YANG-MILLS MINICOURSE NOTES

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Contents

Notation and Conventions

We use Einstein summation notation, i.e. indices that appear on the top and bottom of an expression are implicitly summed over. For example

$$a_i dx^i = \sum_i a_i dx^i$$

For a smooth manifold X, we let Ω^k_X denote the space of differential k-forms. When X is a complex manifold, we let $\mathcal{A}^k_X$ denote the space of smooth complex-valued k-forms, and $\mathcal{A}^{p,q}_X$ the space of smooth (p,q)-forms. We reserve Ω^k_X and $\Omega^{p,q}_X$ for the spaces of holomorphic k and (p,q) forms respectively.

1. Principal Bundles and Connections

Good references for this material would be [5] and [7]

Fix a close manifold X (compact and without boundary) and a Lie group G.

Definition 1.1. A ***principal* G*-bundle*** is a fiber bundle $\pi : P \to M$ with a smooth right G action such that:

(1) The action of G preserves the fibers of π, and gives each fiber $P_x := \pi^{-1}(x)$ the structure of a ***right* G*-torsor***, i.e. the action of G on P_x is free and transitive.
(2) For every point $x \in X$, there exists a ***local trivialization*** of P, i.e. a diffeomorphism $\varphi : P|_U := \pi^{-1}(U) \to U \times G$ that is G-equivariant (where the action on $U \times G$ is

right multiplication with the second factor) and the following diagram commutes:

$$\begin{tikzcd} P|_U \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times G \arrow[dl] \\ & U & \end{tikzcd}$$

where the map $U \times G \to U$ is projection onto the first factor.

Example 1.2. Let $E \to X$ be a real vector bundle of rank k. For $x \in X$, let $\mathcal{B}_x$ denote the set of all bases of the fiber E_x, i.e. the set of linear isomorphisms $\mathbb{R}^k \to E_x$. This has a natural right action of $\mathrm{GL}_k\mathbb{R}$ by precomposition. Furthermore, this action is free and transitive, giving $\mathcal{B}_x$ the structure of a $\mathrm{GL}_k\mathbb{R}$-torsor. Then let

$$\mathcal{B}_{\mathrm{GL}_k\mathbb{R}}(E) := \coprod_{x \in X} \mathcal{B}_x$$

Using local trivializations of the vector bundle E, we equip $\mathcal{B}_{\mathrm{GL}_k\mathbb{R}}(E)$ with the structure of a smooth manifold such that the map $\pi : \mathcal{B}_{\mathrm{GL}_k\mathbb{R}}(E) \to X$ taking $\mathcal{B}_x$ to x is a submersion. This gives $\pi : \mathcal{B}_{GL_k\mathbb{R}} \to X$ the structure of a principal $\mathrm{GL}_k\mathbb{R}$-bundle, called the ***frame bundle*** of E, where the local trivializations are defined in terms of local trivializations of E.

Example 1.3. Let $E \to X$ be a rank k vector bundle equipped with a fiber metric, i.e. a smoothly varying inner product on the fibers E_x. Then the ***orthonormal frame bundle*** of E, denoted $\mathcal{B}_{\mathrm{O}}(E)$, is the principal O_k-bundle where the fiber over $x \in X$ is the O_k-torsor of orthonormal bases for E_x.

A near identical story holds for complex vector bundles – from any complex vector bundle we get a principal $\mathrm{GL}_k\mathbb{C}$-bundle of frames, and if we fix a Hermitian fiber metric, we get a principal U_k-bundle of orthonormal frames.

Principal bundles can be thought bundles of symmetries of some other fiber bundle, which can be made precise using the notion of an associated bundle, which allows one to construct fiber bundles out of principal bundles.

Definition 1.4. Let $P \to X$ be a principal G-bundle, and let F be a smooth manifold with right G action. The ***associated fiber bundle***, denoted $P \times_G F$ (sometimes denoted $P \times^G F$) is the space

$$P \times_G F := (P \times F)/G$$

where the right G-action on F is the diagonal action, i.e. $(p, f) \cdot g = (p \cdot g, f \cdot g)$.

If instead we have a left G-action on F, we can turn it into a right action by defining $f \cdot g := g^{-1} \cdot f$. As the name suggests, $P \times_G F$ is a fiber bundle.

Exercise 1.5. Let $\pi : P \to X$ a principal bundle. Given a smooth right action of G on F, use local trivializations of $P \to X$ to show that the map taking an equivalence class $[p, f]$ to $\pi(p)$ gives $P \times_G F$ the structure of a fiber bundle over X with model fiber F.

In the case that the model fiber is a vector space V, and the action is linear, the associated bundle $P \times_G V$ is a vector bundle.

Exercise 1.6.

(1) Let $E \to X$ be a rank k vector bundle, and $\mathcal{B}_{\mathrm{GL}_k\mathbb{R}}(E)$ be its $\mathrm{GL}_k\mathbb{R}$-bundle of frames. Let $\rho : \mathrm{GL}_k\mathbb{R} \to \mathrm{GL}_k\mathbb{R}$ be the defining representation (i.e. the identity map). Show that the associated bundle $\mathcal{B}_{\mathrm{GL}_k\mathbb{R}}(E) \times_{\mathrm{GL}_k\mathbb{R}} \mathbb{R}^k$ is isomorphic to E.
(2) Further suppose that E comes equipped with a fiber metric, and let $\mathcal{B}_{\mathrm{O}}(E)$ be its orthonormal frame bundle. The associated bundle $\mathcal{B}_{\mathrm{O}}(E) \times_{\mathrm{O}_k} \mathbb{R}^k$ is isomorphic to E by a near identical proof as the previous part. How can one recover the fiber metric?
(3) Let $\rho^* : \mathrm{GL}_k\mathbb{R} \to \mathrm{GL}_k\mathbb{R}$ denote the dual representation of the defining representation ρ, so $\rho(A) = (A^{-1})^T$. Show that the associated bundle is isomorphic to the dual bundle E^*. In particular, this should illuminate the distinction between the tangent and cotangent bundles.

Example 1.7. There are two important examples of associated bundles that we'll need to discuss the Yang-Mills equations.

(1) The bundle $\mathrm{Ad}\, P := P \times_G G$ where G acts on G by conjugation.
(2) The bundle $\mathrm{ad}\, P := P \times_G \mathfrak{g}$ (also denoted $\mathfrak{g}_P$) where the action is the adjoint action.

Confusingly, the former is sometimes called the "Adjoint bundle" and the latter is sometimes called the "adjoint bundle," which makes it admittedly hard to distuinguish between them when speaking.

Associated bundles have another nice feature – their sections have a nice interpretation in terms of G-equivariant maps.

Proposition 1.8. *Let $E = P \times_G F$ be an associated fiber bundle, and let $\Gamma(X, E)$ denote the space of global sections, i.e. the space of smooth maps $f : X \to E$ such that $\pi \circ f = \mathrm{id}_X$, where $\pi : E \to X$ denotes the projection map. Then there is a bijective correspondence*

$$\Gamma(X, E) \longleftrightarrow \{G - \textit{equivariant maps } P \to F\}$$

Proof. Let $\sigma : X \to E$ be a section. Then define the map $\widetilde{\sigma} : P \to F$ as follows: for $x \in X$, let (p, f) be a representative for $\sigma(x)$. Then define $\widetilde{\sigma}(x) := f$.

In the other direction, let $\widetilde{\varphi} : P \to F$ be an equivariant map. Then define the section $\varphi : X \to E$ by $\varphi(x) = [p, \widetilde{\varphi}(p)]$ for any choice of $p \in P_x$. ∎

Exercise 1.9. Verify that the map φ defined above is well-defined. Verify the two constructions above are inverses to each other.

The main takeaway from the proposition is the motto that "G-equivariant objects on P descend to objects on X."

Before we discuss connections on principal bundles, we introduce the concept of vector bundled valued forms.

Definition 1.10. Let $E \to X$ be a vector bundle. An ***E-valued differential k-form*** is a section of $\Lambda^k T^*X \otimes E$. We denote the space of E-valued k-forms by $\Omega^k_X(E)$. If V is a fixed vector space, a ***V-valued differential k-form*** is a $X \times V$-valued k-form, and we let $\Omega^k_X(V)$ denote the space of V-valued k-forms.

In a local frame $\{e_i\}$ for E, an E-valued k-form ω can be written uniquely as

$$\omega = \omega^i \otimes e_i$$

for k-forms $\omega^i \in \Omega^k_X$,so an E-valued k-form can be thought of as a vector of k-forms. We will usually omite the tensor symbol, and simply write $\omega = \omega^i e_i$. However, this does not transform tensorially with respect to coordinate changes on X unless E is a trivial bundle. The components of the vector transform tensorially with respect to coordinate changes, but the vector itself changes according to the transition functions of the vector bundle E. Given E-valued forms $\omega \in \Omega^k_X(E)$ and $\eta \in \Omega^\ell_X(E)$, we define their wedge product in a local trivialization to be

$$\omega \wedge \eta := (\omega^i \wedge \eta^j) e_i \otimes e_j$$

which is an element of $\Omega^{k+\ell}_X(E \otimes E)$.

For the most part, we will be concerned with Lie algebra valued forms, which are just $\mathfrak{g}$-valued forms for a fixed Lie algebra $\mathfrak{g}$. These forms have some additional operations coming from the Lie algebra structure of $\mathfrak{g}$. Fix a basis $\{\xi_i\}$ for $\mathfrak{g}$. This determines a global trivialization of the trivial bundle $X \times \mathfrak{g}$, so any $\mathfrak{g}$-valued K-form $\omega \in \Omega^k_X(\mathfrak{g})$ can be uniquely written as $\omega = \omega^i \xi_i$. Let $\omega \in \Omega^k_X(\mathfrak{g})$ and $\eta \in \Omega^\ell_X(\mathfrak{g})$. Then define their bracket to be

$$[\omega \wedge \eta] := (\omega^i \wedge \eta^j)[\xi_i, \xi_j]$$

In other words, it is the composition

$$\Omega^k_X(\mathfrak{g}) \otimes \Omega^k_X(\mathfrak{g}) \longrightarrow \Omega^{k+\ell}_X(\mathfrak{g} \otimes \mathfrak{g}) \longrightarrow \Omega^{k+\ell}(\mathfrak{g})$$

where the first map is the wedge product, and the second map is induced by the Lie bracket. Finally, the usual exterior derivative $d : \Omega^k_X \to \Omega^{k+1}_X$ extends to an operator on $\Omega^k_X(\mathfrak{g})$, given by $d\omega = d\omega^i \xi_i$.

