Thesis

Jeffrey Jiang

Contents

Acknowledgments

Introduction

CHAPTER 1

Clifford Algebras and Spin Groups

1. Clifford Algebras

Definition 1.1. Let V be a real finite dimensional vector space with a nondegenerate symmetric bilinear form $b : V \times V \to \mathbb{R}$. Then the ***Clifford Algebra*** of V is the data of a unital associative $\mathbb{R}$-algebra $\mathrm{Cliff}(V,b)$ and a linear map $i : V \to \mathrm{Cliff}(V,b)$ satisfying the following universal property: Given any linear map $\varphi : V \to A$ of V into any unital associative $\mathbb{R}$-algebra A satisfying $\varphi(v)^2 = b(v,v)$, there exists a unique map $\tilde{\varphi} : \mathrm{Cliff}(V,b) \to A$ such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[d, "i"'] \arrow[dr, "\varphi"] & \\ \mathrm{Cliff}(V,b) \arrow[r, "\tilde{\varphi}"'] & A \end{tikzcd}$$

❖

This universal property uniquely characterizes the Clifford algebra $\mathrm{Cliff}(V,b)$ up to unique isomorphism.

Theorem 1.2. *The Clifford algebra is unique up to unique isomorphism, i.e. given another unital associative algebra C equipped with a linear map $j : V \to C$ satisfying the universal property, there exists a unique algebra isomorphism $\varphi : \mathrm{Cliff}(V,b) \to C$.*

Proof. Given such an algebra C with a map $j : V \to C$, the map j satisfies the Clifford relation $j(v)^2 = b(v,v)$, so it induces a unique map $\varphi : \mathrm{Cliff}(V,b) \to C$. The map $i : V \to \mathrm{Cliff}(V,b)$ also satisfies the Clifford relation, so it induces a map $\psi : C \to \mathrm{Cliff}(V,b)$. We claim that these maps are inverses. To do so, we show the compositions $\varphi \circ \psi$ and $\psi \circ \varphi$ are identity. Using the universal property of the Clifford algebra once more, the map i induces a unique map $\mathrm{Cliff}(V,b) \to \mathrm{Cliff}(V,b)$ such that

$$\begin{tikzcd} V \arrow[d, "i"'] \arrow[dr, "i"] & \\ \mathrm{Cliff}(V,b) \arrow[r] & \mathrm{Cliff}(V,b) \end{tikzcd}$$

commutes. The identity map makes this diagram commute, so by uniqueness, this is the induced map. The map $\psi \circ \varphi$ also makes this diagram commute, so it must be identity by uniqueness. An identical proof shows that $\varphi \circ \psi$ is the identity map id_C. ■

Explicitly, $\mathrm{Cliff}(V,b)$ is realized as a quotient of the tensor algebra

$$\mathcal{T}(V) = \bigoplus_{n \in \mathbb{Z}^{\geq 0}} V^{\otimes n}$$

where we quotient by the left ideal generated by elements of the form $v \otimes v - b(v,v)$, and the linear map $i : V \to \mathcal{T}(V)/(v \otimes v - b(v,v))$ is given by the inclusion $V \hookrightarrow \mathcal{T}(V)$ followed by the quotient map. We identify V with its image $i(V)$ as a subspace of $\mathrm{Cliff}(V,b)$.

DEFINITION 1.3. Define the bilinear form $b : \mathbb{R}^{p+q} \times \mathbb{R}^{p+q} \to \mathbb{R}$ by

$$b(v,w) = \sum_{i=1}^{p} v^i w^i - \sum_{i=p+1}^{p+1} v^i w^i$$

where the v^i and w^i are the components of v and w with respect to the standard basis on $\mathbb{R}^{p+q}$. We denote the vector space $\mathbb{R}^{p+q}$ equipped with this bilinear form as $\mathbb{R}^{p,q}$. ❖

Let V be a vector space equipped with a nondegenerate bilinear form b and fix a basis for V. Then the bilinear form b is given by a symmetric invertible matrix B, which is conjugate to a diagonal matrix where all the diagonal entries are either 1 or -1. If after conjugation B has p positive entries and q negative entries, we say b has signature (p,q). Any bilinear form b of signature (p,q) admits a basis $\{e_i\}$ satisfying

(1) For $1 \leq i \leq p$, we have $b(e_i, e_i) = 1$
(2) For $p+1 \leq j \leq p+q$, we have $b(e_j, e_j) = -1$
(3) For $i \neq j$, we have $b(e_i, e_j) = 0$

Any such basis then determines an isomorphism $(V,b) \to \mathbb{R}^{p,q}$. and we call such a basis an ***orthogonal basis*** for (V,b). In addition, we get a basis for $\mathrm{Cliff}(V,b)$, given by

$$\{e_{i_1}e_{i_2}\cdots e_{i_k} \;:\; 0 \leq k \leq n,\ 1 \leq i_j \leq n\}$$

where we interpret the product of 0 basis vectors to be the unit element 1. This then implies the dimension of $\mathrm{Cliff}(V,b)$ as a vector space is $2^{\dim V}$. This basis also determines an isomorphism $\mathrm{Cliff}(V,b) \to \mathrm{Cliff}_{p,q}(\mathbb{R})$, where $\mathrm{Cliff}_{p,q}(\mathbb{R})$ is the Clifford algebra for $\mathbb{R}^{p,q}$. Given $v, w \in V$, we write v and w in these bases as $v^i e_i$ and $w^i e_i$ (using Einstein summation convention), and derive the useful relation

$$\begin{aligned} vw + wv &= v^i w^j e_i e_j + v^i w^j e_j e_i \\ &= v^i w^i (e_i^2 + e_i^2) \\ &= 2b(v,w) \end{aligned}$$

where we use the fact that the e_i are orthgonal to deduce that $e_i e_j = -e_j e_i$ if $i \neq j$.

DEFINITION 1.4. [1] Let V be a vector space. A $\mathbb{Z}/2\mathbb{Z}$ ***grading*** on V is a direct sum decomposition $V = V^0 \oplus V^1$. Elements of V^0 are said to be ***even*** and elements of V^1 are said to be ***odd***. Elements of the even and odd subspaces are said to be ***homogeneous***. Given an homogeneous element $v \in V$, define its ***parity***, denoted $|v|$ by

$$|v| = \begin{cases} 0 & v \in V^0 \\ 1 & v \in V^1 \end{cases}$$

Equivalently, it is the data of a linear map $\varepsilon : V \to V$ such that ε acts by identity on a subspace V^0 of V and negative identity on a complementary subspace V^1, which gives us the desired direct sum decomposition of V as the ± 1 eigenspaces of ε. The map ε is called the ***grading operator***.

❖

DEFINITION 1.5. A $\mathbb{Z}/2\mathbb{Z}$ ***graded algebra*** A over $\mathbb{R}$ (often called a superalgebra) is an $\mathbb{R}$-algebra A equipped with a grading $A = A^0 \oplus A^1$ such that the multiplication respects the grading, i.e. given homogeneous elements $a, b \in A$, Their product ab is an element of $A^{|a|+|b|}$ where the addition is done mod 2. ❖

EXAMPLE 1.6.

(1) Any $\mathbb{R}$-algebra A can be made into a graded algebra where we let $A^0 = A$ and $A^1 = 0$.
(2) The exterior algebra $\Lambda^\bullet V$ of a vector space V is a $\mathbb{Z}/2\mathbb{Z}$ graded algebra (in fact, it is $\mathbb{Z}$ graded), where the grading is $\Lambda^\bullet V = \Lambda^{\text{even}} V \oplus \Lambda^{\text{odd}} V$ where $\Lambda^0 V$ is the subspace spanned by even products of vectors and $\Lambda^1 V$ is the subspace spanned by odd products of vectors.

◀

[1] It is common in the literature to refer to $\mathbb{Z}/2\mathbb{Z}$ graded vector spaces as super vector spaces. The “super" prefix often refers to a $\mathbb{Z}/2\mathbb{Z}$ grading on the relevant object.

For the most part, the algebras we work with will be $\mathbb{Z}/2\mathbb{Z}$ graded, so the term "graded" may be used in lieu of "$\mathbb{Z}/2\mathbb{Z}$ graded." In the case of ambiguity, we will specify the grading.
The Clifford algebra $\mathrm{Cliff}(V,b)$ is naturally a $\mathbb{Z}/2\mathbb{Z}$ graded algebra. Fix a basis $\{e_i\}$ for $\mathrm{Cliff}(V,b)$. We then define the grading

$$\mathrm{Cliff}(V,b) = \mathrm{Cliff}^0(V,b) \oplus \mathrm{Cliff}^1(V,b)$$

Where $\mathrm{Cliff}^0(V,b)$ is the $\mathbb{R}$-span of all even products of basis vectors, and $\mathrm{Cliff}^1(V,b)$ is the $\mathbb{R}$-span of all odd products of basis vectors. Since the product of even elements is again even, the subspace $\mathrm{Cliff}^0(V,b)$ forms a subalgebra, and is called the ***even subalgebra***. There is an extremely nice relationship between a Clifford algebra and its even subalgebra.

THEOREM 1.7. *The even subalgebra* $\mathrm{Cliff}^0_{p,q}(\mathbb{R})$ *is isomorphic to both* $\mathrm{Cliff}_{q,p-1}$ *and* $\mathrm{Cliff}_{p,q-1}$ *as ungraded algebras (as long as $p-1>0$ or $q-1>0$.)*

PROOF. Fix a basis $\left\{e_1^+, \dots e_p^+, e_1^- \dots e_q^-\right\}$ for $\mathbb{R}^{p,q}$, where $(e_i^+)^2 = 1$ and $(e_i^-)^2 = -1$. We then compute

$$(e_i^+e_j^+)^2 = -(e_i^+)^2(e_j^+)^2 = -1$$
$$(e_i^-e_j^-)^2 = -(e_i^-)^2(e_j^-)^2 = -1$$
$$(e_i^+e_j^-)^2 = -(e_i^+)^2(e_i^-)^2 = 1$$
$$(e_i^-e_j^+)^2 = -(e_i^-)^2(e_j^+)^2 = 1$$

Assume $q \neq 0$. Then a generating set for $\mathrm{Cliff}^0_{p,q}(\mathbb{R})$ is

$$\left\{e_1^-e_j^+ \ : \ 1 \leq j \leq p\right\} \cup \{e_1^-e_k^- \ : \ 2 \leq k \leq q\}$$

All the elements in the first set square to 1, and all the elements in the second set square to -1. We then get an isomorphism $\mathrm{Cliff}^0_{p,q}(\mathbb{R}) \to \mathrm{Cliff}_{p,q-1}$ via the mappings

$$e_1^-e_j^+ \mapsto e_j^+$$
$$e_1^-e_k^- \mapsto e_{k-1}^-$$

In the case where $p \neq 0$, we have that an equally good generating set for $\mathrm{Cliff}^0_{p,q}(\mathbb{R})$ is

$$\left\{e_1^+e_j^+ \ : \ 2 \leq j \leq p\right\} \cup \{e_1^+e_i^- \ : \ 1 \leq i \leq q\}$$

Where the elements in the first set square to -1 and the elements of the second set square to 1. Then the mappings

$$e_1^+e_j^+ \mapsto e_{j-1}^-$$
$$e_1^+e_j^- \mapsto e_j^+$$

gives the isomorphism $\mathrm{Cliff}^0_{p,q}(\mathbb{R}) \to \mathrm{Cliff}_{q,p-1}$. ∎

Given two $\mathbb{R}$-algebras A and B, we can form their tensor product $A \otimes B$, which has $A \otimes B$ as the underlying vector space, and the multiplication is defined as

$$(a \otimes b)(c \otimes d) = ac \otimes bd$$

In the case that both A and B are $\mathbb{Z}/2\mathbb{Z}$ graded algebras, we have an alternate version of the tensor product, where the underlying vector space is also $A \otimes B$. The grading on the tensor product is the decomposition

$$A \otimes B = (A^0 \otimes B^0 \oplus A^1 \otimes A^1) \oplus (A^0 \otimes B^1 \oplus A^1 \otimes B^0)$$

and the multiplication of homogeneous elements is given by

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|}(ac \otimes bd)$$

We see that in the multiplication, we are formally commuting the elements of b and c, and we want to introduce a sign whenever elements are moved past each other. This is the ***Koszul sign rule***. Another concept that needs a slight modification in the graded case is the opposite algebra. In the normal case, given an $\mathbb{R}$-algebra

A, the ***opposite algebra*** is the algebra A^{op} with the same underlying vector space, but the multiplication in A^{op} is given by $a * b = ba$, where ba is the multiplication in A. In doing so, we are formally commuting a and b, so in the graded situation, we invoke the Koszul sign rule when defining multiplication in the opposite algebra, and define the multiplication of homogeneous elements in A^{op} to be $a * b = (-1)^{|a||b|}ba$.

