Thesis

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CHAPTER 1

Preliminaries

1. Clifford Algebras

Definition 1.1. Let V be a real finite dimensional vector space with a nondegenerate symmetric bilinear form $b : V \times V \to \mathbb{R}$. Then the ***Clifford Algebra*** of V is the data of a unital associative $\mathbb{R}$-algebra $\mathrm{Cliff}(V, b)$ and a linear map $i : V \to \mathrm{Cliff}(V, b)$ satisfying the following universal property: Given any linear map $\varphi : V \to A$ of V into any unital associative $\mathbb{R}$-algebra A satisfying $\varphi(v)^2 = b(v, v)$, there exists a unique map $\tilde{\varphi} : \mathrm{Cliff}(V, b) \to A$ such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[d, "i"'] \arrow[dr, "\varphi"] & \\ \mathrm{Cliff}(V,b) \arrow[r, "\tilde{\varphi}"'] & A \end{tikzcd}$$

Explicitly, we can construct $\mathrm{Cliff}(V, b)$ as a quotient of the tensor algebra

$$\mathcal{T}(V) = \bigoplus_{n\in\mathbb{Z}^{\geq 0}} V^{\otimes n}$$

by the left ideal generated by elements of the form $v \otimes v - b(v, v)$, and the map $i : V \to \mathcal{T}(V)/(v \otimes v - b(v, v))$ is given by the inclusion $V \hookrightarrow \mathcal{T}(V)$ followed by the quotient map. We identify V with its image $i(V)$ as a subspace of $\mathrm{Cliff}(V, b)$.

Fix a basis for V. Then the bilinear form b is given by a symmetric invertible matrix B, which is conjugate to a diagonal matrix where all the diagonal entries are either 1 or -1. If after conjugation B has p positive entries and q negative entries, we say b has signature (p, q). Any bilinear form b of signature (p, q) admits a basis $\{e_i\}$ satisfying

(1) For $1 \leq i \leq p$, we have $b(e_i, e_i) = 1$
(2) For $p + 1 \leq j \leq p + q$, we have $b(e_j, e_j) = -1$
(3) For $i \neq j$, we have $b(e_i, e_j) = 0$

Any such basis then determines an isomorphism $(V, b) \to \mathbb{R}^{p|q}$, where $\mathbb{R}^{p|q}$ denotes $\mathbb{R}^{p+q}$ with the bilinear form given by the matrix

$$\begin{pmatrix} \mathrm{id}_{\mathbb{R}^p} & 0 \\ 0 & -\mathrm{id}_{\mathbb{R}^q} \end{pmatrix}$$

In addition, we get a basis for $\mathrm{Cliff}(V, b)$, given by

$$\left\{e_{i_1}e_{i_2}\cdots e_{i_k} \;:\; 0 \leq k \leq n,\ 1 \leq i_j \leq n\right\}$$

where we interpret the product of 0 basis vectors to be the unit element 1. which then implies the dimension of $\mathrm{Cliff}(V, b)$ as a vector space is $2^{\dim V}$. This basis also determines an isomorphism $\mathrm{Cliff}(V, b) \to \mathrm{Cliff}_{p,q}(\mathbb{R})$, where $\mathrm{Cliff}_{p,q}(\mathbb{R})$ is the Clifford algebra for $\mathbb{R}^{p|q}$. Given $v, w \in V$, we write v and w in these bases as $v^i e_i$ and $w^i e_i$ (using Einstein summation convention), and derive the useful relation

$$\begin{aligned} vw + wv &= v^i w^j e_i e_j + v^i w^j e_j e_i \\ &= v^i w^i (e_i^2 + e_i^2 \\ &= 2b(v, w) \end{aligned}$$

where we use the fact that the e_i are orthgonal to deduce that $e_ie_j = -e_je_i$ if $i \neq j$. Finally, this basis allows us to see that $\mathrm{Cliff}(V, b)$ has a $\mathbb{Z}/2\mathbb{Z}$ grading

$$\mathrm{Cliff}(V, b) = \mathrm{Cliff}^0(V, b) \oplus \mathrm{Cliff}^1(V, b)$$

Where $\mathrm{Cliff}^0(V, b)$ is the $\mathbb{R}$-span of all even products of basis vectors, and $\mathrm{Cliff}^1(V, b)$ is the $\mathbb{R}$-span of all odd products of basis vectors. In particular, $\mathrm{Cliff}^0(V, b)$ forms a subalgebra, and is called the ***even subalgebra***, and we say that the elements of $\mathrm{Cliff}^0(V, b)$ are even. We then call $\mathrm{Cliff}^1(V, b)$ the odd subspace, and say that its elements are odd. Elements that are contained in the odd or even subspace are called ***homogeneous***, and given a homogeneous element $a \in \mathrm{Cliff}(V, b)$, we define it's ***parity*** $|a|$ as

$$|a| = \begin{cases} 0 & a \in \mathrm{Cliff}^0(V, b) \\ 1 & a \in \mathrm{Cliff}^1(V, b) \end{cases}$$

There is an extremely nice relationship between a Clifford algebra and it's even subalgebra.

