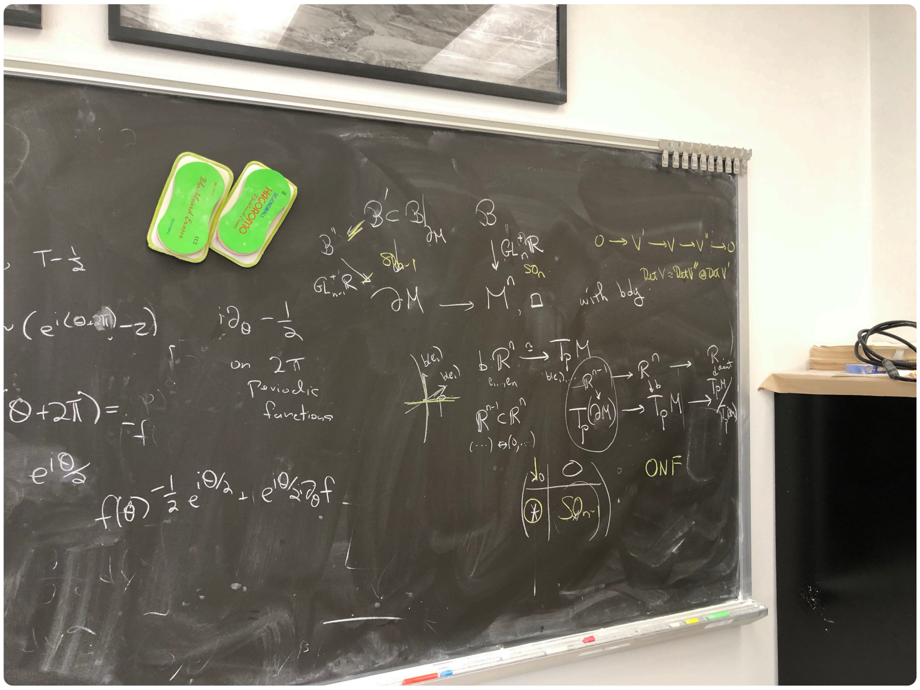
1. Identify $\mathrm{Pin}_1^{\pm}$ structures on S^1
2. Think about Spin_2 structures on 2-mflds.
3. Given a spin structure on M, how does this give a spin structure on ∂M?
   - Do this with $S^1 = \partial D^2$.
   - Which spin structures on S^1 arise from spin structures on D^2
4. Repeat with $\mathrm{Pin}^{\pm}$ (Do on D^2)
5. Find more patterns in the chessboard. Especially regarding graded modules.



How does a $Spin_n$ structure on M induce a $Spin_{n-1}$ structure on ∂M?

With SO_n

Given an orientable mfld M, we can reduce the structure group to SO_n

$$\begin{array}{ccc} B_{SO}(M) & \circlearrowleft_{SO_n} \\ \downarrow & \\ M & \end{array}$$

The inclusion $i : \partial M \hookrightarrow M$ gives a pullback bundle

$$\begin{tikzcd}
\mathcal{B}_{SO}(M)\big|_{\partial M} \arrow[r, hook] \arrow[d] & \mathcal{B}_{SO}(M) \circlearrowleft SO_n \arrow[d] \\
\partial M \arrow[r, hook] & M
\end{tikzcd}$$

Fixing the outward normal vector field N as the first basis vector for each $T_p\partial M$, we get a subbundle $\mathcal{B}_{SO}(\partial M)$ by picking all bases such that

$b(e_1) = N_p$. Giving

$$\begin{tikzcd}
 & \mathcal{B}_{SO}(M)\big|_{\partial M} \circlearrowleft SO_n \arrow[r, hook] \arrow[d] & \mathcal{B}_{SO}(M) \circlearrowleft SO_n \arrow[d] \\
SO_{n-1} \circlearrowright \mathcal{B}_{SO}(\partial M) \arrow[ur, hook] \arrow[r] & \partial M \arrow[r, hook] & M
\end{tikzcd}$$

How about for spin manifolds?