We now discuss connections. Let $\pi : P \to X$ be a principal G-bundle, and let $\mathfrak{g}$ be the Lie algebra of G. The projection map π is a submersion, so it is constant rank. Therefore, the subset $V \subset TP$ where the fiber over p is $\ker d\pi_p$ is a subbundle, called the ***vertical distribution*** of P, giving us an exact sequence of vector bundles over P

$$0 \longrightarrow V \longrightarrow TP \longrightarrow \pi^* TX \longrightarrow 0$$

Definition 1.11. A ***connection*** on P is a distribution $H \subset TP$ such that

(1) $V \oplus H = TP$

(2) $H_{p \cdot g} = d(R_g)_p H_p$, where $R_g : P \to P$ is the map $p \mapsto p \cdot g$.

The distribution H is also called the ***horizontal distribution***. We let $\mathscr{A}(P)$ denote the space of connections on P.

Equivalently, it is a choice of G-invariant splitting of the exact sequence. The perspective of viewing a connection as a horizontal distribution is useful at times, but it is often more convenient for computations to rephrase a connection in terms of $\mathfrak{g}$-valued forms. Let $\exp : \mathfrak{g} \to G$ denote the exponential map. Given $X \in \mathfrak{g}$ and $p \in P$, the exponential map determines a curve γ_X with $\gamma_X(0) = p$ where

$$\gamma_X(t) := p \cdot \exp(tX)$$

Since the action of G preserves the fiber $P_{\pi(p)}$, the tangent vector

$$\dot{\gamma}_X := \left.\frac{d}{dt}\right|_{t=0} \gamma_X(t)$$

lies in V_p. Furthermore, since the action of G on P is free, we have that $\dot{\gamma}_X = 0$ if and only if $X = 0$. Finally, the mapping $X \mapsto \dot{\gamma}_X$ is linear, so we have that this gives an isomorphism $\mathfrak{g} \to V_p$ by a dimension count. Doing this over all $p \in P$, this gives an isomorphism of V with the trivial bundle $P \times \mathfrak{g}$. Because of this, we will implicitly identify elements of $\mathfrak{g}$ with the vertical vector fields they determine. One thing to note is how these vector fields transform with respect to the action of G.

Proposition 1.12. *Let $X \in \mathfrak{g}$, and let $\widetilde{X}$ denote the vertical vector field on P induced by X. For $g \in G$, let $R_g : P \to P$ be map given by the action of g. Then*

$$(R_g)_* \widetilde{X} = \widetilde{\mathrm{Ad}_{g^{-1}} X}$$

Proof. We compute

$$\begin{aligned}
((R_g)_* \widetilde{X}_p) &= (R_g)_* \left(\left.\frac{d}{dt}\right|_{t=0} p \cdot \exp(tX) \right) \\
&= \left.\frac{d}{dt}\right|_{t=0} p \cdot (\exp(tX) g) \\
&= (p \cdot g)(g^{-1} \exp(tX) g) \\
&= (\widetilde{\mathrm{Ad}_{g^{-1}} X})_{p \cdot g}
\end{aligned}$$

■

Furthermore, the identification of the vertical distribution V with the trivial bundle $X \times \mathfrak{g}$ gives us a nice characterization of E-valued forms, when $E = P \times_G W$ is an associated bundle coming from a linear representation $\rho : G \to \mathrm{GL}(W)$.

Proposition 1.13. *Let $P \to M$ be a principal bundle and E the associated bundle coming from a linear representation $\rho : G \to \mathrm{GL}(W)$. Then there is a bijective correspondence*

$$\Omega^k_X(E) \leftrightarrow \left\{ \alpha \in \Omega^k_P(W) \ : \ R_g^* \alpha = \rho(g^{-1})\alpha \ , \ \iota_X \alpha = 0 \forall X \in \mathfrak{g} \right\}$$

where we identify $X \in \mathfrak{g}$ with its vertical vector field and ι_ξ denotes interior multiplication.

Morally, the correspondence comes from the fact that a form on P descending to X should satisfy G-invariance, and should be invariant in the vertical directions.

Exercise 1.14. Prove the previous proposition.

Exercise 1.15. Using the proposition above, prove that the space $\mathscr{A}(P)$ of connections is an affine space over $\Omega^1_X(\mathfrak{g}_P)$, i.e. show that the difference $A_1 - A_2$ between two connections is an element of $\Omega^1_X(\mathfrak{g}_P)$.

Now suppose we have a horizontal distribution $H \subset TP$. The decomposition $TP = V \oplus H$ gives us a projection map $TP \to V$ with kernel H. Identifying V with $P \times \mathfrak{g}$, the projection map can be idenfied with a $\mathfrak{g}$-valued 1-form $A \in \Omega^1_P(\mathfrak{g})$, called the ***connection***

1-***form***. Using the transformation law for the vertical vector fields determined by $\mathfrak{g}$, we get the following transformation law for the connection 1-form A.

Proposition 1.16. *A connection* 1*-form* $A \in \Omega^1_P(\mathfrak{g})$ *satisfies*

$$R_g^* A = \operatorname{Ad}_{g^{-1}} A$$

Proof. For $v \in T_p P$, decompose $v = X + h$ with $X \in \mathfrak{g}$ and $h \in H_p$. We then compute

$$\begin{aligned}(R_g^* A)(v) &= (R_g^* A)(X + h) \\ &= A_{p \cdot g}((R_g)_* X + (R_g)_* h) \\ &= A_{p \cdot g}(\operatorname{Ad}_{g^{-1}} X) \\ &= (\operatorname{Ad}_{g^{-1}} A)_p (X + h)\end{aligned}$$

where we use the fact that H_p is the the kernel of A_p and the fact that H is G-invariant. ■

Furthermore, since A is given by projection onto the vertical distribution, we have that $\iota_X A = X$ for all $X \in \mathfrak{g}$. This gives us an identification of $\mathscr{A}(P)$ with the subset of $\Omega^1_P(\mathfrak{g})$ satisfying the conditions

(1) $R_g^* A = \operatorname{Ad}_{g^{-1}} A$
(2) $\iota_X A = X$ for all $X \in \mathfrak{g}$.

The second condition can be rephrased in terms of the ***Maurer-Cartan form*** $\theta \in \Omega^1_G(\mathfrak{g})$, which is defined by

$$\theta_g(v) = (dL_{g^{-1}})_g(v)$$

where $L_{g^{-1}} : G \to G$ is left multiplication by g^{-1}. The action of G on any G-torsor X gives us a Maurer-Cartan form on X.

Exercise 1.17. Show that the second condition is equivalent to $A|_{P_x} = \theta$ for any $x \in X$.

The Maurer-Cartan form θ satisfies the ***Maurer-Cartan equation***

$$d\theta + \frac{1}{2}[\theta \wedge \theta] = 0$$

Definition 1.18. Let $A \in \mathscr{A}(P)$ be a connection. Then the ***curvature of*** A, denoted F_A, is the $\mathfrak{g}$-valued 2 form

$$F_A := dA = \frac{1}{2}[A \wedge A]$$

Proposition 1.19.

(1) $R_g^* F_A = \operatorname{Ad}_{g^{-1}} F_A$*.*
(2) $\iota_X F_A = 0$ *for all* $X \in \mathfrak{g}$*.*

Proof.

(1) We compute

$$\begin{aligned} R^*F_A &= R_g^* dA + \frac{1}{2}R_g^*[A \wedge A] \\ &= d(\operatorname{Ad}_{g^{-1}} A) + \frac{1}{2}[R_g^*A \wedge R_G^*A] \\ &= \operatorname{Ad}_{g^{-1}} dA + \frac{1}{2}[\operatorname{Ad}_{g^{-1}} A \wedge \operatorname{Ad}_{g^{-1}} A] \\ &= \operatorname{Ad}_{g^{-1}} F_A \end{aligned}$$

(2) For this, we use a lemma.

Lemma (***Cartan's magic formula***)**.** *Let X be a vector field, and ω a k-form. Let $\mathcal{L}_X$ denote the Lie derivative along X. Then*

$$\mathcal{L}_X = d\iota_X\omega + \iota_X d\omega$$

Let $X \in \mathfrak{g}$, interpreted as a vertical vector field on P. Then we compute

$$\iota_X F_A = \iota_X dA + \frac{1}{2}\iota_X[A \wedge A]$$

We compute the two terms separately. Cartan's magic formula gives us that

$$\mathcal{L}_X A = d\iota_X A + \iota_X dA$$

Since $\iota_X F_A$ is the constant function with value X, we have that $d\iota_X F_A$ is 0, so we get $\mathcal{L}_X F_A = \iota_X dF_A$. Using the definition of the Lie derivative, we compute

$$\begin{aligned} \mathcal{L}_X A &= \frac{d}{dt}\bigg|_{t=0} R^*_{\exp tX} A \\ &= \frac{d}{dt}\bigg|_{t=0} \operatorname{Ad}_{\exp(tX)^{-1}} A \\ &= [-X, A] \end{aligned}$$

For the other term, we compute

$$\frac{1}{2}\iota_X[A \wedge A] = \frac{1}{2}[\iota_X A \wedge A] = [X, A]$$

adding these together gives us the desired result.

■

In other words, the curvature F_A descends to a $\mathfrak{g}_P$-valued 2-form on the base manifold X.

Exercise 1.20. Given a connection A on a principal bundle P, prove that the curvature F_A vanishes if and only if horizontal distribution H defined by the kernel of A is integrable. To prove this, reformulate Frobenius' theorem (a distribution is integrable if and only if it is involutive) in terms of the vanishing of a tensor, and show that this tensor (up to sign) is equal to F_A.

Definition 1.21. A connection $A \in \mathscr{A}(P)$ is ***flat*** if $F_A = 0$.

For vector bundles, a more familiar definition of a connection involves a first order operator on sections satisfying a Leibniz rule. Using the language of principal bundles and associated bundles, we recover this notion with the exterior covariant derivative.