THEOREM 1.2. *The even subalgebra* $\mathrm{Cliff}^0_{p,q}(\mathbb{R})$ *is isomorphic to both* $\mathrm{Cliff}_{q,p-1}$ *and* $\mathrm{Cliff}_{p,q-1}$ *as ungraded algebras (as long as* $p - 1 > 0$ *or* $q - 1 > 0$*.)*

PROOF. Fix a basis $\left\{e_1^+, \dots e_p^+, e_1^- \dots e_q^-\right\}$ for $\mathbb{R}^{p|q}$, where $(e_i^+)^2 = 1$ and $(e_i^-)^2 = -1$. Then a quick computation shows that

$$\begin{aligned} (e_i^+e_j^+)^2 &= -(e_i^+)^2(e_j^+)^2 = -1 \\ (e_i^-e_j^-)^2 &= -(e_i^-)^2(e_j^-)^2 = -1 \\ (e_i^+e_j^-)^2 &= -(e_i^+)^2(e_i^-)^2 = 1 \\ (e_i^-e_j^+)^2 &= -(e_i^-)^2(e_j^+)^2 = 1 \end{aligned}$$

Assume $q \neq 0$. Then a generating set for $\mathrm{Cliff}^0_{p,q}(\mathbb{R})$ is

$$\left\{e_1^-e_j^+ \ : \ 1 \leq j \leq p\right\} \cup \left\{e_1^-e_k^- \ : \ 2 \leq k \leq q\right\}$$

All the elements in the first set square to 1, and all the elements in the second set square to -1. We then get an isomorphism $\mathrm{Cliff}^0_{p,q}(\mathbb{R}) \to \mathrm{Cliff}_{p,q-1}$ via the mappings

$$\begin{aligned} e_1^-e_j^+ &\mapsto e_j^+ \\ e_1^-e_k^- &\mapsto e_{k-1}^- \end{aligned}$$

In the case where $p \neq 0$, we have that an equally good generating set for $\mathrm{Cliff}^0_{p,q}(\mathbb{R})$ is

$$\left\{e_1^+e_j^+ \ : \ 2 \leq j \leq p\right\} \cup \left\{e_1^+e_i^- \ : \ 1 \leq i \leq q\right\}$$

Where the elements in the first set square to -1 and the elements of the second set square to 1. Then the mappings

$$\begin{aligned} e_1^+e_j^+ &\mapsto e_{j-1}^- \\ e_1^+e_j^- &\mapsto e_j^+ \end{aligned}$$

gives the isomorphism $\mathrm{Cliff}^0_{p,q}(\mathbb{R}) \to \mathrm{Cliff}_{q,p-1}$. ∎

Given two $\mathbb{R}$-algebras A and B, we can form their tensor product $A \otimes B$, which has $A \otimes B$ as the underlying vector space, and the multiplication is defined as

$$(a \otimes b)(c \otimes d) = ac \otimes bd$$

In the case that both A and B are $\mathbb{Z}/2\mathbb{Z}$ graded algebras, we have an alternate version of the tensor product, where the underlying vector space is also $A \otimes B$, but the multiplication is given by

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|}(ac \otimes bd)$$

We see that in the multiplication, we are formally commuting the elements of b and c, and we want to introduce a sign whenever elements are moved past each other. This is the ***Koszul sign rule***. Another concept that needs a slight modification in the graded case is the opposite algebra. In the normal case, given an $\mathbb{R}$-algebra A, the ***opposite algebra*** is the algebra A^{op} with the same underlying vector space, but the multiplication in A^{op} is given by $a * b = ba$, where ba is the multiplication in A. In doing so, we are formally commuting a and b, so in the graded situation, we invoke the Koszul sign rule when defining the opposite algebra, and define the multiplication in A^{op} to be $a * b = (-1)^{|a||b|} ba$.

One remarkable fact is that Clifford algebras are closed under the graded tensor product, i.e. the graded tensor products of two Clifford algebras is another Clifford algebra. Likewise, the graded opposite algebra of a Clifford algebra is again a Clifford algebra. For the remainder of this section, we will let $\otimes$ denote the graded tensor product, and the superscript op will denote the graded opposite algebra.

THEOREM 1.3. $\mathrm{Cliff}_{p+t,q+s}(\mathbb{R}) \cong \mathrm{Cliff}_{p,q}(\mathbb{R}) \otimes \mathrm{Cliff}_{t,s}(\mathbb{R})$

PROOF. To give a map $\varphi : \mathrm{Cliff}_{p+t,q+s}(\mathbb{R}) \to \mathrm{Cliff}_{p,q}(\mathbb{R}) \otimes \mathrm{Cliff}_{t,s}(\mathbb{R})$, it is sufficient to specify it's action on $\mathbb{R}^{p+t|q+s}$, and checking that the Clifford relations hold. Let $\left\{b_1^+, \dots, b_{p+t}^+, b_1^-, \dots b_{q+s}^-\right\}$ denote the standard orthogonal basis for $\mathbb{R}^{p+t|q+s}$ where $(b_i^+)^2 = 1$ and $(b_i^-)^2 = -1$. We then define the bases $\{e_i^\pm\}$ and $\{f_i^\pm\}$ analogously for $\mathbb{R}^{p|q}$ and $\mathbb{R}^{t|s}$ respectively. Then define φ by

$$\varphi(b_i^+) = \begin{cases} e_i^+ \otimes 1 & 1 \le i \le p \\ 1 \otimes f_i^+ & p+1 \le i \le p+t \end{cases}$$

$$\varphi(b_i^-) = \begin{cases} e_i^- \otimes 1 & 1 \le i \le q \\ 1 \otimes f_i^- & q+1 \le i \le q+s \end{cases}$$

This map is injective on generators, so if we show that this satisfies the Clifford relations, then the map given by extending the map to all of $\mathrm{Cliff}_{p+t,q+s}(\mathbb{R})$ will be an isomorphism by dimension reasons. Showing the Clifford relations amounts to showing

(1) $\varphi(b_i^+)^2 = 1$
(2) $\varphi(b_i^-)^2 = -1$
(3) The images of any pair of distinct basis vectors anticommute.

