Clifford Algebras and Spin Geometry

Senior Honors Thesis: supervised by Professor Daniel Freed

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Acknowledgments

The mathematician does not study pure mathematics because it is useful; he studies it because he delights in it and he delights in it because it is beautiful.
– Henri Poincaré

I would first like to thank my family for their unceasing love and support. You all have provided a warm and nurturing environment for me to develop into who I am today. Without you, none of this would have been possible.

I would also like to thank my DRP mentor, Arun Debray. It has been an absolute pleasure reading with you, and I have learned an enormous amount of mathematics over the course of all our conversations. Your help in writing this thesis has been invaluable, and your puns and jokes will be missed.

To all my friends – you guys are the reason that my undergraduate experience has been so wonderful. I've loved staying up late doing homework with you and all the conversations we've had about mathematics. I hope our mathematical careers will continue to intersect in the upcoming years, and I wish you the best in your graduate studies, and in any other pursuits.

My interest in pure mathematics was almost non-existent before I started my undergraduate career here at the University of Texas at Austin. I'd like to thank the mathematics department here for providing an environment for me appreciate mathematics and grow into an aspiring mathematician. In particular, I'd like to thank Professor Andy Neitzke for his invaluable advice and mentorship, Professors Sean Keel and Cameron Gordon for teaching some of my favorite classes, and Professor Daniel Allcock for the many wonderful conversations I've had from popping unannounced into his office.

Last but not least, I would like to thank Professor Dan Freed for supervising this senior thesis, in addition to all of his guidance during my time here at UT. You have been and inexhaustible source of advice and insight, and it is safe to say that I would not be anywhere near where I am now without your mentorship. You have been an enormous influence on the kind of mathematics I want to pursue, and have played an integral part in shaping how I view geometry and topology. You are one of my mathematical role models, and I would be extremely lucky to even be an infinitesimal affine approximation of you.

Introduction

Every morning in the early part of October 1843, on my coming down to breakfast, your brother William Edwin and yourself used to ask me: "Well, Papa, can you multiply triples?" Whereto I was always obliged to reply, with a sad shake of the head, "No, I can only add and subtract them."
– William Rowan Hamilton

On October 16, 1843, William Rowan Hamilton famously carved the equations

$$i^2 = j^2 = k^2 = ijk = -1$$

into Broom Bridge, marking his discovery of the quaternions with an act of mathematical vandalism. It was well known at the time that the algebraic operations on the complex numbers $\mathbb{C}$ encoded a great deal of the geometry of the plane $\mathbb{R}^2$ – addition corresponded to vector translation, and the multiplication of complex numbers corresponded to scalings and rotations. After fruitlessly attempting to construct an analogous algebraic structure on $\mathbb{R}^3$, Hamilton came to the realization that he needed an additional dimension to encode the rotations of $\mathbb{R}^3$, thereby constructing the quaternions $\mathbb{H}$.

Later in the 19^{th} century, William Clifford further generalized the algebras $\mathbb{C}$ and $\mathbb{H}$ to higher dimensions when he constructed his geometric algebras, which now bear his name. Clifford's general construction created an $\mathbb{R}$-algebra that encoded the geometry of a vector space V equipped with a nondegenerate quadratic form $Q : V \to \mathbb{R}$. Clifford algebras would prove to play a central part in the development of geometry, topology, and physics in the 20^{th} century.

Mathematicians quickly found uses for Clifford algebras. In the early 20^{th} century, mathematicians were puzzled by an anomaly regarding linear representations of the special orthogonal groups SO_n. When studying the representations of the Lie algebras $\mathfrak{so}_n$, they realized that some of the Lie algebra representations failed to exponentiate to linear representations of the group SO_n, and instead gave rise to projective representations. The root of the problem was that the special orthogonal groups are not simply connected – for $n > 2$, the fundamental group $\pi_1(SO_n)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, indicating the existence of a simply connected double cover. [1] These double covers are the sources of the missing representations. At this point, Clifford algebras came to the rescue, and these double covers were realized as subgroups of the multiplicative groups of Clifford algebras. These groups were dubbed the Spin groups, and the mysterious missing representations were named the Spin representations.

In 1928, Paul Dirac was developing a relativistic theory of the electron when he realized he needed a first order differential operator that squared to the Laplacian. No first order operator

[1] This can be illustrated in 3 dimensions with the well-known belt trick, in which a 360 degree rotation of a belt buckle leaves the belt twisted

with scalar coefficients was able to satisfy this, but Dirac realized the the problem could be solved with the use of matrices, and constructed his gamma matrices. Remarkably, the relations

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} I$$

satisfied by Dirac's gamma matrices are exactly the defining relations for the Clifford algebra for Minkowski spacetime $\mathbb{R}^{1,3}$. Using these matrices, Dirac derived his famous equation

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

where ψ was an element of the Spin representation of the group $\text{Spin}_{1,3}$. Spin groups and Clifford algebras would prove to be vital elements in the development of quantum field theory.

Not to be outdone by the physicists, mathematicians also made extensive use of Clifford algebras and Spin groups to produce astounding results in geometry and topology. Through the work of many prominent mathematicians such as Michael Atiyah, Isadore Singer, and Raoul Bott, Clifford algebras and Spin groups saw remarkable applications in fields such as K-theory, homotopy theory, and gauge theory, producing numerous seminal theorems like the Atiyah-Singer index theorem and the Bott periodicity theorem. In particular, Atiyah and Singer generalized Dirac's differential operator to manifolds equipped with a Spin structure – the Dirac operator. The methods and tools developed would prove to be fundamental to the rapid advancement in the fields of geometry and topology at the time, and remain important to this day.

In this thesis we aim to explore and develop some of the theory of Clifford algebras, Spin and Pin groups, and Spin/Pin structures on manifolds. In particular, we plan to construct and classify Clifford algebras and their modules, and transport these constructions to the nonlinear world of manifolds. We will develop some of the general theory for Spin and Pin structures on manifolds, and will give several examples. Building upon this, we investigate Dirac operators on Spin manifolds, and will see several important results relating Dirac operators to Laplacians and curvature. Furthermore, we will explore the rich field of index theory, as well as its relationship with Dirac operators

CHAPTER 1

Clifford Algebras and Spin Groups

No one fully understands spinors. Their algebra is formally understood, but their geometrical significance is mysterious. In some sense they describe the "square root" of geometry and, just as understanding the concept of $\sqrt{-1}$ took centuries, the same might be true of spinors.
– Sir Michael Atiyah

1. Clifford Algebras

Definition 1.1. Let V be a real finite dimensional vector space with a nondegenerate symmetric bilinear form $b : V \times V \to \mathbb{R}$. Then the ***Clifford Algebra*** of V is the data of a unital associative $\mathbb{R}$-algebra $\mathrm{Cliff}(V, b)$ and a linear map $i : V \to \mathrm{Cliff}(V, b)$ satisfying the following universal property: Given any linear map $\varphi : V \to A$ of V into any unital associative $\mathbb{R}$-algebra A satisfying $\varphi(v)^2 = b(v, v)$, there exists a unique algebra homomorphism $\tilde{\varphi} : \mathrm{Cliff}(V, b) \to A$ such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[d, "i"'] \arrow[dr, "\varphi"] & \\ \mathrm{Cliff}(V,b) \arrow[r, "\tilde{\varphi}"'] & A \end{tikzcd}$$

❖

This universal property uniquely characterizes the Clifford algebra $\mathrm{Cliff}(V, b)$ up to unique isomorphism.

Theorem 1.2. *The Clifford algebra is unique up to unique isomorphism, i.e. given another unital associative algebra C equipped with a linear map $j : V \to C$ satisfying the universal property, there exists a unique algebra isomorphism $\varphi : \mathrm{Cliff}(V, b) \to C$.*

Proof. Given such an algebra C with a linear map $j : V \to C$, the map j satisfies the Clifford relation $j(v)^2 = b(v, v)$, so it induces a unique algebra homomorphism $\varphi : \mathrm{Cliff}(V, b) \to C$. The linear map $i : V \to \mathrm{Cliff}(V, b)$ also satisfies the Clifford relation, so it induces an algebra homomorphism $\psi : C \to \mathrm{Cliff}(V, b)$. We claim that these maps are inverses. To do so, we show the compositions $\varphi \circ \psi$ and $\psi \circ \varphi$ are identity. Using the universal property of the Clifford algebra once more, the map i induces a unique map $\mathrm{Cliff}(V, b) \to \mathrm{Cliff}(V, b)$ such that

$$\begin{tikzcd} V \arrow[d, "i"'] \arrow[dr, "i"] & \\ \mathrm{Cliff}(V,b) \arrow[r] & \mathrm{Cliff}(V,b) \end{tikzcd}$$

commutes. The identity map makes this diagram commute, so by uniqueness, this is the induced map. The map $\psi \circ \varphi$ also makes this diagram commute, so it must be identity by uniqueness. An identical proof shows that $\varphi \circ \psi$ is the identity map id_C. ■

Explicitly, $\mathrm{Cliff}(V,b)$ is realized as a quotient of the tensor algebra

$$\mathcal{T}(V) = \bigoplus_{n \in \mathbb{Z}^{\geq 0}} V^{\otimes n}$$

where we quotient by the two-sided ideal generated by elements of the form $v \otimes v - b(v,v)$, and the linear map $i : V \to \mathcal{T}(V)/(v \otimes v - b(v,v))$ is given by the inclusion $V \hookrightarrow \mathcal{T}(V)$ followed by the quotient map. We identify V with its image $i(V)$ as a subspace of $\mathrm{Cliff}(V,b)$.

THEOREM 1.3. *The Clifford algebra is functorial in the following way: Given vector spaces V and V' equipped with nondegenerate symmetric bilinear forms b and b' respectively and a linear map $T : V \to V'$ such that $b(v,w) = b'(Tv,Tw)$ for all $v,w \in V$, there exists a unique algebra homomorphism $T_* : \mathrm{Cliff}(V,b) \to \mathrm{Cliff}(V',b')$ such that*

$$\begin{CD} V @>T>> V' \\ @V{i}VV @VV{i'}V \\ \mathrm{Cliff}(V,b) @>>{T_*}> \mathrm{Cliff}(V',b') \end{CD}$$

commutes, where i and i' are the inclusions of V and V' into their respective Clifford algebras.

PROOF. We use the realization of $\mathrm{Cliff}(V,b)$ as the quotient

$$\mathrm{Cliff}(V,b) = \mathcal{T}(V)/(v \otimes v - b(v,v))$$

The map T induces a map $\tilde{T} : \mathcal{T}(V) \to \mathcal{T}(V')$ where the action on homogeneous elements is given by

$$\tilde{T}(v_1 \otimes \cdots \otimes v_k) = Tv_1 \otimes \cdots \otimes Tv_k$$

The fact that $b(v,w) = b'(Tv,Tw)$ implies that $\tilde{T}$ maps the ideal $(v \otimes v - b(v,v))$ to the ideal $(v' \otimes v' - b'(v',b'))$, so the map $\tilde{T}$ descends to the quotients, giving a map $T_* : \mathrm{Cliff}(V,b) \to \mathrm{Cliff}(V',b')$. The fact that the diagram commutes follows from the fact that the maps i and i' are the inclusions of V and V' into their respective tensor algebras, followed by the quotient maps into $\mathrm{Cliff}(V,b)$ and $\mathrm{Cliff}(V',b')$ respectively. ■

DEFINITION 1.4. Define the bilinear form $b : \mathbb{R}^{p+q} \times \mathbb{R}^{p+q} \to \mathbb{R}$ by

$$b(v,w) = \sum_{i=1}^{p} v^i w^i - \sum_{i=p+1}^{p+q} v^i w^i$$

where the v^i and w^i are the components of v and w with respect to the standard basis on $\mathbb{R}^{p+q}$. We denote the vector space $\mathbb{R}^{p+q}$ equipped with this bilinear form as $\mathbb{R}^{p,q}$. ❖

Let V be a vector space equipped with a nondegenerate bilinear form b and fix a basis for V. Then the bilinear form b is given by a symmetric invertible matrix B, which is conjugate to a diagonal matrix where all the diagonal entries are either 1 or -1. If after conjugation, B has p positive entries and q negative entries, we say b has signature (p,q). Any bilinear form b of signature (p,q) admits a basis $\{e_i\}$ satisfying

(1) For $1 \leq i \leq p$, we have $b(e_i, e_i) = 1$
(2) For $p+1 \leq j \leq p+q$, we have $b(e_j, e_j) = -1$
(3) For $i \neq j$, we have $b(e_i, e_j) = 0$

Any such basis then determines an isomorphism $(V, b) \to \mathbb{R}^{p,q}$, and we call such a basis an ***orthogonal basis*** for (V, b). In addition, we get a basis for $\mathrm{Cliff}(V, b)$ given by

$$\{e_{i_1} e_{i_2} \cdots e_{i_k} \,:\, 0 \leq k \leq n,\, 1 \leq i_j \leq n\}$$

where we interpret the product of 0 basis vectors to be the unit element 1. This then implies the dimension of $\mathrm{Cliff}(V, b)$ as a vector space is $2^{\dim V}$. This basis also determines an isomorphism $\mathrm{Cliff}(V, b) \to \mathrm{Cliff}_{p,q}(\mathbb{R})$, where $\mathrm{Cliff}_{p,q}(\mathbb{R})$ is the Clifford algebra for $\mathbb{R}^{p,q}$. Given $v, w \in V$, we write v and w in these bases as $v^i e_i$ and $w^i e_i$ (using Einstein summation convention), and derive the useful relation

$$\begin{aligned} vw + wv &= v^i w^j e_i e_j + v^i w^j e_j e_i \\ &= v^i w^i (e_i^2 + e_i^2) \\ &= 2b(v, w) \end{aligned}$$

where we use the fact that the e_i are orthogonal to deduce that $e_i e_j = -e_j e_i$ if $i \neq j$.

Definition 1.5. [1] Let V be a vector space. A $\mathbb{Z}/2\mathbb{Z}$ ***grading*** on V is a direct sum decomposition $V = V^0 \oplus V^1$. Elements of V^0 are said to be ***even*** and elements of V^1 are said to be ***odd***. Elements of the even and odd subspaces are said to be ***homogeneous***. Given an homogeneous element $v \in V$, define its ***parity***, denoted $|v|$ by

$$|v| = \begin{cases} 0 & v \in V^0 \\ 1 & v \in V^1 \end{cases}$$

Equivalently, it is the data of a linear map $\varepsilon : V \to V$ such that ε acts by identity on a subspace V^0 of V and negative identity on a complementary subspace V^1, which gives the direct sum decomposition of V as the ± 1 eigenspaces of ε. The map ε is called the ***grading operator***.

❖

Definition 1.6. A $\mathbb{Z}/2\mathbb{Z}$ ***graded algebra*** A over $\mathbb{R}$ (often called a superalgebra) is an $\mathbb{R}$-algebra A equipped with a grading $A = A^0 \oplus A^1$ such that the multiplication respects the grading, i.e. given homogeneous elements $a, b \in A$, their product ab is an element of $A^{|a|+|b|}$ where the addition is done mod 2. ❖

Example 1.7.

(1) Any $\mathbb{R}$-algebra A can be made into a graded algebra where we let $A^0 = A$ and $A^1 = 0$.
(2) The exterior algebra $\Lambda^\bullet V$ of a vector space V is a $\mathbb{Z}/2\mathbb{Z}$ graded algebra (in fact, it has a $\mathbb{Z}$ grading as well), where the grading is $\Lambda^\bullet V = \Lambda^{\mathrm{even}} V \oplus \Lambda^{\mathrm{odd}} V$ where $\Lambda^{\mathrm{even}} V$ is the subspace spanned by even products of vectors and $\Lambda^{\mathrm{odd}} V$ is the subspace spanned by odd products of vectors.
(3) Let $V = V^0 \oplus V^1$ be a $\mathbb{Z}/2\mathbb{Z}$ graded vector space. Then the algebra of endomorphisms $\mathrm{End}\, V$ can be endowed with the structure of a graded algebra, where the even subspace consists of linear maps preserving the decomposition $V^0 \oplus V^1$, and the odd subspace consists of linear maps T reversing the decomposition, i.e. $T(V^i) = V^{i+1 \mod 2}$. In a ordered basis where the first elements are all even and the last elements are all odd, the even elements of $\mathrm{End}\, V$ are block diagonal, while the odd elements are block off-diagonal. ◂

For the most part, the algebras we work with will be $\mathbb{Z}/2\mathbb{Z}$ graded, so the term "graded" may be used in lieu of "$\mathbb{Z}/2\mathbb{Z}$ graded." In the case of ambiguity, we will specify the grading.

[1] It is common in the literature to refer to $\mathbb{Z}/2\mathbb{Z}$ graded vector spaces as super vector spaces. The "super" prefix often refers to a $\mathbb{Z}/2\mathbb{Z}$ grading on the relevant object.

The Clifford algebra $\mathrm{Cliff}(V,b)$ is naturally a $\mathbb{Z}/2\mathbb{Z}$ graded algebra. Fix a basis $\{e_i\}$ for V. We then define the grading

$$\mathrm{Cliff}(V,b) = \mathrm{Cliff}^0(V,b) \oplus \mathrm{Cliff}^1(V,b)$$

Where $\mathrm{Cliff}^0(V,b)$ is the $\mathbb{R}$-span of all even products of basis vectors, and $\mathrm{Cliff}^1(V,b)$ is the $\mathbb{R}$-span of all odd products of basis vectors. Since the product of even elements is again even, the subspace $\mathrm{Cliff}^0(V,b)$ forms a subalgebra, and is called the ***even subalgebra***.

As $\mathbb{Z}/2\mathbb{Z}$-graded *vector spaces*, $\mathrm{Cliff}(V,b)$ is *naturally* isomorphic to the exterior algebra $\Lambda^\bullet(V)$. To construct the isomorphism, we first construct an algebra homomorphism $\mathrm{Cliff}(V,b) \to \mathrm{End}(\Lambda^\bullet(V))$. where we map $v \in V$ to $\varepsilon(v) + 1/2\iota(v)$, where

$$\begin{aligned}
\varepsilon(v)(\omega) &= v \wedge \omega \\
\iota(v)(v_1 \wedge \cdots \wedge v_k) &= \sum_{i=1}^{k} (-1)^{k+1} \langle v, v_i \rangle v_1 \wedge \cdots \widehat{v_i} \cdots \wedge v_k
\end{aligned}$$

where $\widehat{v_i}$ indicates that the i^{th} term in the wedge product is missing. This map satisfies the Clifford relations, so it extends to a unique map $\varphi : \mathrm{Cliff}(V,b) \to \mathrm{End}(\Lambda^\bullet(V))$. Then the mapping $v \mapsto \varphi(v)(1)$ is the desired isomorphism $\mathrm{Cliff}(V,b) \to \Lambda^\bullet(V)$. In particular, if e_i is an orthogonal basis for V,

$$\varphi(e_{i_1} e_{i_2} \cdots e_{i_k}) = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}$$

Of course, $\mathrm{Cliff}(V,b)$ and $\Lambda^\bullet(V)$ are noncanonically isomorphic, since they are the same dimension and the even and odd subspaces are the same dimensions, but the key point here is that the isomorphism is canonical. Since them map is not an algebra homomorphism, a useful thing to note is how multiplication in the Clifford algebra looks like in the exterior algebra under the isomorphism φ.

Proposition 1.8. *For $v \in V$ and $\eta \in \mathrm{Cliff}(V,b)$*

$$\varphi(v\eta) = v \wedge \eta + \iota(v)(\eta)$$

Proof. Fix an orthogonal basis $\{e_i^\pm\}$ where $(e_i^+)^2 = 1$, $(e_i^-)^2 = -1$. It suffices to check that the identity holds when v is a basis vector and η is a product of basis vectors. We compute

$$e_i^\pm e_{i_1}^\pm \cdots e_{i_k}^\pm = \begin{cases} \pm e_{i_2} \cdots e_{i_k} & e_{i_1}^\pm = e_i^\pm \\ e_1 e_{i_1} \cdots e_{i_k} & e_i^\pm \neq e_{i_1}^\pm \end{cases}$$

In addition, we have that

$$e_i^\pm \wedge (e_{i_1}^\pm \wedge \cdots \wedge e_{i_k}^\pm) + \iota(e_i^\pm)(e_{i_1}^\pm \wedge \cdots \wedge e_{i_k}^\pm) = \begin{cases} 0 \pm e_{i_2}^\pm \wedge \cdots \wedge e_{i_k}^\pm & e_1^\pm = e_{i_1}^\pm \\ e_i^\pm \wedge e_{i_1}^\pm \wedge \cdots \wedge e_{i_k}^\pm, & e_i^\pm \neq e_{i_1}^\pm \end{cases}$$

Then using the fact that φ sends products of orthogonal vectors to wedges of orthogonal vectors, we see that the desired identity holds on a basis, completing the proof. ∎

There is an extremely nice relationship between a Clifford algebra and its even subalgebra.

Theorem 1.9. *The even subalgebra $\mathrm{Cliff}^0_{p,q}(\mathbb{R})$ is isomorphic to both $\mathrm{Cliff}_{q,p-1}$ and $\mathrm{Cliff}_{p,q-1}$ as ungraded algebras (as long as $p-1>0$ or $q-1>0$.)*

Proof. Fix a basis $\left\{e_1^+, \dots e_p^+, e_1^- \dots e_q^-\right\}$ for $\mathbb{R}^{p,q}$, where $(e_i^+)^2 = 1$ and $(e_i^-)^2 = -1$. We then compute

$$\begin{aligned}
(e_i^+ e_j^+)^2 &= -(e_i^+)^2(e_j^+)^2 = -1 \\
(e_i^- e_j^-)^2 &= -(e_i^-)^2(e_j^-)^2 = -1 \\
(e_i^+ e_j^-)^2 &= -(e_i^+)^2(e_i^-)^2 = 1 \\
(e_i^- e_j^+)^2 &= -(e_i^-)^2(e_j^+)^2 = 1
\end{aligned}$$

Assume $q \neq 0$. Then a generating set for $\mathrm{Cliff}^0_{p,q}(\mathbb{R})$ is

$$\left\{e_1^- e_j^+ \ : \ 1 \leq j \leq p\right\} \cup \left\{e_1^- e_k^- \ : \ 2 \leq k \leq q\right\}$$

All the elements in the first set square to 1, and all the elements in the second set square to -1. We then get an isomorphism $\mathrm{Cliff}^0_{p,q}(\mathbb{R}) \to \mathrm{Cliff}_{p,q-1}$ via the mappings

$$\begin{aligned}
e_1^- e_j^+ &\mapsto e_j^+ \\
e_1^- e_k^- &\mapsto e_{k-1}^-
\end{aligned}$$

In the case where $p \neq 0$, we have that an equally good generating set for $\mathrm{Cliff}^0_{p,q}(\mathbb{R})$ is

$$\left\{e_1^+ e_j^+ \ : \ 2 \leq j \leq p\right\} \cup \left\{e_1^+ e_i^- \ : \ 1 \leq i \leq q\right\}$$

Where the elements in the first set square to -1 and the elements of the second set square to 1. Then the mappings

$$\begin{aligned}
e_1^+ e_j^+ &\mapsto e_{j-1}^- \\
e_1^+ e_j^- &\mapsto e_j^+
\end{aligned}$$

gives the isomorphism $\mathrm{Cliff}^0_{p,q}(\mathbb{R}) \to \mathrm{Cliff}_{q,p-1}$. ■

Given two $\mathbb{R}$-algebras A and B, we can form their tensor product $A \otimes B$, which has $A \otimes B$ as the underlying vector space, and the multiplication is defined as

$$(a \otimes b)(c \otimes d) = ac \otimes bd$$

In the case that both A and B are $\mathbb{Z}/2\mathbb{Z}$ graded algebras, we have an alternate version of the tensor product, where the underlying vector space is also $A \otimes B$. The grading on the tensor product is the decomposition

$$A \otimes B = (A^0 \otimes B^0 \oplus A^1 \otimes A^1) \oplus (A^0 \otimes B^1 \oplus A^1 \otimes B^0)$$

and the multiplication of homogeneous elements is given by

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|}(ac \otimes bd)$$

We see that in the multiplication, we are formally commuting the elements of b and c, and we want to introduce a sign whenever elements are moved past each other. This is the ***Koszul sign rule***. Another concept that needs a slight modification in the graded case is the opposite algebra. In the ungraded case, given an $\mathbb{R}$-algebra A, the ***opposite algebra*** is the algebra A^{op} with the same underlying vector space, but the multiplication in A^{op} is given by $a * b = ba$, where ba is the multiplication in A. In doing so, we are formally commuting a and b, so in the graded situation, we invoke the Koszul sign rule when defining multiplication in the opposite algebra, and define the multiplication of homogeneous elements in A^{op} to be $a * b = (-1)^{|a||b|}ba$.

One remarkable fact is that Clifford algebras are closed under the graded tensor product, i.e. the graded tensor products of two Clifford algebras is another Clifford algebra. Likewise, the graded opposite algebra of a Clifford algebra is again a Clifford algebra. For the remainder of this section, we will let $\otimes$ denote the graded tensor product, and the superscript op will denote the graded opposite algebra.

Theorem 1.10. $\mathrm{Cliff}_{p+t,q+s}(\mathbb{R}) \cong \mathrm{Cliff}_{p,q}(\mathbb{R}) \otimes \mathrm{Cliff}_{t,s}(\mathbb{R})$

Proof. To give a map $\varphi : \mathrm{Cliff}_{p+t,q+s}(\mathbb{R}) \to \mathrm{Cliff}_{p,q}(\mathbb{R}) \otimes \mathrm{Cliff}_{t,s}(\mathbb{R})$, it is sufficient to specify its action on $\mathbb{R}^{p+t,q+s}$ and to check that the Clifford relations hold. Let $\left\{b_1^+, \dots, b_{p+t}^+, b_1^-, \dots b_{q+s}^-\right\}$ denote the standard orthogonal basis for $\mathbb{R}^{p+t,q+s}$ where $(b_i^+)^2 = 1$ and $(b_i^-)^2 = -1$. We then define the bases $\{e_i^\pm\}$ and $\{f_i^\pm\}$ analogously for $\mathbb{R}^{p,q}$ and $\mathbb{R}^{t,s}$ respectively. Then define φ by

$$\varphi(b_i^+) = \begin{cases} e_i^+ \otimes 1 & 1 \le i \le p \\ 1 \otimes f_i^+ & p+1 \le i \le p+t \end{cases}$$

$$\varphi(b_i^-) = \begin{cases} e_i^- \otimes 1 & 1 \le i \le q \\ 1 \otimes f_i^- & q+1 \le i \le q+s \end{cases}$$

This map is injective on generators, so if we show that this satisfies the Clifford relations, the map given by extending the map to all of $\mathrm{Cliff}_{p+t,q+s}(\mathbb{R})$ will be an isomorphism by a dimension count. Showing the Clifford relations amounts to showing

(1) $\varphi(b_i^+)^2 = 1$
(2) $\varphi(b_i^-)^2 = -1$
(3) The images of any pair of distinct basis vectors anticommute.

The first two are relations are clear from how we defined φ. To show that the images of distinct basis vectors anticommute, there are serveral cases to consider. Given b_i^+ and b_j^+ where $1 \le i, j \le p$, they anticommute, because e_i^+ and e_j^+ anticommute. In the case where $1 \le i \le p$ and $p+1 \le j \le p+t$, we compute

$$\begin{aligned} \varphi(b_i^+)\varphi(b_j^+) + \varphi(b_j^+)\varphi(b_i+) &= (e_i^+ \otimes 1)(1 \otimes f_j^+) + (1 \otimes f_j^+)(e_i^+ \otimes 1) \\ &= e_i^+ \otimes f_j^+ - e_i^+ \otimes f_j^+ \end{aligned}$$

where we use the Koszul sign rule for the second term, noting that f_j+ and e_i^+ are both odd elements. The proof that the images of the b_i^- anti commute with each other, as well as the proof that the images of the b_i^+ and b_i^- anticommute are exactly the same. ∎

Theorem 1.11. *The graded opposite algebra* $\mathrm{Cliff}_{p,q}^{op}(\mathbb{R})$ *is isomorphic to* $\mathrm{Cliff}_{q,p}(\mathbb{R})$.

Proof. Fix an orthogonal basis $\{e_i^\pm\}$ for $\mathbb{R}^{p,q}$, where $(e_i^\pm)^2 = \pm 1$. We note that since the $e_i^\pm$ are odd elements, they square to ∓ 1 in the opposite algebra. Indeed, the mapping $e_i^\pm \to e_i^\mp$ defines the isomorphism $\mathrm{Cliff}_{p,q}^{\mathrm{op}} \to \mathrm{Cliff}_{q,p}$. ∎

Because of these theorems, once we compute a few of the lower dimensional Clifford algebras, we will have enough data to fully classify all Clifford algebras over $\mathbb{R}$.

