algebra. Likewise, the graded opposite algebra of a Clifford algebra is again a Clifford algebra. For the remainder of this section, we will let  $\otimes$  denote the graded tensor product, and the superscript  $\text{op}$  will denote the graded opposite algebra.

**THEOREM 1.10.**  $\text{Cliff}_{p+t,q+s}(\mathbb{R}) \cong \text{Cliff}_{p,q}(\mathbb{R}) \otimes \text{Cliff}_{t,s}(\mathbb{R})$

**PROOF.** To give a map  $\varphi : \text{Cliff}_{p+t,q+s}(\mathbb{R}) \rightarrow \text{Cliff}_{p,q}(\mathbb{R}) \otimes \text{Cliff}_{t,s}(\mathbb{R})$ , it is sufficient to specify its action on  $\mathbb{R}^{p+t,q+s}$  and to check that the Clifford relations hold. Let  $\{b_1^+, \dots, b_{p+t}^+, b_1^-, \dots, b_{q+s}^-\}$  denote the standard orthogonal basis for  $\mathbb{R}^{p+t,q+s}$  where  $(b_i^+)^2 = 1$  and  $(b_i^-)^2 = -1$ . We then define the bases  $\{e_i^\pm\}$  and  $\{f_i^\pm\}$  analogously for  $\mathbb{R}^{p,q}$  and  $\mathbb{R}^{t,s}$  respectively. Then define  $\varphi$  by

$$\begin{aligned} \varphi(b_i^+) &= \begin{cases} e_i^+ \otimes 1 & 1 \leq i \leq p \\ 1 \otimes f_i^+ & p+1 \leq i \leq p+t \end{cases} \\ \varphi(b_i^-) &= \begin{cases} e_i^- \otimes 1 & 1 \leq i \leq q \\ 1 \otimes f_i^- & q+1 \leq i \leq q+s \end{cases} \end{aligned}$$

This map is injective on generators, so if we show that this satisfies the Clifford relations, the map given by extending the map to all of  $\text{Cliff}_{p+t,q+s}(\mathbb{R})$  will be an isomorphism by a dimension count. Showing the Clifford relations amounts to showing

- (1)  $\varphi(b_i^+)^2 = 1$
- (2)  $\varphi(b_i^-)^2 = -1$
- (3) The images of any pair of distinct basis vectors anticommute.

The first two are relations are clear from how we defined  $\varphi$ . To show that the images of distinct basis vectors anticommute, there are several cases to consider. Given  $b_i^+$  and  $b_j^+$  where  $1 \leq i, j \leq p$ , they anticommute, because  $e_i^+$  and  $e_j^+$  anticommute. In the case where  $1 \leq i \leq p$  and  $p+1 \leq j \leq p+t$ , we compute

$$\begin{aligned} \varphi(b_i^+)\varphi(b_j^+) + \varphi(b_j^+)\varphi(b_i^+) &= (e_i^+ \otimes 1)(1 \otimes f_j^+) + (1 \otimes f_j^+)(e_i^+ \otimes 1) \\ &= e_i^+ \otimes f_j^+ - e_i^+ \otimes f_j^+ \end{aligned}$$

where we use the Koszul sign rule for the second term, noting that  $f_j^+$  and  $e_i^+$  are both odd elements. The proof that the images of the  $b_i^-$  anticommute with each other, as well as the proof that the images of the  $b_i^+$  and  $b_i^-$  anticommute are exactly the same. ■

**THEOREM 1.11.** *The graded opposite algebra  $\text{Cliff}_{p,q}^{\text{op}}(\mathbb{R})$  is isomorphic to  $\text{Cliff}_{q,p}(\mathbb{R})$ .*

**PROOF.** Fix an orthogonal basis  $\{e_i^\pm\}$  for  $\mathbb{R}^{p,q}$ , where  $(e_i^\pm)^2 = \pm 1$ . We note that since the  $e_i^\pm$  are odd elements, they square to  $\mp 1$  in the opposite algebra. Indeed, the mapping  $e_i^\pm \rightarrow e_i^\mp$  defines the isomorphism  $\text{Cliff}_{p,q}^{\text{op}} \rightarrow \text{Cliff}_{q,p}$ . ■

Because of these theorems, once we compute a few of the lower dimensional Clifford algebras, we will have enough data to fully classify all Clifford algebras over  $\mathbb{R}$ .

**EXAMPLE 1.12** (*Some low dimensional examples*).

- (1) The Clifford algebra  $\text{Cliff}_{0,0}(\mathbb{R})$  is isomorphic to  $\mathbb{R}$ .
- (2) As ungraded algebras, the Clifford algebra  $\text{Cliff}_{0,1}(\mathbb{R})$  is isomorphic to  $\mathbb{C}$ , where the isomorphism is given by  $e_1 \mapsto i$ .
- (3) As ungraded algebras,  $\text{Cliff}_{0,2}(\mathbb{R})$  is isomorphic to the quaternions  $\mathbb{H}$ , where the isomorphism is given by  $e_1 \mapsto i$  and  $e_2 \mapsto j$ .

- (4) As graded algebras,  $\text{Cliff}_{1,1}(\mathbb{R})$  is isomorphic to  $\text{End}(\mathbb{R}^{1|1})$ , where  $\mathbb{R}^{1|1}$  denotes the  $\mathbb{Z}/2\mathbb{Z}$  graded vector space  $\mathbb{R} \oplus \mathbb{R}$ . The isomorphism is given by

$$e_1^+ \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad e_1^- \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- (5) As ungraded algebras  $\text{Cliff}_{1,0}(\mathbb{R})$  is isomorphic to the product algebra  $\mathbb{R} \times \mathbb{R}$ , where  $e_1 \mapsto (1, -1)$ .
- (6) As ungraded algebras,  $\text{Cliff}_{2,0}(\mathbb{R})$  is isomorphic to the algebra  $M_2\mathbb{R}$  of  $2 \times 2$  matrices with coefficients in  $\mathbb{R}$ . The isomorphism is given by

$$e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad e_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

◀

To classify all Clifford algebras as ungraded algebras, it suffices to know the following table, which is derived by identifying a few low dimensional Clifford algebras and using the fact that the graded tensor product of Clifford algebras is another Clifford algebra.

7	$M_8\mathbb{C}$	$M_8\mathbb{H}$	$M_8\mathbb{H} \times M_8\mathbb{H}$	$M_{16}\mathbb{H}$	$M_{32}\mathbb{C}$	$M_{64}\mathbb{R}$	$M_{64}\mathbb{R} \times M_{64}\mathbb{R}$	$M_{128}\mathbb{R}$
6	$M_4\mathbb{H}$	$M_4\mathbb{H} \times M_4\mathbb{H}$	$M_8\mathbb{H}$	$M_{16}\mathbb{C}$	$M_{32}\mathbb{R}$	$M_{32}\mathbb{R} \times M_{32}\mathbb{R}$	$M_{64}\mathbb{R}$	$M_{64}\mathbb{C}$
5	$M_2\mathbb{H} \times M_2\mathbb{H}$	$M_4\mathbb{H}$	$M_8\mathbb{C}$	$M_{16}\mathbb{R}$	$M_{16}\mathbb{R} \times M_{16}\mathbb{R}$	$M_{32}\mathbb{R}$	$M_{32}\mathbb{C}$	$M_{32}\mathbb{H}$
4	$M_2\mathbb{H}$	$M_4\mathbb{C}$	$M_8\mathbb{R}$	$M_8\mathbb{R} \times M_8\mathbb{R}$	$M_{16}\mathbb{R}$	$M_{16}\mathbb{C}$	$M_{16}\mathbb{H}$	$M_{16}\mathbb{H} \times M_{16}\mathbb{H}$
3	$M_2\mathbb{C}$	$M_4\mathbb{R}$	$M_4\mathbb{R} \times M_4\mathbb{R}$	$M_8\mathbb{R}$	$M_8\mathbb{C}$	$M_8\mathbb{H}$	$M_8\mathbb{H} \times M_8\mathbb{H}$	$M_{16}\mathbb{H}$
2	$M_2\mathbb{R}$	$M_2\mathbb{R} \times M_2\mathbb{R}$	$M_4\mathbb{R}$	$M_4\mathbb{C}$	$M_4\mathbb{H}$	$M_4\mathbb{H} \times M_4\mathbb{H}$	$M_8\mathbb{H}$	$M_{16}\mathbb{C}$
1	$\mathbb{R} \times \mathbb{R}$	$M_2\mathbb{R}$	$M_2\mathbb{C}$	$M_2\mathbb{H}$	$M_2\mathbb{H} \times M_2\mathbb{H}$	$M_4\mathbb{H}$	$M_8\mathbb{C}$	$M_{16}\mathbb{R}$
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \times \mathbb{H}$	$M_2\mathbb{H}$	$M_4\mathbb{C}$	$M_8\mathbb{R}$	$M_8\mathbb{R} \times M_8\mathbb{R}$
p \ q	0	1	2	3	4	5	6	7

To read the table, the bottom left entry is  $\text{Cliff}_{0,0} \cong \mathbb{R}$ , and moving to the right increments the signature from  $(p, q)$  to  $(p, q + 1)$ , and moving up increments the signature  $(p, q)$  to  $(p + 1, q)$ . Any other Clifford algebra can be obtained from an algebra on this table by tensoring with  $M_{16}\mathbb{R}$ , since incrementing the signature by 8 (by adding to either  $p$  or  $q$ ) results in tensoring with  $M_{16}\mathbb{R}$ .

## 2. Clifford Modules

**DEFINITION 2.1.** A (left) *Clifford module* for the Clifford algebra  $\text{Cliff}_{p,q}(\mathbb{R})$  is a module for  $\text{Cliff}_{p,q}(\mathbb{R})$  in the usual sense i.e. a real vector space  $V$  equipped with an algebra action  $\bullet : \text{Cliff}_{p,q}(\mathbb{R}) \times V \rightarrow V$  satisfying the following properties

- (1) Every element of  $\text{Cliff}_{p,q}(\mathbb{R})$  acts linearly on  $V$ .
- (2)  $(AB) \cdot v = A \cdot (B \cdot v)$  for all  $v \in V$ .
- (3)  $(A + B) \cdot v = A \cdot v + B \cdot v$  for all  $v \in V$ .

Equivalently, it is the data of a real vector space  $V$  and an algebra homomorphism  $\text{Cliff}_{p,q}(\mathbb{R}) \rightarrow \text{End}(V)$ . ❖

**DEFINITION 2.2.** A Clifford module is *irreducible* if there exist no proper nontrivial submodules. ❖

From the classification of Clifford algebras, all the Clifford algebras are either matrix algebras  $M_n\mathbb{F}$ , where  $\mathbb{F} = \mathbb{R}, \mathbb{C}$ , or  $\mathbb{H}$ , or products  $M_n\mathbb{F} \times M_n\mathbb{F}$  of two copies of the same matrix algebra. This is sufficient to conclude that Clifford algebras are semisimple, so all Clifford modules are direct sums of irreducible modules. Therefore, classifying all Clifford modules reduces to classifying the irreducible Clifford modules.

**THEOREM 2.3.** *Let  $\mathbb{F} = \mathbb{R}, \mathbb{C}$ , or  $\mathbb{H}$ . Then any nontrivial irreducible module for  $M_n\mathbb{F}$  is isomorphic to  $\mathbb{F}^n$  with the standard action.*

**PROOF.** We first note that  $M_n\mathbb{F}$  acts transitively on  $\mathbb{F}^n$ , which implies that it is irreducible. We then must show that  $\mathbb{F}^n$  is, up to isomorphism, the only irreducible  $M_n\mathbb{F}$  module. The matrix algebra  $M_n\mathbb{F}$  admits an increasing chain of left ideals

$$0 = I_0 \subset I_1 \subset \dots \subset I_n = M_n\mathbb{F}$$

where  $I_k$  is the set of matrices where only the first  $k$  columns are nonzero. These ideals have the property that the quotient  $I_k/I_{k-1}$  is isomorphic to  $\mathbb{F}^n$  as a left  $M_n\mathbb{F}$  module. Then let  $M$  be some nontrivial irreducible  $M_n\mathbb{F}$  module, and fix  $m \in M$ . Then the orbit  $M_n\mathbb{F} \cdot m$  of  $m$  under the algebra action is a nonzero submodule, so it must be all of  $M$ . Then the map  $\varphi : M_n\mathbb{F} \rightarrow M$  given by  $A \mapsto A \cdot m$  is a surjective map of left  $M_n\mathbb{F}$  modules. Then there must exist some smallest  $k$  such that  $\varphi(I_k)$  is nonzero, and by construction,  $\varphi|_{I_k}$  factors through the quotient  $I_k/I_{k-1}$ , which is isomorphic to  $\mathbb{F}^n$  with the standard action. Then since  $\mathbb{F}^n$  is irreducible, this gives us a nonzero map between irreducible modules, which is an isomorphism by Schur's Lemma.  $\blacksquare$

**THEOREM 2.4.** *Any nontrivial irreducible module for  $M_n\mathbb{F} \times M_n\mathbb{F}$  is isomorphic to either  $\mathbb{F}^n$  where the left factor acts in the usual way, and the right factor acts by 0, or  $\mathbb{F}^n$  where the left factor acts by 0 and the right factor acts in the usual way.*

**PROOF.** Let  $R$  denote  $\mathbb{F}^n$  where the right factor acts nontrivially, and let  $L$  denote  $\mathbb{F}^n$  where the left factor acts nontrivially. Both  $L$  and  $R$  are irreducible since  $M_n\mathbb{F} \times M_n\mathbb{F}$  acts transitively on them. To show that they are the only irreducible modules up to isomorphism, we use a similar technique as above. Let  $J_k$  denote the chain of increasing ideals in  $M_n\mathbb{F}$ , as we used above. Then  $M_n\mathbb{F} \times M_n\mathbb{F}$  admits a chain of increasing left ideals  $J_k$

$$0 = J_0 \subset I_1 \times \{0\} \subset \dots \subset M_n\mathbb{F} \times \{0\} \subset M_n\mathbb{F} \times I_1 \subset \dots \subset M_n\mathbb{F} \times M_n\mathbb{F} = J_{2n}$$

We note that for  $1 \leq k \leq n$ , we have that  $J_k/J_{k-1}$  is isomorphic to  $L$ , and for  $n+1 \leq k \leq 2n$ , we have that  $J_k/J_{k-1}$  is isomorphic to  $R$ . Then given a nontrivial irreducible module  $M$  and a nonzero element  $m$ , we get a surjective map  $\varphi : M_n\mathbb{F} \times M_n\mathbb{F} \rightarrow M$  where  $A \mapsto A \cdot m$ . Like before, there exists some smallest  $k$  such that  $\varphi(J_k)$  is nonzero, which then factors through to an isomorphism  $J_k/J_{k-1} \rightarrow M$ , so  $M$  is either isomorphic to  $R$  or  $L$ .  $\blacksquare$

This then gives a full classification of the irreducible ungraded Clifford modules.

Since the Clifford algebras are  $\mathbb{Z}/2\mathbb{Z}$  graded, we have an analogous notion of a graded Clifford module.

**DEFINITION 2.5.** Let  $A = A^0 \oplus A^1$  be a  $\mathbb{Z}/2\mathbb{Z}$  graded algebra. Then a  $\mathbb{Z}/2\mathbb{Z}$  **graded module** for  $A$  is a module  $M = M^0 \oplus M^1$  for  $A$  such that  $A^i \cdot M^j \subset M^{i+j \bmod 2}$ .  $\blacklozenge$

A **graded Clifford module** is then a  $\mathbb{Z}/2\mathbb{Z}$  graded module for a Clifford algebra  $\text{Cliff}_{p,q}(\mathbb{R})$ . We already have all the ingredients we need to classify all of the graded Clifford modules – we have a complete classification of the ungraded Clifford modules, and we have the relationship between  $\text{Cliff}_{p,q}(\mathbb{R})$  and its even subalgebra  $\text{Cliff}_{p,q}^0(\mathbb{R})$ .

**THEOREM 2.6.** *There is a bijective correspondence*

$$\left\{ \text{Ungraded Clifford modules over } \text{Cliff}_{p,q}^0(\mathbb{R}) \right\} \longleftrightarrow \left\{ \text{Graded Clifford modules over } \text{Cliff}_{p,q}(\mathbb{R}) \right\}$$

**PROOF.** We provide maps in both directions. Let  $M = M^0 \oplus M^1$  be a graded module for  $\text{Cliff}_{p,q}(\mathbb{R})$ . The action of  $\text{Cliff}_{p,q}^0(\mathbb{R})$  preserves the even subspace  $M^0$ , which gives  $M^0$  the structure



of an ungraded  $\text{Cliff}_{p,q}^0(\mathbb{R})$  module. Let  $r$  denote the mapping  $M \mapsto M^0$ .

In the other direction, let  $V$  be an ungraded module over  $\text{Cliff}_{p,q}^0(\mathbb{R})$ . Then let  $t$  denote the mapping  $V \mapsto V' = \text{Cliff}_{p,q}(\mathbb{R}) \otimes_{\text{Cliff}_{p,q}^0(\mathbb{R})} V$ . Where we treat  $\text{Cliff}_{p,q}(\mathbb{R})$  as a left module under multiplication by elements of  $\text{Cliff}_{p,q}^0(\mathbb{R})$ . We then endow  $V'$  the structure of a  $\text{Cliff}_{p,q}(\mathbb{R})$  module, where  $\text{Cliff}_{p,q}(\mathbb{R})$  acts on the  $\text{Cliff}_{p,q}(\mathbb{R})$  factor by left multiplication. We further impose the grading on  $V'$  where

$$V' = (\text{Cliff}_{p,q}^0(\mathbb{R}) \otimes_{\text{Cliff}_{p,q}^0(\mathbb{R})} V) \oplus (\text{Cliff}_{p,q}^1(\mathbb{R}) \otimes_{\text{Cliff}_{p,q}^1(\mathbb{R})} V)$$

We then claim that these two maps are inverses, which we will verify by showing that the compositions  $r \circ t$  and  $t \circ r$  are identity. We first show that  $r \circ t$  is identity. Let  $V$  be an ungraded module over  $\text{Cliff}_{p,q}^0(\mathbb{R})$ . Then the even subalgebra of  $t(V) = \text{Cliff}_{p,q} \otimes_{\text{Cliff}_{p,q}^0(\mathbb{R})} V$  is  $\text{Cliff}_{p,q}^0(\mathbb{R}) \otimes_{\text{Cliff}_{p,q}^0(\mathbb{R})} V$ , which is naturally isomorphic to  $V$  itself. In the other direction, let  $M = M^0 \otimes M^1$  be a module for  $\text{Cliff}_{p,q}(\mathbb{R})$ . Showing that  $t \circ r$  is the identity then amounts to showing that  $\text{Cliff}_{p,q}(\mathbb{R}) \otimes_{\text{Cliff}_{p,q}^0(\mathbb{R})} M^0$  is isomorphic to  $M$ . The desired isomorphism is explicitly determined by the mapping  $a \otimes m \mapsto a \cdot m$ . ■

From this theorem, understanding graded Clifford modules is equivalent to understanding the ungraded modules, since the even subalgebra of a Clifford algebra is isomorphic as ungraded algebras with a smaller Clifford algebra. Putting everything together gives us the complete classification of graded Clifford modules.

Given a ring homomorphism  $\varphi : A \rightarrow B$ , this induces a pullback map  ${}_B\text{Mod} \rightarrow {}_A\text{Mod}$  of left modules. Given a  $B$  module  $M$ , we get an  $A$  module via  $\varphi$  by defining the ring action on the underlying abelian group of  $M$  to be  $a \cdot m = \varphi(a)m$ , where the right hand side is the ring action from  $B$ . This pullback map reveals a beautiful periodicity among the Clifford algebras and their modules, which is one of the many forms of **Bott periodicity**. The Clifford modules over a fixed Clifford algebra  $\text{Cliff}_{p,q}(\mathbb{R})$  form a commutative monoid under the direct sum, which we denote  $\mathcal{M}_{p,q}$ . The inclusion  $\mathbb{R}^{p,q} \hookrightarrow \mathbb{R}^{p,q+1}$  induces an inclusion  $\text{Cliff}_{p,q}(\mathbb{R}) \hookrightarrow \text{Cliff}_{p,q+1}(\mathbb{R})$ , which then induces a monoid homomorphism  $\mathcal{M}_{p,q+1} \rightarrow \mathcal{M}_{p,q}$ . Likewise, the inclusion  $\mathbb{R}^{p,q} \hookrightarrow \mathbb{R}^{p+1,q}$  induces a monoid homomorphism  $\mathcal{M}_{p+1,q} \rightarrow \mathcal{M}_{p,q+1}$ . Since these monoid maps are induced by inclusions, they represent the restriction of Clifford modules. We can then compute the cokernels of these restrictions, which computes the degree to which Clifford modules of  $\text{Cliff}_{p,q}(\mathbb{R})$  fail to extend to Clifford modules over  $\text{Cliff}_{p+1,q}(\mathbb{R})$  and  $\text{Cliff}_{p,q+1}(\mathbb{R})$ . To compute these cokernels, it suffices to compute it for the table, since tensoring with  $M_{16}\mathbb{R}$  will result in the pattern in the chessboard to repeat. From the Clifford chessboard, we observe that the inclusions of  $\text{Cliff}_{p,q}(\mathbb{R})$  into  $\text{Cliff}_{p+1,q}(\mathbb{R})$  and  $\text{Cliff}_{p,q+1}(\mathbb{R})$  all fall into one of the following four cases.

- (1)  $M_n\mathbb{F} \hookrightarrow M_n\mathbb{F}'$  where  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ , and  $\mathbb{F}' = \mathbb{C}$  or  $\mathbb{H}$  is the division algebra of twice the dimension of  $\mathbb{F}$  as a vector space over  $\mathbb{R}$ .
- (2)  $M_n\mathbb{F} \times M_n\mathbb{F} \hookrightarrow M_{2n}\mathbb{F}$ .
- (3)  $M_n\mathbb{F}' \hookrightarrow M_{2n}\mathbb{F}$ , where  $\mathbb{F}$  and  $\mathbb{F}'$  are defined as above.
- (4)  $M_n\mathbb{F} \hookrightarrow M_n\mathbb{F} \times M_n\mathbb{F}$ .

Using semisimplicity of the Clifford algebras, along with our classification of the ungraded Clifford modules, we then compute the cokernels for all of these cases.

- (1) In this case, the irreducible module for  $M_n\mathbb{F}'$  is  $(\mathbb{F}')^n$ , which is twice the dimension of  $\mathbb{F}^n$  as an  $\mathbb{R}$  vector space. Then since both  $M_n\mathbb{F}$  and  $M_n\mathbb{F}'$  admit only a single irreducible module, the monoid homomorphism is a map  $\mathbb{Z}^{\geq 0} \rightarrow \mathbb{Z}^{\geq 0}$  where  $1 \mapsto 2$ . The cokernel is then the group  $\mathbb{Z}/2\mathbb{Z}$ .

(2) The algebra  $M_n\mathbb{F} \times M_n\mathbb{F}$  injects into  $M_{2n}\mathbb{F}$  as block matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

with  $A, B \in M_n\mathbb{F}$ . From this, we see that the irreducible module  $\mathbb{F}^{2n}$  decomposes into a direct sum of the two irreducible modules for  $M_n\mathbb{F} \times M_n\mathbb{F}$ . The monoid homomorphism  $\mathbb{Z}^{\geq 0} \rightarrow \mathbb{Z}^{\geq 0} \times \mathbb{Z}^{\geq 0}$  is then given by  $1 \mapsto (1, 1)$ , and the cokernel is the group  $\mathbb{Z}$ .

(3) In this case, the irreducible modules  $(\mathbb{F}')^n$  and  $\mathbb{F}^{2n}$  are the same dimension as vector spaces over  $\mathbb{R}$ , so the monoid homomorphism is given by  $1 \mapsto 1$ , so the cokernel is the trivial group 0.

(4) In this case, both irreducible modules for the product  $M_n\mathbb{F} \times M_n\mathbb{F}$  are the same dimension as the irreducible module for  $M_n\mathbb{F}$ , so the monoid homomorphism  $\mathbb{Z}^{\geq 0} \times \mathbb{Z}^{\geq 0} \rightarrow \mathbb{Z}^{\geq 0}$  is given by  $(1, 0) \mapsto 1$  and  $(0, 1) \mapsto 1$ , so the cokernel is again the trivial group 0.