The first two are relations are clear from how we defined φ. To show that the images of distinct basis vectors anticommute, there are serveral cases to consider. Given b_i^+ and b_j^+ where $1 \le i, j \le p$, they anticommute, because e_i^+ and e_j^+ anticommute. In the case where $1 \le i \le p$ and $p+1 \le j \le p+t$, we compute

$$\begin{aligned} \varphi(b_i^+)\varphi(b_j^+) + \varphi(b_j^+)\varphi(b_i+) &= (e_i^+ \otimes 1)(1 \otimes f_j^+) + (1 \otimes f_j^+)(e_i^+ \otimes 1) \\ &= e_i^+ \otimes f_j^+ - e_i^+ \otimes f_j^+ \end{aligned}$$

where we use the Koszul sign rule for the second term, noting that f_j+ and e_i^+ are both odd. The proof that the images of the b_i^- anti commute with each other, as well as the proof that the images of the b_i^+ and b_i^- anticommute are exactly the same. ∎

THEOREM 1.4. *The graded opposite algebra* $\mathrm{Cliff}_{p,q}^{op}$ *is isomorphic to* $\mathrm{Cliff}_{q,p}$.

PROOF. Fix an orthgonal basis $\{e_i^\pm\}$ for $\mathbb{R}^{p,q}$, where $(e_i^\pm)^2 = \pm 1$. We note that since the $e_i^\pm$ are odd elements, they square to $e_i^\mp$ in the opposite algebra. Indeed, the mapping $e_i^\pm \to e_i^\mp$ defines the isomorphism $\mathrm{Cliff}_{p,q}^{\text{op}} \to \mathrm{Cliff}_{q,p}$. ∎

Because of these theorems, once we compute a few of the lower dimensional Clifford algebras, we will have enough data to fully classify all Clifford algebras over $\mathbb{R}$. In the case $p = q = 0$, we let $\mathrm{Cliff}_{0,0}(\mathbb{R}) = \mathbb{R}$.

EXAMPLE 1.5 (***Some low dimensional examples***).

(1) As ungraded algebras, the Clifford algebra $\mathrm{Cliff}_{0,1}(\mathbb{R})$ is isomorphic to $\mathbb{C}$, where the isomorphism is given by $e_1 \mapsto i$.

(2) As ungraded algebras, $\mathrm{Cliff}_{0,2}(\mathbb{R})$ is isomorphic to the quaternions $\mathbb{H}$, where the isomorphism is given by $e_1 \mapsto i$ and $e_2 \mapsto j$.

(3) As a graded algebra, $\mathrm{Cliff}_{1,1}(\mathbb{R})$ is isomorphic to $\mathrm{End}(\mathbb{R}^{1|1})$. The isomorphism is given by

$$e_1^+ \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad e_1^- \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(4) As ungraded algebras $\mathrm{Cliff}_{1,0}(\mathbb{R})$ is isomorphic to the product algebra $\mathbb{R} \times \mathbb{R}$, where $e_1 \mapsto (1, -1)$.

(5) As ungraded algebras, $\mathrm{Cliff}_{2,0}(\mathbb{R})$ is isomorphic to $M_2\mathbb{R}$. The isomorphism is given by

$$e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad e_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

To classify all Clifford algebras as ungraded algebras, it suffices to know the following table:

$M_8\mathbb{C}$	$M_8\mathbb{H}$	$M_8\mathbb{H} \times M_8\mathbb{H}$	$M_{16}\mathbb{H}$	$M_{32}\mathbb{C}$	$M_{64}\mathbb{R}$	$M_{64}\mathbb{R} \times M_{64}\mathbb{R}$	$M_{128}\mathbb{R}$
$M_4\mathbb{H}$	$M_4\mathbb{H} \times M_4\mathbb{H}$	$M_8\mathbb{H}$	$M_{16}\mathbb{C}$	$M_{32}\mathbb{R}$	$M_{32}\mathbb{R} \times M_{32}\mathbb{R}$	$M_{64}\mathbb{R}$	$M_{64}\mathbb{C}$
$M_2\mathbb{H} \times M_2\mathbb{H}$	$M_4\mathbb{H}$	$M_8\mathbb{C}$	$M_{16}\mathbb{R}$	$M_{16}\mathbb{R} \times M_{16}\mathbb{R}$	$M_{32}\mathbb{R}$	$M_{32}\mathbb{C}$	$M_{32}H$
$M_2\mathbb{H}$	$M_4\mathbb{C}$	$M_8\mathbb{R}$	$M_8\mathbb{R} \times M_8\mathbb{R}$	$M_{16}\mathbb{R}$	$M_{16}\mathbb{C}$	$M_{16}\mathbb{H}$	$M_{16}\mathbb{H} \times M_{16}\mathbb{H}$
$M_2\mathbb{C}$	$M_4\mathbb{R}$	$M_4\mathbb{R} \times M_4\mathbb{R}$	$M_8\mathbb{R}$	$M_8\mathbb{C}$	$M_8\mathbb{H}$	$M_8\mathbb{H} \times M_8\mathbb{H}$	$M_{16}\mathbb{H}$
$M_2\mathbb{R}$	$M_2\mathbb{R} \times M_2\mathbb{R}$	$M_4\mathbb{R}$	$M_4\mathbb{C}$	$M_4\mathbb{H}$	$M_4\mathbb{H} \times M_4\mathbb{H}$	$M_8\mathbb{H}$	$M_{16}\mathbb{C}$
$\mathbb{R} \times \mathbb{R}$	$M_2\mathbb{R}$	$M_2\mathbb{C}$	$M_2\mathbb{H}$	$M_2\mathbb{H} \times M_2\mathbb{H}$	$M_4\mathbb{H}$	$M_8\mathbb{C}$	$M_{16}\mathbb{R}$
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \times \mathbb{H}$	$M_2\mathbb{H}$	$M_4\mathbb{C}$	$M_8\mathbb{R}$	$M_8\mathbb{R} \times M_8\mathbb{R}$

To read the table, the bottom right entry is $\mathrm{Cliff}_{0,0} \cong \mathbb{R}$, and moving to the right increments the signature from (p, q) to $(p, q+1)$, and moving up increments the signature (p, q) to $(p+1, q)$. Any other Clifford algebra can be obtained from an algebra on this table by tensoring with $M_{16}\mathbb{R}$, since incremeting the signature by 8 (by adding to either p or q) results in tensoring with $M_{16}\mathbb{R}$.

