YANG-MILLS MINICOURSE NOTES

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Contents

Notation and Conventions

We use Einstein summation notation, i.e. indices that appear on the top and bottom of an expression are implicitly summed over. For example

$$a_i dx^i = \sum_i a_i dx^i$$

For a smooth manifold X, we let Ω^k_X denote the space of differential k-forms. When X is a complex manifold, we let $\mathcal{A}^k_X$ denote the space of smooth complex-valued k-forms, and $\mathcal{A}^{p,q}_X$ the space of smooth (p,q)-forms. We reserve Ω^k_X and $\Omega^{p,q}_X$ for the spaces of holomorphic k and (p,q) forms respectively.

1. Principal Bundles and Connections

Good references for this material would be [5] and [7]

Fix a close manifold X (compact and without boundary) and a Lie group G.

Definition 1.1. A ***principal G-bundle*** is a fiber bundle $\pi : P \to M$ with a smooth right G action such that:

(1) The action of G preserves the fibers of π, and gives each fiber $P_x := \pi^{-1}(x)$ the structure of a ***right G-torsor***, i.e. the action of G on P_x is free and transitive.
(2) For every point $x \in X$, there exists a ***local trivialization*** of P, i.e. a diffeomorphism $\varphi : P|_U := \pi^{-1}(U) \to U \times G$ that is G-equivariant (where the action on $U \times G$ is right multiplication with the second factor) and the following diagram commutes:

$$\begin{tikzcd} P|_U \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times G \arrow[dl] \\ & U & \end{tikzcd}$$

where the map $U \times G \to U$ is projection onto the first factor.

Example 1.2. Let $E \to X$ be a real vector bundle of rank k. For $x \in X$, let $\mathcal{B}_x$ denote the set of all bases of the fiber E_x, i.e. the set of linear isomorphisms $\mathbb{R}^k \to E_x$. This has a natural right action of $\mathrm{GL}_k\mathbb{R}$ by precomposition. Furthermore, this action is free and transitive, giving $\mathcal{B}_x$ the structure of a $\mathrm{GL}_k\mathbb{R}$-torsor. Then let

$$\mathcal{B}_{\mathrm{GL}_k\mathbb{R}}(E) := \coprod_{x \in X} \mathcal{B}_x$$

Using local trivializations of the vector bundle E, we equip $\mathcal{B}_{\mathrm{GL}_k\mathbb{R}}(E)$ with the structure of a smooth manifold such that the map $\pi : \mathcal{B}_{\mathrm{GL}_k\mathbb{R}}(E) \to X$ taking $\mathcal{B}_x$ to x is a submersion. This gives $\pi : \mathcal{B}_{GL_k\mathbb{R}} \to X$ the structure of a principal $\mathrm{GL}_k\mathbb{R}$-bundle, called the ***frame bundle*** of E, where the local trivializations are defined in terms of local trivializations of E.

Example 1.3. Let $E \to X$ be a rank k vector bundle equipped with a fiber metric, i.e. a smoothly varying inner product on the fibers E_x. Then the ***orthonormal frame bundle*** of E, denoted $\mathcal{B}_{\mathrm{O}}(E)$, is the principal O_k-bundle where the fiber over $x \in X$ is the O_k-torsor of orthonormal bases for E_x.

A near identical story holds for complex vector bundles – from any complex vector bundle we get a principal $\mathrm{GL}_k\mathbb{C}$-bundle of frames, and if we fix a Hermitian fiber metric, we get a principal U_k-bundle of orthonormal frames.

Principal bundles can be thought bundles of symmetries of some other fiber bundle, which can be made precise using the notion of an associated bundle, which allows one to construct fiber bundles out of principal bundles.

Definition 1.4. Let $P \to X$ be a principal G-bundle, and let F be a smooth manifold with right G action. The ***associated fiber bundle***, denoted $P \times_G F$ (sometimes denoted $P \times^G F$) is the space

$$P \times_G F := (P \times F)/G$$

where the right G-action on F is the diagonal action, i.e. $(p, f) \cdot g = (p \cdot g, f \cdot g)$.

If instead we have a left G-action on F, we can turn it into a right action by defining $f \cdot g := g^{-1} \cdot f$. As the name suggests, $P \times_G F$ is a fiber bundle.

