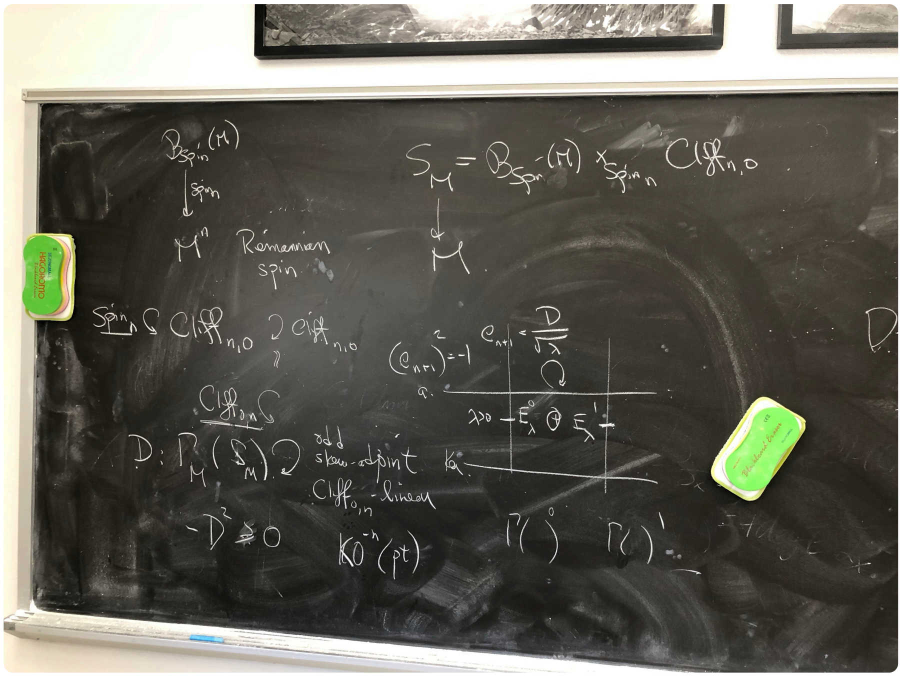

and connection things.

$\pi : E \to M$ a fiber bundle with model fiber F

$$\begin{tikzcd} S \arrow[d] \\ E \end{tikzcd}$$

fiber over $e \in E$ is the space of splittings of

$$0 \to V_e \to T_e E \to T_{\pi(e)} M \to 0$$

which is affine over $\mathrm{Hom}(T_{\pi(e)} M, V_e)$

Let $\varphi : \pi^{-1}(U) \to U \times F$ be a local trivialization. This gives a identificatn $T\pi^{-1}(U) \cong T(U \times F) = TM \oplus TF$

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr] \arrow[dr] & & U \times F \arrow[dl] \\ & U & \end{tikzcd}$$

A left splitting is equivalent to a right splitting

$$0 \to A \xrightarrow{\varphi} B \underset{i}{\overset{\psi}{\rightleftarrows}} C \to 0$$

Given i, produce a $j: B \to C$ s.t.

$\varphi \circ j = id_A$.

Since φ is injective, $\exists\ \varphi^{-1}: \varphi(A) \to A$

Let $\pi: B \to \varphi(A)$ be the projection.

$B_{spin}(M) \times Cliff_{n,0}(\mathbb{R}) \circlearrowleft Cliff_{n,0}(\mathbb{R})$

Does this descend to

$$B_{spin}(M) \underset{Spin_n}{\times} Cliff_{n,0} = B_{spin}(M) \times Cliff_{n,0} \Big/ (p,v) \sim (p \cdot g, g^{-1} \cdot v)$$

let $\varphi \in Cliff_{n,0}$

$$(p, v\varphi) \sim (p \cdot g, g^{-1} \cdot v \varphi) \quad \checkmark$$

Let

$$\begin{array}{ll} P \circlearrowleft G & \text{be a principal } G\text{-bundle} \\ \downarrow \pi & \text{with connection 1-form} \\ M & \omega \in \Omega^1_P(\mathfrak{g}) \end{array}$$

Then $R_g^* \omega = Ad_{g^{-1}} \omega$

$$Ad_{g^{-1}}(\omega)(v) = Ad_{g^{-1}}(\omega(v))$$

Proof :

First check on vertical vectors $V_p \cong \mathfrak{g}$. The isomorphism $\mathfrak{g} \to V_p$ is given by

$$X \longmapsto \left.\frac{d}{dt}\right|_{t=0} p \cdot \exp(tX) = \tilde{X}$$

Let $X \in \mathfrak{g}$. Then

$$\begin{aligned}
(R_g)_*(\tilde{X})_{p\cdot g} &= d(R_g)_p(\tilde{X}_p) \\
&= d(R_g)_p\left(\left.\frac{d}{dt}\right|_{t=0} p \cdot \exp(tX)\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} p \cdot \exp(t) \cdot g \\
&= \left.\frac{d}{dt}\right|_{t=0} p \cdot g \left(g^{-1} \exp(t) \cdot g\right)
\end{aligned}$$

$$= \frac{d}{dt}\Big|_{t=0} p \cdot g \exp\left(Ad_{g^{-1}} \exp(tX)\right)$$

$$= \widetilde{Ad_{g^{-1}}(X)}\Big|_{p\cdot g}$$

Then any vector $v \in T_pP$ can be uniquely written as $v = \tilde{X} + h$ $X \in \mathfrak{g}$ $h \in H_p$.