With these cokernels in hand, we can fill in the table

7	$\begin{smallmatrix} 0 \\ \mathbb{Z}/2\mathbb{Z} \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} \mathbb{Z} \\ \mathbb{Z} \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} \mathbb{Z}/2\mathbb{Z} \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} \mathbb{Z}/2\mathbb{Z} \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} \mathbb{Z} \\ \mathbb{Z} \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ \mathbb{Z}/2\mathbb{Z} \end{smallmatrix}$
6	$\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} \mathbb{Z} \\ \mathbb{Z} \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} \mathbb{Z}/2\mathbb{Z} \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} \mathbb{Z}/2\mathbb{Z} \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} \mathbb{Z} \\ \mathbb{Z} \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ \mathbb{Z}/2\mathbb{Z} \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ \mathbb{Z}/2\mathbb{Z} \end{smallmatrix}$
5	$\begin{smallmatrix} \mathbb{Z} \\ \mathbb{Z} \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} \mathbb{Z}/2\mathbb{Z} \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} \mathbb{Z}/2\mathbb{Z} \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} \mathbb{Z} \\ \mathbb{Z} \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ \mathbb{Z}/2\mathbb{Z} \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ \mathbb{Z}/2\mathbb{Z} \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$
4	$\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} \mathbb{Z}/2\mathbb{Z} \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} \mathbb{Z}/2\mathbb{Z} \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} \mathbb{Z} \\ \mathbb{Z} \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ \mathbb{Z}/2\mathbb{Z} \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ \mathbb{Z}/2\mathbb{Z} \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} \mathbb{Z} \\ \mathbb{Z} \end{smallmatrix}$
3	$\begin{smallmatrix} \mathbb{Z}/2\mathbb{Z} \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} \mathbb{Z}/2\mathbb{Z} \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} \mathbb{Z} \\ \mathbb{Z} \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ \mathbb{Z}/2\mathbb{Z} \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ \mathbb{Z}/2\mathbb{Z} \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} \mathbb{Z} \\ \mathbb{Z} \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$
2	$\begin{smallmatrix} \mathbb{Z}/2\mathbb{Z} \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} \mathbb{Z} \\ \mathbb{Z} \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ \mathbb{Z}/2\mathbb{Z} \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ \mathbb{Z}/2\mathbb{Z} \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} \mathbb{Z} \\ \mathbb{Z} \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} \mathbb{Z}/2\mathbb{Z} \\ 0 \end{smallmatrix}$
1	$\begin{smallmatrix} \mathbb{Z} \\ \mathbb{Z} \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ \mathbb{Z}/2\mathbb{Z} \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ \mathbb{Z}/2\mathbb{Z} \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} \mathbb{Z} \\ \mathbb{Z} \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} \mathbb{Z}/2\mathbb{Z} \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} \mathbb{Z}/2\mathbb{Z} \\ 0 \end{smallmatrix}$
0	$\begin{smallmatrix} 0 \\ \mathbb{Z}/2\mathbb{Z} \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ \mathbb{Z}/2\mathbb{Z} \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} \mathbb{Z} \\ \mathbb{Z} \end{smallmatrix}$	$\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} \mathbb{Z}/2\mathbb{Z} \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} \mathbb{Z}/2\mathbb{Z} \\ 0 \end{smallmatrix}$	$\begin{smallmatrix} \mathbb{Z} \\ \mathbb{Z} \end{smallmatrix}$
p q	0	1	2	3	4	5	6	7

where the top left corner of each box is the cokernel for the monoid homomorphism induced by the inclusion  $\text{Cliff}_{p,q}(\mathbb{R}) \hookrightarrow \text{Cliff}_{p+1,q}(\mathbb{R})$ , and the bottom right corner is the cokernel for the monoid homomorphism induced by the inclusion  $\text{Cliff}_{p,q}(\mathbb{R}) \hookrightarrow \text{Cliff}_{p,q+1}(\mathbb{R})$ . There is an analogous table for the graded modules, where everything is shifted right one square. The periodicity patterns in this table are one of the many forms of **Bott periodicity**, which has wide ranging implications in homotopy theory and K-theory. The sequence

$$\mathbb{Z}/2\mathbb{Z} \quad \mathbb{Z}/2\mathbb{Z} \quad 0 \quad \mathbb{Z} \quad 0 \quad 0 \quad 0 \quad \mathbb{Z}$$

is affectionately called the **Bott song**. The groups in the table are part of an extremely rich and deep theory called **K-Theory**, and are called the **real K-Theory groups** of a point, denoted  $KO^n(\text{pt})$ .

### 3. The Pin and Spin Groups

The group of invertible elements in  $\text{Cliff}_{p,q}(\mathbb{R})$ , denoted  $\text{Cliff}_{p,q}^\times(\mathbb{R})$  contains a group  $\text{Pin}_{p,q}$ , which is a double cover of the group  $O_{p,q}$  of matrices preserving the standard bilinear form  $\langle \cdot, \cdot \rangle$  on  $\mathbb{R}^{p,q}$ . Inside of  $\text{Pin}_{p,q}$ , there exists a subgroup  $\text{Spin}_{p,q} \subset \text{Pin}_{p,q}$ , which double covers the group  $SO_{p,q}$ , which consists of the subgroup of  $O_{p,q}$  where all the elements have determinant equal to 1.

**DEFINITION 3.1.** The **Pin group**  $\text{Pin}_{p,q}$  is the subgroup of  $\text{Cliff}_{p,q}^\times(\mathbb{R})$  generated by the set

$$\{v \in \mathbb{R}^{p,q} : v^2 = \pm 1\}$$

The *Spin group*  $\text{Spin}_{p,q}$  is the subgroup of  $\text{Pin}_{p,q}$  generated by even products of basis vectors, i.e.

$$\text{Spin}_{p,q} = \text{Pin}_{p,q} \cap \text{Cliff}_{p,q}^0(\mathbb{R})$$

In indefinite signatures.  $SO_{p,q}$  has multiple components. Some conventions let  $\text{Spin}_{p,q}$  denote the double cover of the identity component  $SO_{p,q}^+$ , which is the identity component of  $\text{Spin}_{p,q}$ , denoted  $\text{Spin}_{p,q}^0$ . In the case that the bilinear form is definite, we let  $\text{Pin}_n^+ = \text{Pin}_{n,0}$  and  $\text{Pin}_n^- = \text{Pin}_{0,n}$ . There is no such distinction for the Spin groups in definite signatures.  $\blacklozenge$

**THEOREM 3.2.**  $\text{Spin}_{p,q} \cong \text{Spin}_{q,p}$ .

**PROOF.** Recall that we have an isomorphism  $\text{Cliff}_{p,q}^{\text{op}} \rightarrow \text{Cliff}_{q,p}$  where  $e_i^\pm \mapsto e_i^\mp$ . In addition, the even subalgebra  $\text{Cliff}_{q,p}^0$  is isomorphic to the (ungraded) opposite algebra of the even subalgebra  $\text{Cliff}_{p,q}^0$ . Therefore, the Spin group  $\text{Spin}_{q,p} \subset \text{Cliff}_{q,p}$  is isomorphic to the opposite group  $\text{Spin}_{p,q}^{\text{op}}$ . We then know that every group is isomorphic to its opposite group via the map  $g \mapsto g^{-1}$ , giving us the desired isomorphism.  $\blacksquare$

In particular, this implies that the Spin groups in definite signatures are isomorphic, so we will henceforth denote them as  $\text{Spin}_n$ . To show that the Pin and Spin groups cover  $O_{p,q}$  and  $\text{Spin}_{p,q}$ , we make a short digression. Given a vector  $v \in \mathbb{R}^{p,q}$ , we can define a reflection map  $R_v : \mathbb{R}^{p,q} \rightarrow \mathbb{R}^{p,q}$  given by  $R_v(w) = w - 2\langle v, w \rangle v$ , which will reflect across the hyperplane  $v^\perp$ .

**THEOREM 3.3 (Cartan-Dieudonné).** Any orthogonal transformation  $A \in O_{p,q}$  is the composition of at most  $p + q$  hyperplane reflections, where we interpret the identity map as the composition of 0 reflections.

**PROOF.** We prove this by induction on  $n = p + q$ . The case  $n = 1$  is trivial, since  $O_1 = \{\pm 1\}$ . Then given  $A \in O_{p,q}$ , fix some nonzero  $v \in \mathbb{R}^{p,q}$ . Then define  $R : \mathbb{R}^{p,q} \rightarrow \mathbb{R}^{p,q}$  by

$$R(w) = w - 2 \frac{\langle Av - v, w \rangle}{\langle Av - v, Av - v \rangle} (Av - v)$$

Then  $R$  is a reflection about the hyperplane orthogonal to  $Av - v$ , and will interchange  $v$  and  $Av$ . Therefore,  $RA$  is an orthogonal transformation fixing  $v$ . Since  $RA$  is orthogonal, it will also fix the orthogonal complement  $v^\perp$ , so it will restrict to an orthogonal transformation on  $v^\perp$ . The orthogonal complement  $v^\perp$  is 1 dimension lower than  $\mathbb{R}^{p,q}$ , and restricting the bilinear form to  $v^\perp$ , we know by the inductive hypothesis that  $RA|_{v^\perp}$  can be written as at most  $n - 1$  hyperplane reflections in  $v^\perp$ . Since  $RA$  fixes  $v$ , we can extend all of these transformations to a hyperplane reflection on all of  $\mathbb{R}^{p,q}$ , by taking the span of each hyperplane with  $v$ , giving us that  $RA$  is a composition of at most  $n - 1$  reflections. Finally, composing  $RA$  with  $R$  gives us that  $A$  can be written as a composition of at most  $n$  hyperplane reflections.  $\blacksquare$

The Cartan-Dieudonné theorem will be the central piece for showing that the Pin and Spin groups are double covers of the orthogonal groups. The Clifford algebra has an automorphism  $\alpha : \text{Cliff}_{p,q}(\mathbb{R}) \rightarrow \text{Cliff}_{p,q}(\mathbb{R})$  that extends the mapping  $v \mapsto v-$ . The action of  $\alpha$  on a product  $v_1 \cdots v_k$  of vectors  $v_i \in \mathbb{R}^{p,q}$  is

$$\alpha(v_1 \cdots v_k) = (-1)^k v_1 \cdots v_k$$

the automorphism  $\alpha$  gives another way to realize the grading on  $\text{Cliff}_{p,q}$  – it is the grading operator. The  $+1$ -eigenspace of  $\alpha$  is exactly  $\text{Cliff}_{p,q}^0(\mathbb{R})$ , and the  $-1$ -eigenspace is the odd subspace  $\text{Cliff}_{p,q}^1(\mathbb{R})$ .

**THEOREM 3.4.** *There exist 2-to-1 group homomorphisms  $\text{Pin}_{p,q} \rightarrow O_{p,q}$  and  $\text{Spin}_{p,q} \rightarrow SO_{p,q}$ , i.e. there exist short exact sequences of groups*

$$0 \longrightarrow \{\pm 1\} \longrightarrow \text{Pin}_{p,q} \longrightarrow O_{p,q} \longrightarrow 0$$

$$0 \longrightarrow \{\pm 1\} \longrightarrow \text{Spin}_{p,q} \longrightarrow SO_{p,q} \longrightarrow 0$$

**PROOF.** We first consider the case of  $\text{Pin}_{p,q}$ . To do this, we need to construct a group action where  $\text{Pin}_{p,q}$  acts on  $\mathbb{R}^{p,q}$  by orthogonal transformations. We note that for a vector  $v \in \mathbb{R}^{p,q}$  (identifying  $\mathbb{R}^{p,q}$  as a subspace of  $\text{Cliff}_{p,q}(\mathbb{R})$ ), satisfying  $\langle v, v \rangle = \pm 1$ , we have that  $v^{-1} = \pm v$ . Then given  $g \in \text{Pin}_{p,q}$ , and  $v \in \mathbb{R}^{p,q}$ , we claim that the left action

$$g \cdot v = \alpha(g)vg^{-1}$$

defines the group action we desire. To show this, we must show that this indeed maps  $\mathbb{R}^{p,q}$  back into itself, and that the group elements act by orthogonal transformations. We first compute this actions on vectors  $v \in \mathbb{R}^{p,q}$  with  $\langle v, v \rangle = \pm 1$ . In either case, since  $v \in \mathbb{R}^{p,q}$ , we have that  $\alpha(v) = -v$ . First assume that  $\langle v, v \rangle = 1$ . Then given  $w \in \mathbb{R}^{p,q}$ , we compute

$$\begin{aligned} -vww^{-1} &= -vww \\ &= (wv - 2\langle v, w \rangle)v \\ &= w - 2\langle v, w \rangle v \end{aligned}$$

Which is hyperplane reflection about the orthogonal complement of  $v$ . In the case that  $\langle v, v \rangle = -1$ , we compute

$$\begin{aligned} -vww^{-1} &= -vw(-v) \\ &= (2\langle -v, w \rangle + vw)(-v) \\ &= w - 2\langle -v, w \rangle(-v) \end{aligned}$$

which is hyperplane reflection about the orthogonal complement of  $-v$ , which is the same as the orthogonal complement of  $v$ . Then given two vectors  $v_1, v_2 \in \mathbb{R}^{p,q}$ , we have that  $\alpha(v_1v_2) = v_1v_2$ , so given  $w \in \mathbb{R}^{p,q}$ ,

$$\alpha(v_1v_2)w(v_1v_2)^{-1} = (-v_1)(-v_2)wv_2^{-1}v_1^{-1}$$

which is exactly the composition of hyperplane reflection about  $v_2^\perp$ , with hyperplane reflection about  $v_1^\perp$ . Therefore,  $\text{Pin}_{p,q}$  acts by orthogonal transformations, giving us a homomorphism  $\text{Pin}_{p,q} \rightarrow O_{p,q}$ . This map is surjective by the Cartan-Dieudonné theorem, and it can be verified that the kernel is  $\{\pm 1\}$ . In addition, an even number of hyperplane reflections is orientation preserving, which gives a surjection  $\text{Spin}_{p,q} \rightarrow SO_{p,q}$ , by restricting the map  $\text{Pin}_{p,q} \rightarrow O_{p,q}$ . In addition, the kernel  $\{\pm 1\}$  is contained in  $\text{Spin}_{p,q}$ , so this map is also a double covering. ■

We also have the complex Pin and Spin groups, denoted  $\text{Pin}_n\mathbb{C}$  and  $\text{Spin}_n\mathbb{C}$ , which double cover the complex orthogonal groups  $O_n\mathbb{C}$  and  $SO_n\mathbb{C}$  respectively.

Two simple examples of spin groups occur in dimensions 2 and 3. Since  $SO_2 \cong \mathbb{T}$ , where

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$$

we have that  $\text{Spin}_2 \cong \mathbb{T}$ , where the covering map is given by  $z \mapsto z^2$ . In the case of  $SO_3$ , we consider the unit quaternions, which form a Lie group isomorphic to the group  $SU_2$ . Then given  $q \in SU_2$ ,

we define the map  $\varphi_q : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  where  $\varphi_q(v) = qv\bar{q}$ , where  $\bar{q}$  is the quaternionic conjugate of  $q$ , i.e.

$$\overline{a + bi + cj + dk} = a - bi - cj - dk$$

and we identify  $v = v^i e_i$  with  $v^1 i + v^2 j + v^3 k \in \mathbb{H}$ . The mapping  $q \mapsto \varphi_q$  then gives a double cover  $SU_2 \rightarrow SO_3$ . In particular,  $SU_2$  is diffeomorphic to the sphere  $S^3$ , so the double covering realizes  $SO_3$  as the quotient of  $S^3$  by the antipodal map, giving us that  $SO_3$  is diffeomorphic to  $\mathbb{RP}^3$ .

Many examples of low dimensional Spin groups arise from investigating the relationship between a 4 dimensional complex vector space  $V$  and its second exterior power  $\Lambda^2 V$ . Fix a volume form  $\mu \in \Lambda^4 V^*$ . This then induces a symmetric, nondegenerate bilinear form  $\langle \cdot, \cdot \rangle$  on  $\Lambda^2 V$  by

$$\langle \alpha, \beta \rangle = \langle \alpha \wedge \beta, \mu \rangle$$

where  $\langle \alpha \wedge \beta, u \rangle$  denotes the natural pairing of the vector space  $\Lambda^4 V$  with its dual  $\Lambda^4 V^*$ . Fix a basis  $\{e_i\}$  for  $V$  where  $\mu(e_1 \wedge e_2 \wedge e_3 \wedge e_4) = 1$ . In this basis, we see that the group of transformations  $\text{Aut}(V, \mu)$  preserving  $\mu$  is isomorphic to the group  $SL_4 \mathbb{C}$ . In addition, each map  $T \in \text{Aut}(V, \mu)$  induces a map  $\Lambda^2 T : \Lambda^2 V \rightarrow \Lambda^2 V$ , which is determined by the formula  $\Lambda^2 T(v \wedge w) = Tv \wedge Tw$ . For any  $T \in \text{Aut}(V, \mu)$ , the induced map  $\Lambda^2 T$  preserves the bilinear form on  $\Lambda^2 V$ , so the mapping  $T \mapsto \Lambda^2 T$  determines a group homomorphism  $\text{Aut}(V, \mu) \rightarrow \text{Aut}(\Lambda^2 V, \langle \cdot, \cdot \rangle)$ , where  $\text{Aut}(\Lambda^2 V, \langle \cdot, \cdot \rangle)$  denotes the group of linear automorphisms preserving the bilinear form. The kernel of this map is  $\{\pm \text{id}_V\}$ , and fixing an orthogonal basis for  $\langle \cdot, \cdot \rangle$  gives us that this map is a double cover  $SL_4 \mathbb{C} \rightarrow SO_6 \mathbb{C}$ , so  $SL_4 \mathbb{C}$  is isomorphic to the complex spin group  $\text{Spin}_4 \mathbb{C}$ .

If we then fix a hermitian inner product  $h : V \times V \rightarrow \mathbb{C}$ , we can consider the automorphisms  $\text{Aut}(V, \mu, h)$  preserving  $h$  and  $\mu$ , which is isomorphic to the group  $SU_4$ . The bilinear form  $h$  induces a hermitian inner product (which we also denote  $h$ ) on  $\Lambda^2 V$  defined by

$$h(v_1 \wedge v_2, v_3 \wedge v_4) = \det \begin{pmatrix} h(v_1, v_3) & h(v_1, v_4) \\ h(v_2, v_3) & h(v_2, v_4) \end{pmatrix}$$

Then if  $T \in \text{Aut}(V, \mu, h)$ ,  $\Lambda^2 T$  preserves the bilinear form  $\langle \cdot, \cdot \rangle$  induced by  $\mu$  as well as the hermitian inner product induced by  $h$ . The group that preserves both of these structures is isomorphic to  $SO_6 \mathbb{C} \cap U_6$ , which is  $SO_6 \mathbb{R}$ . This gives us that  $SU_4 \cong \text{Spin}_6$ .

In general, one can play the game of fixing additional structure on  $V$  (e.g. a real structure, quaternionic structure, symplectic form) and look for the induced structure on  $\Lambda^2 V$ . This then gives a map from automorphisms of  $V$  preserving this additional structure to automorphisms of  $\Lambda^2 V$  preserving the induced structure. Playing this game then determines several other low dimensional Spin groups.

$$\begin{aligned} \text{Spin}_5 \mathbb{C} &\cong Sp_4 \mathbb{C} & \text{Spin}_4 &\cong Sp(4) & \text{Spin}_4 \mathbb{C} &\cong SL_2 \mathbb{C} \times SL_2 \mathbb{C} \\ \text{Spin}_{1,3}^0 &\cong SL_2 \mathbb{C} & \text{Spin}_{1,2}^0 &\cong SL_2 \mathbb{R} & \text{Spin}_{1,5}^0 &\cong SL_2 \mathbb{H} \end{aligned}$$

Where  $Sp_4 \mathbb{C}$  denotes the group of  $4 \times 4$  matrices preserving a symplectic form,  $Sp(4) = Sp_4 \mathbb{C} \cap U_4$ , and  $SL_2 \mathbb{H}$  denotes the automorphisms of a 2 dimensional quaternionic vector space with determinant 1 when regarded as  $4 \times 4$  complex matrices.

**DEFINITION 3.5.** Given a Pin group  $\text{Pin}_{p,q}$ , the **Pinor representations** are representations of  $\text{Pin}_{p,q}$  that arise from an irreducible Clifford module  $M$  (i.e. the action of  $\text{Pin}_{p,q}$  can be extended to an action of  $\text{Cliff}_{p,q}(\mathbb{R})$ ). The **Spinor representations** are defined analogously for the group  $\text{Spin}_{p,q}$ .  $\blacklozenge$

From the classification of Clifford modules, we get a classification of all the Pinor representations. From the relationship between a Clifford algebra and its even subalgebra, we also get a complete classification of all the Spinor representations.

#### 4. Projective Spin Representations

Using the isomorphisms of the Clifford algebras with matrix algebras or products of matrix algebras along with the identification of the even subalgebra with another Clifford algebra, we have a complete classification of the irreducible modules over the even subalgebras  $\text{Cliff}_n^0(\mathbb{R})$ . Restricting to the Spin group  $\text{Spin}_n \subset \text{Cliff}_n^0(\mathbb{R})$ , this gives us the Spin representations. These Spin representations define projective representations of the group  $SO_n$ . Given a Spin representation  $S$ , we have that the elements  $\pm 1$  act by scalars since they arise by restricting the action of a Clifford algebra on an irreducible module. Then since  $SO_n$  is the quotient of  $\text{Spin}_n$  by the subgroup  $\{\pm 1\}$  of the center, we get a projective representation of  $SO_n$  on the projectivization  $\mathbb{P}S$  in the following way: Given an element  $A \in SO_n$ , we have that  $A$  lifts to two elements  $\{\pm \tilde{A}\} \subset \text{Spin}_n$  which differ by  $-1$ . Since  $-1$  acts by a scalar on  $S$ , these elements determine the same action on  $\mathbb{P}S$ , giving us a well defined action of  $SO_n$  on  $\mathbb{P}S$ .

The Spin representation  $S$  is not realized canonically, while the projective Spin representation  $\mathbb{P}S$  is canonical. Given an isomorphism  $\varphi : V \rightarrow W$ , this induces a unique algebra isomorphism  $\text{End } V \rightarrow \text{End } W$  where  $A \in \text{End } V$  is mapped to  $\varphi \circ A \circ \varphi^{-1}$ . However, the converse is not true.

**PROPOSITION 4.1.** *An algebra isomorphism  $\varphi : \text{End } V \rightarrow \text{End } W$  induces a  $Z(\mathbb{F})$ -torsor of maps  $V \rightarrow W$ , where  $Z(\mathbb{F})$  is the center of  $\mathbb{F}$ .*

To prove this, we need a lemma.

**LEMMA 4.2.** *The group of algebra automorphisms  $\text{Aut}(M_n\mathbb{F})$  is isomorphic to the projective general linear group  $\text{PGL}_n\mathbb{F} = \text{GL}_n\mathbb{F}/Z(\text{GL}_n\mathbb{F})$ .*

**PROOF.** Let  $\alpha : M_n\mathbb{F} \rightarrow M_n\mathbb{F}$  be an algebra automorphism. We know that  $M_n\mathbb{F}$  admits a single irreducible module  $M$  up to isomorphism, which is  $\mathbb{F}^n$  with the standard action. Then  $\alpha$  defines another module  $M^\alpha$ , which is the same underlying vector space as  $M$  with the algebra action given by  $T \cdot v = \alpha(T)v$ , where the right hand side is the action of  $\alpha(T)$  on the module  $M$ . Since  $\alpha$  is an algebra automorphism,  $M_n\mathbb{F}$  acts transitively on  $M^\alpha$ , so it is also an irreducible module, which must be isomorphic to  $M$ . Therefore, there exists a module isomorphism  $A : M \rightarrow M^\alpha$ . Since  $M$  and  $M^\alpha$  are the same underlying vector space,  $A$  is also a linear isomorphism  $A : M \rightarrow M$ , thought of as a vector space instead of a module. Then since  $A$  is a module homomorphism, we have that for any  $T \in M_n\mathbb{F}$ ,

$$A \circ T = \alpha(T) \circ A \implies A \circ T \circ A^{-1} = \alpha(T)$$

so  $\alpha$  is given by conjugation by  $A \in \text{GL}(M)$ . In a basis, this tells us that the map  $\text{GL}_n\mathbb{F} \rightarrow \text{Aut}(M_n\mathbb{F})$  given by conjugation is surjective, and the kernel of this map is the center of  $\text{GL}_n\mathbb{F}$ , so by the first isomorphism theorem, we have that  $\text{Aut}(M_n\mathbb{F}) \cong \text{PGL}_n\mathbb{F}$ . ■

**PROOF OF PROPOSITION.** Fix bases for  $V$  and  $W$ . These bases then induce isomorphisms  $M_n\mathbb{F} \rightarrow V$  and  $M_n\mathbb{F} \rightarrow W$ . In these bases, the algebra isomorphism  $\varphi$  is given by an automorphism  $M_n\mathbb{F} \rightarrow M_n\mathbb{F}$ . From the lemma, we know that this automorphism is determined by an element of  $\text{PGL}_n\mathbb{F} = \text{GL}_n\mathbb{F}/Z(\text{GL}_n\mathbb{F})$ . The center  $Z(\text{GL}_n\mathbb{F})$  consists of scalar matrices  $\lambda I$  with  $\lambda \in Z(\mathbb{F})$ , which is acted on freely and transitively by  $Z(\mathbb{F})$  by multiplication, giving it the structure of a  $Z(\mathbb{F})$ -torsor. ■

However, an isomorphism  $\varphi : \text{End } V \rightarrow \text{End } W$  does induce a unique isomorphism  $\mathbb{P}V \rightarrow \mathbb{P}W$  of projective spaces. To see, this we make an identification between 1 dimensional subspaces of  $V$  with maximal left ideals of  $\text{End } V$ .

PROPOSITION 4.3. *There is a bijection*

$$\{\text{Maximal left ideals of } \text{End } V\} \longleftrightarrow \mathbb{P}V$$

PROOF. Given a line  $L \in \mathbb{P}V$ , the **annihilator** of  $L$  is the set

$$\text{Ann}(L) = \{M \in \text{End } V : M(L) = 0\}$$

In fact,  $\text{Ann}(L)$  is a left ideal in  $\text{End } V$ , since given  $A \in \text{End } V$  and  $M \in \text{Ann}(L)$ ,  $L$  lies in the kernel of  $A \circ M$ . We claim that  $\text{Ann}(L)$  is maximal. Suppose  $\text{Ann}(L) \subset I$  is properly contained in a left ideal  $I$ . Fix an ordered basis for  $V$  in which the first basis element is a nonzero element of  $L$ , then elements of  $\text{Ann}(L)$  are represented in this basis by matrices with all zeroes in the first column. Then since  $\text{Ann}(L)$  is properly contained in  $I$ , there exists some  $M \in I$  such that  $M \notin \text{Ann}(L)$ , which implies that as a matrix, the first column of  $M$  is nonzero. Then pick a matrix  $A \in \text{Ann}(L)$  in which the nonzero columns complete the first column into a basis for  $\mathbb{R}^n$ . Then  $A + M$  is an invertible element of  $\text{End } V$ , so  $I$  must be all of  $\text{End } V$ . Therefore  $\text{Ann}(L)$  is maximal. To show that the mapping  $L \mapsto \text{Ann}(L)$  is a bijection, we produce an inverse. Let  $I \subset \text{End } V$  be a maximal ideal. Then we claim that the subspace

$$\mathbb{V}(I) = \bigcap_{M \in I} \ker M$$

is a 1 dimensional subspace of  $V$ . We note that  $\mathbb{V}(I)$  cannot be trivial, since this would imply that  $I$  would contain an invertible element, contradicting that it is a proper ideal. We also see that it cannot be higher than 2 dimensional, since otherwise,  $I$  would be contained in the annihilator of a proper nontrivial subspace of  $\mathbb{V}(I)$ , contradicting maximality of  $I$ . We then claim that these two mappings are inverses. We certainly have that  $\text{Ann}(\mathbb{V}(I)) \supset I$ , so by maximality, this must be  $I$ . In addition it is clear that  $\mathbb{V}(\text{Ann}(L)) = L$  by the definition of  $\mathbb{V}(I)$  and the annihilator. Therefore, these mappings are inverses. ■

Therefore, given an algebra isomorphism  $\varphi : \text{End } V \rightarrow \text{End } W$ , this induces a map  $\mathbb{P}V \rightarrow \mathbb{P}W$  since the image of a maximal left ideal under an isomorphism is again a maximal left ideal. In addition, the induced map is a bijection, since it has an inverse given by the induced map of  $\varphi^{-1}$ . In addition, the group of units  $GL(V)$  acts on  $\mathbb{P}V$  by right multiplication – given a maximal ideal  $I$  and  $A \in GL(V)$ , the ideal  $I \cdot A$  is also a maximal left ideal.

