(1) Classify pointed G-bundles over S^1
for G-discrete.

$$\begin{tikzcd} p \in P \arrow[rr, dashed, "?"] \arrow[dr] & & P' \arrow[dl] \\ & x \in S^1 & \end{tikzcd}$$

(Use monodromy)

(2) $Pin_n^{\pm}$ structure on M gives a Spin structure on the orientation double cover

(3) Orientation double cover of Möbius band
(Cylinder?)

(4) $Pin_2^{\pm}$ structure on möbius band, restriction to S^1

(5) $Pin_2^{\pm}$ structures on the disk,
restriction to S^1

Pointed G-bundles over S^1, G-discrete.

The setup:
$$\begin{array}{c} P \\ \downarrow \pi \\ S^1 \end{array} \quad \text{a principal } G\text{-bundle.}$$

$x \in S^1$ $p \in P$ with $\pi(p) = x$.

Goal: Classify using the holonomy. Given

$[\gamma] \in \pi_1(S^1)$, We can lift γ to a path $\tilde{\gamma}$ in P, with $\tilde{\gamma}(0) = p$. Then $\tilde{\gamma}(1) \in \pi^{-1}(x)$
$\Rightarrow \tilde{\gamma}(1) = p \cdot g$ for some $g \in G$, which

gives a group homorphism
$$\pi_1(S^1) \to G. \quad \leftarrow \text{The \underline{holonomy}}$$
Since $\pi_1(S^1) \cong \mathbb{Z}$, it is entirely determined by the image of the generator $\alpha(t) = e^{it}$.

Does every such homomorphism determine a Principal G-bundle?

Do this in more generality.

Let X be a space (sufficiently nice?)
G a discrete group.

Claim. There is a bijection

$$\left\{ \begin{array}{c} \text{Pointed principal } G\text{-bundles} \\ (P, p_0) \longrightarrow (X, x_0) \end{array} \right\}$$

$$\updownarrow$$

$$\left\{ \text{Group homomorphisms } \pi_1(X, x_0) \longrightarrow G \right\}$$

Given a G-bundle $\begin{array}{c} P \\ \downarrow \\ M \end{array}$ we get an

action on each fiber by $\pi_1(X, x_0)$ by path lifting, which gives a homomorphism

$\pi_1(X, x_0) \to G.$

Conversely, let $\varphi : \pi_1(X, x_0) \to G$ be a group homomorphism.

We want to construct a principal G-bundle

$$\begin{array}{c} P, p_0 \\ \downarrow \;\; \downarrow \\ X \;\; x_0 \end{array}$$

with holonomy φ.

Let $\tilde{X}$ be the universal cover of X, $\tilde{x}_0$ a lift of x_0. Then

$$\begin{array}{c} (\tilde{X}, \tilde{x}_0) \\ \downarrow \\ (X, x_0) \end{array}$$

is a (pointed) principal $\pi_1(X, x_0)$-bundle over (X, x_0)

Define P as the associated bundle

$$(\tilde{X}, \tilde{x}_0) \underset{\pi_1(X, x_0)}{\times} (G, e)$$

Explicitly,

$$P = \tilde{X} \times G \Big/ (x,g) \sim (g \cdot \gamma, \varphi(\gamma)^{-1} \cdot g)$$

$$p_0 = (\tilde{x}_0, e)$$

Is this a principal G-bundle?

$\tilde{X} \times G$ has a right G-action by right multiplication. Does this descend to P?

Want $[x, g] \cdot h = [x, gh]$

Well-defined?

$$[x\cdot\gamma,\ \varphi(\gamma)^{-1}\cdot g]h = [x\cdot\gamma,\ \varphi(\gamma^{-1})\cdot g\cdot h]$$

Yep!

So P is a principal G bundle!

What is the holonomy?

$$p_0\cdot\gamma = [\tilde{x}_0\cdot\gamma, h] \sim [x_0, \varphi(\gamma)\cdot h]$$

Holonomy is φ!

Let $\mathrm{Bun}_G(X) = \left\{ \begin{array}{c} \text{Isomorphism classes of} \\ \text{Pointed } G\text{-bundles} \\ (P, p_0) \to (X, x_0) \end{array} \right\}$

$\mathrm{Bun}_G(X) \circlearrowleft G$ by permuting the basepoints

$$(P, p_0)\cdot g = (P, p_0\cdot g)$$

If we quotient by this group action, this identifies all bundles that are isomorphic without basepoint.

How is this group action represented wrt homomorphisms? $\pi_1(X, x_0) \to G$

Given

$$\begin{array}{c} (P, p_0) \\ \downarrow \\ (X, x_0) \end{array}$$

with holonomy $\varphi: \pi_1(X, x_0) \to G$

What happens to φ under the action of $g \in G$?

