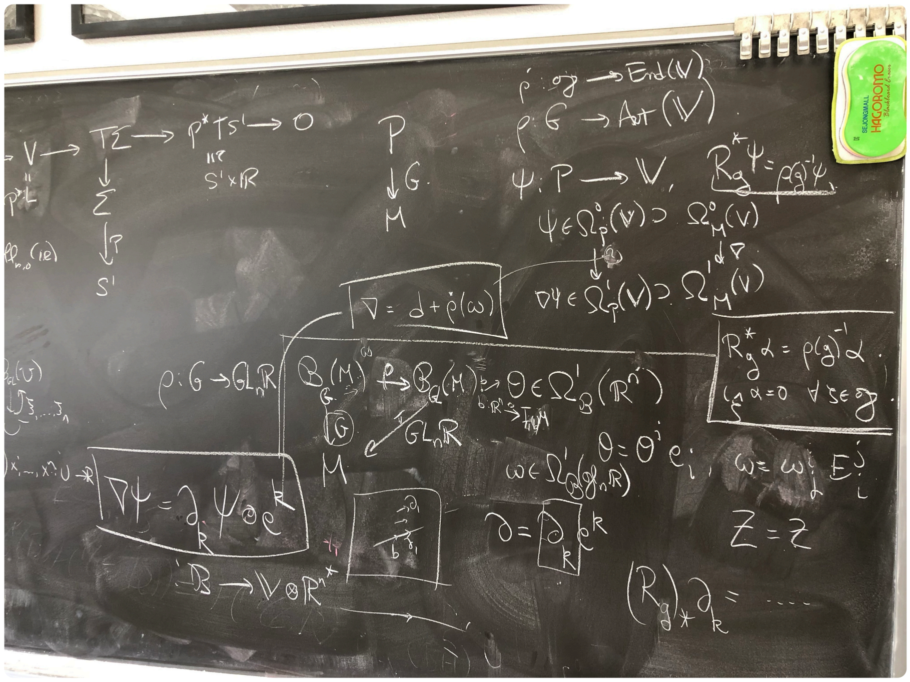


Suppose we have a principal G-bundle with connection H, connection 1-form

$$\Theta \in \Omega^1_P(\mathfrak{g})$$

$$\begin{array}{l} H \\ \downarrow \\ P \circlearrowleft G \\ \downarrow \pi \\ M \end{array}$$

and a vector space V with a G-representation

$$\varphi : G \to GL(V)$$

(Left action)

We get an associated vector bundle

$$\begin{array}{c} E = P \times_G V. \\ \downarrow \\ M \end{array}$$

and a connection $\tilde{H} \subset TE$

$$\Omega^0_M(E) = \Gamma_M(E) \longleftrightarrow \left\{ \begin{array}{l} \varphi : P \to V \\ \varphi(p \cdot g) = \rho(g)^{-1}\varphi(p) \end{array} \right\}$$

$$\cap$$

$$\Omega^0_P(V)$$

Claim: There is a bijection

$$\Omega^k_M(E) \longleftrightarrow \left\{ \begin{array}{l} \alpha \in \Omega^k_P(V) \\ R_g^* \alpha = \alpha \quad , \; i_\xi(\alpha) = 0 \\ \qquad \text{for } \xi \in \mathfrak{g} \end{array} \right\}$$

Proof:

We provide maps in both directions.

Let $\omega \in \Omega^K_M(E)$. want to produce

$\tilde{\omega} \in \Omega^K_P(V)$. $\pi^*\omega \in \Omega^K_P(E)$

$\omega_p(v) = [q, a]$ $q \in \pi_E^{-1}(p)$

Suppose $\pi^*\omega_p(v) = [q, a]$

$q = p \cdot g$ for a unique $g \in G$.

so $[q, a]$ has a unique representative

$[p, \tilde{a}]$

Define $\tilde{\omega}_p(v) = \tilde{a}$

Need to check ① $R_g^*\tilde{\omega} = \tilde{\omega}$

② $i_\xi(\omega) = 0$ for $\xi \in \mathfrak{g}$

① $\left(R_g^* \tilde{\omega}\right)_p(v) = \tilde{\omega}_{p\cdot g}\left((R_g)_* v\right)$

$$\begin{array}{lll}
\rho : G \longrightarrow GL(V) & P \circlearrowleft G & E = P \times_G V. \\
\dot{\rho} : \mathfrak{g} \longrightarrow End(V) & \downarrow \pi & \downarrow \\
 & M & M
\end{array}$$

Define the covariant derivative

$$\nabla : \Omega^k_\rho(V) \longrightarrow \Omega^{k+1}_\rho(V)$$

$$\nabla \varphi = \underbrace{d\varphi + \dot{\rho}(\Theta) \wedge \varphi.}_{\text{What does this mean?}}$$

$d\varphi$, apply d to each term

$$\left(\dot{\rho}(\Theta) \wedge \varphi\right)(v_1, \dots v_k)$$

$$\text{"Alt"}\left[\underbrace{\dot{\rho}(\Theta(v_1))}_{\in \, End\, V}\Big(\underbrace{\varphi(v_2 \dots v_k)}_{\in \, V}\Big)\right] \in V$$

In a local trivialization,

can write $\dot{\rho}(\Theta)$ as a matrix

$\dot{\rho}(\Theta)^i_j$ of 1 forms. and

φ as a vector φ^i of K. forms.