Definition 1.22. Let $E = P \times_G W$ be the associated vector bundle obtained from a linear representation $\rho : G \to \mathrm{GL}(W)$, and let $\dot{\rho} : \mathfrak{g} \to \mathrm{End}(W)$ denote the associated Lie algebra representation. The ***exterior covariant derviative*** is the map

$$\begin{aligned} d_A : \Omega^p_X(E) &\to \Omega^{p+1}_X(E) \\ \psi &\mapsto d\psi + \dot{\rho}(A) \wedge \psi \end{aligned}$$

Exercise 1.23. Recall that a connection on a vector bundle $E \to X$ is given in a local trivialization by $d + A$ for some $\mathrm{End}(E)$-valued 1-form A. Show that when $P = \mathcal{B}_{\mathrm{GL}_k \mathbb{R}}(E)$ is the frame bundle of a vector bundle E, the exterior covariant derivative on sections agrees with this definition.

For the most part, we will be concerned with situation when the vector bundle is $\mathfrak{g}_P$, in which case, the formula is given by

$$d_A \psi = d\psi + [A, \psi]$$

Since $\mathscr{A}(P)$ is an affine space over $\Omega^1_X(\mathfrak{g}_P)$, given a connection $A \in \mathscr{A}(P)$ and a $\mathfrak{g}_P$-valued 1-form $\eta \in \Omega^1_X(\mathfrak{g}_P)$, we have that $A + \eta$ is also a connection. It can be shown that the curvature of $A + \eta$ is given by

$$F_{A+\eta} = F_A + \frac{1}{2}[\eta \wedge \eta] + d_A \eta$$

In particular, if we take a line of connections $A + t\eta$ with $t \in \mathbb{R}$, we have

$$\left.\frac{d}{dt}\right|_{t=0} F_{A+t\eta} = \left.\frac{d}{dt}\right|_{t=0} F_A + \frac{t^2}{2}[\eta \wedge \eta] + t d_A \eta = d_A \eta$$

So the exterior covariant derivative on $\mathfrak{g}_P$ measures the infitesimal change of the curvature of A in the direction η.

2. The Yang-Mills Equations

To discuss the Yang-Mills equations, we will restrict to compact Lie groups G. As before, X will denote an n-dimensional closed smooth manifold.

Since G is compact, its Lie algebra $\mathfrak{g}$ is semisimple, so the Killing form $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ is nondegenerate. For the rest of our discussion, $\langle \cdot, \cdot \rangle$ can be replaced by any inner product invariant under the Adjoint action, though it does us no harm to assume that it is the Killing form.

Lemma 2.1. *Let $\langle \cdot, \cdot \rangle$ denote any Adjoint invariant inner product on $\mathfrak{g}$. Then for $X_1, X_2, X_3 \in \mathfrak{g}$, we have*

$$\langle [X_1, X_2], X_3 \rangle = \langle X_1, [X_2, X_3] \rangle$$

Proof. We compute

$$\begin{aligned}\langle [X_1, X_2], X_3\rangle &= \langle [-X_2, X_1], X_3\rangle \\ &= \frac{d}{dt}\bigg|_{t=0} \langle \mathrm{Ad}_{\exp(-tX_2)} X_1, X_3\rangle \\ &= \frac{d}{dt}\bigg|_{t=0} \langle \mathrm{Ad}_{\exp tX_2} \mathrm{Ad}_{\exp(-tX_2)} X_1, \mathrm{Ad}_{\exp(tX_2)} X_3\rangle \\ &= \langle X_1, [X_2, X_3]\rangle\end{aligned}$$

■

The form $\langle \cdot, \cdot \rangle$ induces a fiber metric on $P \times \mathfrak{g}$, and invariance under the Adjoint action tells us that this fiber metric descends to a fiber metric on $\mathfrak{g}_P$. This gives us pairings

$$\begin{aligned}\Omega^k_X(\mathfrak{g}_P) \otimes \Omega^\ell_X(\mathfrak{g}_P) &\to \Omega^{k+\ell}_X \\ \omega \otimes \eta &\mapsto \langle \omega, \eta\rangle\end{aligned}$$

We now fix an orientation and a Riemannian metric g on X. This gives us:

(1) A Riemannian volume form $\mathrm{Vol}_g \in \Omega^n_X$.
(2) A Hodge star operator $\star : \Omega^k_X \to \Omega^{n-k}_X$.
(3) Fiber metrics $\langle \cdot, \cdot \rangle_g$ on the bundles $\Lambda^k T^* X$.

The Hodge star extends to $\mathfrak{g}_P$-valued forms, which gives us inner products on $\Omega^k_X(\mathfrak{g}_P)$ given by

$$(\omega, \eta) := \int_X \langle \omega, \star\eta\rangle$$

We let $\|\cdot\|$ denote the norm induced by these inner products.

We now introduce the gauge group of a prinicipal G-bundle $P \to X$.

Definition 2.2. Let $\pi : P \to X$ be a principal G-bundle. The ***gauge group***, denoted $\mathscr{G}(P)$, is the group of automorphisms of P, i.e. G-equivariant diffeomorphisms $\varphi : P \to P$ such that $\pi = \pi \circ \varphi$. An element of $\mathscr{G}(P)$ is called a ***gauge transformation***.

Proposition 2.3. *The group $\mathscr{G}(P)$ is isomorphic to the group of sections $\Gamma(X, \mathrm{Ad}\, P)$, where the group operation is pointwise multiplication.*

Proof. We provide maps in both directions. Suppose we have an automorphism $\varphi : P \to P$. Since $\pi = \pi \circ \varphi$, the map φ preserves the fibers of π. Therefore, for any $p \in P$, we have that p and $\varphi(p)$ differ by the action of some $g_p \in G$. The mapping $g_\varphi : P \to G$ taking $p \mapsto g_p$ is easilty verified to be equivariant with respect to the conjugation action of G, so it defines a section of $\mathrm{Ad}\, P$

In the other direction, given a G-equivariant map $f : P \to G$, we get a bundle automorphism $\varphi_f : P \to P$ where $\varphi_f(p) = p \cdot f(p)$. The two maps we constructed are clearly inverse to each other, giving us the desired correspondence. ■

The gauge group $\mathscr{G}(P)$ acts on the space $\Omega^1_P(\mathfrak{g})$ of $\mathfrak{g}$-valued forms by pullback. We claim that it preserves the subspace $\mathscr{A}(P) \subset \Omega^1_P(\mathfrak{g})$.

Proposition 2.4. *For a connection $A \in \mathscr{A}(P)$ and a gauge transformation $\varphi : P \to P$, we have*

(1) $R_g^* A = \mathrm{Ad}_{g^{-1}} \varphi^* A$.
(2) $\iota_X \varphi^* A = X$ *for all* $X \in \mathfrak{g}$.

Equivalently, if we let $g_\varphi : P \to G$ denote the equivariant map associated to φ, we have

$$\varphi^* A = \mathrm{Ad}_{g_\varphi^{-1} A} + g_\varphi^* \theta$$

where $\theta \in \Omega^1_G(\mathfrak{g})$ is the Maurer-Cartan form.

Exercise 2.5. Prove the previous proposition.

Definition 2.6. Two connections A_1 and A_2 are ***gauge equivalent*** if there exists a gauge transformation $\varphi \in \mathscr{G}(P)$ such that $\varphi^* A_1 = A_2$.

Proposition 2.7. *Let $A \in \mathscr{A}(P)$ be a connection, $\varphi : P \to P$ a gauge transformation, and $g_\varphi : P \to G$ the associated equivariant map. Then*

$$F_{\varphi^* A} = \mathrm{Ad}_{g_\varphi^{-1}} F_A$$

Proof. We compute

$$\begin{aligned}
F_{\varphi^* A} &= d(\mathrm{Ad}_{g_\varphi^{-1}} A + g_\varphi^* \theta) + \frac{1}{2}[\mathrm{Ad}_{g_\varphi^{-1}} A + g_\varphi^* \theta \wedge \mathrm{Ad}_{g_\varphi^{-1}} A + g_\varphi^* \theta] \\
&= \mathrm{Ad}_{g_\varphi^{-1}} dA + g_\varphi^* d\theta + \frac{1}{2} \Big([\mathrm{Ad}_{g_\varphi^{-1}} A \wedge \mathrm{Ad}_{g_\varphi^{-1}} A] + [\mathrm{Ad}_{g_\varphi^{-1}} A \wedge g_\varphi^* \theta] + [g_\varphi^* \theta \wedge \mathrm{Ad}_{g_\varphi^{-1}} A] + [g_\varphi^* \theta \wedge g_\varphi^* \theta] \Big) \\
&= \mathrm{Ad}_{g_\varphi^{-1}} dA + \frac{1}{2}[\mathrm{Ad}_{g_\varphi^{-1}} A, \mathrm{Ad}_{g_\varphi^{-1}} A]] \\
&= \mathrm{Ad}_{g_\varphi^{-1} F_A}
\end{aligned}$$

The term

$$g_\varphi^* d\theta + \frac{1}{2}[g_\varphi^* \theta \wedge g_\varphi^* \theta]$$

vanishes due to the Maurer-Cartan equation. The term

$$\frac{1}{2} \Big([\mathrm{Ad}_{g_\varphi^{-1}} A \wedge g_\varphi^* \theta] + [g_\varphi^* \theta \wedge \mathrm{Ad}_{g_\varphi^{-1}} A] \Big)$$

vanishes due to the fact that $[\cdot, \cdot]$ is skew symmetric on 1-forms. ■

With some of the preliminary results established, we arrive at the Yang-Mills functional.

Definition 2.8. The ***Yang-Mills functional*** is the map $L : \mathscr{A}(P) \to \mathbb{R}$ given by

$$L(A) := \|F_A\|^2 = \int_X \langle F_A \wedge \star F_A \rangle$$

We note that for any gauge transformation $\varphi \in \mathscr{G}(P)$, we have $L(\varphi^* A) = L(A)$, since we have

$$L(\varphi^* A) = \int_X \langle \mathrm{Ad}_{g_\varphi^{-1}} F_A \wedge \star \mathrm{Ad}_{g_\varphi^{-1}} F_A \rangle = \int_X \langle F_A \wedge \star F_A \rangle = L(A)$$

because of this we say that L is ***gauge invariant***.

The Yang-Mills equations are the variational equations for the Yang-Mills functional.