Exercise 1.5. Let $\pi : P \to X$ a principal bundle. Given a smooth right action of G on F, use local trivializations of $P \to X$ to show that the map taking an equivalence class $[p, f]$ to $\pi(p)$ gives $P \times_G F$ the structure of a fiber bundle over X with model fiber F.

In the case that the model fiber is a vector space V, and the action is linear, the associated bundle $P \times_G V$ is a vector bundle.

Exercise 1.6.

(1) Let $E \to X$ be a rank k vector bundle, and $\mathcal{B}_{\mathrm{GL}_k\mathbb{R}}(E)$ be its $\mathrm{GL}_k\mathbb{R}$-bundle of frames. Let $\rho : \mathrm{GL}_k\mathbb{R} \to \mathrm{GL}_k\mathbb{R}$ be the defining representation (i.e. the identity map). Show that the associated bundle $\mathcal{B}_{\mathrm{GL}_k\mathbb{R}}(E) \times_{\mathrm{GL}_k\mathbb{R}} \mathbb{R}^k$ is isomorphic to E.

(2) Further suppose that E comes equipped with a fiber metric, and let $\mathcal{B}_{\mathrm{O}}(E)$ be its orthonormal frame bundle. The associated bundle $\mathcal{B}_{\mathrm{O}}(E) \times_{\mathrm{O}_k} \mathbb{R}^k$ is isomorphic to E by a near identical proof as the previous part. How can one recover the fiber metric?

(3) Let $\rho^* : \mathrm{GL}_k\mathbb{R} \to \mathrm{GL}_k\mathbb{R}$ denote the dual representation of the defining representation ρ, so $\rho(A) = (A^{-1})^T$. Show that the associated bundle is isomorphic to the dual bundle E^*. In particular, this should illuminate the distinction between the tangent and cotangent bundles.

Example 1.7. There are two important examples of associated bundles that we'll need to discuss the Yang-Mills equations.

(1) The bundle $\mathrm{Ad}\, P := P \times_G G$ where G acts on G by conjugation.

(2) The bundle $\mathrm{ad}\, P := P \times_G \mathfrak{g}$ (also denoted $\mathfrak{g}_P$) where the action is the adjoint action.

Confusingly, the former is sometimes called the "Adjoint bundle" and the latter is sometimes called the "adjoint bundle," which makes it admittedly hard to distuinguish between them when speaking.

Associated bundles have another nice feature – their sections have a nice interpretation in terms of G-equivariant maps.

Proposition 1.8. *Let $E = P \times_G F$ be an associated fiber bundle, and let $\Gamma(X, E)$ denote the space of global sections, i.e. the space of smooth maps $f : X \to E$ such that $\pi \circ f = \mathrm{id}_X$, where $\pi : E \to X$ denotes the projection map. Then there is a bijective correspondence*

$$\Gamma(X, E) \longleftrightarrow \{G - \textit{equivariant maps } P \to F\}$$

Proof. Let $\sigma : X \to E$ be a section. Then define the map $\widetilde{\sigma} : P \to F$ as follows: for $x \in X$, let (p, f) be a representative for $\sigma(x)$. Then define $\widetilde{\sigma}(x) := f$.

In the other direction, let $\widetilde{\varphi} : P \to F$ be an equivariant map. Then define the section $\varphi : X \to E$ by $\varphi(x) = [p, \widetilde{\varphi}(p)]$ for any choice of $p \in P_x$. ∎

Exercise 1.9. Verify that the map φ defined above is well-defined. Verify the two constructions above are inverses to each other.

The main takeaway from the proposition is the motto that "G-equivariant objects on P descend to objects on X."

Before we discuss connections on principal bundles, we introduce the concept of vector bundled valued forms.

Definition 1.10. Let $E \to X$ be a vector bundle. An ***E-valued differential k-form*** is a section of $\Lambda^k T^*X \otimes E$. We denote the space of E-valued k-forms by $\Omega^k_X(E)$. If V is a fixed vector space, a ***V-valued differential k-form*** is a $X \times V$-valued k-form, and we let $\Omega^k_X(V)$ denote the space of V-valued k-forms.