Then

$$(\dot{\rho}(\Theta) \wedge \varphi)^i = \dot{\rho}(\Theta)^i_j \wedge \varphi^j$$

<u>Claim:</u> Given $\varphi \in \Omega_\rho^{K,G}(V)$,

$$\nabla\varphi = d\varphi + \dot{\rho}(\Theta) \wedge \varphi \in \Omega_\rho^{K,G}(V)$$

<u>Lemma</u>: (Cartan's Magic Formula)

$$\mathcal{L}_V \omega = i_V(d\omega) + d(i_V(\omega))$$

To show the claim, we must show:

① $i_X(\nabla\varphi) = 0$ for X vertical

② $R_g^* \nabla\varphi = \rho(g)^{-1} \nabla\varphi$

For ① we first claim that $i_X(d\varphi^i) = 0 \ \forall i$

By Cartan's magic formula,

$$\underbrace{\mathcal{L}_X \varphi^i}_{=\,0} = i_X(d\varphi^i) + d(\underbrace{i_X(\varphi^i)}_{=\,0})$$

Since φ^i is invariant in the vertical direction

0 since $\varphi \in \Omega_\rho^{k,G}(V)$

so $\quad i_X(d\varphi^i) = 0$

We next need to show that

$i_X(\dot\rho(\Theta) \wedge \varphi) = 0$.

suppose $k=1$, $\dim V = 2$.

Then $\dot{p}(\Theta)$ is a matrix of 1 forms

$$\dot{p}(\Theta) = \begin{pmatrix} \omega^1_1 & \omega^1_2 \\ \omega^2_1 & \omega^2_2 \end{pmatrix}$$

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$$

$$\dot{p}(\Theta) \wedge \psi = \begin{pmatrix} \omega^1_1 \wedge \psi^1 + \omega^1_2 \wedge \psi^2 \\ \omega^2_1 \wedge \psi^1 + \omega^2_2 \wedge \psi^2 \end{pmatrix}$$

Let X be vertical and V any vector field.

Then

$$\left(\dot{p}(\Theta) \wedge \psi\right)(X, V) = \begin{pmatrix} \omega^1_1 \wedge \psi^1(X,V) + \omega^1_2 \wedge \psi^2(X,V) \\ \omega^2_1 \wedge \psi^1(X,V) + \omega^2_2 \psi^2(X,V) \end{pmatrix}$$

$$= \begin{pmatrix} \det\begin{pmatrix} \omega^1_1(X) & \omega^1_1(V) \\ \psi^1(X) & \psi^1(V) \end{pmatrix} + \det\begin{pmatrix} \omega^1_2(X) & \omega^1_2(V) \\ \psi^2(X) & \psi^2(V) \end{pmatrix} \\ \det\begin{pmatrix} \omega^2_1(X) & \omega^2_1(V) \\ \psi^1(X) & \psi^1(V) \end{pmatrix} + \det\begin{pmatrix} \omega^2_2(X) & \omega^2_2(V) \\ \psi^2(X) & \psi^2(V) \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \omega^1_1(x)\varphi^1(v) - \underset{\|\atop 0}{\varphi^1(x)\omega^1_1(v)} + \omega^1_2(x)\varphi^2(v) - \underset{\|\atop 0}{\varphi^2(x)\omega^1_2(v)} \\ \\ \omega^2_1(x)\varphi^1(v) - \underset{\|\atop 0}{\varphi^1(x)\omega^2_1(v)} + \omega^2_2(x)\varphi^2(v) - \underset{\|\atop 0}{\varphi^2(v)\omega^2_2(v)} \end{pmatrix}$$

$$= \begin{pmatrix} \omega^1_1(x)\varphi^1(v) + \omega^1_2(x)\varphi^2(v) \\ \omega^2_1(x)\varphi^1(v) + \omega^2_2(x)\varphi^2(v) \end{pmatrix}$$

If $X \in \mathfrak{g}$, $\dot{\rho}(\Theta(X)) = \dot{\rho}(X)$

Covariant derivative on sections

$$\nabla : \Omega^0_M(E) \longrightarrow \Omega^1_M(E)$$

$$(\nabla\varphi)_x(v) = \frac{d}{dt}\Big|_{t=0} \tau_t^{-1}(\varphi(\gamma(t))$$

w τ_t is parallel transport along γ with $\gamma'(0) = v$.