Proposition 2.9 (***The first variation***)**.** *Let A be a local extremum of L. Then we have*

$$d_A \star F_A = 0$$

Proof. Let $\eta \in \Omega^1_X(\mathfrak{g}_P)$. We then compute

$$\begin{aligned} L(A) &= \int_X \langle F_{A+t\eta} \wedge \star F_{A+t\eta} \\ &= \int_X \langle F_A + \frac{t^2}{2}[\eta \wedge \eta] + t d_A \eta \wedge \star (F_A + \frac{t^2}{2}[\eta \wedge \eta] + t d_A \eta) \end{aligned}$$

The term linear in t is

$$\int_X \langle F_A \wedge \star d_A \eta + \langle d_A \eta \wedge \star F_A \rangle = 2(F_A, d_A \eta)$$

Then let $d_A^* = (-1)^{2n+1} \star d_A \star$ denote the formal adjoint to d_A. Since A is a local extremum, the term linear in t must vanish, so for every η, we must have

$$(F_A, d_A \eta) = (d_A^* F_A, \eta) = 0$$

Then since up to sign $d_A^* = \star d_A \star$ and $\star$ is an isomorphism, we have $d_A \star F_A = 0$. ∎

The first variation gives us what are referred to as the ***Yang-Mills equations***

$$\begin{aligned} d_A F_A &= 0 \\ d_A^* F_A &= 0 \end{aligned}$$

Definition 2.10. A ***Yang-Mills connection*** is a connection $A \in \mathscr{A}(P)$ satisfying the Yang-Mills equations, i.e. a local extremum of L.

Exercise 2.11. In the case that $G = \mathrm{U}_1$, show that the curvature of a connection A can be identified as an element of Ω^2_X. Show that A is a Yang-Mills connection if and only if F_A is a harmonic form, i.e. $\Delta F_A = 0$, where $\Delta = dd^* + d^* d$ is the Hodge Laplacian. Use this to show that the space of Yang-Mills connections on a principal U_1-bundle P is a torsor over the vector space of closed 1-forms on X.

The first equation is simply the ***Bianchi identity*** and the second comes from the first variation.

Proposition 2.12 (***The second variation***)**.** *Let A be a Yang-Mills connection. Then for every $\eta \in \Omega^1_X(\mathfrak{g}_P)$, we have*

$$\left.\frac{d}{dt}\right|_{t=0} d^*_{A+t\eta} F_{A+t\eta} = d_A^* d_A \eta + \star[\eta \wedge \star F_A]$$

The proof of this is similar to the proof of the first variation, and involves expanding out $d^*_{A+t\eta} F_{A+t\eta}$ and then taking the term linear in t. If we think of L as a Morse function on $\mathscr{A}(P)$, for a Yang-Mills connection A, the operator $d_A^* d_A + \star[\cdot \wedge \star F_A]$ can be interpreted as the Hessian of L at the critical point A. In particular, if η is tangent to the critical submanifold of Yang-Mills connections, one can use the Atiyah-Singer index theorem with this operator to compute the dimension of the space of Yang-Mills connections.

We now restrict ourselves to the case where X is a Riemann surface. Let $\Gamma_{\mathbb{R}}$ denote the central extension of $\pi_1(X)$ by $\mathbb{R}$ where if we let J denote a generator for $\mathbb{R}$, we have the relation $\prod_i [a_i, b_i] = J$ where the a_i and b_i are the generators for the usual presentation of a closed surface of genus g. Using this group one can prove the following theorems, though we will omit the proofs.

Theorem 2.13. *Every principal G-bundle $P \to X$ admits a Yang-Mills connection.*

Theorem 2.14. *There is a bijective correspondence*

$$\mathrm{Hom}(\Gamma_{\mathbb{R}}, G)/G \longleftrightarrow \{\textit{Principal } G\textit{-bundles } P \to X \textit{with a Yang-Mills connection}\} / \sim$$

where the action of G is conjugation and the equivalence relation is gauge equivalence.

The second theorem should be thought of an analogue of the classical Riemann-Hilbert correspondence.

Exercise 2.15. The classical Riemann-Hilbert correspondence gives a bijection

$$\mathrm{Hom}(\pi_1(X), G)/G \leftrightarrow \{\text{Principal } G\text{-bundles } P \to X \text{ equipped with a flat connection } A\} / \sim$$

where the action of G on $\mathrm{Hom}(\pi_1(X), G)$ is by conjugation, and the equivalence relation is gauge equivalence of conenctions. The correspondence assigns to $\rho \in \mathrm{Hom}(\pi_1(X), G)$ the associated bundle

$$\widetilde{X} \times_{\pi_1(X)} G$$

where $\widetilde{X}$ is the universal cover of X, and the connection is the one induced by descending the trivial connection on $\widetilde{X} \times G$ to the quotient. In the other direction, the holonomy of a flat connection defines (up to conjugation by G) a homomorphism $\pi_1(X) \to G$.

A principal U_1-bundle $P \to X$ corresponds to a Hermitian line bundle $L \to X$ by taking the associated bundle $P \times_{\mathrm{U}_1} \mathbb{C}$ with the standard action of U_1 on $\mathbb{C}$. Using this correspondence and the classical Riemann-Hilbert correspondence, show that there is a bijection

$$\mathrm{Hom}(\Gamma_{\mathbb{R}}, \mathrm{U}_1)/\mathrm{U}_1 \longleftrightarrow \{\text{Principal } \mathrm{U}_1\text{-bundles } P \to X \text{with a Yang-Mills connection}\} / \sim$$

For the rest of our discussion, we will restrict to case where $G = \mathrm{U}_n$. We first make a remark involving the proofs of the two preceding theorems. As with the U_1 case, the data of a principal U_n-bundle $P \to X$ is equivalent to a rank n complex vector bundle $E \to X$ equipped with a Hermitian metric. In the proofs, one shows that a Yang-Mills connection A is equivalent to the choice of a Lie algebra element $X \in \mathfrak{u}_n$. Writing $X = -2\pi i \Lambda$ for a Hermitian matrix Λ, the Yang-Mills condition implies that the trace of Λ is equal to the first Chern class of E, thought of as an integer by integrating over X. If we let λ_i denote the i^{th}-eigenvalue (arranged in ascending order) and n_i the multiplicity of λ_i, one can show that $n_i\lambda_i$ must also be integral. These observations will be useful when we relate Yang-Mills connections with holomorphic vector bundles.

3. Holomorphic Vector Bundles and Yang-Mills Connections

A good references for this material would be [4] and [6]

Let X denote a complex manifold.

Definition 3.1. A ***holomorphic vector bundle*** is a complex vector bundle $\pi : E \to X$ such that the total space E is a complex manifold and π is holomorphic.

Given a holomorphic vector bundle $E \to X$, we can find a trivialization of E such that the transition functions are holomorphic. In a neighborhood $U \subset X$ such that $E|_U$ is trivial, the smooth sections can be identified with functions $U \to \mathbb{C}^n$, and the holomorphic sections can be identified with the holomorphic functions $U \to \mathbb{C}^n$. We have a local operator $\overline{\partial}$, which we can apply componentwise to a local section to get an operator on smooth sections over U. Furthermore, since $\overline{\partial}$ annihilates holomorphic functions and the transition functions are holomorphic, we have that $\overline{\partial}$ glues to a well defined operator $\overline{\partial}_E : \mathcal{A}^0_X(E) \to \mathcal{A}^{0,1}_X(E)$. The holomorphic sections of E are then exactly the sections annihilated by $\overline{\partial}_E$. Furthermore, the operator $\overline{\partial}_E$ extends to operators $\overline{\partial}_E : \mathcal{A}^k_X(E) \to \mathcal{A}^{k+1}_X(E)$, and satifies the condition $\overline{\partial}_E^2 = 0$, since $\overline{\partial}^2 = 0$. The punchline is that the holomorphic structure on E is entirely determined by this operator.

Theorem 3.2. *Let $\pi : E \to X$ be a C^∞ complex vector bundle, and let $D : \mathcal{A}^0_X(E) \to \mathcal{A}^{0,1}(E)$ be an operator satifying $D^2 = 0$. Then there exists a unique complex structure on E such that π is holomorphic and D coincides with the operator $\overline{\partial}_E$.*

This can be seen as a linearized version of the Newlander-Nirenberg theorem. In particular, a holomorphic vector bundle $E \to X$ can be thought of as a smooth vector bundle along with a choice of operator $\overline{\partial}_E$. Since the operator $\overline{\partial}$ satisfies a Leibniz rule, the operator $\overline{\partial}_E$ behaves like a connection. In a *smooth* local trivialization, we can write

$$\overline{\partial}_E = \overline{\partial} + B$$

where B is a smooth $\mathrm{M}_n\mathbb{C}$-valued $(0,1)$-form. Indeed, we have that the space of holomorphic structures on a smooth vector bundle $E \to X$ is an affine space over $\mathcal{A}^{1,0}(\mathrm{End}\, E)$. We let $\mathscr{C}(E)$ denote the space of holomorphic structures on E.

We now restrict to the case where X is a Riemann surface.

Definition 3.3. The ***slope*** of a holomorphic vector bundle $E \to M$ is

$$\mu(E) := \frac{c_1(E)}{\mathrm{rank}(E)}$$

where we think of $c_1(E) \in H^2(X, \mathbb{Z})$ as an integer via integration over X.

Sometimes the integer $c_1(E)$ is also referred to as the ***degree*** of E. One thing to note is that the slope of a holomorphic vector bundle is independent of the holomorphic structure – both the degree and rank are topological invariants, and only depend on the underlying C^∞ complex vector bundle.

Definition 3.4. A holomorphic vector bundle $E \to X$ is

(1) ***Stable*** if for every holomorphic subbundle $F \subset E$, we have $\mu(F) < \mu(E)$.
(2) ***Semistable*** if for every holomorphic subbundle $F \subset E$, we have $\mu(F) \leq \mu(E)$.
(3) ***Unstable*** if E is not semistable.

While the slope is a topological invariant, stability is not, since we only consider holomorphic subbundles – which depend on the holomorphic structure. We also note that both the degree and rank are additive in exact sequences, which immediately gives us:

Proposition 3.5. *Suppose we have the short exact sequence of holomorphic bundles*

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$$

Then we have

$$\mu(F) = \frac{\deg(E) + \deg(G)}{\operatorname{rank}(E) + \operatorname{rank}(G)}$$

Corollary 3.6. *Given a short exact sequence of holomorphic bundles*

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$$

If $\mu(E) \geq \mu(F)$, then $\mu(F) \geq \mu(G)$. Likewise, if $\mu(E) \leq \mu(F)$, then $\mu(F) \leq \mu(G)$.

