YANG-MILLS MINICOURSE NOTES

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Contents

1. Principal Bundles and Connections

Fix a compact manifold X and a Lie group G.

Definition 1.1. A ***principal*** G***-bundle*** is a fiber bundle $\pi : P \to M$ with a smooth right G action such that:

(1) The action of G preserves the fibers of π, and gives each fiber $P_x := \pi^{-1}(x)$ the structure of a ***right*** G***-torsor***, i.e. the action of G on P_x is free and transitive.

(2) For every point $x \in X$, there exists a ***local trivialization*** of P, i.e. a diffeomorphism $\varphi : P|_U := \pi^{-1}(U) \to U \times G$ that is G-equivariant (where the action on $U \times G$ is right multiplication with the second factor) and the following diagram commutes:

$$\begin{tikzcd} P|_U \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times G \arrow[dl] \\ & U & \end{tikzcd}$$

where the map $U \times G \to U$ is projection onto the first factor.

Example 1.2. Let $E \to X$ be a real vector bundle of rank k. For $x \in X$, let $\mathcal{B}_x$ denote the set of all bases of the fiber E_x, i.e. the set of linear isomorphisms $\mathbb{R}^k \to E_x$. This has a natural right action of $\mathrm{GL}_k\mathbb{R}$ by precomposition. Furthermore, this action is free and transitive, giving $\mathcal{B}_x$ the structure of a $\mathrm{GL}_k\mathbb{R}$-torsor. Then let

$$\mathcal{B}_{\mathrm{GL}_k\mathbb{R}}(E) := \coprod_{x \in X} \mathcal{B}_x$$

Using local trivializations of the vector bundle E, we equip $\mathcal{B}_{\mathrm{GL}_k\mathbb{R}}(E)$ with the structure of a smooth manifold such that the map $\pi : \mathcal{B}_{\mathrm{GL}_k\mathbb{R}}(E) \to X$ taking $\mathcal{B}_x$ to x is a submersion. This gives $\pi : \mathcal{B}_{GL_k\mathbb{R}} \to X$ the structure of a principal $\mathrm{GL}_k\mathbb{R}$-bundle, called the ***frame bundle*** of E, where the local trivializations are defined in terms of local trivializations of E.

Example 1.3. Let $E \to X$ be a rank k vector bundle equipped with a fiber metric, i.e. a smoothly varying inner product on the fibers E_x. Then the ***orthonormal frame bundle*** of E, denoted $\mathcal{B}_{\mathrm{O}}(E)$, is the principal O_k-bundle where the fiber over $x \in X$ is the O_k-torsor of orthonormal bases for E_x.

A near identical story holds for complex vector bundles – from any complex vector bundle we get a principal $\mathrm{GL}_k\mathbb{C}$-bundle of frames, and if we fix a Hermitian fiber metric, we get a principal U_k-bundle of orthonormal frames.

Principal bundles can be thought bundles of symmetries of some other fiber bundle, which can be made precise using the notion of an associated bundle, which allows one to construct fiber bundles out of principal bundles.

Definition 1.4. Let $P \to X$ be a principal G-bundle, and let F be a smooth manifold with right G action. The ***associated fiber bundle***, denoted $P \times_G F$ (sometimes denoted $P \times^G F$) is the space

$$P \times_G F := (P \times F)/G$$

where the right G-action on F is the diagonal action, i.e. $(p, f) \cdot g = (p \cdot g, f \cdot g)$.

If instead we have a left G-action on F, we can turn it into a right action by defining $f \cdot g := g^{-1} \cdot f$. As the name suggests, $P \times_G F$ is a fiber bundle.

Exercise 1.5. Let $\pi : P \to X$ a principal bundle. Given a smooth right action of G on F, use local trivializations of $P \to X$ to show that the map taking an equivalence class $[p, f]$ to $\pi(p)$ gives $P \times_G F$ the structure of a fiber bundle over X with model fiber F.

In the case that the model fiber is a vector space V, and the action is linear, the associated bundle $P \times_G V$ is a vector bundle.

Exercise 1.6.

(1) Let $E \to X$ be a rank k vector bundle, and $\mathcal{B}_{\mathrm{GL}_k\mathbb{R}}(E)$ be its $\mathrm{GL}_k\mathbb{R}$-bundle of frames. Let $\rho : \mathrm{GL}_k\mathbb{R} \to \mathrm{GL}_k\mathbb{R}$ be the defining representation (i.e. the identity map). Show that the associated bundle $\mathcal{B}_{\mathrm{GL}_k\mathbb{R}}(E) \times_{\mathrm{GL}_k\mathbb{R}} \mathbb{R}^k$ is isomorphic to E.

(2) Further suppose that E comes equipped with a fiber metric, and let $\mathcal{B}_{\mathrm{O}}(E)$ be its orthonormal frame bundle. The associated bundle $\mathcal{B}_{\mathrm{O}}(E) \times_{\mathrm{O}_k} \mathbb{R}^k$ is isomorphic to E by a near identical proof as the previous part. How can one recover the fiber metric?

(3) Let $\rho^* : \mathrm{GL}_k \to \mathrm{GL}_k$ denote the dual representation of the defining representation ρ. Show that the associated bundle is isomorphic to the dual bundle E^*. In particular, this should illuminate the distinction between the tangent and cotangent bundles.

Associated bundles have another nice feature – their sections have a nice interpretation in terms of G-equivariant maps.

Proposition 1.7. *Let $E = P \times_G F$ be an associated fiber bundle, and let $\Gamma(X, E)$ denote the space of global sections, i.e. the space of smooth maps $f : X \to E$ such that $\pi \circ f = \mathrm{id}_X$, where $\pi : E \to X$ denotes the projection map. Then there is a bijective correspondence*

$$\Gamma(X, E) \longleftrightarrow \{G - \textit{equivariant maps } P \to F\}$$

Proof. Let $\sigma : X \to E$ be a section. Then define the map $\widetilde{\sigma} : P \to F$ as follows: for $x \in X$, let (p, f) be a representative for $\sigma(x)$. Then define $\widetilde{\sigma}(x) := f$.

In the other direction, let $\widetilde{\varphi} : P \to F$ be an equivariant map. Then define the section $\varphi : X \to E$ by $\varphi(x) = [p, \widetilde{\varphi}(p)]$. ∎

Exercise 1.8. Verify that the map φ defined above is well-defined. Verify the two constructions above are inverses to each other.

References

[1] M. F. Atiyah and R. Bott. "The Yang-Mills Equations over Riemann Surfaces". In: *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences* 308.1505 (1983), pp. 523–615. ISSN: 00804614. URL: http://www.jstor.org/stable/37156.