What is the holonomy $\varphi_g : \pi_1(X, x_0) \to G$ of $(P, p_0 \cdot g)$?

given $[\gamma] \in \pi_1(X, x_0)$. Let $\tilde{\gamma}$ be the lift at p_0. Then let $\tilde{\gamma}'$ be the lift starting at $p_0 \cdot g$

By uniqueness, it must be $\tilde{\gamma}\cdot g$ i.e

$$\tilde{\gamma}'(t) = \tilde{\gamma}(t)\cdot g$$

Then $\tilde{\gamma}'(1) = \tilde{\gamma}(1)\cdot g$

$$= p_0 \cdot \varphi(\gamma)\cdot g$$

$$\Rightarrow \quad \varphi_g(\gamma) = g^{-1}\varphi(\gamma)\, g$$

since $(p_0 \cdot g)\cdot \varphi_g(\gamma) = p_0 \cdot g \cdot g^{-1}\cdot \varphi(\gamma)\cdot g$

$$\qquad\qquad\qquad\qquad\qquad\qquad \parallel$$

$$\qquad\qquad\qquad\qquad p_0 \cdot \varphi(\gamma)\cdot g$$

Re: $\mathrm{Pin}^{\pm}$-structures on S^1

Recall $\mathrm{Pin}_1^+ \cong \mathbb{Z}/2 \times \mathbb{Z}/2$

$\mathrm{Pin}_1^- \cong \mathbb{Z}/4$

A $\mathrm{pin}_1^{\pm}$ structure is a principal bundle

$$\begin{array}{l} P \circlearrowleft \mathrm{Pin}^{\pm}_1 \\ \downarrow \\ M \end{array}$$

with a $\mathrm{Pin}^{\pm}$-equivariant

$$P \to B_O(M).$$

How many Pin_1^+-bundles are there over S^1?

Homomorphisms $\mathbb{Z} \longrightarrow \mathrm{Pin}_1^+$ up to conjugation

Since Pin_1^+ is abelian, this is just

Homomorphisms $\mathbb{Z} \longrightarrow \mathrm{Pin}_1^+$

There are 4: determined by image of

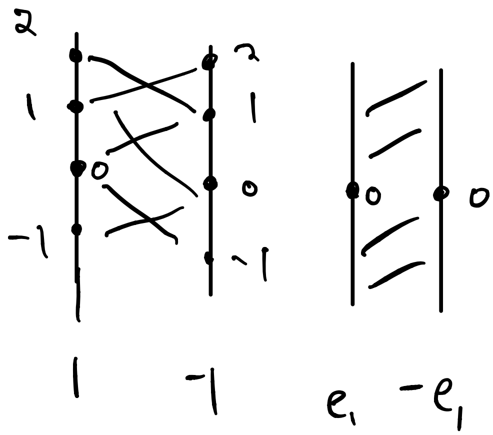
1

$1 \longmapsto 1 \iff$ 4 connected components

$1 \longmapsto -1 \iff$ 2 components.