In other words, slopes behave monotonically in short exact sequences. The terminology comes from Geometric Invariant Theory (GIT). The main result will use is:

Theorem 3.7 (***The Harder-Narasimhan Filtration***)**.** *Let $E \to X$ be a holomorphic vector bundle. Then E admits a canonical filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

by holomorphic subbundles E_i such that E_i/E_{i-1} is semistable and

$$\mu(E_1/E_0) > \mu(E_2/E_1) > \cdots > \mu(E_n/E_{n-1})$$

The proof of the above theorem is not extremely difficult, but we omit it. The main idea is that any holomorphic vector bundle has a unique maximal semistable subbundle, which we take to be E_1. We then take E_2 to be the preimage of the maximal semistable bundle of E_1/E_0 under the quotient map, and continue inductively. The slopes $\mu_i := \mu(E_i/E_{i-1})$ gives us n rational numbers. If k denotes the rank of E, then we construct an element of $\mathbb{Q}^k$ by arranging the μ_i in order, and repeating the entry μ_i a total of $\operatorname{rank}(E_i/E_{i-1})$ times. We call this vector the ***Harder-Narasimhan type*** of E.

Our ultimate goal will be to relate moduli spaces of holomorphic vector bundles over X to Yang-Mills connections. To see this, let $E \to X$ be a C^∞ complex vector bundle of rank n, and fix a Hermitian metric on E. Then let $P \to X$ denote the principal U_n-bundle of frames for E. We abbreviate the gauge group $\mathscr{G}(P)$ as $\mathscr{G}$.

Proposition 3.8. *There is a bijection $\mathscr{A}(P) \leftrightarrow \mathscr{C}(E)$.*

Proof. We provide maps in both directions. Suppose we have a connection $A \in \mathscr{A}(P)$. Then A induces a covariant derivative $d_A : \mathcal{A}^0_X(E) \to \mathcal{A}^1_X(E)$. The $(0,1)$ part of d_A automatically satisfies $(d_A^{0,1})^2 = 0$, since $\mathcal{A}^2_X = 0$ by dimension reasons. Therefore, $d_A^{0,1}$ defines a holomorphic structure on E.

In the other direction, given a holomorphic structure $\overline{\partial}_E$, there exists a unique Hermitian connection A such that $d_A^{1,0} = \overline{\partial}_E$ called the ***Chern connection***, which is a sort of analogue to the Levi-Civita connection in Riemannian geometry. ■

Let $\mathscr{G}_{\mathbb{C}}$ denote the group of smooth bundle automorphisms of E. Though both $\mathscr{G}_{\mathbb{C}}$ and $\mathscr{G}$ are both infinite dimensional, the former can be seen as the complexification of the latter. The space $\mathscr{C}(E)$ has a natural action by $\mathscr{G}_{\mathbb{C}}$ by conjugation. Furthermore, the orbits under this action are exactly the isomorphism classes of holomorphic structures on E. This is most easily seen by characterizing an isomorphism $\varphi : E \to F$ of holomorphic vector bundles as a smooth bundle isomorphism intertwining $\overline{\partial}_E$ and $\overline{\partial}_F$. However, the naïve quotient $\mathscr{C}(E)/\mathscr{G}_{\mathbb{C}}$ is poorly behaved (for example, it is not Hausdorff). To remedy this, we restrict our attention to semistable bundles.

The relationship between $\mathscr{G}_{\mathbb{C}}$ and $\mathscr{G}$ as well as the identification of $\mathscr{A}(P)$ and $\mathscr{C}(E)$ suggests that isomorphism classes of holomorphic bundles should have something to do with gauge equivalence classes of connections on P. This is turns out to be true, and is an infinite dimensional version of the relationship between a GIT quotient and a symplectic quotient. To investigate further, we make a short digression regarding this relationship.

Let G be a reductive complex group, and X a Kähler manifold with Kähler metric ω, equipped with a "nice" action of G. In the usual setting, X is a smooth projective variety with a fixed embedding $X \hookrightarrow \mathbb{CP}^N$, the Kähler metric ω is the restriction of the Fubini-Study form, and the G-action is induced by a homomorphism $G \to \mathrm{GL}_{N+1}(\mathbb{C})$. In general, the naïve quotient X/G is not well behaved, and one restricts the action to a subset X_{ss} consisting of ***semistable points*** to construct the ***GIT quotient*** X_{ss}/G.

Then let $K \subset G$ denote the maximal compact subgroup, which has the property that its complexification is isomorphic to G. Suppose that the action of K on X is symplectic, i.e. the action of any $k \in K$ preserves the Kähler metric on X. Let $\mathfrak{k}$ denote the Lie algebra of K. Then the infinitesimal action of K is given by the Lie algebra homomorphism $\mathfrak{k} \to \mathfrak{X}(X)$ (where $\mathfrak{X}(X)$ denotes the space of vector fields on X) defined by $\xi \mapsto X_\xi$ where

$$(X_\xi)_p := \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi) \cdot p$$

Definition 3.9. A symplectic action of K on X is ***Hamiltonian*** if for each $\xi \in \mathfrak{k}$, there exists a function $H_\xi : X \to \mathbb{R}$ such that for all $p \in X$ and $v \in T_pX$, we have

$$\omega_p((X_\xi)_p, v) = (dH_\xi)_p(v)$$

and the mapping $\xi \mapsto H_\xi$ is K-equivariant with respect to the right actions of K on $\mathfrak{k}$ by the Adjoint action and precomposition with left translation L_k on $C^\infty(X)$. The functions H_ξ are called ***Hamiltonian functions***.

Definition 3.10. Suppose we have a Hamiltonian action of K on X. A ***moment map*** for the action is a K-equivariant map $X \to \mathfrak{k}^*$ (where the action on $\mathfrak{k}$ is the coadjoint action) such that for any $p \in X$, $v \in T_pX$, and $\xi \in \mathfrak{k}$, we have

$$d\mu_p(v)(\xi) = \omega_p((X_\xi)_p, v)$$

One things to note is that the Hamiltonian functions can be recovered by the moment maps. If a Hamiltonian action admits a moment map, then

$$H_\xi(p) = \mu(p)(\xi)$$

The let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathfrak{k}^*$ that is invariant under the coadjoint action, and $\|\cdot\|$ the induced norm. Since X is compact, the map $\|\mu\|^2 : X \to \mathbb{R}$ attains its minimum, and WLOG we assume that the minimum value is 0.

Definition 3.11. The ***symplectic quotient*** of X by K is the quotient space

$$\mu^{-1}(0)/K$$

The symplectic quotient can also be referred to as the ***symplectic reduction*** or the ***Marsden-Weinstein quotient***.

Theorem 3.12. *The symplectic quotient of X by K admits a unique Kähler structure such that the Kähler metric on $\mu^{-1}(0)/K$ is induced by the Kähler metric on X.*

The relationship between the GIT quotient and the symplectic quotient is given by the Kempf-Ness theorem.

Theorem 3.13 (***Kempf-Ness***)**.** *Suppose a complex reductive group G acts on a Kähler manifold X such that the action of the maximal compact subgroup $K \subset G$ is Hamiltonian and admits a moment map $\mu : X \to \mathfrak{k}^*$. Then the G-orbit of any semistable point contains a unique K-orbit minimizing $\|\mu\|^2$. This establishes a homeomorphism*

$$X_{ss}/G \longleftrightarrow \mu^{-1}(0)/K$$

We now want to relate the previous discussion to our situation. Using the identification of $\mathscr{A}(P)$ and $\mathscr{C}(E)$, we want the action of $\mathscr{G}_{\mathbb{C}}$ to play the role of the complex reductive group G and the gauge group $\mathscr{G}$ to play the role of the maximal compact subgroup. Since the space $\mathscr{A}(P)$ is infinite dimensional, along with the groups $\mathscr{G}_{\mathbb{C}}$ and $\mathscr{G}$, we are working in an infinite dimensional setting, but we will gloss over the analytic details and work with them formally.

Our first task is to realize $\mathscr{A}(P)$ as a "Kähler manifold." Since X is a surface, the Hodge star maps $\mathcal{A}^1_X(\mathfrak{g}_P)$ to itself and squares to -1, so it defines a "complex structure" on $\mathscr{A}(P)$, where we use the fact that $\mathscr{A}(P)$ is affine over the vector space $\mathcal{A}^1_X(\mathfrak{g}_P)$ to identify the "tangent space" of $\mathscr{A}(P)$ at a connection A with $\mathcal{A}^1_X(\mathfrak{g}_P)$. Furthermore, the fact that for 1-forms $\omega, \eta \in \mathcal{A}^1_X(\mathfrak{g}_P)$ the pairing $\langle \omega \wedge \eta \rangle$ is skew-symmetric, we can identify the pairing

$$\omega \otimes \eta \mapsto \int_X \langle \omega \wedge \eta \rangle$$

as a "symplectic form" on $\mathscr{A}(P)$. Together, these give $\mathscr{A}(P)$ the structure of a "Kähler manifold."

Our next task is to show that the action of $\mathscr{G}$ on $\mathscr{A}(P)$ is "Hamiltonian" with respect to this Kähler structure. One can identify the "Lie algebra" of $\mathscr{G}$ with the space of sections $\Gamma(X, \mathfrak{g}_P)$.

Proposition 3.14. *The infinitesimal action of $\phi \in \Gamma(X, \mathfrak{g}_P)$ on $\mathscr{A}(P)$ is given by the mapping $A \mapsto d_A\phi$.*

Proof. We compute the vector field at a connection $A \in \mathscr{A}(P)$ to be

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_{\exp(t\phi)^{-1}} A + \exp(t\phi)^*\theta &= -[\phi, A] + \frac{d}{dt}\bigg|_{t=0} (dL_{\exp(-t\phi)} d(\exp(t\phi))) \\
&= [A, \phi] + \left(\frac{d}{dt}\bigg|_{t=0} dL_{\exp(-t\phi)}\right) d(\exp(0)) + dL_{\exp(0)} \left(\frac{d}{dt}\bigg|_{t=0} d(\exp(t\phi))\right) \\
&= [A, \phi] + d\phi \\
&= d_A\phi
\end{aligned}$$

where for the third equality we use the product rule, and in the fourth equality we use the fact that $\exp(0) = \mathrm{id}$ and that the derivative of $\exp(t\phi)$ as $t \to 0$ is ϕ. ∎

Proposition 3.15. *Let $\phi \in \Gamma(X, \mathfrak{g}_P)$. Then the function*

$$\begin{aligned}
H_\phi : \mathscr{A}(P) &\to \mathbb{R} \\
A &\mapsto \int_X \langle F_A \wedge \phi \rangle
\end{aligned}$$

is a Hamiltonian function for ϕ.

