UNDERGRADUATE THESIS NOTES

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Week 1

Exercise 1.1. Prove a small lemma

Lemma. *Let G be a group, and V a finite dimensional irreducible complex representation. Then elements of $Z(G)$ act by scalars.*

Proof. Let $z \in Z(G)$, then since z commutes with the action of G, it determines a G-module isomorphism $\varphi_z : V \to V$. Then φ_z admits some eigenvalue λ, and the λ eigenspace of φ_z is G-invariant, so φ_z must be the map $\lambda \,\mathrm{id}_V$. ■

Exercise 1.2. Consider the groups $\mathrm{Pin}_{p,q}$ and $\mathrm{Spin}_{p,q}$, which lie in the Clifford algebra $\mathrm{Cliff}_{p,q}$. A ***Pin representation*** is a representation V of $\mathrm{Pin}_{p,q}$ that extends to an irreducible Clifford module. Likewise, a ***Spin representation*** is a representation V of $\mathrm{Spin}_{p,q}$ that extends to an irreducible $\mathrm{Cliff}^0_{p,q}$ module. Find some of these representations.

We'll deal with the easier case first, where we complexify $\mathrm{Cliff}_{p,q}$, giving us the complex Clifford algebra $\mathrm{Cliff}(p+q,\mathbb{C})$. A reference for this is Fulton and Harris.

The complex Clifford algebra $\mathrm{Cliff}(n,\mathbb{C})$, is the Clifford algebra generated by $\mathbb{C}^n$ with the *bilinear* (not sesquilinear!) form $\langle \cdot, \cdot \rangle$, where

$$\langle v, w \rangle = \sum_i v^i w^i$$

We deal with the even dimensional case first.

Claim. There exists a basis $\{e_1, \dots, e_n, f_1, \dots f_n\}$ for $\mathbb{C}^{2n}$ such that

(1) $\langle e_i, e_i \rangle = \langle f_i, f_i \rangle = 0$
(2) $\langle e_i, f_j \rangle = \delta_{ij}$

Proof. For convenience, let $\{a_i\}$ denote the first n standard basis vectors for $\mathbb{C}^{2n}$, and let $\{b_i\}$ denote the last n standard basis vectors. Then the basis $\{a_1, \dots a_n, ib_1, \dots ib_n\}$ has the property that $\langle a_i, a_i \rangle = 1$ and $\langle ib_j, ib_j \rangle = -1$. Then setting $e_j = a_j + ib_j$ and $f_j = a_j - ib_j$ has the first property, but satisfies $\langle e_i, f_j \rangle = 2\delta_{ij}$. Normalizing these vectors then gives the second property. ■

We then plan to use this basis to construct the irreducible $\mathrm{Cliff}(2n,\mathbb{C})$ module. Write

$$\mathbb{C}^{2n} = W \oplus W'$$

where $W = \mathrm{span}\,\{e_i\}$. We claim that $\mathrm{Cliff}(2n,\mathbb{C}) \cong \mathrm{End}(\Lambda^\bullet W)$. where $\Lambda^\bullet W$ denotes the exterior algebra of W. To construct a map $\mathrm{Cliff}(2n,\mathbb{C}) \to \mathrm{End}(\Lambda^\bullet W)$, we need to provide maps $\varphi : W \to \mathrm{End}(\Lambda^\bullet W)$ and $\varphi' : W' \to \mathrm{End}(\Lambda^\bullet W)$ satisfying the Clifford relations, i.e.