$$\mathbb{R} \times_{\mathbb{Z}} Pin_1^+$$

$$(x,g) \sim (x+z, \varphi(z)^{-1} g)$$



$$(0,1) \sim (1,-1) \sim (2,1)$$

2 fold cover -1 acts

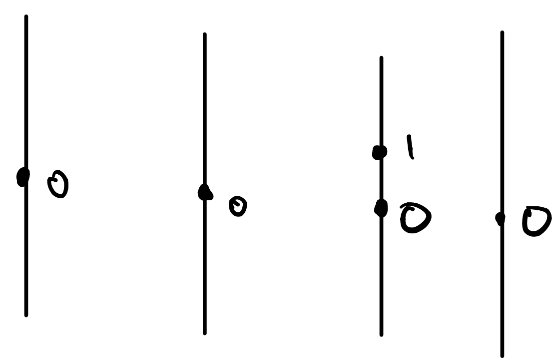
$$(0,1)(-1) = (0,-1) \sim (1,1)$$

by rotating by π

$$(x,1)e_1 = (x,e_1) \sim (x+1,-e_1)$$

$1 \longmapsto e_1$

$(x, g) \sim (x+2, \varphi(2)^{-1} g)$



$1 \qquad -1 \qquad e_1 \quad -e_1$

$(0,1) \sim (1, e_1) \sim (2, e_1)$

2 fold cover $\quad e_1$ adds π

-1 permutes the circles

$(0,-1) \sim (1, -e_1)$

$1 \longmapsto -e_1$

$(0,1) \sim (1, -e_1) \sim (2, 1)$

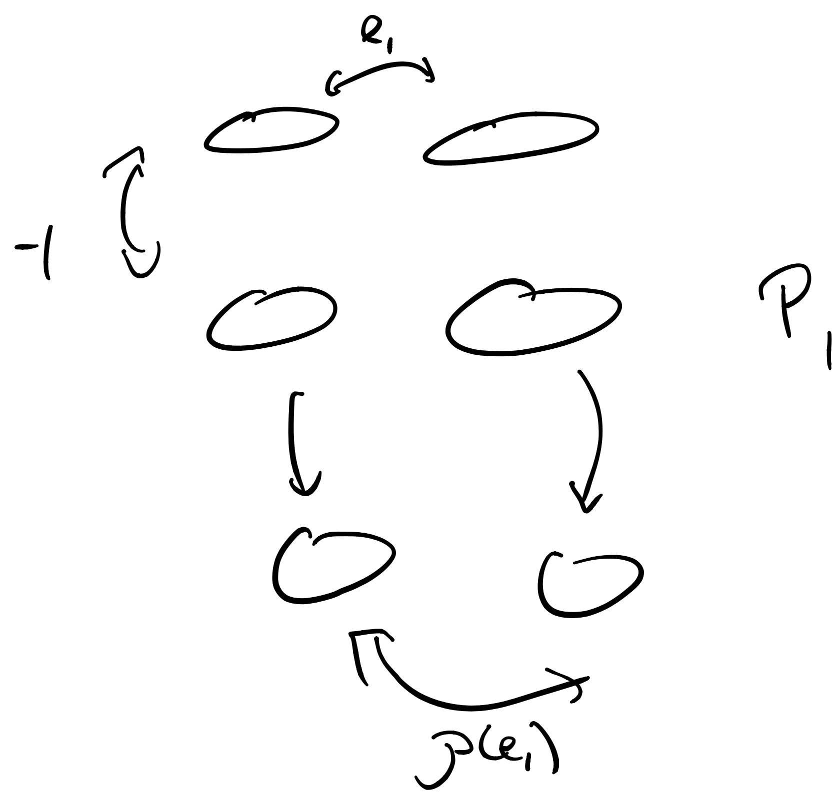
$-e_1$ adds π

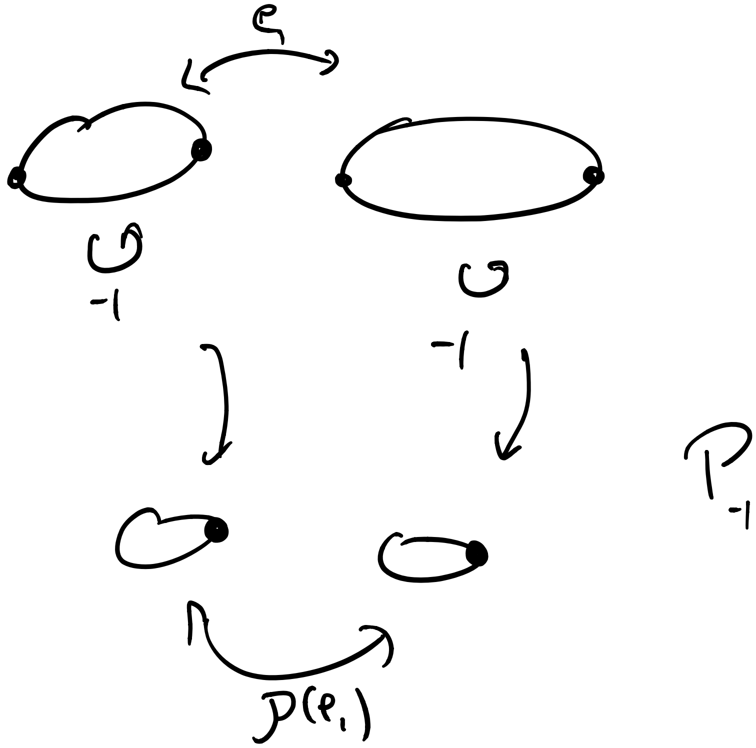
-1 permutes circle

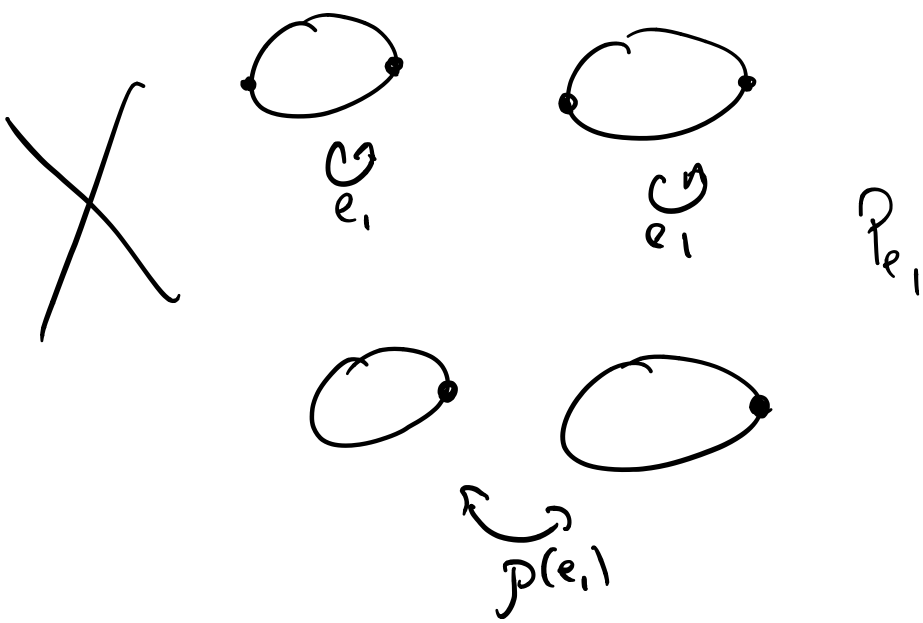
$B_0(S^1)$ is $S^1 \amalg S^1$ permuted by the action of $\mathbb{Z}$.