Example 1.12 (***Some low dimensional examples***).

(1) The Clifford algebra $\mathrm{Cliff}_{0,0}(\mathbb{R})$ is isomorphic to $\mathbb{R}$.
(2) As ungraded algebras, the Clifford algebra $\mathrm{Cliff}_{0,1}(\mathbb{R})$ is isomorphic to $\mathbb{C}$, where the isomorphism is given by $e_1 \mapsto i$.
(3) As ungraded algebras, $\mathrm{Cliff}_{0,2}(\mathbb{R})$ is isomorphic to the quaternions $\mathbb{H}$, where the isomorphism is given by $e_1 \mapsto i$ and $e_2 \mapsto j$.

(4) As graded algebras, $\mathrm{Cliff}_{1,1}(\mathbb{R})$ is isomorphic to $\mathrm{End}(\mathbb{R}^{1|1})$, where $\mathbb{R}^{1|1}$ denotes the $\mathbb{Z}/2\mathbb{Z}$ graded vector space $\mathbb{R} \oplus \mathbb{R}$. The isomorphism is given by

$$e_1^+ \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad e_1^- \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(5) As ungraded algebras $\mathrm{Cliff}_{1,0}(\mathbb{R})$ is isomorphic to the product algebra $\mathbb{R} \times \mathbb{R}$, where $e_1 \mapsto (1, -1)$.

(6) As ungraded algebras, $\mathrm{Cliff}_{2,0}(\mathbb{R})$ is isomorphic to the algebra $M_2\mathbb{R}$ of 2×2 matrices with coefficients in $\mathbb{R}$. The isomorphism is given by

$$e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad e_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

◀

To classify all Clifford algebras as ungraded algebras, it suffices to know the following table, which is derived by identifying a few low dimensional Clifford algebras and using the fact that the graded tensor product of Clifford algebras is another Clifford algebra.

7	$M_8\mathbb{C}$	$M_8\mathbb{H}$	$M_8\mathbb{H} \times M_8\mathbb{H}$	$M_{16}\mathbb{H}$	$M_{32}\mathbb{C}$	$M_{64}\mathbb{R}$	$M_{64}\mathbb{R} \times M_{64}\mathbb{R}$	$M_{128}\mathbb{R}$
6	$M_4\mathbb{H}$	$M_4\mathbb{H} \times M_4\mathbb{H}$	$M_8\mathbb{H}$	$M_{16}\mathbb{C}$	$M_{32}\mathbb{R}$	$M_{32}\mathbb{R} \times M_{32}\mathbb{R}$	$M_{64}\mathbb{R}$	$M_{64}\mathbb{C}$
5	$M_2\mathbb{H} \times M_2\mathbb{H}$	$M_4\mathbb{H}$	$M_8\mathbb{C}$	$M_{16}\mathbb{R}$	$M_{16}\mathbb{R} \times M_{16}\mathbb{R}$	$M_{32}\mathbb{R}$	$M_{32}\mathbb{C}$	$M_{32}\mathbb{H}$
4	$M_2\mathbb{H}$	$M_4\mathbb{C}$	$M_8\mathbb{R}$	$M_8\mathbb{R} \times M_8\mathbb{R}$	$M_{16}\mathbb{R}$	$M_{16}\mathbb{C}$	$M_{16}\mathbb{H}$	$M_{16}\mathbb{H} \times M_{16}\mathbb{H}$
3	$M_2\mathbb{C}$	$M_4\mathbb{R}$	$M_4\mathbb{R} \times M_4\mathbb{R}$	$M_8\mathbb{R}$	$M_8\mathbb{C}$	$M_8\mathbb{H}$	$M_8\mathbb{H} \times M_8\mathbb{H}$	$M_{16}\mathbb{H}$
2	$M_2\mathbb{R}$	$M_2\mathbb{R} \times M_2\mathbb{R}$	$M_4\mathbb{R}$	$M_4\mathbb{C}$	$M_4\mathbb{H}$	$M_4\mathbb{H} \times M_4\mathbb{H}$	$M_8\mathbb{H}$	$M_{16}\mathbb{C}$
1	$\mathbb{R} \times \mathbb{R}$	$M_2\mathbb{R}$	$M_2\mathbb{C}$	$M_2\mathbb{H}$	$M_2\mathbb{H} \times M_2\mathbb{H}$	$M_4\mathbb{H}$	$M_8\mathbb{C}$	$M_{16}\mathbb{R}$
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \times \mathbb{H}$	$M_2\mathbb{H}$	$M_4\mathbb{C}$	$M_8\mathbb{R}$	$M_8\mathbb{R} \times M_8\mathbb{R}$
p \ q	0	1	2	3	4	5	6	7

To read the table, the bottom left entry is $\mathrm{Cliff}_{0,0} \cong \mathbb{R}$, and moving to the right increments the signature from (p, q) to $(p, q+1)$, and moving up increments the signature (p, q) to $(p+1, q)$. Any other Clifford algebra can be obtained from an algebra on this table by tensoring with $M_{16}\mathbb{R}$, since incrementing the signature by 8 (by adding to either p or q) results in tensoring with $M_{16}\mathbb{R}$.

2. Clifford Modules

Definition 2.1. A (left) ***Clifford module*** for the Clifford algebra $\mathrm{Cliff}_{p,q}(\mathbb{R})$ is a module for $\mathrm{Cliff}_{p,q}(\mathbb{R})$ in the usual sense i.e. a real vector space V equipped with an algebra action $\bullet : \mathrm{Cliff}_{p,q}(\mathbb{R}) \times V \to V$ satisfying the following properties

(1) Every element of $\mathrm{Cliff}_{p,q}(\mathbb{R})$ acts linearly on V.
(2) $(AB) \cdot v = A \cdot (B \cdot v)$ for all $v \in V$.
(3) $(A + B) \cdot v = A \cdot v + B \cdot V$ for all $v \in V$.

Equivalently, it is the data of a real vector space V and an algebra homomorphism $\mathrm{Cliff}_{p,q}(\mathbb{R}) \to \mathrm{End}(V)$. ❖

Definition 2.2. A Clifford module is ***irreducible*** if there exist no proper nontrivial submodules. ❖

From the classification of Clifford algebras, all the Clifford algebras are either matrix algebras $M_n\mathbb{F}$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, or products $M_n\mathbb{F} \times M_n\mathbb{F}$ of two copies of the same matrix algebra. This is sufficient to conclude that Clifford algebras are semisimple, so all Clifford modules are direct sums of irreducible modules. Therefore, classifying all Clifford modules reduces to classifying the irreducible Clifford modules.

Theorem 2.3. *Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. Then any nontrivial irreducible module for $M_n\mathbb{F}$ is isomorphic to $\mathbb{F}^n$ with the standard action.*

Proof. We first note that $M_n\mathbb{F}$ acts transitively on $\mathbb{F}^n$, which implies that it is irreducible. We then must show that $\mathbb{F}^n$ is, up to isomorphism, the only irreducible $M_n\mathbb{F}$ module. The matrix algebra $M_n\mathbb{F}$ admits an increasing chain of left ideals

$$0 = I_0 \subset I_1 \subset \ldots \subset I_n = M_n\mathbb{F}$$

where I_k is the set of matrices where only the first k columns are nonzero. These ideals have the property that the quotient I_k/I_{k-1} is isomorphic to $\mathbb{F}^n$ as a left $M_n\mathbb{F}$ module. Then let M be some nontrivial irreducible $M_n\mathbb{F}$ module, and fix $m \in M$. Then the orbit $M_n\mathbb{F} \cdot m$ of m under the algebra action is a nonzero submodule, so it must be all of M. Then the map $\varphi : M_n\mathbb{F} \to M$ given by $A \mapsto A \cdot m$ is a surjective map of left $M_n\mathbb{F}$ modules. Then there must exist some smallest k such that $\varphi(I_k)$ is nonzero, and by construction, $\varphi|_{I_k}$ factors through the quotient I_k/I_{k-1}, which is isomorphic to $\mathbb{F}^n$ with the standard action. Then since $\mathbb{F}^n$ is irreducible, this gives us a nonzero map between irreducible modules, which is an isomorphism by Schur's Lemma. ■

Theorem 2.4. *Any nontrivial irreducible module for $M_n\mathbb{F} \times M_n\mathbb{F}$ is isomorphic to either $\mathbb{F}^n$ where the left factor acts in the usual way, and the right factor acts by 0, or $\mathbb{F}^n$ where the left factor acts by 0 and the right factor acts in the usual way.*

Proof. Let R denote $\mathbb{F}^n$ where the right factor acts nontrivially, and let L denote $\mathbb{F}^n$ where the left factor acts nontrivially. Both L and R are irreducible since $M_n\mathbb{F} \times M_n\mathbb{F}$ acts transitively on them. To show that they are the only irreducible modules up to isomorphism, we use a similar technique as above. Let I_k denote the chain of increasing ideals in $M_n\mathbb{F}$, as we used above. Then $M_n\mathbb{F} \times M_n\mathbb{F}$ admits a chain of increasing left ideals J_k

$$0 = J_0 \subset I_1 \times \{0\} \subset \ldots \subset M_n\mathbb{F} \times \{0\} \subset M_n\mathbb{F} \times I_1 \subset \ldots \subset M_n\mathbb{F} \times M_n\mathbb{F} = J_{2n}$$

We note that for $1 \leq k \leq n$, we have that J_k/J_{k-1} is isomorphic to L, and for $n+1 \leq k \leq 2n$, we have that J_k/J_{k-1} is isomorphic to R. Then given a nontrivial irreducible module M and a nonzero element m, we get a surjective map $\varphi : M_n\mathbb{F} \times M_n\mathbb{F}$ where $A \mapsto A \cdot m$. Like before, there exists some smallest k such that $\varphi(J_k)$ is nonzero, which then factors through to an isomorphism $J_k/J_{k-1} \to M$, so M is either isomorphic to R or L. ■

This then gives a full classification of the irreducible ungraded Clifford modules.

Since the Clifford algebras are $\mathbb{Z}/2\mathbb{Z}$ graded, we have an analogous notion of a graded Clifford module.

Definition 2.5. Let $A = A^0 \oplus A^1$ be a $\mathbb{Z}/2\mathbb{Z}$ graded algebra. Then a $\mathbb{Z}/2\mathbb{Z}$ ***graded module*** for A is a module $M = M^0 \oplus M^1$ for A such that $A^i \cdot M^j \subset M^{i+j \mod 2}$. ❖

A ***graded Clifford module*** is then a $\mathbb{Z}/2\mathbb{Z}$ graded module for a Clifford algebra $\mathrm{Cliff}_{p,q}(\mathbb{R})$. We already have all the ingredients we need to classify all of the graded Clifford modules – we have a complete classification of the ungraded Clifford modules, and we have the relationship between $\mathrm{Cliff}_{p,q}(\mathbb{R})$ and its even subalgebra $\mathrm{Cliff}^0_{p,q}(\mathbb{R})$.

Theorem 2.6. *There is a bijective correspondence*

$$\left\{\textit{Ungraded Clifford modules over } \mathrm{Cliff}^0_{p,q}(\mathbb{R})\right\} \longleftrightarrow \left\{\textit{Graded Clifford modules over } \mathrm{Cliff}_{p,q}(\mathbb{R})\right\}$$

Proof. We provide maps in both directions. Let $M = M^0 \oplus M^1$ be a graded module for $\mathrm{Cliff}_{p,q}(\mathbb{R})$. The action of $\mathrm{Cliff}^0_{p,q}(\mathbb{R})$ preserves the even subspace M^0, which gives M^0 the structure of an ungraded $\mathrm{Cliff}^0_{p,q}(\mathbb{R})$ module. Let r denote the mapping $M \mapsto M^0$.

In the other direction, let V be an ungraded module over $\mathrm{Cliff}^0_{p,q}(\mathbb{R})$. Then let t denote the mapping $V \mapsto V' = \mathrm{Cliff}_{p,q}(\mathbb{R}) \otimes_{\mathrm{Cliff}^0_{p,q}(\mathbb{R})} V$. Where we treat $\mathrm{Cliff}_{p,q}(\mathbb{R})$ as a left module under multiplication by elements of $\mathrm{Cliff}^0_{p,q}(\mathbb{R})$. We then endow V' the structure of a $\mathrm{Cliff}_{p,q}(\mathbb{R})$ module, where $\mathrm{Cliff}_{p,q}(\mathbb{R})$ acts on the $\mathrm{Cliff}_{p,q}(\mathbb{R})$ factor by left multiplication. We further impose the grading on V' where

$$V' = (\mathrm{Cliff}^0_{p,q}(\mathbb{R}) \otimes_{\mathrm{Cliff}^0_{p,q}(\mathbb{R})} V) \oplus (\mathrm{Cliff}^1_{p,q}(\mathbb{R}) \otimes_{\mathrm{Cliff}^1_{p,q}(\mathbb{R})} V)$$

We then claim that these two maps are inverses, which we will verify by showing that the compositions $r \circ t$ and $t \circ r$ are identity. We first show that $r \circ t$ is identity. Let V be an ungraded module over $\mathrm{Cliff}^0_{p,q}(\mathbb{R})$. Then the even subalgebra of $t(V) = \mathrm{Cliff}_{p,q} \otimes_{\mathrm{Cliff}^0_{p,q}(\mathbb{R})} V$ is $\mathrm{Cliff}^0_{p,q}(\mathbb{R}) \otimes_{\mathrm{Cliff}^0_{p,q}(\mathbb{R})} V$, which is naturally isomorphic to V itself. In the other direction, let $M = M^0 \otimes M^1$ be a module for $\mathrm{Cliff}_{p,q}(\mathbb{R})$. Showing that $t \circ r$ is the identity then amounts to showing that $\mathrm{Cliff}_{p,q}(\mathbb{R}) \otimes_{\mathrm{Cliff}^0_{p,q}(\mathbb{R})} M^0$ is isomorphic to M. The desired isomorphism is explicitly determined by the mapping $a \otimes m \mapsto a \cdot m$. ■

From this theorem, understanding graded Clifford modules is equivalent to understanding the ungraded modules, since the even subalgebra of a Clifford algebra is isomorphic as ungraded algebras with a smaller Clifford algebra. Putting everything together gives us the complete classification of graded Clifford modules.

Given a ring homomorphism $\varphi : A \to B$, this induces a pullback map ${}_B\mathsf{Mod} \to {}_A\mathsf{Mod}$ of left modules. Given a B module M, we get an A module via φ by defining the ring action on the underlying abelian group of M to be $a \cdot m = \varphi(a)m$, where the right hand side is the ring action from B. This pullback map reveals a beautiful periodicity among the Clifford algebras and their modules, which is one of the many forms of ***Bott periodicity***. The Clifford modules over a fixed Clifford algebra $\mathrm{Cliff}_{p,q}(\mathbb{R})$ form a commutative monoid under the direct sum, which we denote $\mathcal{M}_{p,q}$. The inclusion $\mathbb{R}^{p,q} \hookrightarrow \mathbb{R}^{p,q+1}$ induces an inclusion $\mathrm{Cliff}_{p,q}(\mathbb{R}) \hookrightarrow \mathrm{Cliff}_{p,q+1}(\mathbb{R})$, which then induces a monoid homomorphism $\mathcal{M}_{p,q+1} \to \mathcal{M}_{p,q}$. Likewise, the inclusion $\mathbb{R}^{p,q} \hookrightarrow \mathbb{R}^{p+1,q}$ induces a monoid homomorphism $\mathcal{M}_{p+1,q} \to \mathcal{M}_{q,p+1}$. Since these monoid maps are induced by inclusions, they represent the restriction of Clifford modules. We can then compute the cokernels of these restrictions, which computes the degree to which Clifford modules of $\mathrm{Cliff}_{p,q}(\mathbb{R})$ fail to extend to Clifford modules over $\mathrm{Cliff}_{p+1,q}(\mathbb{R})$ and $\mathrm{Cliff}_{p,q+1}(\mathbb{R})$. To compute these cokernels, it suffices to compute it for the table, since tensoring with $M_{16}\mathbb{R}$ will result in the pattern in the chessboard to repeat. From the Clifford chessboard, we observe that the inclusions of $\mathrm{Cliff}_{p,q}(\mathbb{R})$ into $\mathrm{Cliff}_{p+1,q}(\mathbb{R})$ and $\mathrm{Cliff}_{p,q+1}$ all fall into one of the following four cases.

(1) $M_n\mathbb{F} \hookrightarrow M_n\mathbb{F}'$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $\mathbb{F}' = \mathbb{C}$ or $\mathbb{H}$ is the division algebra of twice the dimension of $\mathbb{F}$ as a vector space over $\mathbb{R}$.
(2) $M_n\mathbb{F} \times M_n\mathbb{F} \hookrightarrow M_{2n}\mathbb{F}$.
(3) $M_n\mathbb{F}' \hookrightarrow M_{2n}\mathbb{F}$, where $\mathbb{F}$ and $\mathbb{F}'$ are defined as above.
(4) $M_n\mathbb{F} \hookrightarrow M_n\mathbb{F} \times M_n\mathbb{F}$.

Using semisimplicity of the Clifford algebras, along with out classification of the ungraded Clifford modules, we then compute the cokernels for all of these cases.

(1) In this case, the irreducible module for $M_n\mathbb{F}'$ is $(\mathbb{F}')^n$, which is twice the dimension of $\mathbb{F}^n$ as an $\mathbb{R}$ vector space. Then since both $M_n\mathbb{F}$ and $M_n\mathbb{F}'$ admit only a single irreducible module, the monoid homomorphism is a map $\mathbb{Z}^{\geq 0} \to \mathbb{Z}^{\geq 0}$ where $1 \mapsto 2$. The cokernel is then the group $\mathbb{Z}/2\mathbb{Z}$.

(2) The algebra $M_n\mathbb{F} \times M_n\mathbb{F}$ injects into $M_{2n}\mathbb{F}$ as block matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

with $A, B \in M_n\mathbb{F}$. From this, we see that the irreducible module $\mathbb{F}^{2n}$ decomposes into a direct sum of the two irreducible modules for $M_n\mathbb{F} \times M_n\mathbb{F}$. The monoid homomorphism $\mathbb{Z}^{\geq 0} \to \mathbb{Z}^{\geq 0} \times \mathbb{Z}^{\geq 0}$ is then given by $1 \mapsto (1, 1)$, and the cokernel is the group $\mathbb{Z}$.

(3) In this case, the irreducible modules $(\mathbb{F}')^n$ and $\mathbb{F}^{2n}$ are the same dimension as vector spaces over $\mathbb{R}$, so the monoid homomorphism is given by $1 \mapsto 1$, so the cokernel is the trivial group 0.

(4) In this case, both irreducible modules for the product $M_n\mathbb{F} \times M_n\mathbb{F}$ are the same dimension as the irreducible module for $M_n\mathbb{F}$, so the monoid homomorphism $\mathbb{Z}^{\geq 0} \times \mathbb{Z}^{\geq 0} \to \mathbb{Z}^{\geq 0}$ is given by $(1, 0) \mapsto 1$ and $(0, 1) \mapsto 1$, so the cokernel is again the trivial group 0.

With these cokernels in hand, we can fill in the table

p \ q	0	1	2	3	4	5	6	7
7	0 / $\mathbb{Z}/2\mathbb{Z}$	0 / 0	$\mathbb{Z}$ / $\mathbb{Z}$	0 / 0	$\mathbb{Z}/2\mathbb{Z}$ / 0	$\mathbb{Z}/2\mathbb{Z}$ / 0	$\mathbb{Z}$ / $\mathbb{Z}$	0 / $\mathbb{Z}/2\mathbb{Z}$
6	0 / 0	$\mathbb{Z}$ / $\mathbb{Z}$	0 / 0	$\mathbb{Z}/2\mathbb{Z}$ / 0	$\mathbb{Z}/2\mathbb{Z}$ / 0	$\mathbb{Z}$ / $\mathbb{Z}$	0 / $\mathbb{Z}/2\mathbb{Z}$	0 / $\mathbb{Z}/2\mathbb{Z}$
5	$\mathbb{Z}$ / $\mathbb{Z}$	0 / 0	$\mathbb{Z}/2\mathbb{Z}$ / 0	$\mathbb{Z}/2\mathbb{Z}$ / 0	$\mathbb{Z}$ / $\mathbb{Z}$	0 / $\mathbb{Z}/2\mathbb{Z}$	0 / $\mathbb{Z}/2\mathbb{Z}$	0 / 0
4	0 / 0	$\mathbb{Z}/2\mathbb{Z}$ / 0	$\mathbb{Z}/2\mathbb{Z}$ / 0	$\mathbb{Z}$ / $\mathbb{Z}$	0 / $\mathbb{Z}/2\mathbb{Z}$	0 / $\mathbb{Z}/2\mathbb{Z}$	0 / 0	$\mathbb{Z}$ / $\mathbb{Z}$
3	$\mathbb{Z}/2\mathbb{Z}$ / 0	$\mathbb{Z}/2\mathbb{Z}$ / 0	$\mathbb{Z}$ / $\mathbb{Z}$	0 / $\mathbb{Z}/2\mathbb{Z}$	0 / $\mathbb{Z}/2\mathbb{Z}$	0 / 0	$\mathbb{Z}$ / $\mathbb{Z}$	0 / 0
2	$\mathbb{Z}/2\mathbb{Z}$ / 0	$\mathbb{Z}$ / $\mathbb{Z}$	0 / $\mathbb{Z}/2\mathbb{Z}$	0 / $\mathbb{Z}/2\mathbb{Z}$	0 / 0	$\mathbb{Z}$ / $\mathbb{Z}$	0 / 0	$\mathbb{Z}/2\mathbb{Z}$ / 0
1	$\mathbb{Z}$ / $\mathbb{Z}$	0 / $\mathbb{Z}/2\mathbb{Z}$	0 / $\mathbb{Z}/2\mathbb{Z}$	0 / 0	$\mathbb{Z}$ / $\mathbb{Z}$	0 / 0	$\mathbb{Z}/2\mathbb{Z}$ / 0	$\mathbb{Z}/2\mathbb{Z}$ / 0
0	0 / $\mathbb{Z}/2\mathbb{Z}$	0 / $\mathbb{Z}/2\mathbb{Z}$	0 / 0	$\mathbb{Z}$ / $\mathbb{Z}$	0 / 0	$\mathbb{Z}/2\mathbb{Z}$ / 0	$\mathbb{Z}/2\mathbb{Z}$ / 0	$\mathbb{Z}$ / $\mathbb{Z}$

where the top left corner of each box is the cokernel for the monoid homomorphism induced by the inclusion $\mathrm{Cliff}_{p,q}(\mathbb{R}) \hookrightarrow \mathrm{Cliff}_{p+1,q}(\mathbb{R})$, and the bottom right corner is the cokernel for the monoid homomorphism induced by the inclusion $\mathrm{Cliff}_{p,q}(\mathbb{R}) \hookrightarrow \mathrm{Cliff}_{p,q+1}(\mathbb{R})$. There is an analogous table for the graded modules, where everything is shifted right one square. The periodicity patterns in this table are one of the many forms of ***Bott periodicity***, which has wide ranging implications in homotopy theory and K-theory. The sequence

$$\mathbb{Z}/2\mathbb{Z} \qquad \mathbb{Z}/2\mathbb{Z} \qquad 0 \qquad \mathbb{Z} \qquad 0 \qquad 0 \qquad 0 \qquad \mathbb{Z}$$

is affectionately called the ***Bott song***. The groups in the table are part of an extremely rich and deep theory called ***K-Theory***, and are called the ***real K-Theory groups*** of a point, denoted $KO^n(\mathrm{pt})$.

3. The Pin and Spin Groups

The group of invertible elements in $\mathrm{Cliff}_{p,q}(\mathbb{R})$, denoted $\mathrm{Cliff}^{\times}_{p,q}(\mathbb{R})$ contains a group $\mathrm{Pin}_{p,q}$, which is a double cover of the group $O_{p,q}$ of matrices preserving the standard bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{p,q}$. Inside of $\mathrm{Pin}_{p,q}$, there exists a subgroup $\mathrm{Spin}_{p,q} \subset \mathrm{Pin}_{p,q}$, which double covers the group $SO_{p,q}$, which consists of the subgroup of $O_{p,q}$ where all the elements have determinant equal to 1.

Definition 3.1. The ***Pin group*** $\mathrm{Pin}_{p,q}$ is the subgroup of $\mathrm{Cliff}^{\times}_{p,q}(\mathbb{R})$ generated by the set

$$\{v \in \mathbb{R}^{p,q} \ : \ v^2 = \pm 1\}$$

The ***Spin group*** $\mathrm{Spin}_{p,q}$ is the subgroup of $\mathrm{Pin}_{p,q}$ generated by even products of basis vectors, i.e.

$$\mathrm{Spin}_{p,q} = \mathrm{Pin}_{p,q} \cap \mathrm{Cliff}^0_{p,q}(\mathbb{R})$$

In indefinite signatures. $SO_{p,q}$ has multiple components. Some conventions let $\mathrm{Spin}_{p,q}$ denote the double cover of the identity component $SO^+_{p,q}$, which is the identity component of $\mathrm{Spin}_{p,q}$, denoted $\mathrm{Spin}^0_{p,q}$. In the case that the bilinear form is definite, we let $\mathrm{Pin}^+_n = \mathrm{Pin}_{n,0}$ and $\mathrm{Pin}^-_n = \mathrm{Pin}_{0,n}$. There is no such distinction for the Spin groups in definite signatures. ❖

THEOREM 3.2. $\mathrm{Spin}_{p,q} \cong \mathrm{Spin}_{q,p}$.

PROOF. Recall that we have an isomorphism $\mathrm{Cliff}^{\mathrm{op}}_{p,q} \to \mathrm{Cliff}_{q,p}$ where $e^{\pm}_i \mapsto e^{\mp}_i$. In addition, the even subalgebra $\mathrm{Cliff}^0_{q,p}$ is isomorphic to the (ungraded) opposite algebra of the even subalgebra $\mathrm{Cliff}^0_{p,q}$. Therefore, the Spin group $\mathrm{Spin}_{q,p} \subset \mathrm{Cliff}_{q,p}$ is isomorphic to the opposite group $\mathrm{Spin}^{\mathrm{op}}_{p,q}$. We then know that every group is isomorphic to its opposite group via the map $g \mapsto g^{-1}$, giving us the desired isomorphism. ■

In particular, this implies that the Spin groups in definite signatures are isomorphic, so we will henceforth denote them as Spin_n. To show that the Pin and Spin groups cover $O_{p,q}$ and $\mathrm{Spin}_{p,q}$, we make a short digression. Given a vector $v \in \mathbb{R}^{p,q}$, we can define a reflection map $R_v : \mathbb{R}^{p,q} \to \mathbb{R}^{p,q}$ given by $R_v(w) = w - 2\langle v, w\rangle v$, which will reflect across the hyperplane $v^\perp$.

THEOREM 3.3 (***Cartan-Dieudonné***). *Any orthogonal transformation $A \in O_{p,q}$ is the composition of at most $p+q$ hyperplane reflections, where we interpret the identity map as the composition of* 0 *reflections.*

PROOF. We prove this by induction on $n = p+q$. The case $n = 1$ is trivial, since $O_1 = \{\pm 1\}$. Then given $A \in O_{p,q}$, fix some nonzero $v \in \mathbb{R}^{p,q}$. Then define $R : \mathbb{R}^{p,q} \to \mathbb{R}^{p,q}$ by

$$R(w) = w - 2\frac{\langle Av - v, w\rangle}{\langle Av - v, Av - v\rangle}(Av - v)$$

Then R is a reflection about the hyperplane orthogonal to $Av - v$, and will interchange v and Av. Therefore, RA is an orthogonal transformation fixing v. Since RA is orthogonal, it will also fix the orthogonal complement $v^\perp$, so it will restrict to an orthgonal transformation on $v^\perp$. The orthogonal complement $v^\perp$ is 1 dimension lower than $\mathbb{R}^{p,q}$, and restricting the bilinear form to $v^\perp$, we know by the inductive hypotehsis that $RA|_{v^\perp}$ can be written as at most $n-1$ hyperplane reflections in $v^\perp$. Since RA fixes v, we can extend all of these transformations to a hyperplane reflection on all of $\mathbb{R}^{p,q}$, by taking the span of each hyperplane with v, giving us that RA is a composition of at most $n-1$ reflections. Finally, composing RA with R gives us that A can be written as a composition of at most n hyperplane reflections. ■

The Cartan-Dieudonné theorem will be the central piece for showing that the Pin and Spin groups are double covers of the orthogonal groups. The Clifford algebra has an automorphism $\alpha : \mathrm{Cliff}_{p,q}(\mathbb{R}) \to \mathrm{Cliff}_{p,q}(\mathbb{R})$ that extends the mapping $v \mapsto v-$. The action of α on a product $v_1 \cdots v_k$ of vectors $v_i \in \mathbb{R}^{p,q}$ is

$$\alpha(v_1 \cdots v_k) = (-1)^k v_1 \cdots v_k$$

the automorphism α gives another way to realize the grading on $\mathrm{Cliff}_{p,q}$ – it is the grading operator. The $+1$-eigenspace of α is exactly $\mathrm{Cliff}^0_{p,q}(\mathbb{R})$, and the -1-eigenspace is the odd subspace $\mathrm{Cliff}^0_{p,q}(\mathbb{R})$.