Using the identification

$$\Omega^k_M(E) \simeq \Omega^{k,G}_P(V)$$

Given $\varphi \in \Omega^{0,G}_P(V)$

$$\nabla\varphi = d\varphi + \dot{\rho}(\Theta)\varphi$$

$$(\nabla\varphi)'_x(v) = [\pi^{-1}(x), \nabla\varphi(v)]$$

$$= \left[\pi^{-1}(x), d\varphi(v) + \dot{\rho}(\Theta)(v)\left(\varphi(\pi^{-1}(x))\right)\right]$$

$$\begin{matrix} \uparrow & & \uparrow \\ V & & V \end{matrix}$$

Let $\gamma : I \to M$ with $\gamma'(0) = v$.

And $\tau_t : P_x \to P_{\gamma(t)}$ the parallel transport maps. Want to compute

$$\left.\frac{d}{dt}\right|_{t=0} \tau_t^{-1}(\varphi(\gamma(t)))$$

Need to figue out how the parallel trasport maps look like. How it behaves with thinking of $\varphi : P \to V$.

Given $[p, v] \in E_x$, $\tau_t[x,v] = [\tilde{\gamma}_p(t), v]$

Where $\tilde{\gamma}_p : I \to P$ is the lift of γ based at $p \in P$.

$$\frac{d}{dt}\Big|_{t=0} \tau_t^{-1}(\psi(\gamma(t)))$$ Let $\varphi(\gamma(t)) = [p_t, v_t]$

$p_t \in P_{\gamma(t)}$ $v_t \in V$.

φ induces a map $\tilde{\varphi} : P \to V$, which satisfies

$$\tilde{\varphi}(p_t) = v_t$$
$$\tilde{\varphi}(p_t \cdot g) = g^{-1} \cdot v$$

How to interpret $\tau_t(\psi(\gamma(t)))$?

$$\underset{V}{\tau_t}(\varphi(\gamma(t))] = v_t \text{ ? How to chose?}$$

Pick $p \in P_x$ Let γ_p be the lift of γ to p. Then

$$\psi(\gamma(t)) = [\gamma_p(t), v_t)$$

and let $\tau_t(\tilde{\varphi}(\gamma(t)) = v_t = \tilde{\varphi}(\gamma(t))$

Then what is $\tau_t^{-1}(\tilde{\Psi}(\gamma(t)))$? Just identity! (?)

Can we interpret τ_t as $V \to V$?

Just the identity map.

$E_x \to E_{\gamma(t)}$

$[p, v] \mapsto [\gamma_p(t), v_t]$

Under the identification of the fibers with V, we can compute

$$\frac{d}{dt}\Big|_{t=0} \Psi(\gamma(t))$$

What is the identificate $\varphi_t : E_{\gamma(t)} \to V$?

$$[\gamma_p(t), v] \mapsto v$$

Aside $P \circlearrowleft G$ $\rho : G \to GL(V)$

$$\begin{array}{c} P \\ \downarrow \pi \end{array}$$

$E = P \times_G V$ the associated bundle.

Each $p \in P_x$ determines a linear isomorphism

$\varphi_p : V \longrightarrow E_x$ by

$v \longmapsto [p, v]$

This gives a map $\varphi : P \to B(E)$

$d\varphi : \mathfrak{g} \oplus H \longrightarrow TB(E)$

$(\xi, h) \longmapsto \dot{\rho}(\xi) + 0$

$$\frac{d}{dt}\Big|_{t=0} \tau_t^{-1}\left(\psi(\gamma(t))\right)$$

$d(fg) = g\,df + f\,dg$

$$= \frac{d}{dt}\Big|_{t=0} \varphi_{\gamma(t)}\left(\psi(\gamma(t))\right)$$

In a basis the i^{th} components is

$$\frac{d}{dt}\Big|_{t=0} \left(\varphi_{\gamma(t)}\right)^i_j \psi(\gamma(t))^j$$

$$= \psi^j(0)\, d\varphi^i_j\Big|_0 + \left(\varphi_p\right)^i_j\, d\psi^j\Big|_0$$

$$= \rho(\Theta)^i_j(v)(x) + d\varphi^i|_0$$

$$\varphi : P \longrightarrow V \qquad E = P \times_G V$$

$$\nabla\varphi = \partial_i\varphi \otimes e^i$$

$$\varphi \in \Omega^0_P(V) \quad \nabla\varphi \in \Omega^1_P(V) \sim \Omega^1_M(E)$$

How to interpret this?

$$\Omega^1_P(V) = \Gamma_P(V \otimes T^*P)$$

$$\Omega^1_M(E) = \Gamma_M(E \otimes T^*M)$$

$$= \Gamma_M(E \otimes (P \times_{Gl_n\mathbb{R}} (\mathbb{R}^n)^*))$$

$$\wr$$

$Gl_n\mathbb{R}$ equivariant

$$\alpha : B(M) \to V \otimes \mathbb{R}^{n*}$$

A, B $GL_n\mathbb{R}$ representations.