One remarkable fact is that Clifford algebras are closed under the graded tensor product, i.e. the graded tensor products of two Clifford algebras is another Clifford algebra. Likewise, the graded opposite algebra of a Clifford algebra is again a Clifford algebra. For the remainder of this section, we will let $\otimes$ denote the graded tensor product, and the superscript op will denote the graded opposite algebra.

THEOREM 1.8. $\mathrm{Cliff}_{p+t,q+s}(\mathbb{R}) \cong \mathrm{Cliff}_{p,q}(\mathbb{R}) \otimes \mathrm{Cliff}_{t,s}(\mathbb{R})$

PROOF. To give a map $\varphi : \mathrm{Cliff}_{p+t,q+s}(\mathbb{R}) \to \mathrm{Cliff}_{p,q}(\mathbb{R}) \otimes \mathrm{Cliff}_{t,s}(\mathbb{R})$, it is sufficient to specify its action on $\mathbb{R}^{p+t,q+s}$ and to check that the Clifford relations hold. Let $\left\{b_1^+, \dots, b_{p+t}^+, b_1^-, \dots b_{q+s}^-\right\}$ denote the standard orthogonal basis for $\mathbb{R}^{p+t|q+s}$ where $(b_i^+)^2 = 1$ and $(b_i^-)^2 = -1$. We then define the bases $\{e_i^\pm\}$ and $\{f_i^\pm\}$ analogously for $\mathbb{R}^{p,q}$ and $\mathbb{R}^{t,s}$ respectively. Then define φ by

$$\varphi(b_i^+) = \begin{cases} e_i^+ \otimes 1 & 1 \le i \le p \\ 1 \otimes f_i^+ & p+1 \le i \le p+t \end{cases}$$

$$\varphi(b_i^-) = \begin{cases} e_i^- \otimes 1 & 1 \le i \le q \\ 1 \otimes f_i^- & q+1 \le i \le q+s \end{cases}$$

This map is injective on generators, so if we show that this satisfies the Clifford relations, then the map given by extending the map to all of $\mathrm{Cliff}_{p+t,q+s}(\mathbb{R})$ will be an isomorphism by a dimension count. Showing the Clifford relations amounts to showing

(1) $\varphi(b_i^+)^2 = 1$
(2) $\varphi(b_i^-)^2 = -1$
(3) The images of any pair of distinct basis vectors anticommute.

The first two are relations are clear from how we defined φ. To show that the images of distinct basis vectors anticommute, there are serveral cases to consider. Given b_i^+ and b_j^+ where $1 \le i, j \le p$, they anticommute, because e_i^+ and e_j^+ anticommute. In the case where $1 \le i \le p$ and $p+1 \le j \le p+t$, we compute

$$\begin{aligned} \varphi(b_i^+)\varphi(b_j^+) + \varphi(b_j^+)\varphi(b_i+) &= (e_i^+ \otimes 1)(1 \otimes f_j^+) + (1 \otimes f_j^+)(e_i^+ \otimes 1) \\ &= e_i^+ \otimes f_j^+ - e_i^+ \otimes f_j^+ \end{aligned}$$

where we use the Koszul sign rule for the second term, noting that f_j+ and e_i^+ are both odd elements. The proof that the images of the b_i^- anti commute with each other, as well as the proof that the images of the b_i^+ and b_i^- anticommute are exactly the same. ■

THEOREM 1.9. *The graded opposite algebra* $\mathrm{Cliff}_{p,q}^{op}$ *is isomorphic to* $\mathrm{Cliff}_{q,p}$.

PROOF. Fix an orthgonal basis $\{e_i^\pm\}$ for $\mathbb{R}^{p,q}$, where $(e_i^\pm)^2 = \pm 1$. We note that since the $e_i^\pm$ are odd elements, they square to ∓ 1 in the opposite algebra. Indeed, the mapping $e_i^\pm \to e_i^\mp$ defines the isomorphism $\mathrm{Cliff}_{p,q}^{\mathrm{op}} \to \mathrm{Cliff}_{q,p}$. ■

Because of these theorems, once we compute a few of the lower dimensional Clifford algebras, we will have enough data to fully classify all Clifford algebras over $\mathbb{R}$.

EXAMPLE 1.10 (***Some low dimensional examples***).

(1) The Clifford algebra $\mathrm{Cliff}_{0,0}(\mathbb{R})$ is isomorphic to $\mathbb{R}$.
(2) As ungraded algebras, the Clifford algebra $\mathrm{Cliff}_{0,1}(\mathbb{R})$ is isomorphic to $\mathbb{C}$, where the isomorphism is given by $e_1 \mapsto i$.
(3) As ungraded algebras, $\mathrm{Cliff}_{0,2}(\mathbb{R})$ is isomorphic to the quaternions $\mathbb{H}$, where the isomorphism is given by $e_1 \mapsto i$ and $e_2 \mapsto j$.

(4) As graded algebras, $\mathrm{Cliff}_{1,1}(\mathbb{R})$ is isomorphic to $\mathrm{End}(\mathbb{R}^{1|1})$. The isomorphism is given by

$$e_1^+ \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad e_1^- \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(5) As ungraded algebras $\mathrm{Cliff}_{1,0}(\mathbb{R})$ is isomorphic to the product algebra $\mathbb{R} \times \mathbb{R}$, where $e_1 \mapsto (1, -1)$.

(6) As ungraded algebras, $\mathrm{Cliff}_{2,0}(\mathbb{R})$ is isomorphic to the algebra $M_2\mathbb{R}$ of 2×2 matrices with coefficients in $\mathbb{R}$. The isomorphism is given by

$$e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad e_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

◂

To classify all Clifford algebras as ungraded algebras, it suffices to know the following table:

$M_8\mathbb{C}$	$M_8\mathbb{H}$	$M_8\mathbb{H} \times M_8\mathbb{H}$	$M_{16}\mathbb{H}$	$M_{32}\mathbb{C}$	$M_{64}\mathbb{R}$	$M_{64}\mathbb{R} \times M_{64}\mathbb{R}$	$M_{128}\mathbb{R}$
$M_4\mathbb{H}$	$M_4\mathbb{H} \times M_4\mathbb{H}$	$M_8\mathbb{H}$	$M_{16}\mathbb{C}$	$M_{32}\mathbb{R}$	$M_{32}\mathbb{R} \times M_{32}\mathbb{R}$	$M_{64}\mathbb{R}$	$M_{64}\mathbb{C}$
$M_2\mathbb{H} \times M_2\mathbb{H}$	$M_4\mathbb{H}$	$M_8\mathbb{C}$	$M_{16}\mathbb{R}$	$M_{16}\mathbb{R} \times M_{16}\mathbb{R}$	$M_{32}\mathbb{R}$	$M_{32}\mathbb{C}$	$M_{32}\mathbb{H}$
$M_2\mathbb{H}$	$M_4\mathbb{C}$	$M_8\mathbb{R}$	$M_8\mathbb{R} \times M_8\mathbb{R}$	$M_{16}\mathbb{R}$	$M_{16}\mathbb{C}$	$M_{16}\mathbb{H}$	$M_{16}\mathbb{H} \times M_{16}\mathbb{H}$
$M_2\mathbb{C}$	$M_4\mathbb{R}$	$M_4\mathbb{R} \times M_4\mathbb{R}$	$M_8\mathbb{R}$	$M_8\mathbb{C}$	$M_8\mathbb{H}$	$M_8\mathbb{H} \times M_8\mathbb{H}$	$M_{16}\mathbb{H}$
$M_2\mathbb{R}$	$M_2\mathbb{R} \times M_2\mathbb{R}$	$M_4\mathbb{R}$	$M_4\mathbb{C}$	$M_4\mathbb{H}$	$M_4\mathbb{H} \times M_4\mathbb{H}$	$M_8\mathbb{H}$	$M_{16}\mathbb{C}$
$\mathbb{R} \times \mathbb{R}$	$M_2\mathbb{R}$	$M_2\mathbb{C}$	$M_2\mathbb{H}$	$M_2\mathbb{H} \times M_2\mathbb{H}$	$M_4\mathbb{H}$	$M_8\mathbb{C}$	$M_{16}\mathbb{R}$
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \times \mathbb{H}$	$M_2\mathbb{H}$	$M_4\mathbb{C}$	$M_8\mathbb{R}$	$M_8\mathbb{R} \times M_8\mathbb{R}$

To read the table, the bottom left entry is $\mathrm{Cliff}_{0,0} \cong \mathbb{R}$, and moving to the right increments the signature from (p, q) to $(p, q+1)$, and moving up increments the signature (p, q) to $(p+1, q)$. Any other Clifford algebra can be obtained from an algebra on this table by tensoring with $M_{16}\mathbb{R}$, since incremeting the signature by 8 (by adding to either p or q) results in tensoring with $M_{16}\mathbb{R}$.

2. Clifford Modules

Definition 2.1. A (left) ***Clifford module*** for the Clifford algebra $\mathrm{Cliff}_{p,q}(\mathbb{R})$ is a module for $\mathrm{Cliff}_{p,q}(\mathbb{R})$ in the usual sense i.e. a real vector space V equipped with an algebra action $\bullet : \mathrm{Cliff}_{p,q} \times V \to V$ satisfying

(1) Every element of $\mathrm{Cliff}_{p,q}(\mathbb{R})$ acts linearly on V.
(2) $(AB) \cdot v = A \cdot (B \cdot v)$ for all $v \in V$.
(3) $(A + B) \cdot v = A \cdot v + B \cdot V$ for all $v \in V$.

Equivalently, it is the data of a real vector space V and a homomorphism $\mathrm{Cliff}_{p,q}(\mathbb{R}) \to \mathrm{End}(V)$. ❖

Definition 2.2. A Clifford module is ***irreducible*** if there exist no proper nontrivial submodules. ❖

From the classification of Clifford algebras, all the Clifford algebras are either matrix algebras $M_n\mathbb{F}$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, or products $M_n\mathbb{F} \times M_n\mathbb{F}$ of two copies of the same matrix algebra. This is sufficient to conclude that Clifford algebras are semisimple, so all Clifford modules will be direct sums of irreducible modules. Therefore, classifying all Clifford modules reduces to classifying the irreducible Clifford modules.