Exercise 3.16. Prove the previous proposition.

Since $\langle \cdot, \cdot \rangle$ is invariant under the adjoint action, the mapping $\phi \mapsto H_\phi$ is clearly $\mathscr{G}$ equivariant, so this tells us that the action is Hamiltonian. Furthermore, the computation we made identifies the mapping $A \mapsto F_A$ as the moment map for this action. To summarize, we have the following analogies

$$\begin{aligned}
\text{Kähler manifold } X &\longleftrightarrow \mathscr{A}(P) \\
\text{Complex reductive group } G &\longleftrightarrow \mathscr{G}_{\mathbb{C}} \\
\text{Maximal compact subgroup } K \subset G &\longleftrightarrow \mathscr{G} \\
\text{Moment map } \mu &\longleftrightarrow A \mapsto F_A \\
\text{Norm square of the moment map } \|\mu\|^2 &\longleftrightarrow L
\end{aligned}$$

The last missing piece is something analogous to the Kempf-Ness theorem.

Theorem 3.17 (***Narasimhan-Seshadri***)**.** *Let $\mathscr{A}_s(P) \subset \mathscr{A}(P)$ denote the subspace of connections that are absolute minimal for the Yang-Mills functional, and correspond to irreducible representations $\Gamma_{\mathbb{R}} \to \mathrm{U}_n$. Let $\mathscr{C}_s(E)$ denote the subspace of stable holomorphic structures on E. The isomorphism classes of holomorphic bundles in $\mathscr{C}_s(E)$ admit unique Yang-Mills connections (up to gauge equivalence) minimizing the Yang-Mills functional. In other words, there is a homeomorphism*

$$\mathscr{A}_s(P)/\mathscr{G} \longleftrightarrow \mathscr{C}_s(E)/\mathscr{G}_{\mathbb{C}}$$

Remark. The original proof is more algebraic in flavor. A proof more in the spirit of the Atiyah-Bott paper was given by Donaldson in [2]. The spirit of this proof is carried on by the proof of Hermitian-Yang-Mills and the nonabelian Hodge theorem, which were both grew out of the developments from the Atiyah-Bott paper.

One issue is that the Narasimhan-Seshadri theorem only works for stable bundles. However, in the case that the rank and degree of E are coprime, stability and semistability coincide for numerical reasons. Our computations for the cohomology of the moduli space of holomorphic bundles of rank n and degree k will include the assumption that the rank and degree of E are coprime.

In the finite dimensional case, when things are sufficiently, nice, the function $f = \|\mu\|^2$ is an equivariant Morse-Bott function, which gives a stratification of the space X. Using Mayer-Vietoris, along with some other algebraic topology and equivariant cohomology, one can use this stratification to compute the equivariant cohomology of the space X with respect to the equivariant cohomology of the strata. When the G action is free, this tell us the regular cohomology of the quotient space X/G. In our situation, the Yang-Mills functional plays the role of the norm-squared of the moment map, and one would hope that the analysis needed to do equivariant Morse theory isn't too hard. Unfortunately, the analysis is *very* hard. However, not all is lost. If we can find a nice stratification of $\mathscr{C}(E) = \mathscr{A}(P)$ that *looks like* it came from a nice equivariant Morse function, then we can still do the cohomology computations. In our case, we have a candidate for such a stratification. For a fixed Harder-Narasimhan type μ, let $\mathscr{C}_\mu(E)$ denote the subspace of holomorphic structures on E whose Harder-Narasimhan filtration has type μ. This subspace is preserved by the action of $\mathscr{G}_\mathbb{C}$, and together all these subspaces give a stratification of the space $\mathscr{C}(E)$ called the ***Harder-Narasimhan stratification***.

4. Equivariant Cohomology

References for this material include [8] and [3].

Our goal now is to compute the cohomology of the moduli space of semistable holomorphic bundles over X of rank n and degree k, denoted $N(n,k)$, in the case that n and k are coprime. Recall that $N(n,k)$ has a global quotient description as $N(n,k) = \mathscr{C}_{ss}(E)/\mathscr{G}_\mathbb{C}$. The computation of the cohomology of $N(n,k)$ uses the concept of ***equivariant cohomology***, which is a cohomology theory for spaces with the action of a group. Before doing so, we give a bit of exposition on where equivariant cohomology fits into our story.

Recall that we mentioned that a U_n-Yang-Mills connection is equivalent to the choice of some Hermitian matrix, with some conditions on the eigenvalues. Using the correspondence between unitary connections and holomorphic structures, these conditions actually reflect the Harder-Narasimhan filtration of the corresponding holomorphic bundle. Using the perspective of unitary connections, Atiyah and Bott managed to show that the Harder-Narasimhan stratification is ***equivariantly perfect***, i.e. looks like it came from an equivariantly perfect Morse function. In other words, we can understand the equivariant cohomology of $\mathscr{C}(E)$ as being built from a simple formula involving the equivariant cohomology of the strata $\mathscr{C}_\mu(E)$. This gives us a formula for the equivariant cohomology of the semistable strata, from which we can compute the ordinary cohomology of the quotient $N(n,k)$.

Definition 4.1. Let G be a Lie group. A ***classifying space*** for G is a topological space BG such that we have a functorial correspondence

$$\{\text{Principal } G\text{-bundles } P \to X\} \longleftrightarrow \{\text{Homotopy classes of maps } X \to BG\}$$

Suppose EG is a contractible space with a free action of G. Then the quotient map $EG \to EG/G$ is a principal G-bundle.

Theorem 4.2. *The quotient space $BG = EG/G$ is a classifying space for G, which assigns to a homotopy class of maps $[f] : X \to BG$ the principal G-bundle f^*EG, which is independent of our choice of representative of f. Furthermore, The spaces EG and EG/G are unique up to homotopy equivalence.*

Since the spaces EG and BG are only well-defined up to homotopy equivalence, we will call a specific choice of EG and BG a ***model*** for $EG \to BG$.

Definition 4.3. Let G be a Lie group and X a smooth manfold with an action of G. The ***equivariant cohomology*** of X, denoted $H^\bullet_G(X, \mathbb{Z})$, is the ordinary cohomology of the total space of the associated bundle $EG \times_G X \to BG$, i.e.

$$H^\bullet_G(X, \mathbb{Z}) := H^\bullet(EG \times_G X, \mathbb{Z})$$

We note that since any two models for $EG \to BG$ are homotopy equivalent, this is well defined. In addition, we can define the equivariant cohomology over any other coefficient group we want. Intuitively, the equivariant cohomology of X is is something like the regular cohomology of the quotient space X/G, but this isn'texactly true when the action isn't nice. The difference is essentially that the equivariant cohomology keeps track of stabilizer subgroups. To see this, note that we have a map $EG \times_G X \to X/G$ taking an equivalence class $[p, x] \in EG \times_G X$ to the orbit $[x] \in X/G$.

Proposition 4.4. *The fiber of $EG \times_G X \to X/G$ over $[x]$ is the quotient space EG/G_x, where G_x is the stabilizer of any representative of $[x]$.*

Proof. The fiber over $[x]$ is the subset $\{[p, x \cdot g] \ : \ g \in G\} \subset EG \times_G X$. equivalently, this is the subset $\{[p \cdot g^{-1}, x] \ : \ g \in G\}$. Using representatives of this form then gives the desired result. ∎

We note that since EG is contractible and the action of $G_x \subset G$ is free on EG, the fiber over $[x]$ is a model for BG_x.

Corollary 4.5. *If the action of G on X is free, then the map $EG \times_G X \to X/G$ is a homotopy equivalence.*

Corollary 4.6. *If X is contractible, then $H^\bullet_G(X, \mathbb{Z}) = H^\bullet(BG, \mathbb{Z})$.*

Proof. The map $EG \times_G X \to X/G$ is a fiber bundle with contractible fiber EG. ∎

We now return to the computation of the cohomology of $N(n, k)$. The heavy lifting for this computation comes from the following theorems:

Theorem 4.7. *Let $E \to X$ be a holomorphic vector bundle with Harder-Narasimhan filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_r = E$$

Choose a C^∞ splitting of the Harder-Narasimhan filtration, giving us a direct sum decomposition as C^∞ bundles

$$E = \bigoplus_i D_i$$

such that

$$E_i = \bigoplus_{j<i} D_j$$

Then we have

$$H^\bullet_{\mathscr{G}_\mathbb{C}}(\mathscr{C}_\mu(E),\mathbb{Q}) \cong \bigotimes_{i=1}^r H^\bullet_{\mathscr{G}_\mathbb{C}(D_i)}(\mathscr{C}_{ss}(D_i),\mathbb{Q})$$

where $\mathscr{G}_\mathbb{C}(D_i)$ denotes the group of smoot bundle automorphisms of D_i.

Theorem 4.8. *Let k_μ denote the real codimension of $\mathscr{C}_\mu(E)$ inside of $\mathscr{C}(E)$, and let $P_{t,\mathscr{G}_\mathbb{C}}(X)$ denote the $\mathscr{G}_\mathbb{C}$-equivariant Poincaré series for a space X, i.e.*

$$P_{t,\mathscr{G}_\mathbb{C}}(X) := \sum_i (\dim H^\bullet_{\mathscr{G}_\mathbb{C}}(X,\mathbb{Q}))t^i$$

Then we have

$$P_{t,\mathscr{G}_\mathbb{C}}(\mathscr{C}(E)) = \sum_\mu t^{k_\mu} P_{t,\mathscr{G}_\mathbb{C}}(\mathscr{C}_\mu(E))$$

Theorem 4.9. *The Poincaré seires for $B\mathscr{G}_\mathbb{C}$ is*

$$P_t(\mathscr{G}_\mathbb{C}) = \frac{\prod_{k=1}^n (1+t^{2k-1})^2 g}{(1-t^2 n)\prod_{k=1}^{n-1}(1-t^{2k})^2}$$

where g is the genus of X.

