Every element of SO_3 is rotation about an axis

Given an isomorphism

$\text{End } V \xrightarrow{\Phi} \text{End } W$, does there exist an associated isomorphism

① $V \xrightarrow{\tilde{\Phi}} W$ inducing Φ?

② How about an isomorphism

$\mathbb{P}V \longrightarrow \mathbb{P}W$ (what is an isomorphism of $\mathbb{P}V \to \mathbb{P}W$?)

(What is a projective structure?

1) Fill in a table of the cokernels.

2) Isogenies of Spin — Compute the Centers.

Given an isomorphism $\varphi: V \to W$, this induces an isomorphism

$$\tilde{\varphi}\quad \mathrm{End}(V) \longrightarrow \mathrm{End}(W)$$
$$A \longmapsto \varphi \circ A \circ \varphi^{-1}$$

Given an isomorphism $\psi: \mathrm{End}(V) \longrightarrow \mathrm{End}(W)$, can we construct an isomorphism $\varphi: V \to W$ inducing ψ?

Fix bases $b_V: \mathbb{R}^n \to V$, $b_W: \mathbb{R}^n \to W$. this induces maps $\mathcal{B}_V: M_n\mathbb{R} \to \mathrm{End}(V)$, $\mathcal{B}_W: M_n\mathbb{R} \to \mathrm{End}(W)$

$$\begin{tikzcd} V \arrow[r, dashed] & W \\ \mathbb{R}^n \arrow[u, "b_V"] \arrow[r, dashed] & \mathbb{R}^n \arrow[u, "b_W"'] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{End}\,V \arrow[r, "\psi"] & \mathrm{End}\,W \arrow[d, "\mathcal{B}_W^{-1}"] \\ M_n\mathbb{R} \arrow[u, "\mathcal{B}_V"] \arrow[r, "\tilde{\psi}"'] & M_n\mathbb{R} \end{tikzcd}$$

ψ induces an automorphism of $M_n\mathbb{R}$.
does an automorphism of $M_n\mathbb{R}$ determine a automorphism $\mathbb{R}^n \to \mathbb{R}^n$?

<u>Fact</u> : Any automorphism of $M_n\mathbb{R}$ is conjugation by $A \in GL_n\mathbb{R}$ (TODO?)

So in these bases, $\tilde{\varphi}$ is given by conjugation by some matrix A, up to scalars

So if we fix a basis, yes

Any automorphism of $End(V)$ is conjugation by $A \in GL(V)$

$End(V)$ has a unique irreducible module V. An automorphism α of $End\ V$ determines a new module V^α where $M \cdot v = \alpha(M) \cdot v$.

We then note the V^α is also an irreducible module, so it is isomorphic to V as an $End\,V$ module

$\Rightarrow$ there exists an isomorphism

$A : V \to V^\alpha$

and $A \in GL(V)$, since V and V^α have the same underlying vector space.

Since A is a module homomorphism, for $M \in \operatorname{End} V$,

$$AM = \alpha(M) A$$
$$\Downarrow$$
$$AMA^{-1} = \alpha(M)$$

$$\Downarrow$$

$$\operatorname{Aut}(\operatorname{End} V) \cong PGL(V)$$

So an isomorphism $\operatorname{End} V \to \operatorname{End} W$ induces a family of isomorph. $V \longrightarrow M$ differing by scalars.

What does this imply about projective representations of SO_n?

Suppose $Cliff_n \cong M_n \mathbb{R}$

Then this implies that $Cliff_n$ "$\cong$" $\operatorname{End}(\$)$ where $\$$ is only determined up to "projective equivalence" (What does this mean?)

In what sense is the projective representation "Canonical"