Theorem 2.3. *Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. Then any nontrivial irreducible module for $M_n\mathbb{F}$ is isomorphic to $\mathbb{F}^n$ with the standard action.*

Proof. We first note that $M_n\mathbb{F}$ acts transitively on $\mathbb{F}^n$, which implies that it is irreducible. We then must show that $\mathbb{F}^n$ is, up to isomorphism, the only irreducible $M_n\mathbb{F}$ module. The matrix algebra $M_n\mathbb{F}$ admits an increasing chain of left ideals

$$0 = I_0 \subset I_1 \subset \ldots \subset I_n = M_n\mathbb{F}$$

where I_k is the set of matrices where only the first k columns are nonzero. These ideals have the property that the quotient I_k/I_{k-1} is isomorphic to $\mathbb{F}^n$ as a left $M_n\mathbb{F}$ module. Then let M be some nontrivial irreducible $M_n\mathbb{F}$ module, and fix $m \in M$. Then the orbit $M_n\mathbb{F} \cdot m$ of m under the algebra action is a nonzero submodule, so it must be all of M. Then the map $\varphi : M_n\mathbb{F} \to M$ given by $A \mapsto A \cdot m$ is a surjective map of left $M_n\mathbb{F}$ modules. Then there must exist some smallest k such that $\varphi(I_k)$ is nonzero, and by construction, $\varphi|_{I_k}$ factors through the quotient I_k/I_{k-1}, which is isomorphic to $\mathbb{F}^n$ with the standard action. Then since $\mathbb{F}^n$ is irreducible, this gives us a nontrivial map between irreducible modules, which is an isomorphism by Schur's Lemma. ■

Theorem 2.4. *Any nontrivial irreducible module for $M_n\mathbb{F} \times M_n\mathbb{F}$ is isomorphic to either $\mathbb{F}^n$ where the left factor acts in the usual way, and the right factor acts by 0, or $\mathbb{F}^n$ where the left factor acts by 0 and the right factor acts in the usual way.*

Proof. Let R denote $\mathbb{F}^n$ where the right factor acts nontrivially, and let L denote $\mathbb{F}^n$ where the left factor acts nontrivially. Both L and R are irreducible since $M_n\mathbb{F} \times M_n\mathbb{F}$ acts transitively on them. To show that they are the only irreducible modules up to isomorphism, we use a similar techinique as above. Let I_k denote the chain of increasing ideals in $M_n\mathbb{F}$, as we used above. Then $M_n\mathbb{F} \times M_n\mathbb{F}$ admits a chain of increasing left ideals J_k

$$0 = J_0 \subset I_1 \times \{0\} \subset \ldots \subset M_n\mathbb{F} \times \{0\} \subset M_n\mathbb{F} \times I_1 \subset \ldots \subset M_n\mathbb{F} \times M_n\mathbb{F} = J_{2n}$$

We note that for $1 \leq k \leq n$, we have that J_k/J_{k-1} is isomorphic to L, and for $n+1 \leq k \leq 2n$, we have that J_k/J_{k-1} is isomorphic to R. Then given a nontrivial irreducible module M and a nonzero element m, we get a surjective map $\varphi : M_n\mathbb{F} \times M_n\mathbb{F}$ where $A \mapsto A \cdot m$. Like before, there exists some smallest k such that $\varphi(J_k)$ is nonzero, which then factors through to an isomorphism $J_k/J_{k-1} \to M$, so M is either isomorphic to R or L. ∎

This then gives a full classification of the irreducible ungraded Clifford modules.

3. The Pin and Spin Groups

The group of invertible elements in $\mathrm{Cliff}_{p,q}(\mathbb{R})$, denoted $\mathrm{Cliff}_{p,q}^{\times}(\mathbb{R})$ contains a group $\mathrm{Pin}_{p,q}$, which a double cover of the group $O_{p,q}$ of matrices preserving the standard bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{p,q}$. Inside of $\mathrm{Pin}_{p,q}$, there exists a subgroup $\mathrm{Spin}_{p,q} \subset \mathrm{Pin}_{p,q}$, which double covers the group $SO_{p,q}$, which consists of the subgroup of $O_{p,q}$ where all the elements have determinant equal to 1.

Definition 3.1. The ***Pin group*** $\mathrm{Pin}_{p,q}$ is the subgroup of $\mathrm{Cliff}_{p,q}^{\times}(\mathbb{R})$ generated by the set

$$\left\{ v \in \mathbb{R}^{p,q} \ : \ v^2 = \pm 1 \right\}$$

The ***Spin group*** $\mathrm{Spin}_{p,q}$ is the subgroup of $\mathrm{Pin}_{p,q}$ generated by even products of basis vectors, i.e. $\mathrm{Spin}_{p,q} = \mathrm{Pin}_{p,q} \cap \mathrm{Cliff}_{p,q}^{0}(\mathbb{R})$. I n the case that the bilinear form is definite, we let $\mathrm{Pin}_n^+ = \mathrm{Pin}_{n,0}$ and $\mathrm{Pin}_n^- = \mathrm{Pin}_{0,n}$. There is no such distinction for the Spin groups in definite signatures. ❖

Theorem 3.2. $\mathrm{Spin}_{0,n} \cong \mathrm{Spin}_{n,0}$.

Proof. Recall that we have an isomorphism $\mathrm{Cliff}_{p,q}^{\mathrm{op}} \to \mathrm{Cliff}_{q,p}$ where $e_i^{\pm} \mapsto e_i^{\mp}$. In addition, the even subalgebra $\mathrm{Cliff}_{q,p}^{0}$ is isomorphic to the (ungraded) opposite algebra of the even subalgebra $\mathrm{Cliff}_{p,q}^{0}$. Therefore, the Spin group $\mathrm{Spin}_{q,p} \subset \mathrm{Cliff}_{q,p}$ is isomorphic to the opposite group $\mathrm{Spin}_{p,q}^{\mathrm{op}}$. We then know that every group is isomorphic to its opposite group via the map $g \mapsto g^{-1}$, giving us the desired isomorphism. ∎

In particular, this implies that the Spin groups in definite signatures are isomorphic, so we will henceforth denote them as Spin_n. To show that the Pin and Spin groups cover $O_{p,q}$ and $\mathrm{Spin}_{p,q}$, we make a short digression. Given a vector $v \in \mathbb{R}^{p,q}$, we can define a reflection map $R_v : \mathbb{R}^{p,q} \to \mathbb{R}^{p,q}$ given by $R_v(w) = w - 2\langle v, w \rangle v$, which will reflect across the hyperplane $v^{\perp}$.

Theorem 3.3 (***Cartan-Dieudonné***). *Any orthogonal transformation $A \in O_{p,q}$ can be written as the composition of at most $p+q$ hyperplane reflections, where we interpret the identity map as the composition of 0 reflections.*

Proof. We prove this by induction on $n = p + q$. The case $n = 1$ is trivial, since $O_1 = \{\pm 1\}$. Then given $A \in O_{p,q}$, fix some nonzero $v \in \mathbb{R}^{p,q}$. Then define $R : \mathbb{R}^{p,q} \to \mathbb{R}^{p,q}$ by

$$R(w) = w - 2\frac{\langle Av - v, w \rangle}{\langle Av - v, Av - v \rangle}(Av - v)$$

Then R is a reflection about the hyperplane orthogonal to $Av - v$, and will interchange v and Av. Therefore, RA is an orthogonal transformation fixing v. Since RA is orthogonal, it will also fix the orthogonal complement $v^{\perp}$, so it will restrict to an orthgonal transformation on $v^{\perp}$. The orthogonal complement $v^{\perp}$ is 1 dimension lower than $\mathbb{R}^{p,q}$, and restricting the bilinear form to $v^{\perp}$, we know by the inductive hypotehsis that $RA|_{v^\perp}$ can be written as at most $n-1$ hyperplane reflections in $v^\perp$. Since RA fixes v, we can extend all of these transformations to a hyperplane reflection on all of $\mathbb{R}^{p,q}$, by taking the span of each hyperplane with v, giving us that RA is a composition of at most $n-1$ reflections. Finally, composing RA with R gives us that A can be written as a composition of at most n hyperplane reflections. ∎

The Cartan-Dieudonné theorem will be the central piece for showing that the Pin and Spin groups are double covers of the orthogonal groups.

Theorem 3.4. *There exist* 2*-to-*1 *group homomorphisms* $\mathrm{Pin}_{p,q} \to O_{p,q}$ *and* $\mathrm{Spin}_{p,q} \to SO_{p,q}$*, i.e. there exist short exact sequences of groups*

$$0 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Pin}_{p,q} \longrightarrow O_{p,q} \longrightarrow 0$$

$$0 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}_{p,q} \longrightarrow SO_{p.q} \longrightarrow 0$$

Proof. We first consider the case of $\mathrm{Pin}_{p,q}$. To do this, we need to construct a group action where $\mathrm{Pin}_{p,q}$ acts on $\mathbb{R}^{p,q}$ by orthogonal transformations. We note that for a vector $v \in \mathbb{R}^{p,q}$, identifying $\mathbb{R}^{p,q}$ as a subspace of $\mathrm{Cliff}_{p,q}(\mathbb{R})$, satisfying $\langle v, v\rangle = \pm 1$, we have that $v^{-1} = \pm v$. Then given $g \in \mathrm{Pin}_{p,q}$, and $v \in \mathbb{R}^{p,q}$, we claim that the left action

$$g \cdot v = -gvg^{-1}$$

defines the group action we desire. To show this, we must show that this indeed maps $\mathbb{R}^{p,q}$ back into itself, and that the group elements act by orthogonal trasnformations. It suffices to check this on the generating set of elements v with $\langle v, v\rangle = \pm 1$. First assume that $\langle v, v\rangle = 1$. Then given $w \in \mathbb{R}^{p,q}$, we compute

$$\begin{aligned} -vwv^{-1} &= -vwv \\ &= (wv - 2\langle v, w\rangle)v \\ &= w - 2\langle v, w\rangle v \end{aligned}$$

Which is hyperplane reflection about the orthogonal complement of v. In the case that $\langle v, v\rangle = -1$, we compute

$$\begin{aligned} -vwv^{-1} &= -vw(-v) \\ &= (2\langle -v, w\rangle + wv)(-v) \\ &= w - 2\langle -v, w\rangle(-v) \end{aligned}$$

which is hyperplane reflection about the orthogonal complement of $-v^\perp$, which is the same as the orthogonal complement of $v^\perp$. Therefore, $\mathrm{Pin}_{p,q}$ acts by orthogonal transformations, giving us a homomorphism $\mathrm{Pin}_{p,q} \to O_{p,q}$. This map is surjective by the Cartan-Dieudonné theorem, and it can be verified that the kernel is $\{\pm 1\}$. ∎

We also have the complex Pin and Spin groups, denoted $\mathrm{Pin}_n\mathbb{C}$ and $\mathrm{Spin}_n\mathbb{C}$, which double cover the complex orthogonal groups $O_n\mathbb{C}$ and $SO_n\mathbb{C}$ respectively.