THEOREM 3.4. *There exist* 2*-to-*1 *group homomorphisms* $\mathrm{Pin}_{p,q} \to O_{p,q}$ *and* $\mathrm{Spin}_{p,q} \to SO_{p,q}$*, i.e. there exist short exact sequences of groups*

$$0 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Pin}_{p,q} \longrightarrow O_{p,q} \longrightarrow 0$$

$$0 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}_{p,q} \longrightarrow SO_{p.q} \longrightarrow 0$$

PROOF. We first consider the case of $\mathrm{Pin}_{p,q}$. To do this, we need to construct a group action where $\mathrm{Pin}_{p,q}$ acts on $\mathbb{R}^{p,q}$ by orthogonal transformations. We note that for a vector $v \in \mathbb{R}^{p,q}$ (identifying $\mathbb{R}^{p,q}$ as a subspace of $\mathrm{Cliff}_{p,q}(\mathbb{R})$), satisfying $\langle v, v\rangle = \pm 1$, we have that $v^{-1} = \pm v$. Then given $g \in \mathrm{Pin}_{p,q}$, and $v \in \mathbb{R}^{p,q}$, we claim that the left action

$$g \cdot v = \alpha(g) v g^{-1}$$

defines the group action we desire. To show this, we must show that this indeed maps $\mathbb{R}^{p,q}$ back into itself, and that the group elements act by orthogonal transformations. We first compute this actions on vectors $v \in \mathbb{R}^{p,q}$ with $\langle v, v\rangle = \pm 1$. In either case, since $v \in \mathbb{R}^{p,q}$, we have that $\alpha(v) = -v$. First assume that $\langle v, v\rangle = 1$. Then given $w \in \mathbb{R}^{p,q}$, we compute

$$\begin{aligned} -vwv^{-1} &= -vwv \\ &= (wv - 2\langle v, w\rangle)v \\ &= w - 2\langle v, w\rangle v \end{aligned}$$

Which is hyperplane reflection about the orthogonal complement of v. In the case that $\langle v, v\rangle = -1$, we compute

$$\begin{aligned} -vwv^{-1} &= -vw(-v) \\ &= (2\langle -v, w\rangle + wv)(-v) \\ &= w - 2\langle -v, w\rangle(-v) \end{aligned}$$

which is hyperplane reflection about the orthogonal complement of $-v$, which is the same as the orthogonal complement of v. Then given two vectors $v_1, v_2 \in \mathbb{R}^{p,q}$, we have that $\alpha(v_1 v_2) = v_1 v_2$, so given $w \in \mathbb{R}^{p,q}$,

$$\alpha(v_1 v_2) w (v_1 v_2)^{-1} = (-v_1)(-v_2) w v_2^{-1} v_1^{-1}$$

which is exactly the composition of hyperplane reflection about $v_2^{\perp}$, with hyperplane reflection about $v_1^{\perp}$. Therefore, $\mathrm{Pin}_{p,q}$ acts by orthogonal transformations, giving us a homomorphism $\mathrm{Pin}_{p,q} \to O_{p,q}$. This map is surjective by the Cartan-Dieudonné theorem, and it can be verified that the kernel is $\{\pm 1\}$. In addition, an even number of hyperplane reflections is orientation preserving, which gives a surjection $\mathrm{Spin}_{p,q} \to SO_{p,q}$, by restricting the map $\mathrm{Pin}_{p,q} \to O_{p,q}$. In addition, the kernel $\{\pm 1\}$ is contained in $\mathrm{Spin}_{p,q}$, so this map is also a double covering. ■

We also have the complex Pin and Spin groups, denoted $\mathrm{Pin}_n\mathbb{C}$ and $\mathrm{Spin}_n\mathbb{C}$, which double cover the complex orthogonal groups $O_n\mathbb{C}$ and $SO_n\mathbb{C}$ respectively.

Two simple examples of spin groups occur in dimensions 2 and 3. Since $SO_2 \cong \mathbb{T}$, where

$$\mathbb{T} = \{z \in \mathbb{C} \ : \ |z| = 1\}$$

we have that $\mathrm{Spin}_2 \cong \mathbb{T}$, where the covering map is given by $z \mapsto z^2$. In the case of SO_3, we consider the unit quaternions, which form a Lie group isomorphic to the group SU_2. Then given $q \in SU_2$, we define the map $\varphi_q : \mathbb{R}^3 \to \mathbb{R}^3$ where $\varphi_q(v) = qv\bar{q}$, where $\bar{q}$ is the quaternionic conjugate of q, i.e.

$$\overline{a + bi + cj + dk} = a - bi - cj - dk$$

and we identify $v = v^i e_i$ is with $v^1 i + v^2 j + v^3 k \in \mathbb{H}$. The mapping $q \mapsto \varphi_q$ then gives a double cover $SU_2 \to SO_3$. In particular, SU_2 is diffeomorphic to the sphere S^3, so the double covering realizes SO_3 as the quotient of S^3 by the antipodal map, giving us that SO_3 is diffeomorphic to $\mathbb{RP}^3$.

Many examples of low dimensional Spin groups arise from investigating the relationship between a 4 dimensional complex vector space V and its second exterior power $\Lambda^2 V$. Fix a volume form $\mu \in \Lambda^4 V^*$. This then induces a symmetric, nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ on $\Lambda^2 V$ by

$$\langle \alpha, \beta \rangle = \langle \alpha \wedge \beta, \mu \rangle$$

where $\langle \alpha \wedge \beta,\, u \rangle$ denotes the natural pairing of the vector space $\Lambda^4 V$ with its dual $\Lambda^4 V^*$. Fix a basis $\{e_i\}$ for V where $\mu(e_1 \wedge e_2 \wedge e_3 \wedge e_4) = 1$. In this basis, we see that the group of transformations $\mathrm{Aut}(V, \mu)$ preserving μ is isomorphic to the group $SL_4\mathbb{C}$. In addition, each map $T \in \mathrm{Aut}(V, \mu)$ induces a map $\Lambda^2 T : \Lambda^2 V \to \Lambda^2 V$, which is determined by the formula $\Lambda^2 T(v \wedge w) = Tv \wedge Tw$. For any $T \in \mathrm{Aut}(V, \mu)$, the induced map $\Lambda^2 T$ preserves the bilinear form on $\Lambda^2 V$, so the mapping $T \mapsto \Lambda^2 V$ determines a group homomorphism $\mathrm{Aut}(V, \mu) \to \mathrm{Aut}(\Lambda^2 V, \langle \cdot, \cdot \rangle)$, where $\mathrm{Aut}(\Lambda^2 V, \langle \cdot, \cdot \rangle)$ denotes the group of linear automorphisms preserving the bilinear form. The kernel of this map is $\{\pm\, \mathrm{id}_V\}$, and fixing an orthogonal basis for $\langle \cdot, \cdot \rangle$ gives us that this map is a double cover $SL_4\mathbb{C} \to SO_6\mathbb{C}$, so $SL_4\mathbb{C}$ is isomorphic to the complex spin group $\mathrm{Spin}_4\mathbb{C}$

If we then fix a hermitian inner product $h : V \times V \to \mathbb{C}$, we can consider the automorphisms $\mathrm{Aut}(V, \mu, h)$ preserving h and μ, which is isomorphic to the group SU_4. The bilinear form h induces a hermitian inner product (which we also denote h) on $\Lambda^2 V$ defined by

$$h(v_1 \wedge v_2, v_3 \wedge v_4) = \det \begin{pmatrix} h(v_1, v_3) & h(v_1, v_4) \\ h(v_2, v_3) & h(v_2, v_4) \end{pmatrix}$$

Then if $T \in \mathrm{Aut}(V, \mu, h)$, $\Lambda^2 T$ preserves the bilinear form $\langle \cdot, \cdot \rangle$ induced by μ as well as the hermitian inner product induced by h. The group that preserves both of these structures is isomorphic to $SO_6\mathbb{C} \cap U_6$, which is $SO_6\mathbb{R}$. This gives us that $SU_4 \cong \mathrm{Spin}_6$.

In general, one can play the game of fixing additional structure on V (e.g. a real structure, quaternionic structure, symplectic form) and look for the induced structure on $\Lambda^2 V$. This then gives a map from automorphisms of V preserving this additional structure to automorphisms of $\Lambda^2 V$ preserving the induced structure. Playing this game then determines several other low dimensional Spin groups.

$$\begin{array}{lll} \mathrm{Spin}_5\mathbb{C} \cong Sp_4\mathbb{C} & \mathrm{Spin}_4 \cong Sp(4) & \mathrm{Spin}_4\mathbb{C} \cong SL_2\mathbb{C} \times SL_2\mathbb{C} \\ \mathrm{Spin}^0_{1,3} \cong SL_2\mathbb{C} & \mathrm{Spin}^0_{1,2} \cong SL_2\mathbb{R} & \mathrm{Spin}^0_{1,5} \cong SL_2\mathbb{H} \end{array}$$

Where $Sp_4\mathbb{C}$ denotes the group of 4×4 matrices preserving a symplectic form, $Sp(4) = Sp_4\mathbb{C} \cap U_4$, and $SL_2\mathbb{H}$ denotes the automorphisms of a 2 dimensional quaternionic vector space with determinant 1 when regarded as 4×4 complex matrices.

Definition 3.5. Given a Pin group $\mathrm{Pin}_{p,q}$, the ***Pinor representations*** are representations of $\mathrm{Pin}_{p,q}$ that arise from an irreducible Clifford module M (i.e. the action of $\mathrm{Pin}_{p,q}$ can be extended to an action of $\mathrm{Cliff}_{p,q}(\mathbb{R})$). The ***Spinor representations*** are defined analogously for the group $\mathrm{Spin}_{p,q}$. ❖

From the classification of Clifford modules, we get a classification of all the Pinor representations. From the relationship between a Clifford algebra and its even subalgebra, we also get a complete classification of all the Spinor representations.

4. Projective Spin Representations

Using the isomorphisms of the Clifford algebras with matrix algebras or products of matrix algebras along with the identification of the even subalgebra with another Clifford algebra, we have a complete classification of the irreducible modules over the even subalgebras $\mathrm{Cliff}_n^0(\mathbb{R})$. Restricting to the Spin group $\mathrm{Spin}_n \subset \mathrm{Cliff}_n^0(\mathbb{R})$, this gives us the Spin representations. These Spin representations define projective representations of the group SO_n. Given a Spin representation $\mathbb{S}$, we have that the elements ± 1 act by scalars since they arise by restricting the action of a Clifford algebra on an irreducible module. Then since SO_n is the quotient of Spin_n by the subgroup $\{\pm 1\}$ of the center, we get a projective representation of SO_n on the projectivization $\mathbb{PS}$ in the following way: Given an element $A \in SO_n$, we have that A lifts to two elements $\{\pm \tilde{A}\} \subset \mathrm{Spin}_n$ which differ by -1. Since -1 acts by a scalar on $\mathbb{S}$, these elements determine the same action on $\mathbb{PS}$, giving us a well defined action of SO_n on $\mathbb{PS}$.

The Spin representation $\mathbb{S}$ is not realized canonically, while the projective Spin representation $\mathbb{PS}$ is canonical. Given an isomorphism $\varphi : V \to W$, this induces a unique algebra isomorphism $\mathrm{End}\, V \to \mathrm{End}\, W$ where $A \in \mathrm{End}\, V$ is mapped to $\varphi \circ A \circ \varphi^{-1}$. However, the converse is not true.

Proposition 4.1. *An algebra isomorphism $\varphi : \mathrm{End}\, V \to \mathrm{End}\, W$ induces a $Z(\mathbb{F})$-torsor of maps $V \to W$, where $Z(\mathbb{F})$ is the center of $\mathbb{F}$.*

To prove this, we need a lemma.

Lemma 4.2. *The group of algebra automorphisms $\mathrm{Aut}(M_n\mathbb{F})$ is isomorphic to the projective general linear group $PGL_n\mathbb{F} = GL_n\mathbb{F}/Z(GL_n\mathbb{F})$.*

Proof. Let $\alpha : M_n\mathbb{F} \to M_n\mathbb{F}$ be an algebra automorphism. We know that $M_n\mathbb{F}$ admits a single irreducible module M up to isomorphism, which is $\mathbb{F}^n$ with the standard action. Then α defines another module M^α, which is the same underlying vector space as M with the algebra action given by $T \cdot v = \alpha(T)v$, where the right hand side is the action of $\alpha(T)$ on the module M. Since α is an algebra automorphism, $M_n\mathbb{F}$ acts transitively on M^α, so it is also an irreducible module, which must be isomorphic to M. Therefore, there exists a module isomorphism $A : M \to M^\alpha$. Since M and M^α are the same underlying vector space, A is also a linear isomorphism $A : M \to M$, thought of as a vector space instead of a module. Then since A is a module homomorphism, we have that for any $T \in M_n\mathbb{F}$,

$$A \circ T = \alpha(T) \circ A \implies A \circ T \circ A^{-1} = \alpha(T)$$

so α is given by conjugation by $A \in GL(M)$. In a basis, this tells us that the map $GL_n\mathbb{F} \to \mathrm{Aut}(M_n\mathbb{F})$ given by conjugation is surjective, and the kernel of this map is the center of $GL_n\mathbb{F}$, so by the first isomorphism theorem, we have that $\mathrm{Aut}(M_n\mathbb{F}) \cong PGL_n\mathbb{F}$. ∎

Proof of Proposition. Fix bases for V and W. These bases then induce isomorphisms $M_n\mathbb{F} \to V$ and $M_n\mathbb{F} \to W$. In these bases, the algebra isomorphism φ is given by an automorphism $M_n\mathbb{F} \to M_n\mathbb{F}$. From the lemma, we know that this automorphism is determined by an element of $PGL_n\mathbb{F} = GL_n\mathbb{F}/Z(GL_n\mathbb{F})$. The center $Z(\mathrm{GL}_n\mathbb{F})$ consists of scalar matrices λI with $\lambda \in Z(\mathbb{F})$, which is acted on freely and transitively by $Z(\mathbb{F})$ by multiplication, giving it the structure of a $Z(\mathbb{F})$-torsor. ∎

However, an isomorphism $\varphi : \mathrm{End}\, V \to \mathrm{End}\, W$ does induce a unique isomorphism $\mathbb{P}V \to \mathbb{P}W$ of projective spaces. To see, this we make an identification between 1 dimensional subspaces of V with maximal left ideals of $\mathrm{End}\, V$.

Proposition 4.3. *There is a bijection*

$$\{\textit{Maximal left ideals of } \operatorname{End} V\} \longleftrightarrow \mathbb{P}V$$

Proof. Given a line $L \in \mathbb{P}V$, the ***annihilator*** of L is the set

$$\operatorname{Ann}(L) = \{M \in \operatorname{End} V \ : \ M(L) = 0\}$$

In fact, $\operatorname{Ann}(L)$ is a left ideal in $\operatorname{End} V$, since given $A \in \operatorname{End} V$ and $M \in \operatorname{Ann}(L)$, L lies in the kernel of $A \circ M$. We claim that $\operatorname{Ann}(L)$ is maximal. Suppose $\operatorname{Ann}(L) \subset I$ is properly contained in a left ideal I. Fix an ordered basis for V in which the first basis element is a nonzero element of L, then elements of $\operatorname{Ann}(L)$ are represented in this basis by matrices with all zeroes in the first column. Then since $\operatorname{Ann}(L)$ is properly contained in I, there exists some $M \in I$ such that $M \notin \operatorname{Ann}(L)$, which implies that as a matrix, the first column of M is nonzero. Then pick a matrix $A \in \operatorname{Ann}(L)$ in which the nonzero columns complete the first column into a basis for $\mathbb{R}^n$. Then $A + M$ is an invertible element of $\operatorname{End} V$, so I must be all of $\operatorname{End} V$. Therefore $\operatorname{Ann}(L)$ is maximal. To show that the mapping $L \mapsto \operatorname{Ann}(L)$ is a bijection, we produce an inverse. Let $I \subset \operatorname{End} V$ be a maximal ideal. Then we claim that the subspace

$$\mathbb{V}(I) = \bigcap_{M \in I} \ker M$$

is a 1 dimensional subspace of V. We note that $\mathbb{V}(I)$ cannot be trivial, since this would imply that I would contain an invertible element, contradicting that it is a proper ideal. We also see that it cannot be higher than 2 dimensional, since otherwise, I woud be contained in the annihilator of a proper nontrivial subspace of $\mathbb{V}(I)$, contradicting maximality of I. We then claim that these two mappings are inverses. We certainly have that $\operatorname{Ann}(\mathbb{V}(I)) \supset I$, so by maximality, this must be I. In addition it is clear that $\mathbb{V}(\operatorname{Ann}(L)) = L$ by the definition of $\mathbb{V}(I)$ and the annihilator. Therefore, these mappings are inverses. ■

Therefore, given an algebra isomorphism $\varphi : \operatorname{End} V \to \operatorname{End} W$, this induces a map $\mathbb{P}V \to \mathbb{P}W$ since the image of a maximal left ideal under an isomorphism is again a maximal left ideal. In addition, the induced map is a bijection, since it has an inverse given by the induced map of φ^{-1}. In addition, the group of units $GL(V)$ acts on $\mathbb{P}V$ by right multiplication – given a maximal ideal I and $A \in GL(V)$, the ideal $I \cdot A$ is also a maximal left ideal.

This gives us a canonical realization of the Spin representations. In the case that the even subalgebra is isomorphic to a matrix algebra $M_n\mathbb{F}$, the projective Spin representation is restriction of the action of $\mathrm{Cliff}_n^0(\mathbb{R})$ on maximal left ideals of $\mathrm{Cliff}_n^0(\mathbb{F})$. In the case that the even subalgebra is isomorphic to a product $M_n\mathbb{F} \times M_n\mathbb{F}$, the irreducible modules identify the subalgebras L and R isomorphic to $M_n\mathbb{F} \times \{0\}$ and $\{0\} \times M_n\mathbb{F}$ by singling out the maximal subalgebra that acts nontrivially. Looking at the maximal left ideals of these subalgebras then identifies the two projective Spin representations. In addition, since -1 acts trivially on left ideals, these projective Spin representions descend to the quotient $\mathrm{Spin}_n / \{\pm 1\} \cong SO_n$, giving us the projective representations of SO_n.

CHAPTER 2

Spin Structures on Manifolds

Mathematics is a part of physics. Physics is an experimental science, a part of natural science. Mathematics is the part of physics where experiments are cheap.
– V.I. Arnold

1. Fiber Bundles

In geometry and topology, we often want to consider families of objects (e.g. manifolds, vector spaces) that vary smoothly along a manifold like the tangent spaces of a manifold. This is best formalized in the construction of a fiber bundle.

Definition 1.1. Let M and F be smooth manifolds. Then a ***fiber bundle*** over M with model fiber F is a the data of a smooth manifold E with a smooth map $\pi : E \to M$ such that for every point $p \in M$, there is a neighborhood $U \subset M$ containing p and a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ such that the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "p_U"] \\ & U & \end{tikzcd}$$

commutes, where p_U denotes projection onto the first factor. The map φ is called a ***local trivialization*** of the fiber bundle $\pi : E \to M$. The manifold M is called the ***base space***, while the manifold E is called the ***total space***. ❖

Given a fiber bundle $\pi : E \to M$ we often denote the fiber $\pi^{-1}(p)$ by E_p.

Definition 1.2. let $\pi : E \to M$ and $p : L \to M$ be fiber bundles with model fiber F over M. A ***bundle homomorphism*** is a smooth map $\varphi : E \to L$ such that the following diagram commutes

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & L \arrow[dl, "p"] \\ & M & \end{tikzcd}$$

❖

An important property of fiber bundles is that they pull back.

Definition 1.3. Let $\pi : E \to M$ be a fiber bundle with model fiber F and let $f : X \to M$ be a smooth map. Then the ***pullback*** of E by f is the data of a smooth manifold

$$f^*E = \left\{(x, p) \ : \ x \in X, p \in \pi^{-1}(f(x))\right\}$$

along with the projection $p : f^*E \to M$ given by $(x, p) \mapsto x$, giving $f^*E \to X$ the structure of a fiber bundle over X with model fiber F. The bundle f^*E also comes equipped with a natural map $\alpha : f^*E \to E$ where $\alpha(x, p) = p$. Pullbacks give rise to the commutative diagram

$$\begin{tikzcd} f^*E \arrow[r, "\alpha"] \arrow[d, "p"'] & E \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & M \end{tikzcd}$$

and are an instance of a general construction called the ***fiber product***. ❖

Definition 1.4. Let $\pi : E \to M$ be a fiber bundle. A ***local section*** of $\pi : E \to M$ is a smooth map $\sigma : U \to E$ of an open set $U \subset M$ such that $\pi \circ \sigma = \mathrm{id}_U$. If $U = M$, the section is called a ***global section***. Equivalently, it is the smooth assignment of an element in E_p to each point $p \in U$. We denote the set of sections of $\pi : E \to M$ over a set U as $\Gamma_U(E)$. ❖

A fiber bundle is a very general construction in which the fibers F do not necessarily admit extra structure. An important special case of a fiber bundle is a vector bundle, where the fibers are vector spaces.

Definition 1.5. Let M be a smooth manifold. A ***vector bundle*** over M is fiber bundle $\pi : E \to M$ with model fiber $\mathbb{R}^n$ (or $\mathbb{C}^n$) such that the local trivializations $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ (or $\mathbb{C}^n$) restrict to linear isomorphisms on the fibers, i.e. for all $p \in U$, the restriction $\varphi|_{\pi^{-1}(p)} : \pi^{-1}(p) \to \{p\} \times \mathbb{R}^n$ (or $\mathbb{C}^n$) is an isomorphism. The dimension n of the fibers is called the ***rank*** of the vector bundle. A ***vector bundle homomorphism*** $\varphi : E \to L$ is a bundle homomorphism with the added stipulation that the restrictions to the fibers $\varphi|_{E_x}$ are linear maps. ❖

Example 1.6.

(1) Given a smooth manifold M, the tangent bundle $TM = \coprod_{p \in M} T_pM$ is a vector bundle, where the projection $\pi : TM \to M$ is the mapping $(p, v) \mapsto p$. The rank of TM is the dimension of M. Local coordinates on (x^i) on $U \subset M$ induce maps $\pi^{-1}(U) \to U \times \mathbb{R}^n$, where $(p, v) \mapsto (x^1(p), \dots x^n(p), v^1, \dots v^n)$, where the v^i are the components of v with respect to the coordinate bases ∂_i. These maps define a basis for a topology on TM, which we then give a smooth structure by declaring the projection $\pi : TM \to M$ to be smooth. In addition, these maps define local trivializations for TM, showing that it is a vector bundle.

(2) A point in $\mathbb{RP}^n$ is a line ℓ in $\mathbb{R}^{n+1}$. The ***tautological bundle*** over $\mathbb{RP}^n$ is the bundle where the fiber over ℓ is the subspace ℓ itself. An analogous construction can be done for Grassmannians $\mathrm{Gr}_k(V)$, which parameterize k-dimensional subspaces of a vector space V. ◂

Definition 1.7. A ***Lie group*** is a smooth manifold G with a group structure such that the multiplication map $(g, h) \mapsto gh$ and the inversion map $g \mapsto g^{-1}$ are smooth. ❖

Almost all of the groups we have discussed previously are Lie groups.

Example 1.8.

(1) The orthogonal groups O_n and special orthogonal groups SO_n are Lie groups.
(2) The unitary groups U_n and special unitary groups SU_n are Lie groups. ◂

Another important class of fiber bundles are principal bundles. Given a Lie group G, the fibers of a principal G-bundle are G-torsors – manifolds with a smooth free and transitive right (or left) G-action. Given a G torsor M and a point p in M, there is a diffeomorphism $G \to M$, where $g \mapsto p \cdot g$. A G-torsor can be thought of as G without a distinguished identity element.

Definition 1.9. Let G be a Lie group, and M a smooth manifold. A ***principal G-bundle*** over M is the data of

(1) A smooth manifold P with a map $\pi : P \to M$.
(2) A smooth right G-action on P that preserves the fibers of π and is free and transitive on the fibers of π.
(3) For every point $p \in M$, a neighborhood $U \subset M$ containing p and a G-equivariant diffeomorphism $\varphi : \pi^{-1}(U) \to U \times G$ (where the right action on $U \times G$ is right multiplication on the second factor) such that we get the commutative diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "p_U"] \\ & U & \end{tikzcd}$$

where p_U denotes projection onto the first factor.

A ***principal bundle homomorphism*** is a bundle homomorphism $\varphi : P \to Q$ that is G-equivariant. ❖

Example 1.10. Given a smooth manifold M and a point $p \in M$, a basis of the tangent space is a linear isomorphism $b : \mathbb{R}^n \to T_pM$. The group $GL_n\mathbb{R}$ acts freely and transitively on the set of bases $\mathcal{B}_p$ on the right by $b \cdot g = b \circ g$. Then the ***frame bundle*** of M, denoted $\mathcal{B}(M)$ is the disjoint union

$$\mathcal{B}(M) = \coprod_{p \in M} \mathcal{B}_p$$

where π is the projection map $(p, b) \mapsto p$. Like with the tangent bundle, we use local coordinates on M to give maps $\pi^{-1}(U) \to U \times GL_n\mathbb{R}$. These maps can then be used to define a topology and smooth structure, and form local trivializations of $\mathcal{B}(M)$, giving it the structure of a principal $GL_n\mathbb{R}$-bundle over M. ◀

Example 1.11. Given a smooth manifold M, a Riemannian metric g induces an inner product g_p on each tangent space T_pM, where g_p denotes the metric g evaluated at p. Then the set of orthonormal bases of T_pM is the set of all linear isometries $(\mathbb{R}^n, \langle \cdot, \cdot \rangle) \to (M, g_p)$ where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$. Then taking the disjoint union over all points $p \in M$ of orthonormal bases for the tangent spaces T_pM forms the ***orthonormal frame bundle*** $\mathcal{B}_O(M)$, which is a principal O_n bundle. ◀

Definition 1.12. Let $\pi : P \to M$ be a principal G-bundle over M, and let F be a smooth manifold with a smooth left G action. Then the ***associated fiber bundle***, denoted $P \times_G F$, is the set

$$P \times_G F = P \times G/(p, g) \sim (p \cdot h, h^{-1}g)$$

Since the group action on P preserves the fibers, the projection $p_1 : P \times F \to P$ composed with the projection $\pi : P \to M$ descends to the quotient, giving us a projection map $\Phi : P \times_G F \to M$. ❖

The first thing to check is that $P \times_G F$ is a fiber bundle, justifying the name.

Proposition 1.13. *Let $\pi : P \to M$ be a principal G-bundle and F a manifold with a left G action. then the associated bundle $\Phi : P \times_G F \to M$ is a fiber bundle with model fiber F.*

Proof. We wish to provide local trivializations $\Phi^{-1}(U) \to U \times F$ for the associated bundle. Fix a local trivialization $\psi : \pi^{-1}(U) \to U \times G$. Then ψ is of the form $\psi(p) = (\pi(p), \tilde{\psi}(p))$ for some $\tilde{\psi} : \pi^{-1}(U) \to G$, satisfying $\tilde{\psi}(p \cdot g) = \tilde{\psi}(p)g$. Define

$$\begin{aligned} \varphi : \Phi^{-1}(U) &\to U \times F \\ [p, f] &\mapsto (\pi(p), \tilde{\psi}(p) \cdot f) \end{aligned}$$

We first note that this is well defined on equivalence classes, since

$$[p \cdot g, g^{-1} \cdot f] \mapsto (\pi(p \cdot g), \tilde{\psi}(p \cdot g) \cdot g^{-1} \cdot f) = (\pi(p), \psi(p) \cdot f)$$

So φ is well defined. We also note that $\Phi(\varphi[p, f]) = \pi(p)$ by how Φ was defined, so φ is a local trivialization provided it is a homeomorphism. To show this, we construct an inverse. Define $\alpha : U \times F \to \Phi^{-1}(U)$ by $\alpha(u, f) = [\psi^{-1}(u, e), f]$, where e denotes the identity element of G. We then claim that α is the inverse. We compute

$$\begin{aligned}(\varphi \circ \alpha)(u, f) &= \varphi[\psi^{-1}(u, e), f] \\ &= (u, e \cdot f) \\ &= (u, f)\end{aligned}$$

In the other direction, we compute

$$\begin{aligned}(\alpha \circ \varphi)[p, f] &= \alpha(\pi(p), \tilde{\psi}(p)) \\ &= [\psi^{-1}(\pi(p), e), f] \\ &= [p, f]\end{aligned}$$

Therefore, φ is a local trivialization, giving us that $\Phi : P \times_G F \to M$ is a fiber bundle with model fiber F. ■

There is a correspondence between sections of an associated bundle and G-equivariant maps $P \to F$.