Definition 1.6. A (left) ***Clifford module*** for the Clifford algebra $\mathrm{Cliff}_{p,q}(\mathbb{R})$ is a module for $\mathrm{Cliff}_{p,q}(\mathbb{R})$ in the usual sense i.e. a real vector space V equipped with an algebra action $\bullet : \mathrm{Cliff}_{p,q} \times V \to V$ satisfying

(1) Every element of $\mathrm{Cliff}_{p,q}(\mathbb{R})$ acts linearly on V.
(2) $(AB) \cdot v = A \cdot (B \cdot v)$ for all $v \in V$.
(3) $(A + B) \cdot v = A \cdot v + B \cdot V$ for all $v \in V$.

Equivalently, it is the data of a real vector space V and a homomorphism $\mathrm{Cliff}_{p,q}(\mathbb{R}) \to \mathrm{End}(V)$.

Definition 1.7. A Clifford module is ***irreducible*** if there exist no proper nontrivial submodules.

From the classification of Clifford algebras, all the Clifford algebras are either matrix algebras $M_n\mathbb{F}$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, or products $M_n\mathbb{F} \times M_n\mathbb{F}$ of two copies of the same matrix algebra. This is sufficient to conclude that Clifford algebras are semisimple, so all Clifford modules will be direct sums of irreducible modules. Therefore, classifying all Clifford modules reduces to classifying the irreducible Clifford modules.

Theorem 1.8. *Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. Then any nontrivial irreducible module for $M_n\mathbb{F}$ is isomorphic to $\mathbb{F}^n$ with the standard action.*

Proof. We first note that $M_n\mathbb{F}$ acts transitively on $\mathbb{F}^n$, which implies that it is irreducible. We then must show that $\mathbb{F}^n$ is, up to isomorphism, the only irreducible $M_n\mathbb{F}$ module. The matrix algebra $M_n\mathbb{F}$ admits an increasing chain of left ideals

$$0 = I_0 \subset I_1 \subset \ldots \subset I_n = M_n\mathbb{F}$$

where I_k is the set of matrices where only the first k columns are nonzero. These ideals have the property that the quotient I_k/I_{k-1} is isomorphic to $\mathbb{F}^n$ as a left $M_n\mathbb{F}$ module. Then let M be some nontrivial irreducible $M_n\mathbb{F}$ module, and fix $m \in M$. Then the orbit $M_n\mathbb{F} \cdot m$ of m under the algebra action is a nonzero submodule, so it must be all of M. Then the map $\varphi : M_n\mathbb{F} \to M$ given by $A \mapsto A \cdot m$ is a surjective map of left $M_n\mathbb{F}$ modules. Then there must exist some smallest k such that $\varphi(I_k)$ is nonzero, and by construction, $\varphi|_{I_k}$ factors through the quotient I_k/I_{k-1}, which is isomorphic to $\mathbb{F}^n$ with the standard action. Then since $\mathbb{F}^n$ is irreducible, this gives us a nontrivial map between irreducible modules, which is an isomorphism by Schur's Lemma. ■

THEOREM 1.9. *Any nontrivial irreducible module for $M_n\mathbb{F} \times M_n\mathbb{F}$ is isomorphic to either $\mathbb{F}^n$ where the left factor acts in the usual way, and the right factor acts by 0, or $\mathbb{F}^n$ where the left factor acts by 0 and the right factor acts in the usual way.*

PROOF. Let R denote $\mathbb{F}^n$ where the right factor acts nontrivially, and let L denote $\mathbb{F}^n$ where the left factor acts nontrivially. Both L and R are irreducible since $M_n\mathbb{F} \times M_n\mathbb{F}$ acts transitively on them. To show that they are the only irreducible modules up to isomorphism, we use a similar techinique as above. Let I_k denote the chain of increasing ideals in $M_n\mathbb{F}$, as we used above. Then $M_n\mathbb{F} \times M_n\mathbb{F}$ admits a chain of increasing left ideals J_k

$$0 = J_0 \subset I_1 \times \{0\} \subset \ldots \subset M_n\mathbb{F} \times \{0\} \subset M_n\mathbb{F} \times I_1 \subset \ldots \subset M_n\mathbb{F} \times M_n\mathbb{F} = J_{2n}$$

We note that for $1 \leq k \leq n$, we have that J_k/J_{k-1} is isomorphic to L, and for $n+1 \leq k \leq 2n$, we have that J_k/J_{k-1} is isomorphic to R. Then given a nontrivial irreducible module M and a nonzero element m, we get a surjective map $\varphi : M_n\mathbb{F} \times M_n\mathbb{F}$ where $A \mapsto A \cdot m$. Like before, there exists some smallest k such that $\varphi(J_k)$ is nonzero, which then factors through to an isomorphism $J_k/J_{k-1} \to M$, so M is either isomorphic to R or L. ■

This then gives a full classification of the irreducible ungraded Clifford modules.

2. Complex Clifford Algebras

Much of the discussion regarding Clifford algebras can be reconstructed using complex vector spaces. However, there is an important distinction to be made. Over $\mathbb{C}$, the notion of signature no longer makes sense when discussing bilinear forms. Given a bilinear form $B : V \times V \to \mathbb{C}$ and a vector $v \in V$ with $B(v,v) = 1$, we have that $B(iv, iv) = -1$. Therefore, the complex Clifford algebras generated by V with any nondegenerate bilinear form are entirely determined by their dimension. In the case of $\mathbb{C}^n$ with the standard bilinear form

$$\langle v, w \rangle = \sum_i v^i w^i$$

we denote the Clifford algebra by $\mathrm{Cliff}_n(\mathbb{C})$. In the complex case, the classification is much simpler, and is determined by the parity of the dimension.

