YANG-MILLS MINICOURSE NOTES

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Contents

Notation and Conventions

We use Einstein summation notation, i.e. indices that appear on the top and bottom of an expression are implicitly summed over. For example

$$a_i dx^i = \sum_i a_i dx^i$$

For a smooth manifold X, we let Ω^k_X denote the space of differential k-forms.

1. Principal Bundles and Connections

Good references for this material would be [2] and [3]

Fix a compact manifold X and a Lie group G.

Definition 1.1. A ***principal*** G***-bundle*** is a fiber bundle $\pi : P \to M$ with a smooth right G action such that:

(1) The action of G preserves the fibers of π, and gives each fiber $P_x := \pi^{-1}(x)$ the structure of a ***right*** G***-torsor***, i.e. the action of G on P_x is free and transitive.
(2) For every point $x \in X$, there exists a ***local trivialization*** of P, i.e. a diffeomorphism $\varphi : P|_U := \pi^{-1}(U) \to U \times G$ that is G-equivariant (where the action on $U \times G$ is right multiplication with the second factor) and the following diagram commutes:

$$\begin{tikzcd} P|_U \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times G \arrow[dl] \\ & U & \end{tikzcd}$$

where the map $U \times G \to U$ is projection onto the first factor.

Example 1.2. Let $E \to X$ be a real vector bundle of rank k. For $x \in X$, let $\mathcal{B}_x$ denote the set of all bases of the fiber E_x, i.e. the set of linear isomorphisms $\mathbb{R}^k \to E_x$. This has a natural right action of $\mathrm{GL}_k\mathbb{R}$ by precomposition. Furthermore, this action is free and transitive, giving $\mathcal{B}_x$ the structure of a $\mathrm{GL}_k\mathbb{R}$-torsor. Then let

$$\mathcal{B}_{\mathrm{GL}_k\mathbb{R}}(E) := \coprod_{x\in X} \mathcal{B}_x$$

Using local trivializations of the vector bundle E, we equip $\mathcal{B}_{\mathrm{GL}_k\mathbb{R}}(E)$ with the structure of a smooth manifold such that the map $\pi : \mathcal{B}_{\mathrm{GL}_k\mathbb{R}}(E) \to X$ taking $\mathcal{B}_x$ to x is a submersion. This gives $\pi : \mathcal{B}_{GL_k\mathbb{R}} \to X$ the structure of a principal $\mathrm{GL}_k\mathbb{R}$-bundle, called the ***frame bundle*** of E, where the local trivializations are defined in terms of local trivializations of E.

Example 1.3. Let $E \to X$ be a rank k vector bundle equipped with a fiber metric, i.e. a smoothly varying inner product on the fibers E_x. Then the ***orthonormal frame bundle*** of E, denoted $\mathcal{B}_{\mathrm{O}}(E)$, is the principal O_k-bundle where the fiber over $x \in X$ is the O_k-torsor of orthonormal bases for E_x.

A near identical story holds for complex vector bundles – from any complex vector bundle we get a principal $\mathrm{GL}_k\mathbb{C}$-bundle of frames, and if we fix a Hermitian fiber metric, we get a principal U_k-bundle of orthonormal frames.

Principal bundles can be thought bundles of symmetries of some other fiber bundle, which can be made precise using the notion of an associated bundle, which allows one to construct fiber bundles out of principal bundles.

Definition 1.4. Let $P \to X$ be a principal G-bundle, and let F be a smooth manifold with right G action. The ***associated fiber bundle***, denoted $P \times_G F$ (sometimes denoted $P \times^G F$) is the space

$$P \times_G F := (P \times F)/G$$

where the right G-action on F is the diagonal action, i.e. $(p, f) \cdot g = (p \cdot g, f \cdot g)$.

If instead we have a left G-action on F, we can turn it into a right action by defining $f \cdot g := g^{-1} \cdot f$. As the name suggests, $P \times_G F$ is a fiber bundle.

Exercise 1.5. Let $\pi : P \to X$ a principal bundle. Given a smooth right action of G on F, use local trivializations of $P \to X$ to show that the map taking an equivalence class $[p, f]$ to $\pi(p)$ gives $P \times_G F$ the structure of a fiber bundle over X with model fiber F.

In the case that the model fiber is a vector space V, and the action is linear, the associated bundle $P \times_G V$ is a vector bundle.

Exercise 1.6.

(1) Let $E \to X$ be a rank k vector bundle, and $\mathcal{B}_{\mathrm{GL}_k\mathbb{R}}(E)$ be its $\mathrm{GL}_k\mathbb{R}$-bundle of frames. Let $\rho : \mathrm{GL}_k\mathbb{R} \to \mathrm{GL}_k\mathbb{R}$ be the defining representation (i.e. the identity map). Show that the associated bundle $\mathcal{B}_{\mathrm{GL}_k\mathbb{R}}(E) \times_{\mathrm{GL}_k\mathbb{R}} \mathbb{R}^k$ is isomorphic to E.