This gives us a canonical realization of the Spin representations. In the case that the even subalgebra is isomorphic to a matrix algebra  $M_n\mathbb{F}$ , the projective Spin representation is restriction of the action of  $\text{Cliff}_n^0(\mathbb{R})$  on maximal left ideals of  $\text{Cliff}_n^0(\mathbb{F})$ . In the case that the even subalgebra is isomorphic to a product  $M_n\mathbb{F} \times M_n\mathbb{F}$ , the irreducible modules identify the subalgebras  $L$  and  $R$  isomorphic to  $M_n\mathbb{F} \times \{0\}$  and  $\{0\} \times M_n\mathbb{F}$  by singling out the maximal subalgebra that acts non-trivially. Looking at the maximal left ideals of these subalgebras then identifies the two projective Spin representations. In addition, since  $-1$  acts trivially on left ideals, these projective Spin representations descend to the quotient  $\text{Spin}_n / \{\pm 1\} \cong SO_n$ , giving us the projective representations of  $SO_n$ .





## CHAPTER 2

# Spin Structures on Manifolds



*Mathematics is a part of physics. Physics is an experimental science, a part of natural science. Mathematics is the part of physics where experiments are cheap.*

– V.I. Arnold



### 1. Fiber Bundles

In geometry and topology, we often want to consider families of objects (e.g. manifolds, vector spaces) that vary smoothly along a manifold like the tangent spaces of a manifold. This is best formalized in the construction of a fiber bundle.

**DEFINITION 1.1.** Let  $M$  and  $F$  be smooth manifolds. Then a **fiber bundle** over  $M$  with model fiber  $F$  is the data of a smooth manifold  $E$  with a smooth map  $\pi : E \rightarrow M$  such that for every point  $p \in M$ , there is a neighborhood  $U \subset M$  containing  $p$  and a diffeomorphism  $\varphi : \pi^{-1}(U) \rightarrow U \times F$  such that the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times F \\ & \searrow \pi \quad \swarrow p_U & \\ & U & \end{array}$$

commutes, where  $p_U$  denotes projection onto the first factor. The map  $\varphi$  is called a **local trivialization** of the fiber bundle  $\pi : E \rightarrow M$ . The manifold  $M$  is called the **base space**, while the manifold  $E$  is called the **total space**. ❖

Given a fiber bundle  $\pi : E \rightarrow M$  we often denote the fiber  $\pi^{-1}(p)$  by  $E_p$ .

**DEFINITION 1.2.** let  $\pi : E \rightarrow M$  and  $p : L \rightarrow M$  be fiber bundles with model fiber  $F$  over  $M$ . A **bundle homomorphism** is a smooth map  $\varphi : E \rightarrow L$  such that the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & L \\ & \searrow \pi \quad \swarrow p & \\ & M & \end{array}$$



An important property of fiber bundles is that they pull back.

**DEFINITION 1.3.** Let  $\pi : E \rightarrow M$  be a fiber bundle with model fiber  $F$  and let  $f : X \rightarrow M$  be a smooth map. Then the **pullback** of  $E$  by  $f$  is the data of a smooth manifold

$$f^*E = \left\{ (x, p) : x \in X, p \in \pi^{-1}(f(x)) \right\}$$

along with the projection  $p : f^*E \rightarrow M$  given by  $(x, p) \mapsto x$ , giving  $f^*E \rightarrow X$  the structure of a fiber bundle over  $X$  with model fiber  $F$ . The bundle  $f^*E$  also comes equipped with a natural map  $\alpha : f^*E \rightarrow E$  where  $\alpha(x, p) = p$ . Pullbacks give rise to the commutative diagram

$$\begin{array}{ccc} f^*E & \xrightarrow{\alpha} & E \\ p \downarrow & & \downarrow \pi \\ X & \xrightarrow{f} & M \end{array}$$

and are an instance of a general construction called the **fiber product**. ❖

**DEFINITION 1.4.** Let  $\pi : E \rightarrow M$  be a fiber bundle. A **local section** of  $\pi : E \rightarrow M$  is a smooth map  $\sigma : U \rightarrow E$  of an open set  $U \subset M$  such that  $\pi \circ \sigma = \text{id}_U$ . If  $U = M$ , the section is called a **global section**. Equivalently, it is the smooth assignment of an element in  $E_p$  to each point  $p \in U$ . We denote the set of sections of  $\pi : E \rightarrow M$  over a set  $U$  as  $\Gamma_U(E)$ . ❖

A fiber bundle is a very general construction in which the fibers  $F$  do not necessarily admit extra structure. An important special case of a fiber bundle is a vector bundle, where the fibers are vector spaces.

**DEFINITION 1.5.** Let  $M$  be a smooth manifold. A **vector bundle** over  $M$  is fiber bundle  $\pi : E \rightarrow M$  with model fiber  $\mathbb{R}^n$  (or  $\mathbb{C}^n$ ) such that the local trivializations  $\varphi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$  (or  $\mathbb{C}^n$ ) restrict to linear isomorphisms on the fibers, i.e. for all  $p \in U$ , the restriction  $\varphi|_{\pi^{-1}(p)} : \pi^{-1}(p) \rightarrow \{p\} \times \mathbb{R}^n$  (or  $\mathbb{C}^n$ ) is an isomorphism. The dimension  $n$  of the fibers is called the **rank** of the vector bundle. A **vector bundle homomorphism**  $\varphi : E \rightarrow L$  is a bundle homomorphism with the added stipulation that the restrictions to the fibers  $\varphi|_{E_x}$  are linear maps. ❖

**EXAMPLE 1.6.**

- (1) Given a smooth manifold  $M$ , the tangent bundle  $TM = \coprod_{p \in M} T_p M$  is a vector bundle, where the projection  $\pi : TM \rightarrow M$  is the mapping  $(p, v) \mapsto p$ . The rank of  $TM$  is the dimension of  $M$ . Local coordinates on  $(x^i)$  on  $U \subset M$  induce maps  $\pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$ , where  $(p, v) \mapsto (x^1(p), \dots, x^n(p), v^1, \dots, v^n)$ , where the  $v^i$  are the components of  $v$  with respect to the coordinate bases  $\partial_i$ . These maps define a basis for a topology on  $TM$ , which we then give a smooth structure by declaring the projection  $\pi : TM \rightarrow M$  to be smooth. In addition, these maps define local trivializations for  $TM$ , showing that it is a vector bundle.
- (2) A point in  $\mathbb{RP}^n$  is a line  $\ell$  in  $\mathbb{R}^{n+1}$ . The **tautological bundle** over  $\mathbb{RP}^n$  is the bundle where the fiber over  $\ell$  is the subspace  $\ell$  itself. An analogous construction can be done for Grassmannians  $\text{Gr}_k(V)$ , which parameterize  $k$ -dimensional subspaces of a vector space  $V$ . ◀

**DEFINITION 1.7.** A **Lie group** is a smooth manifold  $G$  with a group structure such that the multiplication map  $(g, h) \mapsto gh$  and the inversion map  $g \mapsto g^{-1}$  are smooth. ❖

Almost all of the groups we have discussed previously are Lie groups.

**EXAMPLE 1.8.**

- (1) The orthogonal groups  $O_n$  and special orthogonal groups  $SO_n$  are Lie groups.
- (2) The unitary groups  $U_n$  and special unitary groups  $SU_n$  are Lie groups. ◀

Another important class of fiber bundles are principal bundles. Given a Lie group  $G$ , the fibers of a principal  $G$ -bundle are  $G$ -torsors – manifolds with a smooth free and transitive right (or left)  $G$ -action. Given a  $G$  torsor  $M$  and a point  $p$  in  $M$ , there is a diffeomorphism  $G \rightarrow M$ , where  $g \mapsto p \cdot g$ . A  $G$ -torsor can be thought of as  $G$  without a distinguished identity element.

**DEFINITION 1.9.** Let  $G$  be a Lie group, and  $M$  a smooth manifold. A **principal  $G$ -bundle** over  $M$  is the data of

- (1) A smooth manifold  $P$  with a map  $\pi : P \rightarrow M$ .
- (2) A smooth right  $G$ -action on  $P$  that preserves the fibers of  $\pi$  and is free and transitive on the fibers of  $\pi$ .
- (3) For every point  $p \in M$ , a neighborhood  $U \subset M$  containing  $p$  and a  $G$ -equivariant diffeomorphism  $\varphi : \pi^{-1}(U) \rightarrow U \times G$  (where the right action on  $U \times G$  is right multiplication on the second factor) such that we get the commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times G \\ & \searrow \pi \quad \swarrow p_U & \\ & U & \end{array}$$

where  $p_U$  denotes projection onto the first factor.

A **principal bundle homomorphism** is a bundle homomorphism  $\varphi : P \rightarrow Q$  that is  $G$ -equivariant.  $\blacklozenge$

EXAMPLE 1.10. Given a smooth manifold  $M$  and a point  $p \in M$ , a basis of the tangent space is a linear isomorphism  $b : \mathbb{R}^n \rightarrow T_p M$ . The group  $GL_n \mathbb{R}$  acts freely and transitively on the set of bases  $\mathcal{B}_p$  on the right by  $b \cdot g = b \circ g$ . Then the **frame bundle** of  $M$ , denoted  $\mathcal{B}(M)$  is the disjoint union

$$\mathcal{B}(M) = \coprod_{p \in M} \mathcal{B}_p$$

where  $\pi$  is the projection map  $(p, b) \mapsto p$ . Like with the tangent bundle, we use local coordinates on  $M$  to give maps  $\pi^{-1}(U) \rightarrow U \times GL_n \mathbb{R}$ . These maps can then be used to define a topology and smooth structure, and form local trivializations of  $\mathcal{B}(M)$ , giving it the structure of a principal  $GL_n \mathbb{R}$ -bundle over  $M$ .  $\blacktriangleleft$

EXAMPLE 1.11. Given a smooth manifold  $M$ , a Riemannian metric  $g$  induces an inner product  $g_p$  on each tangent space  $T_p M$ , where  $g_p$  denotes the metric  $g$  evaluated at  $p$ . Then the set of orthonormal bases of  $T_p M$  is the set of all linear isometries  $(\mathbb{R}^n, \langle \cdot, \cdot \rangle) \rightarrow (M, g_p)$  where  $\langle \cdot, \cdot \rangle$  is the standard inner product on  $\mathbb{R}^n$ . Then taking the disjoint union over all points  $p \in M$  of orthonormal bases for the tangent spaces  $T_p M$  forms the **orthonormal frame bundle**  $\mathcal{B}_O(M)$ , which is a principal  $O_n$  bundle.  $\blacktriangleleft$

DEFINITION 1.12. Let  $\pi : P \rightarrow M$  be a principal  $G$ -bundle over  $M$ , and let  $F$  be a smooth manifold with a smooth left  $G$  action. Then the **associated fiber bundle**, denoted  $P \times_G F$ , is the set

$$P \times_G F = P \times G / (p, g) \sim (p \cdot h, h^{-1}g)$$

Since the group action on  $P$  preserves the fibers, the projection  $p_1 : P \times F \rightarrow P$  composed with the projection  $\pi : P \rightarrow M$  descends to the quotient, giving us a projection map  $\Phi : P \times_G F \rightarrow M$ .  $\blacklozenge$

The first thing to check is that  $P \times_G F$  is a fiber bundle, justifying the name.

PROPOSITION 1.13. Let  $\pi : P \rightarrow M$  be a principal  $G$ -bundle and  $F$  a manifold with a left  $G$  action. then the associated bundle  $\Phi : P \times_G F \rightarrow M$  is a fiber bundle with model fiber  $F$ .

PROOF. We wish to provide local trivializations  $\Phi^{-1}(U) \rightarrow U \times F$  for the associated bundle. Fix a local trivialization  $\psi : \pi^{-1}(U) \rightarrow U \times G$ . Then  $\psi$  is of the form  $\psi(p) = (\pi(p), \tilde{\psi}(p))$  for some  $\tilde{\psi} : \pi^{-1}(U) \rightarrow G$ , satisfying  $\tilde{\psi}(p \cdot g) = \tilde{\psi}(p)g$ . Define

$$\begin{aligned} \varphi : \Phi^{-1}(U) &\rightarrow U \times F \\ [p, f] &\mapsto (\pi(p), \tilde{\psi}(p) \cdot f) \end{aligned}$$

We first note that this is well defined on equivalence classes, since

$$[p \cdot g, g^{-1} \cdot f] \mapsto (\pi(p \cdot g), \tilde{\psi}(p \cdot g) \cdot g^{-1} \cdot f) = (\pi(p), \psi(p) \cdot f)$$

So  $\varphi$  is well defined. We also note that  $\Phi(\varphi[p, f]) = \pi(p)$  by how  $\Phi$  was defined, so  $\varphi$  is a local trivialization provided it is a homeomorphism. To show this, we construct an inverse. Define  $\alpha : U \times F \rightarrow \Phi^{-1}(U)$  by  $\alpha(u, f) = [\psi^{-1}(u, e), f]$ , where  $e$  denotes the identity element of  $G$ . We then claim that  $\alpha$  is the inverse. We compute

$$\begin{aligned} (\varphi \circ \alpha)(u, f) &= \varphi[\psi^{-1}(u, e), f] \\ &= (u, e \cdot f) \\ &= (u, f) \end{aligned}$$

In the other direction, we compute

$$\begin{aligned} (\alpha \circ \varphi)[p, f] &= \alpha(\pi(p), \tilde{\psi}(p)) \\ &= [\psi^{-1}(\pi(p), e), f] \\ &= [p, f] \end{aligned}$$

Therefore,  $\varphi$  is a local trivialization, giving us that  $\Phi : P \times_G F \rightarrow M$  is a fiber bundle with model fiber  $F$ . ■

There is a correspondence between sections of an associated bundle and  $G$ -equivariant maps  $P \rightarrow F$ .

**PROPOSITION 1.14.** *Let  $\pi : P \rightarrow M$  be a principal  $G$ -bundle. Then there is a bijection*

$$\{G\text{-equivariant maps } P \rightarrow F\} \longleftrightarrow \Gamma_M(P \times_G F)$$

**PROOF.** Since  $F$  has a left  $G$ -action, we use it to define a right  $G$ -action by  $f \cdot g = g^{-1} \cdot f$ . Then what we mean by a  $G$ -equivariant map is a map  $\varphi : P \rightarrow F$  such that

$$\varphi(p \cdot g) = g^{-1} \cdot \varphi(p)$$

We then wish to use  $G$ -equivariant map  $\varphi$  to produce a section  $\tilde{\varphi} : M \rightarrow P \times_G F$  of the associated bundle. For a point  $x \in M$ , pick any  $p \in \pi^{-1}(x)$  in the fiber. Then define

$$\tilde{\varphi}(x) = [p, \varphi(p)]$$

where  $[p, \varphi(p)]$  denotes the equivalence class of  $(p, \varphi(p))$  in  $P \times_G F$ . We first claim that this map is well defined, i.e. it is independent of our choice of point  $p \in \pi^{-1}(x)$ . We know that  $G$  acts freely and transitively on  $\pi^{-1}(x)$ , so all points in the fiber are of the form  $p \cdot g$  for a unique  $g \in G$ . Then we have that for any  $g \in G$

$$(p \cdot g, \varphi(p \cdot g)) = (p \cdot g, g^{-1} \cdot \varphi(p)) \sim (p, \varphi(p))$$

where we use the  $G$ -equivariance of  $\varphi$  and the definition of the equivalence relation on the associated bundle. In addition, this induced map is a section of  $P \times_G F \rightarrow M$ , since the image is represented by an element of  $P \times F$  with an element of the fiber  $\pi^{-1}(x)$  in the first factor.

Conversely, given a section  $\sigma : M \rightarrow P \times_G F$ , we wish to produce a  $G$ -equivariant map  $P \rightarrow F$ . Given such a section, and a point  $x \in M$ , we have that  $\sigma(x) = [p, f]$  for some  $p \in P$  and  $f \in F$ . Then define  $\tilde{\sigma} : P \rightarrow F$  such that  $\tilde{\sigma}(p) = f$ , and  $\tilde{\sigma}(p \cdot g) = g^{-1} \cdot f$ . Since  $G$  acts freely and transitively on the fibers, this specifies the map on every point of  $P$ . In addition,  $\tilde{\sigma}$  is  $G$ -equivariant by construction. Then the maps

$$\{G\text{-equivariant maps } P \rightarrow F\} \longleftrightarrow \Gamma_M(P \times_G F)$$

we provided are easily verified to be inverses to each other, giving us the correspondence. ■

In some sense, the geometry of the associated fiber bundle  $P \times_G F$  is controlled by the group  $G$ , as  $G$  determines a distinguished group of symmetries of the fiber  $F$ . Given an element  $p \in P$ , the element  $p$  determines a diffeomorphism  $\varphi_p : F \rightarrow (P \times_G F)_{\pi(p)}$  via the mapping  $f \mapsto [p, f]$ , and the group action of  $G$  on  $P$  gives a transformation law on the associated bundle via the group action of  $G$  on  $F$ .

EXAMPLE 1.15 (The tangent bundle as an associated bundle). Given a manifold  $M$ , we can take the frame bundle  $\pi : \mathcal{B}(M) \rightarrow M$ , which is a principal  $GL_n \mathbb{R}$  bundle. The group  $GL_n \mathbb{R}$  acts linearly on  $\mathbb{R}^n$  in the standard way, giving us an associated vector bundle  $\mathcal{B}(M) \times_{GL_n \mathbb{R}} \mathbb{R}^n$ . We claim that this bundle is isomorphic to the tangent bundle  $TM$ , i.e. there exists an diffeomorphism  $\varphi : \mathcal{B}(M) \times_{GL_n \mathbb{R}} \mathbb{R}^n \rightarrow TM$  that restricts to linear isomorphisms on the fibers and the diagram

$$\begin{array}{ccc} \mathcal{B}(M) \times_{GL_n \mathbb{R}} \mathbb{R}^n & \xrightarrow{\varphi} & TM \\ & \searrow & \swarrow \\ & M & \end{array}$$

commutes, where the maps to  $M$  are the bundle projections. Recall that elements of  $\mathcal{B}(M) \times_{GL_n \mathbb{R}} \mathbb{R}^n$  are represented by pairs  $(b, v)$ , where  $b : \mathbb{R}^n \rightarrow T_{\pi(b)} M$  is a linear isomorphism, and  $v$  is a vector in  $\mathbb{R}^n$ . Then define  $\varphi$  by

$$\varphi[b, v] = (\pi(b), b(v))$$

This is well defined, since  $\varphi[b \circ g, g^{-1}(v)] = (\pi(b \circ g), (b \circ g)(g^{-1}(v))) = (\pi(b), b(v))$ . This is an isomorphism, where the inverse mapping maps  $(p, v) \in TM$  to  $(b, \tilde{v})$  where  $b$  is any basis of  $T_p M$  and  $\tilde{v}$  is the coordinate representation of  $v$  in the basis  $b$ . In this example, we see that the associated bundle identifies the same vector under different coordinate transformations, which defines the symmetries of  $TM$ .  $\blacktriangleleft$

In a similar fashion, the cotangent bundle  $T^*M$  to a manifold  $M$  is naturally isomorphic to the associated bundle  $\mathcal{B}(M) \times_{GL_n \mathbb{R}} (\mathbb{R}^n)^*$ , where  $GL_n \mathbb{R}$  acts on  $(\mathbb{R}^n)^*$  by the dual of the standard representation (i.e.  $A \in GL_n \mathbb{R}$  acts by  $(A^{-1})^*$ .)

In general, given a rank  $n$  vector bundle  $E \rightarrow M$ , we can construct the frame bundle  $\mathcal{B}(E)$  for  $E$ , where the fiber over  $x \in M$  is the  $GL_n \mathbb{R}$ -torsor of bases for the fiber  $E_x$ . We can then recover  $E$  by taking the associated bundle  $\mathcal{B}(E) \times_{GL_n \mathbb{R}} \mathbb{R}^n$ , so the process of taking frames and constructing associated bundles are inverses of each other. In general, if  $E$  is an associated vector bundle  $P \times_G V$  obtained via a linear representation  $\rho : G \rightarrow GL(V)$ , the elements of  $P$  have a natural interpretation as bases of the fibers of  $E$ . In this case, each element  $p \in P$  determines a linear isomorphism  $\varphi_p : V \rightarrow E_p$ , giving a  $G$ -equivariant  $\varphi : P \rightarrow \mathcal{B}(E)$  of  $P$  to the frame bundle of  $E$ , where  $G$  acts on  $\mathcal{B}(E)$  through the representation  $\rho$ .

DEFINITION 1.16. Let  $\pi : P \rightarrow M$  be a principal  $G$ -bundle, and  $\rho : H \rightarrow G$  a group homomorphism. The map  $\rho$  gives  $G$  a left  $H$  action where  $h \cdot p = \rho(h) \cdot p$ . A **reduction of structure group** is the data of a principal  $H$  bundle  $\varphi : Q \rightarrow M$  and an  $H$ -equivariant bundle homomorphism  $F : Q \rightarrow P$ .  $\blacklozenge$

The map  $F : Q \rightarrow P$  induces a map  $\tilde{F} : Q \times_H G \rightarrow P$ , where we map the equivalence class  $[q, g]$  to  $F(q)g$ . This is well defined on equivalence classes since

$$(q \cdot h, \rho(h)^{-1}g) \mapsto F(q \cdot h)\rho(h)^{-1}g = F(q)\rho(h)\rho(h)^{-1}g = F(q)g$$

EXAMPLE 1.17 (Reduction from  $GL_n \mathbb{R}$  to  $O_n$ ). Let  $M$  be a smooth manifold, and  $\pi : \mathcal{B}(M) \rightarrow M$  its bundle of frames. The inclusion map  $\iota : O_n \hookrightarrow GL_n \mathbb{R}$  gives an action of  $O_n$  on  $\mathcal{B}(M)$ , where given  $b \in \mathcal{B}(M)$  and  $T \in O_n$ ,  $b \cdot T = b \cdot \iota(T)$ . We then take the quotient by this  $O_n$  action, giving us a quotient map  $q : \mathcal{B}(M) \rightarrow \mathcal{B}(M)/O_n$ . Since the inclusion is injective, the  $O_n$  action is free on

$\mathcal{B}(M)$ , so this gives  $q : \mathcal{B}(M) \rightarrow \mathcal{B}(M)/O_n$  the structure of a  $O_n$  bundle. In addition, the action of  $O_n$  preserves the fibers of  $\pi : \mathcal{B}(M) \rightarrow M$ , so  $\pi$  descends to the quotient, so  $\mathcal{B}(M)/O_n \rightarrow M$  is a fiber bundle over  $M$  with model fiber  $GL_n\mathbb{R}/O_n$ . Since  $GL_n\mathbb{R}$  deformation retracts onto  $O_n$  via the Gram-Schmidt algorithm,  $GL_n\mathbb{R}/O_n$  is contractible, so the fiber bundle  $\mathcal{B}(M)/O_n \rightarrow M$  admits global sections. Then given a section  $\sigma : M \rightarrow \mathcal{B}(M)/O_n$ , this gives an  $O_n$  bundle over  $M$  via the pullback  $\sigma^*\mathcal{B}(M)$ . In addition, we get a  $O_n$ -equivariant map  $\sigma^*\mathcal{B}(M) \rightarrow \mathcal{B}(M)$  given by  $(p, b) \mapsto (p, \iota(b))$ . The bundle  $\sigma^*\mathcal{B}(M)$  can be thought of as the bundle of orthonormal frames with respect to a Riemannian metric on  $M$ , and the fact that  $\mathcal{B}(M)/O_n$  admits sections corresponds to the fact that every manifold admits a Riemannian metric.  $\blacktriangleleft$

Principal bundles can be thought of as a generalization of covering spaces. For discrete groups  $G$ , the data of a principal  $G$ -bundle  $P \rightarrow M$  is the data of a (possibly disconnected) cover of  $M$  with a deck transformation group isomorphic to  $G$  acting freely and transitively on the fibers. If  $G$  is finite, this is a  $|G|$ -fold covering, and if  $G$  is countable, the cover is an infinitely sheeted cover. Discreteness of  $G$  allows for a very complete description of all principal  $G$ -bundles over a given manifold  $M$ .

**DEFINITION 1.18.** Let  $(M, x_0)$  be a pointed topological space (in our case, usually a manifold), i.e. a space  $M$  with a choice of distinguished basepoint  $x_0 \in M$ . Then a **pointed principal  $G$ -bundle** over  $M$  is the data of a principal  $G$ -bundle  $\pi : P \rightarrow M$ , along with a choice of basepoint  $p_0 \in \pi^{-1}(x_0)$ . We denote this as  $\pi : (P, p_0) \rightarrow (M, x_0)$ . We denote the set of isomorphism classes of pointed principal  $G$ -bundles over  $(M, x_0)$  as  $\text{Bun}_G(M, x_0)$ .  $\blacklozenge$

We now recall some properties of covering spaces. A pointed covering space  $\pi : (\tilde{M}, \tilde{x}_0) \rightarrow (M, x_0)$  has the **path lifting** property, i.e. given a path  $\gamma : I \rightarrow M$  with  $\gamma(0) = x_0$ , there exists a unique path  $\tilde{\gamma} : I \rightarrow \tilde{M}$  with  $\tilde{\gamma}(0) = \tilde{x}_0$  such that  $\pi \circ \tilde{\gamma} = \gamma$ . In addition, the covering space has the **homotopy lifting property**. Given paths  $\gamma_1, \gamma_2 : I \rightarrow M$  with  $\gamma_i(0) = x_0$  and a homotopy  $F : I \times I \rightarrow M$  from  $\gamma_1$  to  $\gamma_2$ , there exists a unique homotopy  $\tilde{F} : I \times I \rightarrow \tilde{M}$  between  $\tilde{\gamma}_1$  and  $\tilde{\gamma}_2$ . These two facts define an action of  $\pi_1(M, x_0)$  on the fiber  $\pi^{-1}(x_0)$  where given a homotopy class  $[\gamma]$  of loops, the action of  $[\gamma]$  on  $p \in \pi^{-1}(x_0)$  is the endpoint of  $\tilde{\gamma}(1)$  of the lifted loop  $\tilde{\gamma}$ . The unique path lifting and homotopy lifting properties guarantees that this defines a group action, called the **holonomy** of the covering space. In the special case that our covering space is a principal  $G$ -bundle with a discrete  $G$ , the point  $\tilde{\gamma}(1)$  is equal to  $\tilde{x}_0 \cdot g$  for a unique group element  $g \in G$ , so the holonomy action determines a group homomorphism  $\pi_1(M, x_0) \rightarrow G$ . In fact, the holonomy completely determines the bundle.