Proposition 1.14. *Let $\pi : P \to M$ be a principal G-bundle. Then there is a bijection*

$$\{G\text{-equivariant maps } P \to F\} \longleftrightarrow \Gamma_M(P \times_G F)$$

Proof. Since F has a left G-action, we use it to define a right G-action by $f \cdot g = g^{-1} \cdot f$. Then what we mean by a G-equivariant map is a map $\varphi : P \to F$ such that

$$\varphi(p \cdot g) = g^{-1} \cdot \varphi(p)$$

We then wish to use G-equivariant map φ to produce a section $\tilde{\varphi} : M \to P \times_G F$ of the associated bundle. For a point $x \in M$, pick any $p \in \pi^{-1}(x)$ in the fiber. Then define

$$\tilde{\varphi}(x) = [p, \varphi(p)]$$

where $[p, \varphi(p)]$ denotes the equivalence class of $(p, \varphi(p))$ in $P \times_G F$. We first claim that this map is well defined, i.e. it is independent of our choice of point $p \in \pi^{-1}(x)$. We know that G acts freely and transitively on $\pi^{-1}(x)$, so all points in the fiber are of the form $p \cdot g$ for a unique $g \in G$. Then we have that for any $g \in G$

$$(p \cdot g, \varphi(p \cdot g)) = (p \cdot g, g^{-1} \cdot \varphi(p)) \sim (p, \varphi(p))$$

where we use the G-equivariance of φ and the definition of the equivalence relation on the associated bundle. In addition, this induced map is a section of $P \times_G F \to M$, since the image is represented by an element of $P \times F$ with an element of the fiber $\pi^{-1}(x)$ in the first factor.

Conversely, given a section $\sigma : M \to P \times_G F$, we wish to produce a G-equivariant map $P \to F$. Given such a section, and a point $x \in M$, we have that $\sigma(x) = [p, f]$ for some $p \in P$ and $f \in F$. Then define $\tilde{\sigma} : P \to F$ such that $\tilde{\sigma}(p) = f$, and $\tilde{\sigma}(p \cdot g) = g^{-1} \cdot f$. Since G acts freely and transitively on the fibers, this specifies the map on every point of P. In addition, $\tilde{\sigma}$ is G-equivariant by construction. Then the maps

$$\{G\text{-equivariant maps } P \to F\} \longleftrightarrow \Gamma_M(P \times_G F)$$

we provided are easily verified to be inverses to each other, giving us the correspondence. ■

In some sense, the geometry of the associated fiber bundle $P \times_G F$ is controlled by the group G, as G determines a distinguished group of symmetries of the fiber F. Given an element $p \in P$, the element p determines a diffeomorphism $\varphi_p : F \to (P \times_G F)_{\pi(p)}$ via the mapping $f \mapsto [p, f]$, and the group action of G on P gives a transformation law on the associated bundle via the group action of G on F.

Example 1.15 (The tangent bundle as an associated bundle). Given a manifold M, we can take the frame bundle $\pi : \mathcal{B}(M) \to M$, which is a principal $GL_n\mathbb{R}$ bundle. The group $GL_n\mathbb{R}$ acts linearly on $\mathbb{R}^n$ in the standard way, giving us an associated vector bundle $\mathcal{B}(M) \times_{GL_n\mathbb{R}} \mathbb{R}^n$. We claim that this bundle is isomorphic to the tangent bundle TM, i.e. there exists an diffeomorphism $\varphi : \mathcal{B}(M) \times_{GL_n\mathbb{R}} \mathbb{R}^n \to TM$ that restricts to linear isomorphisms on the fibers and the diagram

$$\begin{tikzcd} \mathcal{B}(M) \times_{GL_n\mathbb{R}} \mathbb{R}^n \arrow[rr, "\varphi"] \arrow[dr] & & TM \arrow[dl] \\ & M & \end{tikzcd}$$

commutes, where the maps to M are the bundle projections. Recall that elements of $\mathcal{B}(M) \times_{GL_n\mathbb{R}} \mathbb{R}^n$ are represented by pairs (b, v), where $b : \mathbb{R}^n \to T_{\pi(b)}M$ is a linear isomorphism, and v is a vector in $\mathbb{R}^n$. Then define φ by

$$\varphi[b, v] = (\pi(b), b(v))$$

This is well defined, since $\varphi[b \circ g, g^{-1}(v)] = (\pi(b \circ g), (b \circ g)(g^{-1}(v))) = (\pi(b), b(v))$. This is an isomorphism, where the inverse mapping maps $(p, v) \in TM$ to $(b, \tilde{v})$ where b is any basis of T_pM and $\tilde{v}$ is the coordinate representation of v in the basis b. In this example, we see that the associated bundle identifies the same vector under different coordinate transformations, which defines the symmetries of TM. ◄

In a similar fashion, the cotangent bundle T^*M to a manifold M is naturally isomorphic to the associated bundle $\mathcal{B}(M) \times_{GL_n\mathbb{R}} (\mathbb{R}^n)^*$, where $GL_n\mathbb{R}$ acts on $(\mathbb{R}^n)^*$ by the dual of the standard representation (i.e. $A \in GL_n\mathbb{R} acts by (A^{-1})^*$.)

In general, given a rank n vector bundle $E \to M$, we can construct the frame bundle $\mathcal{B}(E)$ for E, where the fiber over $x \in M$ is the $GL_n\mathbb{R}$-torsor of bases for the fiber E_x. We can then recover E by taking the associated bundle $\mathcal{B}(E) \times_{GL_n\mathbb{R}} \mathbb{R}^n$, so the process of taking frames and constructing associated bundles are inverses of each other. In general, if E is an associated vector bundle $P \times_G V$ obtained via a linear representation $\rho : G \to GL(V)$, the elements of P have a natural interpretation as bases of the fibers of E. In this case, each element $p \in P$ determines a linear isomorphism $\varphi_p : V \to E_p$, giving a G-equivariant $\varphi : P \to \mathcal{B}(E)$ of P to the frame bundle of E, where G acts on $\mathcal{B}(E)$ through the representation ρ.

Definition 1.16. Let $\pi : P \to M$ be a principal G-bundle, and $\rho : H \to G$ a group homomorphism. The map ρ gives G a left H action where $h \cdot p = \rho(h) \cdot p$. A ***reduction of structure group*** is the data of a principal H bundle $\varphi : Q \to M$ and an H-equivariant bundle homomorphism $F : Q \to P$. ❖

The map $F : Q \to P$ induces a map $\tilde{F} : Q \times_H G \to P$, where we map the equivalence class $[q, g]$ to $F(q)g$. This is well defined on equivalences classes since

$$(q \cdot h, \rho(h)^{-1}g) \mapsto F(q \cdot h)\rho(h)^{-1}g = F(q)\rho(h)\rho(h)^{-1}g = F(q)g$$

Example 1.17 (Reduction from $GL_n\mathbb{R}$ to O_n). Let M be a smooth manifold, and $\pi : \mathcal{B}(M) \to M$ its bundle of frames. The inclusion map $\iota : O_n \hookrightarrow GL_n\mathbb{R}$ gives an action of O_n on $\mathcal{B}(M)$, where given $b \in \mathcal{B}(M)$ and $T \in O_n$, $b \cdot T = b \cdot \iota(T)$. We then take the quotient by this O_n action, giving us a quotient map $q : \mathcal{B}(M) \to \mathcal{B}(M)/O_n$. Since the inclusion is injective, the O_n action is free on

$\mathcal{B}(M)$, so this gives $q : \mathcal{B}(M) \to \mathcal{B}(M)/O_n$ the structure of a O_n bundle. In addition, the action of O_n preserves the fibers of $\pi : \mathcal{B}(M) \to M$, so π descends to the quotient, so $\mathcal{B}(M)/O_n \to M$ is a fiber bundle over M with model fiber $GL_n\mathbb{R}/O_n$. Since $GL_n\mathbb{R}$ deformation retracts onto O_n via the Gram-Schmidt algorithm, $GL_n\mathbb{R}/O_n$ is contractible, so the fiber bundle $\mathcal{B}(M)/O_n \to M$ admits global sections. Then given a section $\sigma : M \to \mathcal{B}(M)/O_n$, this gives an O_n bundle over M via the pullback $\sigma^*\mathcal{B}(M)$. In addition, we get a O_n-equivariant map $\sigma^*\mathcal{B}(M) \to \mathcal{B}(M)$ given by $(p, b) \mapsto (p, \iota(b))$. The bundle $\sigma^*\mathcal{B}(M)$ can be thought of as the bundle of orthonormal frames with respect to a Riemannian metric on M, and the fact that $\mathcal{B}(M)/O_n$ admits sections corresponds to the fact that every manifold admits a Riemannian metric. ◂

Principal bundles can be thought of as a generalization of covering spaces. For discrete groups G, the data of a principal G-bundle $P \to M$ is the data of a (possibly disconnected) cover of M with a deck transformation group isomorphic to G acting freely and transitively on the fibers. If G is finite, this is a $|G|$-fold covering, and if G is countable, the cover is an infinitely sheeted cover. Discreteness of G allows for a very complete description of all principal G-bundles over a given manifold M.

Definition 1.18. Let (M, x_0) be a pointed topological space (in our case, usually a manifold), i.e. a space M with a choice of distinguished basepoint $x_0 \in M$. Then a ***pointed principal G-bundle*** over M is the data of a principal G-bundle $\pi : P \to M$, along with a choice of basepoint $p_0 \in \pi^{-1}(x_0)$. We denote this as $\pi : (P, p_0) \to (M, x_0)$. We denote the set of isomorphism classes of pointed principal G-bundles over (M, x_0) as $\mathsf{Bun}_G(M, x_0)$. ❖

We now recall some properties of covering spaces. A pointed covering space $\pi : (\tilde{M}, \tilde{x_0}) \to (M, x_0)$ has the ***path lifting*** property, i.e. given a path $\gamma : I \to M$ with $\gamma(0) = x_0$, there exists a unique path $\tilde{\gamma} : I \to \tilde{M}$ with $\gamma(0) = \tilde{x_0}$ such that $\pi \circ \tilde{\gamma} = \gamma$. In addition, the covering space has the ***homotopy lifting property***. Given paths $\gamma_1, \gamma_2 : I \to M$ with $\gamma_i(0) = x_0$ and a homotopy $F : I \times I \to M$ from γ_1 to γ_2, there exists a unique homotopy $\tilde{F} : I \times I \to \tilde{M}$ between $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$. These two facts define an action of $\pi_1(M, x_0)$ on the fiber $\pi^{-1}(x_0)$ where given a homotopy class $[\gamma]$ of loops, the action of $[\gamma]$ on $p \in \pi^{-1}(x_0)$ is the endpoint of $\tilde{\gamma}(1)$ of the lifted loop $\tilde{\gamma}$. The unique path lifting and homotopy lifting properties guarantees that this defines a group action, called the ***holonomy*** of the covering space. In the special case that our covering space is a principal G-bundle with a discrete G, the point $\tilde{\gamma}(1)$ is equal to $\tilde{x_0} \cdot g$ for a unique group element $g \in G$, so the holonomy action determines a group homomorphism $\pi_1(M, x_0) \to G$. In fact, the holonomy completely determines the bundle.

Theorem 1.19. *For a discrete group G and a connected manifold M, there is a bijection*

$$\mathsf{Bun}_G(M, x_0) \longleftrightarrow \mathsf{Hom}(\pi_1(M, x_0), G)$$

Proof. We once more provide maps in both directions. Every pointed principal G-bundle $(P, p_0) \to (M, x_0)$ determines a group homomorphism $\pi_1(M, x_0) \to G$ by the holonomy action. In the other direction, given a group homomorphism $\varphi : \pi_1(M, x_0) \to G$, we want to construct a pointed principal G bundle $(P, p_0) \to (M, x_0)$ with holonomy φ. Every connected pointed manifold (M, x_0) has a ***universal cover*** $(\tilde{M}, \tilde{x}_0) \to (M, x_0)$, which has a free and transitive action by $\pi_1(M, x_0)$, making it a pointed principal $\pi_1(M, x_0)$ bundle over (M, x_0). In addition, the homomorphism $\varphi : \pi_1(M, x_0) \to G$ determines a left action of $\pi_1(M, x_0)$ on G by $[\gamma] \cdot g = \varphi[\gamma]g$. This allows us to define the associated bundle

$$P = \tilde{M} \times_{\pi_1(M, x_0)} G = \tilde{M} \times G/(x, g) \sim (x \cdot [\gamma], \varphi[\gamma]^{-1}g)$$

where our distinguished basepoint $p_0 \in P$ is $p_0 = [\tilde{x}_0, e]$. This is a fiber bundle with model fiber G, and the right action of G on $\tilde{M} \times G$ descends to a free an transitive action on the quotient, making it a principal G-bundle. We then check the holonomy action of a loop $[\gamma] \in \pi_1(M, x_0)$. We compute for p in the fiber over x_0

$$p \cdot [\gamma] = [\tilde{x_0} \cdot [\gamma], h] \sim [x_0, \varphi[\gamma] \cdot h]$$

So the holonomy is φ, giving us the desired correspondence. ■

The set $\mathrm{Bun}_G(M, x_0)$ comes equipped with an action of G, where the action of $g \in G$ on a pointed bundle $(P, p_0) \to (M, x_0)$ is by permuting basepoints, i.e. $(P, p_0) \cdot g = (P, p_0 \cdot g)$, where the covering map is left untouched. Quotienting by this group action gives us the identification of $\mathrm{Bun}_G(M, x_0)/G$ with isomorphism classes of principal G-bundles $P \to M$. where the isomorphisms are without basepoints. From the perspective of pointed bundles $(P, p_0) \to (M, x_0)$ as homomorphisms $\varphi : \pi_1(M, x_0) \to G$, the group action of G on $\mathrm{Bun}_G(M, x_0)$ is an action by conjugation. Given $(P, p_0) \to (M, x_0)$ with holonomy $\varphi : \pi_1(M, x_0) \to G$, we want to know the holonomy φ_g of the bundle $(P, p_0 \cdot g) \to (M, x_0)$. Given a loop γ based at x_0, let $\tilde{\gamma}$ denote its lift to (P, p_0) by uniqueness of path lifting, the lift of γ to $(P, p_0 \cdot g)$ is the path $\tilde{\gamma}_g(t) = \tilde{\gamma}(t) \cdot g$. Then

$$\begin{aligned}
\tilde{\gamma_g}(1) &= \tilde{\gamma}(1) \cdot g \\
&= p_0 \cdot \varphi[\gamma] \cdot g \\
&= p_0 \cdot g \cdot g^{-1} \cdot \varphi[\gamma] \cdot g
\end{aligned}$$

So $\varphi_g = g^{-1} \cdot \varphi \cdot g$.

2. Dirac Operators in $\mathbb{R}^n$

One important application of Clifford algebras and Spin groups comes from physics. In the process of developing a relativistic equation for the electron, Paul Dirac saw the need for a first order differential operator D such that D squared to the Laplace operator.[1]

$$\Delta = -\sum_i \frac{\partial^2}{\partial (x^i)^2}$$

If D were the be first order, it would have to be written as

$$D = a^i \frac{\partial}{\partial x^i}$$

for some coefficients a^i. However, it is clear that choosing scalar coefficients for the a^i will not suffice. For example, in $\mathbb{R}^2$ with the standard coordinates x and y, any first order operator $D = a^1\partial_x + a^2\partial_y$ satisfies

$$D^2 = \left(a^1 \frac{\partial}{\partial x} + a^2 \frac{\partial}{\partial y}\right)^2 = (a^1)^2 \frac{\partial^2}{\partial x^2} + a^1 a^2 \frac{\partial^2}{\partial x \partial y} + a^2 a^1 \frac{\partial^2}{\partial y \partial x} + (a^2)^2 \frac{\partial^2}{\partial y^2}$$

From this equation, we see that the a^i must square to -1, and we must also have that $a^1a^2 + a^2a^1 = 0$ in order for the mixed partial terms to vanish. This is not possible if the a^i are scalars (in either $\mathbb{R}$ or $\mathbb{C}$). However, the required relations are exactly the relations between orthogonal basis vectors in the Clifford algebra $\mathrm{Cliff}_{0,n}(\mathbb{R})$!

Definition 2.1. Let $\{e_i\}$ be the standard basis for $\mathbb{R}^n$, $\{e^i\}$ [2] its dual basis, and $x^1, \ldots, x^n$ the standard coordinates on $\mathbb{R}^n$. The ***Dirac operator*** on $\mathbb{R}^n$ is the first order differential operator

$$D = e^i \frac{\partial}{\partial x^i}$$

[1] One reason for choosing this sign convention for Δ is that the spectrum of Δ is positive with this choice of sign

[2] The choice of using the dual basis elements e^i instead of the e_i is so that D behaves tensorially with respect to coordinate change.

❖

It is not clear from the definition what function space D should act on. The partial derivative operators make sense for any vector valued function, but multiplication by e^i does not make sense an arbitrary vector space – it must be a Clifford module. Therefore, D is an operator $D : C^\infty(\mathbb{R}^n, M) \to C^\infty(\mathbb{R}^n, M)$ on smooth functions from $\mathbb{R}^n$ to a Clifford module M. In fuller generality, functions from $\mathbb{R}^n \to M$ are equivalent to smooth sections of $\mathbb{R}^n \times M \to \mathbb{R}^n$, which is a bundle of Clifford modules over $\mathbb{R}^n$. This viewpoint generalizes more naturally, and the Dirac operator defined on a general Riemannian manifold is an operator on sections of some bundle.

The Dirac operators in dimensions 1, 2, and 4 exhibit some extremely interesting behavior, corresponding to the appearances of $\mathbb{C}$ and $\mathbb{H}$ when $\mathbb{R}^n$ is given a negative definite bilinear form. In 1 dimension, the Clifford algebra $\mathrm{Cliff}_{0,1}(\mathbb{R})$ admits the ordered basis $(1, e^1)$, which then gives a basis for $C^\infty(\mathbb{R}, \mathrm{Cliff}_{0,1}(\mathbb{R}))$ as a $C^\infty(\mathbb{R})$ module where for any $f : \mathbb{R} \to \mathrm{Cliff}_{0,1}(\mathbb{R})$, we have the decomposition $f(x) = u(x) + e^1 v(x)$, giving a column vector representation

$$f = \begin{pmatrix} u \\ v \end{pmatrix}$$

The Dirac operator is $e^1\partial_x$, which is represented in matrix form as

$$D = \begin{pmatrix} 0 & -\partial_x \\ \partial_x & 0 \end{pmatrix}$$

Note that the matrix is block off-diagonal. The vector space $C^\infty(\mathbb{R}, \mathrm{Cliff}_{0,1}(\mathbb{R}))$ has a natural grading where the even elements are maps $\mathbb{R} \to \mathrm{Cliff}^0_{0,1}(\mathbb{R})$ and the odd elements are maps $\mathbb{R} \to \mathrm{Cliff}^1_{0,1}(\mathbb{R})$. Since we picked the ordered basis $(1, e^1)$ for $\mathrm{Cliff}_{0,1}(\mathbb{R})$, whose elements are even and odd respectively, the bottom left block of the matrix is action of D as a map from the even subspace to the odd subspace, and the top right block of the matrix is the action of D as a map from the odd subspace to the even subspace. This shows that D is an odd operator, as it reverses the grading on $C^\infty(\mathbb{R}, \mathrm{Cliff}_{0,1})$.

In 2 dimensions, $\mathrm{Cliff}_{0,2}(\mathbb{R})$ has the ordered basis $(1, e^1e^2, e_1, e_2)$, where we choose the ordering by the parity of the elements. Again, we can write any function $f : \mathbb{R}^2 \to \mathrm{Cliff}_{0,2}(\mathbb{R})$ in terms of this basis as

$$f(x, y) = f_0(x, y) + e^1e^2 f_{12}(x, y) + e^1 f_1(x, y) + e^2 f_2(x, y)$$

where the component functions are all elements of $C^\infty(\mathbb{R}^2)$. In this basis, the action of the Dirac operator $D = e^1\partial_x + e^2\partial_y$ is represented by the matrix of differential operators

$$D = \begin{pmatrix} 0 & 0 & -\partial_x & -\partial_y \\ 0 & 0 & -\partial_y & \partial_x \\ \partial_x & \partial_y & 0 & 0 \\ \partial_y & -\partial_x & 0 & 0 \end{pmatrix}$$

which is actually a familiar differential operator in disguise. We make the identification of $\mathrm{Cliff}_{0,2}(\mathbb{R})$ with the quaternions $\mathbb{H}$, which have a direct sum decomposition $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}$, giving a $\mathbb{Z}/2\mathbb{Z}$-grading. The mappings $e_1e_2 \mapsto i$ and $e_2 \mapsto i$ determine vector space isomorphisms of the even and odd subspaces of $\mathrm{Cliff}_{0,2}(\mathbb{R})$ with $\mathbb{C}$, giving an isomorphism of $\mathrm{Cliff}_{0,2}(\mathbb{R})$ with $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}$. The operator in the top right block is then exactly twice the differential operator

$$-\frac{\partial}{\partial z} = -\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

on functions $\mathbb{R}^2 \to \mathbb{C}$, and the operator in the bottom block is the twice the operator

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

So D is more compactly represented under this identification as

$$D = \begin{pmatrix} 0 & 2\partial_z \\ 2\partial_{\overline{z}} & 0 \end{pmatrix}$$

In 4 dimensions, the irreducible module over $\mathrm{Cliff}_{0,3} \cong M_2\mathbb{H}$ is $\mathbb{H}^2$. This has a direct sum decomposition as $\mathbb{H}^2 = \mathbb{H} \oplus \mathbb{H}$, allowing us to represent functions $f : \mathbb{R}^4 \to \mathbb{H}^2$ as pairs of functions $\mathbb{R}^4 \to \mathbb{H}$, which are the maps into the even and odd subspaces of $\mathbb{H}^2 = \mathbb{H} \oplus \mathbb{H}$. The quaternions admit operators similar to the operators ∂_z and $\partial_{\overline{z}}$, which are

$$\frac{\partial}{\partial q} = \frac{1}{4}\left(\frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^2} - j\frac{\partial}{\partial x^3} - k\frac{\partial}{\partial x^4}\right)$$

$$\frac{\partial}{\partial \overline{q}} = \frac{1}{4}\left(\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2} + j\frac{\partial}{\partial x^3} + k\frac{\partial}{\partial x^4}\right)$$

Using these identifications, the Dirac operator is compactly represented as

$$D = \begin{pmatrix} 0 & -4\partial_q \\ 4\partial_{\overline{q}} & 0 \end{pmatrix}$$

which encodes quaternionic analogues of the Cauchy-Riemann equations.

3. Spin Structures

Every manifold M admits a Riemannian metric, so there always exists a reduction of structure group from $GL_n\mathbb{R}$ to O_n. If M is orientable, over each point $p \in M$, we have a preferred set of orthonormal bases of T_pM, which correspond to a choice of component of the O_n-torsor of orthonormal frames of T_pM. Doing this for each point $p \in M$, this determines a subbundle P of $\mathcal{B}_O(M)$. In addition, each fiber has a free and transitive action of SO_n via the inclusion $SO_n \hookrightarrow O_n$, giving it the structure of a principal SO_n-bundle over M. In addition, the inclusion $P \hookrightarrow \mathcal{B}_O(M)$ is SO_n-equivariant, so it defines a reduction of structure group from O_n to SO_n. We denote $P = \mathcal{B}_{SO}(M)$, and call it the ***bundle of oriented orthonormal frames***.

A ***Spin structure*** on M is a further reduction of structure group to Spin_n with respect to the double cover $\mathrm{Spin}_n \to SO_n$, i.e. the data of a principal Spin_n bundle $P \to M$, along with a Spin_n-equivariant map $P \to \mathcal{B}_{SO}(M)$, where $\mathcal{B}_{SO}(M)$ denotes the bundle of positively oriented orthonormal frames and Spin_n acts on $\mathcal{B}_{SO}(M)$ through the double cover $\mathrm{Spin}_n \to SO_n$. In particular, this map is a double cover, and fiberwise is the double covering $\mathrm{Spin}_n \to SO_n$. Depending on the value n, there exist either one or two irreducible representations for Spin_n. Let $\mathbb{S}$ denote the direct sum all of all the irreducible Spin representations for Spin_n, i.e. $\mathbb{S}$ is equal to the single irreducible Spin representation in the case that Spin_n only admits one irreducible representation, and $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ in the case that Spin_n admits two inequivalent irreducible representations $\mathbb{S}^+$ and $\mathbb{S}^-$. Then given a Spin structure on M with a principal Spin_n bundle P, the ***spinor bundle*** over M is the associated vector bundle $P \times_{\mathrm{Spin}_n} \mathbb{S}$, which can be thought of as a Spin analogue to the tangent bundle of M.

EXAMPLE 3.1 (Spin structures on S^1). The group Spin_1 is equal to $\{\pm 1\}$, so a Spin structure on S^1. is a double cover $\pi : P \to S^1$ along with a Spin_1-equivariant map $P \to \mathcal{B}_{SO_1}(S^1)$. Since $SO_1 = 1$, $\mathcal{B}_{SO}(S^1)$ is the trivial bundle $S^1 \times \{1\}$. Therefore, specifying a Spin_1-equivariant map $P \to S^1 \times \{1\}$ is no additional data, since we are forced to map all of the fiber $\pi^{-1}(x)$ to $(x, 1)$ for any $x \in S^1$. Consequently, all double covers give rise to a reduction of structure group to Spin_1. There are only two double covers of S^1 up to isomorphism. One of them is the disconnected double cover, which is the disjoint union $S^1 \coprod S^1$, which we denote as $\pi_1 : P_1 \to S^1$. The other is the connected double cover, which the circle covering itself via the map $z \mapsto z^2$, which we denote as $\pi_2 : P_2 \to S^1$. For convenience, we parameterize P_2 with angles $\theta \in [0, 4\pi)$, so the covering map is given by $\theta \mapsto e^{i\theta}$. The Spin representation is the sign representation on $\mathbb{R}$, where -1 acts by multiplication by -1, and the complexifying gives us an action on $\mathbb{C}$ where -1 acts by multiplication by -1, giving us two spinor bundles $P_1 \times_{\mathrm{Spin}_1} \mathbb{C}$ and $P_2 \times_{\mathrm{Spin}_1} \mathbb{C}$.

In the first case, the associated bundle is a trivial bundle. Using the identification with Spin_1-equivariant maps $P_1 \to \mathbb{C}$ with sections of the associated bundle, it suffices to find such a map to produce a global section of $\pi_1 : P_1 \to S^1$. Write P_1 as the disjoint union $S_1 \coprod S_1$, with the circles parameterized by angles $\theta, \varphi \in [0, 2\pi)$. The action of -1 is given by $\theta \mapsto \varphi$ and $\varphi \mapsto \theta$. Then the mappings $\theta \mapsto e^{i\theta}$ and $\varphi \mapsto e^{i\varphi}$ define a Spin_1-equivariant map $P_1 \to \mathbb{C}$, giving us a trivialization of the associated bundle $P_1 \times_{\mathrm{Spin}_1} \mathbb{C}$. In addition, we see that sections of $P_1 \times_{\mathrm{Spin}_1} \mathbb{C}$ are equivalent data to maps $S^1 \to \mathbb{C}$, since once we map one of the components of P_1 into $\mathbb{C}$, this entirely determines how we need to map the other component in order to remain Spin_1-equivariant. This further allows us to identify sections of the spinor bundle with 2π-periodic functions $\mathbb{R} \to \mathbb{C}$.