In the even case of $\mathbb{C}^{2n}$, we first prove a lemma.

LEMMA 2.1. *There exists a basis $\{e_1, \ldots e_n, f_1, \ldots, f_n\}$ for $\mathbb{C}^{2n}$ satisfying*

(1) $\langle e_i, e_i \rangle = \langle f_i, f_i \rangle = 0$
(2) $\langle e_i, f_j \rangle = \delta_{ij}$

where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$.

PROOF. Let $\{a_i\}$ denote the first n standard basis vectors for $\mathbb{C}^{2n}$, and let $\{b_i\}$ denote the last n standard basis vectors. Then setting $e_i = a_i + ib_i$ and $f_i = a_i - ib_i$, we get a basis $\{e_1, \ldots e_n, f_1, \ldots, f_n\}$ for $\mathbb{C}^{2n}$. We then compute

$$\begin{aligned}
\langle e_i, e_i \rangle &= \langle a_i + ib_i, a_i + ib_i \rangle \\
&= \langle a_i, a_i \rangle + 2\langle a_i, ib_i \rangle + \langle ib_i, ib_i \rangle \\
&= 1 + 0 + -1 \\
&= 0 \\
\langle f_i, f_i \rangle &= \langle a_i - ib_i, a_i - ib_i \rangle \\
&= \langle a_i, a_i \rangle - 2\rangle a_i, ib_i \rangle + \langle ib_i, ib_i \rangle \\
&= 0 \\
\langle e_i, f_j \rangle &= \langle a_i + ib_i, a_j - ib_j \rangle \\
&= \langle a_i, a_j \rangle - \langle a_i, ib_j \rangle + \langle a_j, ib_i \rangle - \langle ib_i, ib_j \rangle \\
&= \delta_{ij} + 0 + 0 + \delta ij \\
&= 2\delta_{ij}
\end{aligned}$$

So normalizing the e_i and f_i by dividing by $\sqrt{2}$ gives the desired basis. ■

This basis gives a direct sum decomposition $\mathbb{C}^{2n} = W \oplus W'$ where W is the span of the e_i and W' is the span of the f_i. We then claim that $\mathrm{Cliff}_{2n}(\mathbb{C})$ is isomorphic to the endomorphism algebra $\mathrm{End}(\Lambda^\bullet W)$. To give a map $\mathrm{Cliff}_{2n}(\mathbb{C}) \to \mathrm{End}(\Lambda^\bullet W)$, we need to specify two maps $\varphi : W \to \mathrm{End}(\Lambda^\bullet W)$ and $\varphi' : W' \to \mathrm{End}(\Lambda^\bullet W)$ such that for all $w, p \in W$ and $w', p' \in W'$ we have

(1) $\varphi(w) \circ \varphi(p) + \varphi(p) \circ \varphi(w) = 0$
(2) $\varphi'(w') \circ \varphi(p') + \varphi(p') \circ \varphi(w') = 0$
(3) $\varphi(w) \circ \varphi'(w') + \varphi'(w') \circ \varphi(w) = 2\langle w, w' \rangle$

where we use the fact that $\langle \cdot, \cdot \rangle$ vanishes on W and W'. For notational convenience, we will denote $\varphi(w)$ as φ_w, and will do the same with φ'. Define $\varphi_w : \Lambda^\bullet W \to \Lambda^\bullet W$ by $\varphi_w(\alpha) = w \wedge \alpha$ and $\varphi'(w')$ by

$$\varphi'_{w'}(v_1 \wedge \cdots \wedge v_k) = 2\sum_i (-1)^{i-1} \langle w', v_i \rangle v_1 \wedge \cdots \widehat{v_i} \cdots \wedge v_k$$

which will satisfy these relations, and these two maps define the desired isomorphism $\mathrm{Cliff}_{2n}(\mathbb{C}) \to \mathrm{End}(\Lambda^\bullet W)$.

In the odd dimensional case of $\mathbb{C}^{2n+1}$, we can decompose $\mathbb{C}^{2n+1}$ as $\mathbb{C}^{2n+1} = W \oplus W' \oplus U$, where W and W' are the same as in the decomposition in the even case, and U is the orthogonal complement to $W \oplus W'$. We then define the maps from W and W' into $\mathrm{End}(\Lambda^\bullet W)$, and then define a map $U \to \mathrm{End}(\Lambda^\bullet W)$ where a unit vector in U acts by identity on the odd elements of $\Lambda^\bullet W$ and by negative identity on the even elements. This then defines a map $\varphi : \mathrm{Cliff}_{2n+1}(\mathbb{C}) \to \mathrm{End}(\Lambda^\bullet W)$. Repeating this with $\mathrm{End}(\Lambda^\bullet W')$ gives another map $\psi : \mathrm{Cliff}_{2n+1}(\mathbb{C}) \to \mathrm{End}(\Lambda^\bullet W')$. Then the product map $\varphi \times \psi : \mathrm{Cliff}_{2n+1}(\mathbb{C}) \to \mathrm{End}(\Lambda^\bullet W) \times \mathrm{End}(\Lambda^\bullet W')$ is the desired isomorphism.

3. The Pin and Spin Groups

The group of invertible elements in $\mathrm{Cliff}_{p,q}(\mathbb{R})$, denoted $\mathrm{Cliff}^\times_{p,q}(\mathbb{R})$ contains a group $\mathrm{Pin}_{p,q}$, which a double cover of the group $O_{p,q}$ of matrices preserving the standard bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{p|q}$. Inside of $\mathrm{Pin}_{p,q}$, there exists a subgroup $\mathrm{Spin}_{p,q} \subset \mathrm{Pin}_{p,q}$, which double covers the group $SO_{p,q}$, which consists of the subgroup of $O_{p,q}$ where all the elements have determinant equal to 1.

