$\mathrm{Cliff}_{n,0} \circlearrowright \mathrm{Cliff}_{n,0}$. This becomes a left $\mathrm{Cliff}_{0,n}$ action by

$$e_i \cdot m = m \cdot \alpha(e_i)$$

Does the Dirac operator commute with the left $\mathrm{Cliff}_{0,n}$ action?

$D = c(e^k)\partial_k$ action on sections.

$$\Psi \in \Gamma_M(\$(M)) = \Omega_P^{0,\mathrm{spin}}(\mathrm{Cliff}_{n,0})$$

$D\Psi = e^k \partial_k \Psi$ Let $e_i \in \mathrm{Cliff}_{0,n}$

$$e_i(D\Psi) \quad \text{vs} \quad D(e_i\Psi)$$

$$e_i(D\Psi) = (D\Psi)(-e_i)$$

$$D(e_i\Psi) = D(\Psi(-e_i))$$

$$\begin{pmatrix} 0 & -D_1 \\ D_0 & 0 \end{pmatrix}\begin{pmatrix} 0 & D_1 \\ D_0 & 0 \end{pmatrix} = \begin{pmatrix} -D_1D_0 & 0 \\ 0 & D_0D_1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & D_1 \\ D_0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -D_1 \\ D_0 & 0 \end{pmatrix} = \begin{pmatrix} D_1D_0 & 0 \\ 0 & -D_0D_1 \end{pmatrix}$$

Does not commute

$$\begin{pmatrix} D_1D_0 & 0 \\ 0 & D_0D_1 \end{pmatrix}\begin{pmatrix} 0 & -D_1 \\ D_0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -D_1D_0D_1 \\ D_0D_1D_0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -D_1 \\ D_0 & 0 \end{pmatrix}\begin{pmatrix} D_1D_0 & 0 \\ 0 & D_0D_1 \end{pmatrix} = \begin{pmatrix} 0 & -D_1D_0D_1 \\ D_0D_1D_0 & 0 \end{pmatrix}$$

Commutes!

$$\begin{pmatrix} 0 & -D_1 \\ D_0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -D_1 \\ D_0 & 0 \end{pmatrix} = \begin{pmatrix} -D_1D_0 & 0 \\ 0 & D_0 -D_1 \end{pmatrix}$$

Small computation $\qquad \nabla_v w - \nabla_w v = [v,w]$

$$\nabla^2_{v,w} - \nabla^2_{wv} = R_{v,w}$$

$$\|$$

$$\left(\nabla_v \nabla_w - \nabla_{\nabla_{v,w}}\right) - \left(\nabla_w \nabla_v - \nabla_{\nabla_{w,v}}\right)$$

$$\|$$

$$\nabla_v \nabla_w - \nabla_w \nabla_v - \nabla_{\nabla_v w} - \nabla_{\nabla_w v}$$

$$= \nabla_v \nabla_w - \nabla_v \nabla_w - \nabla_{\nabla_v v - \nabla_w v} = R_{v,w}$$

Low dimensional index

On S^1. We have 2-spin structures and two different Clifford bundles.

What is $KO^{-1}(pt)$? $\mathbb{Z}/2$

Consider the bounding spin structure on S^1

The spinor fields on S^1 are equivalent to $\underset{\mathbb{Z}/2}{\underset{\text{\rotatebox{90}{$\cong$}}}{Spin_1}}$-equivariant maps $S^1 \to \mathbb{C}$

$\Leftrightarrow$ 2π-antiperiodic maps $\mathbb{R} \to \mathbb{C}$

Dirac operator on $\$(S^1)$ is

$$D = i\partial_\theta - \frac{1}{2}$$

No kernel $\Rightarrow$ trivial in $KO^{-1}(pt) \cong \mathbb{Z}/2\mathbb{Z}$

On the disconnected, the kernel is nontrivial $\Rightarrow$ nontrivial in $KO^{-1}(pt)$

Volume element of $Cliff_n$. relation with $KO^n(pt)$

Spin Structure on S^2

$$\begin{array}{ccccc} B_{spin}(S^2) & \longrightarrow & B_{SO}(S^2) & \longrightarrow & S^2 \\ \wr\downarrow & & \wr\downarrow & & \downarrow \\ Spin_3 & \longrightarrow & SO_3 & \longrightarrow & S^2 \end{array}$$

the Hopf fibration.

$Cliff_2 \cong \mathbb{H}$ has a single irrep, itself.

$$Cliff_2^0 = \mathbb{R} \oplus e_1e_2\mathbb{R} = \mathbb{C}$$

$$e_1e_2e_1e_2 = -e_1e_1e_2e_2 = -1$$

Gives grading $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$

$\mathrm{Spin}_2 \cong \mathbb{T} \subset \mathbb{C}$

acts on $\mathbb{C}$ in the standard way

acts on $j\mathbb{C}$ by the conjugate action

by picking an appropriate basis, this is just the standard rep.