In a local frame $\{e_i\}$ for E, an E-valued k-form ω can be written uniquely as

$$\omega = \omega^i \otimes e_i$$

for k-forms $\omega^i \in \Omega^k_X$,so an E-valued k-form can be thought of as a vector of k-forms. We will usually omite the tensor symbol, and simply write $\omega = \omega^i e_i$. However, this does not transform tensorially with respect to coordinate changes on X unless E is a trivial bundle. The components of the vector transform tensorially with respect to coordinate changes, but the vector itself changes according to the transition functions of the vector bundle E. Given E-valued forms $\omega \in \Omega^k_X(E)$ and $\eta \in \Omega^\ell_X(E)$, we define their wedge product in a local trivialization to be

$$\omega \wedge \eta := (\omega^i \wedge \eta^j) e_i \otimes e_j$$

which is an element of $\Omega^{k+\ell}_X(E \otimes E)$.

For the most part, we will be concerned with Lie algebra valued forms, which are just $\mathfrak{g}$-valued forms for a fixed Lie algebra $\mathfrak{g}$. These forms have some additional operations coming from the Lie algebra structure of $\mathfrak{g}$. Fix a basis $\{\xi_i\}$ for $\mathfrak{g}$. This determines a global trivialization of the trivial bundle $X \times \mathfrak{g}$, so any $\mathfrak{g}$-valued K-form $\omega \in \Omega^k_X(\mathfrak{g})$ can be uniquely written as $\omega = \omega^i \xi_i$. Let $\omega \in \Omega^k_X(\mathfrak{g})$ and $\eta \in \Omega^\ell_X(\mathfrak{g})$. Then define their bracket to be

$$[\omega \wedge \eta] := (\omega^i \wedge \eta^j)[\xi_i, \xi_j]$$

In other words, it is the composition

$$\Omega^k_X(\mathfrak{g}) \otimes \Omega^k_X(\mathfrak{g}) \longrightarrow \Omega^{k+\ell}_X(\mathfrak{g} \otimes \mathfrak{g}) \longrightarrow \Omega^{k+\ell}(\mathfrak{g})$$

where the first map is the wedge product, and the second map is induced by the Lie bracket. Finally, the usual exterior derivative $d : \Omega^k_X \to \Omega^{k+1}_X$ extends to an operator on $\Omega^k_X(\mathfrak{g})$, given by $d\omega = d\omega^i \xi_i$.

We now discuss connections. Let $\pi : P \to X$ be a principal G-bundle, and let $\mathfrak{g}$ be the Lie algebra of G. The projection map π is a submersion, so it is constant rank. Therefore, the subset $V \subset TP$ where the fiber over p is $\ker d\pi_p$ is a subbundle, called the ***vertical distribution*** of P, giving us an exact sequence of vector bundles over P

$$0 \longrightarrow V \longrightarrow TP \longrightarrow \pi^* TX \longrightarrow 0$$

Definition 1.11. A ***connection*** on P is a distribution $H \subset TP$ such that

(1) $V \oplus H = TP$

(2) $H_{p \cdot g} = d(R_g)_p H_p$, where $R_g : P \to P$ is the map $p \mapsto p \cdot g$.

The distribution H is also called the ***horizontal distribution***. We let $\mathscr{A}(P)$ denote the space of connections on P.

Equivalently, it is a choice of G-invariant splitting of the exact sequence. The perspective of viewing a connection as a horizontal distribution is useful at times, but it is often more convenient for computations to rephrase a connection in terms of $\mathfrak{g}$-valued forms. Let $\exp : \mathfrak{g} \to G$ denote the exponential map. Given $X \in \mathfrak{g}$ and $p \in P$, the exponential map determines a curve γ_X with $\gamma_X(0) = p$ where

$$\gamma_X(t) := p \cdot \exp(tX)$$

Since the action of G preserves the fiber $P_{\pi(p)}$, the tangent vector

$$\dot{\gamma}_X := \left.\frac{d}{dt}\right|_{t=0} \gamma_X(t)$$

lies in V_p. Furthermore, since the action of G on P is free, we have that $\dot{\gamma}_X = 0$ if and only if $X = 0$. Finally, the mapping $X \mapsto \dot{\gamma}_X$ is linear, so we have that this gives an isomorphism $\mathfrak{g} \to V_p$ by a dimension count. Doing this over all $p \in P$, this gives an isomorphism of V with the trivial bundle $P \times \mathfrak{g}$. Because of this, we will implicitly identify elements of $\mathfrak{g}$ with the vertical vector fields they determine. One thing to note is how these vector fields transform with respect to the action of G.