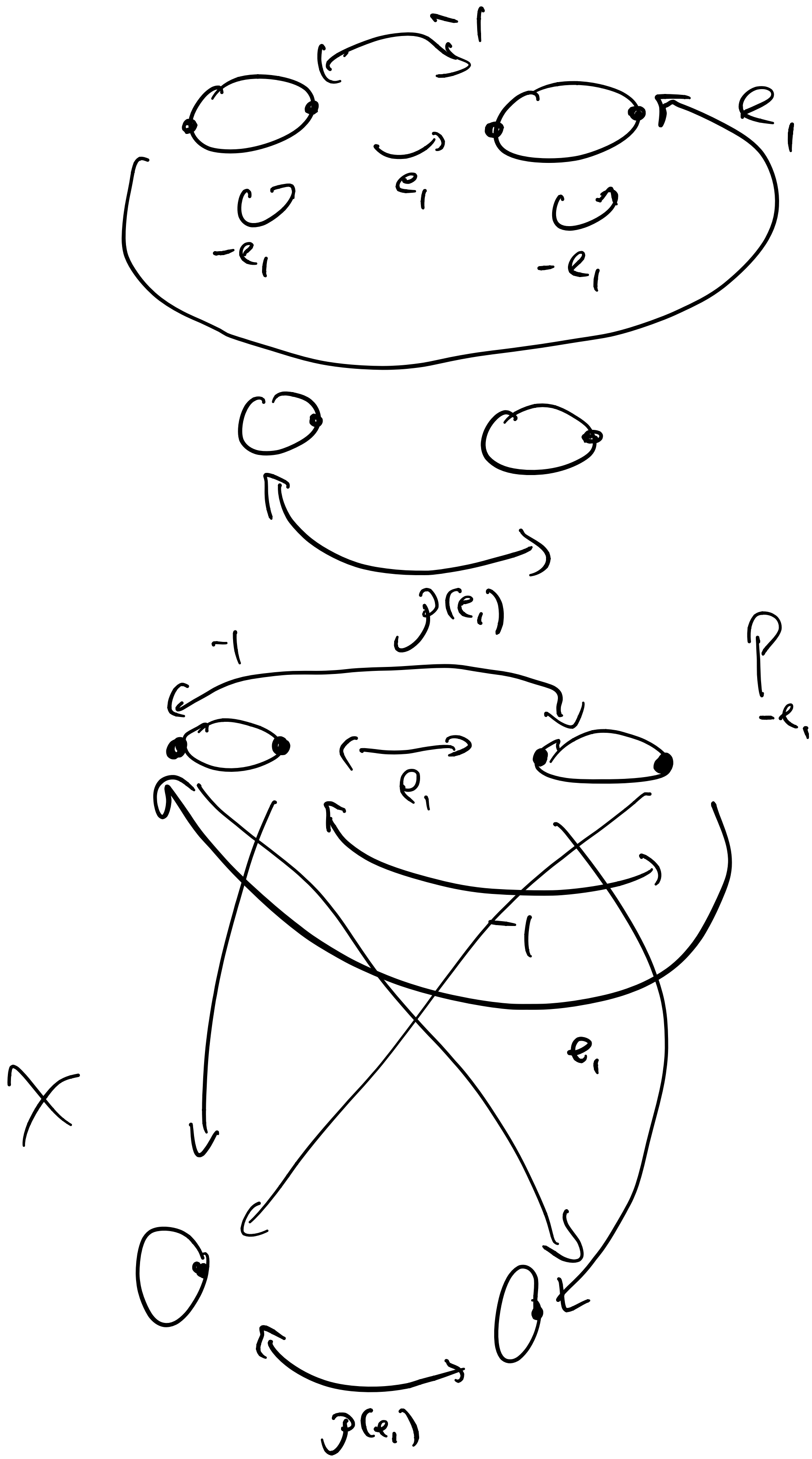
Disconnected bundle

$$\begin{array}{c} P \\ \downarrow \\ S^1 \end{array} \quad \text{wants} \quad \begin{array}{c} P \\ \downarrow \\ B_0(S^1) \\ \downarrow \\ S^1 \end{array}$$




e_1
-1
-1
P_{-1}
$p(e_1)$


X
e_1
e_1
P_{e_1}
$p(e_1)$


-1
e1
e1
-e1
-e1
p(e1)
-1
P-e1
e1
-1
e1
X
p(e1)