**THEOREM 1.19.** For a discrete group  $G$  and a connected manifold  $M$ , there is a bijection

$$\text{Bun}_G(M, x_0) \longleftrightarrow \text{Hom}(\pi_1(M, x_0), G)$$

**PROOF.** We once more provide maps in both directions. Every pointed principal  $G$ -bundle  $(P, p_0) \rightarrow (M, x_0)$  determines a group homomorphism  $\pi_1(M, x_0) \rightarrow G$  by the holonomy action. In the other direction, given a group homomorphism  $\varphi : \pi_1(M, x_0) \rightarrow G$ , we want to construct a pointed principal  $G$  bundle  $(P, p_0) \rightarrow (M, x_0)$  with holonomy  $\varphi$ . Every connected pointed manifold  $(M, x_0)$  has a **universal cover**  $(\tilde{M}, \tilde{x}_0) \rightarrow (M, x_0)$ , which has a free and transitive action by  $\pi_1(M, x_0)$ , making it a pointed principal  $\pi_1(M, x_0)$  bundle over  $(M, x_0)$ . In addition, the homomorphism  $\varphi : \pi_1(M, x_0) \rightarrow G$  determines a left action of  $\pi_1(M, x_0)$  on  $G$  by  $[\gamma] \cdot g = \varphi[\gamma]g$ . This allows us to define the associated bundle

$$P = \tilde{M} \times_{\pi_1(M, x_0)} G = \tilde{M} \times G / (x, g) \sim (x \cdot [\gamma], \varphi[\gamma]^{-1}g)$$

where our distinguished basepoint  $p_0 \in P$  is  $p_0 = [\tilde{x}_0, e]$ . This is a fiber bundle with model fiber  $G$ , and the right action of  $G$  on  $\tilde{M} \times G$  descends to a free and transitive action on the quotient, making it

a principal  $G$ -bundle. We then check the holonomy action of a loop  $[\gamma] \in \pi_1(M, x_0)$ . We compute for  $p$  in the fiber over  $x_0$

$$p \cdot [\gamma] = [\tilde{x}_0 \cdot [\gamma], h] \sim [x_0, \varphi[\gamma] \cdot h]$$

So the holonomy is  $\varphi$ , giving us the desired correspondence. ■

The set  $\text{Bun}_G(M, x_0)$  comes equipped with an action of  $G$ , where the action of  $g \in G$  on a pointed bundle  $(P, p_0) \rightarrow (M, x_0)$  is by permuting basepoints, i.e.  $(P, p_0) \cdot g = (P, p_0 \cdot g)$ , where the covering map is left untouched. Quotienting by this group action gives us the identification of  $\text{Bun}_G(M, x_0)/G$  with isomorphism classes of principal  $G$ -bundles  $P \rightarrow M$ , where the isomorphisms are without basepoints. From the perspective of pointed bundles  $(P, p_0) \rightarrow (M, x_0)$  as homomorphisms  $\varphi : \pi_1(M, x_0) \rightarrow G$ , the group action of  $G$  on  $\text{Bun}_G(M, x_0)$  is an action by conjugation. Given  $(P, p_0) \rightarrow (M, x_0)$  with holonomy  $\varphi : \pi_1(M, x_0) \rightarrow G$ , we want to know the holonomy  $\varphi_g$  of the bundle  $(P, p_0 \cdot g) \rightarrow (M, x_0)$ . Given a loop  $\gamma$  based at  $x_0$ , let  $\tilde{\gamma}$  denote its lift to  $(P, p_0)$  by uniqueness of path lifting, the lift of  $\gamma$  to  $(P, p_0 \cdot g)$  is the path  $\tilde{\gamma}_g(t) = \tilde{\gamma}(t) \cdot g$ . Then

$$\begin{aligned} \gamma_g(1) &= \tilde{\gamma}(1) \cdot g \\ &= p_0 \cdot \varphi[\gamma] \cdot g \\ &= p_0 \cdot g \cdot g^{-1} \cdot \varphi[\gamma] \cdot g \end{aligned}$$

So  $\varphi_g = g^{-1} \cdot \varphi \cdot g$ .

## 2. Dirac Operators in $\mathbb{R}^n$

One important application of Clifford algebras and Spin groups comes from physics. In the process of developing a relativistic equation for the electron, Paul Dirac saw the need for a first order differential operator  $D$  such that  $D$  squared to the Laplace operator.<sup>1</sup>

$$\Delta = - \sum_i \frac{\partial^2}{\partial (x^i)^2}$$

If  $D$  were the be first order, it would have to be written as

$$D = a^i \frac{\partial}{\partial x^i}$$

for some coefficients  $a^i$ . However, it is clear that choosing scalar coefficients for the  $a^i$  will not suffice. For example, in  $\mathbb{R}^2$  with the standard coordinates  $x$  and  $y$ , any first order operator  $D = a^1 \partial_x + a^2 \partial_y$  satisfies

$$D^2 = \left( a^1 \frac{\partial}{\partial x} + a^2 \frac{\partial}{\partial y} \right)^2 = (a^1)^2 \frac{\partial^2}{\partial x^2} + a^1 a^2 \frac{\partial^2}{\partial x \partial y} + a^2 a^1 \frac{\partial^2}{\partial y \partial x} + (a^2)^2 \frac{\partial^2}{\partial y^2}$$

From this equation, we see that the  $a^i$  must square to  $-1$ , and we must also have that  $a^1 a^2 + a^2 a^1 = 0$  in order for the mixed partial terms to vanish. This is not possible if the  $a^i$  are scalars (in either  $\mathbb{R}$  or  $\mathbb{C}$ ). However, the required relations are exactly the relations between orthogonal basis vectors in the Clifford algebra  $\text{Cliff}_{0,n}(\mathbb{R})$ !

**DEFINITION 2.1.** Let  $\{e_i\}$  be the standard basis for  $\mathbb{R}^n$ ,  $\{e^i\}$ <sup>2</sup> its dual basis, and  $x^1, \dots, x^n$  the standard coordinates on  $\mathbb{R}^n$ . The **Dirac operator** on  $\mathbb{R}^n$  is the first order differential operator

$$D = e^i \frac{\partial}{\partial x^i}$$

<sup>1</sup>One reason for choosing this sign convention for  $\Delta$  is that the spectrum of  $\Delta$  is positive with this choice of sign

<sup>2</sup>The choice of using the dual basis elements  $e^i$  instead of the  $e_i$  is so that  $D$  behaves tensorially with respect to coordinate change.



It is not clear from the definition what function space  $D$  should act on. The partial derivative operators make sense for any vector valued function, but multiplication by  $e^i$  does not make sense on an arbitrary vector space – it must be a Clifford module. Therefore,  $D$  is an operator  $D : C^\infty(\mathbb{R}^n, M) \rightarrow C^\infty(\mathbb{R}^n, M)$  on smooth functions from  $\mathbb{R}^n$  to a Clifford module  $M$ . In fuller generality, functions from  $\mathbb{R}^n \rightarrow M$  are equivalent to smooth sections of  $\mathbb{R}^n \times M \rightarrow \mathbb{R}^n$ , which is a bundle of Clifford modules over  $\mathbb{R}^n$ . This viewpoint generalizes more naturally, and the Dirac operator defined on a general Riemannian manifold is an operator on sections of some bundle.

The Dirac operators in dimensions 1, 2, and 4 exhibit some extremely interesting behavior, corresponding to the appearances of  $\mathbb{C}$  and  $\mathbb{H}$  when  $\mathbb{R}^n$  is given a negative definite bilinear form. In 1 dimension, the Clifford algebra  $\text{Cliff}_{0,1}(\mathbb{R})$  admits the ordered basis  $(1, e^1)$ , which then gives a basis for  $C^\infty(\mathbb{R}, \text{Cliff}_{0,1}(\mathbb{R}))$  as a  $C^\infty(\mathbb{R})$  module where for any  $f : \mathbb{R} \rightarrow \text{Cliff}_{0,1}(\mathbb{R})$ , we have the decomposition  $f(x) = u(x) + e^1 v(x)$ , giving a column vector representation

$$f = \begin{pmatrix} u \\ v \end{pmatrix}$$

The Dirac operator is  $e^1 \partial_x$ , which is represented in matrix form as

$$D = \begin{pmatrix} 0 & -\partial_x \\ \partial_x & 0 \end{pmatrix}$$

Note that the matrix is block off-diagonal. The vector space  $C^\infty(\mathbb{R}, \text{Cliff}_{0,1}(\mathbb{R}))$  has a natural grading where the even elements are maps  $\mathbb{R} \rightarrow \text{Cliff}_{0,1}^0(\mathbb{R})$  and the odd elements are maps  $\mathbb{R} \rightarrow \text{Cliff}_{0,1}^1(\mathbb{R})$ . Since we picked the ordered basis  $(1, e^1)$  for  $\text{Cliff}_{0,1}(\mathbb{R})$ , whose elements are even and odd respectively, the bottom left block of the matrix is action of  $D$  as a map from the even subspace to the odd subspace, and the top right block of the matrix is the action of  $D$  as a map from the odd subspace to the even subspace. This shows that  $D$  is an odd operator, as it reverses the grading on  $C^\infty(\mathbb{R}, \text{Cliff}_{0,1})$ .

In 2 dimensions,  $\text{Cliff}_{0,2}(\mathbb{R})$  has the ordered basis  $(1, e^1 e^2, e_1, e_2)$ , where we choose the ordering by the parity of the elements. Again, we can write any function  $f : \mathbb{R}^2 \rightarrow \text{Cliff}_{0,2}(\mathbb{R})$  in terms of this basis as

$$f(x, y) = f_0(x, y) + e^1 e^2 f_{12}(x, y) + e^1 f_1(x, y) + e^2 f_2(x, y)$$

where the component functions are all elements of  $C^\infty(\mathbb{R}^2)$ . In this basis, the action of the Dirac operator  $D = e^1 \partial_x + e^2 \partial_y$  is represented by the matrix of differential operators

$$D = \begin{pmatrix} 0 & 0 & -\partial_x & -\partial_y \\ 0 & 0 & -\partial_y & \partial_x \\ \partial_x & \partial_y & 0 & 0 \\ \partial_y & -\partial_x & 0 & 0 \end{pmatrix}$$

which is actually a familiar differential operator in disguise. We make the identification of  $\text{Cliff}_{0,2}(\mathbb{R})$  with the quaternions  $\mathbb{H}$ , which have a direct sum decomposition  $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}$ , giving a  $\mathbb{Z}/2\mathbb{Z}$ -grading. The mappings  $e_1 e_2 \mapsto i$  and  $e_2 \mapsto i$  determine vector space isomorphisms of the even and odd subspaces of  $\text{Cliff}_{0,2}(\mathbb{R})$  with  $\mathbb{C}$ , giving an isomorphism of  $\text{Cliff}_{0,2}(\mathbb{R})$  with  $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}$ . The operator in the top right block is then exactly twice the differential operator

$$-\frac{\partial}{\partial z} = -\frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$$



on functions  $\mathbb{R}^2 \rightarrow \mathbb{C}$ , and the operator in the bottom block is the twice the operator

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

So  $D$  is more compactly represented under this identification as

$$D = \begin{pmatrix} 0 & 2\partial_z \\ 2\partial_{\bar{z}} & 0 \end{pmatrix}$$

In 4 dimensions, the irreducible module over  $\text{Cliff}_{0,3} \cong M_2\mathbb{H}$  is  $\mathbb{H}^2$ . This has a direct sum decomposition as  $\mathbb{H}^2 = \mathbb{H} \oplus \mathbb{H}$ , allowing us to represent functions  $f : \mathbb{R}^4 \rightarrow \mathbb{H}^2$  as pairs of functions  $\mathbb{R}^4 \rightarrow \mathbb{H}$ , which are the maps into the even and odd subspaces of  $\mathbb{H}^2 = \mathbb{H} \oplus \mathbb{H}$ . The quaternions admit operators similar to the operators  $\partial_z$  and  $\partial_{\bar{z}}$ , which are

$$\begin{aligned} \frac{\partial}{\partial q} &= \frac{1}{4} \left( \frac{\partial}{\partial x^1} - i \frac{\partial}{\partial x^2} - j \frac{\partial}{\partial x^3} - k \frac{\partial}{\partial x^4} \right) \\ \frac{\partial}{\partial \bar{q}} &= \frac{1}{4} \left( \frac{\partial}{\partial x^1} + i \frac{\partial}{\partial x^2} + j \frac{\partial}{\partial x^3} + k \frac{\partial}{\partial x^4} \right) \end{aligned}$$

Using these identifications, the Dirac operator is compactly represented as

$$D = \begin{pmatrix} 0 & -4\partial_q \\ 4\partial_{\bar{q}} & 0 \end{pmatrix}$$

which encodes quaternionic analogues of the Cauchy-Riemann equations.

### 3. Spin Structures

Every manifold  $M$  admits a Riemannian metric, so there always exists a reduction of structure group from  $GL_n\mathbb{R}$  to  $O_n$ . If  $M$  is orientable, over each point  $p \in M$ , we have a preferred set of orthonormal bases of  $T_pM$ , which correspond to a choice of component of the  $O_n$ -torsor of orthonormal frames of  $T_pM$ . Doing this for each point  $p \in M$ , this determines a subbundle  $P$  of  $\mathcal{B}_O(M)$ . In addition, each fiber has a free and transitive action of  $SO_n$  via the inclusion  $SO_n \hookrightarrow O_n$ , giving it the structure of a principal  $SO_n$ -bundle over  $M$ . In addition, the inclusion  $P \hookrightarrow \mathcal{B}_O(M)$  is  $SO_n$ -equivariant, so it defines a reduction of structure group from  $O_n$  to  $SO_n$ . We denote  $P = \mathcal{B}_{SO}(M)$ , and call it the **bundle of oriented orthonormal frames**.

A **Spin structure** on  $M$  is a further reduction of structure group to  $\text{Spin}_n$  with respect to the double cover  $\text{Spin}_n \rightarrow SO_n$ , i.e. the data of a principal  $\text{Spin}_n$  bundle  $P \rightarrow M$ , along with a  $\text{Spin}_n$ -equivariant map  $P \rightarrow \mathcal{B}_{SO}(M)$ , where  $\mathcal{B}_{SO}(M)$  denotes the bundle of positively oriented orthonormal frames and  $\text{Spin}_n$  acts on  $\mathcal{B}_{SO}(M)$  through the double cover  $\text{Spin}_n \rightarrow SO_n$ . In particular, this map is a double cover, and fiberwise is the double covering  $\text{Spin}_n \rightarrow SO_n$ . Depending on the value  $n$ , there exist either one or two irreducible representations for  $\text{Spin}_n$ . Let  $S$  denote the direct sum of all the irreducible Spin representations for  $\text{Spin}_n$ , i.e.  $S$  is equal to the single irreducible Spin representation in the case that  $\text{Spin}_n$  only admits one irreducible representation, and  $S = S^+ \oplus S^-$  in the case that  $\text{Spin}_n$  admits two inequivalent irreducible representations  $S^+$  and  $S^-$ . Then given a Spin structure on  $M$  with a principal  $\text{Spin}_n$  bundle  $P$ , the **spinor bundle** over  $M$  is the associated vector bundle  $P \times_{\text{Spin}_n} S$ , which can be thought of as a Spin analogue to the tangent bundle of  $M$ .

**EXAMPLE 3.1** (Spin structures on  $S^1$ ). The group  $\text{Spin}_1$  is equal to  $\{\pm 1\}$ , so a Spin structure on  $S^1$  is a double cover  $\pi : P \rightarrow S^1$  along with a  $\text{Spin}_1$ -equivariant map  $P \rightarrow \mathcal{B}_{SO_1}(S^1)$ . Since  $SO_1 = 1$ ,  $\mathcal{B}_{SO}(S^1)$  is the trivial bundle  $S^1 \times \{1\}$ . Therefore, specifying a  $\text{Spin}_1$ -equivariant map  $P \rightarrow S^1 \times \{1\}$  is no additional data, since we are forced to map all of the fiber  $\pi^{-1}(x)$  to  $(x, 1)$  for any  $x \in S^1$ . Consequently, all double covers give rise to a reduction of structure group to  $\text{Spin}_1$ . There are only two double covers of  $S^1$  up to isomorphism. One of them is the disconnected double cover, which

is the disjoint union  $S^1 \amalg S^1$ , which we denote as  $\pi_1 : P_1 \rightarrow S^1$ . The other is the connected double cover, which the circle covering itself via the map  $z \mapsto z^2$ , which we denote as  $\pi_2 : P_2 \rightarrow S^1$ . For convenience, we parameterize  $P_2$  with angles  $\theta \in [0, 4\pi)$ , so the covering map is given by  $\theta \mapsto e^{i\theta}$ . The Spin representation is the sign representation on  $\mathbb{R}$ , where  $-1$  acts by multiplication by  $-1$ , and the complexifying gives us an action on  $\mathbb{C}$  where  $-1$  acts by multiplication by  $-1$ , giving us two spinor bundles  $P_1 \times_{\text{Spin}_1} \mathbb{C}$  and  $P_2 \times_{\text{Spin}_1} \mathbb{C}$ .

In the first case, the associated bundle is a trivial bundle. Using the identification with  $\text{Spin}_1$ -equivariant maps  $P_1 \rightarrow \mathbb{C}$  with sections of the associated bundle, it suffices to find such a map to produce a global section of  $\pi_1 : P_1 \rightarrow S^1$ . Write  $P_1$  as the disjoint union  $S_1 \amalg S_1$ , with the circles parameterized by angles  $\theta, \varphi \in [0, 2\pi)$ . The action of  $-1$  is given by  $\theta \mapsto \varphi$  and  $\varphi \mapsto \theta$ . Then the mappings  $\theta \mapsto e^{i\theta}$  and  $\varphi \mapsto e^{i\varphi}$  define a  $\text{Spin}_1$ -equivariant map  $P_1 \rightarrow \mathbb{C}$ , giving us a trivialization of the associated bundle  $P_1 \times_{\text{Spin}_1} \mathbb{C}$ . In addition, we see that sections of  $P_1 \times_{\text{Spin}_1} \mathbb{C}$  are equivalent data to maps  $S^1 \rightarrow \mathbb{C}$ , since once we map one of the components of  $P_1$  into  $\mathbb{C}$ , this entirely determines how we need to map the other component in order to remain  $\text{Spin}_1$ -equivariant. This further allows us to identify sections of the spinor bundle with  $2\pi$ -periodic functions  $\mathbb{R} \rightarrow \mathbb{C}$ .

In the second case, the bundle is also trivial! We again construct a trivialization for the associated bundle by providing a  $\text{Spin}_1$ -equivariant map  $\sigma : P_2 \rightarrow \mathbb{C}$ . Using the parameterization of  $P_2$  with angles  $\theta \in [0, 4\pi)$ , the  $\text{Spin}_1$  action on  $P_2$  is given by  $-1 \cdot \theta = \theta + 2\pi \bmod 4\pi$ . Then define  $\sigma$  by  $\sigma(\theta) = e^{i\theta/2}$ . This map is  $\text{Spin}_1$ -equivariant, so it produces a trivialization of the spinor bundle  $P_2 \times_{\text{Spin}_1} \mathbb{C}$ . In addition, we see that sections of the spinor bundle correspond to  $2\pi$ -antiperiodic functions, i.e. functions  $\psi : \mathbb{R} \rightarrow \mathbb{C}$  such that  $\psi(\theta) = -\psi(\theta + 2\pi)$ , using the fact that we parameterized  $P_2$  with angles from  $[0, 4\pi)$ . Another way to write a  $2\pi$ -antiperiodic map  $\psi$  is as a product  $\psi(\theta) = e^{i\theta/2} f(\theta)$  for a  $2\pi$ -periodic function  $f$ , which is the representation of the section  $\psi$  with respect to the trivialization  $\sigma$  defined above.

The two different spin structures produced two isomorphic vector bundles, but there is still a way to distinguish between the two – their Dirac operators. The Clifford algebra  $\text{Cliff}_{0,1}(\mathbb{R})$  is isomorphic to  $\mathbb{C}$  as an  $\mathbb{R}$ -algebra via the mappings  $1 \mapsto 1$ , and  $e_1 \mapsto i$ , so the Dirac operator on  $\mathbb{R}$  can also be written as  $i \frac{d}{dt}$ . We then use our identifications of sections of the spinor bundles with functions  $\mathbb{R} \rightarrow \mathbb{C}$  to investigate the Dirac operators on each bundle. In the case of the disconnected double cover  $P_1$ , we have the identifications of sections of  $P_1 \times_{\text{Spin}_1} \mathbb{C}$  with  $2\pi$ -periodic functions  $\mathbb{R} \rightarrow \mathbb{C}$ . Then given a section  $\psi$ , we can identify it as a function  $\mathbb{R} \rightarrow \mathbb{C}$ , and use the Dirac operator in  $\mathbb{R}$ , which will again be a  $2\pi$ -periodic function, giving us another section, giving us that the Dirac operator  $D_1$  on sections of  $P_1 \times_{\text{Spin}_1} \mathbb{C}$  is just  $i \frac{d}{d\theta}$ . For the connected double cover, we used the global section associated to  $\sigma(\theta) = e^{i\theta/2}$  to identify sections of  $P_2 \times_{\text{Spin}_1} \mathbb{C}$  as products  $\psi(\theta) = e^{i\theta/2} f(\theta)$  for a  $2\pi$ -periodic function  $f : \mathbb{R} \rightarrow \mathbb{C}$ . Then applying the Dirac operator from  $\mathbb{R}$  to this function, we get

$$\begin{aligned} D\psi(\theta) &= i \frac{d}{d\theta} \left( e^{i\theta/2} f(\theta) \right) \\ &= e^{i\theta/2} \frac{df}{d\theta} - \frac{1}{2} e^{i\theta/2} \end{aligned}$$

So in the local trivialization  $\sigma(\theta) = e^{i\theta/2}$ , the operator  $D_2$  operates on  $2\pi$ -periodic functions, just like  $D_1$ , and is given by  $D_2 = i\partial_\theta - \frac{1}{2}$ . In particular, the first operator  $D_1$  has integer spectrum, and the spectrum of  $D_2$  is the spectrum of  $D_1$  shifted by  $\frac{1}{2}$ , which allows us to distinguish to two Spin structures on  $S^1$  by their Dirac operators.  $\blacktriangleleft$

Given an oriented manifold  $M$ , we get a reduction of structure group of the frame bundle  $\mathcal{B}(M)$  to  $SO_n$ , giving us a principal  $SO_n$  bundle  $\mathcal{B}_{SO}(M)$ . If  $M$  has a nonempty boundary, this induces an orientation on  $\partial M$  in the following way: by first reducing the structure group to  $O_n$ , we get a

Riemannian metric  $g$  on  $M$ . Then given a point  $p \in \partial M$ , the tangent space of the boundary  $T_p \partial M$  is a codimension 1 subspace of  $T_p M$ . The Riemannian metric  $g$  allows us to pick a distinguished complementary subspace to  $T_p \partial M$  – the orthogonal complement  $(T_p \partial M)^\perp$ . From this subspace, we have a distinguished choice of vector – the outward normal vector. In appropriate coordinates, the inclusion of the tangent space  $T_p \partial M$  is given locally by the inclusion

$$\begin{aligned} \mathbb{R}^{n-1} &\hookrightarrow \mathbb{R}^n \\ (x^1, \dots, x^{n-1}) &\mapsto (0, x^1, \dots, x^{n-1}) \end{aligned}$$

and the outward normal is the unit length vector with a positive first component. This defines a vector field  $N$  along  $\partial M$ , where the value  $N_p$  of  $N$  at the point  $p$  is the outward normal vector in  $T_p M$ . On the boundary  $\partial M$ , we restrict the frame bundle  $\mathcal{B}_{SO}(M)$  to  $\partial M$  by pulling back by the inclusion  $\partial M \hookrightarrow M$ , giving us the restricted bundle  $\mathcal{B}_{SO}(M)|_{\partial M}$ , which is a principal  $SO_n$ -bundle over  $\partial M$ . An element  $b \in \mathcal{B}_{SO}(M)$  is an orientation preserving linear isometry  $(\mathbb{R}, \langle \cdot, \cdot \rangle) \rightarrow (T_p M, g_p)$ , and using the normal vector, we define a subbundle  $\mathcal{B}_{SO}(\partial M) \subset \mathcal{B}_{SO}(M)|_{\partial M}$  by

$$\mathcal{B}_{SO}(\partial M) = \{(p, b) \in \mathcal{B}_{SO}(M)|_{\partial M} : b(e_1) = N_p\}$$

where  $e_1$  denotes the first standard basis vector of  $\mathbb{R}^n$ , and  $\pi$  is the bundle projection. We then get an  $SO_{n-1}$  action on  $\mathcal{B}_{SO}(\partial M)$ , where we include  $SO_{n-1} \hookrightarrow SO_n$  as matrices of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$$

where  $A \in SO_{n-1}$ . This then acts on each fiber of  $\mathcal{B}_{SO}(\partial M)$  by precomposition. This action is free and transitive, so this gives  $\mathcal{B}_{SO}(\partial M)$  the structure of a principal  $SO_{n-1}$ -bundle over  $\partial M$ . In addition, this bundle comes with a natural map to  $\mathcal{B}_O(\partial M)$ , which is just the inclusion map, so it is a reduction of structure group of  $O_{n-1}$  to  $SO_{n-1}$ . In this way, we see that an orientation on  $M$  determines an orientation on the boundary. The preceding discussion is summarized in the diagram

$$\begin{array}{ccccc} \mathcal{B}_{SO}(\partial M) & \hookrightarrow & \mathcal{B}_{SO}(M)|_{\partial M} & \hookrightarrow & \mathcal{B}_{SO}(M) \\ & \searrow & \downarrow & & \downarrow \\ & & \partial M & \hookrightarrow & M \end{array}$$