Proposition 1.12. *Let $X \in \mathfrak{g}$, and let $\widetilde{X}$ denote the vertical vector field on P induced by X. For $g \in G$, let $R_g : P \to P$ be map given by the action of g. Then*

$$(R_g)_* \widetilde{X} = \widetilde{\mathrm{Ad}_{g^{-1}} X}$$

Proof. We compute

$$\begin{aligned}
((R_g)_* \widetilde{X}_p) &= (R_g)_* \left(\frac{d}{dt}\bigg|_{t=0} p \cdot \exp(tX) \right) \\
&= \frac{d}{dt}\bigg|_{t=0} p \cdot (\exp(tX) g) \\
&= (p \cdot g)(g^{-1} \exp(tX) g) \\
&= (\widetilde{\mathrm{Ad}_{g^{-1}} X})_{p \cdot g}
\end{aligned}$$

∎

Furthermore, the identification of the vertical distribution V with the trivial bundle $X \times \mathfrak{g}$ gives us a nice characterization of E-valued forms, when $E = P \times_G W$ is an associated bundle coming from a linear representation $\rho : G \to \mathrm{GL}(W)$.

Proposition 1.13. *Let $P \to M$ be a principal bundle and E the associated bundle coming from a linear representation $\rho : G \to \mathrm{GL}(W)$. Then there is a bijective correspondence*

$$\Omega^k_X(E) \leftrightarrow \left\{ \alpha \in \Omega^k_P(W) \ : \ R_g^* \alpha = \rho(g^{-1})\alpha \ , \ \iota_X \alpha = 0 \forall X \in \mathfrak{g} \right\}$$

where we identify $X \in \mathfrak{g}$ with its vertical vector field and ι_ξ denotes interior multiplication.

Morally, the correspondence comes from the fact that a form on P descending to X should satisfy G-invariance, and should be invariant in the vertical directions.

Exercise 1.14. Prove the previous proposition.

Exercise 1.15. Using the proposition above, prove that the space $\mathscr{A}(P)$ of connections is an affine space over $\Omega^1_X(\mathfrak{g}_P)$, i.e. show that the difference $A_1 - A_2$ between two connections is an element of $\Omega^1_X(\mathfrak{g}_P)$.

Now suppose we have a horizontal distribution $H \subset TP$. The decomposition $TP = V \oplus H$ gives us a projection map $TP \to V$ with kernel H. Identifying V with $P \times \mathfrak{g}$, the projection map can be idenfied with a $\mathfrak{g}$-valued 1-form $A \in \Omega^1_P(\mathfrak{g})$, called the ***connection* 1*-form***. Using the transformation law for the vertical vector fields determined by $\mathfrak{g}$, we get the following transformation law for the connection 1-form A.

Proposition 1.16. *A connection* 1*-form $A \in \Omega^1_P(\mathfrak{g})$ satisfies*

$$R_g^* A = \mathrm{Ad}_{g^{-1}} A$$

Proof. For $v \in T_pP$, decompose $v = X + h$ with $X \in \mathfrak{g}$ and $h \in H_p$. We then compute

$$\begin{aligned}(R_g^*A)(v) &= (R_g^*A)(X+h) \\ &= A_{p\cdot g}((R_g)_*X + (R_g)_*h) \\ &= A_{p\cdot g}(\mathrm{Ad}_{g^{-1}} X) \\ &= (\mathrm{Ad}_{g^{-1}} A)_p(X+h)\end{aligned}$$

where we use the fact that H_p is the the kernel of A_p and the fact that H is G-invariant. ■

Furthermore, since A is given by projection onto the vertical distribution, we have that $\iota_X A = X$ for all $X \in \mathfrak{g}$. This gives us an identification of $\mathscr{A}(P)$ with the subset of $\Omega^1_P(\mathfrak{g})$ satisfying the conditions

(1) $R_g^*A = \mathrm{Ad}_{g^{-1}} A$
(2) $\iota_X A = X$ for all $X \in \mathfrak{g}$.

The second condition can be rephrased in terms of the ***Maurer-Cartan form*** $\theta \in \Omega^1_G(\mathfrak{g})$, which is defined by

$$\theta_g(v) = (dL_{g^{-1}})_g(v)$$

where $L_{g^{-1}} : G \to G$ is left multiplication by g^{-1}. The action of G on any G-torsor X gives us a Maurer-Cartan form on X.

Exercise 1.17. Show that the second condition is equivalent to $A|_{P_x} = \theta$ for any $x \in X$.