$M_8\mathbb{C}$	$M_8\mathbb{H}$	$M_8\mathbb{H}\times M_8\mathbb{H}$	$M_{16}\mathbb{H}$	$M_{32}\mathbb{C}$	$M_{64}\mathbb{R}$	$M_{64}\mathbb{R}\times M_{64}\mathbb{R}$	$M_{128}\mathbb{R}$
$M_4\mathbb{H}$	$M_4\mathbb{H}\times M_4\mathbb{H}$	$M_8\mathbb{H}$	$M_{16}\mathbb{C}$	$M_{32}\mathbb{R}$	$M_{32}\mathbb{R}\times M_{32}\mathbb{R}$	$M_{64}\mathbb{R}$	$M_{64}\mathbb{C}$
$M_2\mathbb{H}\times M_2\mathbb{H}$	$M_4\mathbb{H}$	$M_8\mathbb{C}$	$M_{16}\mathbb{R}$	$M_{16}\mathbb{R}\times M_{16}\mathbb{R}$	$M_{32}\mathbb{R}$	$M_{32}\mathbb{C}$	$M_{32}H$
$M_2\mathbb{H}$	$M_4\mathbb{C}$	$M_8\mathbb{R}$	$M_8\mathbb{R}\times M_8\mathbb{R}$	$M_{16}\mathbb{R}$	$M_{16}\mathbb{C}$	$M_{16}\mathbb{H}$	$M_{16}\mathbb{H}\times M_{16}\mathbb{H}$
$M_2\mathbb{C}$	$M_4\mathbb{R}$	$M_4\mathbb{R}\times M_4\mathbb{R}$	$M_8\mathbb{R}$	$M_8\mathbb{C}$	$M_8\mathbb{H}$	$M_8\mathbb{H}\times M_8\mathbb{H}$	$M_{16}\mathbb{H}$
$M_2\mathbb{R}$	$M_2\mathbb{R}\times M_2\mathbb{R}$	$M_4\mathbb{R}$	$M_4\mathbb{C}$	$M_4\mathbb{H}$	$M_4\mathbb{H}\times M_4\mathbb{H}$	$M_8\mathbb{H}$	$M_{16}\mathbb{C}$
$\mathbb{R}\times\mathbb{R}$	$M_2\mathbb{R}$	$M_2\mathbb{C}$	$M_2\mathbb{H}$	$M_2\mathbb{H}\times M_2\mathbb{H}$	$M_4\mathbb{H}$	$M_8\mathbb{C}$	$M_{16}\mathbb{R}$
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\times\mathbb{H}$	$M_2\mathbb{H}$	$M_4\mathbb{C}$	$M_8\mathbb{R}$	$M_8\mathbb{R}\times M_8\mathbb{R}$

0 / $\mathbb{Z}/2$	0 / 0	$\mathbb{Z}$ / $\mathbb{Z}$	0 / 0	$\mathbb{Z}/2$ / 0	$\mathbb{Z}/2$ / 0	$\mathbb{Z}$ / $\mathbb{Z}$	0 / $\mathbb{Z}/2$
0 / 0	$\mathbb{Z}$ / $\mathbb{Z}$	0 / 0	$\mathbb{Z}/2$ / 0	$\mathbb{Z}/2$ / 0	$\mathbb{Z}$ / $\mathbb{Z}$	0 / $\mathbb{Z}/2$	0 / $\mathbb{Z}/2$
$\mathbb{Z}$ / $\mathbb{Z}$	0 / 0	$\mathbb{Z}/2$ / 0	$\mathbb{Z}/2$ / 0	$\mathbb{Z}$ / $\mathbb{Z}$	0 / $\mathbb{Z}/2$	0 / $\mathbb{Z}/2$	0 / 0
0 / 0	$\mathbb{Z}/2$ / 0	$\mathbb{Z}/$ / 0	$\mathbb{Z}$ / $\mathbb{Z}$	0 / $\mathbb{Z}/2$	0 / $\mathbb{Z}/2$	0 / 0	$\mathbb{Z}$ / $\mathbb{Z}$
$\mathbb{Z}/2$ / 0	$\mathbb{Z}/2$ / 0	$\mathbb{Z}$ / $\mathbb{Z}$	0 / $\mathbb{Z}/2$	0 / $\mathbb{Z}2$	0 / 0	$\mathbb{Z}$ / $\mathbb{Z}$	0 / 0
$\mathbb{Z}/2\mathbb{Z}$ / 0	$\mathbb{Z}$ / $\mathbb{Z}$	0 / $\mathbb{Z}/2$	0 / $\mathbb{Z}/2$	0 / 0	$\mathbb{Z}$ / $\mathbb{Z}$	0 / 0	$\mathbb{Z}/2$ / 0
$\mathbb{Z}$ / $\mathbb{Z}$	0 / $\mathbb{Z}/2$	0 / $\mathbb{Z}/2$	0 / 0	$\mathbb{Z}$ / $\mathbb{Z}$	0 / 0	$\mathbb{Z}/2$ / 0	$\mathbb{Z}/2$ / 0
0 / $\mathbb{Z}/2$	0 / $\mathbb{Z}/2$	0 / 0	$\mathbb{Z}$ / $\mathbb{Z}$	0 / 0	$\mathbb{Z}/2$ / 0	$\mathbb{Z}/2$ / 0	$\mathbb{Z}$ / $\mathbb{Z}$

Graded Clifford Modules (Irreducible)

Claim

$$\begin{tikzcd} \{\text{ungraded } Cliff^0_{p,q} \text{ modules}\} \arrow[d, shift right, "Cliff_{p,q} \otimes_{Cliff^0_{p,q}} M"'] \arrow[d, leftrightarrow] \\ \{\text{graded } Cliff_{p,q} \text{ modules}\} \arrow[u, shift right, "M \mapsto M^0"'] \end{tikzcd}$$

Recall - a graded $Cliff_{p,q}$ module is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{R}$-vector space $M = M^0 \oplus M^1$ s.t.