Two simple examples of spin groups occur in dimensions 2 and 3. Since $SO_2 \cong \mathbb{T}$, where

$$\mathbb{T} = \{z \in \mathbb{C} \,:\, |z| = 1\}$$

we have that $\mathrm{Spin}_2 \cong \mathbb{T}$, where the covering map is given by $z \mapsto z^2$. In the case of SO_3, we consider the unit quaternions, which form a Lie group isomorphic to the group SU_2. Then given $q \in SU_2$, we define the map $\varphi_q : \mathbb{R}^3 \to \mathbb{R}^3$ where $\varphi_q(v) = qv\bar{q}$, where $\bar{q}$ is the quaternionic conjugate of q (e.g. $\overline{a + bi + cj + dk} = a - bi - cj - dk$), and $v = v^i e_i$ is idenitified with $v^1 i + v^2 j + v^3 k \in \mathbb{H}$. The mapping $q \mapsto \varphi_q$ then gives a double cover $SU_2 \to SO_3$. In particular, SU_2 is diffeomorphic to the sphere S^3, so the double covering realizes SO_3 as the quotient of S^3 by the antipodal map, giving us that SO_3 is diffeomorphic to $\mathbb{RP}^3$

Many examples of low dimensional Spin groups arise from investigating the relationship between a 4 dimensional complex vector space V and its second exterior power $\Lambda^2 V$. Fix a volume form $\mu \in \Lambda^4 V^*$. This then induces a symmetric, nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ on $\Lambda^2 V$ by

$$\langle \alpha, \beta \rangle = \langle \alpha \wedge \beta, \mu \rangle$$

where $\langle \alpha \wedge \beta, u \rangle$ denotes the natural pairing of the vector space $\Lambda^4 V$ with its dual $\Lambda^4 V^*$. Fix a basis $\{e_i\}$ for V where $\mu(e_1 \wedge e_2 \wedge e_3 \wedge e_4) = 1$. In this basis, we see that the group of transformations $\mathrm{Aut}(V, \mu)$ perserving μ is isomorphic to the group $SL_4\mathbb{C}$. In addition, each map $T \in \mathrm{Aut}(V, \mu)$ induces a map $\Lambda^2 T : \Lambda^2 V \to \Lambda^2 V$, which is determined by the formula $\Lambda^2 T(v \wedge w) = Tv \wedge Tw$. For any $T \in \mathrm{Aut}(V, \mu)$, the induced map $\Lambda^2 T$ preserves the bilinear form on $\Lambda^2 V$, so the mapping $T \mapsto \Lambda^2 V$ determines a group homomorphism $\mathrm{Aut}(V, \mu) \to \mathrm{Aut}(\Lambda^2 V, \langle \cdot, \cdot \rangle)$, where $\mathrm{Aut}(\Lambda^2 V, \langle \cdot, \cdot \rangle)$ denotes the group of linear automorphisms perserving the bilinear form. The kernel of this map is $\{\pm \mathrm{id}_V\}$, and fixing an orthogonal basis for $\langle \cdot, \cdot \rangle$ gives us that this map is a double cover $SL_4\mathbb{C} \to SO_6\mathbb{C}$, so $SL_4\mathbb{C}$ is isomorphic to the complex spin group $\mathrm{Spin}_4\mathbb{C}$

If we then fix a hermitian inner product $h : V \times V \to \mathbb{C}$, we can consider the automorphisms $\mathrm{Aut}(V, \mu, h)$ preserving h and μ, which is isomorphic to the group SU_4. The bilinear form h induces a hermitian inner product (which we also denote h) on $\Lambda^2 V$ defined by

$$h(v_1 \wedge v_2, v_3 \wedge v_4) = \det \begin{pmatrix} h(v_1, v_3) & h(v_1, v_4) \\ h(v_2, v_3) & h(v_2, v_4) \end{pmatrix}$$

Then if $T \in \mathrm{Aut}(V, \mu, h)$, $\Lambda^2 T$ preserves the bilinear form $\langle \cdot, \cdot \rangle$ induced by μ as well as the hermitian inner product induced by h. The group that preserves both of these structures is isomorphic to $SO_6\mathbb{C} \cap U_6$, which is $SO_6\mathbb{R}$. This gives us that $SU_4 \cong \mathrm{Spin}_6$.

In general, one can play the game of fixing additional structure on V (e.g. a real stucture, quaternionic structure, symplectic form) and look for the induced structure on $\Lambda^2 V$. This then gives a map from automorphisms of V preserving this additional structure to automorphisms of $\Lambda^2 V$ preserving the induced structure. Playing this game then determines several other low dimensional Spin groups.

$$\begin{array}{lll} \mathrm{Spin}_5\mathbb{C} \cong Sp_4\mathbb{C} & \mathrm{Spin}_4 \cong Sp(4) & \mathrm{Spin}_4\mathbb{C} \cong SL_2\mathbb{C} \times SL_2\mathbb{C} \\ \mathrm{Spin}_{1,3} \cong SL_2\mathbb{C} & \mathrm{Spin}_{1,2} \cong SL_2\mathbb{R} & \mathrm{Spin}_{1,5} \cong SL_2\mathbb{H} \end{array}$$

Where $Sp_4\mathbb{C}$ denotes the group of 4×4 matrices preserving a symplectic form, $Sp(4) = Sp_4\mathbb{C} \cap U_4$, and $SL_2\mathbb{H}$ denotes the automorphisms of a 2 dimensional quaternionic vector space with determinant 1 when regarded as 4×4 complex matrices.

Definition 3.5. Given a Pin group $\mathrm{Pin}_{p,q}$, the ***Pinor representations*** are representations of $\mathrm{Pin}_{p,q}$ that arise from an irreducible Clifford module M (i.e. the action of $\mathrm{Pin}_{p,q}$ can be extended to an action of $\mathrm{Cliff}_{p,q}(\mathbb{R})$). The ***Spinor representations*** are defined analogously for the group $\mathrm{Spin}_{p,q}$. ❖

From the classification of Clifford modules, we get a classification of all the Pinor representations. From the relationship between a Clifford algebra and its even subalgebra, we also get a complete classification of all the Spinor representations.

4. Projective Spin Representations

Using the isomorphisms of the Clifford algebras with matrix algebras or products of matrix algebras along with the identification of the even subalgebra with another Clifford algebra, we have a complete classification of the irreducible modules over the even subalgebras $\mathrm{Cliff}^0_n(\mathbb{R})$. Restricting to the Spin group $\mathrm{Spin}_n \subset \mathrm{Cliff}^0_n(\mathbb{R})$, this gives us the Spin representations. These Spin representations define projective representations of the group SO_n.

Proposition 4.1. *Let G be a group, and V a finite dimensional irreducible representation of G. Then every element of the center acts by scalars, i.e. for $g \in Z(G)$, there exists a scalar λ_g such that for all v in V, we have*

$$g \cdot v = \lambda_g v$$

Proof. Let $g \in Z(G)$. Then for every $h \in G$ and $v \in V$, we have that

$$h \cdot (g \cdot v) = g \cdot (h \cdot v)$$

Therefore, the action of g defines a G-equivariant map $V \to V$, which by Schur's Lemma must necessarily be a scalar multiple of the identity. ∎

Given a Spin representation $\mathbb{S}$, we have that that the elements ± 1 act by scalars. Then since SO_n is the quotient of Spin_n by the subgroup $\{\pm 1\}$ of the center, we get a projective representation of SO_n on the projectivization $\mathbb{PS}$ in the following way: Given an element $A \in SO_n$, we have that A lifts to two elements $\{\pm \tilde{A}\} \subset \mathrm{Spin}_n$ which differ by -1. Since -1 acts by a scalar on $\mathbb{S}$, these elements determine the same action on $\mathbb{PS}$, giving us a well defined action of SO_n on $\mathbb{PS}$.

The Spin representation $\mathbb{S}$ is not realized canonically, while the projective Spin representation $\mathbb{PS}$ is canonical. Given an isomorphism $\varphi : V \to W$, this induces a unique algebra isomorphism $\mathrm{End}\, V \to \mathrm{End}\, W$ where $A \in \mathrm{End}\, V$ is mapped to $\varphi \circ A \circ \varphi^{-1}$. However, the converse is not true.

Proposition 4.2. *Let $\varphi : \mathrm{End}\, V \to \mathrm{End}\, W$ be an algebra isomorphism. Then there does not exist a unique isomorphism $\psi : V \to W$ inducing φ.*

To prove this, we need a lemma.

Lemma 4.3. *The group of algebra automorphisms $\mathrm{Aut}(M_n\mathbb{F})$ is isomorphic to the projective general linear group $PGL_n\mathbb{F} = GL_n\mathbb{F}/Z(GL_n\mathbb{F})$.*

Proof. Let $\alpha : M_n\mathbb{F} \to M_n\mathbb{F}$ be an algebra automorphism. We know that $M_n\mathbb{F}$ admits a single irreducible module M up to isomorphism, which is $\mathbb{F}^n$ with the standard action. Then α defines another module M^α, which is the same underlying vector space as M with the algebra action given by $T \cdot v = \alpha(T)v$, where the right hand side is the action of $\alpha(T)$ on the module M. Since α is an algebra automorphism, $M_n\mathbb{F}$ acts transitively on M^α, so it is also an irreducible module, which must be isomorphic to M. Therefore, there exists a module isomorphism $A : M \to M^\alpha$. Since M and M^α are the same underlying vector space, A is also a linear isomorphism $A : M \to M$, thought of as a vector space instead of a module. Then since A is a module homomorphism. we have that for any $T \in M_n\mathbb{F}$,

$$A \circ T = \alpha(T) \circ A \implies A \circ T \circ A^{-1} = \alpha(T)$$

so α is given by conjugation by $A \in GL(M)$. In a basis, this tells us that the map $GL_n\mathbb{F} \to M_n\mathbb{F}$ given by conjugation is surjective, and the kernel of this map is the center of $GL_n\mathbb{F}$, so by the first isomorphism theorem, we have that $\mathrm{Aut}(M_n\mathbb{F}) \cong PGL_n\mathbb{F}$. ∎

Proof of Proposition. Fix bases for V and W. These bases then induces isomorphisms $M_n\mathbb{F} \to V$ and $M_n\mathbb{F} \to W$. Then in these bases, the algebra isomorphism φ is given by an automorphism $M_n\mathbb{F} \to M_n\mathbb{F}$, and the question now translates to asking whether this automorphism induces an isomorphism $\mathbb{F}^n \to \mathbb{F}^n$. From the lemma, we know this is false – an automorphism of $M_n\mathbb{F}$ only determines an element of $PGL_n\mathbb{F}$, so it is induced by an entire family of automorphisms differing by $Z(GL_n\mathbb{F})$. ∎

However, an isomorphism $\varphi : \mathrm{End}\, V \to \mathrm{End}\, W$ does induce an isomorphism $\mathbb{P}V \to \mathbb{P}W$ of projective spaces. To see, this we make an identification between 1 dimensional subspaces of V with maximal left ideals of $\mathrm{End}\, V$.