Pin_1^- structures on S

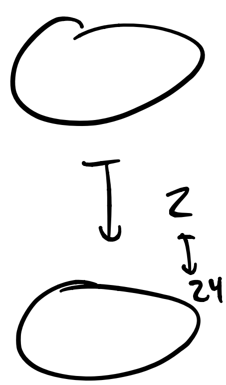
4 bundles, classified by homomorphisms

$$\mathbb{Z} \longrightarrow Pin_1^- \cong \mathbb{Z}/4$$

$$\{1, e_1, -1, -e_1\}$$

$$1 \mapsto e_1$$

$$(x, g) \sim (x+1, -e_1 g) \sim (x+2, -1 g) \sim (x+3, e_1 g) \sim (x+4, g)$$



$1 \mapsto -1$

$$(x, g) \sim (x+1, -g) \sim (x+2, g)$$

Cleaner way of showing

$$\begin{tikzcd}
B_{spin}(\partial M) \arrow[r, hook] \arrow[d] & B_{spin}(M)|_{\partial M} \arrow[d] \\
B_{SO}(\partial M) \arrow[r, hook] \arrow[dr] & B_{SO}(M)|_{\partial M} \arrow[d] \\
& \partial M
\end{tikzcd}$$

has structure group $Spin_{n-1}$

$SO_{n-1} \circlearrowright B_{SO}(\partial M)$ by $\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$

the preimage of $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \middle| A \in SO_{n-1} \right\}$

under $Spin_n \to SO_n$ is isomorphic to

$Spin_{n-1}$

This acts on $B_{spin}(M)\big|_{\partial M}$ via the $Spin_n$ action

and on $B_{so}(\partial M)$ by the double cover

so we get an action

$$B_{so}(\partial M) \times B_{spin}(M)\big|_{\partial M} \circlearrowleft Spin_{n-1}$$

and determines an action on the pullback $B_{spin}(\partial M)$

We want

$$\begin{tikzcd}
B_{spin}(\partial M) \arrow[d] \arrow[dd, bend left, "q"] \\
B_{so}(\partial M) \arrow[d] \\
M
\end{tikzcd}$$

where q is the quotient by the $Spin_{n-1}$ action.

Spin_{n-1} is almost tautologically the subgroup preserving

$$B_{Spin}(M)\big|_{\partial M}$$

Since Spin_n acts freely and transitively, Spin_{n-1} acts transitively, so the quotient map is ∂M.

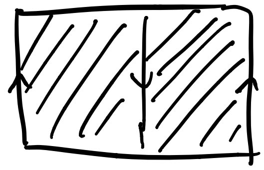
Orientation Double cover of Möbius band

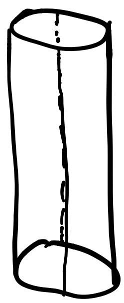
Suffices to find a double cover

$$M = S^1 \times [-1,1] \big/ (x,-1) \sim (-x,1)$$

M

$\Downarrow$

$S^1 \times [-1,1] =$ 

 $\circlearrowleft \mathbb{Z}/2$ $\qquad S^1 \times [-1,1] \big/ \mathbb{Z}/2$

$$\begin{array}{c} S^1 \times [-1,1] \big/ \mathbb{Z}/2 \\ \text{\rotatebox{90}{$\cong$}?} \\ M \end{array}$$

Pin structures on M?

$$\mathrm{Pin}_2^+ \subset \mathrm{Cliff}_{2,0} \cong M_2\mathbb{R}$$

$$\mathrm{span}\{1, e_1, e_2, e_1e_2\}$$

Topologically, Pin_2^+ is two disjoint circles.

$$\mathrm{Spin}_2 \subset \mathrm{Pin}_2^+$$

$$\mathrm{Spin}_2 = \{\cos\theta + \sin\theta\, e_1e_2\}$$

action on $e_1, e_2 \in \mathbb{R}^2$

$$(\cos\theta + \sin\theta\, e_1e_2)\quad e_1 \quad (\cos\theta - \sin\theta\, e_1e_2)$$

$$\boxed{\begin{gathered}(\cos\theta + \sin\theta\, e_1e_2)(\cos\theta - \sin\theta\, e_1e_2) \\ \parallel \\ \cos^2\theta - \cos\theta\sin\theta\, e_1e_2 + \cos\theta\sin\theta\, e_1e_2 + \sin^2\theta\end{gathered}}$$

$$= (\cos\theta e_1 - \sin\theta e_2)(\cos\theta - \sin\theta\, e_1e_2)$$

$$\parallel$$

$$\cos^2\theta e_1 - \cos\theta\sin\theta e_2 - \cos\theta\sin\theta e_2 - \sin^2\theta e_1$$

$$= \underbrace{(\cos^2\theta - \sin^2\theta)}_{\cos 2\theta} e_1 - \underbrace{2\cos\theta\sin\theta}_{\sin 2\theta} e_2$$

$$(\cos\theta + \sin\theta e_1e_2)\, e_2\, (\cos\theta - \sin\theta e_1e_2)$$

$$= (\cos\theta e_2 + \sin\theta e_1)(\cos\theta - \sin\theta e_1 e_2)$$

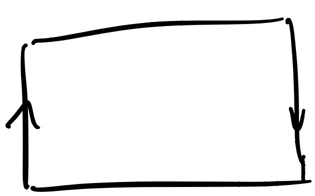
$$= (\cos^2\theta e_2 + \cos\theta\sin\theta e_1 + \cos\theta\sin\theta e_1 - \sin^2\theta e_2)$$

$$= \sin 2\theta e_1 + \cos 2\theta e_2$$

so $\cos\theta + \sin\theta e_1e_2 \longmapsto \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}$