$Cliff^0_{p,q}$ preserves the grading and

$Cliff^1_{p,q}$ reverses the grading

This implies that M^0 restricts to a $Cliff^0_{p,q}$ module.

Given a $Cliff^0_{p,q}$ module M^0, can we recover M^1? Apparently, $Cliff_{p,q} \otimes_{Cliff^0_{p,q}} M^0 \cong M$

$Cliff_{p,q}$ acts on $Cliff_{p,q} \otimes_{Cliff^0_{p,q}} M^0$

by $a \cdot (c \otimes m) = ac \otimes m$

Left multiplication

What is the grading on $\mathrm{Cliff}_{p,q} \otimes_{\mathrm{Cliff}^0_{p,q}} M^0$?

$$\left(\mathrm{Cliff}^0_{p,q} \oplus \mathrm{Cliff}^1_{p,q}\right) \otimes M^0$$

$$\cong$$

$$\underbrace{\left(\mathrm{Cliff}^0_{p,q} \otimes M^0\right)}_{\text{Even}} \oplus \underbrace{\left(\mathrm{Cliff}^1_{p,q} \otimes M^0\right)}_{\text{odd}}$$

$\mathrm{Cliff}_{p,q}$ acts on the left factor by left multiplication.

In addition, the even part is isomorphic to M^0, so

$$\{\mathrm{Cliff}^0_{p,q} \text{ modules}\} \to \{\mathrm{Cliff}_{p,q} \text{ modules}\} \to \{\mathrm{Cliff}^0 \text{ mod}\}$$

is identity

then for $M = M^0 \oplus M^1$ a $\mathrm{Cliff}_{p,q}$ module,

is $\mathrm{Cliff}_{p,q} \otimes_{\mathrm{Cliff}^0_{p,q}} M^0 \cong M$?

Define a map

$$\varphi : \mathrm{Cliff}_{p,q} \otimes_{\mathrm{Cliff}^0_{p,q}} M^0 \longrightarrow M$$

by $c \otimes m \longmapsto cm$

map is nonzero, and a module homomorphism, and is surjective since M is irreducible.

WTS its an isomorphism by dimension reasons.

M^0 is half the dimension of M,

Does $\mathrm{Cliff}_{p,q} \otimes_{\mathrm{Cliff}^0_{p,q}} M^0$ double the dimension?

Maybe just check for a basis?

Assume this for now

of Graded modules

$M_8\mathbb{C}$	$M_8\mathbb{H}$	$M_8\mathbb{H} \times M_8\mathbb{H}$	$M_{16}\mathbb{H}$	$M_{32}\mathbb{C}$	$M_{64}\mathbb{R}$	$M_{64}\mathbb{R} \times M_{64}\mathbb{R}$	$M_{128}\mathbb{R}$
$M_4\mathbb{H}$	$M_4\mathbb{H} \times M_4\mathbb{H}$	$M_8\mathbb{H}$	$M_{16}\mathbb{C}$	$M_{32}\mathbb{R}$	$M_{32}\mathbb{R} \times M_{32}\mathbb{R}$	$M_{64}\mathbb{R}$	$M_{64}\mathbb{C}$
$M_2\mathbb{H} \times M_2\mathbb{H}$	$M_4\mathbb{H}$	$M_8\mathbb{C}$	$M_{16}\mathbb{R}$	$M_{16}\mathbb{R} \times M_{16}\mathbb{R}$	$M_{32}\mathbb{R}$	$M_{32}\mathbb{C}$	$M_{32}H$
$M_2\mathbb{H}$	$M_4\mathbb{C}$	$M_8\mathbb{R}$	$M_8\mathbb{R} \times M_8\mathbb{R}$	$M_{16}\mathbb{R}$	$M_{16}\mathbb{C}$	$M_{16}\mathbb{H}$	$M_{16}\mathbb{H} \times M_{16}\mathbb{H}$
$M_2\mathbb{C}$	$M_4\mathbb{R}$	$M_4\mathbb{R} \times M_4\mathbb{R}$	$M_8\mathbb{R}$	$M_8\mathbb{C}$	$M_8\mathbb{H}$	$M_8\mathbb{H} \times M_8\mathbb{H}$	$M_{16}\mathbb{H}$
$M_2\mathbb{R}$	$M_2\mathbb{R} \times M_2\mathbb{R}$	$M_4\mathbb{R}$	$M_4\mathbb{C}$	$M_4\mathbb{H}$	$M_4\mathbb{H} \times M_4\mathbb{H}$	$M_8\mathbb{H}$	$M_{16}\mathbb{C}$
$\mathbb{R} \times \mathbb{R}$	$M_2\mathbb{R}$	$M_2\mathbb{C}$	$M_2\mathbb{H}$	$M_2\mathbb{H} \times M_2\mathbb{H}$	$M_4\mathbb{H}$	$M_8\mathbb{C}$	$M_{16}\mathbb{R}$
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \times \mathbb{H}$	$M_2\mathbb{H}$	$M_4\mathbb{C}$	$M_8\mathbb{R}$	$M_8\mathbb{R} \times M_8\mathbb{R}$