(2) Further suppose that E comes equipped with a fiber metric, and let $\mathcal{B}_{\mathrm{O}}(E)$ be its orthonormal frame bundle. The associated bundle $\mathcal{B}_{\mathrm{O}}(E) \times_{\mathrm{O}_k} \mathbb{R}^k$ is isomorphic to E by a near identical proof as the previous part. How can one recover the fiber metric?

(3) Let $\rho^* : \mathrm{GL}_k\mathbb{R} \to \mathrm{GL}_k\mathbb{R}$ denote the dual representation of the defining representation ρ, so $\rho(A) = (A^{-1})^T$. Show that the associated bundle is isomorphic to the dual bundle E^*. In particular, this should illuminate the distinction between the tangent and cotangent bundles.

Example 1.7. There are two important examples of associated bundles that we'll need to discuss the Yang-Mills equations.

(1) The bundle $\mathrm{Ad}\, P := P \times_G G$ where G acts on G by conjugation.
(2) The bundle $\mathrm{ad}\, P := P \times_G \mathfrak{g}$ (also denoted $\mathfrak{g}_P$) where the action is the adjoint action.

Confusingly, the former is sometimes called the "Adjoint bundle" and the latter is sometimes called the "adjoint bundle," which makes it admittedly hard to distuinguish between them when speaking.

Associated bundles have another nice feature – their sections have a nice interpretation in terms of G-equivariant maps.

Proposition 1.8. *Let $E = P \times_G F$ be an associated fiber bundle, and let $\Gamma(X, E)$ denote the space of global sections, i.e. the space of smooth maps $f : X \to E$ such that $\pi \circ f = \mathrm{id}_X$, where $\pi : E \to X$ denotes the projection map. Then there is a bijective correspondence*

$$\Gamma(X, E) \longleftrightarrow \{G - \textit{equivariant maps } P \to F\}$$

Proof. Let $\sigma : X \to E$ be a section. Then define the map $\widetilde{\sigma} : P \to F$ as follows: for $x \in X$, let (p, f) be a representative for $\sigma(x)$. Then define $\widetilde{\sigma}(x) := f$.

In the other direction, let $\widetilde{\varphi} : P \to F$ be an equivariant map. Then define the section $\varphi : X \to E$ by $\varphi(x) = [p, \widetilde{\varphi}(p)]$ for any choice of $p \in P_x$. ■

Exercise 1.9. Verify that the map φ defined above is well-defined. Verify the two constructions above are inverses to each other.

The main takeaway from the proposition is the motto that "G-equivariant objects on P descend to objects on X."

Before we discuss connections on principal bundles, we introduce the concept of vector bundled valued forms.

Definition 1.10. Let $E \to X$ be a vector bundle. An ***E-valued differential k-form*** is a section of $\Lambda^k T^*X \otimes E$. We denote the space of E-valued k-forms by $\Omega^k_X(E)$. If V is a fixed vector space, a ***V-valued differential k-form*** is a $X \times V$-valued k-form, and we let $\Omega^k_X(V)$ denote the space of V-valued k-forms.

In a local frame $\{e_i\}$ for E, an E-valued k-form ω can be written uniquely as

$$\omega = \omega^i \otimes e_i$$

for k-forms $\omega^i \in \Omega^k_X$,so an E-valued k-form can be thought of as a vector of k-forms. We will usually omite the tensor symbol, and simply write $\omega = \omega^i e_i$. However, this does not transform tensorially with respect to coordinate changes on X unless E is a trivial bundle. The components of the vector transform tensorially with respect to coordinate changes, but the vector itself changes according to the transition functions of the vector bundle E.

Given E-valued forms $\omega \in \Omega^k_X(E)$ and $\eta \in \Omega^\ell_X(E)$, we define their wedge product in a local trivialization to be

$$\omega \wedge \eta := (\omega^i \wedge \eta^j) e_i \otimes e_j$$

which is an element of $\Omega^{k+\ell}_X(E \otimes E)$.