Proposition 4.4. *There is a bijection*

$$\{\textit{Maximal left ideals of } \mathrm{End}\, V\} \longleftrightarrow \mathbb{P}V$$

Proof. Given a line $L \in \mathbb{P}V$, the ***annihilator*** of L is the set

$$\mathrm{Ann}(L) = \{M \in \mathrm{End}\, V \ : \ M(L) = 0\}$$

In fact, $\mathrm{Ann}(L)$ is a left ideal in $\mathrm{End}\, V$, since given $A \in \mathrm{End}\, V$ and $M \in \mathrm{Ann}(L)$, L lies in the kernel of $A \circ M$. We claim that $\mathrm{Ann}(L)$ is maximal. Suppose $\mathrm{Ann}(L) \subset I$ is properly contained in a left ideal I. Fix an ordered basis for V in which the first basis element is a nonzero element of L, then elements of $\mathrm{Ann}(L)$ are represented in this basis by matrices with all zeroes in the first column. Then since $\mathrm{Ann}(L)$ is properly contained in I, there exists some $M \in I$ such that $M \notin \mathrm{Ann}(L)$, which implies that as a matrix, the first column of M is nonzero. Then pick a matrix $A \in \mathrm{Ann}(L)$ in which the nonzero columns complete the first column into a basis for $\mathbb{R}^n$. Then $A+M$ is an invertible element of End V, so I must be all of End V. Therefore $\mathrm{Ann}(L)$ is maximal. To show that the mapping $L \mapsto \mathrm{Ann}(L)$ is a bijection, we produce an inverse. Let $I \subset$ End V be a maximal ideal. Then we claim that the subspace

$$\mathbb{V}(I) = \bigcap_{M \in I} \ker M$$

is a 1 dimensional subspace of V. We note that $\mathbb{V}(I)$ cannot be trivial, since this would imply that I would contain an invertible element, contradicting that it is a proper ideal. We also see that it cannot be higher than 2 dimensional, since otherwise, I woud be contained in the annihilator of a proper nontrivial subspace of $\mathbb{V}(I)$, contradicting maximality of I. We then claim that these two mappings are inverses. We certainly have that $\mathrm{Ann}(\mathbb{V}(I)) \supset I$, so by maximality, this must be I. In addition it is clear that $\mathbb{V}(\mathrm{Ann}(L)) = L$ by the definition of $\mathbb{V}(I)$ and the annihilator. Therefore, these mappings are inverses. ∎

Therefore, given an algebra isomorphism $\varphi : \mathrm{End}\, V \to \mathrm{End}\, W$, this induces a map $\mathbb{P}V \to \mathbb{P}W$ since the image of a maximal ideal under an isomorphism is a maximal left ideal. In addition, the induced map is a bijection, since it has an inverse given by the induced map of φ^{-1}. In addition, the group of units $GL(V)$ acts on $\mathbb{P}V$ by right multiplication – given a maximal ideal I and $A \in GL(V)$, the ideal $I \cdot A$ is also a maximal left ideal.

This gives us a canonical realization of the Spin representations. In the case that the even subalgebra is isomorphic to a matrix algebra $M_n\mathbb{F}$, the projective Spin representation is restriction of the action of $\mathrm{Cliff}_n^0(\mathbb{R})$ on maximal left ideals of $\mathrm{Cliff}_n^0(\mathbb{F})$. In the case that the even subalgebra is isomorphic to a product $M_n\mathbb{F} \times M_n\mathbb{F}$, the irreducible modules identify the subalgebras L and R isomorphic to $M_n\mathbb{F} \times \{0\}$ and $\{0\} \times M_n\mathbb{F}$ by singling out the maximal subalgebra that acts nontrivially. Looking at the maximal left ideals of these subalgebras then identifies the two projective Spin representations. In addition, since -1 acts trivially on left ideals, these projective Spin represenions descend to the quotient $\mathrm{Spin}_n / \{\pm 1\} \cong SO_n$, giving us the projective representations of SO_n.

CHAPTER 2

Spin Structures on Manifolds

1. Fiber Bundles

DEFINITION 1.1. Let M and F be smooth manifolds. Then a ***fiber bundle*** over M with model fiber F is a the data of a smooth manifold E with a smooth map $\pi : E \to M$ such that for every point $p \in M$, there is a neighborhood $U \subset M$ containing p such that there exists an diffeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ such that the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "p_U"] \\ & U & \end{tikzcd}$$

commutes, where P_U denotes projection onto the first factor. The map φ is called a ***local trivialization*** of the fiber bundle $\pi : E \to M$. ❖

Given a fiber bundle $\pi : E \to M$ we often denote the fiber $\pi^{-1}(p)$ by E_p.

DEFINITION 1.2. let $\pi : E \to M$ and $p : L \to M$ be fiber bundles with model fiber F over M. A ***bundle homomorphism*** is a smooth map $\varphi : E \to L$ such that the following diagram commutes

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & L \arrow[dl, "p"] \\ & M & \end{tikzcd}$$

❖

An important property of fiber bundles is that they pull back.

DEFINITION 1.3. Let $\pi : E \to M$ be a fiber bundle with model fiber F and let $f : X \to M$ be a smooth map. Then the ***pullback*** of E by f is the data of a smooth manifold

$$f^*E = \left\{ (x, p) \ : \ x \in X, p \in \pi^{-1}(f(x)) \right\}$$

along with the projection $p : f^*E \to M$ given by $(x, p) \mapsto x$, giving $f^*E \to X$ the structure of a fiber bundle over X with model fiber F. The bundle f^*E also comes equipped with a natural map $\alpha : f^*E \to E$ where $\alpha(x, p) = p$. Pullbacks give rise to the diagram

$$\begin{tikzcd} f^*E \arrow[r, "\alpha"] \arrow[d, "p"'] & E \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & M \end{tikzcd}$$

and are an instance of a general construction called the ***fibered product***. ❖

Definition 1.4. Let $\pi : E \to M$ be a fiber bundle. A ***local section*** of $\pi : E \to M$ is a smooth map $\sigma : U \to E$ of an open set $U \subset M$ such that $\pi \circ \sigma = \mathrm{id}_U$. If $U = M$, the section is called a ***global section***. Equivalently, it is the smooth assignment of an element in E_p to each point $p \in U$. We denote the set of sections of $\pi : E \to M$ over a set U as $\Gamma_U(E)$. ❖

A fiber bundle is a very general construction in which the fibers F do not necessarily admit extra structure. An important special case of a fiber bundle is a vector bundle, where the fibers are vector spaces.

Definition 1.5. Let M be a smooth manifold. A ***vector bundle*** over M is fiber bundle $\pi : E \to M$ with model fiber $\mathbb{R}^n$ (or $\mathbb{C}^n$) such that the local trivializations $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ (or $\mathbb{C}^n$) restrict to linear isomorphisms on the fibers, i.e. for all $p \in U$, the restriction $\varphi|_{\pi^{-1}(p)} : \pi^{-1}(p) \to \{p\} \times \mathbb{R}^n$ (or $\mathbb{C}^n$) is an isomorphism. The dimension n of the fibers is called the ***rank*** of the vector bundle. A ***vector bundle homomorphism*** $\varphi : E \to L$ is a bundle homomorphism with the added stipulation that the restrictions to the fibers $\varphi|_{E_x}$ are linear maps. ❖

Example 1.6.

(1) Given a smooth manifold M, the tangent bundle $TM = \coprod_{p \in M} T_pM$ is a vector bundle, where the rank is the dimension of M.
(2) The ***tautological bundle*** over $\mathbb{RP}^n$ is the vector bundle that assigns to each subspace $\ell \in \mathbb{RP}^n$ itself as its fiber. An analgous construction defines the tautological bundle over the Grassmannian $\mathrm{Gr}_k(\mathbb{R}^n)$.

◀

Definition 1.7. A ***Lie group*** is a smooth manifold G with a group structure such that the multiplication map $(g, h) \mapsto gh$ and the inversion map $g \mapsto g^{-1}$ are smooth. ❖

Example 1.8.

(1) The group $GL_n\mathbb{R}$ of invertible $n \times n$ matrices is an open subset of $M_n\mathbb{R}$, and therefore a smooth n^2 dimensional manifold.
(2) The group $SL_n\mathbb{R}$ of $n \times n$ matrices with determinant 1 is a closed submanifold of $GL_{\mathbb{R}}$.
(3) The orthogonal groups O_n and special orthogonal groups SO_n are Lie groups.
(4) The unitary groups U_n and special unitary groups SU_n are Lie groups.

◀

Another important class of fiber bundles are principal bundles, in which the fibers have the structure of G-torsors.

Definition 1.9. Let G be a Lie group, and M a smooth manifold. A ***Principal G-bundle*** over M is the data of

(1) A smooth manifold P with a map $\pi : P \to M$.
(2) A smooth right G-action on P that is free and transitive on the fibers of π.
(3) For every point $p \in M$, a neighborhood $U \subset M$ containing p and a G-equivariant diffeomorphism $\varphi : \pi^{-1}(U) \to U \times G$ (where the right action on $U \times G$ is right multiplication on the second factor) such that we get the commutative diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "p_U"] \\ & U & \end{tikzcd}$$

where p_U denotes projection onto the first factor.

A ***principal bundle homomorphism*** is a bundle homomorphism $\varphi : P \to Q$ that is G-equivariant. ❖

Example 1.10. Given a smooth manifold M and a point $p \in M$, a basis of the tangent space is a linear isomorphism $b : \mathbb{R}^n \to T_pM$. The group $GL_n\mathbb{R}$ acts freely and transitively on the set of bases $\mathcal{B}_p$ on the right by $b \cdot g = b \circ g$. Then the ***frame bundle*** of M, denoted $\mathcal{B}(M)$ is the disjoint union

$$\mathcal{B}(M) = \coprod_{p \in M} \mathcal{B}_p$$

where π is the projection map $(p, b) \mapsto p$. Then $\mathcal{B}(M)$ is a principal $GL_n\mathbb{R}$ bundle over M. ⋖

EXAMPLE 1.11. Given a smooth manifold M, a Riemannian metric g induces an inner product g_p on each tangent space T_pM, where g_p denotes the metric g evaluated at p. Then the set of orthonormal bases of T_pM is the set of all linear isometries $(\mathbb{R}^n, \langle\cdot,\cdot\rangle) \to (M, g_p)$ where $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{R}^n$. Then taking the disjoint union over all points $p \in M$ of orthonormal bases for the tangent spaces T_pM forms the ***orthonormal frame bundle*** $\mathcal{B}_O(M)$, which is a principal O_n bundle. ⋖

DEFINITION 1.12. Let $\pi : P \to M$ be a principal G-bundle over M, and let F be a smooth manifold with a smooth left G action. Then the ***associated fiber bundle***, denoted $P \times_G F$, is the set

$$P \times_G F = P \times G/(p, g) \sim (p \cdot h, h^{-1}g)$$

Since the group action on P preserves the fibers, the projection $p_1 : P \times F \to P$ composed with the projection $\pi : P \to M$ descends to the quotient, givng us a projection map $\Phi : P \times_G F \to M$. ❖

The first thing to check is that $P \times_G F$ is a fiber bundle, justifying the name.