In a similar fashion, a Spin structure on  $M$  will also induce a Spin structure on  $\partial M$ , though the process is slightly more involved. Given a Spin manifold  $M$ , it comes equipped with a principal  $\text{Spin}_n$ -bundle  $\mathcal{B}_{\text{Spin}}(M)$  along with a  $\text{Spin}_n$ -equivariant map  $\mathcal{B}_{\text{Spin}}(M) \rightarrow \mathcal{B}_{SO}(M)$ , where the Spin action on  $\mathcal{B}_{SO}(M)$  is induced by the double cover  $\text{Spin}_n \rightarrow SO_n$ . Just as before, we pull back both  $\mathcal{B}_{SO}(M)$  and  $\mathcal{B}_{\text{Spin}}(M)$  along the inclusion  $\partial M \hookrightarrow M$ . In addition, we construct the  $SO_{n-1}$ -bundle  $\mathcal{B}_{SO}(\partial M)$ , which gives us the diagram

$$\begin{array}{ccccc} & & \mathcal{B}_{\text{Spin}}(M)|_{\partial M} & \hookrightarrow & \mathcal{B}_{\text{Spin}}(M) \\ & & \downarrow & & \downarrow \\ \mathcal{B}_{SO}(\partial M) & \hookrightarrow & \mathcal{B}_{SO}(M)|_{\partial M} & \hookrightarrow & \mathcal{B}_{SO}(M) \\ & \searrow & \downarrow & & \downarrow \\ & & \partial M & \hookrightarrow & M \end{array}$$

This diagram tells us the exact ingredients we need to construct the  $\text{Spin}_{n-1}$ -bundle over  $\partial M$  – it must be the pullback of  $\mathcal{B}_{\text{Spin}}(M)|_{\partial M}$  along the inclusion  $\mathcal{B}_{SO}(\partial M) \hookrightarrow \mathcal{B}_{SO}(M)|_{\partial M}$ , so it fits into

the commutative square

$$\begin{array}{ccc} \mathcal{B}_{\text{Spin}}(\partial M) & \hookrightarrow & \mathcal{B}_{\text{Spin}}(M)|_{\partial M} \\ \downarrow & & \downarrow \\ \mathcal{B}_{SO}(\partial M) & \hookrightarrow & \mathcal{B}_{SO}(M)|_{\partial M} \end{array}$$

In addition, it comes equipped with a map  $p : \mathcal{B}_{\text{Spin}}(\partial M) \rightarrow \partial M$  by composing the map  $\mathcal{B}_{\text{Spin}}(\partial M) \rightarrow \mathcal{B}_{SO}(\partial M)$  with the projection  $\pi : \mathcal{B}_{SO}(\partial M) \rightarrow \partial M$ . However, it is not immediately clear that the pullback bundle, (which we will suggestively denote by  $\mathcal{B}_{\text{Spin}}(\partial M)$ ) is indeed a principal  $\text{Spin}_{n-1}$ -bundle over  $\partial M$ . We have a  $SO_{n-1}$  action on  $\mathcal{B}_{SO}(\partial M) \subset \mathcal{B}_{SO}(M)|_{\partial M}$  via the subgroup of matrices of the form

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} : A \in SO_{n-1} \right\}$$

The preimage of this subgroup under the double covering  $\text{Spin}_n \rightarrow SO_n$  is a subgroup isomorphic to  $\text{Spin}_{n-1}$ , giving us an action of  $\text{Spin}_{n-1}$  on  $\mathcal{B}_{\text{Spin}}(M)|_{\partial M}$  by restriction. In addition, this action preserves the preimage of  $\mathcal{B}_{SO}(M)|_{\partial M}$  under the double covering  $\mathcal{B}_{\text{Spin}}(M)|_{\partial M} \rightarrow \mathcal{B}_{SO}(M)|_{\partial M}$ , which is exactly the pullback  $\mathcal{B}_{\text{Spin}}(\partial M)$ . Then since the action of  $\text{Spin}_n$  is free, the restricted action of  $\text{Spin}_{n-1}$  on  $\mathcal{B}_{\text{Spin}}(\partial M) \subset \mathcal{B}_{\text{Spin}}(M)|_{\partial M}$  is free, and is transitive by construction. Therefore,  $\mathcal{B}_{\text{Spin}}(\partial M)$  is the principal  $\text{Spin}_{n-1}$ -bundle over  $\partial M$  that we desire. All in all, the construction is summarized by the diagram

$$\begin{array}{ccccc} \mathcal{B}_{\text{Spin}}(\partial M) & \hookrightarrow & \mathcal{B}_{\text{Spin}}(M)|_{\partial M} & \hookrightarrow & \mathcal{B}_{\text{Spin}}(M) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{B}_{SO}(\partial M) & \hookrightarrow & \mathcal{B}_{SO}(M)|_{\partial M} & \hookrightarrow & \mathcal{B}_{SO}(M) \\ & \searrow & \downarrow & & \downarrow \\ & & \partial M & \hookrightarrow & M \end{array}$$

**EXAMPLE 3.2** (The induced Spin structure on  $\partial D^2$ ). Let  $D^2 = \{v \in \mathbb{R}^2 : |v| \leq 1\}$  be the 2-disk, equipped with the Riemannian metric inherited from  $\mathbb{R}^2$ . We have that  $\partial D^2 = S^1$ . Since  $D^2$  is contractible, both the oriented orthonormal frame bundle  $\mathcal{B}_{SO}(D^2)$  and the spin bundle  $\mathcal{B}_{\text{Spin}}(D^2)$  inherited from  $\mathbb{R}^2$  are trivial bundles, so their restrictions onto  $\partial D^2$  are also trivial. In addition, we have that  $\text{Spin}_2 \cong SO_2$ , and the covering map is given by  $g \mapsto g^2$ . Using the orientation on  $D^2$  inherited from  $\mathbb{R}^2$ , we need to construct the induced orientation on  $\partial D^2$ . To do so, we parameterize  $\partial D^2$  by angles  $\theta \in [0, 2\pi)$ . Then the outward normal at each point  $\theta \in \partial D^2$  is the vector  $(\cos \theta, \sin \theta)$ , where we use the canonical identification of  $T_p D^2$  with  $\mathbb{R}^2$ . Then the bundle  $\mathcal{B}_{SO}(\partial D^2) \subset \mathcal{B}_{SO}(D^2)|_{\partial D^2}$  is the bundle where the fiber of  $\theta \in \partial D^2$  under the map  $\mathcal{B}_{SO}(\partial D^2)$  is the matrix

$$b_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

Then pulling back  $\mathcal{B}_{\text{Spin}}(D^2)|_{\partial D^2}$  by the inclusion  $\mathcal{B}_{SO}(\partial D^2) \hookrightarrow \mathcal{B}_{SO}(D^2)|_{\partial D^2}$ , we have that

$$\mathcal{B}_{\text{Spin}}(\partial D^2) = \{(\theta, g) : g^2 = b_\theta\}$$

Explicitly, this means that the fiber of  $\mathcal{B}_{SO}(\partial D^2) \rightarrow \partial D^2$  over a point  $\theta \in \partial D^2$  is the two point set

$$\left\{ \begin{pmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix}, \begin{pmatrix} -\cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & -\cos \frac{\theta}{2} \end{pmatrix} \right\}$$

which tells us that the  $\text{Spin}_1$ -bundle  $\mathcal{B}_{\text{Spin}}(\partial D^2)$  is the connected double cover given by  $g \mapsto g^2$ .  $\blacktriangleleft$

4.  $\text{Pin}_n^\pm$  Structures

The same general construction for Spin structures works for  $\text{Pin}_n^\pm$  as well. When we refer to  $\text{Pin}_n^\pm$ , we refer to a fixed choice of  $\text{Pin}_n^+$  or  $\text{Pin}_n^-$ .

DEFINITION 4.1. Let  $M$  be a smooth manifold. Then a  $\text{Pin}_n^\pm$  **structure** on  $M$  is a reduction of structure group for  $O_n$  to  $\text{Pin}_n^\pm$ .  $\diamond$

Just like with Spin, a  $\text{Pin}_n^\pm$  structure on  $M$  induces a  $\text{Pin}_{n-1}^\pm$  on the boundary using the outward unit normal, which gives the analogous diagram

$$\begin{array}{ccccc}
 \mathcal{B}_{\text{Pin}^\pm}(\partial M) & \hookrightarrow & \mathcal{B}_{\text{Pin}^\pm}(M)|_{\partial M} & \hookrightarrow & \mathcal{B}_{\text{Pin}^\pm}(M) \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathcal{B}_O(\partial M) & \hookrightarrow & \mathcal{B}_O(M)|_{\partial M} & \hookrightarrow & \mathcal{B}_O(M) \\
 & \searrow & \downarrow & & \downarrow \\
 & & \partial M & \hookrightarrow & M
 \end{array}$$

EXAMPLE 4.2 (Pin structures on  $S^1$ ). There are two problems to discuss here, since  $\text{Pin}_1^+$  and  $\text{Pin}_1^-$  are different groups, namely

$$\begin{aligned}
 \text{Pin}_1^+ &\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \\
 \text{Pin}_1^- &\cong \mathbb{Z}/4\mathbb{Z}
 \end{aligned}$$

The isomorphism  $\text{Pin}_1^+ \rightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  maps  $-1 \mapsto (1, 0)$  and  $e_1 \mapsto (0, 1)$ , and the isomorphism  $\text{Pin}_1^- \rightarrow \mathbb{Z}/4\mathbb{Z}$  maps  $e_1 \mapsto 1$ .

We do the  $\text{Pin}_1^+$  case first. In this case, the Clifford algebra  $\text{Cliff}_{1,0}(\mathbb{R})$  is isomorphic to  $\mathbb{R} \times \mathbb{R}$ , where on the basis  $\{1, e_1\}$ , we map  $1 \mapsto 1$  and  $e_1 \mapsto (-1, 1)$ . There are two irreducible modules corresponding to projection onto one of the factors, so the Pinor representations are the one dimensional real representations  $\mathbb{P}^+$  and  $\mathbb{P}^-$ , where the action of  $e_1$  on  $\mathbb{P}^\pm$  is by  $\pm 1$ . Let  $\mathbb{P} = \mathbb{P}^+ \oplus \mathbb{P}^-$ . We then take inventory of all the principal  $\text{Pin}_1^+$ -bundles over  $S^1$ . From before, we know that these are classified by homomorphisms  $\pi_1(S^1, \theta_0) \rightarrow \text{Pin}_1^+$  up to conjugation. Since  $\text{Pin}_1^+$  is abelian, the conjugation action is trivial, so principal  $\text{Pin}_1^+$ -bundles are classified by homomorphisms  $\pi_1(S^1, \theta_0) \rightarrow \text{Pin}_1^+$ . There are 4 such homomorphisms, which are all determined by the image of a generator  $\alpha$  of  $\pi_1(S^1, \theta_0)$ , since the fundamental group is infinite cyclic. We denote all the bundles  $P_g \rightarrow S^1$ , where  $g$  is the image of  $\alpha$  under the corresponding homomorphism.

- (1) The bundle  $P_1 \rightarrow S^1$  has 4 connected components, each diffeomorphic to a circle. The map  $P - 1 \rightarrow S^1$  maps each component diffeomorphically onto  $S^1$ , and the action of  $\text{Pin}_1^+$  permutes these 4 components.
- (2) The bundle  $P_{-1} \rightarrow S^1$  has two components, each diffeomorphic to a circle and double covering  $S^1$ . The element  $-1$  acts by rotating both circles by  $\pi$ , and the element  $e_1$  exchanges the two circles.
- (3) The bundle  $P_{e_1} \rightarrow S^1$  has two components, each diffeomorphic to a circle and double covering  $S^1$ . The action of  $e_1$  rotates both circles by  $\pi$ , and  $-1$  exchanges the two circles.
- (4) The bundle  $P_{-e_1} \rightarrow S^1$ , has two components, each diffeomorphic to a circle and double covering  $S^1$ . The element  $-e_1$  acts by rotation by  $\pi$ , and the element  $-1$  exchanges the two circles.

We then must determine which bundles admit  $\text{Pin}_1^+$ -equivariant maps  $P_g \rightarrow \mathcal{B}_O(S^1)$ , which is a pair of disjoint circles permuted by  $O_1 = \{\pm 1\}$ . Under the double covering, the elements  $\pm 1 \in \text{Pin}_1^+$  act trivially, while the elements  $\pm e_1$  act by  $-1$ .

- (1) The bundle  $P_1 \rightarrow S^1$  admits a  $\text{Pin}_1^+$ -equivariant map  $P_1 \rightarrow \mathcal{B}_O(S^1)$ . After mapping one of the components diffeomorphically onto one of the components of  $\mathcal{B}_O(S^1)$ , we know how to map the other 3 components to make the map  $\text{Pin}_1^+$ -equivariant.
- (2) The bundle  $P_{-1} \rightarrow S^1$  can also define a  $\text{Pin}_1^+$  structure on  $S^1$ . Since  $-1$  rotates both circles by  $\pi$ , we map each component to a component of  $\mathcal{B}_O(S^1)$  via the double cover  $z \mapsto z^2$ .
- (3) The bundle  $P_{e_1} \rightarrow S^1$  cannot determine a  $\text{Pin}_1^+$  structure. The action of  $e_1$  rotates both components of  $P_{e_1}$  by  $\pi$ , but permutes the components of  $\mathcal{B}_O(S^1)$ . Any  $\text{Pin}_1^+$ -equivariant map  $P_{e_1} \rightarrow \mathcal{B}_O(S^1)$  must map one component of  $P_{e_1}$  onto two components of  $\mathcal{B}_O(S^1)$ , which is impossible.
- (4) The bundle  $P_{-e_1} \rightarrow S^1$  also cannot determine a  $\text{Pin}_1^+$  structure by the same reason that  $P_{e_1}$  cannot.

One thing to note is that an orientation of  $S^1$  and a  $\text{Pin}_1^+$  structure on  $S^1$  determine a Spin structure. A choice of component of  $\mathcal{B}_O(S^1)$  determines an orientation, and taking the preimage of that component under the maps  $P_{\pm 1} \rightarrow \mathcal{B}_O(S^1)$ , we recover a Spin structure. In particular, the bundle  $P_1$  corresponds to the Spin structure with Dirac operator  $i\partial_\theta$  and the bundle  $P_{-1}$  corresponds to the Spin structure with Dirac operator  $i\partial_\theta - \frac{1}{2}$ . Indeed, the associated bundles  $P_{\pm 1} \times_{\text{Pin}_1^+} \mathbb{P}$  have the same Dirac operators as their corresponding Spin structures.

For the case of  $\text{Pin}_1^-$ , there are again 4 different principal  $\text{Pin}_1^-$  bundles, classified by homomorphisms  $\pi_1(S^1, \theta_0) \rightarrow \text{Pin}_1^-$ . As before, we let  $\alpha$  denote a generator for  $\pi_1(S^1, \theta_0)$ , and let  $P_g \rightarrow S^1$  denote the principal  $\text{Pin}_1^-$  bundle corresponding to the homomorphism mapping  $\alpha \mapsto g$ .

- (1) The bundle  $P_1 \rightarrow S^1$  has 4 components cyclically permuted by the action of  $e_1$ .
- (2) The bundle  $P_{-1} \rightarrow S^1$  has 2 components, both double covering  $S^1$ . the element  $-1$  acts by rotation by  $\pi$  on both factors, and the action of  $e_1$  permutes the two components.
- (3) The bundle  $P_{e_1} \rightarrow S^1$  corresponds to the 4-fold covering  $S^1 \rightarrow S^1$  given by  $z \mapsto z^4$ . The action of  $e_1$  acts by rotation by  $\pi/2$ .
- (4) The bundle  $P_{-e_1} \rightarrow S^1$  also corresponds to the 4-fold covering,  $S^1 \rightarrow S^1$ , where the action of  $-e_1$  is given by rotation by  $\pi/2$ .

Again, we need to figure out which bundles can determine  $\text{Pin}_1^-$  structures on  $S^1$ .

- (1) The bundle  $P_1$  admits a map  $P_1 \rightarrow \mathcal{B}_O(S^1)$ . After mapping any component diffeomorphically onto a component of  $\mathcal{B}_O(S^1)$ , the rest of the map is determined.
- (2) The bundle  $P_{-1}$  admits a map  $P_{-1} \rightarrow \mathcal{B}_O(S^1)$ , where each component maps onto a component of  $\mathcal{B}_O(S^1)$  via the double cover  $z \mapsto z^2$ .
- (3) This bundle cannot determine a  $\text{Pin}_1^-$  structure, since there cannot exist a continuous surjective map from a connected space to a disconnected space.
- (4) This bundle cannot determine a  $\text{Pin}_1^-$  structure by the same reasoning as above.

Again, we see that the  $\text{Pin}_1^-$  structures on  $S^1$  correspond to Spin structures once we fix an orientation, and the associated bundles to these  $\text{Pin}_1^-$ -bundles will have the same Dirac operator as their corresponding spin structures.  $\blacktriangleleft$

On the circle, we observed that a  $\text{Pin}^\pm$  structure and an orientation are equivalent data to a Spin structure. This is true in general via the same construction. A  $\text{Pin}^\pm$  structure is the data of a principal  $\text{Pin}_n^\pm$ -bundle  $\mathcal{B}_{\text{Pin}^\pm}(M) \rightarrow M$  along with a  $\text{Pin}_n^\pm$ -equivariant map  $\mathcal{B}_{\text{Pin}^\pm}(M) \rightarrow \mathcal{B}_O(M)$ . An orientation determines a subset  $\mathcal{B}_{SO}(M) \subset \mathcal{B}_O(M)$  that is a principal  $SO_n$ -bundle over  $M$ , and taking the preimage of  $\mathcal{B}_{SO}(M)$  under the map  $\mathcal{B}_{\text{Pin}^\pm}(M) \rightarrow \mathcal{B}_O(M)$  is a principal  $\text{Spin}_n$ -bundle over  $M$ , since the preimage of  $SO_n \subset O_n$  under the double covers  $\text{Pin}_n^\pm \rightarrow O_n$  is a subgroup isomorphic to  $\text{Spin}_n$ . The restriction of  $\mathcal{B}_{\text{Pin}^\pm}(M) \rightarrow \mathcal{B}_O(M)$  then determines the desired Spin structure.

Unlike  $\text{Spin}$  structures,  $\text{Pin}^\pm$  structures do not require an orientation, and can be defined on nonorientable manifolds.

EXAMPLE 4.3 ( $\text{Pin}^\pm$  structures on the Möbius band). Let  $M$  denote the Möbius band, which is the quotient space

$$M = [-1, 1] \times [-1, 1] / (-1, x) \sim (1, -x)$$

The square  $S = [-1, 1] \times [-1, 1]$  inherits a Riemannian metric from  $\mathbb{R}^2$ , and the orthonormal frame bundle  $\mathcal{B}_O(S)$  is isomorphic to the product bundle  $S \times O_2$ . This then descends to  $M$  after we specify gluing data for the identified edges of the square. The identification of the edges reverses the direction in the second factor, which corresponds to a transformation by the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

giving us that the orthonormal frame bundle  $\mathcal{B}_O(M)$  of  $M$  induced by the metric on  $S$  is isomorphic to

$$S \times O_2 / (-1, x, b) \sim (1, -x, A \cdot b)$$

We do the  $\text{Pin}_2^+$  case first. Inside of  $\text{Cliff}_{2,0} \cong M_2\mathbb{R}$ , the group  $\text{Pin}_2^+$  is topologically two disjoint circles. The identity component is

$$\text{Spin}_2 = \{\cos \theta + \sin \theta e_1 e_2 : \theta \in \mathbb{R}\}$$

and the second component is the subset  $e_1 \cdot \text{Spin}_2$ . The square  $S$  admits a single principal  $\text{Pin}_2^+$ -bundle, which is the trivial bundle  $S \times \text{Pin}_2^+$ , and any principal  $\text{Pin}_2^+$ -bundle over  $M$  will come by specifying gluing data of  $S \times \text{Pin}_2^+$  on the identified edges of  $S$ . More explicitly, any principal  $\text{Pin}_2^+$ -bundle over  $M$  will arise as a quotient

$$S \times \text{Pin}_2^+ / (-1, x, g) \sim (1, -x, \psi(g))$$

where  $\psi : \text{Pin}_2^+ \rightarrow \text{Pin}_2^+$  is a  $\text{Pin}_2^+$ -equivariant diffeomorphism. From this characterization, we can determine which maps  $\psi$  determine a principal  $\text{Pin}_2^+$ -bundle  $P \rightarrow M$  that admit a  $\text{Pin}_2^+$ -equivariant map  $P \rightarrow \mathcal{B}_O(M)$ . For points that are not identified, i.e. points on  $S$  with first component not equal to  $\pm 1$ , we can specify  $P \rightarrow \mathcal{B}_O(M)$  fiberwise by the standard double cover  $\text{Pin}_2^+ \rightarrow O_2$ . On the identified edges, in order for the map to be  $\text{Pin}_2^+$ -equivariant, we require that the image of the equivalence class  $[-1, x, g] \in P$  to be  $[-1, x, \rho(g)]$ , where  $\rho : \text{Pin}_2^+ \rightarrow O_2$  is the double cover. However, we have that  $[-1, x, g] = [1, -x, \psi(g)]$  and  $[-1, x, \rho(g)] = [1, -x, A \cdot \rho(g)]$ , so we need  $\psi(g)$  to be one of the preimages of  $A \cdot \rho(g)$  under the double cover in order for this to be well defined. Since  $A$  is an orthogonal transformation, it has two preimages under the double cover, which are  $\rho^{-1}(A) = \pm e_1$ . Therefore,  $\psi$  must be left multiplication by either  $e_1$  or  $-e_1$ , giving us two  $\text{Pin}^+$  structures on  $M$ .

We then want to determine the induced  $\text{Pin}_1^+$  structure on the boundary. Using the standard coordinates on  $S$  inherited from  $\mathbb{R}^2$ , the outward normal on the top edge of the square points upwards, while the outward normal points downwards on the bottom edge of the square. Therefore, the fiber of a point on the top edge under  $\mathcal{B}_O(M) \rightarrow M$  is the two point set

$$\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

and the fiber over a point on the bottom edge is

$$\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\}$$

and on the endpoints, we have the identifications

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

We then need to compute the preimages of the matrices under the double cover  $\rho : \text{Pin}_2^+ \rightarrow O_2$  to find the fiber under  $\mathcal{B}_{\text{Pin}^+}(M) \rightarrow M$ . We compute

$$\begin{aligned} \rho^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &= \left\{ \pm \frac{\sqrt{2}}{2}(e_1 + e_2) \right\} \\ \rho^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} &= \left\{ \pm \frac{\sqrt{2}}{2}(-1 + e_1 e_2) \right\} \\ \rho^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= \left\{ \pm \frac{\sqrt{2}}{2}(1 + e_1 e_2) \right\} \\ \rho^{-1} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} &= \left\{ \pm \frac{\sqrt{2}}{2}(-e_1 + e_2) \right\} \end{aligned}$$

Depending on our choice of  $\pm e_1$  for the gluing map at the endpoints, we get different identifications on the endpoints of the top edge and the endpoints on the bottom edge, but for either we choose, the resulting bundle over  $\partial M \cong S^1$  has four connected components, where each of the 4 elements in the fibers of the top edge of the square pair with one of the elements in the fibers of the bottom edge, giving 4 circles. If we orient the boundary of  $M$ , this gives us the nonbounding Spin structure corresponding to the disconnected double cover of  $S^1$ .

We now do the case of  $\text{Pin}_2^-$ . Most of the discussion with  $\text{Pin}_2^+$  carries over to  $\text{Pin}_2^-$ , up until the point where we compute the preimages of the matrices under double cover. Let  $\varphi : \text{Pin}_2^- \rightarrow O_2$  be the double cover. We compute the preimages

$$\begin{aligned} \varphi^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &= \left\{ \pm \frac{\sqrt{2}}{2}(e_1 + e_2) \right\} \\ \varphi^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} &= \left\{ \pm \frac{\sqrt{2}}{2}(1 + e_1 e_2) \right\} \\ \varphi^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= \left\{ \pm \frac{\sqrt{2}}{2}(-1 + e_1 e_2) \right\} \\ \varphi^{-1} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} &= \left\{ \pm \frac{\sqrt{2}}{2}(-e_1 + e_2) \right\} \end{aligned}$$

This defines a  $\text{Pin}_2^-$  structure on  $\partial M$  that has two connected components. To see this, we glue the fiber over top right corner with the fiber over the bottom left corner via left multiplication by  $e_1$ , giving the identifications

$$\begin{aligned} \pm \frac{\sqrt{2}}{2}(e_1 + e_2) &\sim \pm \frac{\sqrt{2}}{2}(-1 + e_1 e_2) \\ \pm \frac{\sqrt{2}}{2}(1 + e_1 e_2) &\sim \mp \frac{\sqrt{2}}{2}(-e_1 + e_2) \end{aligned}$$



The identifications of the fiber over the bottom right corner with the fiber over the top left corner are

$$\begin{aligned}\pm \frac{\sqrt{2}}{2}(-1 + e_1 e_2) &\sim \mp \frac{\sqrt{2}}{2}(e_1 + e_2) \\ \pm \frac{\sqrt{2}}{2}(-e_1 + e_2) &\sim \pm \frac{\sqrt{2}}{2}(1 + e_1 e_2)\end{aligned}$$

The two flipped signs connects two of the components of the fiber together, giving two components instead of the four components in the case with  $\text{Pin}_2^+$ , so after fixing an orientation of  $\partial M$ , we get the Spin structure corresponding to the bounding Spin structure, i.e. the Spin structure associated to the connected double cover of  $S^1$ .  $\blacktriangleleft$

To every smooth manifold  $M$ , there is an associated oriented manifold called the orientation double cover, which is a principal  $\mathbb{Z}/2\mathbb{Z}$ -bundle encoding information about the orientability of  $M$ .