Definition 3.1. The ***Pin group*** $\mathrm{Pin}_{p,q}$ is the subgroup of $\mathrm{Cliff}^\times_{p,q}(\mathbb{R})$ generated by the set

$$\left\{ v \in \mathbb{R}^{p,q} \ : \ v^2 = \pm 1 \right\}$$

The ***Spin group*** $\mathrm{Spin}_{p,q}$ is the subgroup of $\mathrm{Pin}_{p,q}$ generated by even products of basis vectors. In other words, $\mathrm{Spin}_{p,q} = \mathrm{Pin}_{p,q} \cap \mathrm{Cliff}^0_{p,q}(\mathbb{R})$. In the case that the bilinear form is definite, we let $\mathrm{Pin}^+_n = \mathrm{Pin}_{n,0}$ and $\mathrm{Pin}^-_n = \mathrm{Pin}_{0,n}$.

Theorem 3.2. $\mathrm{Spin}_{0,n} \cong \mathrm{Spin}_{n,0}$.

Proof. Recall that we have an isomorphism $\mathrm{Cliff}^{\mathrm{op}}_{p,q} \to \mathrm{Cliff}_{q,p}$ where $e_i^\pm \mapsto e_i^\mp$. In addition, the even subalgebra $\mathrm{Cliff}^0_{q,p}$ is isomorphic to the (ungraded) opposite algebra of the even subalgebra $\mathrm{Cliff}^0_{p,q}$. Therefore, the Spin group $\mathrm{Spin}_{q,p} \subset \mathrm{Cliff}_{q,p}$ is isomorphic to the opposite group $\mathrm{Spin}^{\mathrm{op}}_{p,q}$. We then know that every group is isomorphic to its opposite group via the map $g \mapsto g^{-1}$, giving us the desired isomorphism. ∎

In particular, this implies that the Spin groups in definite signatures are isomorphic, so we will henceforth denote them as Spin_n. To show that the Pin and Spin groups cover $O_{p,q}$ and $\mathrm{Spin}_{p,q}$, we make a short digression. Given a vector $v \in \mathbb{R}^{p,q}$, we can define a reflection map $R_v : \mathbb{R}^{p,q} \to \mathbb{R}^{p,q}$ given by $R_v(w) = w - 2\langle v, w \rangle v$, which will reflect across the hyperplane $v^\perp$.

Theorem 3.3 (***Cartan-Dieudonné***). *Any orthogonal transformation $A \in O_{p,q}$ can be written as the composition of at most $p+q$ hyperplane reflections, where we interpret the identity map as the composition of* 0 *reflections.*

Proof. We prove this by induction on $n = p + q$. The case $n = 1$ is trivial, since $O_1 = \{\pm 1\}$. Then given $A \in O_{p,q}$, fix some nonzero $v \in \mathbb{R}^{p|q}$. Then define $R : \mathbb{R}^{p|q} \to \mathbb{R}^{p|q}$ by

$$R(w) = w - 2\frac{\langle Av - v, w \rangle}{\langle Av - v, Av - v \rangle}(Av - v)$$

Then R will be a reflection about the hyperplane orthogonal to $Av - v$, and will interchange v and Av. Therefore, RA is an orthogonal transformation fixing v. Since RA is orthogonal, it will also fix the orthogonal complement $v^\perp$, so it will restrict to an orthgonal transformation on $v^\perp$. We then see that $v^\perp$ is 1 dimension lower thant $\mathbb{R}^{p|q}$, and restricting the bilinear form to $v^\perp$, we know by the inductive hypotehsis that $RA|_{v^\perp}$ can be written as at most $n-1$ hyperplane reflections in $v^\perp$. Since RA fixes v, we can extend all of these transformations to a hyperplane reflection on all of $\mathbb{R}^{p|q}$, by taking the span of each hyperplane wiht v, giving us that RA is a composition of at most $n-1$ reflections. Finally, composing RA with R gives us that A can be written as a composition of at most n hyperplane reflections. ∎

The Cartan-Dieudonné theorem will be the central piece for showing that the Pin and Spin groups cover the orthogonal groups.

Theorem 3.4. *There exist* 2*-to-*1 *group homomorphisms* $\mathrm{Pin}_{p,q} \to O_{p,q}$ *and* $\mathrm{Spin}_{p,q} \to SO_{p,q}$*, i.e. there exist short exact sequences*

$$0 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Pin}_{p,q} \longrightarrow O_{p,q} \longrightarrow 0$$

$$0 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}_{p,q} \longrightarrow SO_{p.q} \longrightarrow 0$$

Proof. We first consider the case of $\mathrm{Pin}_{p,q}$. To do this, we need to construct a group action where $\mathrm{Pin}_{p,q}$ acts on $\mathbb{R}^{p|q}$ by orthogonal transformations. There exists an involution $T : \mathrm{Cliff}_{p,q}(\mathbb{R}) \to \mathrm{Cliff}_{p,q}(\mathbb{R})$, where given the standard orthogonal basis $\{e_1, \dots, e_{p+1}\}$, we define

$$T(e_{i_1} \cdots e_{i_k}) = e_{i_k} \cdots e_{i_1}$$

and extending linearly to the remainder of $\mathrm{Cliff}_{p,q}(\mathbb{R})$. Given $a \in \mathrm{Cliff}_{p,q}(\mathbb{R})$, we denote $T(a)$ by a^T. We note that for a vector $v \in \mathbb{R}^{p|q}$, identifying $\mathbb{R}^{p|q}$ as a subspace of $\mathrm{Cliff}_{p,q}(\mathbb{R})$, satisfying $\langle v, v\rangle = \pm 1$, we have that $v^T = v$ and $v^{-1} = \pm v$. Then given $g \in \mathrm{Pin}_{p,q}$, and $v \in \mathbb{R}^{p|q}$, we claim that the left action