PROPOSITION 1.13. *Let $\pi : P \to M$ be a principal G-bundle and F a manifold with a left G action. then the associated bundle $\Phi : P \times_G F \to M$ is a fiber bundle with model fiber F.*

PROOF. We wish to provide local trivializations $\Phi^{-1}(U) \to U \times F$ for the associated bundle. Fix a local trivialization $\psi : \pi^{-1}(U) \to U \times G$. Then ψ is of the form $\psi(p) = (\pi(p), \tilde{\psi}(p))$ for some $\tilde{\psi} : \pi^{-1}(U) \to G$, satisfying $\tilde{\psi}(p \cdot g) = \tilde{\psi}(p)g$. Define

$$\begin{aligned} \varphi : \Phi^{-1}(U) &\to U \times F \\ [p, f] &\mapsto (\pi(p), \tilde{\psi}(p) \cdot f) \end{aligned}$$

We first note that this is well defined on equivalence classes, since

$$[p \cdot g, g^{-1} \cdot f] \mapsto (\pi(p \cdot g), \tilde{\psi}(p \cdot g) \cdot g^{-1} \cdot f) = (\pi(p), \psi(p) \cdot f)$$

So φ is well defined. We also note that $\Phi(\varphi[p, f]) = \pi(p)$ by how Φ was defined, so φ is a local trivialization provided it is a homeomorphism. To show this, we construct an inverse. Define $\alpha : U \times F \to \Phi^{-1}(U)$ by $\alpha(u, f) = [\psi^{-1}(u, e), f]$, where e denotes the identity element of G. We then claim that α is the inverse. We compute

$$\begin{aligned} (\varphi \circ \alpha)(u, f) &= \varphi[\psi^{-1}(u, e), f] \\ &= (u, e \cdot f) \\ &= (u, f) \end{aligned}$$

In the other direction, we compute

$$\begin{aligned} (\alpha \circ \varphi)[p, f] &= \alpha(\pi(p), \tilde{\psi}(p)) \\ &= [\psi^{-1}(\pi(p), e), f] \\ &= [p, f] \end{aligned}$$

Therefore, φ is a local trivialization, giving us that $\Phi : P \times_G F \to M$ is a fiber bundle with model fiber F. ■

There is a correspondence between sections of an associated bundle and G-equivariant maps $P \to F$.

PROPOSITION 1.14. *Let $\pi : P \to M$ be a principal G-bundle. Then there is a bijection*

$$\{G\text{-equivariant maps } P \to F\} \longleftrightarrow \Gamma_M(P \times_G F)$$

PROOF. Since F has a left G-action, we first convert this to a right G-action by $f \cdot g = g^{-1} \cdot f$. Then what we mean by a G-equivariant map is a map $\varphi : P \to F$ such that

$$\varphi(p \cdot g) = g^{-1} \cdot \varphi(p)$$

We then wish to use G-equivariant map φ to produce a section $\tilde{\varphi} : M \to P \times_G F$ of the associated bundle. For a point $x \in M$, pick any $p \in \pi^{-1}(x)$ in the fiber. Then define

$$\tilde{\varphi}(x) = [p, \varphi(p)]$$

where $[p, \varphi(p)]$ denotes the equivalence class of $(p, \varphi(p))$ in $P \times_G F$. We first claim that this map is well defined, i.e. it is independent of our choice of point $p \in \pi^{-1}(x)$. We know that G acts freely and transitively on $\pi^{-1}(x)$, so all points in the fiber are of the form $p \cdot g$ for a unique $g \in G$. Then we have that for any $g \in G$

$$(p \cdot g, \varphi(p \cdot g)) = (p \cdot g, g^{-1} \cdot \varphi(p)) \sim (p, \varphi(p))$$

where we use the G-equivariance of φ and the definition of the equivalence relation on the associated bundle. In addition, this induced map is a section of $P \times_G F \to M$, since the image is represented by an element of $P \times F$ with an element of the fiber $\pi^{-1}(x)$ in the first factor.

Conversely, given a section $\sigma : M \to P \times_G F$, we wish to produce a G-equivariant map $P \to F$. Given such a section, and a point $x \in M$, we have that $\sigma(x) = [p, f]$ for some $p \in P$ and $f \in F$. Then define $\tilde{\sigma} : P \to F$ such that $\tilde{\sigma}(p) = f$, and $\tilde{\sigma}(p \cdot g) = g^{-1} \cdot f$. Since G acts freely and transitively on the fibers, this is well defined, and specifies the map on every point of P. In addition, $\tilde{\sigma}$ is G-equivariant by construction. Then the maps

$$\{G\text{-equivariant maps } P \to F\} \longleftrightarrow \Gamma_M(P \times_G F)$$

we provided are easily verified to be inverses to each other, giving us the correspondence. ■

In some sense, the geometry of the associated fiber bundle $P \times_G F$ is controlled by the group G, as G determines a disinguished group of symmetries of the fiber F.

Example 1.15 (The tangent bundle as an associated bundle). Given a manifold M, we can take the frame bundle $\pi : \mathcal{B}(M) \to M$, which is a principal $GL_n\mathbb{R}$ bundle. The group $GL_n\mathbb{R}$ acts linearly on $\mathbb{R}^n$ in the standard way, giving us an associated vector bundle $\mathcal{B}(M) \times_{GL_n\mathbb{R}} \mathbb{R}^n$. We claim that this bundle is isomorphic to the tangent bundle TM, i.e. there exists an diffeomorphism $\varphi : \mathcal{B}(M) \times_{GL_n\mathbb{R}} \mathbb{R}^n \to TM$ that restricts to linear isomorphisms on the fibers and the diagram

$$\begin{tikzcd} \mathcal{B}(M) \times_{GL_n\mathbb{R}} \mathbb{R}^n \arrow[rr, "\varphi"] \arrow[dr] & & TM \arrow[dl] \\ & M & \end{tikzcd}$$

commutes, where the maps to M are the bundle projections. Recall that elements of $\mathcal{B}(M) \times_{GL_n\mathbb{R}} R^n$ are represented by pairs (b, v), where $b : \mathbb{R}^n \to T_{\pi(b)}M$ is a linear isomorphism, and v is a vector in $\mathbb{R}^n$. Then define φ by

$$\varphi[b, v] = (\pi(b), b(v))$$

This is well defined, since $\varphi[b \circ g, g^{-1}(v)] = (\pi(b \circ g), (b \circ g)(g^{-1}(v))) = (\pi(b), b(v))$. This is an isomorphism, where the inverse mapping maps $(p, v) \in TM$ to $(b, \tilde{v})$ where b is any basis of T_pM and $\tilde{v}$ is the coordinate representation of v in the basis b. In this example, we see that the associated bundle identifies the same vector under different coordinate transformations, which defines the symmetries of TM. ◂

In general, given a rank n vector bundle $E \to M$, we can construct the frame bundle $\mathcal{B}(E)$ for E and recover E by taking the associated bundle $\mathcal{B}(E) \times_{GL_n\mathbb{R}} \mathbb{R}^n$, so the process of taking frames and constructing associated bundles are inverses.

Definition 1.16. Let $\pi : P \to M$ be a principal G-bundle, and $\rho : H \to G$ a group homomorphism. The map ρ gives P a left H action where $h \cdot p = \rho(h) \cdot p$. A ***reduction of structure group*** is the data of a principal H bundle $\varphi : Q \to M$ and an H-equivariant bundle homomorphism $F : Q \to P$. ❖

The map $F : Q \to P$ induces a map $\tilde{F} : Q \times_H G \to P$, where we map the equivalence class $[q, g]$ to $F(q)g$. This is well defined on equivalences classes since

$$(q \cdot h, \rho(h)^{-1}g) \mapsto F(q \cdot h)\rho(h)^{-1}g = F(q)\rho(h)\rho(h)^{-1}g = F(q)g$$

Example 1.17 (Reduction from $GL_n\mathbb{R}$ to O_n). Let M be a smooth manifold, and $\pi : \mathcal{B}(M) \to M$ its bundle of frames. The inclusion map $\iota : O_n \hookrightarrow GL_n\mathbb{R}$ gives an action of O_n on $\mathcal{B}(M)$, where given $b \in \mathcal{B}(M)$ and $T \in O_n$, $b \cdot T = b \cdot \iota(T)$. We then take the quotient by this O_n action, giving us a quotient map $q : \mathcal{B}(M)/O_n$. Since the inclusion is injective, the O_n action is free on $\mathcal{B}(M)$, so this gives $q : \mathcal{B}(M) \to \mathcal{B}(M)/O_n$ the structure of a O_n bundle. In addition, the action of O_n preserves the fibers of $\pi : \mathcal{B}(M) \to M$, so π descends to the quotient, so $\mathcal{B}(M)/O_n \to M$ is a fiber bundle with model fiber $GL_n\mathbb{R}/O_n$. Since $GL_n\mathbb{R}$ deformation retracts onto O_n via the Gram-Schmidit algorithm, $GL_n\mathbb{R}/O_n$ is contractible, so the fiber bundle $\mathcal{B}(M)/O_n \to M$ admits global sections. Then given a section $\sigma : M \to \mathcal{B}(M)/O_n$, this gives an O_n bundle over M via the pullback $\sigma^*\mathcal{B}(M)$. In addition, we get a O_n-equivariant map $\sigma^*\mathcal{B}(M) \to \mathcal{B}(M)$ given by $(p, b) \mapsto (p, \iota(b))$. The bundle $\sigma^*\mathcal{B}(M)$ can be thought of as the bundle of orthonormal frames with respect to some Riemannian metric on M, and the fact that $\mathcal{B}(M)/O_n$ admits sections corresponds to the fact that every manifold admits a Riemannian metric. ◁

2. Dirac Operators in $\mathbb{R}^n$

One important application of Clifford algebras and Spin groups comes from physics. In the process of developing a relativistic equation for the electron, Paul Dirac saw the need for a first order differential operator D such that D squared to the Laplace operator[1].

$$\Delta = -\sum_i \frac{\partial^2}{\partial(x^i)^2}$$

If D were the be first order, it would have to be written as

$$D = a^i \frac{\partial}{\partial x^i}$$

for some coefficients a^i. However, it is clear that choosing scalar coefficients for the a^i will not suffice. For example, in $\mathbb{R}^2$ with the standard coordinates x and y, we see that given any first order operator $D = a^1\partial_x + a^2\partial_y$ satisfies

$$D^2 = \left(a^1 \frac{\partial}{\partial x} + a^2 \frac{\partial}{\partial y}\right)^2 = (a^1)^2 \frac{\partial^2}{\partial x^2} + a^1 a^2 \frac{\partial^2}{\partial x \partial y} + a^2 a^1 \frac{\partial^2}{\partial y \partial x} + (a^2)^2 \frac{\partial^2}{\partial y^2}$$

From this equation, we see that the a^i must square to -1, and we must also have that $a^1a^2 + a^2a^1 = 0$ in order for the mixed partial terms to vanish. This is not possible if the a^i are scalars (in either $\mathbb{R}$ or $\mathbb{C}$). However, the required relations are exactly the relations between orthogonal basis vectors in the Clifford algebra $\mathrm{Cliff}_{0,n}(\mathbb{R})$!

Definition 2.1. Let $\{e_i\}$ be the standard basis for $\mathbb{R}^n$, $\{e^i\}$ its dual basis, and $x^1, \ldots, x^n$ the standard coordinates on $\mathbb{R}^n$. The ***Dirac operator*** on $\mathbb{R}^n$ is the first order differential operator

$$D = e^i \frac{\partial}{\partial x^i}$$

❖

It is not clear from the definition what function space D should act on. The partial derivative operators make sense for any vector valued function, but multiplcation by e^i does not make sense an arbitrary vector space – it must be a Clifford module. Therefore, D is an operator $D : C^\infty(\mathbb{R}^n, M) \to C^\infty(\mathbb{R}^n, M)$ on smooth functions from $\mathbb{R}^n$ to a Clifford module M.