2	1	1	1	2	1	1	1	2
1	1	1	2	1	1	1	2	1
1	1	2	1	1	1	2	1	1
1	2	1	1	1	2	1	1	1
2	1	1	1	2	1	1	1	2
1	1	1	2	1	1	1	2	1
1	1	2	1	1	1	2	1	1
1	2	1	1	1	2	1	1	1
?	1	1	1	2	1	1	1	2

TODO Figure out what these modules are

Cokernels w/ graded Modules.

$M_8\mathbb{C}$	$M_8\mathbb{H}$	$M_8\mathbb{H} \times M_8\mathbb{H}$	$M_{16}\mathbb{H}$	$M_{32}\mathbb{C}$	$M_{64}\mathbb{R}$	$M_{64}\mathbb{R} \times M_{64}\mathbb{R}$	$M_{128}\mathbb{R}$
$M_4\mathbb{H}$	$M_4\mathbb{H} \times M_4\mathbb{H}$	$M_8\mathbb{H}$	$M_{16}\mathbb{C}$	$M_{32}\mathbb{R}$	$M_{32}\mathbb{R} \times M_{32}\mathbb{R}$	$M_{64}\mathbb{R}$	$M_{64}\mathbb{C}$
$M_2\mathbb{H} \times M_2\mathbb{H}$	$M_4\mathbb{H}$	$M_8\mathbb{C}$	$M_{16}\mathbb{R}$	$M_{16}\mathbb{R} \times M_{16}\mathbb{R}$	$M_{32}\mathbb{R}$	$M_{32}\mathbb{C}$	$M_{32}H$
$M_2\mathbb{H}$	$M_4\mathbb{C}$	$M_8\mathbb{R}$	$M_8\mathbb{R} \times M_8\mathbb{R}$	$M_{16}\mathbb{R}$	$M_{16}\mathbb{C}$	$M_{16}\mathbb{H}$	$M_{16}\mathbb{H} \times M_{16}\mathbb{H}$
$M_2\mathbb{C}$	$M_4\mathbb{R}$	$M_4\mathbb{R} \times M_4\mathbb{R}$	$M_8\mathbb{R}$	$M_8\mathbb{C}$	$M_8\mathbb{H}$	$M_8\mathbb{H} \times M_8\mathbb{H}$	$M_{16}\mathbb{H}$
$M_2\mathbb{R}$	$M_2\mathbb{R} \times M_2\mathbb{R}$	$M_4\mathbb{R}$	$M_4\mathbb{C}$	$M_4\mathbb{H}$	$M_4\mathbb{H} \times M_4\mathbb{H}$	$M_8\mathbb{H}$	$M_{16}\mathbb{C}$
$\mathbb{R} \times \mathbb{R}$	$M_2\mathbb{R}$	$M_2\mathbb{C}$	$M_2\mathbb{H}$	$M_2\mathbb{H} \times M_2\mathbb{H}$	$M_4\mathbb{H}$	$M_8\mathbb{C}$	$M_{16}\mathbb{R}$
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \times \mathbb{H}$	$M_2\mathbb{H}$	$M_4\mathbb{C}$	$M_8\mathbb{R}$	$M_8\mathbb{R} \times M_8\mathbb{R}$

0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$