In the second case, the bundle is also trivial! We again contruct a trivialization for the associated bundle by providing a Spin_1-equivariant map $\sigma : P_2 \to \mathbb{C}$. Using the parameterization of P_2 with angles $\theta \in [0, 4\pi)$, the Spin_1 action on P_2 is given by $-1 \cdot \theta = \theta + 2\pi \mod 4\pi$. Then define σ by $\sigma(\theta) = e^{i\theta/2}$. This map is Spin_1-equivariant, so it produces a trivialization of the spinor bundle $P_2 \times_{\mathrm{Spin}_1} \mathbb{C}$. In addition, we see that sections of the spinor bundle correspond to 2π-antiperiodic functions, i.e. functions $\psi : \mathbb{R} \to \mathbb{C}$ such that $\psi(\theta) = -\psi(\theta + 2\pi)$, using the fact that we parameterized P_2 with angles from $[0, 4\pi)$. Another way to write a 2π-antiperiodic map ψ is as a product $\psi(\theta) = e^{i\theta/2} f(\theta)$ for a 2π-periodic function f, which is the representation of the section ψ with respect to the trivialization σ defined above.

The two different spin structures produced two isomorphic vector bundles, but there is still a way to distinguish between the two – their Dirac operators. The Clifford algebra $\mathrm{Cliff}_{0,1}(\mathbb{R})$ is isomorphic to $\mathbb{C}$ as an $\mathbb{R}$-algebra via the mappings $1 \mapsto 1$, and $e_1 \mapsto i$, so the Dirac operator on $\mathbb{R}$ can also be written as $i\frac{d}{dt}$. We then use our identifications of sections of the spinor bundles with functions $\mathbb{R} \to \mathbb{C}$ to investigate the Dirac operators on each bundle. In the case of the disconnected double cover P_1, we have the identifications of sections of $P_1 \times \mathrm{Spin}_1\mathbb{C}$ with 2π-periodic functions $\mathbb{R} \to \mathbb{C}$. Then given a section ψ, we can identify it as a function $\mathbb{R} \to \mathbb{C}$, and use the Dirac operator in $\mathbb{R}$, which will again be a 2π-periodic function, giving us another section, giving us that the Dirac operator D_1 on sections of $P_1 \times_{\mathrm{Spin}_1} \mathbb{C}$ is just $i\frac{d}{d\theta}$. For the connected double cover, we used the global section associated to $\sigma(\theta) = e^{i\theta/2}$ to identify sections of $P_2 \times_{\mathrm{Spin}_1} \mathbb{C}$ as products $\psi(\theta) = e^{i\theta/2} f(\theta)$ for a 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$. Then applying the Dirac operator from $\mathbb{R}$ to this function, we get

$$\begin{aligned} D\psi(\theta) &= i\frac{d}{d\theta}\left(e^{i\theta/2} f(\theta)\right) \\ &= e^{i\theta/2}\frac{df}{d\theta} - \frac{1}{2}e^{i\theta/2} \end{aligned}$$

So in the local trivialization $\sigma(\theta) = e^{i\theta/2}$, the operator D_2 operates on 2π-periodic functions, just like D_1, and is given by $D_2 = i\partial_\theta - \frac{1}{2}$. In particular, the first operator D_1 has integer spectrum, and the spectrum of D_2 is the spectrum of D_1 shifted by $\frac{1}{2}$, which allows us to distinguish to two Spin structures on S^1 by their Dirac operators. ◁

In fact, every sphere is a Spin manifold

Theorem 3.2. *For all $n \in \mathbb{Z}^{\geq 0}$, S^n is a Spin manifold.*

Proof. We use the standard embedding of S^n into $\mathbb{R}^{n+1}$. Given a point $p \in S^n$, the tangent space $T_pS^n = p^\perp$, where we take the orthogonal complement in $\mathbb{R}^{n+1}$ equipped with the standard inner product, and has an orientation induced by the standard orientation on $\mathbb{R}^{n+1}$. We use this to define a principal SO_n-bundle $\pi : SO_{n+1} \to S^n$. Given a matrix $A \in SO_{n+1}$, the first column determines a point in $p \in S^n$, and the remaining columns determine an oriented basis b_A of $p^\perp = T_pS^n$. Then the mapping $\pi(A) = b_A$ is the oriented orthonormal frame bundle $\mathcal{B}_{SO}(S^n)$, and the total space is diffeomorphic to SO_{n+1}. Then let $\rho : \mathrm{Spin}_{n+1} \to SO_{n+1}$ denote the standard double cover. The composition $\rho \circ \pi : \mathrm{Spin}_{n+1} \to S^n$ is then a principal Spin_n bundle over S^n, giving the desired Spin structure. ■

In particular, the Spin structure over S^2 is the Hopf fibration $S^3 \to S^2$, since $SO_3 \cong \mathbb{RP}^3$.

Given an oriented manifold M, we get a reduction of structure group of the frame bundle $\mathcal{B}(M)$ to SO_n, giving us a principal SO_n bundle $\mathcal{B}_{SO}(M)$. If M has a nonempty boundary, this induces an orientation on ∂M in the following way: by first reducing the structure group to O_n, we get a Riemannian metric g on M. Then given a point $p \in \partial M$, the tangent space of the boundary $T_p\partial M$ is a codimension 1 subspace of T_pM. The Riemannian metric g allows us to pick a distinguished complementary subspace to $T_p\partial M$ – the orthogonal complement $(T_p\partial M)^\perp$. From this subspace, we have a distinguished choice of vector – the outward normal vector. In appropriate coordinates, the inclusion of the tangent space $T_p\partial M$ is given locally by the inclusion

$$\begin{gathered}\mathbb{R}^{n-1} \hookrightarrow \mathbb{R}^n \\ (x^1, \ldots, x^{n-1}) \mapsto (0, x^1, \ldots, x^{n-1})\end{gathered}$$

and the outward normal is the unit length vector with a positive first component. This defines a vector field N along ∂M, where the value N_p of N at the point p is the outward normal vector in T_pM. On the boundary ∂M, we restrict the frame bundle $\mathcal{B}_{SO}(M)$ to ∂M by pulling back by the inclusion $\partial M \hookrightarrow M$, giving us the restricted bundle $\mathcal{B}_{SO}(M)|_{\partial M}$, which is a principal SO_n-bundle over ∂M. An element $b \in \mathcal{B}_{SO}(M)$ is an orientation preserving linear isometry $(\mathbb{R}, \langle\cdot,\cdot\rangle) \to (T_pM, g_p)$, and using the normal vector, we define a subbundle $\mathcal{B}_{SO}(\partial M) \subset \mathcal{B}_{SO}(M)|_{\partial M}$ by

$$\mathcal{B}_{SO}(\partial M) = \left\{(p, b) \in \mathcal{B}_{SO}(M)|_{\partial M} \ : \ b(e_1) = N_p\right\}$$

where e_1 denotes the first standard basis vector of $\mathbb{R}^n$, and π is the bundle projection. We then get an SO_{n-1} action on $\mathcal{B}_{SO}(\partial M)$, where we include $SO_{n-1} \hookrightarrow SO_n$ as matrices of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$$

where $A \in SO_{n-1}$. This then acts on each fiber of $\mathcal{B}_{SO}(\partial M)$ by precomposition. This action is free and transitive, so this gives $\mathcal{B}_{SO}(\partial M)$ the structure of a principal SO_{n-1}-bundle over ∂M. In addition, this bundle comes with a natural map to $\mathcal{B}_O(\partial M)$, which is just the inclusion map, so it is a reduction of structure group of O_{n-1} to SO_{n-1} In this way, we see that an orientation on M determines an orientation on the boundary. The preceding discussion is summarized in the diagram

$$\begin{tikzcd}
\mathcal{B}_{SO}(\partial M) \arrow[r, hook] \arrow[dr] & \mathcal{B}_{SO}(M)|_{\partial M} \arrow[r, hook] \arrow[d] & \mathcal{B}_{SO}(M) \arrow[d] \\
& \partial M \arrow[r, hook] & M
\end{tikzcd}$$

In a similar fashion, a Spin structure on M will also induce a Spin structure on ∂M, though the process is slightly more involved. Given a Spin manifold M, it comes equipped with a principal Spin_n-bundle $\mathcal{B}_{\mathrm{Spin}}(M)$ along with a Spin_n-equivariant map $\mathcal{B}_{\mathrm{Spin}}(M) \to \mathcal{B}_{SO}(M)$, where the Spin action on $\mathcal{B}_{SO}(M)$ is induced by the double cover $\mathrm{Spin}_n \to SO_n$. Just as before, we pull back both $\mathcal{B}_{SO}(M)$ and $\mathcal{B}_{\mathrm{Spin}}(M)$ along the inclusion $\partial M \hookrightarrow M$. In addition, we construct the SO_{n-1}-bundle $\mathcal{B}_{SO}(\partial M)$, which gives us the diagram

$$\begin{tikzcd}
 & \mathcal{B}_{\mathrm{Spin}}(M)|_{\partial M} \arrow[r, hook] \arrow[d] & \mathcal{B}_{\mathrm{Spin}}(M) \arrow[d] \\
\mathcal{B}_{SO}(\partial M) \arrow[r, hook] \arrow[dr] & \mathcal{B}_{SO}(M)|_{\partial M} \arrow[r, hook] \arrow[d] & \mathcal{B}_{SO}(M) \arrow[d] \\
 & \partial M \arrow[r, hook] & M
\end{tikzcd}$$

This diagram tells us the exact ingredients we need to construct the Spin_{n-1}-bundle over ∂M – it must be the pullback of $\mathcal{B}_{\mathrm{Spin}}(M)|_{\partial M}$ along the inclusion $\mathcal{B}_{SO}(\partial M) \hookrightarrow \mathcal{B}_{SO}(M)|_{\partial M}$, so it fits into the commutative square

$$\begin{tikzcd}
\mathcal{B}_{\mathrm{Spin}}(\partial M) \arrow[r, hook] \arrow[d] & \mathcal{B}_{\mathrm{Spin}}(M)|_{\partial M} \arrow[d] \\
\mathcal{B}_{SO}(\partial M) \arrow[r, hook] & \mathcal{B}_{\mathrm{Spin}}(M)|_{\partial M}
\end{tikzcd}$$

In addition, it comes equipped with a map $p : \mathcal{B}_{\mathrm{Spin}}(\partial M) \to \partial M$ by composing the map $\mathcal{B}_{\mathrm{Spin}}(\partial M) \to \mathcal{B}_{SO}(\partial M)$ with the projection $\pi : \mathcal{B}_{SO}(\partial M) \to \partial M$. However, it is not immediately clear that the pullback bundle, (which we will suggestively denote by $\mathcal{B}_{\mathrm{Spin}}(\partial M)$) is indeed a principal Spin_{n-1}-bundle over ∂M. We have a SO_{n-1} action on $\mathcal{B}_{SO}(\partial M) \subset \mathcal{B}_{SO}(M)|_{\partial M}$ via the subgroup of matrices of the form

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \ : \ A \in SO_{n-1} \right\}$$

The preimage of this subgroup under the double covering $\mathrm{Spin}_n \to SO_n$ is a subgroup isomorphic to Spin_{n-1}, giving us an action of Spin_{n-1} on $\mathcal{B}_{\mathrm{Spin}}(M)|_{\partial M}$ by restriction. In addition, this action preserves the preimage of $\mathcal{B}_{SO}(M)|_{\partial M}$ under the double covering $\mathcal{B}_{\mathrm{Spin}}(M)|_{\partial M} \to \mathcal{B}_{SO}(M)|_{\partial M}$, which is exactly the pullback $\mathcal{B}_{\mathrm{Spin}}(\partial M)$. Then since the action of Spin_n is free, the restricted action of Spin_{n-1} on $\mathcal{B}_{\mathrm{Spin}}(\partial M) \subset \mathcal{B}_{\mathrm{Spin}}(M)|_{\partial M}$ is free, and is transitive by construction. Therefore, $\mathcal{B}_{\mathrm{Spin}}(\partial M)$ is the principal Spin_{n-1}-bundle over ∂M that we desire. All in all, the construction is summarized by the diagram

$$\begin{tikzcd}
\mathcal{B}_{\mathrm{Spin}}(\partial M) \arrow[r, hook] \arrow[d] & \mathcal{B}_{\mathrm{Spin}}(M)|_{\partial M} \arrow[r, hook] \arrow[d] & \mathcal{B}_{\mathrm{Spin}}(M) \arrow[d] \\
\mathcal{B}_{SO}(\partial M) \arrow[r, hook] \arrow[dr] & \mathcal{B}_{SO}(M)|_{\partial M} \arrow[r, hook] \arrow[d] & \mathcal{B}_{SO}(M) \arrow[d] \\
 & \partial M \arrow[r, hook] & M
\end{tikzcd}$$

Example 3.3 (The induced Spin structure on ∂D^2). Let $D^2 = \left\{ v \in \mathbb{R}^2 \ : \ |v| \leq 1 \right\}$ be the 2-disk, equipped with the Riemannian metric inherited from $\mathbb{R}^2$. We have that $\partial D^2 = S^1$. Since D^2 is contractible, both the oriented orthonormal frame bundle $\mathcal{B}_{SO}(D^2)$ and the spin bundle $\mathcal{B}_{\mathrm{Spin}}(D^2)$ inherited from $\mathbb{R}^2$ are trivial bundles, so their restrictions onto ∂D^2 are also trivial. In addition, we have that $\mathrm{Spin}_2 \cong SO_2$, and the covering map is given by $g \mapsto g^2$. Using the orientation on D^2 inherited from $\mathbb{R}^2$, we need to construct the induced orientation on ∂D^2. To do so, we parameterize ∂D^2 by angles $\theta \in [0, 2\pi)$. Then the outward normal at each point $\theta \in \partial D^2$ is the vector $(\cos\theta, \sin\theta)$, where we use the canonical identification of T_pD^2 with $\mathbb{R}^2$. Then the bundle $\mathcal{B}_{SO}(\partial D^2) \subset \mathcal{B}_{SO}(D^2)|_{\partial D^2}$ is the bundle where the fiber of $\theta \in \partial D^2$ under the map $\mathcal{B}_{SO}(\partial D^2)$ is the matrix

$$b_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Then pulling back $\mathcal{B}_{\mathrm{Spin}}(D^2)|_{\partial D^2}$ by the inclusion $\mathcal{B}_{SO}(\partial D^2) \hookrightarrow \mathcal{B}_{SO}(D^2)|_{\partial D^2}$, we have that

$$\mathcal{B}_{\mathrm{Spin}}(\partial D^2) = \left\{ (\theta, g) \ : \ g^2 = b_\theta \right\}$$

Explicitly, this means that the fiber of $\mathcal{B}_{SO}(\partial D^2) \to \partial D^2$ over a point $\theta \in \partial D^2$ is the two point set

$$\left\{ \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, \begin{pmatrix} -\cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{pmatrix} \right\}$$

which tells us that the Spin_1-bundle $\mathcal{B}_{\mathrm{Spin}}(\partial D^2)$ is the connected double cover given by $g \mapsto g^2$. ◂

4. $\mathrm{Pin}_n^{\pm}$ Structures

The same general construction for Spin structures works for $\mathrm{Pin}_n^{\pm}$ as well. When we refer to $\mathrm{Pin}_n^{\pm}$, we refer to a fixed choice of Pin_n^{+} or Pin_n^{-}.

Definition 4.1. Let M be a smooth manifold. Then a $\mathrm{Pin}_n^{\pm}$ ***structure*** on M is a reduction of structure group for O_n to $\mathrm{Pin}_n^{\pm}$. ❖

Just like with Spin, a $\mathrm{Pin}_n^{\pm}$ structure on M induces a $\mathrm{Pin}_{n-1}^{\pm}$ on the boundary using the outward unit normal, which gives the analogous diagram

$$\begin{tikzcd}
\mathcal{B}_{\mathrm{Pin}^\pm}(\partial M) \arrow[r, hook] \arrow[d] & \mathcal{B}_{\mathrm{Pin}^\pm}(M)|_{\partial M} \arrow[r, hook] \arrow[d] & \mathcal{B}_{\mathrm{Pin}^\pm}(M) \arrow[d] \\
\mathcal{B}_{O}(\partial M) \arrow[r, hook] \arrow[dr] & \mathcal{B}_{O}(M)|_{\partial M} \arrow[r, hook] \arrow[d] & \mathcal{B}_{O}(M) \arrow[d] \\
 & \partial M \arrow[r, hook] & M
\end{tikzcd}$$

Example 4.2 (Pin structures on S^1). There are two problems to discuss here, since Pin_1^{+} and Pin_1^{-} are different groups, namely

$$\mathrm{Pin}_1^{+} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$
$$\mathrm{Pin}_1^{-} \cong \mathbb{Z}/4\mathbb{Z}$$

The isomorphism $\mathrm{Pin}_1^{+} \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ maps $-1 \mapsto (1,0)$ and $e_1 \mapsto (0,1)$, and the isomorphism $\mathrm{Pin}_1^{-} \to \mathbb{Z}/4\mathbb{Z}$ maps $e_1 \mapsto 1$.

We do the Pin_1^{+} case first. In this case, the Clifford algebra $\mathrm{Cliff}_{1,0}(\mathbb{R})$ is isomorphic to $\mathbb{R} \times \mathbb{R}$, where on the basis $\{1, e_1\}$, we map $1 \mapsto 1$ and $e_1 \mapsto (-1,1)$. There are two irreducible modules corresponding to projection onto one of the factors, so the Pinor representations are the one dimensional real representations $\mathbb{P}^{+}$ and $\mathbb{P}^{-}$, where the action of e_1 on $\mathbb{P}^{\pm}$ is by ± 1. Let $\mathbb{P} = \mathbb{P}^{+} \oplus \mathbb{P}^{-}$. We then take inventory of all the principal Pin_1^{+}-bundles over S^1. From before, we know that these are classified by homomorphisms $\pi_1(S^1, \theta_0) \to \mathrm{Pin}_1^{+}$ up to conjugation. Since Pin_1^{+} is abelian, the conjugation action is trivial, so principal Pin_1^{+}-bundles are classified by homomorphisms $\pi_1(S^1, \theta_0) \to \mathrm{Pin}_1^{+}$. There are 4 such homomorphisms, which are all determined by the image of a generator α of $\pi_1(S^1, \theta_0)$, since the fundamental group is infinite cyclic. We denote all the bundles $P_g \to S^1$, where g is the image of α under the corresponding homomorphism.

(1) The bundle $P_1 \to S^1$ has 4 connected components, each diffeomorphic to a circle. The map $P - 1 \to S^1$ maps each component diffeomorphically onto S^1, and the action of Pin_1^+ permutes these 4 components.
(2) The bundle $P_{-1} \to S^1$ has two components, each diffeomorphic to a circle and double covering S^1. The element -1 acts by rotating both circles by π, and the element e_1 exchanges the two circles.
(3) The bundle $P_{e_1} \to S^1$ has two components, each diffeomorphic to a circle and double covering S^1. The action of e_1 rotates both circles by π, and -1 exchanges the two circles.
(4) The bundle $P_{-e_1} \to S^1$, has two components, each diffeomorphic to a circle and double covering S^1. The element $-e_1$ acts by rotation by π, and the element -1 exchanges the two circles.

We then must determine which bundles admit Pin_1^+-equivariant maps $P_g \to \mathcal{B}_O(S^1)$, which is a pair of disjoint circles permuted by $O_1 = \{\pm 1\}$. Under the double covering, the elements $\pm 1 \in \mathrm{Pin}_1^+$ act trivially, while the elements $\pm e_1$ act by -1.

(1) The bundle $P_1 \to S^1$ admits a Pin_1^+-equivariant map $P_1 \to \mathcal{B}_O(S^1)$. After mapping one of the components diffeomorphically onto one of the components of $\mathcal{B}_O(S^1)$, we know how to map the other 3 components to make the map Pin_1^+-equivariant.
(2) The bundle $P_{-1} \to S^1$ can also define a Pin_1^+ structure on S^1. Since -1 rotates both circles by π, we map each component to a component of $\mathcal{B}_O(S^1)$ via the double cover $z \mapsto z^2$.
(3) The bundle $P_{e_1} \to S^1$ cannot determine a Pin_1^+ structure. The action of e_1 rotates both components of P_{e_1} by π, but permutes the components of $\mathcal{B}_O(S^1)$. Any Pin_1^+-equivariant map $P_{e_1} \to \mathcal{B}_O(S^1)$ must map one component of P_{e_1} onto two components of $\mathcal{B}_O(S^1)$, which is impossible
(4) The bundle $P_{-e_1} \to S^1$ also cannot determine a Pin_1^+ structure by the same reason that P_{e_1} cannot.

One thing to note is that an orientation of S^1 and a Pin_1^+ structure on S^1 determine a Spin structure. A choice of component of $\mathcal{B}_O(S^1)$ determines an orientation, and taking the preimage of that component under the maps $P_{\pm 1} \to \mathcal{B}_O(S^1)$, we recover a Spin structure. In particular, the bundle P_1 corresponds to the Spin structure with Dirac operator $i\partial_\theta$ and the bundle P_{-1} corresponds to the Spin structure with Dirac operator $i\partial_\theta - \frac{1}{2}$. Indeed, the associated bundles $P_{\pm 1} \times_{\mathrm{Pin}_1^+} \mathbb{P}$ have the same Dirac operators as their corresponding Spin structures.

For the case of Pin_1^-, there are again 4 different principal Pin_1^- bundles, classified by homomorphisms $\pi_1(S^1, \theta_0) \to \mathrm{Pin}_1^-$. As before, we let α denote a generator for $\pi_1(S^1, \theta_0)$, and let $P_g \to S^1$ denote the principal Pin_1^- bundle corresponding to the homomorphism mapping $\alpha \mapsto g$.

(1) The bundle $P_1 \to S^1$ has 4 components cyclically permuted by the action of e_1.
(2) The bundle $P_{-1} \to S^1$ has 2 components, both double covering S^1. the element -1 acts by rotation by π on both factors, and the action of e_1 permutes the two components.
(3) The bundle $P_{e_1} \to S^1$ corresponds to the 4-fold covering $S^1 \to S^1$ given by $z \mapsto z^4$. The action of e_1 acts by rotation by $\pi/2$.
(4) The bundle $P_{-e_1} \to S^1$ also corresponds to the 4-fold covering, $S^1 \to S^1$, where the action of $-e_1$ is given by rotation by $\pi/2$.

Again, we need to figure out which bundles can determine Pin_1^- structures on S^1.

(1) The bundle P_1 admits a map $P_1 \to \mathcal{B}_O(S^1)$. After mapping any component diffeomorphically onto a component of $\mathcal{B}_O(S^1)$, the rest of the map is determined.
(2) The bundle P_{-1} admits a map $P_{-1} \to \mathcal{B}_O(S^1)$, where each component maps onto a component of $\mathcal{B}_O(S^1)$ via the double cover $z \mapsto z^2$.

(3) This bundle cannot determine a Pin_1^- structure, since there cannot exist a continuous surjective map from a connected space to a disconnected space.
(4) This bundle cannot determine a Pin_1^- structure by the same reasoning as above.

Again, we see that the Pin_1^- structures on S^1 correspond to Spin structures once we fix an orientation, and the associated bundles to these Pin_1^--bundles will have the same Dirac operator as their corresponding spin structures. ◀

On the circle, we observed that a $\mathrm{Pin}^{\pm}$ structure and and orientation are equivalent data to a Spin structure. This is true in general via the same construction. A $\mathrm{Pin}^{\pm}$ structure is the data of a principal $\mathrm{Pin}_n^{\pm}$-bundle $\mathcal{B}_{\mathrm{Pin}^{\pm}}(M) \to M$ along with a $\mathrm{Pin}_n^{\pm}$-equivariant map $\mathcal{B}_{\mathrm{Pin}^{\pm}}(M) \to \mathcal{B}_O(M)$. An orientation determines a subset $\mathcal{B}_{SO}(M) \subset \mathcal{B}_O(M)$ that is a principal SO_n-bundle over M, and taking the preimage of $\mathcal{B}_{SO}(M)$ under the map $\mathcal{B}_{\mathrm{Pin}^{\pm}}(M) \to \mathcal{B}_O(M)$ is a principal Spin_n-bundle over M, since the preimage of $SO_n \subset O_n$ under the double covers $\mathrm{Pin}_n^{\pm} \to O_n$ is a subgroup isomorphic to Spin_n. The restriction of $\mathcal{B}_{\mathrm{Pin}^{\pm}}(M) \to \mathcal{B}_O(M)$ then determines the desired Spin structure.

Unlike Spin structures, $\mathrm{Pin}^{\pm}$ structures do not require an orientation, and can be defined on nonorientable manifolds.

Example 4.3 ($\mathrm{Pin}^{\pm}$ structures on the Möbius band). Let M denote the Möbius band, which is the quotient space

$$M = [-1,1] \times [-1,1]/(-1,x) \sim (1,-x)$$

The square $S = [-1,1] \times [-1,1]$ inherits a Riemannian metric from $\mathbb{R}^2$, and the orthonormal frame bundle $\mathcal{B}_O(S)$ is isomorphic to the product bundle $S \times O_2$. This then descends to M after we specify gluing data for the identified edges of the square. The identification of the edges reverses the direction in the second factor, which corresponds to a transformation by the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

giving us that the orthonormal frame bundle $\mathcal{B}_O(M)$ of M induced by the metric on S is isomorphic to

$$S \times O_2/(-1,x,b) \sim (1,-x,A\cdot b)$$

We do the Pin_2^+ case first. Inside of $\mathrm{Cliff}_{2,0} \cong M_2\mathbb{R}$, the group Pin_2^+ is topologically two disjoint circles. The identity component is

$$\mathrm{Spin}_2 = \{\cos\theta + \sin\theta e_1 e_2 \ : \ \theta \in \mathbb{R}\}$$

and the second component is the subset $e_1 \cdot \mathrm{Spin}_2$. The square S admits a single principal Pin_2^+-bundle, which is the trivial bundle $S \times \mathrm{Pin}_2^+$, and any principal Pin_2^+-bundle over M will come by specifying gluing data of $S \times \mathrm{Pin}_2^+$ on the identified edges of S. More explicitly, any principal Pin_2^+-bundle over M will arise as a quotient

$$S \times \mathrm{Pin}_2^+/(-1,x,g) \sim (1,-x,\psi(g))$$

where $\psi : \mathrm{Pin}_2^+ \to \mathrm{Pin}_2^+$ is a Pin_2^+-equivariant diffeomorphism. From this characterization, we can determine which maps ψ determine a principal Pin_2^+-bundle $P \to M$ that admit a Pin_2^+-equivariant map $P \to \mathcal{B}_O(M)$. For points that that are not identified, i.e. points on S with first component not equal to ± 1, we can specify $P \to \mathcal{B}_O(M)$ fiberwise by the standard double cover $\mathrm{Pin}_2^+ \to O_2$. On the identified edges, in order for the map to be Pin_2^+-equivariant, we require that the image of the equivalence class $[-1,x,g] \in P$ to be $[-1,x,\rho(g)]$, where $\rho : \mathrm{Pin}_2^+ \to O_2$ is the double cover. However, we have that $[-1,x,g] = [1,-x,\psi(g)]$ and $[-1,x,\rho(g)] = [1,-x,A\cdot\rho(g)]$, so we need $\psi(g)$ to be one of the preimages of $A \cdot \rho(g)$ under the double cover in order for this to be well defined. Since A is an orthogonal transformation, it has two preimages under the double cover, which are

$\rho^{-1}(A) = \pm e_1$. Therefore, ψ must be left multiplication by either e_1 or $-e_1$, giving us two Pin^+ structures on M.

We then want to determine the induced Pin_1^+ structure on the boundary. Using the standard coordinates on S inherited from $\mathbb{R}^2$, the outward normal on the top edge of the square points upwards, while the outward normal points downwards on the bottom edge of the square. Therefore, the fiber of a point on the top edge under $\mathcal{B}_O(M) \to M$ is the two point set

$$\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

and the fiber over a point on the bottom edge is

$$\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\}$$

and on the endpoints, we have the identifications

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

We then need to compute the preimages of the matrices under the double cover $\rho : \mathrm{Pin}_2^+ \to O_2$ to find the fiber under $\mathcal{B}_{\mathrm{Pin}^+}(M) \to M$. We compute

$$\rho^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \left\{ \pm \frac{\sqrt{2}}{2}(e_1 + e_2) \right\}$$

$$\rho^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \left\{ \pm \frac{\sqrt{2}}{2}(-1 + e_1 e_2) \right\}$$

$$\rho^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \left\{ \pm \frac{\sqrt{2}}{2}(1 + e_1 e_2) \right\}$$

$$\rho^{-1} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \left\{ \pm \frac{\sqrt{2}}{2}(-e_1 + e_2) \right\}$$

Depending on our choice of $\pm e_1$ for the gluing map at the endpoints, we get different identifications on the endpoints of the top edge and the endpoints on the bottom edge, but for either we choose, the resulting bundle over $\partial M \cong S^1$ has four connected components, where each of the 4 elements in the fibers of the top edge of the square pair with one of the elements in the fibers of the bottom edge, giving 4 circles. If we orient the boundary of M, this gives us the nonbounding Spin structure corresponding to the disconnected double cover of S^1.