3. Spin Structures

Every manifold M admits a Riemannian metric, so there always exists a reduction of structure group from $GL_n\mathbb{R}$ to O_n. If M is orientable, we can reduce the structure group further to SO_n, which corresponds to a choice of orientation on M. A ***Spin structure*** on M is a further reduction of structure group to Spin_n, with respect to the double cover $\mathrm{Spin}_n \to SO_n$, i.e. the data of a principal Spin_n bundle $P \to M$, along with a Spin_n-equivariant map $P \to \mathcal{B}_{SO}(M)$, where $\mathcal{B}_{SO}(M)$ denotes the bundle of positively oriented orthonomal frames and Spin_n acts on $\mathcal{B}_{SO}(M)$ through the double cover $\mathrm{Spin}_n \to SO_n$. In particular, this map is a double cover, and fiber-wise is the double covering $\mathrm{Spin}_n \to SO_n$. Depending on the value n, there exist either one or two Spin representations for Spin_n. Let $\mathbb{S}$ denote the direct sum all of all the Spin representations for Spin_n, i.e. $\mathbb{S}$ is equal to the single Spin representation in the case that Spin_n only admits one representation, and $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ in the case that Spin_n admits two inequivalent Spin representations $\mathbb{S}^+$ and $\mathbb{S}^-$. Then

[1]Our reasoning for choosing this sign convention for Δ is twofold – one reason is that the spectrum of Δ is positive with this choice of sign, and the second is that this definition coincides with a generalized Laplace operator on a Riemannian manifold.

given a Spin structure on M with a principal Spin_n bundle P, the ***spinor bundle*** over M is the associated vector bundle $P \times_{\mathrm{Spin}_n} \mathbb{S}$.

Example 3.1 (Spin structures on S^1). The group Spin_1 is equal to $\{\pm 1\}$, so a principal Spin_1-bundle over $\mathbb{S}^1$ is a double cover $\pi : P \to \mathbb{S}^1$ along with a Spin_1-equivariant map $P \to \mathcal{B}_{SO_1}(S^1)$. Since $SO_1 = 1$, $\mathcal{B}_{SO}(S^1)$ is the trivial bundle $S^1 \times \{1\}$. Therefore, specifying a Spin_1-equivariant map $P \to \mathbb{S}^1 \times \{1\}$ is no additional data, since we are forced to map all of the fiber $\pi^{-1}(x)$ to $(x, 1)$ for any $x \in S^1$. Consequently, all double covers give rise to a reduction of structure group to Spin_1. There are only two double covers of S^1. One of them is the disconnected double cover, which is the disjoint union $S^1 \coprod S^1$, which we denote as $\pi_1 : P_1 \to S^1$. The other is the connected double cover, which the circle covering itself via the map $z \mapsto z^2$, which we denote as $\pi_2 : P_2 \to S^1$. For convenience, we parameterize P_2 with angles $\theta \in [0, 4\pi)$, so the covering map is given by $\theta \mapsto e^{i\theta}$. The Spin representation is the sign representation on $\mathbb{R}$, where -1 acts by multiplication by -1, and the complexifying gives us an action on $\mathbb{C}$ where -1 acts by multiplication by -1, giving us two spinor bundles $P_1 \times_{\mathrm{Spin}_1} \mathbb{C}$ and $P_2 \times_{\mathrm{Spin}_1} \mathbb{C}$.

In the first case,the associated bundle is a trivial bundle. Using the identification with Spin_1-equivariant maps $P_1 \to \mathbb{C}$ with sections of the associated bundle, it suffices to find such a map to produce a global section of $\pi_1 : P_1 \to S^1$. Write P_1 as the disjoint union $S_1 \coprod S_1$, with the circles parameterized by angles $\theta, \varphi \in [0, 2\pi)$. The Spin_1 action is then given by $\theta \mapsto \varphi$ and $\varphi \mapsto \theta$. Then the mappings $\theta \mapsto e^{i\theta}$ and $\varphi \mapsto e^{i\varphi}$ define a Spin_1-equivariant map $P_1 \to \mathbb{C}$, giving us a trivialization of the associated bundle $P_1 \times_{\mathrm{Spin}_1} \mathbb{C}$. In addition, we see that sections of $P_1 \times_{\mathrm{Spin}_1} \mathbb{C}$ are equivalent data to maps $S^1 \to \mathbb{C}$, since once we map one of the components of P_1 into $\mathbb{C}$, this entirely determines how we need to map the other component in order to remain Spin_1-equivariant. This further allows us to identify sections of the spinor bundle with 2π-periodic functions $\mathbb{R} \to \mathbb{C}$.

In the second case, the bundle is also trivial! We again contruct a trivialization for the associated bundle by providing a Spin_1-equivariant map $\sigma : P_2 \to \mathbb{C}$. Using the parameterization of P_2 with angles $\theta \in [0, 4\pi)$, the Spin_1 action on P_2 is given by $-1 \cdot \theta = \theta + 2\pi \mod 4\pi$. Then define σ by $\sigma(\theta) = e^{i\theta/2}$. This map is Spin_1-equivariant, so it produces a trivialization of the spinor bundle $P_2 \times_{\mathrm{Spin}_1} \mathbb{C}$. In addition, we see that sections of the spinor bundle correspond to 2π-antiperiodic functions, i.e. functions $\psi : \mathbb{R} \to \mathbb{C}$ such that $f(\theta) = -f(\theta + 2\pi)$, using the fact that we parameterized P_2 with angles from $[0, 4\pi)$. Another way to write a 2π-antiperiodic map ψ is as a product $\psi(\theta) = e^{i\theta/2} f(\theta)$ for a 2π-periodic function f, which is the representation of the section ψ with respect to the trivialization σ defined above.

The two different spin structures produced two isomorphic vector bundles, but there is still a way to distinguish between the two – their Dirac operators. The Clifford algebra $\mathrm{Cliff}_{0,1}(\mathbb{R})$ is isomorphic to $\mathbb{C}$ as an $\mathbb{R}$-algebra via the mappings $1 \mapsto 1$, and $e_1 \mapsto i$, so the Dirac operator on $\mathbb{R}$ can also be written as $i\frac{d}{dt}$. We then use our identifications of sections of the spinor bundles with functions $\mathbb{R} \to \mathbb{C}$ to investigate the Dirac operators on each bundle. In the case of the disconnected double cover P_1, we have the identifications of sections of $P_1 \times \mathrm{Spin}_1\mathbb{C}$ with 2π-periodic functions $\mathbb{R} \to \mathbb{C}$. Then given a section ψ, we can identify it as a function $\mathbb{R} \to \mathbb{C}$, and use the Dirac operator in $\mathbb{R}$, which will again be a 2π-periodic function, giving us another section, giving us that the Dirac operator D_1 on sections of $P_1 \times_{\mathrm{Spin}_1} \mathbb{C}$ is just $i\frac{d}{d\theta}$. For the connected double cover, we used the global section associated to $\sigma(\theta) = e^{i\theta/2}$ to identify sections of $P_2 \times_{\mathrm{Spin}_1} \mathbb{C}$ as products $\psi(\theta) = e^{i\theta/2} f(\theta)$ for a 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$. Then applying the Dirac operator from $\mathbb{R}$ to this function, we get

$$\begin{aligned} D\psi(\theta) &= i\frac{d}{d\theta}\left(e^{i\theta/2} f(\theta)\right) \\ &= e^{i\theta/2}\frac{\partial f}{\partial \theta} - \frac{1}{2}e^{i\theta/2} \end{aligned}$$

So in the local trivialization $\sigma(\theta) = e^{i\theta/2}$, the operator D_2 operates on 2π-periodic functions, just like D_1, and is given by $D_2 = i\partial_\theta - \frac{1}{2}$. In particular, the first operator D_1 has integer spectrum, and the spectrum of D_2 is the spectrum of D_1 shifted by $\frac{1}{2}$, which allows us to distinguish to two Spin structures on S^1 by their Dirac operators. ◂

Given an orientable manifold M, we can reduce the structure group of the frame bundle $\mathcal{B}(M)$ to SO_n, giving us a principal SO_n bundle $\mathcal{B}_{SO}(M)$. If M has a nonempty boundary, this induces an orientation on ∂M in the following way: by first reducing the structure group to O_n, we get a Riemannian metric g on M. Then given a point $p \in \partial M$, the tangent space of the boundary $T_p\partial M$ is a codimension 1 subspace of T_pM. The Riemannian metric g allows us to pick a distinguished complementary subspace to $T_p\partial M$ – the orthogonal complement $(T_p\partial M)^\perp$. From this subspace, we have a distinguished choice of vector – the outward normal vector. In appropriate coordinates, the inclusion of the tangent space $T_p\partial M$ is given locally by the inclusion

$$\begin{gathered}\mathbb{R}^{n-1} \hookrightarrow \mathbb{R}^n \\ (x^1, \dots, x^{n-1}) \mapsto (0, x^1, \dots, x^{n-1})\end{gathered}$$

and the outward normal is the unit length vector with a positive first component. This defines a vector field N along ∂M, where the value N_p of N at the point p is the outward normal vector in T_pM. On the boundary ∂M, we restrict the frame bundle $\mathcal{B}_{SO}(M)$ to ∂M by pulling back by the inclusion $\partial M \hookrightarrow M$, giving us the restricted bundle $\mathcal{B}_{SO}(M)|_{\partial M}$, which is a principal SO_n-bundle over ∂M. An element $b \in \mathcal{B}_{SO}(M)$ is an orientation preserving linear isometry $(\mathbb{R}, \langle\cdot,\cdot\rangle) \to (T_pM, g_p)$, and using the normal vector, we define a subbundle $\mathcal{B}_{SO}(\partial M) \subset \mathcal{B}_{SO}(M)|_{\partial M}$ by

$$\mathcal{B}_{SO}(\partial M) = \left\{ b \in \mathcal{B}_{SO}(M)|_{\partial M} \ : \ b(e_1) = N_{\pi(e_1)} \right\}$$

where e_1 denotes the first standard basis vector of $\mathbb{R}^n$, and π is the bundle projection. We then get an SO_{n-1} action on $\mathcal{B}_{SO}(\partial M)$, where we include $SO_{n-1} \hookrightarrow SO_n$ as matrices of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$$

Where $A \in SO_{n-1}$. This then acts on each fiber of $\mathcal{B}_{SO}(\partial M)$ by precomposition. This action is free and transitive, so this gives $\mathcal{B}_{SO}(\partial M)$ the stucture of a principal SO_{n-1}-bundle over ∂M. In this way, we see that an orientation on M determines an orientation on the boundary. The preceding discussion is summarized in the diagram

$$\begin{tikzcd}
\mathcal{B}_{SO}(\partial M) \arrow[r, hook] \arrow[dr] & \mathcal{B}_{SO}(M)|_{\partial M} \arrow[r, hook] \arrow[d] & \mathcal{B}_{SO}(M) \arrow[d] \\
& \partial M \arrow[r, hook] & M
\end{tikzcd}$$

