UNDERGRADUATE THESIS NOTES

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Week 1

Exercise 1.1. Prove a small lemma

Lemma. *Let G be a group, and V a finite dimensional irreducible complex representation. Then elements of $Z(G)$ act by scalars.*

Proof. Let $z \in Z(G)$, then since z commutes with the action of G, it determines a G-module isomorphism $\varphi_z : V \to V$. Then φ_z admits some eigenvalue λ, and the λ eigenspace of φ_z is G-invariant, so φ_z must be the map $\lambda \, \mathrm{id}_V$. ■

Exercise 1.2. Consider the groups $\mathrm{Pin}_{p,q}$ and $\mathrm{Spin}_{p,q}$, which lie in the Clifford algebra $\mathrm{Cliff}_{p,q}$. A ***Pin representation*** is a representation V of $\mathrm{Pin}_{p,q}$ that extends to an irreducible Clifford module. Likewise, a ***Spin representation*** is a representation V of $\mathrm{Spin}_{p,q}$ that extends to an irreducible $\mathrm{Cliff}^0_{p,q}$ module. Find some of these representations.

Since all Clifford algebras arise (as ungraded algebras) as direct sums of matrix algebras over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, it will be useful to characterize the irreducible modules of these matrix algebras. In the case of $\mathbb{H}$, we use the convention that scalar multiplication on a quaternionic vector space acts on the right, so quaternionic matrices can act on the left.

Proposition 1.3.
The only irreducible $M_n\mathbb{R}$ module is $\mathbb{R}^n$.

Proof. We see that there is an increasing chain of left ideals

$$0 = I_0 \subset I_1 \subset \ldots \subset I_n = M_n\mathbb{R}$$

where I_k is the ideal of matrices where all the entries past the k^{th} column are 0. In addition, we have that this chain of ideals has the property that the quotient space $I_{k+1}/I_k \cong \mathbb{R}^n$ as a left $M_n\mathbb{R}$ module. We note that $\mathbb{R}^n$ is most certainly irreducible, since the orbit of any nonzero vector $v \in \mathbb{R}^n$ is all of $\mathbb{R}^n$.

Then let W denote an arbitrary nontrivial irreducible $M_n\mathbb{R}$ module. Fix $w \in W$. Then the orbit of w under the action of $M_n\mathbb{R}$, and since the module is nontrivial, it must be all of W. Therefore, the mapping

$$\begin{aligned} \varphi : M_n\mathbb{R} &\to W \\ M &\mapsto M \cdot w \end{aligned}$$

is a surjective map of left $M_n\mathbb{R}$ modules. Since this map is surjective, there exists some k such that $\varphi(I_k) \neq 0$. Let k denote the smallest such k. Since $\varphi(I_{k-1}) = 0$, this map factors through the quotient I_k/I_{k-1}, which is isomorphic to R^n as a left module. Then since both $\mathbb{R}^n$ and W are irreducible, this implies that the map $I_k/I_{k-1} \to W$ is an isomorphism, so $W \cong \mathbb{R}^n$. ■

This proof carries over for the matrix algebras $M_n\mathbb{C}$ and $M_n\mathbb{H}$. We then need another lemma to fully classify the real Clifford modules.

Lemma 1.4. *The algebra $A = M_n\mathbb{F} \oplus M_n\mathbb{F}$ (where $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$) has two irreducible modules, the first being isomorphic to $\mathbb{F}^n$ where the first factor has the standard action and the second factor has the trivial action, and the second is isomorphic to $\mathbb{F}^n$ where the first factor acts trivially, and the second factor has the standard action.*

Proof. We first note that the left ideals $0 \oplus M_n\mathbb{F}$ and $M_n\mathbb{F} \oplus 0$ both admit a chain of left ideals as we did above, which we denote as I_k and J_k respectively, which satisfy $I_{k+1}/I_k \cong \mathbb{F}^n$ and $J_{k+1}/J_k \cong \mathbb{F}^n$ as A modules. where the former has the action of the right factor, and the latter has the action of the right factor.