For the most part, we will be concerned with Lie algebra valued forms, which are just $\mathfrak{g}$-valued forms for a fixed Lie algebra $\mathfrak{g}$. These forms have some additional operations coming from the Lie algebra structure of $\mathfrak{g}$. Fix a basis $\{\xi_i\}$ for $\mathfrak{g}$. This determines a global trivialization of the trivial bundle $X \times \mathfrak{g}$, so any $\mathfrak{g}$-valued K-form $\omega \in \Omega^k_X(\mathfrak{g})$ can be uniquely written as $\omega = \omega^i \xi_i$. Let $\omega \in \Omega^k_X(\mathfrak{g})$ and $\eta \in \Omega^\ell_X(\mathfrak{g})$. Then define their bracket to be

$$[\omega \wedge \eta] := (\omega^i \wedge \eta^j)[\xi_i, \xi_j]$$

In other words, it is the composition

$$\Omega^k_X(\mathfrak{g}) \otimes \Omega^k_X(\mathfrak{g}) \longrightarrow \Omega^{k+\ell}_X(\mathfrak{g} \otimes \mathfrak{g}) \longrightarrow \Omega^{k+\ell}(\mathfrak{g})$$

where the first map is the wedge product, and the second map is induced by the Lie bracket. Finally, since $X \times \mathfrak{g}$ is a trivial bundle, the usual exterior derivative $d : \Omega^k_X \to \Omega^{k+1}_X$ extends to a well defined operator on $\Omega^k_X(\mathfrak{g})$, given by $d\omega = d\omega^i \xi_i$.

We now discuss connections. Let $\pi : P \to X$ be a principal G-bundle, and let $\mathfrak{g}$ be the Lie algebra of G. The projection map π is a submersion, so it is constant rank. Therefore, the subset $V \subset TP$ where the fiber over p is $\ker d\pi_p$ is a subbundle, called the ***vertical distribution*** of P, giving us an exact sequence of vector bundles over P

$$0 \longrightarrow V \longrightarrow TP \longrightarrow \pi^* TX \longrightarrow 0$$

Definition 1.11. A ***connection*** on P is a distribution $H \subset TP$ such that

(1) $V \oplus H = TP$

(2) $H_{p \cdot g} = d(R_g)_p H_p$, where $R_g : P \to P$ is the map $p \mapsto p \cdot g$.

The distribution H is also called the ***horizontal distribution***. We let $\mathscr{A}(P)$ denote the space of connections on P.

Equivalently, it is a choice of G-invariant splitting of the exact sequence. The perspective of viewing a connection as a horizontal distribution is useful at times, but it is often more convenient for computations to rephrase a connection in terms of $\mathfrak{g}$-valued forms. Let $\exp : \mathfrak{g} \to G$ denote the exponential map. Given $X \in \mathfrak{g}$ and $p \in P$, the exponential map determines a curve γ_X with $\gamma_X(0) = p$ where

$$\gamma_X(t) := p \cdot \exp(tX)$$

Since the action of G preserves the fiber $P_{\pi(p)}$, the tangent vector

$$\dot{\gamma}_X := \left.\frac{d}{dt}\right|_{t=0} \gamma_X(t)$$

lies in V_p. Furthermore, since the action of G on P is free, we have that $\dot{\gamma}_X = 0$ if and only if $X = 0$. Finally, the mapping $X \mapsto \dot{\gamma}_X$ is linear, so we have that this gives an isomorphism

$\mathfrak{g} \to V_p$ by a dimension count. Doing this over all $p \in P$, this gives an isomorphism of V with the trivial bundle $P \times \mathfrak{g}$. Because of this, we will implicitly identify elements of $\mathfrak{g}$ with the vertical vector fields they determine. One thing to note is how these vector fields transform with respect to the action of G.

Proposition 1.12. *Let $X \in \mathfrak{g}$, and let $\widetilde{X}$ denote the vertical vector field on P induced by X. For $g \in G$, let $R_g : P \to P$ be map given by the action of g. Then*

$$(R_g)_* \widetilde{X} = \widetilde{\mathrm{Ad}_{g^{-1}} X}$$

Proof. We compute

$$\begin{aligned}
((R_g)_* \widetilde{X}_p) &= (R_g)_* \left(\frac{d}{dt}\bigg|_{t=0} p \cdot \exp(tX) \right) \\
&= \frac{d}{dt}\bigg|_{t=0} p \cdot (\exp(tX) g) \\
&= (p \cdot g)(g^{-1} \exp(tX) g) \\
&= (\widetilde{\mathrm{Ad}_{g^{-1}} X})_{p \cdot g}
\end{aligned}$$

■

Furthermore, the identification of the vertical distribution V with the trivial bundle $X \times \mathfrak{g}$ gives us a nice characterization of E-valued forms, when $E = P \times_G W$ is an associated bundle coming from a linear representation $\rho : G \to \mathrm{GL}(W)$.

Proposition 1.13. *Let $P \to M$ be a principal bundle and E the associated bundle coming from a linear representation $\rho : G \to \mathrm{GL}(W)$. Then there is a bijective correspondence*

$$\Omega^k_X(E) \leftrightarrow \left\{ \alpha \in \Omega^k_P(W) \ : \ R_g^* \alpha = \rho(g^{-1})\alpha \ , \ \iota_X \alpha = 0 \forall X \in \mathfrak{g} \right\}$$

where we identify $X \in \mathfrak{g}$ with its vertical vector field and ι_ξ denotes interior multiplication.

