UNDERGRADUATE THESIS NOTES

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Week 1

Exercise 1.1. Prove a small lemma

Lemma. *Let G be a group, and V a finite dimensional irreducible complex representation. Then elements of $Z(G)$ act by scalars.*

Proof. Let $z \in Z(G)$, then since z commutes with the action of G, it determines a G-module isomorphism $\varphi_z : V \to V$. Then φ_z admits some eigenvalue λ, and the λ eigenspace of φ_z is G-invariant, so φ_z must be the map $\lambda \operatorname{id}_V$. ■

Exercise 1.2. Consider the groups $\mathrm{Pin}_{p,q}$ and $\mathrm{Spin}_{p,q}$, which lie in the Clifford algebra $\mathrm{Cliff}_{p,q}$. A ***Pin representation*** is a representation V of $\mathrm{Pin}_{p,q}$ that extends to an irreducible Clifford module. Likewise, a ***Spin representation*** is a representation V of $\mathrm{Spin}_{p,q}$ that extends to an irreducible $\mathrm{Cliff}^0_{p,q}$ module. Find some of these representations.

Since all Clifford algebras arise (as ungraded algebras) as direct sums of matrix algebras over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, it will be useful to characterize the irreducible modules of these matrix algebras. In the case of $\mathbb{H}$, we use the convention that scalar multiplication on a quaternionic vector space acts on the right, so quaternionic matrices can act on the left.

Proposition 1.3.
The only irreducible $M_n\mathbb{R}$ module is $\mathbb{R}^n$.

Proof. We see that there is an increasing chain of left ideals

$$0 = I_0 \subset I_1 \subset \ldots \subset I_n = M_n\mathbb{R}$$

where I_k is the ideal of matrices where all the entries past the k^{th} column are 0. In addition, we have that this chain of ideals has the property that the quotient space $I_{k+1}/I_k \cong \mathbb{R}^n$ as a left $M_n\mathbb{R}$ module. We note that $\mathbb{R}^n$ is most certainly irreducible, since the orbit of any nonzero vector $v \in \mathbb{R}^n$ is all of $\mathbb{R}^n$.

Then let W denote an arbitrary nontrivial irreducible $M_n\mathbb{R}$ module. Fix $w \in W$. Then the orbit of w under the action of $M_n\mathbb{R}$, and since the module is nontrivial, it must be all of W. Therefore, the mapping

$$\begin{aligned} \varphi : M_n\mathbb{R} &\to W \\ M &\mapsto M \cdot w \end{aligned}$$

is a surjective map of left $M_n\mathbb{R}$ modules. Since this map is surjective, there exists some k such that $\varphi(I_k) \neq 0$. Let k denote the smallest such k. Since $\varphi(I_{k-1}) = 0$, this map factors through the quotient I_k/I_{k-1}, which is isomorphic to R^n as a left module. Then since both $\mathbb{R}^n$ and W are irreducible, this implies that the map $I_k/I_{k-1} \to W$ is an isomorphism, so $W \cong \mathbb{R}^n$. ■

This proof carries over for the matrix algebras $M_n\mathbb{C}$ and $M_n\mathbb{H}$. We then need another lemma to fully classify the real Clifford modules.

Lemma 1.4. *The algebra $A = M_n\mathbb{F} \oplus M_n\mathbb{F}$ (where $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$) has two irreducible modules, the first being isomorphic to $\mathbb{F}^n$ where the first factor has the standard action and the second factor has the trivial action, and the second is isomorphic to $\mathbb{F}^n$ where the first factor acts trivially, and the second factor has the standard action.*

Proof. We first note that the left ideals $0 \oplus M_n\mathbb{F}$ and $M_n\mathbb{F} \oplus 0$ both admit a chain of left ideals as we did above, which we denote as I_k and J_k respectively, which satisfy $I_{k+1}/I_k \cong \mathbb{F}^n$ and $J_{k+1}/J_k \cong \mathbb{F}^n$ as A modules. where the former has the action of the right factor, and the latter has the action of the right factor. We denote the modules as R and L respectively. We also have an ascending chain of left ideals

$$0 \subset J_1 \oplus 0 \subset \ldots \subset M_n\mathbb{F} \oplus 0 \subset M_n\mathbb{F} \oplus I_1 \subset \ldots \subset M_n\mathbb{F} \oplus M_n\mathbb{F}$$

Se note that for the first half of the chain, we have that each successive quotient is isomorphic to R. and for the second half, the successive quotients are isomorphic to L. Then given a nontrivial irreducible module W and a nonzero $w \in W$, we know that it's orbit under the algebra action must be all of W, so the map $A \to W$ given by $(M, T) \mapsto (M, T) \cdot w$ is a surjective map of left A modules. Then we know for some ideal in the chain, the map is nonzero, and then the map will factor through the quotient of that ideal by the previous ideal, giving an isomorphism with either R or L.