We now do the case of Pin_2^-. Most of the discussion with Pin_2^+ carries over to Pin_2^-, up until the point where we compute the preimages of the matrices under double cover. Let $\varphi : \mathrm{Pin}_2^- \to O_2$ be the double cover. We compute the preimages

$$\varphi^{-1}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \left\{\pm\frac{\sqrt{2}}{2}(e_1+e_2)\right\}$$

$$\varphi^{-1}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = \left\{\pm\frac{\sqrt{2}}{2}(1+e_1e_2)\right\}$$

$$\varphi^{-1}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = \left\{\pm\frac{\sqrt{2}}{2}(-1+e_1e_2)\right\}$$

$$\varphi^{-1}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix} = \left\{\pm\frac{\sqrt{2}}{2}(-e_1+e_2)\right\}$$

This defines a Pin_2^- structure on ∂M that has two connected components. To see this, we glue the fiber over top right corner with the fiber over the bottom left corner via left multiplication by e_1, giving the identifications

$$\pm\frac{\sqrt{2}}{2}(e_1+e_2) \sim \pm\frac{\sqrt{2}}{2}(-1+e_1e_2)$$
$$\pm\frac{\sqrt{2}}{2}(1+e_1e_2) \sim \mp\frac{\sqrt{2}}{2}(-e_1+e_2)$$

The identifications of the fiber over the bottom right corner with the fiber over the top left corner are

$$\pm\frac{\sqrt{2}}{2}(-1+e_1e_2) \sim \mp\frac{\sqrt{2}}{2}(e_1+e_2)$$
$$\pm\frac{\sqrt{2}}{2}(-e_1+e_2) \sim \pm\frac{\sqrt{2}}{2}(1+e_1e_2)$$

The two flipped signs connects two of the components of the fiber together, giving two components instead of the four components in the case with Pin_2^+, so after fixing an orientation of ∂M, we get the Spin structure corresponding to the bounding Spin structure, i.e. the Spin structure associated to the connected double cover of S^1. ◂

To every smooth manifold M, there is an associated oriented manifold called the orientation double cover, which is a principal $\mathbb{Z}/2\mathbb{Z}$-bundle encoding information about the orientability of M.

Definition 4.4. Let M be a smooth manifold. Then the ***orientation double cover*** is the set

$$\tilde{M} = \{(p,o) \;:\; p \in M, o \text{ is an orientation of } T_pM\}$$

This comes with a natural map $\tilde{M} \to M$ mapping $(p,o) \mapsto p$, and a natural $\mathbb{Z}/2\mathbb{Z}$ action where $(p,o) \mapsto (p,-o)$. Local coordinates on M induce local coordinates on $\tilde{M}$, which we can use to define the topology and smooth structure. Under this topology and smooth structure, the map $\tilde{M} \to M$ is a smooth double covering. ❖

The double cover is a local diffeomorphism, so its differential induces identifications of tangent spaces $T_{(p,o)}\tilde{M} \to T_pM$, which allows us to define a canonical orientation on $\tilde{M}$ where we orient $T_{(p,o)}\tilde{M}$ with the orientation o. The orientation double cover detects orientability of M in the following way.

Theorem 4.5.

(1) *If M is orientable, $\tilde{M}$ is to diffeomorphic the disconnected double cover $M \coprod M$.*
(2) *If M is not orientable, $\tilde{M}$ is connected. Furthermore, $\tilde{M}$ is unique in the following way: Given another double cover $\hat{M} \to M$ of an oriented manifold $\hat{M}$ onto M, there is a unique orientation preserving diffeomorphism $\varphi : \hat{M} \to \tilde{M}$ such that*

$$\begin{tikzcd} \hat{M} \arrow[rr, "\varphi"] \arrow[dr] & & \tilde{M} \arrow[dl] \\ & M & \end{tikzcd}$$

commutes.

In particular, given an unorientable manifold M, any oriented double cover of M is isomorphic to the orientation double cover.

Example 4.6. The cylinder $S^1 \times I$ can be realized as the orientation double cover of the Möbius band M. Let $f : S^1 \times I \to S^1 \times I$ be the map flipping the cylinder. Then the quotient space $S^1 \times I/(x \sim f(x))$ is diffeomorphic to the Möbius band, and the quotient map is the double cover. ◂

Theorem 4.7. *A* $\mathrm{Pin}^\pm$ *structure on a manifold induces a Spin structure on the orientation double cover.*

Proof. Let $\pi : \tilde{M} \to M$ denote the orientation double cover. Fix a Riemannian metric g on M. This induces a Riemannian metric on $\tilde{M}$, which is the pullback metric $\pi^* g$. In addition, the orthonormal frame bundle $\mathcal{B}_O(\tilde{M})$ with respect to the metric $\pi^* g$ is isomorphic to the pullback bundle $\pi^*\mathcal{B}_O(M)$. Pulling back $\mathcal{B}_{\mathrm{Pin}^\pm}(M)$ along the map $\pi^*\mathcal{B}_O(\tilde{M})$, we get the a $\mathrm{Pin}_n^\pm$-bundle $\mathcal{B}_{\mathrm{Pin}^\pm}$ over $\tilde{M}$ with a map to $\pi^*\mathcal{B}_O(M) \cong \mathcal{B}_O(\tilde{M})$, so it defines a $\mathrm{Pin}^\pm$ structure on $\tilde{M}$, giving us the diagram

$$\begin{tikzcd} \mathcal{B}_{\mathrm{Pin}^\pm}(\tilde{M}) \arrow[r] \arrow[d] & \mathcal{B}_{\mathrm{Pin}^\pm}(M) \arrow[d] \\ \pi^*\mathcal{B}_O(M) \arrow[r] \arrow[d] & \mathcal{B}_O(M) \arrow[d] \\ \tilde{M} \arrow[r, "\pi"'] & M \end{tikzcd}$$

Then since $\tilde{M}$ is oriented, and an orientation along with a $\mathrm{Pin}^\pm$ structure determines a Spin structure, we get a Spin structure on $\tilde{M}$. ■

CHAPTER 3

Dirac Operators on Manifolds

Not all the geometrical structures are "equal". It would seem that the Riemannian and complex structures, with their contacts with other fields of mathematics and with their richness in results, should occupy a central position in differential geometry. A unifying idea is the notion of a G-structure, which is the modern version of an equivalence problem first emphasized and exploited in its various special cases by Élie Cartan.
– Shiing-Shen Chern

1. Connections

We previously defined the Dirac operator D on $\mathbb{R}^n$ and explored some instances of D in low dimensions. Somewhat surprisingly, the generalization of the Dirac operator to the nonlinear world of manifolds provides an incredibly general framework for first order differential operators on Riemannian manifolds. The construction of D on $\mathbb{R}^n$ implicitly used the Riemannian geometry on $\mathbb{R}^n$, along with its associated Spin structure. To explore the Dirac operator in more generality on manifolds, we need to develop some of the geometry necessary in order to construct all of the necessary pieces to discuss the Dirac operator.

Definition 1.1. Let M be a smooth manifold. A ***distribution*** on X is a vector subbundle $E \subset TX$. ❖

Definition 1.2. let $\pi : E \to M$ be a fiber bundle. The map π determines a distinguished subspace $\ker d\pi_e \subset T_eE$, called the ***vertical subspace*** which defines a distribution V over E called the ***vertical distribution***. A ***connection*** on E is another distribution $H \subset TE$ such that at every point $e \in E$, we have $V_e \oplus H_e = T_eE$. We also refer to the distribution H as a ***horizontal distribution***. In other words a connection is a choice of splitting of the short exact sequence of vector bundles

$$0 \longrightarrow V \longrightarrow TE \longrightarrow \pi^*TM \longrightarrow 0$$

If E is a principal G-bundle, we ask for the horizontal distribution H to be G-invariant, i.e. $H_{p\cdot g} = d(R_g)_p(H_p)$, where $R_g : E \to E$ denotes the right action of $g \in G$. ❖

Note that for a point $e \in E$, the projection map $d\pi_e$ restricted to the horizontal subspace H_e is a linear isomorphism $H_e \to T_{\pi(e)}X$.

Definition 1.3. Let $E \to M$ be a vector bundle. A E***-valued*** k***-form*** is a section of the vector bundle $\Lambda^kT^*M \otimes E \to M$. We denote the space of E-valued k-forms by $\Omega^k_M(E)$. In the case that $E \to M$ is a trivial bundle $E = M \times V$ for some vector space V, we abbreviate this as a V-valued k form, and the space of V-valued k-forms is denoted $\Omega^k_M(V)$. In a local trivialization of E, an element $\omega \in \Omega^k_M(E)$ can be thought of as a vector of k-forms. ❖

We are mostly concerned with connections on principal bundles, which have a richer structure than connections on a general fiber bundle. Let $\pi : P \to M$ be a principal G-bundle, and let $\mathfrak{g}$

denote the Lie algebra of G. Let $\exp : \mathfrak{g} \to G$ denote the exponential map. Fix $x \in M$, a point $p \in \pi^{-1}(x)$, and an element $X \in \mathfrak{g}$. This determines a curve $\gamma_X : I \to \pi^{-1}(x)$ with $\gamma_X(0) = p$ given by

$$\gamma_X(t) = p \cdot \exp(tX)$$

Since the action of G preserves the fiber $\pi^{-1}(x)$, this curve lies entirely in the fiber $\pi^{-1}(x)$, so the tangent vector $\dot{\gamma}_X$ at $t = 0$ is an element of the vertical space. In addition, since the action of G is free, the tangent vector $\dot{\gamma}_X = 0$ if and only if $X = 0$, so the linear mapping $X \mapsto \dot{\gamma}_X$ is injective, and consequently, an isomorphism $\mathfrak{g} \to V_p$. Doing this over all points of the fiber, given $X \in \mathfrak{g}$, we get a vector field $\tilde{X}$ on the fiber $\pi^{-1}(x)$ by differentiating the right action of $\exp(tX)$.

Defining these vector fields fiberwise then gives an isomorphism $\underline{\mathfrak{g}} \to V$, where $\underline{\mathfrak{g}}$ denotes the trivial bundle $P \times \mathfrak{g}$. Under this isomorphism, the exact sequence of vector bundles becomes

$$0 \longrightarrow \underline{\mathfrak{g}} \longrightarrow TP \longrightarrow \pi^* TM \longrightarrow 0$$

One important thing to note is how these vector fields transform under the right action of G.

Proposition 1.4. *Let $X \in \mathfrak{g}$, and let $\tilde{X}$ be the vector field on P induced by X. Let $g \in G$, and let $R_g : P \to P$ denote the diffeomorphism given by right multiplication by g. Then*

$$(R_g)_*(\tilde{X}) = \widetilde{\operatorname{Ad}_{g^{-1}} X}$$

Where $\operatorname{Ad}_{g^{-1}} : \mathfrak{g} \to \mathfrak{g}$ is the linear map obtained by differentiating the conjugation action $h \mapsto g^{-1}hg$ at the identity element of G.

Proof. Let $p \in P$. By definition, we have that

$$\tilde{X}_p = \left.\frac{d}{dt}\right|_{t=0} p \cdot \exp(tX)$$

Therefore, we have that

$$\begin{aligned}
[(R_g)_*(\tilde{X})]_{p \cdot g} &= d(R_g)_p \left(\left.\frac{d}{dt}\right|_{t=0} p \cdot \exp(tX) \right) \\
&= \left.\frac{d}{dt}\right|_{t=0} p \cdot \exp(tX) g \\
&= \left.\frac{d}{dt}\right|_{t=0} p \cdot g (g^{-1} \exp(tX) g) \\
&= \left(\widetilde{\operatorname{Ad}_g^{-1} X} \right)_{p \cdot g}
\end{aligned}$$

■

Over a point $p \in P$, we have a direct sum decomposition $T_pP = V_p \oplus H_p$, giving us a projection map $T_pP \to V_p$. Using the identification of V_p with $\mathfrak{g}$, this then determines a linear map $\omega_p : T_pP \to \mathfrak{g}$, where $\ker \omega_p = H_p$. Doing this over all fibers $\pi^{-1}(p)$, this determines a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1_P(\mathfrak{g})$ called the ***connection* 1*-form***. From the connection 1-form ω, we can recover H as the kernel of ω. In addition, it is clear from the definition that given $X \in \mathfrak{g}$, we have $\omega_p(\tilde{X}_p) = X$.

In the case of a vector bundle $E \to M$, we can also construct a connection 1-form. Given any point $p \in E_x$, we get a linear isomorphism of the fiber E_x with the vertical space $V_p \subset T_pE$ via the mapping

$$v \mapsto \left.\frac{d}{dt}\right|_{t=0} p + tv$$

A horizontal distribution H then gives us a projection map $T_pE \to V_p$ on each tangent space, and composing with the inverse isomorphism $V_p \to E_x$, we get a linear map $\omega_p : T_pP \to E_x$. Doing this over all fibers gives and π^*E valued 1-form $\omega \in \Omega^1_E(\pi^*E)$, which is also called the connection 1-form. As with principal bundles, the horizontal distribution is recovered as the kernel of ω.

Proposition 1.5. *Let $\pi : P \to M$ be a principal G-bundle and $\omega \in \Omega^1_P(\mathfrak{g})$ a connection 1-form on P. Then for $g \in G$, $R_g^*\omega = \operatorname{Ad}_{g^{-1}} \omega$. where R_g^* denotes pullback by the right action of g on P.*

Proof. Fix $p \in P$, and let $v \in T_pP$. Since $T_pP = V_p \oplus H_p$, we have that $v = \tilde{X}_p + h$ for $X \in \mathfrak{g}$ and $h \in H_p$. We then compute

$$\begin{aligned}(R_g^*\omega)_p(v) &= (R_g^*\omega)_p(\tilde{X}_p + h)\\ &= \omega_{p\cdot g}((R_g)_*(\tilde{X}_p) + (R_g)_*h)\end{aligned}$$

Then since $(R_g)_*\tilde{X} = \widetilde{\operatorname{Ad}_{g^{-1}} X}$ and $(R_g)_*h \in H_{p\cdot g}$, we get that

$$\begin{aligned}\omega_{p\cdot g}((R_g)_*(\tilde{X}_p) + (R_g)_*h) &= \omega_{p\cdot g}((\widetilde{\operatorname{Ad}_{g^{-1}} X})_{p\cdot g} + 0\\ &= \operatorname{Ad}_{g^{-1}} X\\ &= \operatorname{Ad}_{g^{-1}} \omega(\tilde{X}_p + h)\end{aligned}$$

■

We mentioned before that principal G-bundles can be thought of as generalizations of covering spaces, since for discrete G, a principal G-bundle is a covering space. One notion we want to generalize is the notion of path lifting. Given a covering space $\pi : P \to M$, a point $x \in M$, a point $p \in \pi^{-1}(x)$ in the fiber, and a curve $\gamma : I \to M$ with $\gamma(0) = x$, we use the fact that p lies in a unique sheet $U_\alpha \subset \pi^{-1}(U)$ for some evenly covered neighborhood U of x to lift γ to a path $\tilde{\gamma} : I \to P$ starting at p. This relies on the fact that π restricted to U_α is a diffeomorphism onto U. If G is not discrete, we have no hope of this working without additional data since P has a larger dimension than M. The connection $H \subset TP$ is exactly the data we need for path lifting. Using the same setup as above, the curve γ determines a vector field V along its image given by its velocity $V_{\gamma(t)} = \dot{\gamma}(t)$. Then since $d\pi_p|_{H_p} : H_p \to T_{\pi(p)}M$ is an isomorphism, there exists a unique horizontal vector $h_p \in T_pP$ at each point $p \in \pi^{-1}(\gamma(t))$ satisfying $d\pi_p(h_p) = V_{\gamma(t)}$, which determines a vector field along the pullback bundle γ^*TP, which can naturally be thought of as a subset of TP. Integrating these vector fields gives a flow along the pullback bundle, and following the flow line starting at our favorite point in the fiber $\pi^{-1}(x)$ gives us a path $\tilde{\gamma} : I \to P$ such that $\pi \circ \tilde{\gamma} = \gamma$, i.e. a lift of γ. One important thing to note is how path lifting interacts with the group action.

Proposition 1.6. *Let $\pi : P \to M$ be a principal G-bundle equipped with a connection $H \subset TP$, and let $\gamma : I \to M$ be a path in M with $\gamma(0) = x$. Then let $\tilde{\gamma} : I \to P$ be the lift of γ starting at p. Then the lift of γ starting at $p \cdot g$ is $\gamma \circ R_g$, where R_g is the right action of $g \in G$.*

Proof. Since the action of R_g preserves the fibers of π, $\tilde{\gamma} \circ R_g$ satisfies $\pi \circ \gamma \circ R_g = \gamma$. By uniqueness of integral curves horizontal lift of the velocity vector field, $\gamma \circ R_g$ must be the lift of γ based at $p \cdot g$. ■

Path lifting gives us a notion of ***parallel transport*** – an identification of the fibers along a path on M.

Definition 1.7. Let $\pi : P \to M$ be a principal G-bundle with $H \subset TP$ a connection. Let $\gamma : I \to M$ be a path starting at $x \in M$. Let $\gamma_p : I \to P$ denote the lift of γ starting at $p \in \pi^{-1}(x)$. Define the ***parallel transport*** maps $\tau_t : P_x \to P_{\gamma(t)}$ by $\tau_t(p) = \gamma_p(t)$ ❖

We note that the maps τ_t are isomorphisms, since the inverse map is given by parallel transport along the curve $\overline{\gamma}(t) = \gamma(1-t)$. The parallel transport maps justify the naming of a connection – a connection provides the necessary data to connect nearby fibers along a curve γ. This notion of path lifting carries over to any associated bundle $P \times_G F$, which is one way to see that the connection on P induces a connection on every associated bundle. As before, let $\gamma : I \to M$ be a path with $\gamma(0) = x$, and let $[p, f] \in P \times_G F$ in the fiber over x, i.e. $p \in \pi^{-1}(x)$. Let $\tilde{\gamma} : I \to P$ denote the lift of γ to P starting at p. Then define $\widehat{\gamma} : I \to P \times_G F$ by $\widehat{\gamma}(t) = [\tilde{\gamma}(t), f]$, which is visibly a lift of γ to $P \times_G F$ starting at $[p, f]$. However, we should check that this is independent of our choice of representative $[p, f]$. Let $[p \cdot g, g^{-1} \cdot f]$ be another representative of the point $[p, f]$. Then the lift of γ to P starting at $p \cdot g$ is the curve $\tilde{\gamma} \cdot g$, so using this representative, we would define $\widehat{\gamma}(t) = [\tilde{\gamma} \cdot g, g^{-1} \cdot f] = [\tilde{\gamma}(t), f]$, so this definition does not rely on our choice of representative. Another way to see how the connection on P induces a connection on the associated bundle $P \times_G F$ is by looking at the horizontal distribution $H \subset TP$. This includes into the tangent bundle $T(P \times F)$ as the subbundle $H \times \{0\}$, where 0 denotes the zero section of TF. The G-invariance of H then guarantees that H descends to to quotient $P \times F/G = P \times_G F$, giving a connection on $P \times_G F$.

This induced connection has a very nice interpretation when we have an associated vector bundle. Let $\rho : G \to GL(W)$ be a linear representation. Then let $E = P \times_G W$ denote the associated vector bundle induced by ρ. Recall from before that there is a bijective correspondence

$$\Gamma_M(E) \longleftrightarrow \left\{ \alpha : P \to W \ : \ \alpha(p \cdot g) = \rho(g)^{-1} \cdot \alpha(p) \right\}$$

We now generalize the correspondence $\Gamma_M(E)$ with G-equivariant maps $P \to W$. Thinking of functions $P \to W$ as 0-forms on P valued in W and sections of E as 0-forms on M valued in E, we can write this correspondence more suggestively as

$$\Omega^0_M(E) \longleftrightarrow \left\{ \alpha \in \Omega^0_P(W) \ : \ R_g^* \psi = \rho(g)^{-1} \alpha \right\} \subset \Omega^0_P(W)$$

Where R_g^* denotes the pullback of α along right multiplication by $g \in G$. This notation suggests that this correspondence should generalize to arbitrary k-forms.

Proposition 1.8. *There is a bijective correspondence*

$$\Omega^k_M(E) \longleftrightarrow \left\{ \alpha \in \Omega^k_P(W) \ : \ R_g^* \alpha = \rho(g)^{-1} \alpha \ \forall g \in G, \ \iota_{\widehat{\xi}}(\alpha) = 0 \ \forall \xi \in \mathfrak{g} \right\} \subset \Omega^k_P(W)$$

Where $\iota_{\widehat{\xi}}(\alpha)$ denotes the interior multiplication of α with the vector field $\widehat{\xi}$ on P induced by ξ, i.e.

$$\iota_{\widehat{\xi}}(\alpha)(V_1, \ldots, V_{k-1}) = \alpha(\widehat{\xi}, V_1, \ldots, V_{k-1})$$

for vector fields $V_1, \ldots, V_{k-1}$. For convenience, we denote this subset of k-forms on P by $\Omega^{k,G}_P(W)$, where the superscript G reminds us that these k-forms are G-invariant in the appropriate sense.

Remark. Note that the condition $\iota_{\widehat{\xi}}(\alpha)$ hold vacuously when α is a 0-form, so this agrees with our earlier correspondence.

Proof. We provide maps in both directions. Let $\omega \in \omega^k_M(E)$. We want to define $\widetilde{\omega} \in \Omega^{k,G}_P(W)$. Using π, we pullback ω to obtain a k-form $\pi^*\omega \in \Omega^k_P(E)$, and define $\tilde{\omega}$ as follows. Given a point $p \in P$ and $v \in T_pP$, $(\pi^*\omega)_p(v) = [p, a]$, where $[p, a]$ is the unique representative in it's class with p in the first component. We then define $\tilde{\omega}_p(v) = a$. We then must check that $\tilde{\omega} \in \Omega^{k,G}_P(W)$, so we must check

(1) $\iota_\xi(\tilde{\omega}) = 0$ for any vertical vector field ξ.
(2) $R_g^* \tilde{\omega} = \rho(g)^{-1} \tilde{\omega}$.

For the first condition, Let $\xi \in V$ be a vertical vector field. Then we know that $\iota_\xi(\pi^*\omega) = 0$, since $d\pi_p(\xi_p) = 0$ by the definition of the vertical space. Therefore, $\iota_\xi(\tilde{\omega}) = 0$, verifying the first condition. For the second condition, it suffices to check the action of $\tilde{\omega}$ on vertical and horizontal vectors, since we have the direct sum decomposition $T_pP = V_p \oplus H_p$. We the only need to check the action of $\tilde{\omega}$ on horizontal vectors, since $\iota_\xi(\tilde{\omega}) = 0$ for any vertical vector field ξ and the vertical distribution is invariant under the action of G. We first prove a lemma

Lemma. *Let $v \in H_p$ be a horizontal vector. Then $d\pi_p(v) = d\pi_{p\cdot g}((R_g)_*v)$*

Proof. Let $\gamma : I \to M$ be a curve with $\gamma(0) = \pi(p)$ and velocity vector $\gamma'(0) = d\pi_p(v)$. Then let $\tilde{\gamma} : I \to P$ be the lift of γ starting at p. From before, we know that $R_g \circ \tilde{\gamma}$ is the unique lift of γ starting at $p \cdot g$, and by the chain rule, the velocity vector of the curve $R_g \circ \tilde{\gamma}$ at 0 is the vector $(R_g)_*\tilde{\gamma}'(0) = (R_g)_*v$. Therefore, $(R_g)_*v$ is the unique horizontal lift of $d\pi_p(v)$ based at $p \cdot g$, so $d\pi_{p\cdot g}((R_g)_*v) = d\pi_p(v)$. ∎

Let $v_1, \dots v_k \in H_p$ be horizontal vectors and $g \in G$. Then

$$(R_g^*\tilde{\omega})_p(v_1, \dots v_k) = \tilde{\omega}_{p\cdot g}((R_g)_*v_1, \dots, (R_g)_*v_k)$$

Then if $\tilde{\omega}_p(v_1, \dots v_k) = a$, by definition this means that $(\pi^*\omega)_p(v_1, \dots v_k) = [p, a]$. We also have that $(\pi^*\omega)_{p\cdot g}((R_g)_*v_1, \dots (R_g)_*v_k) = [p, a]$, since $\pi(p \cdot g) = \pi(p)$ and that $d\pi_p(v_i) = d\pi_{p\cdot g}(v_i)$. We know that $[p, a] = [p \cdot g, \rho(g)^{-1}a]$, so we conclude that $R_g^*\tilde{\omega} = \rho(g)^{-1}\tilde{\omega}$.

In the other direction, let $\alpha \in \Omega_P^{k,G}(W)$. We wish to define $\alpha' \in \Omega_M^k(E)$. Let $x \in M$ and $v_1, \dots v_k \in T_xM$. Then define

$$\alpha'_x(v_1, \dots, v_k) = [p, \alpha_p(\tilde{v}_1, \dots, \tilde{v}_k)]$$

where $p \in \pi^{-1}(x)$ and the $\tilde{v}_i$ are the unique horizontal lifts of the v_i at p. We first check that this definition is independent of out choice of $p \in \pi^{-1}(x)$. We know α satisfies $R_g^*\alpha = \rho(g)^{-1}\alpha$. Therefore, if we choose a different preimage $p \cdot g \in \pi^{-1}(x)$, we would define α' by

$$\begin{aligned}
\alpha'_x(v_1, \dots, v_k) &= [p \cdot g, (R_g^*\alpha)_p(\tilde{v}_1, \dots, \tilde{v}_k) \\
&= [p \cdot g, \rho(g)\alpha_p(\tilde{v}_1, \dots \tilde{v}_k)] \\
&= [p, \alpha_p(\tilde{v}_1, \dots, \tilde{v}_k)]
\end{aligned}$$

So this definition is independent of our choice of $p \in \pi^{-1}(x)$. It is then easily verified that the maps are inverses to each other. ∎

This is an example of a relatively general phenomenon, in which appropriately G-invariant objects on the total space P descend to objects on the base space M.

The correspondence $\Omega_M^k(E) \leftrightarrow \Omega_P^{k,G}(W)$ gives us a natural way of discussing and extending covariant differentiation, which gives us a way of taking directional derivatives on E with respect to tangent vectors on the base M.

Definition 1.9. Let $\pi : P \to M$ be a principal G-bundle with connection 1-form $\Theta \in \Omega_P^1(\mathfrak{g})$. Let $\rho : G \to GL(W)$ be a linear representation, and let $\dot{\rho} : \mathfrak{g} \to \operatorname{End} V$ denote the induced Lie algebra map. The ***exterior covariant derivative*** is a family of maps $d_\Theta : \Omega_P^k(W) \to \Omega_P^{k+1}(W)$ given by

$$d_\Theta\psi = d\psi + \dot{\rho}(\Theta) \wedge \psi$$

By $\dot{\rho}(\Theta)$, we mean that its action on a vector field V is

$$\dot{\rho}(\Theta)(V) = \dot{\rho}(\Theta(V))$$

which makes sense since $\Theta_p(V_p)$ is an element of $\mathfrak{g}$. Then to define $\dot{\rho}(\Theta) \wedge \psi$, we interpret $\dot{\rho}(\Theta)$ as a matrix $\dot{\rho}(\Theta)^i_j$ of 1-forms and ψ as a vector ψ^i of k-forms and define the wedge product as the usual matrix product

$$(\dot{\rho}(\Theta) \wedge \psi)_i = \dot{\rho}(\Theta)^i_j \wedge \psi^j$$

❖

We are especially interested in the restriction of the d_Θ to the subspace $\Omega^{k,G}_P(V) \cong \Omega^k_M(E)$, where E denotes the associated vector bundle $P \times_G V$. The first thing to check is that given an element $\psi \in \Omega^{k,G}_P(W)$, its image $d_\Theta \psi$ is an element of $\Omega^{k+1,G}_P(W)$.

Proposition 1.10. *Given $\psi \in \Omega^{k,G}_P(W)$, $d_\Theta\psi \in \Omega^{k+1,G}_P(W)$.*

Proof. We need to check two things:

(1) $\iota_\xi(d_\Theta\psi) = 0$ for any vertical vector field ξ.
(2) $R^*_g d_\Theta \psi = \rho(g)^{-1} d_\Theta \psi$.