Given an oriented manifold M, we can reduce to the bundle of oriented orthonormal frames $\mathcal{B}_{SO}(M) \to M$.
$\circlearrowleft SO_n$

A <u>spin structure</u> on M is a reduction of structure group to $Spin_n$. If we let $\rho: Spin_n \to SO_n$ be the double cover, this is the data of a principal $spin_n$ bundle $P \to M$ and a $Spin_n$-equivariant map

$$\mathcal{B}_{Spin}(M) \longrightarrow \mathcal{B}_{SO}(M)$$

(Which induces a map $\mathcal{B}_{Spin}(M) \times_{Spin_n} SO_n \to \mathcal{B}_{SO}(M)$)

So we get

$$\begin{gathered} \mathcal{B}_{spin}(M) \\ \downarrow \\ \mathcal{B}_{SO}(M) \\ \downarrow \\ M \end{gathered}$$

The inclusion $\partial M \hookrightarrow M$ gives pullbacks

$$\begin{tikzcd}
\mathcal{B}_{spin}(M)\big|_{\partial M} \arrow[r, hook] \arrow[d] & \mathcal{B}_{spin}(M) \arrow[d] \\
\mathcal{B}_{SO}(M)\big|_{\partial M} \arrow[r, hook] \arrow[d] & \mathcal{B}_{SO}(M) \arrow[d] \\
\partial M \arrow[r, hook] & M
\end{tikzcd}$$

Then using the outward normal vector field, we get $\mathcal{B}_{SO}(\partial M)$

$$\begin{tikzcd}
\mathcal{B}_{spin}(\partial M) \arrow[r, hook] \arrow[d] & \mathcal{B}_{spin}(M)\big|_{\partial M} \arrow[r, hook] \arrow[d, "p"] & \mathcal{B}_{spin}(M) \arrow[d] \\
\mathcal{B}_{SO}(\partial M) \arrow[r, hook] \arrow[dr] & \mathcal{B}_{SO}(M)\big|_{\partial M} \arrow[r, hook] \arrow[d] & \mathcal{B}_{SO}(M) \arrow[d] \\
& \partial M \arrow[r, hook] & M
\end{tikzcd}$$

How to construct $\mathcal{B}_{spin}(\partial M)$?

The pullback of

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \mathcal{B}_{spin}(M)\big|_{\partial M} \arrow[d, "p"] \\ \mathcal{B}_{so}(\partial M) \arrow[r] \arrow[d] & \mathcal{B}_{so}(M)\big|_{\partial M} \\ M & \end{tikzcd}$$

is a principal $\mathbb{Z}/2\mathbb{Z}$ bundle over $\mathcal{B}_{so}(\partial M)$

Does this make it a principal $spin_{n-1}$ bundle over ∂M?

We know that

$$\mathcal{B}_{so}(\partial M) = \left\{ b \in \mathcal{B}_{so}(M)\big|_{\partial M} \;\middle|\; b(e_1) = N_{\pi(b)} \right\}$$

$\circlearrowleft_{SO_{n-1}}$

We then have

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \mathcal{B}_{spin}(M)\big|_{\partial M} \circlearrowleft Spin_n \arrow[d, "\Phi"] \\ \mathcal{B}_{SO}(\partial M) \arrow[r] & \mathcal{B}_{SO}(M)\big|_{\partial M} \end{tikzcd}$$

$\circlearrowleft_{SO_{n-1}}$ $\qquad$ $\circlearrowleft_{SO_n}$

$$P = \left\{ b \in \mathcal{B}_{spin}(M)\big|_{\partial M} \;\middle|\; \Phi(b) \in \mathcal{B}_{SO}(\partial M) \right\}$$

$\circlearrowleft$ $Spin_n$ $\qquad$ $\circlearrowleft$ SO_{n-1}

Subgroup of $Spin_n$ that preserves P

How is this $Spin_{n-1}$? Suffices to check on a fiber. Fixing a point on a fibe of $\mathcal{B}_{SO}(\partial M) \to \partial M$ shows the this map is the double cove $Spin_{n-1} \to SO_{n-1}$

Example: $Spin_2$ on D^2

$Spin_2 \cong \mathbb{T}$. Since D^2 is contractible, the only $Spin_2$-bundle over D^2 is the trivial bundle $D^2 \times Spin_2$

How does this restrict to a $Spin_1$ structure on $\partial D^2 \cong S^1$?

$$\begin{tikzcd}
\mathcal{B}_{spin}(S^1) \arrow[r, hook] \arrow[d] & \mathcal{B}_{spin}(D^2)\big|_{S^1} \arrow[r, hook] \arrow[d] & \mathcal{B}_{spin}(D^2) \cong D^2 \times Spin_2 \arrow[d] \\
\mathcal{B}_{SO}(S^1) \arrow[r, hook] \arrow[dr] & \mathcal{B}_{SO}(D^2)\big|_{S^1} \arrow[r, hook] \arrow[d] & \mathcal{B}_{SO}(D^2) \cong D^2 \times SO_2 \arrow[d] \\
& S^1 \arrow[r, hook] & D^2
\end{tikzcd}$$