The Maurer-Cartan form θ satisfies the ***Maurer-Cartan equation***

$$d\theta + \frac{1}{2}[\theta \wedge \theta] = 0$$

Definition 1.18. Let $A \in \mathscr{A}(P)$ be a connection. Then the ***curvature of*** A, denoted F_A, is the $\mathfrak{g}$-valued 2 form

$$F_A := dA = \frac{1}{2}[A \wedge A]$$

Proposition 1.19.

(1) $R_g^*F_A = \mathrm{Ad}_{g^{-1}} F_A$.
(2) $\iota_X F_A = 0$ *for all* $X \in \mathfrak{g}$.

Proof.

(1) We compute

$$\begin{aligned}R^*F_A &= R_g^*dA + \frac{1}{2}R_g^*[A \wedge A] \\ &= d(\mathrm{Ad}_{g^{-1}} A) + \frac{1}{2}[R_g^*A \wedge R_G^*A] \\ &= \mathrm{Ad}_{g^{-1}} dA + \frac{1}{2}[\mathrm{Ad}_{g^{-1}} A \wedge \mathrm{Ad}_{g^{-1}} A] \\ &= \mathrm{Ad}_{g^{-1}} F_A\end{aligned}$$

(2) For this, we use a lemma.

Lemma (***Cartan's magic formula***)**.** *Let X be a vector field, and ω a k-form. Let $\mathcal{L}_X$ denote the Lie derivative along X. Then*

$$\mathcal{L}_X = d\iota_X\omega + \iota_X d\omega$$

Let $X \in \mathfrak{g}$, interpreted as a vertical vector field on P. Then we compute

$$\iota_X F_A = \iota_X dA + \frac{1}{2}\iota_X[A \wedge A]$$

We compute the two terms separately. Cartan's magic formula gives us that

$$\mathcal{L}_X A = d\iota_X A + \iota_X dA$$

Since $\iota_X F_A$ is the constant function with value X, we have that $d\iota_X F_A$ is 0, so we get $\mathcal{L}_X F_A = \iota_X dF_A$. Using the definition of the Lie derivative, we compute

$$\begin{aligned}\mathcal{L}_X A &= \frac{d}{dt}\bigg|_{t=0} R^*_{\exp tX} A \\ &= \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_{\exp(tX)^{-1}} A \\ &= [-X, A]\end{aligned}$$

For the other term, we compute

$$\frac{1}{2}\iota_X[A \wedge A] = \frac{1}{2}[\iota_X A \wedge A] = [X, A]$$

adding these together gives us the desired result.

■

In other words, the curvature F_A descends to a $\mathfrak{g}_P$-valued 2-form on the base manifold X.

Exercise 1.20. Given a connection A on a principal bundle P, prove that the curvature F_A vanishes if and only if horizontal distribution H defined by the kernel of A is integrable. To prove this, reformulate Frobenius' theorem (a distribution is integrable if and only if it is involutive) in terms of the vanishing of a tensor, and show that this tensor (up to sign) is equal to F_A.

Definition 1.21. A connection $A \in \mathscr{A}(P)$ is ***flat*** if $F_A = 0$.

For vector bundles, a more familiar definition of a connection involves a first order operator on sections satisfying a Leibniz rule. Using the language of principal bundles and associated bundles, we recover this notion with the exterior covariant derivative.

Definition 1.22. Let $E = P \times_G W$ be the associated vector bundle obtained from a linear representation $\rho : G \to \mathrm{GL}(W)$, and let $\dot{\rho} : \mathfrak{g} \to \mathrm{End}(W)$ denote the associated Lie algebra representation. The ***exterior covariant derviative*** is the map

$$\begin{aligned}d_A : \Omega^p_X(E) &\to \Omega^{p+1}_X(E) \\ \psi &\mapsto d\psi + \dot{\rho}(A) \wedge \psi\end{aligned}$$

Exercise 1.23. Recall that a connection on a vector bundle $E \to X$ is given in a local trivialization by $d + A$ for some $\mathrm{End}(E)$-valued 1-form A. Show that when $P = \mathcal{B}_{\mathrm{GL}_k\mathbb{R}}(E)$ is the frame bundle of a vector bundle E, the exterior covariant derivative on sections agrees with this definition.