DEFINITION 4.4. Let  $M$  be a smooth manifold. Then the **orientation double cover** is the set

$$\tilde{M} = \{(p, o) : p \in M, o \text{ is an orientation of } T_p M\}$$

This comes with a natural map  $\tilde{M} \rightarrow M$  mapping  $(p, o) \mapsto p$ , and a natural  $\mathbb{Z}/2\mathbb{Z}$  action where  $(p, o) \mapsto (p, -o)$ . Local coordinates on  $M$  induce local coordinates on  $\tilde{M}$ , which we can use to define the topology and smooth structure. Under this topology and smooth structure, the map  $\tilde{M} \rightarrow M$  is a smooth double covering.  $\blacklozenge$

The double cover is a local diffeomorphism, so its differential induces identifications of tangent spaces  $T_{(p,o)}\tilde{M} \rightarrow T_p M$ , which allows us to define a canonical orientation on  $\tilde{M}$  where we orient  $T_{(p,o)}\tilde{M}$  with the orientation  $o$ . The orientation double cover detects orientability of  $M$  in the following way.

THEOREM 4.5.

- (1) If  $M$  is orientable,  $\tilde{M}$  is diffeomorphic the disconnected double cover  $M \sqcup M$ .
- (2) If  $M$  is not orientable,  $\tilde{M}$  is connected. Furthermore,  $\tilde{M}$  is unique in the following way: Given another double cover  $\hat{M} \rightarrow M$  of an oriented manifold  $\hat{M}$  onto  $M$ , there is a unique orientation preserving diffeomorphism  $\varphi : \hat{M} \rightarrow \tilde{M}$  such that

$$\begin{array}{ccc} \hat{M} & \xrightarrow{\varphi} & \tilde{M} \\ & \searrow & \swarrow \\ & M & \end{array}$$

commutes.

In particular, given an unorientable manifold  $M$ , any oriented double cover of  $M$  is isomorphic to the orientation double cover.

EXAMPLE 4.6. The cylinder  $S^1 \times I$  can be realized as the orientation double cover of the Möbius band  $M$ . Let  $f : S^1 \times I \rightarrow S^1 \times I$  be the map flipping the cylinder. Then the quotient space  $S^1 \times I / (x \sim f(x))$  is diffeomorphic to the Möbius band, and the quotient map is the double cover.  $\blacktriangleleft$

THEOREM 4.7. A  $\text{Pin}^\pm$  structure on a manifold induces a Spin structure on the orientation double cover.

PROOF. Let  $\pi : \tilde{M} \rightarrow M$  denote the orientation double cover. Fix a Riemannian metric  $g$  on  $M$ . This induces a Riemannian metric on  $\tilde{M}$ , which is the pullback metric  $\pi^*g$ . In addition, the orthonormal frame bundle  $\mathcal{B}_O(\tilde{M})$  with respect to the metric  $\pi^*g$  is isomorphic to the pullback bundle  $\pi^*\mathcal{B}_O(M)$ . Pulling back  $\mathcal{B}_{\text{Pin}^\pm}(M)$  along the map  $\pi^*\mathcal{B}_O(\tilde{M})$ , we get the a  $\text{Pin}_n^\pm$ -bundle

$\mathcal{B}_{\text{Pin}^\pm}$  over  $\tilde{M}$  with a map to  $\pi^* \mathcal{B}_O(M) \cong \mathcal{B}_O(\tilde{M})$ , so it defines a  $\text{Pin}^\pm$  structure on  $\tilde{M}$ , giving us the diagram

$$\begin{array}{ccc}
 \mathcal{B}_{\text{Pin}^\pm}(\tilde{M}) & \longrightarrow & \mathcal{B}_{\text{Pin}^\pm}(M) \\
 \downarrow & & \downarrow \\
 \pi^* \mathcal{B}_O(M) & \longrightarrow & \mathcal{B}_O(M) \\
 \downarrow & & \downarrow \\
 \tilde{M} & \xrightarrow{\pi} & M
 \end{array}$$

Then since  $\tilde{M}$  is oriented, and an orientation along with a  $\text{Pin}^\pm$  structure determines a  $\text{Spin}$  structure, we get a  $\text{Spin}$  structure on  $\tilde{M}$ . ■

## CHAPTER 3

# Dirac Operators on Manifolds



*Not all the geometrical structures are “equal”. It would seem that the Riemannian and complex structures, with their contacts with other fields of mathematics and with their richness in results, should occupy a central position in differential geometry. A unifying idea is the notion of a  $G$ -structure, which is the modern version of an equivalence problem first emphasized and exploited in its various special cases by Élie Cartan.*

– *Shiing-Shen Chern*



### 1. Connections

We previously defined the Dirac operator  $D$  on  $\mathbb{R}^n$  and explored some instances of  $D$  in low dimensions. Somewhat surprisingly, the generalization of the Dirac operator to the nonlinear world of manifolds provides an incredibly general framework for first order differential operators on Riemannian manifolds. The construction of  $D$  on  $\mathbb{R}^n$  implicitly used the Riemannian geometry on  $\mathbb{R}^n$ , along with its associated Spin structure. To explore the Dirac operator in more generality on manifolds, we need to develop some of the geometry necessary in order to construct all of the necessary pieces to discuss the Dirac operator.

**DEFINITION 1.1.** Let  $M$  be a smooth manifold. A **distribution** on  $X$  is a vector subbundle  $E \subset TX$ . ❖

**DEFINITION 1.2.** let  $\pi : E \rightarrow M$  be a fiber bundle. The map  $\pi$  determines a distinguished subspace  $\ker d\pi_e \subset T_e E$ , called the **vertical subspace** which defines a distribution  $V$  over  $E$  called the **vertical distribution**. A **connection** on  $E$  is another distribution  $H \subset TE$  such that at every point  $e \in E$ , we have  $V_e \oplus H_e = T_e E$ . We also refer to the distribution  $H$  as a **horizontal distribution**. In other words a connection is a choice of splitting of the short exact sequence of vector bundles

$$0 \longrightarrow V \longrightarrow TE \longrightarrow \pi^* TM \longrightarrow 0$$

If  $E$  is a principal  $G$ -bundle, we ask for the horizontal distribution  $H$  to be  $G$ -invariant, i.e.  $H_{p \cdot g} = d(R_g)_p(H_p)$ , where  $R_g : E \rightarrow E$  denotes the right action of  $g \in G$ . ❖

Note that for a point  $e \in E$ , the projection map  $d\pi_e$  restricted to the horizontal subspace  $H_e$  is a linear isomorphism  $H_e \rightarrow T_{\pi(e)} X$ .

**DEFINITION 1.3.** Let  $E \rightarrow M$  be a vector bundle. A  **$E$ -valued  $k$ -form** is a section of the vector bundle  $\Lambda^k T^* M \otimes E \rightarrow M$ . We denote the space of  $E$ -valued  $k$ -forms by  $\Omega_M^k(E)$ . In the case that  $E \rightarrow M$  is a trivial bundle  $E = M \times V$  for some vector space  $V$ , we abbreviate this as a  $V$ -valued  $k$  form, and the space of  $V$ -valued  $k$ -forms is denoted  $\Omega_M^k(V)$ . In a local trivialization of  $E$ , an element  $\omega \in \Omega_M^k(E)$  can be thought of as a vector of  $k$ -forms. ❖

We are mostly concerned with connections on principal bundles, which have a richer structure than connections on a general fiber bundle. Let  $\pi : P \rightarrow M$  be a principal  $G$ -bundle, and let  $\mathfrak{g}$

denote the Lie algebra of  $G$ . Let  $\exp : \mathfrak{g} \rightarrow G$  denote the exponential map. Fix  $x \in M$ , a point  $p \in \pi^{-1}(x)$ , and an element  $X \in \mathfrak{g}$ . This determines a curve  $\gamma_X : I \rightarrow \pi^{-1}(x)$  with  $\gamma_X(0) = p$  given by

$$\gamma_X(t) = p \cdot \exp(tX)$$

Since the action of  $G$  preserves the fiber  $\pi^{-1}(x)$ , this curve lies entirely in the fiber  $\pi^{-1}(x)$ , so the tangent vector  $\dot{\gamma}_X$  at  $t = 0$  is an element of the vertical space. In addition, since the action of  $G$  is free, the tangent vector  $\dot{\gamma}_X = 0$  if and only if  $X = 0$ , so the linear mapping  $X \mapsto \dot{\gamma}_X$  is injective, and consequently, an isomorphism  $\mathfrak{g} \rightarrow V_p$ . Doing this over all points of the fiber, given  $X \in \mathfrak{g}$ , we get a vector field  $\tilde{X}$  on the fiber  $\pi^{-1}(x)$  by differentiating the right action of  $\exp(tX)$ .

Defining these vector fields fiberwise then gives an isomorphism  $\underline{\mathfrak{g}} \rightarrow V$ , where  $\underline{\mathfrak{g}}$  denotes the trivial bundle  $P \times \mathfrak{g}$ . Under this isomorphism, the exact sequence of vector bundles becomes

$$0 \longrightarrow \underline{\mathfrak{g}} \longrightarrow TP \longrightarrow \pi^*TM \longrightarrow 0$$

One important thing to note is how these vector fields transform under the right action of  $G$ .

**PROPOSITION 1.4.** *Let  $X \in \mathfrak{g}$ , and let  $\tilde{X}$  be the vector field on  $P$  induced by  $X$ . Let  $g \in G$ , and let  $R_g : P \rightarrow P$  denote the diffeomorphism given by right multiplication by  $g$ . Then*

$$(R_g)_*(\tilde{X}) = \widetilde{\text{Ad}_{g^{-1}} X}$$

Where  $\text{Ad}_{g^{-1}} : \mathfrak{g} \rightarrow \mathfrak{g}$  is the linear map obtained by differentiating the conjugation action  $h \mapsto g^{-1}hg$  at the identity element of  $G$ .

**PROOF.** Let  $p \in P$ . By definition, we have that

$$\tilde{X}_p = \left. \frac{d}{dt} \right|_{t=0} p \cdot \exp(tX)$$

Therefore, we have that

$$\begin{aligned} [(R_g)_*(\tilde{X})]_{p \cdot g} &= d(R_g)_p \left( \left. \frac{d}{dt} \right|_{t=0} p \cdot \exp(tX) \right) \\ &= \left. \frac{d}{dt} \right|_{t=0} p \cdot \exp(tX)g \\ &= \left. \frac{d}{dt} \right|_{t=0} p \cdot g(g^{-1} \exp(tX)g) \\ &= \left( \widetilde{\text{Ad}_g^{-1} X} \right)_{p \cdot g} \end{aligned}$$

■

Over a point  $p \in P$ , we have a direct sum decomposition  $T_pP = V_p \oplus H_p$ , giving us a projection map  $T_pP \rightarrow V_p$ . Using the identification of  $V_p$  with  $\mathfrak{g}$ , this then determines a linear map  $\omega_p : T_pP \rightarrow \mathfrak{g}$ , where  $\ker \omega_p = H_p$ . Doing this over all fibers  $\pi^{-1}(p)$ , this determines a  $\mathfrak{g}$ -valued 1-form  $\omega \in \Omega_P^1(\mathfrak{g})$  called the **connection 1-form**. From the connection 1-form  $\omega$ , we can recover  $H$  as the kernel of  $\omega$ . In addition, it is clear from the definition that given  $X \in \mathfrak{g}$ , we have  $\omega_p(\tilde{X}_p) = X$ .

In the case of a vector bundle  $E \rightarrow M$ , we can also construct a connection 1-form. Given any point  $p \in E_x$ , we get a linear isomorphism of the fiber  $E_x$  with the vertical space  $V_p \subset T_pE$  via the mapping

$$v \mapsto \left. \frac{d}{dt} \right|_{t=0} p + tv$$

A horizontal distribution  $H$  then gives us a projection map  $T_p E \rightarrow V_p$  on each tangent space, and composing with the inverse isomorphism  $V_p \rightarrow E_x$ , we get a linear map  $\omega_p : T_p P \rightarrow E_x$ . Doing this over all fibers gives and  $\pi^* E$  valued 1-form  $\omega \in \Omega_E^1(\pi^* E)$ , which is also called the connection 1-form. As with principal bundles, the horizontal distribution is recovered as the kernel of  $\omega$ .

**PROPOSITION 1.5.** *Let  $\pi : P \rightarrow M$  be a principal  $G$ -bundle and  $\omega \in \Omega_P^1(\mathfrak{g})$  a connection 1-form on  $P$ . Then for  $g \in G$ ,  $R_g^* \omega = \text{Ad}_{g^{-1}} \omega$ . where  $R_g^*$  denotes pullback by the right action of  $g$  on  $P$ .*

**PROOF.** Fix  $p \in P$ , and let  $v \in T_p P$ . Since  $T_p P = V_p \oplus H_p$ , we have that  $v = \tilde{X}_p + h$  for  $X \in \mathfrak{g}$  and  $h \in H_p$ . We then compute

$$\begin{aligned} (R_g^* \omega)_p(v) &= (R_g^* \omega)_p(\tilde{X}_p + h) \\ &= \omega_{p \cdot g}((R_g)_*(\tilde{X}_p) + (R_g)_*h) \end{aligned}$$

Then since  $(R_g)_* \tilde{X} = \widetilde{\text{Ad}_{g^{-1}} X}$  and  $(R_g)_* h \in H_{p \cdot g}$ , we get that

$$\begin{aligned} \omega_{p \cdot g}((R_g)_*(\tilde{X}_p) + (R_g)_*h) &= \omega_{p \cdot g}(\widetilde{(\text{Ad}_{g^{-1}} X)}_{p \cdot g} + 0) \\ &= \text{Ad}_{g^{-1}} X \\ &= \text{Ad}_{g^{-1}} \omega(\tilde{X}_p + h) \end{aligned}$$

■

We mentioned before that principal  $G$ -bundles can be thought of as generalizations of covering spaces, since for discrete  $G$ , a principal  $G$ -bundle is a covering space. One notion we want to generalize is the notion of path lifting. Given a covering space  $\pi : P \rightarrow M$ , a point  $x \in M$ , a point  $p \in \pi^{-1}(x)$  in the fiber, and a curve  $\gamma : I \rightarrow M$  with  $\gamma(0) = x$ , we use the fact that  $p$  lies in a unique sheet  $U_\alpha \subset \pi^{-1}(U)$  for some evenly covered neighborhood  $U$  of  $x$  to lift  $\gamma$  to a path  $\tilde{\gamma} : I \rightarrow P$  starting at  $p$ . This relies on the fact that  $\pi$  restricted to  $U_\alpha$  is a diffeomorphism onto  $U$ . If  $G$  is not discrete, we have no hope of this working without additional data since  $P$  has a larger dimension than  $M$ . The connection  $H \subset TP$  is exactly the data we need for path lifting. Using the same setup as above, the curve  $\gamma$  determines a vector field  $V$  along its image given by its velocity  $V_{\gamma(t)} = \dot{\gamma}(t)$ . Then since  $d\pi_p|_{H_p} : H_p \rightarrow T_{\pi(p)} M$  is an isomorphism, there exists a unique horizontal vector  $h_p \in T_p P$  at each point  $p \in \pi^{-1}(\gamma(t))$  satisfying  $d\pi_p(h_p) = V_{\gamma(t)}$ , which determines a vector field along the pullback bundle  $\gamma^* TP$ , which can naturally be thought of as a subset of  $TP$ . Integrating these vector fields gives a flow along the pullback bundle, and following the flow line starting at our favorite point in the fiber  $\pi^{-1}(x)$  gives us a path  $\tilde{\gamma} : I \rightarrow P$  such that  $\pi \circ \tilde{\gamma} = \gamma$ , i.e. a lift of  $\gamma$ . One important thing to note is how path lifting interacts with the group action.

**PROPOSITION 1.6.** *Let  $\pi : P \rightarrow M$  be a principal  $G$ -bundle equipped with a connection  $H \subset TP$ , and let  $\gamma : I \rightarrow M$  be a path in  $M$  with  $\gamma(0) = x$ . Then let  $\tilde{\gamma} : I \rightarrow P$  be the lift of  $\gamma$  starting at  $p$ . Then the lift of  $\gamma$  starting at  $p \cdot g$  is  $\gamma \circ R_g$ , where  $R_g$  is the right action of  $g \in G$ .*

**PROOF.** Since the action of  $R_g$  preserves the fibers of  $\pi$ ,  $\tilde{\gamma} \circ R_g$  satisfies  $\pi \circ \tilde{\gamma} \circ R_g = \gamma$ . By uniqueness of integral curves horizontal lift of the velocity vector field,  $\gamma \circ R_g$  must be the lift of  $\gamma$  based at  $p \cdot g$ . ■

Path lifting gives us a notion of **parallel transport** – an identification of the fibers along a path on  $M$ .

**DEFINITION 1.7.** Let  $\pi : P \rightarrow M$  be a principal  $G$ -bundle with  $H \subset TP$  a connection. Let  $\gamma : I \rightarrow M$  be a path starting at  $x \in M$ . Let  $\gamma_p : I \rightarrow P$  denote the lift of  $\gamma$  starting at  $p \in \pi^{-1}(x)$ . Define the **parallel transport** maps  $\tau_t : P_x \rightarrow P_{\gamma(t)}$  by  $\tau_t(p) = \gamma_p(t)$  ❖

We note that the maps  $\tau_t$  are isomorphisms, since the inverse map is given by parallel transport along the curve  $\bar{\gamma}(t) = \gamma(1 - t)$ . The parallel transport maps justify the naming of a connection – a connection provides the necessary data to connect nearby fibers along a curve  $\gamma$ . This notion of path lifting carries over to any associated bundle  $P \times_G F$ , which is one way to see that the connection on  $P$  induces a connection on every associated bundle. As before, let  $\gamma : I \rightarrow M$  be a path with  $\gamma(0) = x$ , and let  $[p, f] \in P \times_G F$  in the fiber over  $x$ , i.e.  $p \in \pi^{-1}(x)$ . Let  $\tilde{\gamma} : I \rightarrow P$  denote the lift of  $\gamma$  to  $P$  starting at  $p$ . Then define  $\hat{\gamma} : I \rightarrow P \times_G F$  by  $\hat{\gamma}(t) = [\tilde{\gamma}(t), f]$ , which is visibly a lift of  $\gamma$  to  $P \times_G F$  starting at  $[p, f]$ . However, we should check that this is independent of our choice of representative  $[p, f]$ . Let  $[p \cdot g, g^{-1} \cdot f]$  be another representative of the point  $[p, f]$ . Then the lift of  $\gamma$  to  $P$  starting at  $p \cdot g$  is the curve  $\tilde{\gamma} \cdot g$ , so using this representative, we would define  $\hat{\gamma}(t) = [\tilde{\gamma} \cdot g, g^{-1} \cdot f] = [\tilde{\gamma}(t), f]$ , so this definition does not rely on our choice of representative. Another way to see how the connection on  $P$  induces a connection on the associated bundle  $P \times_G F$  is by looking at the horizontal distribution  $H \subset TP$ . This includes into the tangent bundle  $T(P \times F)$  as the subbundle  $H \times \{0\}$ , where  $0$  denotes the zero section of  $TF$ . The  $G$ -invariance of  $H$  then guarantees that  $H$  descends to quotient  $P \times F / G = P \times_G F$ , giving a connection on  $P \times_G F$ .

This induced connection has a very nice interpretation when we have an associated vector bundle. Let  $\rho : G \rightarrow GL(W)$  be a linear representation. Then let  $E = P \times_G W$  denote the associated vector bundle induced by  $\rho$ . Recall from before that there is a bijective correspondence

$$\Gamma_M(E) \longleftrightarrow \left\{ \alpha : P \rightarrow W : \alpha(p \cdot g) = \rho(g)^{-1} \cdot \alpha(p) \right\}$$

We now generalize the correspondence  $\Gamma_M(E)$  with  $G$ -equivariant maps  $P \rightarrow W$ . Thinking of functions  $P \rightarrow W$  as 0-forms on  $P$  valued in  $W$  and sections of  $E$  as 0-forms on  $M$  valued in  $E$ , we can write this correspondence more suggestively as

$$\Omega_M^0(E) \longleftrightarrow \left\{ \alpha \in \Omega_P^0(W) : R_g^* \alpha = \rho(g)^{-1} \alpha \right\} \subset \Omega_P^0(W)$$

Where  $R_g^*$  denotes the pullback of  $\alpha$  along right multiplication by  $g \in G$ . This notation suggests that this correspondence should generalize to arbitrary  $k$ -forms.

**PROPOSITION 1.8.** *There is a bijective correspondence*

$$\Omega_M^k(E) \longleftrightarrow \left\{ \alpha \in \Omega_P^k(W) : R_g^* \alpha = \rho(g)^{-1} \alpha \ \forall g \in G, \ \iota_{\hat{\xi}}(\alpha) = 0 \ \forall \xi \in \mathfrak{g} \right\} \subset \Omega_P^k(W)$$

Where  $\iota_{\hat{\xi}}(\alpha)$  denotes the interior multiplication of  $\alpha$  with the vector field  $\hat{\xi}$  on  $P$  induced by  $\xi$ , i.e.

$$\iota_{\hat{\xi}}(\alpha)(V_1, \dots, V_{k-1}) = \alpha(\hat{\xi}, V_1, \dots, V_{k-1})$$

for vector fields  $V_1, \dots, V_{k-1}$ . For convenience, we denote this subset of  $k$ -forms on  $P$  by  $\Omega_P^{k,G}(W)$ , where the superscript  $G$  reminds us that these  $k$ -forms are  $G$ -invariant in the appropriate sense.

**REMARK.** Note that the condition  $\iota_{\hat{\xi}}(\alpha) = 0$  hold vacuously when  $\alpha$  is a 0-form, so this agrees with our earlier correspondence.

**PROOF.** We provide maps in both directions. Let  $\omega \in \Omega_M^k(E)$ . We want to define  $\tilde{\omega} \in \Omega_P^{k,G}(W)$ . Using  $\pi$ , we pullback  $\omega$  to obtain a  $k$ -form  $\pi^* \omega \in \Omega_P^k(E)$ , and define  $\tilde{\omega}$  as follows. Given a point  $p \in P$  and  $v \in T_p P$ ,  $(\pi^* \omega)_p(v) = [p, a]$ , where  $[p, a]$  is the unique representative in its class with  $p$  in the first component. We then define  $\tilde{\omega}_p(v) = a$ . We then must check that  $\tilde{\omega} \in \Omega_P^{k,G}(W)$ , so we must check

- (1)  $\iota_{\hat{\xi}}(\tilde{\omega}) = 0$  for any vertical vector field  $\hat{\xi}$ .
- (2)  $R_g^* \tilde{\omega} = \rho(g)^{-1} \tilde{\omega}$ .

For the first condition, Let  $\zeta \in V$  be a vertical vector field. Then we know that  $\iota_\zeta(\pi^*\omega) = 0$ , since  $d\pi_p(\zeta_p) = 0$  by the definition of the vertical space. Therefore,  $\iota_\zeta(\tilde{\omega}) = 0$ , verifying the first condition. For the second condition, it suffices to check the action of  $\tilde{\omega}$  on vertical and horizontal vectors, since we have the direct sum decomposition  $T_pP = V_p \oplus H_p$ . We the only need to check the action of  $\tilde{\omega}$  on horizontal vectors, since  $\iota_\zeta(\tilde{\omega}) = 0$  for any vertical vector field  $\zeta$  and the vertical distribution is invariant under the action of  $G$ . We first prove a lemma

LEMMA. Let  $v \in H_p$  be a horizontal vector. Then  $d\pi_p(v) = d\pi_{p \cdot g}((R_g)_*v)$

PROOF. Let  $\gamma : I \rightarrow M$  be a curve with  $\gamma(0) = \pi(p)$  and velocity vector  $\gamma'(0) = d\pi_p(v)$ . Then let  $\tilde{\gamma} : I \rightarrow P$  be the lift of  $\gamma$  starting at  $p$ . From before, we know that  $R_g \circ \tilde{\gamma}$  is the unique lift of  $\gamma$  starting at  $p \cdot g$ , and by the chain rule, the velocity vector of the curve  $R_g \circ \tilde{\gamma}$  at 0 is the vector  $(R_g)_*\tilde{\gamma}'(0) = (R_g)_*v$ . Therefore,  $(R_g)_*v$  is the unique horizontal lift of  $d\pi_p(v)$  based at  $p \cdot g$ , so  $d\pi_{p \cdot g}((R_g)_*v) = d\pi_p(v)$ . ■

Let  $v_1, \dots, v_k \in H_p$  be horizontal vectors and  $g \in G$ . Then

$$(R_g^*\tilde{\omega})_p(v_1, \dots, v_k) = \tilde{\omega}_{p \cdot g}((R_g)_*v_1, \dots, (R_g)_*v_k)$$

Then if  $\tilde{\omega}_p(v_1, \dots, v_k) = a$ , by definition this means that  $(\pi^*\omega)_p(v_1, \dots, v_k) = [p, a]$ . We also have that  $(\pi^*\omega)_{p \cdot g}((R_g)_*v_1, \dots, (R_g)_*v_k) = [p \cdot g, a]$ , since  $\pi(p \cdot g) = \pi(p)$  and that  $d\pi_p(v_i) = d\pi_{p \cdot g}(v_i)$ . We know that  $[p, a] = [p \cdot g, \rho(g)^{-1}a]$ , so we conclude that  $R_g^*\tilde{\omega} = \rho(g)^{-1}\tilde{\omega}$ .

In the other direction, let  $\alpha \in \Omega_p^{k,G}(W)$ . We wish to define  $\alpha' \in \Omega_M^k(E)$ . Let  $x \in M$  and  $v_1, \dots, v_k \in T_xM$ . Then define

$$\alpha'_x(v_1, \dots, v_k) = [p, \alpha_p(\tilde{v}_1, \dots, \tilde{v}_k)]$$

where  $p \in \pi^{-1}(x)$  and the  $\tilde{v}_i$  are the unique horizontal lifts of the  $v_i$  at  $p$ . We first check that this definition is independent of our choice of  $p \in \pi^{-1}(x)$ . We know  $\alpha$  satisfies  $R_g^*\alpha = \rho(g)^{-1}\alpha$ . Therefore, if we choose a different preimage  $p \cdot g \in \pi^{-1}(x)$ , we would define  $\alpha'$  by

$$\begin{aligned} \alpha'_x(v_1, \dots, v_k) &= [p \cdot g, (R_g^*\alpha)_p(\tilde{v}_1, \dots, \tilde{v}_k)] \\ &= [p \cdot g, \rho(g)\alpha_p(\tilde{v}_1, \dots, \tilde{v}_k)] \\ &= [p, \alpha_p(\tilde{v}_1, \dots, \tilde{v}_k)] \end{aligned}$$

So this definition is independent of our choice of  $p \in \pi^{-1}(x)$ . It is then easily verified that the maps are inverses to each other. ■

This is an example of a relatively general phenomenon, in which appropriately  $G$ -invariant objects on the total space  $P$  descend to objects on the base space  $M$ .