For the first condition, it suffices to check for $\xi \in \mathfrak{g}$ and $\widehat{\xi}$ its corresponding vertical vector field. We then note that $\iota_{\widehat{\xi}}(\psi^j) = 0$ for all j, since $\psi \in \Omega^{k,G}_P(W)$. To compute $\iota_{\widehat{\xi}}(d\psi)$, we use a useful lemma attributed to Cartan

Lemma (***Cartan's Magic Formula***). *Let ξ be a vector field and ω a k-form, and let $\mathcal{L}_\xi$ denote the Lie derivative along ξ. Then*

$$\mathcal{L}_\xi \omega = d(\iota_\xi(\omega)) + \iota_\xi(d\omega)$$

By the lemma, we have that $\mathcal{L}_{\widehat{\xi}}\psi = d\iota_{\widehat{\xi}}(\psi) + \iota_\xi(d\psi)$. By assumption, $d(\iota_{\widehat{\xi}}(\psi)) = d(0) = 0$, so $\mathcal{L}_{\widehat{\xi}}\psi = \iota_{\widehat{\xi}}(d\psi)$. The flow of ξ is given by right multiplication $R_{\exp(t\xi)}$, so

$$\mathcal{L}_{\widehat{\xi}}\psi = \frac{d}{dt}\bigg|_{t=0} R^*_{\exp(t\xi)}\psi$$

Then since $\psi \in \Omega^{k,G}_P(W)$, we have that $R^*_{\exp(t\xi)}\psi = \rho(g)^{-1}\psi$, and using this we compute

$$\begin{aligned}
\mathcal{L}_{\widehat{\xi}}\psi &= \frac{d}{dt}\bigg|_{t=0} R^*_{\exp(t\xi)}\psi \\
&= \frac{d}{dt}\bigg|_{t=0} \rho(\exp(t\xi))^{-1}\psi \\
&= -\dot{\rho}(\xi)\psi
\end{aligned}$$

So $\mathcal{L}_{\widehat{\xi}}\psi = -\dot{\rho}(\xi)\psi = \iota_\xi(d\psi)$. To compute $\iota_{\widehat{\xi}}(\dot{\rho}(\Theta) \wedge \psi)$, we use another helpful identity regarding interior multiplication and the wedge product.

Lemma. *Let ξ be a vector field, and let ω be a k-form and let η be an ℓ-form. Then*

$$\iota_\xi(\omega \wedge \eta) = \iota_\xi(\omega) \wedge \eta + (-1)^k \omega \wedge \iota_\xi(\eta)$$

Fix a basis for W, so $\rho(\Theta)$ is a matrix of 1-forms and ψ is a vector of k-forms. The i^{th} component of $\dot{\rho}(\Theta) \wedge \psi$ is

$$(\rho(\dot{\Theta}) \wedge \psi)^i = \rho(\Theta)^i_j \wedge \psi^j$$

Then using our lemma, we compute

$$\begin{aligned}
\iota_{\widehat{\xi}}((\dot{\rho}(\Theta) \wedge \psi)^i) &= \iota_{\widehat{\xi}}(\dot{\rho}(\Theta)^i_j) \wedge \psi^j - \dot{\rho}(\Theta)^i_j \wedge \iota_{\widehat{\xi}}(\psi^j) \\
&= \dot{\rho}(\Theta(\widehat{\xi}))^i_j \psi^j \\
&= \dot{\rho}(\xi)\psi
\end{aligned}$$

Where we use the fact that $\Theta(\widehat{\xi}) = \xi$ and that $\iota_{\widehat{\xi}}(\psi^j) = 0$ for all j since $\iota_{\xi}(\psi) = 0$. Therefore, we have that

$$\iota_{\widehat{\xi}}(d_\Theta\psi) = \iota_{\widehat{\xi}}(d\psi) + \iota_{\widehat{\xi}}(\dot{\rho}(\Theta) \wedge \psi) = -\dot{\rho}(\xi)\psi + \dot{\rho}(\xi)\psi = 0$$

We now want to verify the second condition. By definition,

$$R_g^* d_\Theta \psi = R_g^* d\psi + R_g^*(\dot{\rho}(\Theta) \wedge \psi)$$

We compute the pullbacks on each term. Pullback commutes with d, so we have that

$$R_g^* d\psi = d(R_g^*\psi) = d(\rho(g)^{-1}\psi) = \rho(g)^{-1} d\psi$$

For the other term, we fix a basis for W, and we compute the i^{th} component of $R_g^*(\dot{\rho}(\Theta) \wedge \psi)$ to be

$$\begin{aligned}
R_g^*(\dot{\rho}(\Theta) \wedge \psi)^i &= R_g^*(\dot{\rho}(\Theta)^i_j \wedge \psi^j) \\
&= \dot{\rho}(R_g^*\Theta)^i_j \wedge (R_g^*\psi)^j \\
&= \dot{\rho}(\mathrm{Ad}_{g^{-1}}\Theta)^i_j \wedge (\rho(g)^{-1})^j_a \psi^a \\
&= (\rho(g)^{-1})^i_k \dot{\rho}(\Theta)^k_\ell \rho(g)^\ell_j \wedge (\rho(g)^{-1})^j_a \psi^a \\
&= (\rho(g)^{-1})^i_k \dot{\rho}(\Theta)^k_\ell \rho(g)^\ell_j (\rho(g)^{-1})^j_a \wedge \psi^a \\
&= (\rho(g)^{-1})^i_k \dot{\rho}(\Theta)^k_a \wedge \psi^a \\
&= (\rho(g)^{-1}\dot{\rho}(\Theta))^i_a \wedge \psi^a
\end{aligned}$$

Therefore, $R_g^*(\dot{\rho}(\Theta) \wedge \psi) = \rho(g)^{-1}\dot{\rho}(\Theta) \wedge \psi$ ■

We are especially interested in the covariant derivative on sections $\Omega^0_M(E)$, as ∇ is best interpreted as a directional derivative of a section ψ in the direction of a tangent vector $\xi \in T_pM$.

Definition 1.11. Let $\pi : P \to M$, $\Theta \in \Omega^1_P(\mathfrak{g})$, $\rho : G \to GL(W)$, $\dot{\rho} : \mathfrak{g} \to \operatorname{End} W$ and $E = P \times_G W$ be as above. Then given a section $\psi \in \Omega^0_M(E)$, and a vector $\xi \in T_xM$, define the ***covariant derivative*** of ψ in the direction ξ by

$$\nabla\psi(\xi) = \left.\frac{d}{dt}\right|_{t=0} \tau_t^{-1}(\psi(\tilde{\gamma}(t)))$$

where $\tilde{\gamma} : I \to E$ is the lift of a curve $\gamma : I \to M$ where $\gamma(0) = x$ and $\gamma'(0) = \xi$ and τ_t denotes the parallel transport maps along γ. ❖

Note that ∇ is an operator $\Omega^0_M(E) \to \Omega^1_M(E)$ – we've already seen one operator with the same domain and codomain, the exterior covariant derivative d_Θ. As the name suggests, these concepts agree.

Proposition 1.12. *The operator ∇ is equal to $d_\Theta : \Omega^0_M(E) \to \Omega^1_M(E)$.*

Proof. We note that the formulas we gave for ∇ and d_Θ use different interpretations of sections of $E \to M$, so we need to translate one operator into the language of the other. Let $\psi \in \Omega^0_M(E)$, and $\tilde{\psi} \in \Omega^0_P(W)$ its corresponding G-equivariant map $P \to W$. Let $\xi \in T_xM$ be a tangent vector, and $\gamma : I \to M$ a curve with $\gamma(0) = x$ and $\gamma'(0) = \xi$. Then let $\tilde{\gamma}_p : I \to P$ denote the horizontal lift of γ to the point p in the fiber P_x. The image $\psi(\gamma(t)) \in E_{\gamma(t)}$ determines a G-equivariant map $P_{\gamma(t)} \to W$, where we map each element $p \in P_{\gamma(t)}$ to the second component of its representative for $\psi(\gamma(t))$, and this is exactly the restriction $\tilde{\psi}|_{P_{\gamma(t)}}$. Under this identification, the covariant derivative is given at the point $p \in P_x$ by

$$\nabla\tilde{\psi}(\tilde{\xi}) = \left.\frac{d}{dt}\right|_{t=0} \tilde{\psi}(\tilde{\gamma}_p(t))$$

where $\tilde{\xi}$ denotes the horizontal lift of ξ to T_pP. We now verify that ∇ and d_Θ agree.

$$\begin{aligned}\nabla\tilde{\psi}(\tilde{\xi}) &= \left.\frac{d}{dt}\right|_{t=0} \tilde{\psi}(\tilde{\gamma}_p(t)) \\ &= d\tilde{\psi}(\tilde{\xi})\end{aligned}$$

We then note that since $\tilde{\xi}$ is horizontal, $(\dot{\rho}(\Theta)\psi)(\tilde{\xi}) = 0$, so

$$d_\Theta\tilde{\psi}(\tilde{\xi}) = d\tilde{\psi}(\tilde{\xi}) = \nabla\tilde{\psi}(\tilde{\xi})$$

■

We will often denote the covariant derivative of a section ψ in the direction of a tangent vector X as $\nabla_X\psi$, which is the same thing as $\nabla\psi(X)$.

2. Curvature

An important quantity associated to a connection is its ***curvature***, which heuristically measures its deviation from our standard notions of a directional derivative. We first discuss curvature on principal bundles, and will then explore its relationship with associated bundles.

Definition 2.1. Let G be a Lie group with Lie algebra $\mathfrak{g}$, and $\pi : P \to M$ a principal G-bundle. Let $\Theta \in \Omega^1_P(\mathfrak{g})$ be a connection on P. The ***curvature tensor*** is a 2-form $\Omega \in \Omega^2_P(\mathfrak{g})$ defined by the equation

$$\Omega = d\Theta + \frac{1}{2}[\Theta \wedge \Theta]$$

where the action of $[\Theta \wedge \Theta]$ on X, Y is defined to be

$$[\Theta \wedge \Theta](X, Y) = [\Theta(X), \Theta(Y)] - [\Theta(Y), \Theta(X)] = 2[\Theta(X), \Theta(Y)]$$

For this reason, some denote $1/2[\Theta \wedge \Theta]$ as just $[\Theta, \Theta]$. ❖

The first thing to note is that it descends to the base manifold M, i.e. it defines an element of $\Omega^2_M(\mathfrak{g}_P)$, where $\mathfrak{g}_P$ is the ***adjoint bundle*** of P, which is the associated vector bundle $P \times_G \mathfrak{g}$, where the action of G on $\mathfrak{g}$ is the adjoint action. This is due to the following properties of Ω.

Proposition 2.2.
(1) $R_g^*\Omega = \mathrm{Ad}_{g^{-1}}\,\Omega$
(2) $\iota_{\hat{\xi}}(\Omega) = 0$ *for all* $\xi \in \mathfrak{g}$.

Proof. For the first property, we compute

$$\begin{aligned}R_g^*\Omega &= R_g^* d\Theta + \frac{1}{2}R_g^*[\Theta \wedge \Theta] \\ &= d(R_g^*\Theta) + \frac{1}{2}[R_g^*\Theta \wedge R_g^*\Theta] \\ &= \mathrm{Ad}_{g^{-1}}\, d\Theta + \frac{1}{2}[\mathrm{Ad}_{g^{-1}}\,\Theta \wedge \mathrm{Ad}_{g^{-1}}\,\Theta] \\ &= \mathrm{Ad}_g^{-1}\,\Omega\end{aligned}$$

where we use the fact that pullback commutes with exterior derivative and the wedge product, and the fact that the adjoint action of G on $\mathfrak{g}$ commutes with brackets. For the second property, let $\xi \in \mathfrak{g}$. We then have that

$$\iota_{\hat{\xi}}(\Omega) = \iota_{\hat{\xi}}(d\Theta) + \iota_{\hat{\xi}}[\Theta \wedge \Theta]$$

We compute the value of the terms separately. To compute $\iota_{\hat{\xi}}(d\Theta)$, we use Cartan's magic formula again to get

$$\mathcal{L}_{\hat{\xi}}\Theta = d(\iota_{\hat{\xi}}(\Theta)) + \iota_{\hat{\xi}}(d\Theta) = 0 + \iota_{\hat{\xi}}(\Theta)$$

since $\iota_{\hat{\xi}}(\Theta) = \xi$ for any $\xi \in \mathfrak{g}$, and d of a constant form is 0. Therefore, $\mathcal{L}_{\hat{\xi}}\Theta = \iota_{\hat{\xi}}(d\Theta)$. We then note that

$$\begin{aligned}
\mathcal{L}_{\hat{\xi}}\Theta &= \frac{d}{dt}\bigg|_{t=0} R^*_{\exp(t\xi)}\Theta \\
&= \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_{\exp(t\xi)^{-1}}\Theta \\
&= -[\xi, \Theta]
\end{aligned}$$

where the action of $[\xi, \Theta]$ on X is $[\xi, \Theta](X) = [\xi, \Theta(X)]$, and we use the fact that the derivative of Ad is the Lie bracket. We then compute

$$\iota_{\hat{\xi}}[\Theta \wedge \Theta] = [\iota_{\hat{\xi}}(\Theta) \wedge \Theta] = [\xi, \Theta]$$

Therefore, $\iota_{\hat{\xi}}(\Omega) = 0$, so the curvature form descends to a form in $\Omega^2_M(\mathfrak{g}_P)$. ■

Having defined the curvature form, we explore its relationship with associated vector bundles of a principal bundle $\pi : P \to M$ equipped with a connection Θ. Let W be a vector space with a homomorphism $\rho : G \to GL(W)$, and $\dot{\rho} : \mathfrak{g} \to \operatorname{End} W$ its associated Lie algebra map. Then let $E \to M$ be the associated vector bundle $E = P \times_G W$. Recall that the connection Θ on P induces an exterior covariant derivative

$$\begin{aligned}
d_\Theta : \Omega^k_M(E) &\to \Omega^{k+1}_M(E) \\
\psi &\mapsto d\psi + \dot{\rho}(\Theta) \wedge \psi
\end{aligned}$$

which gives us a sequence of maps

$$\Omega^0_M(E) \xrightarrow{d_\Theta} \Omega^1_M(E) \xrightarrow{d_\Theta} \Omega^2_M(E) \xrightarrow{d_\Theta} \cdots$$

However, this does *not* form a complex, i.e. $d^2_\Theta \neq 0$. For $\psi \in \Omega^0_M(E)$, we compute

$$\begin{aligned}
d^2_\Theta\psi &= d_\Theta(d\psi + \dot{\rho}(\Theta)\psi) \\
&= d^2\psi + \dot{\rho}(\Theta) \wedge d\psi + d(\dot{\rho}(\Theta)\psi) + \dot{\rho}(\Theta) \wedge \dot{\rho}(\Theta)\psi \\
&= \dot{\rho}(\Theta) \wedge d\psi + d(\dot{\rho}(\Theta)\psi) + \dot{\rho}(\Theta) \wedge \dot{\rho}(\Theta)\psi \\
&= \dot{\rho} \wedge d\psi + \dot{\rho}(d\Theta)\psi - \dot{\rho} \wedge d\psi + \dot{\rho}(\Theta) \wedge \dot{\rho}(\Theta)\psi \\
&= \dot{\rho}(d\Theta)\psi + \dot{\rho}(\Theta) \wedge \dot{\rho}(\Theta)\psi
\end{aligned}$$

We want to relate this to the curvature form Ω on P. To do this, we use a small lemma.

Lemma 2.3.

$$\dot{\rho}(\Theta) \wedge \dot{\rho}(\Theta) = \frac{1}{2}\dot{\rho}([\Theta \wedge \Theta])$$

The proof of the lemma just involves doing the matrix computation, and using the definition of the wedge of 1-forms when computing the matrix commutator for the left hand side. We then compute

$$\begin{aligned}\dot{\rho}(\Omega) &= \dot{\rho}\left(d\Theta + \frac{1}{2}[\Theta \wedge \Theta]\right)\\ &= d(\dot{\rho}(\Theta)) + \frac{1}{2}\dot{\rho}([\Theta \wedge \Theta])\\ &= d(\dot{\rho}(\Theta)) + \dot{\rho}(\Theta) \wedge \dot{\rho}(\Theta)\\ &= d_\Theta^2 \psi\end{aligned}$$

The from this we see that curvature form Ω on P determines an $\mathrm{End}\, E$ valued $2-$form on any associated vector bundle E, and is in some sense the measurement of the failure of the sequence of maps d_Θ to be a complex, and coincides with the usual notion of curvature on vector bundles.

Definition 2.4. Let P be a principal bundle with connection Θ, and Ω the curvature form. $\rho : G \to GL(W)$ a representation, with corresponding Lie algebra representation $\dot{\rho} : \mathfrak{g} \to \mathrm{End}\, W$. Let $E = P \times_G W$. The ***curvature transformation*** is a 2-form $R \in \Omega^2(\mathrm{End}\, E)$ defined by

$$R = \dot{\rho}(\Omega)$$

Given tangent vectors V, W, we often denote the endomorphism $R(V, W)$ by $R_{V,W}$. ❖

Let (M, g) be a Riemannian manifold. The Riemannian metric G gives a reduction of structure group to O_n, giving us the orthonormal frame bundle $\mathcal{B}_O(M)$. It is a wonderful fact that we have a canonical choice of connection on $\mathcal{B}_O(M)$, where we ask for the torsion tensor

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

to vanish.

Theorem 2.5 (***Fundamental Theorem of Riemannian Geometry***). *There exists a unique torsion-free connection Θ on $\mathcal{B}_O(M)$.*

We call this connection the ***Levi-Civita connection***. If in addition, M is oriented, we have a reduction of structure group to $\mathcal{B}_{SO}(M)$, and the Levi-Civita connection restricts to this bundle. We also refer to this connection as the Levi-Civita connection. Since the torsion tensor vanishes, this implies that the covariant derivative ∇ on TM satisfies the identity

$$\nabla_V W - \nabla_W V = [V, W]$$

Dirac operators have a very intimate relationship with curvature, so we take inventory of several identities and definitions that we will need in the future. The first relates the curvature transformation for a Riemannian manifold with the covariant derivative on TM.

Proposition 2.6. *Let $R \in \Omega^2_M(\mathrm{End}\, TM)$ be the curvature transformation for a Riemannian manifold M i.e. the curvature induced by the Levi-Civita connection on $\mathcal{B}_O(M)$. Then let ∇ denote the covariant derivative on TM, again induced by the Levi-Civita connection. Then for tangent vectors V, W, X, we have*

$$R_{V,W} X = \nabla_V \nabla_W X - \nabla_W \nabla_V X - \nabla_{[V,W]} X$$

Proposition 2.7. *Let R be the curvature transformation for a Riemannian manifold M. Then for any tangent vectors V, W, X, Y*

(1) $R_{V,W} X + R_{X,V} W + R_{W,X} V = 0$
(2) $\langle R_{V,W} X, Y \rangle = \langle R_{X,Y} V, W \rangle$

3. Dirac Operators

To discuss Dirac operators on manifolds, we restrict the previous discussion to the case where the principal bundle we're concerned with is the principal bundle of frames $\mathcal{B}(M)$ (or subbundles like $\mathcal{B}_O(M)$ and $\mathcal{B}_{SO}(M)$). The covariant derivative has a very geometric interpretation when applied to the $GL_n\mathbb{R}$ bundle of frames $\pi : \mathcal{B}(M) \to M$. The natural action of $GL_n\mathbb{R}$ on $\mathbb{R}^n$ allows us to define the associated vector bundle $\mathcal{B}(M) \times_{GL_n\mathbb{R}} \mathbb{R}^n$, which is naturally isomorphic to the tangent bundle TM as we showed before. We can then define a natural 1-form $\theta \in \Omega_{\mathcal{B}(M)}(\mathbb{R}^n)$ called the ***soldering form***. Given $b \in \mathcal{B}(M)$, we know that b is a linear isomorphism $b : \mathbb{R}^n \to T_{\pi(b)}M$. Then given $v \in T_bP$, we define the action of θ by

$$\theta_p(v) = b^{-1}(d\pi_p(v))$$

The soldering form in some sense remembers that $\mathcal{B}(M)$ is the bundle of frames. Then given a connection $\Theta \in \Omega^1_P(\mathfrak{gl}_n\mathbb{R})$ on $\mathcal{B}(M)$, we use the Θ to define special vector fields on P, which we denote ∂_i.[1] Given $b \in P$, we define $\partial_i|_b$ by $\partial_i|_b = \widetilde{b(e_1)}$, where e_1 denotes the first standard basis vector of $\mathbb{R}^n$ and $\widetilde{b(e_1)}$ is the unique horizontal lift of $b(e_1) \in T_{\pi(b)}M$ at b with respect to the connection Θ. The vector fields $\partial_2, \dots \partial_n$ are defined analogously, and together the ∂_i give a framing for the horizontal distribution on $\mathcal{B}(M)$. The connection Θ on $\mathcal{B}(M)$ induces a covariant derivative ∇ on any associated vector bundle $E \to M$, which has a very special relationship with the vector fields ∂_i.

Proposition 3.1. *Let M be a smooth manifold and $\pi : \mathcal{B}(M) \to M$ its $GL_n\mathbb{R}$ bundle of frames. Equip $\mathcal{B}(M)$ with a connection Θ and define the vector fields ∂_i as above. Then given a vector space W with a representation $\rho : GL_n\mathbb{R} \to GL(W)$ let $E \to M$ denote the associated bundle, and let ∇ denote the covariant derivative on E induced by Θ. Then given a section $\psi \in \Omega^0_M(E)$, it has a corresponding $GL_n\mathbb{R}$-equivariant map $\tilde{\psi} : P \to W$. Then the covariant derivative of $\tilde{\psi}$ is given by the formula*

$$\nabla\tilde{\psi} = \partial_i\tilde{\psi} \otimes e^i$$

where $e^i \in (\mathbb{R}^n)^*$ denotes the dual basis to the standard basis e_i.

Proof. We first make sense of the formula. The covariant derivative $\nabla\psi$ is an element of $\Omega^1_M(E)$, which is a section of the bundle $E \otimes T^*M$. The bundle T^*M is naturally isomorphic to the associated bundle $\mathcal{B}(M) \times_{GL_n\mathbb{R}} (\mathbb{R}^n)^*$ given by the dual to the standard representation of $GL_n\mathbb{R}$ on $\mathbb{R}^n$. Therefore, we get a representation of $GL_n\mathbb{R}$ on $W \otimes (\mathbb{R}^n)^*$, and $E \otimes T^*M$ is naturally isomorphic to the vector bundle $\mathcal{B}(M) \times_{GL_n\mathbb{R}} (W \otimes (\mathbb{R}^n)^*)$. Sections of this bundle are equivalent $GL_n\mathbb{R}$-equivariant maps $P \to W \otimes (\mathbb{R}^n)^*$. The action of the ∂_i on $\tilde{\psi}$ produces $GL_n\mathbb{R}$-equivariant maps $\partial_i\psi : P \to W$, so the tensor product $\partial_i\tilde{\psi} \otimes e^i$ then defines a section of $E \otimes T^*M$.

We then want to verify that this formula is true. Fix $x \in X$, $v \in T_xM$, and $b \in \pi^{-1}(x)$. The element b determines a basis $b(e_1), \dots b(e_n)$ for T_xM. In this basis, v has the representation $v^i b(e_i)$. Each of the $b(e_i)$ lift to $\partial_i|_b$, so the horizontal lift of v to b is $v^i\partial_i|_b$. Then let $\tilde{\gamma} : (-\varepsilon, \varepsilon) \to P$ be an integral curve of $v^i\partial_i$. Then $\gamma = \pi \circ \tilde{\gamma}$ is a curve in M with $\gamma'(0) = v$, so the covariant derivative $(\nabla\tilde{\psi})_b(v)$ is given by

$$\begin{aligned}(\nabla\tilde{\psi})_b(v) &= \left.\frac{d}{dt}\right|_{t=0} \tilde{\psi}(\tilde{\gamma}(t)) \\ &= v^i\partial_i\tilde{\psi}\end{aligned}$$

Therefore, $\nabla\psi = \partial_i \otimes e^i$. ∎

[1] Note that these vector fields are not coordinate vector fields, though we have in the past used the notation ∂_i to denote coordinate vector fields.

The vector fields ∂_i on $\mathcal{B}(M)$ also give a recipe for constructing differential operators between sections of associated vector bundles, and we will use this recipe to construct a Dirac operator on a general Spin manifold. Let A and B be linear representations of $GL_n\mathbb{R}$, and let $E, F \to M$ be their associated vector bundles. The action of $GL_n\mathbb{R}$ on A and B induces an action on $\mathsf{Hom}(A, B)$, giving it the structure of a $GL_n\mathbb{R}$ representation. Then given a $GL_n\mathbb{R}$-equivariant map $\sigma : (\mathbb{R}^n)^* \to \mathsf{Hom}(A, B)$, we can define a first order differential operator

$$\begin{aligned} D_\sigma : \Gamma_M(E) &\to \Gamma_M(E) \\ \psi &\mapsto [\sigma(e^k)](\partial_k \psi) \end{aligned}$$

Where we interpret ψ as a $GL_n\mathbb{R}$-equivariant map $P \to A$. This method for constructing differential operator has already manifested itself in several differential operators we have discussed.

Example 3.2. Given a vector space W with a $GL_n\mathbb{R}$ action, which gives an associated bundle $\mathcal{B}(M) \times_{GL_n\mathbb{R}} W$. A connection on $\mathcal{B}(M)$ induces a covariant derivative operator ∇ on sections of the associated bundle. In terms of the horizontal vector fields ∂_i defined on $\mathcal{B}(M)$, ∇ is given by

$$\nabla \psi = \partial_k \psi \otimes e^k$$

Let $T : (R^n)^* \to \mathsf{Hom}((\mathbb{R}^n)^*, W)$ be the map where $T(v)(w) = w \otimes v$. Then $\nabla = T(e^k)\partial_k$ ⋖

Example 3.3. Consider $\Lambda^k(\mathbb{R}^n)^*$ and $\Lambda^{k+1}(\mathbb{R}^n)^*$, which have a natural $GL_n\mathbb{R}$ action, and let $\varepsilon : (\mathbb{R}^n)^* \to \mathsf{Hom}(\Lambda^k(\mathbb{R}^n)^*, \Lambda^{k+1}(\mathbb{R}^n)^*)$ be the exterior multiplication map, i.e.

$$\varepsilon(v)(w) = v \wedge w$$

This map is easily seen to be $GL_n\mathbb{R}$-equivariant. Using our recipe, we get a differential operator

$$\begin{aligned} D_\varepsilon : \Omega^k_M &\to \Omega^{k+1}_M \\ \psi &\mapsto [\varepsilon(e^k)](\partial_k \psi) \end{aligned}$$

which is the de Rham differential d. ⋖

The preceding discussion was all done with $\mathcal{B}(M)$ and $G = GL_n\mathbb{R}$, but the discussion applies to any principal bundle P with a reduction of structure group to $\mathcal{B}(M)$, like $\mathcal{B}_O(M)$, $\mathcal{B}_{SO}(M)$, and $\mathcal{B}_{\mathrm{Spin}}(M)$.

Recall that if (M, g) is an oriented Riemannian manifold, we have a canonical choice of connection on TM – the Levi-Civita connection. Furthermore, if M is Spin, we get a canonical choice of connection on $\mathcal{B}_{\mathrm{Spin}}(M)$.

Definition 3.4. Let (M, g) be a Spin manifold, and let Θ be the Levi-Civita connection on the oriented orthonormal frame bundle $\mathcal{B}_{SO}(M)$. The double cover $\pi : \mathcal{B}_{\mathrm{Spin}} \to \mathcal{B}_{SO}(M)$ gives a natural isomorphism

$$T(\mathcal{B}_{\mathrm{Spin}}(M)) \to \pi^*(T\mathcal{B}_{SO}(M))$$

So the pullback of the horizontal distribution H determined by Θ defines a connection on $\mathcal{B}_{\mathrm{Spin}}(M)$ called the ***Spin connection***. ❖

We saw in $\mathbb{R}^n$ that the Dirac operator D acted on functions $\psi \in C^\infty(\mathbb{R}^n, V)$ for a Clifford module V. Over a manifold M, these Clifford modules might vary over each point of the manifold, so the Dirac operator should act on sections $\Gamma_M(\mathbb{S})$ for some bundle $\mathbb{S} \to M$ of Clifford modules over M.

While we have discussed Spin structures extensively, and Dirac operators are of particular importance to Spin geometry, they are not dependent on a Spin structure. Note that the Dirac operator we defined on $\mathbb{R}^n$ only required a nondegenerate bilinear form, which we used to construct the Clifford algebra. This suggests that the only structure necessary to define a Dirac operator is a Riemannian metric.