2 components,

Spin_2 and $e_1 \cdot \mathrm{Spin}_2$



$$I \times I \times \mathrm{Pin}_2^{\pm}$$

The identification on the boundary is the map $\phi : [-1,1] \to [-1,1]$

$$\phi(x) = -x$$

We have

$$\begin{tikzcd}
{[-1,1] \times Pin_2^+} \arrow[r, "\varphi"] \arrow[d, "\pi_1"'] & {[-1,1] \times Pin_2^+} \arrow[d, "\pi_2"'] \\
{[-1,1]} \arrow[r, "\phi"'] & {[-1,1]}
\end{tikzcd}$$

Want $\varphi : [-1,1] \times Pin_2^+ \to [-1,1] \times Pin_2^+$ s.t.

$\pi_2 \circ \varphi = \phi \circ \pi_1$ and φ is Pin_2^+-equivariant.

Explicitly, we have $\pi_1 = \pi_2 = \mathrm{proj}_1$, and we need

The first component of $\Psi(x,g)$ to be $-x$.

$\Rightarrow \Psi = (\phi, \tilde{\Psi})$ for

$\tilde{\Psi} : Pin_2^+ \to Pin_2^+$ a $Pin_2^{\pm}$ equivariant map.

There's a lot of these ☹

Which ones give a map to $B_0(M)$?

$$B_0(M) \cong [-1,1]^2 \times O_2 \Big/ (-1,x,g) \sim \left(-1,x,g\cdot\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)$$

$$P = [-1,1]^2 \times \mathrm{Pin}_2^+ \Big/ (-1,x,g) \sim (1,-x,\varphi(g))$$

Which φ determine bundles

$$\begin{array}{c} P \\ \downarrow \\ M \end{array}$$

with a Pin_2^+-equivariant map

$$P \longrightarrow B_0(M)$$
$$\shortparallel$$
$$[-1,1]^2 \times O_2 \Big/ (-1,x,g) \sim \left(1,x,g \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

$$\begin{tikzcd}
{[-1,x,g]} \arrow[d] & = & {[1,-x,\varphi(g)]} \arrow[d] \\
{[-1,x,\rho(g)]} & = & {[1,-x,\rho(g)\cdot A]}
\end{tikzcd}$$

$$\Downarrow$$
$$\rho(g)$$

$$\Rightarrow \varphi(g) = g \cdot \rho^{-1}(A)$$

$$P^{-1}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$(\cos\theta e_1 + \sin\theta e_2)(\cos\theta e_1 + \sin\theta e_2)$$

$$=$$

$$\cos^2\theta + \cos\theta\sin\theta e_1e_2 - \cos\theta\sin\theta e_1e_2 + \sin^2\theta$$

$$=$$

$$1$$

$$(\cos\theta e_1 + \sin\theta e_2)e_1(\cos\theta e_1 + \sin\theta e_2)$$

$$=$$

$$(\cos\theta - \sin\theta e_1e_2)(\cos\theta e_1 + \sin\theta e_2)$$

$$=$$

$$\cos^2\theta e_1 + \cos\theta\sin\theta e_2 + \cos\theta\sin\theta e_2 - \sin^2\theta e_1$$

$$=$$

$$\cos 2\theta e_1 + \sin 2\theta e_2$$

$$\left((\cos\theta e_1 + \sin\theta e_2) e_2 (\cos\theta e_1 + \sin\theta e_2)\right)$$

$$\parallel$$

$$(\cos\theta e_1 e_2 + \sin\theta)(\cos\theta e_1 + \sin\theta e_2)$$

$$\parallel$$

$$-\cos^2\theta e_2 + \cos\theta \sin\theta e_1 + \sin\theta\cos\theta e_1 + \sin^2\theta e_2$$

$$\parallel$$

$$\sin 2\theta e_1 - \cos 2\theta e_2$$

$$\Rightarrow \quad \cos\theta e_1 + \sin\theta e_2 \mapsto \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

What structure is induced on the boundary?

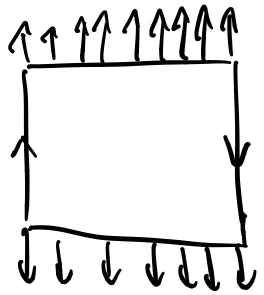
$$B_{pin^+}(M) = [-1,1]^2 \times Pin_2^+ \Big/ (-1, x, g) \sim (1, -x, \pm e_1 \cdot g)$$

What is $B_{pin^+}(\partial M)$?

What is $B_0(\partial M)$?