(1) $\varphi(w)^2 = \varphi'(w')^2 = 0$
(2) $\varphi(w) \circ \varphi'(w') + \varphi'(w') \circ \varphi(w) = 2\langle w, w; \rangle \,\mathrm{id}$
(3) $\varphi(w) \circ \varphi(p) + \varphi(p) \circ \varphi(w) = 0$
(4) $\varphi'(w') \circ \varphi'(p') + \varphi'(p') \circ \varphi(w') = 0$

For notational compactness, we let $\varphi_w = \varphi(w)$, and likewise for φ'. Define φ by $\varphi_w(v) = w \wedge v$, and define φ' by

$$\varphi'_{w'}(v_1 \wedge, \cdot \wedge v_k) = \sum_i (-1)^{i-1} 2\langle w', v_i \rangle v_1 \wedge \cdots \hat{v}_i \cdot \wedge v_k$$

It is clear that $\varphi_w^2 = 0$, but not immediately so for $\varphi'_{w'}$. We compute

$$(\varphi'_{w'})^2(v_1 \wedge \cdot \wedge v_k) = \sum_i (-1)^{i-1} 2\langle w', v_i\rangle \varphi_{w'}(v_1 \wedge \cdots \hat{v}_i \cdot \wedge v_k)$$
$$= \sum_i (-1)^{i-1}\langle w', v_i\rangle \left(\sum_{j<i} (-1)^{j-1} 2\langle w', v_j\rangle v_1 \wedge \cdot \hat{v}_j \cdots \hat{v}_i \cdots \wedge v_k + \sum_{i<j} (-1)^j 2\langle w', v_j\rangle v_1 \wedge \cdot \hat{v}_i \cdots \hat{v}_j \cdots \wedge v_k \right)$$
$$= 0$$

To see that this is 0. we note that each term of the $j < i$ and $i < j$ summatations produce the same element up to a sign, and the sign of each term in one summation is reversed in the other one, so the sums cancel, giving us 0. For the other Clifford relation, we want to show

$$w \wedge \varphi'_{w'}(v_1 \wedge \cdots \wedge v_k) + \varphi_{w'}(w \wedge v_1 \wedge \cdot \wedge v_k) = 2\langle w, w'\rangle v_1 \wedge \cdots \cdot v_k$$

We compute the desired quantity to be

$$\left((-1)^{i-1} 2\langle w', v_i\rangle w \wedge v_1 \wedge \cdots \hat{v}_i \cdots \wedge v_k \right) + \left(2\langle w', w\rangle v_1 \wedge \cdots \wedge v_k \right) + \left(\sum_j (-1)^j 2\langle w', v_j\rangle v_1 \wedge \cdots \hat{v}_j \cdots \wedge v_k \right)$$
$$= 2\langle w' w\rangle v_1 \wedge \cdots \wedge v_k$$

Noting that the first and last summations are identical, except that each term has the opposite sign. Finally, we must also show that $\varphi_w \circ \varphi_p = 0$ and $\varphi'_{w'} \circ \varphi'_{p'} = 0$. The first is clear from the skew symmetry of the wedge product. The second follows from a similar argument that $(\varphi'_{w'})^2 = 0$, which involves two summations being the same, but with opposite signs. Therefore, the maps we have given determine a map $\psi : \mathrm{Cliff}(2n, \mathbb{C}) \to \mathrm{End}(\Lambda^\bullet W)$. This map is an isomorphism (still need to check this), and is in fact an isomorphism of graded algebras, where $\Lambda^\bullet W$ has the grading given by parity (also need to check this). We then see that every even complex Clifford algebra has a unique irreducible module, since it is a matrix algebra.

Example 1.3. The computation for the case $n = 2$ is simple. The set $\{1, e_1\}$ forms a basis for $\Lambda^\bullet W$, and the action of $\mathrm{Cliff}(2, \mathbb{C})$ is given by

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad e_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad f_1 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \qquad e_1 f_1 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

In the case of $n = 4$, we pick the ordered basis $(1, e_1 \wedge e_2, e_1, e_2)$ for $\Lambda^\bullet W$, so the even endormorphisms will be block diagonal, and the odd ones will be block antidiagonal. Then the matrices of the generators will be given by

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad f_1 = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} \qquad f_2 = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We now consider the odd case of $\mathrm{Cliff}(2n+1, \mathbb{C})$. In this case, we have a decomposition

$$\mathbb{C}^{2n+1} = W \oplus W' \oplus U$$

Where we do the same construction as the even case for the subspace $\mathbb{C}^{2n} \subset \mathbb{C}^{2n+1}$, and let U be the span of the last basis vector.