For the most part, we will be concerned with situation when the vector bundle is $\mathfrak{g}_P$, in which case, the formula is given by

$$d_A\psi = d\psi + [A, \psi]$$

Since $\mathscr{A}(P)$ is an affine space over $\Omega^1_X(\mathfrak{g}_P)$, given a connection $A \in \mathscr{A}(P)$ and a $\mathfrak{g}_P$-valued 1-form $\eta \in \Omega^1_X(\mathfrak{g}_P)$, we have that $A + \eta$ is also a connection. It can be shown that the curvature of $A + \eta$ is given by

$$F_{A+\eta} = F_A + \frac{1}{2}[\eta \wedge \eta] + d_A\eta$$

In particular, if we take a line of connections $A + t\eta$ with $t \in \mathbb{R}$, we have

$$\left.\frac{d}{dt}\right|_{t=0} F_{A+t\eta} = \left.\frac{d}{dt}\right|_{t=0} F_A + \frac{t^2}{2}[\eta \wedge \eta] + td_A\eta = d_A\eta$$

So the exterior covariant derivative on $\mathfrak{g}_P$ measures the infitesimal change of the curvature of A in the direction η.

2. The Yang-Mills Equations

To discuss the Yang-Mills equations, we will restrict to compact Lie groups G. As before, X will denote an n-dimensional closed smooth manifold.

Since G is compact, its Lie algebra $\mathfrak{g}$ is semisimple, so the Killing form $\langle\cdot,\cdot\rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ is nondegenerate. For the rest of our discussion, $\langle\cdot,\cdot\rangle$ can be replaced by any inner product invariant under the Adjoint action, though it does us no harm to assume that it is the Killing form.

Lemma 2.1. *Let $\langle\cdot,\cdot\rangle$ denote any Adjoint invariant inner product on $\mathfrak{g}$. Then for $X_1, X_2, X_3 \in \mathfrak{g}$, we have*

$$\langle [X_1, X_2], X_3\rangle = \langle X_1, [X_2, X_3]\rangle$$

Proof. We compute

$$\begin{aligned}
\langle [X_1, X_2], X_3\rangle &= \langle [-X_2, X_1], X_3\rangle \\
&= \left.\frac{d}{dt}\right|_{t=0} \langle \mathrm{Ad}_{\exp(-tX_2)} X_1, X_3\rangle \\
&= \left.\frac{d}{dt}\right|_{t=0} \langle \mathrm{Ad}_{\exp tX_2} \mathrm{Ad}_{\exp(-tX_2)} X_1, \mathrm{Ad}_{\exp(tX_2)} X_3\rangle \\
&= \langle X_1, [X_2, X_3]\rangle
\end{aligned}$$

∎

The form $\langle\cdot,\cdot\rangle$ induces a fiber metric on $P \times \mathfrak{g}$, and invariance under the Adjoint action tells us that this fiber metric descends to a fiber metric on $\mathfrak{g}_P$. This gives us pairings

$$\begin{aligned}\Omega^k_X(\mathfrak{g}_P) \otimes \Omega^\ell_X(\mathfrak{g}_P) &\to \Omega^{k+\ell}_X \\ \omega \otimes \eta &\mapsto \langle \omega, \eta \rangle\end{aligned}$$

We now fix an orientation and a Riemannian metric g on X. This gives us:

(1) A Riemannian volume form $\mathrm{Vol}_g \in \Omega^n_X$.
(2) A Hodge star operator $\star : \Omega^k_X \to \Omega^{n-k}_X$.
(3) Fiber metrics $\langle\cdot,\cdot\rangle_g$ on the bundles $\Lambda^k T^* X$.

The Hodge star extends to $\mathfrak{g}_P$-valued forms, which gives us inner products on $\Omega^k_X(\mathfrak{g}_P)$ given by

$$(\omega, \eta) := \int_X \langle \omega, \star\eta \rangle$$

We let $\|\cdot\|$ denote the norm induced by these inner products.

We now introduce the gauge group of a prinicipal G-bundle $P \to X$.

Definition 2.2. Let $\pi : P \to X$ be a principal G-bundle. The ***gauge group***, denoted $\mathscr{G}(P)$, is the group of automorphisms of P, i.e. G-equivariant diffeomorphisms $\varphi : P \to P$ such that $\pi = \pi \circ \varphi$. An element of $\mathscr{G}(P)$ is called a ***gauge transformation***.

