UNDERGRADUATE THESIS NOTES

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Week 1

Exercise 1.1. Prove a small lemma

Lemma. *Let G be a group, and V a finite dimensional irreducible complex representation. Then elements of $Z(G)$ act by scalars.*

Proof. Let $z \in Z(G)$, then since z commutes with the action of G, it determines a G-module isomorphism $\varphi_z : V \to V$. Then φ_z admits some eigenvalue λ, and the λ eigenspace of φ_z is G-invariant, so φ_z must be the map $\lambda\, \mathrm{id}_V$. ■

Exercise 1.2. Consider the groups $\mathrm{Pin}_{p,q}$ and $\mathrm{Spin}_{p,q}$, which lie in the Clifford algebra $\mathrm{Cliff}_{p,q}$. A ***Pin representation*** is a representation V of $\mathrm{Pin}_{p,q}$ that extends to an irreducible Clifford module. Likewise, a ***Spin representation*** is a representation V of $\mathrm{Spin}_{p,q}$ that extends to an irreducible $\mathrm{Cliff}^0_{p,q}$ module. Find some of these representations.

Since all Clifford algebras arise (as ungraded algebras) as direct sums of matrix algebras over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, it will be useful to characterize the irreducible modules of these matrix algebras. In the case of $\mathbb{H}$, we use the convention that scalar multiplication on a quaternionic vector space acts on the right, so quaternionic matrices can act on the left.

Proposition 1.3.

(1) *The only irreducible $M_n\mathbb{R}$ module is $\mathbb{R}^n$.*
(2) *The only irredu*

Proof. We see that there is an increasing chain of left ideals

$$0 = I_0 \subset I_1 \subset \ldots \subset I_n = M_n\mathbb{R}$$

where I_k is the ideal of matrices where all the entries past the k^{th} column are 0. In addition, we have that this chain of ideals has the property that the quotient space $I_{k+1}/I_k \cong \mathbb{R}^n$ as a left $M_n\mathbb{R}$ module. We note that $\mathbb{R}^n$ is most certainly irreducible, since the orbit of any nonzero vector $v \in \mathbb{R}^n$ is all of $\mathbb{R}^n$.

Then let W denote an arbitrary nontrivial irreducible $M_n\mathbb{R}$ module. Fix $w \in W$. Then the orbit of w under the action of $M_n\mathbb{R}$, and since the module is nontrivial, it must be all of W. Therefore, the mapping

$$\begin{aligned} \varphi : M_n\mathbb{R} &\to W \\ M &\mapsto M \cdot w \end{aligned}$$

is a surjective map of left $M_n\mathbb{R}$ modules. Since this map is surjective, there exists some k such that $\varphi(I_k) \neq 0$. Let k denote the smallest such k. Since $\varphi(I_{k-1})$, this map factors through the quotient I_k/I_{k-1}, which is isomorphic to R^n as a left module. Then since both $\mathbb{R}^n$ and W are irreducible, this implies that the map $I_k/I_{k-1} \to W$ is an isomorphism, so $W \cong \mathbb{R}^n$. ■