$$g \cdot v = -gvg^{-1}$$

defines the group action we desire. To show this, we must show that this indeed maps $\mathbb{R}^{p|q}$ back into itself, and that the group elements act by orthogonal trasnformations. It suffices to check this on the generating set of elements v with $\langle v, v\rangle = \pm 1$. First assume that $\langle v, v\rangle = 1$. Then given $w \in \mathbb{R}^{p|q}$, we compute

$$\begin{aligned} -vwv^{-1} &= -vwv \\ &= (wv - 2\langle v, w\rangle)v \\ &= w - 2\langle v, w\rangle v \end{aligned}$$

Which is hyperplane reflection about the orthogonal complement of v. In the case that $\langle v, v\rangle = -1$, we compute

$$\begin{aligned} -vwv^{-1} &= -vw(-v) \\ &= (2\langle -v, w\rangle + wv)(-v) \\ &= w - 2\langle -v, w\rangle \end{aligned}$$

which is hyperplane reflection about the orthogonal complement of $-v^\perp$, which is the same as the orthogonal complement of $v^\perp$. Therefore, $\mathrm{Pin}_{p,q}$ acts by orthogonal transformations, giving us a homomorphism $\mathrm{Pin}_{p,q} \to O_{p,q}$. This map is surjective by the Cartan-Dieudonné theorem, and it can be verified that the kernel is $\{\pm 1\}$. ∎

Most of the constructions carry over to the complex Clifford algebras, giving us the complex Pin and Spin groups, denoted $\mathrm{Pin}_n\mathbb{C}$ and $\mathrm{Spin}_n\mathbb{C}$, which double cover the complex orthogonal groups $O_n\mathbb{C}$ and $SO_n\mathbb{C}$ respectively.

Two simple examples of spin groups occur in dimensions 2 and 3. Since $SO_2 \cong \mathbb{T}$, where

$$\mathbb{T} = \{z \in \mathbb{C} \;:\; |z| = 1\}$$

we have that $\mathrm{Spin}_2 \cong \mathbb{T}$, where the covering map is given by $z \mapsto z^2$. In the case of SO_3, we consider the unit quaternions, which form a Lie group isomorphic to the group SU_2. Then given $q \in SU_2$, we define the map $\varphi_q : \mathbb{R}^3 \to \mathbb{R}^3$ where $\varphi_q(v) = qv\bar{q}$, where $\bar{q}$ is the quaternionic conjugate of q (e.g. $\overline{a + bi + cj + dk} = a - bi - cj - dk$), and $v = v^i e_i$ is idenitified with $v^1 i + v^2 j + v^3 k \in \mathbb{H}$. The mapping $q \mapsto \varphi_q$ then gives a double cover $SU_2 \to SO_3$. In particular, SU_2 is diffeomorphic to the sphere S^3, so the double covering realizes SO_3 as the quotient of S^3 by the antipodal map, giving us that $SO_3 \cong \mathbb{RP}^3$.

Many examples of low dimensional Spin groups arise from investigating the relationship between a 4 dimensional complex vector space V and it's second exterior power $\Lambda^2 V$. Fix a volume form $\mu \in \Lambda^4 V^*$. This then induces a symmetric, nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ on $\Lambda^2 V$ by

$$\langle \alpha, \beta \rangle = \langle \alpha \wedge \beta, \mu \rangle$$

where $\langle \alpha \wedge \beta, u \rangle$ denotes the natural pairing of the vector space $\Lambda^4 V$ with its dual $\Lambda^4 V^*$. By fixing a basis $\{e_i\}$ for V where $\mu(e_1 \wedge e_2 \wedge e_3 \wedge e_4) = 1$, we see that the group of transformations $\mathrm{Aut}(V, \mu)$ perserving μ is isomorphic to the group $SL_4\mathbb{C}$. In addition, each map $T \in \mathrm{Aut}(V, \mu)$ induces a map $\Lambda^2 T : \Lambda^2 V \to \Lambda^2 V$, which is determined by the formula $\Lambda^2 T(v \wedge w) = Tv \wedge Tw$. For any $T \in \mathrm{Aut}(V, \mu)$, the induced map $\Lambda^2 T$ preserves the bilinear form on $\Lambda^2 V$, so the mapping $T \mapsto \Lambda^2 V$ determines a group homomorphism $\mathrm{Aut}(V, \mu) \to \mathrm{Aut}(\Lambda^2 V, \langle \cdot, \cdot \rangle)$, where $\mathrm{Aut}(\Lambda^2 V, \langle \cdot, \cdot \rangle)$ denotes the group of linear automorphisms perserving the bilinear form. The kernel of this map is $\{\pm \mathrm{id}_V\}$, and fixing an orthogonal basis for $\langle \cdot, \cdot \rangle$ gives us that this map is a double cover $SL_4\mathbb{C} \to SO_6\mathbb{C}$, so $SL_4\mathbb{C}$ is isomorphic to the complex spin group $\mathrm{Spin}_4\mathbb{C}$

If we then fix a hermitian inner product $h : V \times V \to \mathbb{C}$, we can consider the automorphisms $\mathrm{Aut}(V, \mu, h)$ preserving h and μ, which is isomorphic to the group SU_4. The bilinear form h induces a hermitian inner product (which we also denote h) on $\Lambda^2 V$ defined by

$$h(v_1 \wedge v_2, v_3 \wedge v_4) = \det \begin{pmatrix} h(v_1, v_3) & h(v_1, v_4) \\ h(v_2, v_3) & h(v_2, v_4) \end{pmatrix}$$

Then if $T \in \mathrm{Aut}(V, \mu, h)$, $\Lambda^2 T$ preserves the bilinear form $\langle \cdot, \cdot \rangle$ induced by μ as well as the hermitian inner product induced by h. The group that preserves both of these structures is isomorphic to $SO_6\mathbb{C} \cap U_6$, which is $SO_6\mathbb{R}$. This gives us that $SU_4 \cong \mathrm{Spin}_6$.