With associated Bundles $E, F \to M$.

$GL_n\mathbb{R} \circlearrowright (\mathbb{R}^n)^*$ by the dual representation.

Then given a $GL_n\mathbb{R}$ equivariant map

$$\sigma : (\mathbb{R}^n)^* \longrightarrow \mathrm{Hom}(A, B)$$

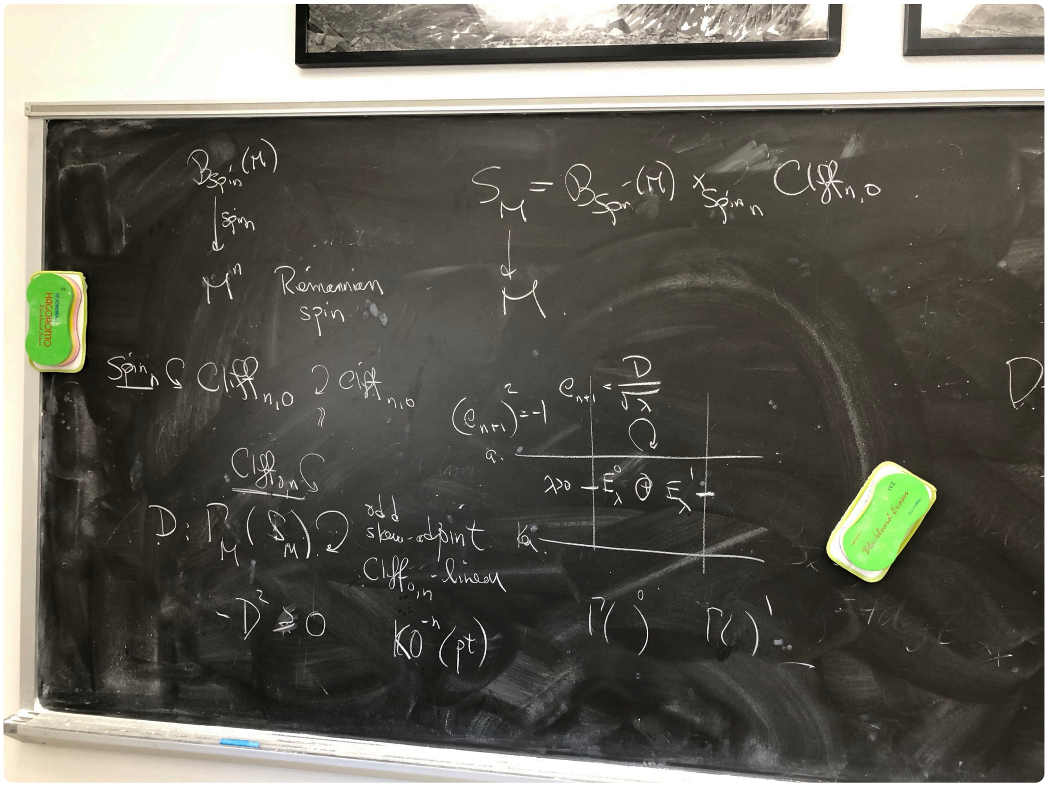
We get a differential operator

$$D_\sigma : \Gamma_M(E) \to \Gamma_M(F)$$

$$\varphi \longmapsto [\sigma(e^k)](\partial_k \varphi)$$

How to interpret? φ is a $GL_n\mathbb{R}$-equivariant map

$B(M) \to A$, and so is $\partial_k \varphi$



Suppose we have a right $\mathrm{Cliff}_{n,0}$ module M

Is this a left $\mathrm{Cliff}_{0,n}$ module?

Let $\alpha : \mathrm{Cliff}_{n,0} \to \mathrm{Cliff}_{n,0}$ be the automorphism extending $v \mapsto -v$

Then define the action of $a \in \mathrm{Cliff}_{0,n}$ on M by

$$a \cdot m = m \cdot \alpha(a).$$

Does this satisfy the Clifford relations?

$$e_i^- e_i^- \cdot m = m \cdot \alpha(e_i^- e_i^-)$$

$$= m \cdot -1$$

$$(e_i^- e_j^- + e_j^- e_i^-) \cdot m$$

$$= e_i^- e_j^- \cdot m + e_j^- e_i^- \cdot m$$

$$= m \cdot e_i e_j + m \cdot e_j e_i$$

$$= m \cdot (e_i e_j + e_j e_i) = 0$$

$\checkmark$.