A principal $O_1 = \{\pm 1\}$ bundle over $\partial M \cong S^1$

$$B_0(M) = [-1,1]^2 \times O_2 \Big/ (-1, x, g) \sim \left(1, -x, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot g\right)$$



B

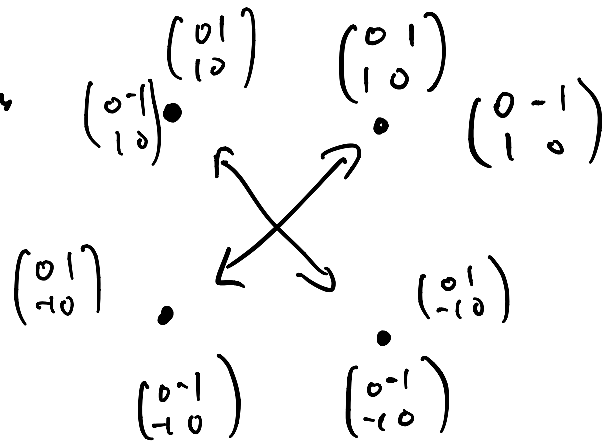
$$\partial M = [-1,1] \times \{1\} \cup [-1,1] \times \{-1\}$$

Fiber over non identified points is

Top $\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$

Bottom $\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\}$

Identification



$B_0(\partial M)$ has 2 components.

What about $B_{pin^+}(\partial M)$?

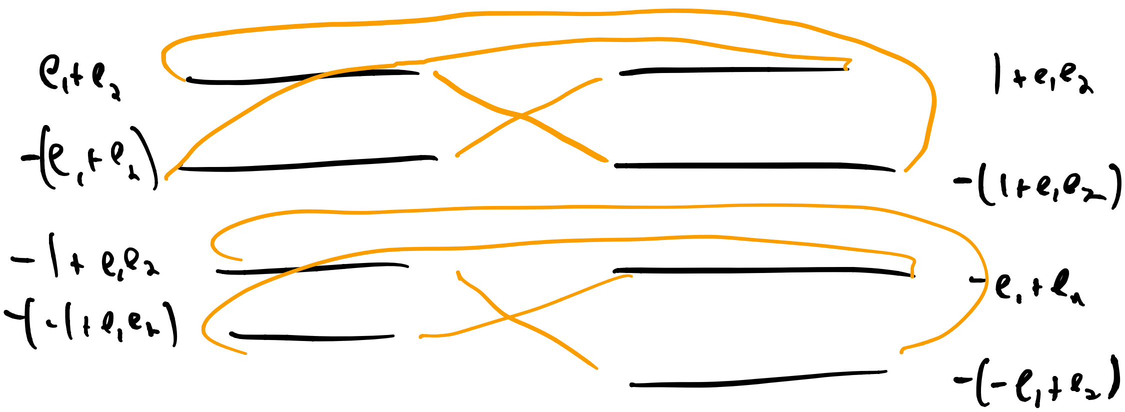
Fiber over top elements

$$\frac{\sqrt{2}}{2}(e_1+e_2),\ -\frac{\sqrt{2}}{2}(e_1+e_2) \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \frac{\sqrt{2}}{2}(-1+e_1e_2),\ -\frac{\sqrt{2}}{2}(-1+e_1e_2) \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Fiber over bottom elements

$$\frac{\sqrt{2}}{2}(1+e_1e_2),\ -\frac{\sqrt{2}}{2}(1+e_1e_2) \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \frac{\sqrt{2}}{2}(-e_1+e_2),\ -\frac{\sqrt{2}}{2}(-e_1+e_2) \longmapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

if we glue with $+e_1$



If we glue with $-e_1$

The case with Pin^-

$$Pin_2^- \subset Cliff_{0,2}$$

$$Spin_2 \subset Pin_2^-$$

$$\| $$

$$\{\cos\theta + \sin\theta\, e_1 e_2\}$$

What is the covering map?

$$(\cos\theta e_1 + \sin\theta e_2)(e_1)(-\cos\theta e_1 - \sin\theta e_2)$$

$$\parallel$$

$$(-\cos\theta - \sin\theta e_1 e_2)(-\cos\theta e_1 - \sin\theta e_2)$$

$$\parallel$$

$$\cos^2\theta e_1 + \cos\theta\sin\theta e_2 + \cos\theta\sin\theta e_2 - \sin^2 e_1$$

$$\parallel$$

$$\cos 2\theta e_1 + \sin 2\theta e_2$$

$$(\cos\theta e_1 + \sin\theta e_2)(e_2)(-\cos\theta e_1 - \sin\theta e_2)$$

$$\parallel$$

$$(\cos\theta e_1 e_2 - \sin\theta)(-\cos\theta e_1 - \sin\theta e_2)$$

$$\parallel$$

$$(-\cos^2\theta e_2 + \cos\theta\sin\theta e_1 + \sin\theta\cos\theta_1 + \sin^2 e_2)$$

$$\parallel$$

$$\sin 2\theta e_1 - \cos 2\theta e_2$$

$$\Rightarrow \cos\theta e_1 + \sin\theta e_2 \longmapsto \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