In general, one can play the game of fixing additional structure on V (e.g. a real stucture, quaternionic structure, symplectic form) and look for the induced structure on $\Lambda^2 V$. This then gives a map from automorphisms of V preserving this additional structure to automorphisms of $\Lambda^2 V$ preserving the induced structure. Playing this game then determines several other low dimensional Spin groups.

$$\begin{array}{lll} \mathrm{Spin}_5\mathbb{C} \cong Sp_4\mathbb{C} & \mathrm{Spin}_4 \cong Sp(4) & \mathrm{Spin}_4\mathbb{C} \cong SL_2\mathbb{C} \times SL_2\mathbb{C} \\ \mathrm{Spin}_{1,3} \cong SL_2\mathbb{C} & \mathrm{Spin}_{1,2} \cong SL_2\mathbb{R} & \mathrm{Spin}_{1,5} \cong SL_2\mathbb{H} \end{array}$$

Where $Sp_4\mathbb{C}$ denotes the group of 4×4 matrices preserving a symplectic form, $Sp(4) = Sp_4\mathbb{C} \cap U_4$, and $SL_2\mathbb{H}$ denotes the automorphisms of a 2 dimensional quaternionic vector space with determinant 1 when regarded as 4×4 complex matrices. Any Clifford module immediately gives a representation of the Pin and Spin groups simply by restricting the action.

Definition 3.5. Given a Pin group $\mathrm{Pin}_{p,q}$, the ***Pinor representations*** are representations of $\mathrm{Pin}_{p,q}$ that arise from an irreducible Clifford module M (i.e. the action of $\mathrm{Pin}_{p,q}$ can be extended to an action of $\mathrm{Cliff}_{p,q}(\mathbb{R})$). The ***Spinor representations*** are defined analogously for the group $\mathrm{Spin}_{p,q}$.

From the classification of Clifford modules, we get a classification of all the Pinor representations. From the relationship between a Clifford algebra and its even subalgebra, we also get a complete classification of all the Spinor representations.

4. Principal Bundles

Definition 4.1. Let M and F be smooth manifolds. Then a ***fiber bundle*** over M with model fiber F is a the data of a smooth manifold E with a smooth map $\pi : E \to M$ such that for every point $p \in M$, there is a neighborhood $U \subset M$ containing p such that there exists an diffeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ such that the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "p_U"] \\ & U & \end{tikzcd}$$

Where P_U denotes projection onto the first factor. The map φ is called a ***local trivialization*** of the fiber bundle $\pi : E \to M$.

Definition 4.2. Let G be a Lie group, and M a smooth manifold. A ***Principal G-bundle*** over M is the data of

(1) A smooth manifold P with a map $\pi : P \to M$.
(2) A smooth right G-action on P that is free and transitive on the fibers of π.
(3) For every point $p \in M$, a neighborhood $U \subset M$ containing p and a G-equivariant diffeomorphism $\varphi : \pi^{-1}(U) \to U \times G$ (where the right action on $U \times G$ is right multiplication on the second factor) such that we get the commutative diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "p_U"] \\ & U & \end{tikzcd}$$

where p_U denotes projection onto the first factor.

Example 4.3. Given a smooth manifold M and a point $p \in M$, a basis of the tangent space is a linear isomorphism $b : \mathbb{R}^n \to T_pM$. The group $GL_n\mathbb{R}$ acts on the set of bases $\mathcal{B}_p$ on the right by $b \cdot g = b \circ g$. Then the ***frame bundle*** of M, denoted $\mathcal{B}(M)$ is the disjoint union

$$\mathcal{B}(M) = \coprod_{p \in M} \mathcal{B}_p$$

where π is the projection map $(p, b) \mapsto p$. Then $\mathcal{B}(M)$ is a principal $GL_n\mathbb{R}$ bundle over M.

Example 4.4. Given a smooth manifold M with a Riemannian metric g, this induces an inner product on each tangent space. Then the set of orthonormal bases of T_pM is the set of all linear isometries $T_pM \to \mathbb{R}^n$, where $\mathbb{R}^n$ is equipped with the standard inner product. Then the disjoint union over all points p of orthonormal bases for the tangent spaces forms the ***orthonormal frame bundle*** $\mathcal{B}_O(M)$, which is a principal O_n bundle.

Definition 4.5. Let $\pi : P \to m$ be a principal G-bundle over M, and let F be a smooth manifold with a smooth left G action. Then the ***associated fiber bundle***, denoted $P \times_G F$, is the set

$$P \times_G F = P \times G/(p, g) \sim (p \cdot h, h^{-1}g)$$

Definition 4.6. Let $\pi : P \to M$ be a principal G-bundle, and $\rho : H \to G$ a group homomorphism. The map ρ gives P a left H action where $h \cdot p = \rho(h) \cdot p$. A ***reduction of structure group*** is the data of a principal H bundle $\varphi : Q \to M$ and an H equivariant map $F : Q \to P$.

The map $F : Q \to P$ induces a map $\tilde{F} : Q \times_H G$, where we map the equivalence class $[q, g]$ to $F(q)g$. This is well defined on equivalences classes since

$$(q \cdot h, \rho(h)^{-1}g) \mapsto F(q \cdot h)\rho(h)^{-1}g = F(q)\rho(h)\rho(h)^{-1}g = F(q)g$$