In a similar fashion, a Spin structure on M will also induce a Spin structure on ∂M, though the process is slightly more involved. Given a Spin manifold M, it comes equipped with a principal Spin_n-bundle $\mathcal{B}_{\mathrm{Spin}}(M)$ along with a Spin_n-equivariant map $\mathcal{B}_{\mathrm{Spin}}(M) \to \mathcal{B}_{SO}(M)$, where the Spin action on $\mathcal{B}_{SO}(M)$ is induced by the double cover $\mathrm{Spin}_n \to SO_n$. Just as before, we pullback both $\mathcal{B}_{SO}(M)$ and $\mathcal{B}_{\mathrm{Spin}}(M)$ along the inclusion $\partial M \hookrightarrow M$. In addition, we construct the SO_{n-1}-bundle $\mathcal{B}_{SO}(\partial M)$, which gives us the diagram

$$\begin{tikzcd}
& \mathcal{B}_{\mathrm{Spin}}(M)|_{\partial M} \arrow[r, hook] \arrow[d] & \mathcal{B}_{\mathrm{Spin}}(M) \arrow[d] \\
\mathcal{B}_{SO}(\partial M) \arrow[r, hook] \arrow[dr] & \mathcal{B}_{SO}(M)|_{\partial M} \arrow[r, hook] \arrow[d] & \mathcal{B}_{SO}(M) \arrow[d] \\
& \partial M \arrow[r, hook] & M
\end{tikzcd}$$

This diagram tells us the exact ingredients we need to construct the Spin_{n-1}-bundle over ∂M – it must be the pullback of $\mathcal{B}_{\mathrm{Spin}}(M)|_{\partial M}$ along the inclusion $\mathcal{B}_{SO}(\partial M) \hookrightarrow \mathcal{B}_{SO}(M)|_{\partial M}$, so it fits into the commutative square

$$\begin{tikzcd}
\mathcal{B}_{\mathrm{Spin}}(\partial M) \arrow[r, hook] \arrow[d] & \mathcal{B}_{\mathrm{Spin}}(M)|_{\partial M} \arrow[d] \\
\mathcal{B}_{SO}(\partial M) \arrow[r, hook] & \mathcal{B}_{\mathrm{Spin}}(M)|_{\partial M}
\end{tikzcd}$$

In addition, it comes equipped with a map $p : \mathcal{B}_{\text{Spin}}(\partial M) \to \partial M$ by composing the map $\mathcal{B}_{\text{Spin}}(\partial M) \to \mathcal{B}_{SO}(\partial M)$ with the projection $\pi : \mathcal{B}_{SO}(\partial M)$. However, it is not immediately clear that the pullback bundle, (which we will suggestively denote by $\mathcal{B}_{\text{Spin}}(\partial M)$) is indeed a principal Spin_{n-1}-bundle over ∂M. As a principal Spin_n-bundle, $\mathcal{B}_{\text{Spin}}(M)|_{\partial M}$ comes equipped with an action of Spin_n. We then take the subgroup $G \subset \text{Spin}_n$ of elements that preserve the fibers of the map p. Since the Spin_n action on $\mathcal{B}_{\text{Spin}}(M)$ is free and transtive on the fibers of $\mathcal{B}_{\text{Spin}}(M)|_{\partial M} \to \partial M$, the action of G on $\mathcal{B}_{\text{Spin}}(\partial M)$ will be as well, giving it the structure of a principal G-bundle over ∂M. We then claim that $G \cong \text{Spin}_{n-1}$. Since the fibers are G-torsors, it suffices to check on a single fiber of the map $\mathcal{B}_{\text{Spin}}(M) \to \partial M$. Fix point $x \in \partial M$, and an oriented orthonormal frame $b \in \pi^{-1}(x)$. This then determines a diffeomorphism $\varphi : SO_{n-1} \to \pi^{-1}(x)$. where $\varphi(g) = g \cdot x$. This diffeomorphism induces a group stucture on the fiber $\pi^{-1}(x)$ such that φ is a group isomorphism. Then the restriction of the double cover $\mathcal{B}_{\text{Spin}}(M)|_{\partial M} \to \mathcal{B}_{SO}(M)|_{\partial M}$ to $\mathcal{B}_{\text{Spin}}(\partial M)$ is also a double cover. Fixing a point b' in the preimage of b under $\mathcal{B}_{\text{Spin}}(\partial M) \to \mathcal{B}_{SO}(\partial M)$ then determines an isomorphism of the fiber with G, which then determines a 2-1 group homomorphism $G \to SO_{n-1}$. Therefore, $G \cong \text{Spin}_{n-1}$ and $\mathcal{B}_{Spin}(\partial M)$ determines a Spin structure on ∂M. All in all, the construction is summarized by the diagram

$$\begin{tikzcd}
\mathcal{B}_{\text{Spin}}(\partial M) \arrow[r, hook] \arrow[d] & \mathcal{B}_{\text{Spin}}(M)|_{\partial M} \arrow[r, hook] \arrow[d] & \mathcal{B}_{\text{Spin}}(M) \arrow[d] \\
\mathcal{B}_{SO}(\partial M) \arrow[r, hook] \arrow[dr] & \mathcal{B}_{SO}(M)|_{\partial M} \arrow[r, hook] \arrow[d] & \mathcal{B}_{SO}(M) \arrow[d] \\
 & \partial M \arrow[r, hook] & M
\end{tikzcd}$$

Example 3.2 (The induced Spin structure on ∂D^2). Let $D^2 = \{v \in \mathbb{R}^2 \ : \ |v| \leq 1\}$ be the 2-disk, equipped with the Riemannian metric inherited from $\mathbb{R}^2$. We have that $\partial D^2 = S^1$. Since D^2 is contractible, both $\mathcal{B}_{SO}(D^2)$ and $\mathcal{B}_{\text{Spin}}(D^2)$ are trivial bundles, so their restrictions onto ∂D^2 are also trivial. In addition, we have that $\text{Spin}_2 \cong SO_2$, and the covering map is given by $g \mapsto g^2$. Using the orientation on D^2 inherited from $\mathbb{R}^2$, we need to construct the induced orientation on ∂D^2. To do so, we parameterize ∂D^2 by angles $\theta \in [0, 2\pi)$. Then the outward normal at each point $\theta \in \partial D^2$ is the vector $(\cos\theta, \sin\theta)$, where we use the canonical identification of T_pD^2 with $\mathbb{R}^2$. Then the bundle $\mathcal{B}_{SO}(\partial D^2) \subset \mathcal{B}_{SO}(D^2)|_{\partial D^2}$ where the fiber over $\theta \in \partial D^2$ is the linear map b_θ given by the matrix

$$b_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Then pulling back $\mathcal{B}_{\text{Spin}}(D^2)|_{\partial D^2}$ by the inclusion $\mathcal{B}_{SO}(\partial D^2) \hookrightarrow \mathcal{B}_{SO}(D^2)|_{\partial D^2}$, we have that

$$\mathcal{B}_{\text{Spin}}(\partial D^2) = \left\{(\theta, g) \ : \ g^2 = b_\theta\right\}$$

Explictly, this means that the fiber over a point θ is the two point set

$$\left\{ \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, \begin{pmatrix} -\cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{pmatrix} \right\}$$

which tells us that the Spin_1-bundle $\mathcal{B}_{\text{Spin}}(\partial D^2)$ is the connected double cover given by $g \mapsto g^2$. ◄

4. $\text{Pin}_n^\pm$ Structures

The same general construction for Spin structures works for $\text{Pin}_n^\pm$ as well.

Definition 4.1. Let M be a smooth manifold. Then a $\text{Pin}_n^\pm$ ***structure*** on M is the data of a principal $\text{Pin}_n^\pm$-bundle $P \to M$, along with a $\text{Pin}_n^\pm$-equivariant map $P \to \mathcal{B}_O(M)$, where the $\text{Pin}_n^\pm$ action on $\mathcal{B}_O(M)$ is given by the double cover $\text{Pin}_n^\pm \to O_n$. ❖

Just like with Spin, a $\mathrm{Pin}_n^\pm$ structure on M induces a $\mathrm{Pin}_{n-1}^\pm$ on the boundary using the outward unit normal, which gives the analogous diagram

$$\begin{tikzcd}
\mathcal{B}_{\mathrm{Pin}^\pm}(\partial M) \arrow[r, hook] \arrow[d] & \mathcal{B}_{\mathrm{Pin}^\pm}(M)|_{\partial M} \arrow[r, hook] \arrow[d] & \mathcal{B}_{\mathrm{Pin}^\pm}(M) \arrow[d] \\
\mathcal{B}_O(\partial M) \arrow[r] \arrow[dr] & \mathcal{B}_O(M)|_{\partial M} \arrow[r, hook] \arrow[d] & \mathcal{B}_O(M) \arrow[d] \\
& \partial M \arrow[r, hook] & M
\end{tikzcd}$$

Example 4.2 (Pin structures on S^1). There are two problems to discuss here, since Pin_1^+ and Pin_1^- are different groups, namely

$$\mathrm{Pin}_1^+ \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$
$$\mathrm{Pin}_1^- \cong \mathbb{Z}/4\mathbb{Z}$$

We do the Pin_1^+ case first. In this case, the Clifford algebra $\mathrm{Cliff}_{1,0}(\mathbb{R})$ is isomorphic to $\mathbb{R} \times \mathbb{R}$, where the isomorphism is determined by $e_1 \mapsto (-1, 1)$. There are two irreducible modules corresponding to projection onto one of the factors, so the Pinor representations are the one dimensional real representations $\mathbb{P}^+$ and $\mathbb{P}^-$, where the action of e_1 on $\mathbb{P}^\pm$ is by ± 1. Let $\mathbb{P} = \mathbb{P}^+ \oplus \mathbb{P}^-$. In addition, we have two principal Pin_1^+-bundles over S^1. One has 4 components that are permuted by the action of Pin_1^+, and the other has two components, which are both the double cover $z \mapsto z^2$.

Then for each Pin_1^+-bundle, we take the associated bundle. For P_1, we have that sections of $P_1 \times_{\mathrm{Pin}_1^+} \mathbb{P}$ are equivalent to Pin_1^+-equivariant maps $P_1 \to P$. Since the action of Pin_1^+ simply permutes the 4 components of P_1 and all the components are diffeomorphic to S^1, we can map a single component to to $\mathbb{P}$, and this determines the map on the other 4 components. Therefore, sections are equivalent data to maps $S^1 \to \mathbb{P}$, i.e. 2π-periodic maps $\mathbb{R} \to \mathbb{P}$. The Dirac operator D for this bundle then given by $D = e_1 \partial_\theta$. For the other bundle, there are two components, which are each the connected double cover of S^1. The element -1 acts in the same way as it did for the connected double cover, and the element e_1 permutes the two components. Then sections of the associated bundle are again equivalent data to 2π-antiperiodic maps $S^1 \to \mathbb{P}$. The Dirac operator is then given by $i\partial_\theta - \frac{1}{2}$.

For Pin_1^-, we also have two principal bundles, which we again denote P_1 and P_2. The first is P_1, which has 4 connected components that are cyclically permuted by the action of Pin_1^-, and P_2 is the connected 4-fold cover $z \mapsto z^4$. There is only a single irreducible module for $\mathrm{Cliff}_{0,1}(\mathbb{R})$, which is $\mathbb{C}$, which is then the single Pinor representation $\mathbb{P}$ where e_1 acts by i. The associated bundle to P_1 is the trivial bundle with the Dirac operator being given by $D = i\partial_\theta$. For the second case, the sections of the associated bundle are determined by functions $\psi : \mathbb{R} \to \mathbb{C}$ where $f(\theta + 2\pi) = if(\theta)$. We can then write these as $\psi = e^{i\theta/4} f$ for a 2π-periodic function f, so under this trivialization of the associated bundle, the Dirac operator is given by $i\partial_\theta - \frac{1}{4}$. $\blacktriangleleft$