Morally, the correspondence comes from the fact that a form on P descending to X should satisfy G-invariance, and should be invariant in the vertical directions.

Exercise 1.14. Prove the previous proposition.

Exercise 1.15. Using the proposition above, prove that the space $\mathscr{A}(P)$ of connections is an affine space over $\Omega^1_X(\mathfrak{g}_P)$, i.e. show that the difference $A_1 - A_2$ between two connections is an element of $\Omega^1_X(\mathfrak{g}_P)$.

Now suppose we have a horizontal distribution $H \subset TP$. The decomposition $TP = V \oplus H$ gives us a projection map $TP \to V$ with kernel H. Identifying V with $P \times \mathfrak{g}$, the projection map can be idenfied with a $\mathfrak{g}$-valued 1-form $A \in \Omega^1_P(\mathfrak{g})$, called the ***connection* 1*-form***. Using the transformation law for the vertical vector fields determined by $\mathfrak{g}$, we get the following transformation law for the connection 1-form A.

Proposition 1.16. *A connection* 1*-form $A \in \Omega^1_P(\mathfrak{g})$ satisfies*

$$R_g^* A = \mathrm{Ad}_{g^{-1}} A$$

Proof. For $v \in T_pP$, decompose $v = X + h$ with $X \in \mathfrak{g}$ and $h \in H_p$. We then compute

$$\begin{aligned}(R_g^*A)(v) &= (R_g^*A)(X+h)\\ &= A_{p\cdot g}((R_g)_*X + (R_g)_*h)\\ &= A_{p\cdot g}(\operatorname{Ad}_{g^{-1}} X)\\ &= (\operatorname{Ad}_{g^{-1}} A)_p(X+h)\end{aligned}$$

where we use the fact that H_p is the the kernel of A_p and the fact that H is G-invariant. ■

Furthermore, since A is given by projection onto the vertical distribution, we have that $\iota_X A = X$ for all $X \in \mathfrak{g}$.

Definition 1.17. Let $A \in \mathscr{A}(P)$ be a connection. Then the ***curvature of*** A, denoted F_A, is the $\mathfrak{g}$-valued 2 form

$$F_A := dA = \frac{1}{2}[A \wedge A]$$

Proposition 1.18.

(1) $R_g^*F_A = \operatorname{Ad}_{g^{-1}} F_A$.
(2) $\iota_X F_A = 0$ *for all* $X \in \mathfrak{g}$.

Proof.

(1) We compute

$$\begin{aligned}R^*F_A &= R_g^*dA + \frac{1}{2}R_g^*[A \wedge A]\\ &= d(\operatorname{Ad}_{g^{-1}} A) + \frac{1}{2}[R_g^*A \wedge R_G^*A]\\ &= \operatorname{Ad}_{g^{-1}} dA + \frac{1}{2}[\operatorname{Ad}_{g^{-1}} A \wedge \operatorname{Ad}_{g^{-1}} A]\\ &= \operatorname{Ad}_{g^{-1}} F_A\end{aligned}$$

(2) For this, we use a lemma.

Lemma (***Cartan's magic formula***)**.** *Let X be a vector field, and ω a k-form. Let $\mathcal{L}_X$ denote the Lie derivative along X. Then*

$$\mathcal{L}_X = d\iota_X\omega + \iota_X d\omega$$

Let $X \in \mathfrak{g}$, interpreted as a vertical vector field on P. Then we compute

$$\iota_X F_A = \iota_X dA + \frac{1}{2}\iota_X[A \wedge A]$$

We compute the two terms separately. Cartan's magic formula gives us that

$$\mathcal{L}_X A = d\iota_X A + \iota_X dA$$

Since $\iota_X F_A$ is the constant function with value X, we have that $d\iota_X F_A$ is 0, so we get $\mathcal{L}_X F_A = \iota_X dF_A$. Using the definition of the Lie derivative, we compute

$$\begin{aligned}\mathcal{L}_X A &= \frac{d}{dt}\bigg|_{t=0} R^*_{\exp tX} A \\ &= \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_{\exp(tX)^{-1}} A \\ &= [-X, A]\end{aligned}$$

For the other term, we compute

$$\frac{1}{2}\iota_X[A \wedge A] = \frac{1}{2}[\iota_X A \wedge A] = [X, A]$$

adding these together gives us the desired result.

■

References

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[2] S. Kobayashi and Nomizu K. *Foundations of Differential Geometry Volume 1*. 1996.

[3] L. Tu. *Differential Geometry*. 2017.