The correspondence  $\Omega_M^k(E) \leftrightarrow \Omega_P^{k,G}(W)$  gives us a natural way of discussing and extending covariant differentiation, which gives us a way of taking directional derivatives on  $E$  with respect to tangent vectors on the base  $M$ .

DEFINITION 1.9. Let  $\pi : P \rightarrow M$  be a principal  $G$ -bundle with connection 1-form  $\Theta \in \Omega_P^1(\mathfrak{g})$ . Let  $\rho : G \rightarrow GL(W)$  be a linear representation, and let  $\dot{\rho} : \mathfrak{g} \rightarrow \text{End } W$  denote the induced Lie algebra map. The *exterior covariant derivative* is a family of maps  $d_\Theta : \Omega_P^k(W) \rightarrow \Omega_P^{k+1}(W)$  given by

$$d_\Theta \psi = d\psi + \dot{\rho}(\Theta) \wedge \psi$$

By  $\dot{\rho}(\Theta)$ , we mean that its action on a vector field  $V$  is

$$\dot{\rho}(\Theta)(V) = \dot{\rho}(\Theta(V))$$

which makes sense since  $\Theta_p(V_p)$  is an element of  $\mathfrak{g}$ . Then to define  $\dot{\rho}(\Theta) \wedge \psi$ , we interpret  $\dot{\rho}(\Theta)$  as a matrix  $\dot{\rho}(\Theta)_j^i$  of 1-forms and  $\psi$  as a vector  $\psi^j$  of  $k$ -forms and define the wedge product as the usual matrix product

$$(\dot{\rho}(\Theta) \wedge \psi)_i = \dot{\rho}(\Theta)_j^i \wedge \psi^j$$

❖

We are especially interested in the restriction of the  $d_\Theta$  to the subspace  $\Omega_p^{k,G}(V) \cong \Omega_M^k(E)$ , where  $E$  denotes the associated vector bundle  $P \times_G V$ . The first thing to check is that given an element  $\psi \in \Omega_p^{k,G}(W)$ , its image  $d_\Theta \psi$  is an element of  $\Omega_p^{k+1,G}(W)$ .

**PROPOSITION 1.10.** *Given  $\psi \in \Omega_p^{k,G}(W)$ ,  $d_\Theta \psi \in \Omega_p^{k+1,G}(W)$ .*

**PROOF.** We need to check two things:

- (1)  $\iota_\xi(d_\Theta \psi) = 0$  for any vertical vector field  $\xi$ .
- (2)  $R_g^* d_\Theta \psi = \rho(g)^{-1} d_\Theta \psi$ .

For the first condition, it suffices to check for  $\xi \in \mathfrak{g}$  and  $\hat{\xi}$  its corresponding vertical vector field. We then note that  $\iota_{\hat{\xi}}(\psi^j) = 0$  for all  $j$ , since  $\psi \in \Omega_p^{k,G}(W)$ . To compute  $\iota_{\hat{\xi}}(d\psi)$ , we use a useful lemma attributed to Cartan

**LEMMA (Cartan's Magic Formula).** *Let  $\xi$  be a vector field and  $\omega$  a  $k$ -form, and let  $\mathcal{L}_\xi$  denote the Lie derivative along  $\xi$ . Then*

$$\mathcal{L}_\xi \omega = d(\iota_\xi(\omega)) + \iota_\xi(d\omega)$$

By the lemma, we have that  $\mathcal{L}_{\hat{\xi}} \psi = d\iota_{\hat{\xi}}(\psi) + \iota_{\hat{\xi}}(d\psi)$ . By assumption,  $d(\iota_{\hat{\xi}}(\psi)) = d(0) = 0$ , so  $\mathcal{L}_{\hat{\xi}} \psi = \iota_{\hat{\xi}}(d\psi)$ . The flow of  $\xi$  is given by right multiplication  $R_{\exp(t\xi)}$ , so

$$\mathcal{L}_{\hat{\xi}} \psi = \left. \frac{d}{dt} \right|_{t=0} R_{\exp(t\xi)}^* \psi$$

Then since  $\psi \in \Omega_p^{k,G}(W)$ , we have that  $R_{\exp(t\xi)}^* \psi = \rho(g)^{-1} \psi$ , and using this we compute

$$\begin{aligned} \mathcal{L}_{\hat{\xi}} \psi &= \left. \frac{d}{dt} \right|_{t=0} R_{\exp(t\xi)}^* \psi \\ &= \left. \frac{d}{dt} \right|_{t=0} \rho(\exp(t\xi))^{-1} \psi \\ &= -\dot{\rho}(\xi) \psi \end{aligned}$$

So  $\mathcal{L}_{\hat{\xi}} \psi = -\dot{\rho}(\xi) \psi = \iota_{\hat{\xi}}(d\psi)$ . To compute  $\iota_{\hat{\xi}}(\dot{\rho}(\Theta) \wedge \psi)$ , we use another helpful identity regarding interior multiplication and the wedge product.

**LEMMA.** *Let  $\xi$  be a vector field, and let  $\omega$  be a  $k$ -form and let  $\eta$  be an  $\ell$ -form. Then*

$$\iota_\xi(\omega \wedge \eta) = \iota_\xi(\omega) \wedge \eta + (-1)^k \omega \wedge \iota_\xi(\eta)$$

Fix a basis for  $W$ , so  $\rho(\Theta)$  is a matrix of 1-forms and  $\psi$  is a vector of  $k$ -forms. The  $i^{\text{th}}$  component of  $\dot{\rho}(\Theta) \wedge \psi$  is

$$(\dot{\rho}(\Theta) \wedge \psi)^i = \rho(\Theta)_j^i \wedge \psi^j$$

Then using our lemma, we compute

$$\begin{aligned} \iota_{\hat{\xi}}((\dot{\rho}(\Theta) \wedge \psi)^i) &= \iota_{\hat{\xi}}(\rho(\Theta)_j^i \wedge \psi^j) = \rho(\Theta)_j^i \wedge \iota_{\hat{\xi}}(\psi^j) \\ &= \dot{\rho}(\Theta(\hat{\xi}))_j^i \psi^j \\ &= \dot{\rho}(\xi) \psi^i \end{aligned}$$



Where we use the fact that  $\Theta(\hat{\xi}) = \xi$  and that  $\iota_{\hat{\xi}}(\psi^j) = 0$  for all  $j$  since  $\iota_{\xi}(\psi) = 0$ . Therefore, we have that

$$\iota_{\hat{\xi}}(d_{\Theta}\psi) = \iota_{\hat{\xi}}(d\psi) + \iota_{\hat{\xi}}(\dot{\rho}(\Theta) \wedge \psi) = -\dot{\rho}(\xi)\psi + \dot{\rho}(\xi)\psi = 0$$

We now want to verify the second condition. By definition,

$$R_g^*d_{\Theta}\psi = R_g^*d\psi + R_g^*(\dot{\rho}(\Theta) \wedge \psi)$$

We compute the pullbacks on each term. Pullback commutes with  $d$ , so we have that

$$R_g^*d\psi = d(R_g^*\psi) = d(\rho(g)^{-1}\psi) = \rho(g)^{-1}d\psi$$

For the other term, we fix a basis for  $W$ , and we compute the  $i^{th}$  component of  $R_g^*(\dot{\rho}(\Theta) \wedge \psi)$  to be

$$\begin{aligned} R_g^*(\dot{\rho}(\Theta) \wedge \psi)^i &= R_g^*(\dot{\rho}(\Theta)_j^i \wedge \psi^j) \\ &= \dot{\rho}(R_g^*\Theta)_j^i \wedge (R_g^*\psi)^j \\ &= \dot{\rho}(\text{Ad}_{g^{-1}}\Theta)_j^i \wedge (\rho(g)^{-1})_a^j \psi^a \\ &= (\rho(g)^{-1})_k^i \dot{\rho}(\Theta)_\ell^k \rho(g)_j^\ell \wedge (\rho(g)^{-1})_a^j \psi^a \\ &= (\rho(g)^{-1})_k^i \dot{\rho}(\Theta)_\ell^k \rho(g)_j^\ell (\rho(g)^{-1})_a^j \wedge \psi^a \\ &= (\rho(g)^{-1})_k^i \dot{\rho}(\Theta)_a^k \wedge \psi^a \\ &= (\rho(g)^{-1} \dot{\rho}(\Theta))_a^i \wedge \psi^a \end{aligned}$$

Therefore,  $R_g^*(\dot{\rho}(\Theta) \wedge \psi) = \rho(g)^{-1} \dot{\rho}(\Theta) \wedge \psi$  ■

We are especially interested in the covariant derivative on sections  $\Omega_M^0(E)$ , as  $\nabla$  is best interpreted as a directional derivative of a section  $\psi$  in the direction of a tangent vector  $\xi \in T_p M$ .

**DEFINITION 1.11.** Let  $\pi : P \rightarrow M$ ,  $\Theta \in \Omega_P^1(\mathfrak{g})$ ,  $\rho : G \rightarrow GL(W)$ ,  $\dot{\rho} : \mathfrak{g} \rightarrow \text{End } W$  and  $E = P \times_G W$  be as above. Then given a section  $\psi \in \Omega_M^0(E)$ , and a vector  $\xi \in T_x M$ , define the **covariant derivative** of  $\psi$  in the direction  $\xi$  by

$$\nabla \psi(\xi) = \left. \frac{d}{dt} \right|_{t=0} \tau_t^{-1}(\psi(\tilde{\gamma}(t)))$$

where  $\tilde{\gamma} : I \rightarrow E$  is the lift of a curve  $\gamma : I \rightarrow M$  where  $\gamma(0) = x$  and  $\gamma'(0) = \xi$  and  $\tau_t$  denotes the parallel transport maps along  $\gamma$ . ❖

Note that  $\nabla$  is an operator  $\Omega_M^0(E) \rightarrow \Omega_M^1(E)$  – we’ve already seen one operator with the same domain and codomain, the exterior covariant derivative  $d_{\Theta}$ . As the name suggests, these concepts agree.

**PROPOSITION 1.12.** *The operator  $\nabla$  is equal to  $d_{\Theta} : \Omega_M^0(E) \rightarrow \Omega_M^1(E)$ .*

**PROOF.** We note that the formulas we gave for  $\nabla$  and  $d_{\Theta}$  use different interpretations of sections of  $E \rightarrow M$ , so we need to translate one operator into the language of the other. Let  $\psi \in \Omega_M^0(E)$ , and  $\tilde{\psi} \in \Omega_P^0(W)$  its corresponding  $G$ -equivariant map  $P \rightarrow W$ . Let  $\xi \in T_x M$  be a tangent vector, and  $\gamma : I \rightarrow M$  a curve with  $\gamma(0) = x$  and  $\gamma'(0) = \xi$ . Then let  $\tilde{\gamma}_p : I \rightarrow P$  denote the horizontal lift of  $\gamma$  to the point  $p$  in the fiber  $P_x$ . The image  $\psi(\gamma(t)) \in E_{\gamma(t)}$  determines a  $G$ -equivariant map  $P_{\gamma(t)} \rightarrow W$ , where we map each element  $p \in P_{\gamma(t)}$  to the second component of its representative for  $\psi(\gamma(t))$ , and this is exactly the restriction  $\tilde{\psi}|_{P_{\gamma(t)}}$ . Under this identification, the covariant derivative is given at the point  $p \in P_x$  by

$$\nabla \tilde{\psi}(\tilde{\xi}) = \frac{d}{dt} \Big|_{t=0} \tilde{\psi}(\tilde{\gamma}_p(t))$$

where  $\tilde{\xi}$  denotes the horizontal lift of  $\xi$  to  $T_p P$ . We now verify that  $\nabla$  and  $d_\Theta$  agree.

$$\begin{aligned} \nabla \tilde{\psi}(\tilde{\xi}) &= \frac{d}{dt} \Big|_{t=0} \tilde{\psi}(\tilde{\gamma}_p(t)) \\ &= d\tilde{\psi}(\tilde{\xi}) \end{aligned}$$

We then note that since  $\tilde{\xi}$  is horizontal,  $(\dot{\rho}(\Theta)\psi)(\tilde{\xi}) = 0$ , so

$$d_\Theta \tilde{\psi}(\tilde{\xi}) = d\tilde{\psi}(\tilde{\xi}) = \nabla \tilde{\psi}(\tilde{\xi})$$

■

We will often denote the covariant derivative of a section  $\psi$  in the direction of a tangent vector  $X$  as  $\nabla_X \psi$ , which is the same thing as  $\nabla \psi(X)$ .

## 2. Curvature

An important quantity associated to a connection is its *curvature*, which heuristically measures its deviation from our standard notions of a directional derivative. We first discuss curvature on principal bundles, and will then explore its relationship with associated bundles.

DEFINITION 2.1. Let  $G$  be a Lie group with Lie algebra  $\mathfrak{g}$ , and  $\pi : P \rightarrow M$  a principal  $G$ -bundle. Let  $\Theta \in \Omega_P^1(\mathfrak{g})$  be a connection on  $P$ . The *curvature tensor* is a 2-form  $\Omega \in \Omega_P^2(\mathfrak{g})$  defined by the equation

$$\Omega = d\Theta + \frac{1}{2}[\Theta \wedge \Theta]$$

where the action of  $[\Theta \wedge \Theta]$  on  $X, Y$  is defined to be

$$[\Theta \wedge \Theta](X, Y) = [\Theta(X), \Theta(Y)] - [\Theta(Y), \Theta(X)] = 2[\Theta(X), \Theta(Y)]$$

For this reason, some denote  $1/2[\Theta \wedge \Theta]$  as just  $[\Theta, \Theta]$ .

◆

The first thing to note is that it descends to the base manifold  $M$ , i.e. it defines an element of  $\Omega_M^2(\mathfrak{g}_P)$ , where  $\mathfrak{g}_P$  is the *adjoint bundle* of  $P$ , which is the associated vector bundle  $P \times_G \mathfrak{g}$ , where the action of  $G$  on  $\mathfrak{g}$  is the adjoint action. This is due to the following properties of  $\Omega$ .

PROPOSITION 2.2.

- (1)  $R_g^* \Omega = \text{Ad}_{g^{-1}} \Omega$
- (2)  $\iota_{\tilde{\xi}}(\Omega) = 0$  for all  $\xi \in \mathfrak{g}$ .

PROOF. For the first property, we compute

$$\begin{aligned} R_g^* \Omega &= R_g^* d\Theta + \frac{1}{2} R_g^* [\Theta \wedge \Theta] \\ &= d(R_g^* \Theta) + \frac{1}{2} [R_g^* \Theta \wedge R_g^* \Theta] \\ &= \text{Ad}_{g^{-1}} d\Theta + \frac{1}{2} [\text{Ad}_{g^{-1}} \Theta \wedge \text{Ad}_{g^{-1}} \Theta] \\ &= \text{Ad}_g^{-1} \Omega \end{aligned}$$

where we use the fact that pullback commutes with exterior derivative and the wedge product, and the fact that the adjoint action of  $G$  on  $\mathfrak{g}$  commutes with brackets. For the second property, let  $\xi \in \mathfrak{g}$ . We then have that

$$\iota_{\tilde{\xi}}(\Omega) = \iota_{\tilde{\xi}}(d\Theta) + \iota_{\tilde{\xi}}[\Theta \wedge \Theta]$$

We compute the value of the terms separately. To compute  $\iota_{\hat{\zeta}}(d\Theta)$ , we use Cartan's magic formula again to get

$$\mathcal{L}_{\hat{\zeta}}\Theta = d(\iota_{\hat{\zeta}}(\Theta)) + \iota_{\hat{\zeta}}(d\Theta) = 0 + \iota_{\hat{\zeta}}(\Theta)$$

since  $\iota_{\hat{\zeta}}(\Theta) = \zeta$  for any  $\zeta \in \mathfrak{g}$ , and  $d$  of a constant form is 0. Therefore,  $\mathcal{L}_{\hat{\zeta}}\Theta = \iota_{\hat{\zeta}}(d\Theta)$ . We then note that

$$\begin{aligned} \mathcal{L}_{\hat{\zeta}}\Theta &= \left. \frac{d}{dt} \right|_{t=0} R_{\exp(t\zeta)}^* \Theta \\ &= \left. \frac{d}{dt} \right|_{t=0} \text{Ad}_{\exp(t\zeta)}^{-1} \Theta \\ &= -[\zeta, \Theta] \end{aligned}$$

where the action of  $[\zeta, \Theta]$  on  $X$  is  $[\zeta, \Theta](X) = [\zeta, \Theta(X)]$ , and we use the fact that the derivative of  $\text{Ad}$  is the Lie bracket. We then compute

$$\iota_{\hat{\zeta}}[\Theta \wedge \Theta] = [\iota_{\hat{\zeta}}(\Theta) \wedge \Theta] = [\zeta, \Theta]$$

Therefore,  $\iota_{\hat{\zeta}}(\Omega) = 0$ , so the curvature form descends to a form in  $\Omega_M^2(\mathfrak{g}_P)$ . ■

Having defined the curvature form, we explore its relationship with associated vector bundles of a principal bundle  $\pi : P \rightarrow M$  equipped with a connection  $\Theta$ . Let  $W$  be a vector space with a homomorphism  $\rho : G \rightarrow GL(W)$ , and  $\dot{\rho} : \mathfrak{g} \rightarrow \text{End } W$  its associated Lie algebra map. Then let  $E \rightarrow M$  be the associated vector bundle  $E = P \times_G W$ . Recall that the connection  $\Theta$  on  $P$  induces an exterior covariant derivative

$$\begin{aligned} d_{\Theta} : \Omega_M^k(E) &\rightarrow \Omega_M^{k+1}(E) \\ \psi &\mapsto d\psi + \dot{\rho}(\Theta) \wedge \psi \end{aligned}$$

which gives us a sequence of maps

$$\Omega_M^0(E) \xrightarrow{d_{\Theta}} \Omega_M^1(E) \xrightarrow{d_{\Theta}} \Omega_M^2(E) \xrightarrow{d_{\Theta}} \dots$$

However, this does *not* form a complex, i.e.  $d_{\Theta}^2 \neq 0$ . For  $\psi \in \Omega_M^0(E)$ , we compute

$$\begin{aligned} d_{\Theta}^2 \psi &= d_{\Theta}(d\psi + \dot{\rho}(\Theta)\psi) \\ &= d^2\psi + \dot{\rho}(\Theta) \wedge d\psi + d(\dot{\rho}(\Theta)\psi) + \dot{\rho}(\Theta) \wedge \dot{\rho}(\Theta)\psi \\ &= \dot{\rho}(\Theta) \wedge d\psi + d(\dot{\rho}(\Theta)\psi) + \dot{\rho}(\Theta) \wedge \dot{\rho}(\Theta)\psi \\ &= \dot{\rho} \wedge d\psi + \dot{\rho}(d\Theta)\psi - \dot{\rho} \wedge d\psi + \dot{\rho}(\Theta) \wedge \dot{\rho}(\Theta)\psi \\ &= \dot{\rho}(d\Theta)\psi + \dot{\rho}(\Theta) \wedge \dot{\rho}(\Theta)\psi \end{aligned}$$

We want to relate this to the curvature form  $\Omega$  on  $P$ . To do this, we use a small lemma.

LEMMA 2.3.

$$\dot{\rho}(\Theta) \wedge \dot{\rho}(\Theta) = \frac{1}{2} \dot{\rho}([\Theta \wedge \Theta])$$

The proof of the lemma just involves doing the matrix computation, and using the definition of the wedge of 1-forms when computing the matrix commutator for the left hand side. We then

compute

$$\begin{aligned}
\dot{\rho}(\Omega) &= \dot{\rho} \left( d\Theta + \frac{1}{2}[\Theta \wedge \Theta] \right) \\
&= d(\dot{\rho}(\Theta)) + \frac{1}{2}\dot{\rho}([\Theta \wedge \Theta]) \\
&= d(\dot{\rho}(\Theta)) + \dot{\rho}(\Theta) \wedge \dot{\rho}(\Theta) \\
&= d_{\Theta}^2 \psi
\end{aligned}$$

The from this we see that curvature form  $\Omega$  on  $P$  determines an  $\text{End } E$  valued 2-form on any associated vector bundle  $E$ , and is in some sense the measurement of the failure of the sequence of maps  $d_{\Theta}$  to be a complex.

**DEFINITION 2.4.** Let  $P$  be a principal bundle with connection  $\Theta$ , and  $\Omega$  the curvature form.  $\rho : G \rightarrow GL(W)$  a representation, with corresponding Lie algebra representation  $\dot{\rho} : \mathfrak{g} \rightarrow \text{End } W$ . Let  $E = P \times_G W$ . The *curvature transformation* is a 2-form  $R \in \Omega^2(\text{End } E)$  defined by

$$R = \dot{\rho}(\Omega)$$

❖

Given tangent vectors  $V, W$ , we often denote the endomorphism  $R(V, W)$  by  $R_{V,W}$ . An identity that will be future use is the following proposition.

**PROPOSITION 2.5.** Let  $R \in \Omega_M^2(\text{End } TM)$  be the curvature transformation of a Riemannian manifold, i.e. the curvature induced by the Levi-Civita connection. Then for any tangent vectors  $V, W, X$

$$R_{V,W}X + R_{X,V}W + R_{W,X}V = 0$$

In other words, the sum over all cyclic permutations of the arguments is 0.

### 3. Dirac Operators

To discuss Dirac operators on manifolds, we restrict the previous discussion to the case where the principal bundle we're concerned with is the principal bundle of frames  $\mathcal{B}(M)$  (or subbundles like  $\mathcal{B}_O(M)$  and  $\mathcal{B}_{SO}(M)$ ). The covariant derivative has a very geometric interpretation when applied to the  $GL_n\mathbb{R}$  bundle of frames  $\pi : \mathcal{B}(M) \rightarrow M$ . The natural action of  $GL_n\mathbb{R}$  on  $\mathbb{R}^n$  allows us to define the associated vector bundle  $\mathcal{B}(M) \times_{GL_n\mathbb{R}} \mathbb{R}^n$ , which is naturally isomorphic to the tangent bundle  $TM$  as we showed before. We can then define a natural 1-form  $\theta \in \Omega_{\mathcal{B}(M)}(\mathbb{R}^n)$  called the *soldering form*. Given  $b \in \mathcal{B}(M)$ , we know that  $b$  is a linear isomorphism  $b : \mathbb{R}^n \rightarrow T_{\pi(b)}M$ . Then given  $v \in T_bP$ , we define the action of  $\theta$  by

$$\theta_p(v) = b^{-1}(d\pi_p(v))$$

The soldering form in some sense remembers that  $\mathcal{B}(M)$  is the bundle of frames. Then given a connection  $\Theta \in \Omega_P^1(\mathfrak{gl}_n\mathbb{R})$  on  $\mathcal{B}(M)$ , we use the  $\Theta$  to define special vector fields on  $P$ , which we denote  $\partial_i$ .<sup>1</sup> Given  $b \in P$ , we define  $\partial_i|_b$  by  $\partial_i|_b = \widetilde{b(e_1)}$ , where  $e_1$  denotes the first standard basis vector of  $\mathbb{R}^n$  and  $\widetilde{b(e_1)}$  is the unique horizontal lift of  $b(e_1) \in T_{\pi(b)}M$  at  $b$  with respect to the connection  $\Theta$ . The vector fields  $\partial_2, \dots, \partial_n$  are defined analogously, and together the  $\partial_i$  give a framing for the horizontal distribution on  $\mathcal{B}(M)$ . The connection  $\Theta$  on  $\mathcal{B}(M)$  induces a covariant derivative  $\nabla$  on any associated vector bundle  $E \rightarrow M$ , which has a very special relationship with the vector fields  $\partial_i$ .

<sup>1</sup>Note that these vector fields are not coordinate vector fields, though we have in the past used the notation  $\partial_i$  to denote coordinate vector fields.

**PROPOSITION 3.1.** *Let  $M$  be a smooth manifold and  $\pi : \mathcal{B}(M) \rightarrow M$  its  $GL_n\mathbb{R}$  bundle of frames. Equip  $\mathcal{B}(M)$  with a connection  $\Theta$  and define the vector fields  $\partial_i$  as above. Then given a vector space  $W$  with a representation  $\rho : GL_n\mathbb{R} \rightarrow GL(W)$  let  $E \rightarrow M$  denote the associated bundle, and let  $\nabla$  denote the covariant derivative on  $E$  induced by  $\Theta$ . Then given a section  $\psi \in \Omega_M^0(E)$ , it has a corresponding  $GL_n\mathbb{R}$ -equivariant map  $\tilde{\psi} : P \rightarrow W$ . Then the covariant derivative of  $\tilde{\psi}$  is given by the formula*

$$\nabla \tilde{\psi} = \partial_i \tilde{\psi} \otimes e^i$$

where  $e^i \in (\mathbb{R}^n)^*$  denotes the dual basis to the standard basis  $e_i$ .

**PROOF.** We first make sense of the formula. The covariant derivative  $\nabla \psi$  is an element of  $\Omega_M^1(E)$ , which is a section of the bundle  $E \otimes T^*M$ . The bundle  $T^*M$  is naturally isomorphic to the associated bundle  $\mathcal{B}(M) \times_{GL_n\mathbb{R}} (\mathbb{R}^n)^*$  given by the dual to the standard representation of  $GL_n\mathbb{R}$  on  $\mathbb{R}^n$ . Therefore, we get a representation of  $GL_n\mathbb{R}$  on  $W \otimes (\mathbb{R}^n)^*$ , and  $E \otimes T^*M$  is naturally isomorphic to the vector bundle  $\mathcal{B}(M) \times_{GL_n\mathbb{R}} (W \otimes (\mathbb{R}^n)^*)$ . Sections of this bundle are equivalent  $GL_n\mathbb{R}$ -equivariant maps  $P \rightarrow W \otimes (\mathbb{R}^n)^*$ . The action of the  $\partial_i$  on  $\tilde{\psi}$  produces  $GL_n\mathbb{R}$ -equivariant maps  $\partial_i \tilde{\psi} : P \rightarrow W$ , so the tensor product  $\partial_i \tilde{\psi} \otimes e^i$  then defines a section of  $E \otimes T^*M$ .