$\underline{Pin_1^{\pm}\text{- structure on } S^1}$

We know that $Pin_1^- \cong \mathbb{Z}/4$

$$Pin_1^+ \cong \mathbb{Z}/2 \times \mathbb{Z}/2$$

So a $Pin_1^{\pm}$ structure on S^1 is a 4-fold cover with the appropriate deck transformation group

$\underline{Pin_1^+}$

There are two such covers, one with 4 components, denoted P_1, and one with 2 components, each covering S^1 by $z \mapsto z^2$, denoted P_2

What are their associated bundles?

$Cliff_{1,0} \cong \mathbb{R} \times \mathbb{R}$ $\qquad$ $Cliff_{0,1} \cong \mathbb{C}$

$$1 \longmapsto (1,1) \qquad \qquad \mapsto 1$$
$$-1 \longmapsto (-1,-1) \qquad \qquad e_1 \mapsto i$$
$$e_1 \longmapsto (-1,1)$$
$$-e_1 \longmapsto (1,-1)$$

$\underline{P_1}$

There are 2 pinor representations, depending on which factor of $\mathbb{R} \times \mathbb{R}$ we project onto

$\mathbb{P}^+ = \mathbb{R}$ where the action of Pin_1^+ is

$$1 \cdot v = v$$
$$-1 \cdot v = -v$$
$$e_1 \cdot v = -v$$
$$-e_1 \cdot v = v$$

$\mathbb{P}^- = \mathbb{R}$ when the Pin_1^+ action is

$$1 \cdot v = v$$
$$-1 \cdot v = -v$$
$$e_1 \cdot v = 1$$
$$-e \cdot v = -1$$

We let $\mathbb{P} = \mathbb{P}^+ \oplus \mathbb{P}^-$. Then the associated bundle

$$P_1 \times_{pin_1^+} \mathbb{P} \quad \text{is a trivial bundle}$$

and we have that

$$\Gamma(P_1 \times_{pin_1^+} \mathbb{P}) \longleftrightarrow \{ P_{in_1}^+\text{-equivariant maps } P_1 \longrightarrow \mathbb{P} \}$$

If we map one component into $\mathbb{P}$, we know where the rest of them go, so

$$\Gamma(P_1 \times_{pin_1^+} \mathbb{P}) = \{ \text{Maps } S^1 \longrightarrow \mathbb{P} \}$$

e_1 acts on $\mathbb{P}$ by $(1,-1)$

Dirac operator is $D = e_1 \partial_\theta$

What about $P_2 \times_{pin_1^+} \mathbb{P}$?

Still a trivial bundle, but what are the sections?

$$\Gamma(P_2 \times_{\mathrm{Pin}_1^+} \mathbb{P}) \longleftrightarrow \{\mathrm{Pin}_1^+\text{-equivariant maps } P_2 \to \mathbb{P}\}$$

$$= \{\text{Antiperiodic functions } S^1 \to \mathbb{P}\}$$

So the Dirac operator looks like

$$e_1\partial_\theta - \frac{1}{2}$$

What about Pin_1^-?

$$\mathrm{Pin}_1^- \subset \mathrm{Cliff}_{0,1}(\mathbb{R}) \cong \mathbb{C}$$

$$\mathrm{Pin}_1^- = \{\pm 1, \pm e_1\}$$

$1 \longmapsto 1$

$e_1 \longmapsto i$

The Pinor rep $\mathbb{P}$ is $\mathbb{C}$ under this identification $\mathrm{Cliff}_{0,1}(\mathbb{R}) \cong \mathbb{C}$

Pin_1^--bundles are Principal $\mathbb{Z}/4$ bundles.

There are again 2 -

$P_1 = 4$ components

$P_2 = 1$ component

$z \mapsto z^4$

Associated bundle on P_1 is trivial

Sections are still equivalent data to maps $S^1 \to \mathbb{C}$, Dirac operator is $i\partial_\theta$

The associated bundle to P_2 if we parameterize with angles $[0, 8\pi)$

$$\Pi \times_{\mathrm{Pin}_1} \mathbb{C} \Big/ (\theta, v) \sim (\theta + 2\pi, iv)$$

Sections $\Leftrightarrow$ $e^{i\theta/4} f$ for a periodic f

Dirac operator on $\overline{\mathrm{Pin}_1}$ is $\partial_\theta - \frac{1}{4}$?