Definition 3.5. Let (M, g) be an oriented Riemannian manifold, giving us a reduction of structure group $\mathcal{B}_{SO}(M) \to \mathcal{B}(M)$. Under the standard representation, SO_n acts on $\mathbb{R}^n$ with the standard inner product by isometries, so every element $A \in SO_n$ induces an algebra automorphism $\rho_A : \mathrm{Cliff}_{n,0}(\mathbb{R}) \to \mathrm{Cliff}_{n,0}(\mathbb{R})$, giving $\mathrm{Cliff}_{n,0}(\mathbb{R})$ the structure of a SO_n representation. The ***Clifford bundle*** of M is the associated vector bundle

$$\mathrm{Cliff}(M) = \mathcal{B}_{SO}(M) \times_{SO_n} \mathrm{Cliff}_{n,0}(\mathbb{R})$$

❖

In fact, the bundle $\mathrm{Cliff}(M) \to M$ is a bundle of Clifford algebras over M, where over each point $p \in M$, the fiber $\mathrm{Cliff}(M)_p$ is $\mathrm{Cliff}(T_pM, g_p)$. Working fiberwise, the natural vector space isomorphism $\mathrm{Cliff}(T_pM, g_p) \to \Lambda^\bullet(T_pM,)$ determines a vector bundle isomorphism $\mathrm{Cliff}(M) \to \Lambda^\bullet(TM)$. Using the Riemannian metric, we get an isomorphism $TM \to T^*M$, and the composition of these isomorphisms determines a vector bundle isomorphism $\mathrm{Cliff}(M) \to \Lambda^\bullet(T^*M)$. In the case that the manifold M admits a Spin structure, we can define further bundles.

Definition 3.6. Let (M, g) be an oriented Riemannian manifold equipped with a Spin structure $\mathcal{B}_{\mathrm{Spin}}(M) \to \mathcal{B}_{SO}(M)$, and let $\mathbb{S}$ be a left Clifford module over $\mathrm{Cliff}_{n,0}(\mathbb{R})$. The inclusion $\mathrm{Spin}_n \hookrightarrow \mathrm{Cliff}_{n,0}(\mathbb{R})$ gives $\mathbb{S}$ the structure of a Spin_n representation. A ***spinor bundle*** over M is the associated vector bundle

$$\mathcal{B}_{\mathrm{Spin}}(M) \times_{\mathrm{Spin}_n} \mathbb{S}$$

❖

The Levi-Civita and Spin connections induce canonical connections on $\mathrm{Cliff}(M)$ and any spinor bundle $\mathbb{S}$. Since the Spin connection is derived from the Levi-Civita connection, the curvature on a spinor bundle $\mathbb{S}$ is closely related to the Riemannian curvature.

Proposition 3.7. *Let (M, g) be a Spin manifold, and $\mathbb{S}$ a spinor bundle over M. Let $R^{\mathbb{S}}$ denote the curvature tensor on $\mathbb{S}$ associated to the Spin connection on $\mathcal{B}_{\mathrm{Spin}}(M)$. Then $R^{\mathbb{S}}$ satisfies*

$$R^{\mathbb{S}}_{V,W} = \frac{1}{4} \sum_{i,j} \langle R_{V,W} e_i, e_j \rangle e_i e_j$$

where R denotes the Riemannian curvature transformation of M.

As one would expect, there is an intimate relationship between the Clifford bundle and a spinor bundle over a Spin manifold. Namely, a spinor bundle $\mathbb{S}$ is a bundle of left Clifford modules over $\mathrm{Cliff}(M)$, i.e. each fiber $\mathbb{S}_p$ is a left Clifford module over the corresponding fiber $\mathrm{Cliff}(M)_p$. In addition, any spinor bundle $\mathcal{B}_{\mathrm{Spin}}(M) \times_{\mathrm{Spin}_n} \mathbb{S}$ admits a fiber metric in which the action of a unit vector in $\mathrm{Cliff}_{n,0}(\mathbb{R})$ is orthogonal. One way to construct a metric is to fix an orthonormal basis of $\mathbb{R}^n$ and average cover the finite group generated by the basis in $\mathrm{Cliff}_{n,0}(\mathbb{R})$. We now have the ingredients to define a general Dirac operator.

Equip $\mathbb{R}^n$ with the standard inner product, and let $b : (\mathbb{R}^n)^* \times (\mathbb{R}^n)^* \to \mathbb{R}$ be the induced inner product on $(\mathbb{R}^n)^*$. Using the double cover $\mathrm{Spin}_n \to SO_n$, the group Spin_n acts on $(\mathbb{R}^n)^*$ through the dual representation of SO_n on $(\mathbb{R}^n)^*$. In addition, Spin_n acts naturally on any Clifford module $\mathbb{S}$ by left multiplication, where we think of Spin_n as a subgroup of the group of units $\mathrm{Cliff}((\mathbb{R}^n)^*, b)^\times$. The ***Clifford multiplication map*** $c : (\mathbb{R}^n)^* \to \mathrm{End}(\mathbb{S}))$ given by $c(v) = va$ is easily verified to be Spin_n-equivariant, so we can apply our recipe for constructing differential operators to construct the Dirac operator.

DEFINITION 3.8. Let (M, g) be a oriented Riemannian manifold equipped with a Spin structure $\varphi : \mathcal{B}_{\mathrm{Spin}}(M) \to \mathcal{B}_{SO}(M)$ and spinor bundle $\mathbb{S} \to M$. Using the Spin connection on $\mathcal{B}_{\mathrm{Spin}}(M)$, define the horizontal vector fields ∂_i on $\mathcal{B}_{\mathrm{Spin}}(M)$ by

$$\partial_i|_p = \widetilde{\varphi(p)(e_i)}$$

where $\widetilde{\varphi(p)(e_i)}$ denotes the horizontal lift of the image of e_i under $\varphi(p) \in \mathcal{B}_{SO}(M)$ to $T(\mathcal{B}_{\mathrm{Spin}}(M))$. The ***Dirac operator*** is the first order differential operator

$$D : \Gamma_M(\mathbb{S}) \to \Gamma_M(\mathbb{S})$$
$$\psi \mapsto c(e^k)\partial_k\psi$$

where we interpret ψ as a Spin_n-equivariant map $\mathcal{B}_{\mathrm{Spin}}(M) \to \mathrm{Cliff}_{n,0}(\mathbb{R})$. ❖

In terms of the covariant derivative ∇ on the spinor bundle induced by the Spin connection, the Dirac operator can be written in an orthonormal frame e_i of $\mathbb{S}$ by

$$D = \sum_i e_i \nabla_{e_i}$$

However, this definition relies on a Spin structure. This construction works for any bundle of modules over $\mathrm{Cliff}(M)$. For any such bundle $E \to M$, equipped with a fiber metric in which the action of $TM \subset \mathrm{Cliff}(M)$ is orthogonal, the Dirac operator on E is defined by the same coordinate formula. Note that this new definition agrees with our previous definitions of Dirac operators on $\mathbb{R}^n$, as well as the ones we defined on S^1. In $\mathbb{R}^n$, the Dirac operator squares to the Laplacian, but on a general manifold, this need not be true. Therefore, we give a name to to operator D^2.

DEFINITION 3.9. The ***Dirac Laplacian*** is the map $D^2 : \Gamma_M(\mathbb{S}) \to \Gamma_M(\mathbb{S})$. ❖

Given compact Riemannian manifold (M, g), the Riemannian metric on M induces an inner product on all the associated tensor bundles of TM and T^*M. In particular, we get an inner product $\langle \cdot, \cdot \rangle$ on the exterior powers $\Lambda^k TM$. Since M is compact, this induces an inner product $(\cdot, \cdot)$ on the space on Ω^k_M given by

$$(\omega, \eta) = \int_M \langle \omega, \eta \rangle \, dV_g$$

where dV_g is the Riemannian volume form of M. Under this inner product, we get a linear map $d^* : \Omega^k_M \to \Omega^{k-1}_M$ that is the adjoint to the exterior derivative d, which is constructed using the Hodge star operator $\star : \Omega^k_M \to \Omega^{n-k}_M$. Again, we can express d^* using our recipe for differential operators. Fittingly, its formula reflects its relationship with the exterior derivative d.

PROPOSITION 3.10. *The operator d^* is given by*

$$d^* = \iota(e^k)\partial_k$$

where $\iota(e^k)(\omega) = \langle e^k, \omega \rangle$ is the inner product of e^k and ω induced by the Riemannian metric.

Using d and d^*, we construct the ***Hodge Laplacian***

$$\Delta = dd^* + d^*d$$

which is a vast generalization of the Laplacian on $\mathbb{R}^n$ and an important tool in studying the geometry and topology of M. Thought of as a map $\Omega^\bullet_M \to \Omega^\bullet_M$, the map $d + d^*$ is a square root of Δ, where we use the fact that $d^2 = (d^*)^2 = 0$. Since Dirac operators were originally conceived to be square roots of the Laplacian, it's no surprise that $d + d^*$ is a Dirac operator.

THEOREM 3.11. *Under the isomorphism* $\mathrm{Cliff}(M) \to \Lambda^\bullet(TM)$, *the Dirac operator D on* $\mathrm{Cliff}(M)$ *is given by $d + d^*$.*

PROOF. Let $\varphi : \text{Cliff}_{n,0}(\mathbb{R}) \to \Lambda^\bullet\mathbb{R}^n$ be the canonical isomorphism. Recall that for $v \in \mathbb{R}^n$ and $\eta \in \text{Cliff}_{n,0}(\mathbb{R})$,

$$\varphi(v\eta) = v \wedge \eta + \iota(v)(\eta)$$

Therefore, under this isomorphism the Clifford multiplication map $c(e^k)$ becomes the map $\varepsilon(e^k) + \iota(e^k)$, so the Dirac operator becomes

$$D = (\varepsilon(e^k) + \iota(e^k))\partial_k = d + d^*$$

■

The Dirac operator has an intimate relationship with another Laplacian operator.

DEFINITION 3.12. Let (M, g) be a oriented Riemannian manifold, equipped with the Levi-Civita connection Θ on $\mathcal{B}_{SO}(M)$, and let ∇ denote the covariant derivative on TM. The ***connection Laplacian*** on M is map $\nabla^*\nabla : \Gamma(TM) \to \Gamma(TM)$ defined by

$$\nabla^*\nabla\psi = \text{trace}(\nabla^2\psi)$$

where ∇^2 is the 2-form defined on vector fields V, W by

$$\nabla^2_{V,W}\psi = \nabla_V\nabla_W\psi - \nabla_{\nabla_V W}\psi$$

for tangent vectors V, W and a vector field ψ. In an orthonormal frame $\{e_i\}$ for TM, $\nabla^*\nabla$ has the coordinate formula

$$\nabla^*\nabla\psi = \sum_i \nabla^2_{e_i,e_i}\psi$$

❖

LEMMA 3.13. *For a Riemannian manifold (M, g), let ∇^2 be defined as above, and let R denote its curvature transformation. Then for vector fields V, W,*

$$\nabla^2_{V,W} - \nabla^2_{W,V} = R_{V,W}$$

PROOF. We compute

$$\begin{aligned}
\nabla^2_{V,W} - \nabla^2_{W,V} &= \nabla_V\nabla_W - \nabla_{\nabla_V W} - (\nabla_W\nabla_V - \nabla_{\nabla_W V}) \\
&= \nabla_V\nabla_W - \nabla_W\nabla_V - (\nabla_{\nabla_V W} - \nabla_{\nabla_W V}) \\
&= \nabla_V\nabla_W - \nabla_W\nabla_V - \nabla_{\nabla_V W - \nabla_W V} \\
&= \nabla_V\nabla_W - \nabla_W\nabla_V - \nabla_{[V,W]} \\
&= R_{V,W}
\end{aligned}$$

where we use the fact that $\nabla_V W - \nabla_W V = [V, W]$ since the Levi-Civita connection is torsion free. ■

We two different "Laplacian type" operators on a manifold – the connection Laplacian $\nabla^*\nabla$, and the Dirac Laplacian D^2. A natural question to ask is how these operators differ, i.e. what is $D^2 - \nabla^*\nabla$? A priori, one might expect the difference to be a second or first order differential operator. The miracle is that the difference is a zeroth order operator, i.e. a tensor, which only involves the curvature of the manifold.

THEOREM 3.14 (***The Weitzenböck Formula***). *Let $\{e_i\}$ denote a local orthonormal frame of any bundle $\mathbb{S} \to M$ of left modules over* Cliff(M) *equipped with a compatible fiber metric. Then*

$$D^2 = \nabla^*\nabla + \frac{1}{2}\sum_{i,j} e_i e_j R^{\mathbb{S}}_{e_i,e_j}$$

where $R^{\mathbb{S}}$ denotes the curvature transformation of the bundle $\mathbb{S}$.

Proof. In the orthonormal frame $\{e_i\}$, the Dirac operator is expressed as $D = \sum_i e_i \nabla_{e_i}$. We then compute

$$\begin{aligned}
D^2 &= \sum_{i,j} (e_i \nabla_{e_i})(e_j \nabla_{e_j}) \\
&= \sum_{i,j} e_i e_j \nabla_{e_i} \nabla_{e_j} \\
&= \sum_i e_i^2 \nabla_{e_i} \nabla_{e_j} + \sum_{j<k} e_j e_k \nabla_{e_j} \nabla_{e_k} \\
&= -\sum_i \nabla^2_{e_i} + \sum_{j<k} e_j e_k (\nabla^2_{e_j,e_k} - \nabla^2_{e_k,e_j}) \\
&= \nabla^* \nabla + \frac{1}{2} \sum_{i,j} e_i e_j R_{e_i,e_j}
\end{aligned}$$

■

The formula relating the Dirac Laplacian to the connection Laplacian and curvature is a sort of "proto-theorem." By applying it to Dirac operators over different manifolds and spinor bundles, we can recover many theorems relating Laplacian-type operators to curvature simply by computing the curvature term. For example, as a corollary, we can express the Hodge Laplacian in terms of the connection Laplacian and some curvature term. Remarkably the curvature term is a familiar friend from Riemannian geometry – the Ricci tensor.

Corollary 3.15. *Let Δ denote the Hodge Laplacian on a Riemannian manifold M. Then*

$$\Delta = \nabla^* \nabla + \mathrm{Ric}$$

where Ric *denotes the Ricci transformation*

$$\mathrm{Ric}(V) = \sum_i R_{e_i,V}(e_i)$$

Proof. Under the identification $\mathrm{Cliff}(M) \to \Lambda^\bullet(TM)$, we know that $D^2 = \Delta$. Using the previous theorem, it then suffices to prove that the curvature term in the Weitzenböck formula under the identification is the Ricci tensor. Let ψ be any vector field. We then compute in an orthonormal frame $\{e_j\}$,

$$\begin{aligned}
\frac{1}{2} \sum_{i,j} e_i e_j R_{e_i,e_j}(\psi) &= \frac{1}{2} \sum_{i,j,k} e_i e_j \langle R_{e_i,e_j}(\psi), e_k \rangle e_k \\
&= \frac{1}{2} \sum_{i,j,k} \langle R_{e_i,e_j}(\psi), e_k \rangle e_i e_j e_k
\end{aligned}$$

We then apply the identity $\langle R_{V,W} X, Y \rangle = \langle R_{X,Y} V, W \rangle$ to each term of the summation, which yields

$$\begin{aligned}
\frac{1}{2} \sum_{i,j,k} \langle R_{\psi,e_k}(e_i), e_j \rangle e_i e_j e_k &= \frac{1}{2} \left(\sum_{i=j=k} \langle R_{\psi,e_k}(e_i), e_j \rangle e_i e_j e_k \;+ \sum_{\substack{i,j,k \\ \text{not all equal}}} \langle R_{\psi,e_k}(e_i), e_j \rangle e_i e_j e_k \right) \\
&= \frac{1}{2} \left(0 + \sum_{\substack{i,j,k \\ \text{not all equal}}} \langle R_{\psi,e_k}(e_i), e_j \rangle e_i e_j e_k \right)
\end{aligned}$$

The first term vanishes since $\langle R_{\psi,e_i}(e_i), e_i)\rangle e_i = \langle R_{e_i,e_i}(\psi), e_i\rangle e_i$, which is zero, since $R_{V,V} = 0$. We then split up the remaining summation into four separate summations, which correspond to the cases

(1) $i = j \neq k$
(2) $i \neq j = k$
(3) $i = k \neq j$
(4) $i \neq j \neq k \neq i$.

We then compute for each case.

(1) In this case, we compute

$$\begin{aligned}\sum_{i=j\neq k} \langle R_{\psi,e_k}(e_i), e_j\rangle e_i e_j e_k &= \sum_{i\neq k} \langle R_{\psi,e_k}(e_i), e_i\rangle e_k \\ &= \sum_{i\neq k} \langle R_{e_i,e_i}(\psi), e_k\rangle e_k \\ &= 0\end{aligned}$$

Since $R_{V,V} = 0$ for any V.

(2) We compute

$$\begin{aligned}\sum_{i\neq j=k} \langle R_{\psi,e_k}(e_i), e_j\rangle e_i e_j e_k &= \sum_{i\neq j} \langle R_{\psi,e_j}(e_i), e_j\rangle e_i \\ &= \sum_{i,j} \langle R_{\psi,e_j}(e_i), e_j\rangle e_i\end{aligned}$$

where we use the fact that the terms where $i = j$ are 0.

(3) This computation is exactly the same as the previous case, giving us $\sum_{i\neq j} \langle R_{\psi,e_j}(e_i), e_j\rangle e_i$ after renaming indices.

(4) This term vanishes, where we use the fact that $e_i e_j = -e_j e_i$.

Therefore, the entire summation is just

$$\frac{1}{2}\left(2\sum_{i,j} \langle R_{\psi,e_j}(e_i), e_j\rangle e_i\right)$$

We then use the fact that R is symmetric in the first pair of indices and in the second pair of indices to conclude that this is equal to

$$\begin{aligned}\sum_{i,j} \langle R_{e_j,\psi}(e_j), e_i\rangle e_i &= \sum_j R_{e_j,\psi}(e_j) \\ &= \mathrm{Ric}(\psi)\end{aligned}$$

■

Furthermore, if M is a Spin manifold, we can go even further.

THEOREM 3.16 (***Lichnerowicz***). *Let (M, g) be a Spin manifold, and let $\mathbb{S}$ be any spinor bundle associated to $\mathcal{B}_{\mathrm{Spin}}(M$, and D the Dirac operator associated to the Spin connection on $\mathbb{S}$. Then*

$$D^2 = \nabla^*\nabla + \frac{1}{4}\kappa$$

where κ denotes the scalar curvature of M, which is obtained by taking the trace of the Ricci tensor. In an orthonormal frame of M, κ is given by the expression

$$\kappa = -\sum_{i,j} \langle R_{e_i,e_j}(e_i), e_j\rangle$$

Proof. Once more, we use the Weitzenböck formula, and compute the curvature term, which involves the curvature $R^{\mathbb{S}}$ of the spinor bundle. Recall that $R^{\mathbb{S}}$ is expressed in terms of the Riemannian curvature by the formula

$$R^{\mathbb{S}}_{V,W}(\psi) = \frac{1}{4}\sum_{i,j}\langle R_{V,W}e_i, e_j\rangle e_i e_j$$

The curvature term in the Weitzenböck formula then becomes

$$\begin{aligned}\frac{1}{2}\sum_{i,j} e_i e_j R^{\mathbb{S}}_{e_i,e_j} &= \frac{1}{8}\sum_{i,j}\sum_{k,\ell} e_i e_j \langle R_{e_i,e_j}(e_k), e_\ell\rangle e_k e_\ell \\ &= \frac{1}{8}\sum_{i,j,k,\ell}\langle R_{e_i,e_j}(e_k), e_\ell\rangle e_i e_j e_k e_\ell\end{aligned}$$

We note that since R antisymmetric in the first pair of indices and in the second pair of indices, any term where $i = j$ or $k = \ell$ vanishes. Using this, we compute

$$\begin{aligned}\frac{1}{8}\sum_{i,j,k,\ell}\langle R_{e_i,e_j}(e_k), e_\ell\rangle e_i e_j e_k e_\ell &= \frac{1}{8}\sum_{\ell}\left(\sum_{i\neq j=k}\langle R_{e_i,e_j}(e_k), e_\ell\rangle e_i e_j e_k + \sum_{i=k\neq j}\langle R_{e_i,e_j}(e_k), e_\ell\rangle e_i e_j e_k\right) e_\ell \\ &= \frac{1}{8}\sum_{\ell}\left(\sum_{i,j}\langle R_{e_i,e_j}(e_j), e_\ell\rangle e_i e_j e_j + \sum_{i,j}\langle R_{e_i,e_j}(e_i), e_\ell\rangle e_i e_j e_i\right) e_\ell \\ &= \frac{1}{8}\left(-\sum_{i,j,\ell}\langle R_{e_j,e_i}(e_j), e_\ell\rangle e_i e_\ell - \sum_{i,j,\ell}\langle R_{e_i,e_j}(e_i), e_\ell\rangle e_j e_\ell\right) \\ &= \frac{1}{8}\left(-\sum_{i,j,\ell}\langle R_{e_i,e_j}(e_i), e_\ell\rangle e_j e_\ell - \sum_{i,j,\ell}\langle R_{e_i,e_j}(e_i), e_\ell\rangle e_j e_\ell\right) \\ &= -\frac{1}{4}\sum_{i,j,\ell}\langle R_{e_i,e_j}(e_i), e_\ell\rangle e_j e_\ell \\ &= \frac{1}{4}\kappa\end{aligned}$$

where the last equality comes from the fact that Clifford multiplication between unit vectors is anticommutative, leaving only the summation over $j = \ell$, and $e_j^2 = 1$. ■

4. Index Theory

A special spinor bundle to consider over a compact Spin manifold M is when the Clifford module is $\mathrm{Cliff}_{n,0}(\mathbb{R})$ itself, giving us a spinor bundle

$$\mathbb{S}(M) = \mathcal{B}_{\mathrm{Spin}}(M) \times_{\mathrm{Spin}_n} \mathrm{Cliff}_{n,0}(\mathbb{R})$$

This is a $\mathbb{Z}/2\mathbb{Z}$-graded spinor bundle, with subbundles

$$\mathbb{S}^0(M) = \mathcal{B}_{\mathrm{Spin}}(M) \times_{\mathrm{Spin}_n} \mathrm{Cliff}^0_{n,0}(\mathbb{R}) \qquad \mathbb{S}^1(M) = \mathcal{B}_{\mathrm{Spin}}(M) \times_{\mathrm{Spin}_n} \mathrm{Cliff}^1_{n,0}(\mathbb{R})$$

In addition, the space of sections $\Gamma_M(\mathbb{S}(M))$ of $\mathbb{S}(M)$ also inherits the structure of a graded Clifford module, where the grading is given by

$$\Gamma^0_M(\mathbb{S}(M)) = \Gamma_M(\mathbb{S}^0(M)) \qquad \Gamma^1_M(\mathbb{S}(M)) = \Gamma_M(\mathbb{S}^1(M))$$

Fix a fiber metric $\langle \cdot, \cdot \rangle$ on $\mathbb{S}(M)$ in which Clifford multiplication is orthogonal. This induces an inner product $(\cdot, \cdot)$ on $\Gamma_M(\mathbb{S}(M))$ defined by

$$(\psi, \varphi) = \int_M \langle \psi, \varphi \rangle dV_g$$

where the inner product is taken fiberwise to obtain a smooth function $\langle \psi, \varphi \rangle$, and dV_g is the Riemannian volume form. Since multiplication by a vector is odd, Clifford multiplication is odd, so the Dirac operator is an odd operator with respect to this grading of $\Gamma_M(\mathbb{S}(M))$, and has the block form

$$\begin{pmatrix} 0 & D_1 \\ D_0 & 0 \end{pmatrix}$$

where D_i is the Dirac operator restricted to the subspace $\Gamma^i_M(\mathbb{S}(M))$. The Clifford algebra $\mathrm{Cliff}_{n,0}$ is also a *right* Clifford module over itself, and this action commutes with the Spin_n action on the left. Therefore, the right action of $\mathrm{Cliff}_{n,0}(\mathbb{R})$ on $\mathcal{B}_{\mathrm{Spin}}(M) \times \mathrm{Cliff}_{n,0}(\mathbb{R})$ descends to a right action on $\mathrm{Cliff}_{n,0}(\mathbb{R})$ on $\mathbb{S}(M)$, giving each fiber the structure of a right Clifford module. The right Clifford action then induces a *left* $\mathrm{Cliff}_{0,n}(\mathbb{R})$ action. Recall that there is an automorphism $\alpha : \mathrm{Cliff}_{p,q} \to \mathrm{Cliff}_{p,q}$ that extends the mapping $v \mapsto -v$ on $\mathbb{R}^{p,q}$. There is a linear isomorphism $A : \mathrm{Cliff}_{n,0}(\mathbb{R}) \to \mathrm{Cliff}_{0,n}(\mathbb{R})$ determined by the identity map $\mathbb{R}^n \to \mathbb{R}^n$, thought of as a map $\mathbb{R}^{n,0} \to \mathbb{R}^{0,n}$. Then the left $\mathrm{Cliff}_{0,n}$ action on $\mathbb{S}(M)$ is defined to be

$$v \cdot m = m \cdot \alpha(A(v))$$

For example, let e_i^- be the standard orthogonal basis for $\mathbb{R}^{0,n}$, and e_i^+ the standard orthogonal basis for $\mathbb{R}^{n,0}$. Then the left action of e_i^- is the right action of $-e_i^+$. In addition, this action respects the grading of $\mathbb{S}(M)$, i.e. multiplication by even elements preserves the grading and multiplication by odd elements reverses the grading. Since right multiplication commutes with left multiplication, and the action of $\mathrm{Cliff}_{0,n}(\mathbb{R})$ is defined in terms of this right action, the Dirac operator is $\mathrm{Cliff}_{0,n}(\mathbb{R})$ linear, i.e. it commutes with the $\mathrm{Cliff}_{0,n}(\mathbb{R})$ action on the left. The fact that D is $\mathrm{Cliff}_{0,n}(\mathbb{R})$ linear implies that for $\psi \in \Gamma^0_M(\mathbb{S}(M))$ and $v \in \mathbb{R}^{0,n} \subset \mathrm{Cliff}_{0,n}(\mathbb{R})$, we have that

$$D(v \cdot \psi) = D_1(v\psi) = v \cdot D\psi = v \cdot D_0\psi$$

The Dirac operator has the special property of being an ***elliptic operator***. We will not define this property, but we will use some consequences of elliptic theory and functional analysis to deduce certain properties of D.

(1) The Dirac operator is skew-adjoint with respect to the inner product on $\Gamma_M(\mathbb{S}(M))$, i.e.

$$(D\psi, \varphi) = (\psi, -D\varphi)$$

With respect to the block decomposition into D_0 and D_1, this implies that $-D_1$ is the adjoint to D_0 with respect to $(\cdot, \cdot)$.

(2) D has finite dimensional kernel.

Recall that D commutes with the left $\mathrm{Cliff}_{0,n}(\mathbb{R})$ action.This then implies that the left action of $\mathrm{Cliff}_{0,n}(\mathbb{R})$ preserves the kernel of D, so it carries a left $\mathrm{Cliff}_{0,n}(\mathbb{R})$ action. In addition, $\ker D$ has a $\mathbb{Z}/2\mathbb{Z}$ grading obtained by intersection with $\mathbb{S}^0(M)$ and $\mathbb{S}^1(M)$, and the $\mathrm{Cliff}_{0,n}(\mathbb{R})$ action respects this grading, giving $\ker D$ the structure of a graded Clifford module. This determines a K-Theory class $[\ker D] \in KO^{-n}(\mathrm{pt})$ called the ***index*** of D.

EXAMPLE 4.1. Recall that we have two Spin structures on S^1, corresponding to the connected and disconnected double covers of the circle. After making appropriate identifications, sections of their corresponding spinor bundles are given by 2π periodic functions $\mathbb{R} \to \mathbb{C}$, but the Dirac operators for the two Spin structures differ. We computed earlier that the Dirac operator for the connected Spin structure is

$$D = i\frac{d}{d\theta} + \frac{1}{2}$$

and the Dirac operator for the disconnected Spin structure is $D = i\partial_\theta$. We then made the observation that the former had no kernel, which implies that the index determines the trivial element in $KO^{-1}(\mathrm{pt}) = \mathbb{Z}/2\mathbb{Z}$. The Dirac operator corresponding to the disconnected Spin structure has a 2 dimensional kernel given by the constant functions $\mathbb{R} \to \mathbb{C}$, so its index determines the nontrivial K-Theory class in $KO^{-1}(\mathrm{pt})$. $\triangleleft$

EXAMPLE 4.2 (***Indices on the torus***). $\triangleleft$

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