Then $\left(R_g^* \omega\right)_p (v) = \left(R_g^* \omega\right)_p (\tilde{X} + h)$

$$= \omega_{p\cdot g}\left(d(R_g)_p(\tilde{X}) + d(R_g)_p(h)\right)$$

$$= \omega_{p\cdot g}\left(Ad_{g^{-1}}(\tilde{X})_{p\cdot g} + h'\right)$$

$$= \omega_{p\cdot g}\left(Ad_{g^{-1}}(\tilde{X})_{p\cdot g}\right)$$

$$= Ad_{g^{-1}}(\tilde{X})$$

Let
$$\begin{array}{c} P \circlearrowleft G \\ \downarrow \pi \\ M \end{array}$$
be a principal bundle with connection H and connection form $\omega \in \Omega^1_P(\mathfrak{g})$

How does a connection induce path lifting?

Fix $x \in M$, $p \in \pi^{-1}(x)$.

Then $T_pP = V_p \oplus H_p$.

$d\pi_p\big|_{H_p} : H_p \longrightarrow T_xM$ is an isomorphism. So $v \in T_xM$ has a unique horizontal preimage

Given $\gamma : I \longrightarrow M$, we can lift the vector field $\gamma'(t)$ to a vector field on P.

Integrating the lift gives a path in P

based at p.

How does H induce path lifting on associated bundles?

Let
$$\begin{array}{c} P \times_G F \\ \pi \big\downarrow \\ M \end{array}$$
be an associated bundle

$\gamma : I \longrightarrow M$ a curve

$[p, f] \in \pi^{-1}(x)$

γ lifts to $\tilde{\gamma} : I \to P$

Define $\hat{\gamma} : I \longrightarrow P \times_G F$ by

$$\hat{\gamma}(t) = [\tilde{\gamma}(t), f]$$

$\hat{\gamma}$ is a lift of γ by construction.

Path lifting on a vector bundle induces

A parallel transport map. Given

$\gamma : I \to M$, define

$$\tau_t : E_{\gamma(0)} \to E_{\gamma(t)}$$

$e \mapsto \tilde{\gamma}(t)$ where $\tilde{\gamma}$ is the lift of

γ with $\tilde{\gamma}(0) = e$.

τ_t is invertible, take the path lift of

$\gamma(-t)$ to get the inverse.

Linear?

$$\tau_t(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \tau_t(v_1) + \lambda_2 \tau_t(v_2)?$$

$$\tau_t(v_1 + v_2) = \gamma_{v_1+v_2}(t)$$

$$\tau_t(v_1) = \gamma_{v_1}(t)$$

$$\tau_t(v_2) = \gamma_{v_2}(t)$$

$\gamma_{v_1+v_2}(t) = \gamma_{v_1}(t) + \gamma_{v_2}(t)$?

Yes by uniqueness of integral curves

Same for scaling.

Given $\xi \in T_pM$, $s \in \Gamma_M(P \times_G V)$

Define the covariant derivative by

$$\nabla_\xi s = \frac{d}{dt}\Big|_{t=0} \tau_t(s(\gamma(t)))$$

Where γ is any curve with $\gamma'(0) = \xi$

TODO - Prove that this satisfies all the properties of the covariant derivative

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Suppose we have
$$\begin{array}{c} E \\ \downarrow \\ M \end{array}$$

$B(E)$ its $GL_n\mathbb{R}$ bundle of frames.

Let $\omega \in \Omega_{B(E)}(\mathfrak{gl}_n\mathbb{R})$ be the connection 1-form, i.e. a 1 form on $B(E)$ valued in $M_n\mathbb{R}$, i.e.

a $n \times n$ matrix of 1-forms.

A local frame $\{e_i\}$ for E is a section

$\sigma : U \longrightarrow B(E)$, so we get a

pullback form $\sigma^*\omega \in \Omega_M(M_n\mathbb{R})$

Define ∇ by $\nabla e_i = (\sigma^*\omega)_i^j \, e_j$

for $\xi \in T_pM$, $\nabla_\xi e_i = (\sigma^*\omega)_i^j(\xi) e_j$

$$\nabla f e_i = df \, e_i + f(\sigma^*\omega)_i^j \, e_j$$