Spinor bundle

$$B_{\mathrm{spin}}(S^2) \times_{\mathrm{Spin}_2} \mathbb{H}$$

$$\text{\rotatebox{90}{$\cong$}}$$

$$S^3 \times_{\mathbb{T}} (\mathbb{C} \oplus \mathbb{C})$$

$$S^3 \times (\mathbb{C}^2) \Big/ (p, v, w) \sim (p \cdot e^{i\theta}, e^{-i\theta} v, e^{i\theta} w)$$

Sections of the bundle are

$\mathbb{T}$-equivariant maps $S^3 \longrightarrow \mathbb{C}^2$

Want this to be isomorphic to $L \oplus L \to \mathbb{CP}^1$

Where L is The tautological bundle $L \to \mathbb{CP}^1$

$S^3 \times_\pi \mathbb{C}$ is a rank 2 bundle over $\mathbb{CP}^1 \cong S^2$. (Complex line bundle)

$$\begin{tikzcd} S^3 \arrow[d] & \sim & \mathbb{C}-\{0\}/\mathbb{C}^* \arrow[d] \\ S^2 & & S^2 \end{tikzcd}$$

$$Ric(\varphi) = \sum_j R_{e_j,\varphi}(e_j)$$

We have the identies

$$① \quad R_{V,W}X + R_{X,V}W + R_{W,V}X = 0$$

$$② \quad \langle R_{V,W}X, Y\rangle = \langle R_{X,Y}V, W\rangle$$

$$R(\varphi) = \frac{1}{2}\sum_{i,j} e_i e_j R_{e_i e_j}(\varphi)$$

$$= \frac{1}{2}\sum_{i,j,k} e_i e_j \langle R_{e_i e_j}(\varphi), e_k\rangle e_k$$

$$= \frac{1}{2}\sum_{i,j,k} \langle R_{e_i e_j}(\varphi), e_k\rangle e_i e_j e_k$$

$$= \frac{1}{2}\sum_{i,j,k} \langle R_{\varphi,e_k}(e_i), e_j\rangle e_i e_j e_k$$

$$= \frac{1}{2}\Big(\sum_j \langle R_{\varphi,e_j}(e_j), e_j\rangle e_j + \sum_{\substack{i,j,k \\ \text{not all equal}}} \langle R_{\varphi e_k} e_i, e_j\rangle e_i e_j e_k$$

$$= \frac{1}{2}\left(Ric(\varphi) + \cancel{\sum_{i \neq j \neq k}} + \sum_{i \neq j = k} + \sum_{i = k \neq j} + \cancel{\sum_{i \neq j \neq k \neq i}} \right)$$

$$= \frac{1}{2}\left(Ric(\varphi) + \sum_{i \neq k} + \sum_{i \neq k}\right.$$

$$\sum_{i \neq j} \langle R_{\varphi e_j}(e_i)\ e_j \rangle e_i = \sum_{i,j} \langle R_{\varphi e_j}(e_i), e_j \rangle e_i$$

$$= 2 \sum_{i < j} \langle R_{\varphi e_j}(e_i), e_j \rangle e_i$$

$$\sum_{i = k \neq j} \langle R_{\varphi e_k}(e_i), e_j \rangle e_k = \sum_{i \neq j} \langle R_{\varphi e_j}(e_i), e_j \rangle e_j$$

.

$$\sum_{i,j} \left\langle R_{\varphi, e_j}(e_i), e_j \right\rangle e_i$$

111	211	311
112	212	312
113	213	313
121	221	321
122	222	322
123	223	323
131	231	331
132	232	332
133	233	333

= $i=j=k$

= $i=k\neq j$

= $i\neq j=k$

= $i=j\neq k$

= $i\neq j\neq k$

$$\sum_{i,j,k,l} \left\langle R_{e_i e_j}(e_k), e_l \right\rangle e_i e_j e_k e_l$$

$i=j$ or $k=l$ terms are 0

$$= \sum_l \left(\sum_{ijk} \left\langle R_{e_i e_j}(e_k), e_l \right\rangle e_i e_j e_k \right) e_l$$

$$= \sum_l \left(\cancel{\sum_{i=j=k}} + \cancel{\sum_{i=j\neq k}} + \sum_{i\neq j=k} + \sum_{i=k\neq j} + \cancel{\sum_{i\neq j\neq k\neq i}} \right)$$

$$= \sum_{\ell} \left(\sum_{i \neq j = k} + \sum_{i = k \neq j} \right)$$

$$= \sum_{\ell} \Bigg(\sum_{i \neq j} \langle R_{e_i e_j}(e_j), e_\ell \rangle e_i e_j e_j + \sum_{i \neq j} \langle R_{e_i e_j}(e_i), e_\ell \rangle e_i e_j e_i e_\ell \Bigg)$$

$$\Vert \qquad\qquad\qquad\qquad \Vert$$

$$- \sum_{i,j,\ell} \langle R_{e_j e_i}(e_j), e_\ell \rangle e_i e_\ell \quad - \sum_{i,j,\ell} \langle R_{e_i e_j}(e_i), e_\ell \rangle e_j e_\ell$$

$$\Vert$$

$$- \sum_{i,j,\ell} \langle R_{e_i e_j}(e_i), e_\ell \rangle e_j e_\ell$$

$$\overset{?}{=} -2 \sum_{i,j,\ell} \langle R_{e_i e_j}(e_i), e_\ell \rangle e_j e_\ell$$