Principal Bundle Stuff

- Sort out reduction of structure group.

Given
$$\begin{array}{c} P \circlearrowleft G \\ \downarrow \\ M \end{array}$$
a principal G-bundle,

A homomorphism $\rho: H \to G$, a reduction of structure group is the data of

1) A principal H bundle $Q \to M$
2) A map $\varphi: Q \to P$ s.t.
$$\varphi(q \cdot h) = \varphi(q) \cdot \rho(h)$$

- φ should induce a map $Q \times_H G \xrightarrow{\tilde{\varphi}} P$

Recall $Q \times_H G = Q \times G \big/ (q,g) \sim (q \cdot h, \rho(h)^{-1} g)$

Associated Bundle

Define $\tilde{\varphi} : Q \times_H G \to P$ by

$$\tilde{\varphi}(q, g) = \varphi(q) \cdot g$$

$$\tilde{\varphi}(q \cdot h, \rho(h)^{-1} g) = \varphi(q)\rho(h)\rho^{-1}(g) = \varphi(q)$$

So $\tilde{\varphi}$ is well-defined

Construct a category of reductions of structure group? (How to distinguish them up to isomorphism?

Pin ⊂ Spin

Define $\mathrm{Pin}^{\pm}_n = \mathrm{Pin}_{(0,n)}$ or $\mathrm{Pin}_{(n,0)}$ depending on $e_i^2 = \pm 1$

<u>Q1</u> Embed $\mathrm{Pin}^{\pm}_n$ into a spin group. Consider doing this by mapping $\mathrm{Pin}^{\pm}_n \hookrightarrow \mathrm{Cliff}^0_{p,q}$ and proving that it lands in $\mathrm{Spin}_{p,q}$

<u>Q2</u> Construct an isomorphism $\mathrm{Spin}_{0,n} \to \mathrm{Spin}_{n,0}$

<u>Q3</u> Investigate <u>the</u> (?) Projective representation of SO_n

spin

Q1 Embedding $\mathrm{Pin}_n^{\pm}$

$\mathrm{Pin}_n^{\pm}$ generated by unit vectors v

v acts on $\mathbb{R}^n$ by hyperplane reflection

~~We know $O_n \hookrightarrow SO_{n+1}$~~

Map $\mathrm{Pin}_n^{\pm} \longrightarrow \mathrm{Cliff}_{p,q}^{0}$

Case of Pin_n^{+}, try mapping into $\mathrm{Cliff}_{n,n}^{0}$

$$e_i \mapsto e_i^+ e_i^- \ ?$$

Lands in $\mathrm{Spin}_{n,n}$. injective

Case of Pin_n^{-}, map into $\mathrm{Cliff}_{(0,2n)}^{0}$

$$e_i \mapsto e_i e_{n+i}$$

A bit overkill, use identification of even subalgebras.

$$\mathrm{Pin}_n^- \subset \mathrm{Cliff}_{(0,n)} \hookrightarrow \mathrm{Cliff}^0_{(0,n+1)}$$

$$e_i^- \longmapsto e_i^- e_{n+1}^-$$

$$\mathrm{Pin}_n^- \subset \mathrm{Cliff}_{(n,0)} \hookrightarrow \mathrm{Cliff}^0_{(1,n)}$$

$$e_i^+ \longrightarrow e_1^+ e_i^-$$

Q: How does Pin act on the spinor rep? ?

Projective representation of SO_n

Representation will factor through PSO_n.

$$\begin{tikzcd} SO_n \arrow[d] & \curvearrowright & \mathbb{R}^3 \arrow[d] \\ PSO_n & \curvearrowright & \mathbb{RP}^{n-1} \end{tikzcd}$$

Low dim examples

$n=3$

Projective rep of SO_3?

SO_3 is centerless, $SO_3 \cong PSO_3$

If we look at $Spin_3 \subset Cl_{0,3} \cong \mathbb{H}$, we have the spinor representation

$Sp_1 \curvearrowright \mathbb{H}$ (unit quaternions multiplication)

± 1 act differently on $\mathbb{H}$

$Sp_1 / \{\pm 1\} \cong SO_3$

In the double cover,

$\pm v \longmapsto$ Reflection about $v^\perp$

Projective representation on $\mathbb{R}P^3 /$ Left multiplication

$p: Sp_1 \longrightarrow SO_3$

$p^{-1}(A)$ has at most 2 points, which differ by -1.

The center of SO_n is either trivial, or ± 1 depending on dimension (TODO prove this).

$$0 \to \{\pm 1\} \to Spin_n \to SO_n \to 0$$

The volume element $\omega = e_1 \cdots e_n$ is central if n is even, since commuting an element will require an even number of transpositions.

In this case, the center of Spin is $\{\pm 1, \pm\omega\}$

Under the double cover, $\pm 1 \mapsto id$

$\pm\omega \mapsto -id$

The spinor reps gives rise to projective reps, since the preimages only differ by a sign.

? Relation between the volume element and the spinor representations?

All elements of center act by scalars on an irrep (maybe only over $\mathbb{C}$?)

what does ω square to? In positive definite,

$$(e_1 e_2)^2 = e_1 e_2 e_1 e_2 = -1$$

Construct an isomorphism $Spin_{0,n} \to Spin_{n,0}$

$Cliff_{p,q}^{op} \cong Cliff_{q,p}$ where the opposite algebra is given by

$$a * b = (-1)^{|a||b|} ba$$

The isomorphism $Cliff_{p,q}^{op} \to Cliff_{q,p}$ is

$$e_i^{\pm} \longmapsto e_i^{\mp}$$

In particular, we have that

$$Cliff_{(n,0)}^{op} \cong Cliff_{(0,n)}$$

$$e_i^{+} \longmapsto e_i^{-}$$

In the graded algebra case, the Even subalgebra $(Cliff_{p,q}^{op})^{0}$ is the opposite algebra $(Cliff_{p,q}^{0})^{op}$, so $spin_{-n} \cong spin_{n}^{op}$.

Any group is isomorphic to its opposite by $g \mapsto g^{-1}$, giving us the desired isomorphism.