We then want to verify that this formula is true. Fix  $x \in X$ ,  $v \in T_x M$ , and  $b \in \pi^{-1}(x)$ . The element  $b$  determines a basis  $b(e_1), \dots, b(e_n)$  for  $T_x M$ . In this basis,  $v$  has the representation  $v^i b(e_i)$ . Each of the  $b(e_i)$  lift to  $\partial_i|_b$ , so the horizontal lift of  $v$  to  $b$  is  $v^i \partial_i|_b$ . Then let  $\tilde{\gamma} : (-\varepsilon, \varepsilon) \rightarrow P$  be an integral curve of  $v^i \partial_i$ . Then  $\gamma = \pi \circ \tilde{\gamma}$  is a curve in  $M$  with  $\gamma'(0) = v$ , so the covariant derivative  $(\nabla \tilde{\psi})_b(v)$  is given by

$$\begin{aligned} (\nabla \tilde{\psi})_b(v) &= \left. \frac{d}{dt} \right|_{t=0} \tilde{\psi}(\tilde{\gamma}(t)) \\ &= v^i \partial_i \tilde{\psi} \end{aligned}$$

Therefore,  $\nabla \psi = \partial_i \otimes e^i$ . ■

The vector fields  $\partial_i$  on  $\mathcal{B}(M)$  also give a recipe for constructing differential operators between sections of associated vector bundles, and we will use this recipe to construct a Dirac operator on a general Spin manifold. Let  $A$  and  $B$  be linear representations of  $GL_n\mathbb{R}$ , and let  $E, F \rightarrow M$  be their associated vector bundles. The action of  $GL_n\mathbb{R}$  on  $A$  and  $B$  induces an action on  $\text{Hom}(A, B)$ , giving it the structure of a  $GL_n\mathbb{R}$  representation. Then given a  $GL_n\mathbb{R}$ -equivariant map  $\sigma : (\mathbb{R}^n)^* \rightarrow \text{Hom}(A, B)$ , we can define a first order differential operator

$$\begin{aligned} D_\sigma : \Gamma_M(E) &\rightarrow \Gamma_M(E) \\ \psi &\mapsto [\sigma(e^k)](\partial_k \psi) \end{aligned}$$

Where we interpret  $\psi$  as a  $GL_n\mathbb{R}$ -equivariant map  $P \rightarrow A$ . This method for constructing differential operator has already manifested itself in several differential operators we have discussed.

**EXAMPLE 3.2.** Given a vector space  $W$  with a  $GL_n\mathbb{R}$  action, which gives an associated bundle  $\mathcal{B}(M) \times_{GL_n\mathbb{R}} W$ . A connection on  $\mathcal{B}(M)$  induces a covariant derivative operator  $\nabla$  on sections of the associated bundle. In terms of the horizontal vector fields  $\partial_i$  defined on  $\mathcal{B}(M)$ ,  $\nabla$  is given by

$$\nabla \psi = \partial_k \psi \otimes e^k$$

Let  $T : (\mathbb{R}^n)^* \rightarrow \text{Hom}((\mathbb{R}^n)^*, W)$  be the map where  $T(v)(w) = w \otimes v$ . Then  $\nabla = T(e^k) \partial_k$  ◀

**EXAMPLE 3.3.** Consider  $\Lambda^k(\mathbb{R}^n)^*$  and  $\Lambda^{k+1}(\mathbb{R}^n)^*$ , which have a natural  $GL_n\mathbb{R}$  action, and let  $\varepsilon : (\mathbb{R}^n)^* \rightarrow \text{Hom}(\Lambda^k(\mathbb{R}^n)^*, \Lambda^{k+1}(\mathbb{R}^n)^*)$  be the exterior multiplication map, i.e.

$$\varepsilon(v)(w) = v \wedge w$$

This map is easily seen to be  $GL_n\mathbb{R}$ -equivariant. Using our recipe, we get a differential operator

$$D_\varepsilon : \Omega_M^k \rightarrow \Omega_M^{k+1}$$

$$\psi \mapsto [\varepsilon(e^k)](\partial_k \psi)$$

which is the de Rham differential  $d$ . ◀

The preceding discussion was all done with  $\mathcal{B}(M)$  and  $G = GL_n\mathbb{R}$ , but the discussion applies to any principal bundle  $P$  with a reduction of structure group to  $\mathcal{B}(M)$ , like  $\mathcal{B}_O(M)$ ,  $\mathcal{B}_{SO}(M)$ , and  $\mathcal{B}_{Spin}(M)$ .

Let  $(M, g)$  be a Riemannian manifold. The Riemannian metric  $G$  gives a reduction of structure group to  $O_n$ , giving us the orthonormal frame bundle  $\mathcal{B}_O(M)$ . It is a wonderful fact that we have a canonical choice of connection on  $\mathcal{B}_O(M)$ , where we ask for a quantity called the torsion tensor to vanish.

**THEOREM 3.4 (Fundamental Theorem of Riemannian Geometry).** *There exists a unique torsion-free connection  $\Theta$  on  $\mathcal{B}_O(M)$ .*

We call this connection the *Levi-Civita connection*. If in addition,  $M$  is oriented, we have a reduction of structure group to  $\mathcal{B}_{SO}(M)$ , and the Levi-Civita connection restricts to this bundle. We also refer to this connection as the Levi-Civita connection. Going further, if  $M$  is Spin, we get a canonical choice of connection on  $\mathcal{B}_{Spin}(M)$ .

**DEFINITION 3.5.** Let  $(M, g)$  be a Spin manifold, and let  $\Theta$  be the Levi-Civita connection on the oriented orthonormal frame bundle  $\mathcal{B}_{SO}(M)$ . The double cover  $\pi : \mathcal{B}_{Spin} \rightarrow \mathcal{B}_{SO}(M)$  gives a natural isomorphism

$$T(\mathcal{B}_{Spin}(M)) \rightarrow \pi^*(T\mathcal{B}_{SO}(M))$$

So the pullback of the horizontal distribution  $H$  determined by  $\Theta$  defines a connection on  $\mathcal{B}_{Spin}(M)$  called the *Spin connection*. ❖

We saw in  $\mathbb{R}^n$  that the Dirac operator  $D$  acted on functions  $\psi \in C^\infty(\mathbb{R}^n, V)$  for a Clifford module  $V$ . Over a manifold  $M$ , these Clifford modules might vary over each point of the manifold, so the Dirac operator should act on sections  $\Gamma_M(\mathcal{S})$  for some bundle  $\mathcal{S} \rightarrow M$  of Clifford modules over  $M$ .

While we have discussed Spin structures extensively, and Dirac operators are of particular importance to Spin geometry, they are not dependent on a Spin structure. Note that the Dirac operator we defined on  $\mathbb{R}^n$  only required a nondegenerate bilinear form, which we used to construct the Clifford algebra. This suggests that the only structure necessary to define a Dirac operator is a Riemannian metric.

**DEFINITION 3.6.** Let  $(M, g)$  be an oriented Riemannian manifold, giving us a reduction of structure group  $\mathcal{B}_{SO}(M) \rightarrow \mathcal{B}(M)$ . Under the standard representation,  $SO_n$  acts on  $\mathbb{R}^n$  with the standard inner product by isometries, so every element  $A \in SO_n$  induces an algebra automorphism  $\rho_A : \text{Cliff}_{n,0}(\mathbb{R}) \rightarrow \text{Cliff}_{n,0}(\mathbb{R})$ , giving  $\text{Cliff}_{n,0}(\mathbb{R})$  the structure of a  $SO_n$  representation. The *Clifford bundle* of  $M$  is the associated vector bundle

$$\text{Cliff}(M) = \mathcal{B}_{SO}(M) \times_{SO_n} \text{Cliff}_{n,0}(\mathbb{R})$$
❖

In fact, the bundle  $\text{Cliff}(M) \rightarrow M$  is a bundle of Clifford algebras over  $M$ , where over each point  $p \in M$ , the fiber  $\text{Cliff}(M)_p$  is  $\text{Cliff}(T_p M, g_p)$ . Working fiberwise, the natural vector space isomorphism  $\text{Cliff}(T_p M, g_p) \rightarrow \Lambda^\bullet(T_p M, )$  determines a vector bundle isomorphism  $\text{Cliff}(M) \rightarrow$

$\Lambda^\bullet(TM)$ . Using the Riemannian metric, we get an isomorphism  $TM \rightarrow T^*M$ , and the composition of these isomorphisms determines a vector bundle isomorphism  $\text{Cliff}(M) \rightarrow \Lambda^\bullet(T^*M)$ . In the case that the manifold  $M$  admits a Spin structure, we can define further bundles.

DEFINITION 3.7. Let  $(M, g)$  be an oriented Riemannian manifold equipped with a Spin structure  $\mathcal{B}_{\text{Spin}}(M) \rightarrow \mathcal{B}_{\text{SO}}(M)$ , and let  $S$  be a left Clifford module over  $\text{Cliff}_{n,0}(\mathbb{R})$ . The inclusion  $\text{Spin}_n \hookrightarrow \text{Cliff}_{n,0}(\mathbb{R})$  gives  $S$  the structure of a  $\text{Spin}_n$  representation. A *spinor bundle* over  $M$  is the associated vector bundle

$$\mathcal{B}_{\text{Spin}}(M) \times_{\text{Spin}_n} S$$

❖

The Levi-Civita and Spin connections induce canonical connections on  $\text{Cliff}(M)$  and any spinor bundle  $S$ .

As one would expect, there is an intimate relationship between the Clifford bundle and a spinor bundle over a Spin manifold. Namely, a spinor bundle  $S$  is a bundle of left Clifford modules over  $\text{Cliff}(M)$ , i.e. each fiber  $S_p$  is a left Clifford module over the corresponding fiber  $\text{Cliff}(M)_p$ . In addition, any spinor bundle  $\mathcal{B}_{\text{Spin}}(M) \times_{\text{Spin}_n} S$  admits a fiber metric in which the action of a unit vector in  $\text{Cliff}_{n,0}(\mathbb{R})$  is orthogonal. One way to construct a metric is to declare the basis of  $\text{Cliff}_{n,0}$  generated by products of the standard basis  $\{e_i\}$  to be an orthonormal basis, and take the induced inner product. This then induces a fiber metric on the associated bundle with the desired property. We now have the ingredients to define a general Dirac operator.

Equip  $\mathbb{R}^n$  with the standard inner product, and let  $b : (\mathbb{R}^n)^* \times (\mathbb{R}^n)^* \rightarrow \mathbb{R}$  be the induced inner product on  $(\mathbb{R}^n)^*$ . Using the double cover  $\text{Spin}_n \rightarrow \text{SO}_n$ , the group  $\text{Spin}_n$  acts on  $(\mathbb{R}^n)^*$  through the dual representation of  $\text{SO}_n$  on  $(\mathbb{R}^n)^*$ . In addition,  $\text{Spin}_n$  acts naturally on any Clifford module  $S$  by left multiplication, where we think of  $\text{Spin}_n$  as a subgroup of the group of units  $\text{Cliff}((\mathbb{R}^n)^*, b)^\times$ . The *Clifford multiplication map*  $c : (\mathbb{R}^n)^* \rightarrow \text{End}(S)$  given by  $c(v) = va$  is easily verified to be  $\text{Spin}_n$ -equivariant, so we can apply our recipe for constructing differential operators to construct the Dirac operator.

DEFINITION 3.8. Let  $(M, g)$  be a oriented Riemannian manifold equipped with a Spin structure  $\varphi : \mathcal{B}_{\text{Spin}}(M) \rightarrow \mathcal{B}_{\text{SO}}(M)$  and spinor bundle  $S \rightarrow M$ . Using the Spin connection on  $\mathcal{B}_{\text{Spin}}(M)$ , define the horizontal vector fields  $\partial_i$  on  $\mathcal{B}_{\text{Spin}}(M)$  by

$$\partial_i|_p = \widetilde{\varphi(p)(e_i)}$$

where  $\widetilde{\varphi(p)(e_i)}$  denotes the horizontal lift of the image of  $e_i$  under  $\varphi(p) \in \mathcal{B}_{\text{SO}}(M)$  to  $T(\mathcal{B}_{\text{Spin}}(M))$ . The *Dirac operator* is the first order differential operator

$$D : \Gamma_M(S) \rightarrow \Gamma_M(S) \\ \psi \mapsto c(e^k) \partial_k \psi$$

where we interpret  $\psi$  as a  $\text{Spin}_n$ -equivariant map  $\mathcal{B}_{\text{Spin}}(M) \rightarrow \text{Cliff}_{n,0}(\mathbb{R})$ .

❖

In terms of the covariant derivative  $\nabla$  on the spinor bundle induced by the Spin connection, the Dirac operator can be written in an orthonormal frame  $e_i$  of  $S$  by

$$D = \sum_i e_i \nabla_{e_i}$$

Note that this agrees with our previous definitions of Dirac operators on  $\mathbb{R}^n$ , as well as the ones we defined on  $S^1$ . In  $\mathbb{R}^n$ , the Dirac operator squares to the Laplacian, but on a general manifold, this need not be true. Therefore, we give a name to the operator  $D^2$ .

DEFINITION 3.9. The **Dirac Laplacian** is the map  $D^2 : \Gamma_M(\mathbb{S}) \rightarrow \Gamma_M(\mathbb{S})$ . ❖

Given compact Riemannian manifold  $(M, g)$ , the Riemannian metric on  $M$  induces an inner product on all the associated tensor bundles of  $TM$  and  $T^*M$ . In particular, we get an inner product  $\langle \cdot, \cdot \rangle$  on the exterior powers  $\Lambda^k TM$ . Since  $M$  is compact, this induces an inner product  $(\cdot, \cdot)$  on the space on  $\Omega_M^k$  given by

$$(\omega, \eta) = \int_M \langle \omega, \eta \rangle dV_g$$

where  $dV_g$  is the Riemannian volume form of  $M$ . Under this inner product, we get a linear map  $d^* : \Omega_M^k \rightarrow \Omega_M^{k-1}$  that is the adjoint to the exterior derivative  $d$ , which is constructed using the Hodge star operator  $\star : \Omega_M^k \rightarrow \Omega_M^{n-k}$ . Again, we can express  $d^*$  using our recipe for differential operators. Fittingly, its formula reflects its relationship with the exterior derivative  $d$ .

PROPOSITION 3.10. The operator  $d^*$  is given by

$$d^* = \iota(e^k) \partial_k$$

where  $\iota(e^k)(\omega) = \langle e^k, \omega \rangle$  is the inner product of  $e^k$  and  $\omega$  induced by the Riemannian metric.

Using  $d$  and  $d^*$ , we construct the **Hodge Laplacian**

$$\Delta = dd^* + d^*d$$

which is a vast generalization of the Laplacian on  $\mathbb{R}^n$  and an important tool in studying the geometry and topology of  $M$ . Thought of as a map  $\Omega_M^\bullet \rightarrow \Omega_M^\bullet$ , the map  $d + d^*$  is a square root of  $\Delta$ , where we use the fact that  $d^2 = (d^*)^2 = 0$ . Since Dirac operators were originally conceived to be square roots of the Laplacian, it's no surprise that  $d + d^*$  is a Dirac operator.

THEOREM 3.11. Under the isomorphism  $\text{Cliff}(M) \rightarrow \Lambda^\bullet(TM)$ , the Dirac operator  $D$  on  $\text{Cliff}(M)$  is given by  $d + d^*$ .

PROOF. Let  $\varphi : \text{Cliff}_{n,0}(\mathbb{R}) \rightarrow \Lambda^\bullet \mathbb{R}^n$  be the canonical isomorphism. Recall that for  $v \in \mathbb{R}^n$  and  $\eta \in \text{Cliff}_{n,0}(\mathbb{R})$ ,

$$\varphi(v\eta) = v \wedge \eta + \iota(v)(\eta)$$

Therefore, under this isomorphism the Clifford multiplication map  $c(e^k)$  becomes the map  $\varepsilon(e^k) + \iota(e^k)$ , so the Dirac operator becomes

$$D = (\varepsilon(e^k) + \iota(e^k)) \partial_k = d + d^*$$
■

The Dirac operator has an intimate relationship with another Laplacian operator.

DEFINITION 3.12. Let  $(M, g)$  be a oriented Riemannian manifold, equipped with the Levi-Civita connection  $\Theta$  on  $\mathcal{B}_{SO}(M)$ , and let  $\nabla$  denote the covariant derivative on  $TM$ . The **connection Laplacian** on  $M$  is map  $\nabla^* \nabla : \Gamma(TM) \rightarrow \Gamma(TM)$  defined by

$$\nabla^* \nabla \psi = \text{trace}(\nabla^2 \psi)$$

where  $\nabla^2$  is the 2-form defined by

$$\nabla_{V,W}^2 \psi = \nabla_V \nabla_W \psi - \nabla_W \nabla_V \psi$$

for tangent vectors  $V, W$  and a vector field  $\psi$ . In an orthonormal frame  $\{e_i\}$  for  $TM$ ,  $\nabla^* \nabla$  has the coordinate formula

$$\nabla^* \nabla \psi = \sum_i \nabla_{e_i, e_i}^2 \psi$$
❖



We two different “Laplacian type” operators on a manifold – the connection Laplacian  $\nabla^*\nabla$ , and the Dirac Laplacian  $D^2$ . A natural question to ask is how these operators differ, i.e. what is  $D^2 - \nabla^*\nabla$ ? A priori, one might expect the difference to be a second or first order differential operator. The miracle is that the difference is a zeroth order operator, i.e. a tensor, which only involves the curvature of the manifold.

**THEOREM 3.13 (The Weitzenböck Formula).** *Let  $\{e_i\}$  denote a local orthonormal frame of a spinor bundle  $S \rightarrow M$ . Then*

$$D^2 = \nabla^*\nabla + \frac{1}{2} \sum_{i,j} e_i e_j R_{e_i, e_j}$$

**PROOF.** In the orthonormal frame  $\{e_i\}$ , the Dirac operator is expressed as  $D = \sum_i e_i \nabla_{e_i}$ . We then compute

$$\begin{aligned} D^2 &= \sum_{i,j} (e_i \nabla_{e_i})(e_j \nabla_{e_j}) \\ &= \sum_{i,j} e_i e_j \nabla_{e_i} \nabla_{e_j} \\ &= \sum_i e_i^2 \nabla_{e_i} \nabla_{e_i} + \sum_{j < k} e_j e_k \nabla_{e_j} \nabla_{e_k} \\ &= - \sum_i \nabla_{e_i}^2 + \sum_{j < k} e_j e_k (\nabla_{e_j, e_k}^2 - \nabla_{e_k, e_j}^2) \\ &= \nabla^*\nabla + \frac{1}{2} \sum_{i,j} e_i e_j R_{e_i, e_j} \end{aligned}$$

■

As a corollary, we can express the Hodge Laplacian in terms of the connection Laplacian and some curvature term. Remarkably the curvature term is a familiar friend from Riemannian geometry – the Ricci tensor.

**COROLLARY 3.14.** *Let  $\Delta$  denote the Hodge Laplacian on a Riemannian manifold  $M$ . Then*

$$\Delta = \nabla^*\nabla + \text{Ric}$$

**PROOF.** Under the identification  $\text{Cliff}(M) \rightarrow \Lambda^\bullet(TM)$ , we know that  $D^2 = \Delta$ . Using the previous theorem, it then suffices to prove that the curvature term in the Weitzenböck formula under the identification is the Ricci tensor. ■

A special spinor bundle to consider over a Spin manifold  $M$  is when the Clifford module is  $\text{Cliff}_{n,0}(\mathbb{R})$  itself, giving us a spinor bundle

$$S(M) = \mathcal{B}_{\text{Spin}}(M) \times_{\text{Spin}_n} \text{Cliff}_{n,0}(\mathbb{R})$$

This is a  $\mathbb{Z}/2\mathbb{Z}$ -graded spinor bundle, with subbundles

$$S^0(M) = \mathcal{B}_{\text{Spin}}(M) \times_{\text{Spin}_n} \text{Cliff}_{n,0}^0(\mathbb{R}) \quad S^1(M) = \mathcal{B}_{\text{Spin}}(M) \times_{\text{Spin}_n} \text{Cliff}_{n,0}^1(\mathbb{R})$$

In addition, the space of sections  $\Gamma_M(S(M))$  of  $S(M)$  also inherits the structure of a graded Clifford module, where the grading is given by

$$\Gamma_M^0(S(M)) = \Gamma_M(S^0(M)) \quad \Gamma_M^1(S(M)) = \Gamma_M(S^1(M))$$

Since multiplication by a vector is odd, Clifford multiplication is odd, so the Dirac operator is an odd operator with respect to this grading of  $\Gamma_M(S(M))$ , and has the block form

$$\begin{pmatrix} 0 & D^1 \\ D^0 & 0 \end{pmatrix}$$

where  $D^i$  is the Dirac operator restricted to the subspace  $\Gamma_M^i(\mathcal{S}(M))$ . The Clifford algebra  $\text{Cliff}_{n,0}$  is also a *right* Clifford module over itself, and this action commutes with the  $\text{Spin}_n$  action on the left. Therefore, the right action of  $\text{Cliff}_{n,0}(\mathbb{R})$  on  $\mathcal{B}_{\text{Spin}}(M) \times \text{Cliff}_{n,0}(\mathbb{R})$  descends to a right action on  $\text{Cliff}_{n,0}(\mathbb{R})$  on  $\mathcal{S}(M)$ , giving each fiber the structure of a right Clifford module. The right Clifford action then induces a *left*  $\text{Cliff}_{0,n}(\mathbb{R})$  action. Recall that there is an automorphism  $\alpha : \text{Cliff}_{p,q} \rightarrow \text{Cliff}_{p,q}$  that extends the mapping  $v \mapsto -v$  on  $\mathbb{R}^{p,q}$ . There is a linear isomorphism  $A : \text{Cliff}_{n,0}(\mathbb{R}) \rightarrow \text{Cliff}_{0,n}(\mathbb{R})$  determined by the identity map  $\mathbb{R}^n \rightarrow \mathbb{R}^n$ , thought of as a map  $\mathbb{R}^{n,0} \rightarrow \mathbb{R}^{0,n}$ . Then the left  $\text{Cliff}_{0,n}$  action on  $\mathcal{S}(M)$  is defined to be

$$v \cdot m = m \cdot \alpha(A(v))$$

For example, let  $e_i^-$  be the standard orthogonal basis for  $\mathbb{R}^{0,n}$ , and  $e_i^+$  the standard orthogonal basis for  $\mathbb{R}^{n,0}$ . Then the left action of  $e_i^-$  is the right action of  $-e_i^+$ . Since right multiplication commutes with left multiplication, and the action of  $\text{Cliff}_{0,n}(\mathbb{R})$  is defined in terms of this right action, the Dirac operator is  $\text{Cliff}_{0,n}(\mathbb{R})$  linear, i.e. it commutes with the  $\text{Cliff}_{0,n}(\mathbb{R})$  action on the left. The fact that  $D$  is  $\text{Cliff}_{0,n}(\mathbb{R})$  linear implies that for  $\psi \in \Gamma_M^0(\mathcal{S}(M))$  and  $v \in \mathbb{R}^{0,n} \subset \text{Cliff}_{0,n}(\mathbb{R})$ , we have that

$$D(v \cdot \psi) = D^1(v\psi) = v \cdot D\psi = v \cdot D^0\psi$$

We can then define a new odd operator  $\tilde{D} : \Gamma_M(\mathcal{S}(M)) \rightarrow \Gamma_M(\mathcal{S}(M))$  by

$$\tilde{D} = \begin{pmatrix} 0 & -D^1 \\ D^0 & 0 \end{pmatrix}$$

We claim that  $\tilde{D}$  anticommutes with the left action of any vector  $v \in \mathbb{R}^{0,n} \subset \text{Cliff}_{0,n}(\mathbb{R})$ . It suffices to show this on the subspaces  $\Gamma_M^0(\mathcal{S}(M))$  and  $\Gamma_M^1(\mathcal{S}(M))$ . In the first case, let  $\psi$  be an even section of  $\mathcal{S}(M)$ . Then for  $v \in \mathbb{R}^{0,n}$ , we compute  $v \cdot \psi \in \Gamma_M^1(\mathcal{S}(M))$ , since multiplication by odd elements is odd, so  $\tilde{D}(v \cdot \psi) = -D^1(v\psi)$ . On the other hand

$$v \cdot \tilde{D}\psi = v \cdot D^0\psi = D^1(v\psi)$$

since  $D$  is  $\text{Cliff}_{0,n}(\mathbb{R})$  linear. The computation in the odd case is similar.

The Dirac operator has the special property of being an *elliptic operator*. We will not define this property, but we will use some consequences of elliptic theory and functional analysis to deduce certain properties of  $D$  and the Dirac Laplacian  $D^2$ . In particular we will use several properties of  $-D^2$ .

- (1)  $-D^2$  is positive and self adjoint, i.e. all the eigenvalues are positive, and

$$(-D^2\psi_1, \psi_2) = (\psi_1, -D^2\psi_2)$$

where the inner product is defined by integrating over  $M$  using the fiber metric on  $\mathcal{S}(M)$ . This does not need elliptic theory to be verified.

- (2)  $-D^2$  has a discrete spectrum – its eigenvalues form a discrete subset of  $\mathbb{R}$ .
- (3)  $-D^2$  has finite dimensional eigenspaces. In particular,  $D$  has a finite dimensional kernel.

With these properties in hand, we make some observations. We first note that any eigenspace of  $-D^2$  is preserved by the left action of  $\text{Cliff}_{0,n}(\mathbb{R})$ , since the left action commutes with  $-D^2$ , and commuting operators preserve eigenspaces. Then the fact that the eigenspaces are finite dimensional implies that they are all finite dimensional  $\text{Cliff}_{0,n}(\mathbb{R})$  modules. In fact, the eigenspaces are graded modules, since they decompose into the even and odd components by intersecting with  $\mathcal{S}^0(M)$  and  $\mathcal{S}^1(M)$ . Let  $E_\lambda$  denote the  $\lambda^2$ -eigenspace of  $-D^2$ . By multiplying out the matrices, it is easy to see that  $\tilde{D}$  commutes with  $-D^2$ , so it preserves  $E_\lambda$ . Then define the map  $F_\lambda = \tilde{D}/\lambda|_{E_\lambda}$ .

## Bibliography

- [1] M. F. Atiyah, R. Bott, and A. Shapiro. “Clifford Modules”. In: *Topology* (1964), pp. 3–38.
- [2] S. Kobayashi and Nomizu K. *Foundations of Differential Geometry Volume 1*. 1996.
- [3] H. B. Lawson and M. L. Michelsohn. *Spin Geometry*. 1998.