$$(\cos\theta + \sin\theta e_1e_2)(e_1)(\cos\theta - \sin\theta e_1e_2)$$

$$\parallel$$

$$(\cos\theta e_1 + \sin\theta e_2)(\cos\theta - \sin\theta e_1e_2)$$

$$\parallel$$

$$\cos^2\theta e_1 + \sin\theta\cos\theta e_2 + \cos\theta\sin\theta e_2 - \sin^2\theta e_1$$

$$\parallel$$

$$\cos2\theta e_1 + \sin2\theta e_2$$

$$(\cos\theta + \sin\theta e_1e_2)(e_2)(\cos\theta - \sin\theta e_1e_2)$$

$$\parallel$$

$$(\cos\theta e_2 - \sin\theta e_1)(\cos\theta - \sin\theta e_1e_2)$$

$$\parallel$$

$$\cos^2\theta e_2 - \cos\theta\sin\theta e_1 - \sin\theta\cos\theta e_1 - \sin^2\theta e_2$$

$$\parallel$$

$$-2\sin\theta e_1 + \cos2\theta e_2$$

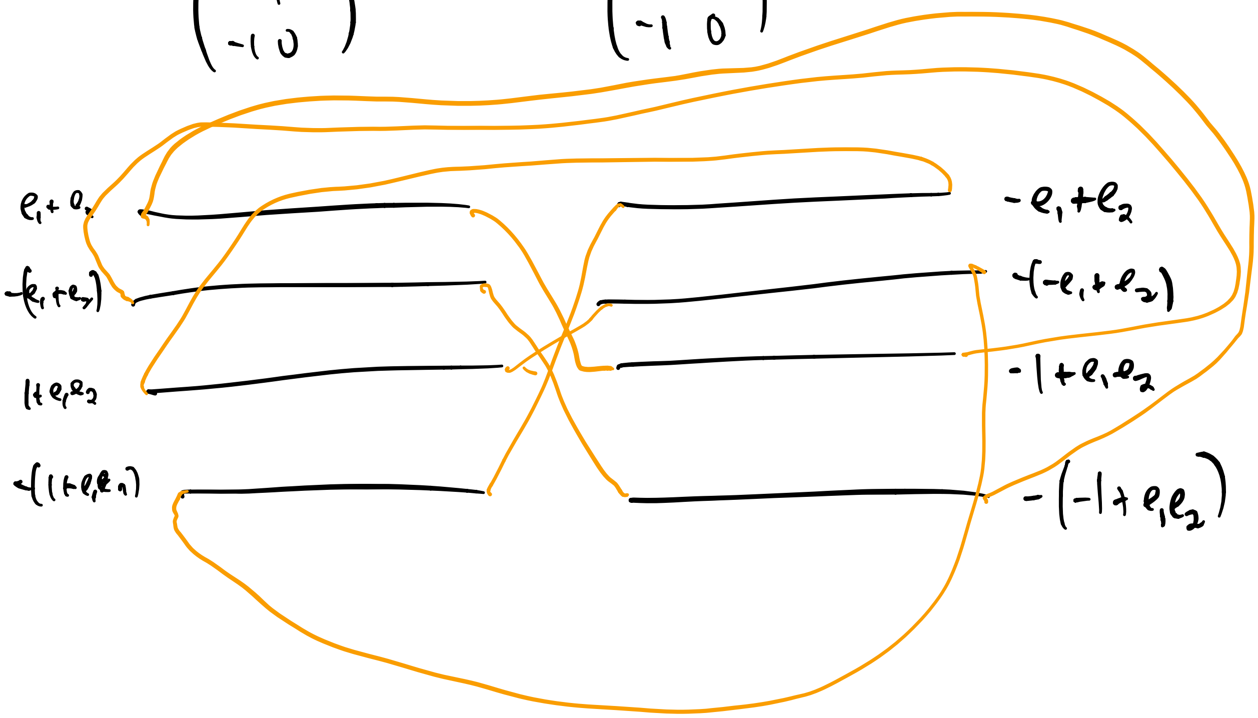
$$\Rightarrow \quad \cos\theta + \sin\theta e_1e_2 \mapsto \begin{pmatrix} \cos2\theta & -\sin2\theta \\ \sin2\theta & \cos2\theta \end{pmatrix}$$

$$\pm\frac{\sqrt{2}}{2}(e_1+e_2) \longrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\pm\frac{\sqrt{2}}{2}(1+e_1e_2) \longrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\pm\frac{\sqrt{2}}{2}(-e_1+e_2) \longrightarrow \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$\pm\frac{\sqrt{2}}{2}(-1+e_1e_2) \longrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$



$$\begin{tikzcd}
\mathcal{B}_{pin}(\tilde{M}) \arrow[r] \arrow[d] & \mathcal{B}_{pin^{\pm}}(M) \arrow[d] \\
\pi^{*}\mathcal{B}_0(\tilde{M}) \arrow[r] \arrow[d] & \mathcal{B}_0(M) \arrow[d] \\
\tilde{M} \arrow[r, "\pi"'] & M
\end{tikzcd}$$