The first theorem tell us that we can understand the $\mathscr{G}_\mathbb{C}$-equivariant cohomology of the strata by understanding the $\mathscr{G}_\mathbb{C}$-equivariant cohomology of the semistable strata for lower dimensional bundles, which will give us an inductive procedure for computing the Poincaré series. The second theorem tells us that the equivariant cohomology of the entire space is a simple expression in terms of the equivariant cohomology of the strata. Since $\mathscr{C}(E)$ is contractible, the third theorem tells us that if we can compute the equivariant cohomology of all the strata except for the semistable locus, then we can compute the equivariant cohomology of the semistable strata.

We first compute the codimension of the strata $\mathscr{C}_\mu(E)$. To do this, we will use some facts regarding infinitesimal variations of holomorphic structures.

Proposition 4.10. *Let $E \to X$ be a holomorphic vector bundle. The infinitesimal variations of the holomorphic structure on E are given by the Dolbeault cohomology group $H^1_{\bar\partial}(X, \mathrm{End}(E))$.*

We are being purposefully vague when we say "infinitesimal variation of holomorphic structure." Our main use for the result is to identify the normal directions of the strata $\mathscr{C}_\mu(E)$. The isomorphism class of a holomorphic structure on a C^∞ vector bundle E is given by a $\mathscr{G}_\mathbb{C}$-orbit in $\mathscr{C}(E)$, and the infinitesimal variation can be interpreted as the normal directions to this orbit. This gives us a way to compute the codimension of the strata $\mathscr{C}_\mu(E)$. From this perspective, the normal directions to $\mathscr{C}_\mu(E)$ consist of infinitesimal variations that change the type of the Harder-Narasimhan filtration. Explicitly, we have a holomorphic subbundle $\mathrm{End}'(E) \to \mathrm{End}(E)$ consisting of holomorphic endomorphisms of E that preserve the Harder-Narasimhan filtration. Then we can identify $H^1_{\bar\partial}(X, \mathrm{End}'(E))$ with the infinitesimal varations consisting of the directions tangent to $\mathscr{C}_\mu(E)$. Furthermore, if we let $\mathrm{End}''(E)$ denote the quotient bundle $\mathrm{End}(E)/\mathrm{End}'(E)$, we can identify $\mathrm{End}''(E)$ with the holomorphic bundle endomorphism that do not preserve the Harder-Narasimhan filtration, which tells us that the complex codimension of $\mathscr{C}_\mu(E)$ in $\mathscr{C}(E)$ is the dimension of $H^1_{\bar\partial}(X, \mathrm{End}''(E))$. To compute this, we use Riemann-Roch.

Theorem 4.11 (***Riemann-Roch***). *Let $E \to X$ be a holomorphic vector bundle, where X is genus g, and let $h^i(E) = \dim H^i_{\bar\partial}(X, E)$. Then*

$$h^0(E) - h^1(E) = c_1(E) + (1-g)\mathrm{rank}(E)$$

Because of Riemann-Roch, it suffices to compute the dimension of $H^0_{\bar\partial}(X, E)$ to compute $H^1_{\bar\partial}(X, E)$, and we want to apply this to the holomorphic bundle $\mathrm{End}''(E)$.

Proposition 4.12.

$$H^0_{\bar\partial}(X, \mathrm{End}''(E)) = 0$$

Proof. An element $g \in H^0_{\bar\partial}(X, \mathrm{End}''(E))$ is a global holomorphic endomorphism of E that does not fix the Harder-Narasimhan filtration. By assuption, there exists some subbundle E_i with $i > 0$ in the filtration such that $g(E_i) \not\subset E_i$. By minimality of i, we have that $g(E_{i-1}) \subset E_{i-1}$. Then let k be the smallest integer such that $g(E_i) \subset E_k$. Then the restriction of g to E_k factors through the quotients to a nontrivial bundle homomorphism $E_i/E_{i-1} \to E_k/E_{k-1}$. We note that both E_i/E_{i-1} and E_k/E_{k-1} are semistable and satisfy $\mu(E_i/E_{i-1}) > \mu(E_k/E_{k-1})$ by the properties of the Harder-Narasimhan filtration. Let $K \subset E_i/E_{i-1}$ be the smallest holomorphic subbundle containing the kernel, and $A \subset E_k/E_{k-1}$ the smallest holomorphic subbundle containing the image, giving us the short exact sequence of holomorphic bundles

$$0 \longrightarrow K \longrightarrow E_i/E_{i-1} \longrightarrow A \longrightarrow 0$$

Semistability of E_i/E_{i-1} implies that $\mu(A) \le \mu(E_k/E_{k-1})$, so $\mu(A) < \mu(E_i/E_{i-1})$. However, semistability of E_i/E_{i-1} also implies that $\mu(K) \le \mu(E_i/E_{i-1})$, which would imply that $\mu(E_i/E_{i-1}) \le \mu(A)$, a contradiction. ■

To use Riemann-Roch, we must identify the rank and degree of $\mathrm{End}''(E)$. Since both of these quantities are topological invariants, we may work in the C^∞ category. We first compute the degree. Since $\mathrm{End}(E) \cong E^* \otimes E$, we have that $\deg(E) = 0$, where we use the fact that the degree of a bundle is the same as the degree of its determinant line, and the formula for the determinant line of a tensor product of bundles. Then since the degree is additive in exact sequences, we get

$$\deg(\mathrm{End}'(E)) + \deg(\mathrm{End}''(E)) = 0$$

We then compute $\deg(\mathrm{End}'(E))$, which will tell us $\deg(\mathrm{End}''(E))$. Fix a smooth splitting of the Harder-Narasimhan filtration, giving us a C^∞ decomposition

$$E = \bigoplus_i D_i$$

This gives us the identification as smooth bundles

$$\mathrm{End}'(E) = \bigoplus_{i \ge j} \mathrm{Hom}(D_i, D_j)$$

Then if we let $\mu(D_i) = k_i/n)i$, we get

$$\deg(\mathrm{End}'(E)) = \sum_{i \ge j} k_j n_i - k_i n_j$$

where we use additivity of degree with respect to direct sums and the identification of $\mathrm{Hom}(D_i, D_j)$ with $D_i^* \otimes D_j$. In the case $i = j$, we get $\mathrm{Hom}(D_i, D_j) = \mathrm{End}(D_i)$, which has degree 0, so we get

$$\deg(\mathrm{End}'(E)) = \sum_{i>j} k_j n_i - k_i n_j$$

Negating this gives the degree of $\mathrm{End}''(E)$.

For the rank, this comes easily from the C^∞ decomposition

$$\mathrm{End}''(E) = \bigoplus_{i>j} \mathrm{Hom}(D_i, D_j)$$

giving us

$$\mathrm{rank}(\mathrm{End}''(E)) = \sum_{i<j} n_i n_j$$

Putting everything together gives us

$$\dim(H^1_{\overline{\partial}}(X, \mathrm{End}''(E))) = \sum_{i>j} n_i k_j - k_i n)j + n_i n_j (g-1)$$

which by our earlier discussion, is the complex codimension of the strata $\mathscr{C}_\mu(E)$.

5. The Cohomology of the Moduli Spaces $N(n,k)$

As before, we let $E \to X$ denote a C^∞ complex vector bundle of rank n and degree k, where n and k are coprime. Recall that this implies the notions of stability and semistability for a holomorphic structure on E coincide in this case.

The theorems in the previous section can be used to compute the equivariant cohomology $H^\bullet_{\mathscr{G}_\mathbb{C}}(\mathscr{C}_\mu(E), \mathbb{Z})$ using an inductive procedure involving the semistable strata for lower dimensional holomorphic bundle, which in turn lets us compute the $\mathscr{G}_\mathbb{C}$-equivariant cohomology of $\mathscr{C}_{ss}(E)$. However, this does not tell the cohomology of the quotient space $N(n,k) := \mathscr{C}_{ss}(E)/\mathscr{G}_\mathbb{C}$, since the action of $\mathscr{G}_\mathbb{C}$ on $\mathscr{C}_{ss}(E)$ isn't free. To compute the cohomology, we must pass to a quotient of $\mathscr{G}_\mathbb{C}$ that acts freely, and then compute the equivariant cohomology with respect to that group.

Proposition 5.1. *Let $\overline{\partial}_E$ be a stable holomorphic structure on E. Then the stabilizer subgroup of $\overline{\partial}_E$ is the central subgroup $C^\times \subset \mathscr{G}_\mathbb{C}$.*

Proof. Clearly $\mathbb{C}^\times$ is contained in the stabilizer of a holomorphic structure, so it suffices to show that any automorphism $g \in \mathscr{G}_\mathbb{C}$ fixing $\overline{\partial}_E$ is multiplication by an element of $\mathbb{C}^\times$. Since g fixes $\overline{\partial}_E$, we get a direct sum decomposition of E as a holomorphic bundle

$$E = E_1 \oplus \cdots \oplus E_\ell$$

where the E_i are eigenbundles of g. We then claim that this decomposition has only one term, which would verify our claim. Since $\overline{\partial}_E$ is stable, $\mu(E_1) < \mu(E)$. Similarly, we have that $\mu(E_2 \oplus \cdots \oplus E_\ell) < \mu(E)$. Furthermore, we have $E_1 \cong E/(E_2 \oplus \cdots \oplus E_k)$. The exact sequence

$$0 \longrightarrow E_2 \oplus \cdots \oplus E_\ell \longrightarrow E \longrightarrow E_1$$

then implies that the slope of E_1 is larger than the slope of E, a contradiction. ∎

This gives us

$$H^\bullet(N(n,k),\mathbb{Z}) = H^\bullet_{\overline{\mathscr{G}}_{\mathbb{C}}}(\mathscr{C}_{ss}(E),\mathbb{Z})$$

where $\overline{\mathscr{G}}_{\mathbb{C}} := \mathscr{G}_{\mathbb{C}}/\mathbb{C}^\times$. To compute the $\overline{\mathscr{G}}_{\mathbb{C}}$-equivariant cohomology, we use:

(1) $\mathscr{G}_{\mathbb{C}}$ deformation retracts onto $\mathscr{G}$.
(2) $\overline{\mathscr{G}}_{\mathbb{C}}$ deformations retracts onto $\overline{\mathscr{G}} := \mathscr{G}/\mathrm{U}_1$

The first point tells us that

$$H^\bullet_{\mathscr{G}_{\mathbb{C}}}(\mathscr{C}_{ss}(E),\mathbb{Z}) \cong H^\bullet_{\mathscr{G}}(\mathscr{C}_{ss}(E),\mathbb{Z})$$

The second point tells us that we may replace $\overline{\mathscr{G}}_{\mathbb{C}}$ with $\overline{\mathscr{G}}$ to compute $H^\bullet(N(n,k),\mathbb{Z})$. To do this, we must first understand the cohomology of the classifying space $B\overline{\mathscr{G}}$. We need the folliwing theorem:

Theorem 5.2 (***Leray-Hirsch***)**.** *Let $E \to X$ be a fiber bundle with model fiber F such that the inclusion $F \hookrightarrow E$ of a fiber induces a surjection in rational cohomology. Then*

$$H^\bullet(E,\mathbb{Q}) \cong H^\bullet(X,\mathbb{Q}) \otimes H^\bullet(F,\mathbb{Q})$$

To use this, we use a functoriality property of classifying spaces. The exact sequence

$$1 \longrightarrow \mathrm{U}_1 \longrightarrow \mathscr{G} \longrightarrow \overline{\mathscr{G}} \longrightarrow 1$$

induces a fibration

$$\begin{tikzcd} B\mathrm{U}_1 \arrow[r] & B\mathscr{G} \arrow[d] \\ & B\overline{\mathscr{G}} \end{tikzcd}$$

To apply Leray-Hirsch, we want to show that the pullback map induced by $B\mathrm{U}_1 \to B\mathscr{G}$ induces a surjection

$$H^\bullet(B\mathscr{G},\mathbb{Q}) \to H^\bullet(B\mathrm{U}_1,\mathbb{Q})$$

To do this, we provide a group homomorphism $\overline{\mathscr{G}} \to \mathrm{U}_1$, such that the composition $\mathrm{U}_1 \hookrightarrow \mathscr{G} \to \mathrm{U}_1$ give maps of classifying spaces $B\mathrm{U}_1 \to B\mathscr{G} \to B\mathrm{U}_1$ inducing an isomorphism $H^\bullet(B\mathrm{U}_1,\mathbb{Q}) \to H^\bullet(B\mathrm{U}_1,\mathbb{Q})$. Fix a point $x \in X$, and let $g \in \mathscr{G}$, which we interpret as a smooth bundle automorphism of E preserving the Hermitian metric. Restricting g to the fiber E_x and taking the derterminant gives us our group homomorphism $\mathscr{G} \to \mathrm{U}_1$. Since E is rank n, this is a degree n map. Then pullback induced map $B\mathrm{U}_1 \to B\mathscr{G} \to B\mathrm{U}_1$ multiplies the generator of $H^\bullet(B\mathrm{U}_1,\mathbb{Q}) \cong \mathbb{Q}[x]$ by n, which is an isomorphism. Therefore, the map $B\mathrm{U}_1 \to B\mathscr{G}$ induces an isomorphism on rational cohomology. We then note that the Poincaré series for $B\mathrm{U}_1$ is

$$P_t(B\mathrm{U}_1) = \frac{1}{1-t^2} = 1 + t^2 + t^4 + \cdots$$

So an application of Leray-Hirsch gives us

$$P_t(B\overline{\mathscr{G}}) = P_t(\mathscr{G})(1-t^2)$$

The next thing to do is to investigate the relationship between $\mathscr{G}$-equivariant cohomology and $\overline{\mathscr{G}}$-equivariant cohomology. Let M be any $\overline{\mathscr{G}}$-space. The quotient map $\mathscr{G} \to \overline{\mathscr{G}}$ gives

M the structure of a $\mathscr{G}$-space. Furthermore, it induces a map $B\mathscr{G} \to B\overline{\mathscr{G}}$, giving us the pullback diagram

$$\begin{tikzcd} E\mathscr{G} \times_{\mathscr{G}} M \arrow[r] \arrow[d] & E\overline{\mathscr{G}} \times_{\overline{\mathscr{G}}} M \arrow[d] \\ B\mathscr{G} \arrow[r] & B\overline{\mathscr{G}} \end{tikzcd}$$

The map $E\mathscr{G} \times_{\mathscr{G}} M \to E\overline{\mathscr{G}} \times_{\overline{\mathscr{G}}} M$ is a $B\mathrm{U}_1$-bundle, giving us the diagram

$$\begin{tikzcd} B\mathrm{U}_1 \arrow[r] \arrow[d, equal] & E\mathscr{G} \times_{\mathscr{G}} M \arrow[r] \arrow[d] & E\overline{\mathscr{G}} \times_{\overline{\mathscr{G}}} M \arrow[d] \\ B\mathrm{U}_1 \arrow[r] & B\mathscr{G} \arrow[r] & B\overline{\mathscr{G}} \end{tikzcd}$$

From this, we can deduce that the map $B\mathrm{U}_1 \to E\mathscr{G} \times_{\mathscr{G}} M$ induces a surjection on rational cohomology, so we can apply Leray-Hirsch to the bundle $E\mathscr{G} \times_{\mathscr{G}} M \to E\overline{\mathscr{G}} \times_{\overline{\mathscr{G}}} M$, giving us

$$H^\bullet_{\mathscr{G}}(M,\mathbb{Q}) \cong H^\bullet(B\mathrm{U}_1,\mathbb{Q}) \otimes H^\bullet_{\overline{\mathscr{G}}}(M,\mathbb{Q})$$

In terms of Poincaré series, we have

$$P_{t,\mathscr{G}}(M) = \frac{P_{t,\overline{\mathscr{G}}}(M)}{1-t^2}$$

In our specific case, letting $M = \mathscr{C}_{ss}(E)$, we get

$$P_t(N(n,k)) = (1-t^2)P_{t,\mathscr{G}}(\mathscr{C}_{ss}(E))$$

In theory, this gives us all the results we need to compute the Ponicaré series for $N(n,k)$. However, it is not immediately clear how the pieces fit together. To get a better idea, we will worth through the case $n = 2$ and $k = 1$. We first take inventory of the facts and formulas we need.

(1) The Poincaré polynomial for the classifying space of the gauge group is

$$P_t(B\mathscr{G}) = \frac{(1+t)^{2g}(1+t^3)^{2g}}{(1-t^4)(1-t^2)^2}$$

(2) Let k_μ denote the real codimension of the strata $\mathscr{C}_\mu(E)$ inside of $\mathscr{C}(E)$. Then

$$P_t(B\mathscr{G}) = \sum_\mu t^{k_\mu} P_{\mathscr{G}}(\mathscr{C}_\mu(E))$$

(3) Let the D_i be the succesive quotients coming from the Harder-Narasimhan filtration of E. Then

$$P_{t,\mathscr{G}}(\mathscr{C}_\mu(E)) = \prod_i P_{t,\mathscr{G}(D_i)}(\mathscr{C}_{ss}(D_i))$$

(4) Let $n_i = \dim D_i$ and $k_i = \deg(D_i)$. Then the codimension of the strata $\mathscr{C}_\mu(E)$ is given by

$$2\sum_{i>j} n_i k_j - k_i n)j + n_i n_j (g-1)$$

We now identify the possible Harder-Narasimhan types for a holomorphic strcture on E. If E is a semistable bundle, then its Harder-Narasimhan filtration is just $0 \subset E$, and the Harder-Narasimhan type is $(1/2, 1/2)$. Otherwise, there exists an rank one subbundle $L \subset E$ with $\mu(L) > \mu(E) = 1/2$, and the Harder-Narasimhan filration is $0 \subset L \subset E$. This means that the Harder-Narasimhan type of E is entirely determined by the degree of L, since we can recover the degree of E/L as $1 - \deg(L)$, so the Harder-Narasimhan type would be $(\deg(L), 1 - \deg(L))$. For notational convenience, let $\mathscr{C}_r(E)$ denote the stratum corresponding to the type $(r+1, -r)$. Then we have

$$P_{t,\mathscr{G}}(\mathscr{C}_r(E)) = P_{t,\mathscr{G}(L)}(\mathscr{C}_{ss}(L)) P_{t,\mathscr{G}(E/L)}(\mathscr{C}_{ss}(E/L))$$

We note that both L and E/L are both line bundles, which are automatically stable, so $\mathscr{C}_{ss}(L) = \mathscr{C}(L)$ and $\mathscr{C}_{ss}(E/L) = \mathscr{C}(E/L)$. Furthmore, our formula for the Poincaré series for the classifying space for the gauge group of a line bundle gives us

$$P_t(B\mathscr{G}(L)) = P_t(B\mathscr{G}(E/L)) = \frac{(1+t)^{2g}}{1-t^2}$$

Therefore, we get

$$P_{t,\mathscr{G}}(\mathscr{C}_r(E)) = \left(\frac{(1+t)^{2g}}{1-t^2} \right)^2$$

We now need to compute the codimensions k_r of the strata $\mathscr{C}_r(E)$. Using the formula we derived earlier, we have

$$k_r = 4r + 2g$$

Putting everything together, we get the following identity

$$\frac{(1+t)^{2g}(1+t^3)^{2g}}{(1-t^4)(1-t^2)^2} = P_{t,\mathscr{G}}(\mathscr{C}_{ss}(E)) + \sum_{r=0}^{\infty} t^{4r+2g} \left(\frac{(1+t)^{2g}}{1-t^2} \right)^2$$

After some manipulations and rearranging, this becomes

$$P_{t,\mathscr{G}}(\mathscr{C}_{ss})(E) = \frac{(1+t)^{2g}(1+t^3)^{2g} - t^{2g}(1+t)^{4g}}{(1-t^4)(1-t^2)^2}$$

Finally, using the relationship between $\mathscr{G}$-equivariant cohomology and $\overline{\mathscr{G}}$-equivariant cohomology, we get

$$\begin{aligned} P_t(N(2,1)) &= (1-t^2) P_{t,\mathscr{G}}(\mathscr{C}_{ss}(E)) \\ &= \frac{(1+t)^{2g}(1+t^3)^{2g} - t^{2g}(1-t)^{4g}}{(1-t^4)(1-t^2)} \end{aligned}$$

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