We then need to show that these modules are not isomorphic. Any such isomorphism would be a linear map $\varphi : \mathbb{F}^n \to \mathbb{F}^n$ satisfying

$$\varphi(Mv) = T\varphi(V)$$

for all matrices M and T. It is clear that no such φ exists, so the two modules are not isomorphic. ∎

This gives a classification of all the irreducible (ungraded) Clifford modules. Recall that the (ungraded) classification of Clifford algebras gives us the Clifford "chessboard"

$M_8\mathbb{C}$	$M_8\mathbb{H}$	$\mathcal{M}_8\mathbb{H} \oplus M_8\mathbb{H}$	$M_{16}\mathbb{H}$	$M_{32}\mathbb{C}$	$M_{64}\mathbb{R}$	$M_{64}\mathbb{R} \oplus M_{64}\mathbb{R}$	$M_{128}\mathbb{R}$
$M_4\mathbb{H}$	$M_4\mathbb{H} \oplus M_4\mathbb{H}$	$M_8\mathbb{H}$	$M_{16}\mathbb{C}$	$M_{32}\mathbb{R}$	$M_{32}\mathbb{R} \oplus M_{32}\mathbb{R}$	$M_{64}\mathbb{R}$	$M_{64}\mathbb{C}$
$M_2\mathbb{H} \oplus M_2\mathbb{H}$	$M_4\mathbb{H}$	$M_8\mathbb{C}$	$M_{16}\mathbb{R}$	$M_{16}\mathbb{R} \oplus M_{16}\mathbb{R}$	$M_{32}\mathbb{R}$	$M_{32}\mathbb{C}$	$M_{32}H$
$M_2\mathbb{H}$	$M_4\mathbb{C}$	$M_8\mathbb{R}$	$M_8\mathbb{R} \oplus M_8\mathbb{R}$	$M_{16}\mathbb{R}$	$M_{16}\mathbb{C}$	$M_{16}\mathbb{H}$	$M_{16}\mathbb{H} \oplus M_{16}\mathbb{H}$
$M_2\mathbb{C}$	$M_4\mathbb{R}$	$M_4\mathbb{R} \oplus M_4\mathbb{R}$	$M_8\mathbb{R}$	$M_8\mathbb{C}$	$M_8\mathbb{H}$	$M_8\mathbb{H} \oplus M_8\mathbb{H}$	$M_{16}\mathbb{H}$
$M_2\mathbb{R}$	$M_2\mathbb{R} \oplus M_2\mathbb{R}$	$M_4\mathbb{R}$	$M_4\mathbb{C}$	$M_4\mathbb{H}$	$M_4\mathbb{H} \oplus M_4\mathbb{H}$	$M_8\mathbb{H}$	$M_{16}\mathbb{C}$
$\mathbb{R} \oplus \mathbb{R}$	$M_2\mathbb{R}$	$M_2\mathbb{C}$	$M_2\mathbb{H}$	$M_2\mathbb{H} \oplus M_2\mathbb{H}$	$M_4\mathbb{H}$	$M_8\mathbb{C}$	$M_{16}\mathbb{R}$
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$M_2\mathbb{H}$	$M_4\mathbb{C}$	$M_8\mathbb{R}$	$M_8\mathbb{R} \oplus M_8\mathbb{R}$

Where the (p, q) element of the table is $\mathrm{Cliff}_{p,q}$ as an ungraded algebra, and the bottom left corner is $\mathrm{Cliff}_{0,0} \cong \mathbb{R}$. All other Clifford algebras can be recovered from this table, since incrementing $p + q$ by 8 results in tensoring with $M_{16}\mathbb{R}$.