Then let W be some nontrivial irreducible A module, and $w \in W$ some nonzero element. Then the map $\varphi : A \to W$ defined by $A \mapsto A \cdot w$ is surjective, and there must exist some smallest k such that $\varphi(I_k)$ or $\varphi(J_k)$ is nonzero. Then φ factors through the quotient I_k/I_{k-1} or J_k/J_{k-1}, which is irreducible, so the map must be an isomorphism.

We then need to show that these modules are not isomorphic. Any such isomorphism would be a linear map $\varphi : \mathbb{F}^n \to \mathbb{F}^n$ satisfying

$$\varphi(Mv) = T\varphi(V)$$

for all matrices M and T. It is clear that no such φ exists, so the two modules are not isomorphic. ■

This gives a classification of all the irreducible (ungraded) Clifford modules. Recall that the (ungraded) classification of Clifford algebras gives us the Clifford "chessboard"

$M_8\mathbb{C}$	$M_8\mathbb{H}$	$\mathcal{M}_8\mathbb{H} \oplus M_8\mathbb{H}$	$M_{16}\mathbb{H}$	$M_{32}\mathbb{C}$	$M_{64}\mathbb{R}$	$M_{64}\mathbb{R} \oplus M_{64}\mathbb{R}$	$M_{128}\mathbb{R}$
$M_4\mathbb{H}$	$M_4\mathbb{H} \oplus M_4\mathbb{H}$	$M_8\mathbb{H}$	$M_{16}\mathbb{C}$	$M_{32}\mathbb{R}$	$M_{32}\mathbb{R} \oplus M_{32}\mathbb{R}$	$M_{64}\mathbb{R}$	$M_{64}\mathbb{C}$
$M_2\mathbb{H} \oplus M_2\mathbb{H}$	$M_4\mathbb{H}$	$M_8\mathbb{C}$	$M_{16}\mathbb{R}$	$M_{16}\mathbb{R} \oplus M_{16}\mathbb{R}$	$M_{32}\mathbb{R}$	$M_{32}\mathbb{C}$	$M_{32}H$
$M_2\mathbb{H}$	$M_4\mathbb{C}$	$M_8\mathbb{R}$	$M_8\mathbb{R} \oplus M_8\mathbb{R}$	$M_{16}\mathbb{R}$	$M_{16}\mathbb{C}$	$M_{16}\mathbb{H}$	$M_{16}\mathbb{H} \oplus M_{16}\mathbb{H}$
$M_2\mathbb{C}$	$M_4\mathbb{R}$	$M_4\mathbb{R} \oplus M_4\mathbb{R}$	$M_8\mathbb{R}$	$M_8\mathbb{C}$	$M_8\mathbb{H}$	$M_8\mathbb{H} \oplus M_8\mathbb{H}$	$M_{16}\mathbb{H}$
$M_2\mathbb{R}$	$M_2\mathbb{R} \oplus M_2\mathbb{R}$	$M_4\mathbb{R}$	$M_4\mathbb{C}$	$M_4\mathbb{H}$	$M_4\mathbb{H} \oplus M_4\mathbb{H}$	$M_8\mathbb{H}$	$M_{16}\mathbb{C}$
$\mathbb{R} \oplus \mathbb{R}$	$M_2\mathbb{R}$	$M_2\mathbb{C}$	$M_2\mathbb{H}$	$M_2\mathbb{H} \oplus M_2\mathbb{H}$	$M_4\mathbb{H}$	$M_8\mathbb{C}$	$M_{16}\mathbb{R}$
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$M_2\mathbb{H}$	$M_4\mathbb{C}$	$M_8\mathbb{R}$	$M_8\mathbb{R} \oplus M_8\mathbb{R}$

Where the (p,q) element of the table is $\mathrm{Cliff}_{p,q}$ as an ungraded algebra, and the bottom left corner is $\mathrm{Cliff}_{0,0} \cong \mathbb{R}$. All other Clifford algebras can be recovered from this table, since incrementing $p+q$ by 8 results in tensoring with $M_{16}\mathbb{R}$.