Proposition 2.3. *The group $\mathscr{G}(P)$ is isomorphic to the group of sections $\Gamma(X, \mathrm{Ad}\, P)$, where the group operation is pointwise multiplication.*

Proof. We provide maps in both directions. Suppose we have an automorphism $\varphi : P \to P$. Since $\pi = \pi \circ \varphi$, the map φ preserves the fibers of π. Therefore, for any $p \in P$, we have that p and $\varphi(p)$ differ by the action of some $g_p \in G$. The mapping $g_\varphi : P \to G$ taking $p \mapsto g_p$ is easilty verified to be equivariant with respect to the conjugation action of G, so it defines a section of $\mathrm{Ad}\, P$

In the other direction, given a G-equivariant map $f : P \to G$, we get a bundle automorphism $\varphi_f : P \to P$ where $\varphi_f(p) = p \cdot f(p)$. The two maps we constructed are clearly inverse to each other, giving us the desired correspondence. ∎

The gauge group $\mathscr{G}(P)$ acts on the space $\Omega^1_P(\mathfrak{g})$ of $\mathfrak{g}$-valued forms by pullback. We claim that it preserves the subspace $\mathscr{A}(P) \subset \Omega^1_P(\mathfrak{g})$.

Proposition 2.4. *For a connection $A \in \mathscr{A}(P)$ and a gauge transformation $\varphi : P \to P$, we have*

(1) $R_g^* A = \mathrm{Ad}_{g^{-1}} \varphi^* A$.
(2) $\iota_X \varphi^* A = X$ *for all* $X \in \mathfrak{g}$.

Equivalently, if we let $g_\varphi : P \to G$ denote the equivariant map associated to φ, we have

$$\varphi^* A = \mathrm{Ad}_{g_\varphi^{-1} A} + g_\varphi^* \theta$$

where $\theta \in \Omega^1_G(\mathfrak{g})$ is the Maurer-Cartan form.

Exercise 2.5. Prove the previous proposition.

Definition 2.6. Two connections A_1 and A_2 are ***gauge equivalent*** if there exists a gauge transformation $\varphi \in \mathscr{G}(P)$ such that $\varphi^* A_1 = A_2$.

Proposition 2.7. *Let $A \in \mathscr{A}(P)$ be a connection, $\varphi : P \to P$ a gauge transformation, and $g_\varphi : P \to G$ the associated equivariant map. Then*

$$F_{\varphi^* A} = \operatorname{Ad}_{g_\varphi^{-1}} F_A$$

Proof. We compute

$$\begin{aligned}
F_{\varphi^* A} &= d(\operatorname{Ad}_{g_\varphi^{-1}} A + g_\varphi^* \theta) + \frac{1}{2}[\operatorname{Ad}_{g_\varphi^{-1}} A + g_\varphi^*\theta \wedge \operatorname{Ad}_{g_\varphi^{-1}} A + g_\varphi^*\theta] \\
&= \operatorname{Ad}_{g_\varphi^{-1}} dA + g_\varphi^* d\theta + \frac{1}{2}\left([\operatorname{Ad}_{g_\varphi^{-1}} A \wedge \operatorname{Ad}_{g_\varphi^{-1}} A] + [\operatorname{Ad}_{g_\varphi^{-1}} A \wedge g_\varphi^*\theta] + [g_\varphi^*\theta \wedge \operatorname{Ad}_{g_\varphi^{-1}} A] + [g_\varphi^*\theta \wedge g_\varphi^*\theta]\right) \\
&= \operatorname{Ad}_{g_\varphi^{-1}} dA + \frac{1}{2}[\operatorname{Ad}_{g_\varphi^{-1}} A, \operatorname{Ad}_{g_\varphi^{-1}} A]] \\
&= \operatorname{Ad}_{g_\varphi^{-1} F_A}
\end{aligned}$$

The term

$$g_\varphi^* d\theta + \frac{1}{2}[g_\varphi^*\theta \wedge g_\varphi^*\theta]$$

vanishes due to the Maurer-Cartan equation. The term

$$\frac{1}{2}\left([\operatorname{Ad}_{g_\varphi^{-1}} A \wedge g_\varphi^*\theta] + [g_\varphi^*\theta \wedge \operatorname{Ad}_{g_\varphi^{-1}} A]\right)$$

vanishes due to the fact that $[\cdot,\cdot]$ is skew symmetric on 1-forms. ∎

With some of the preliminary results established, we arrive at the Yang-Mills functional.

Definition 2.8. The ***Yang-Mills functional*** is the map $L : \mathscr{A}(P) \to \mathbb{R}$ given by

$$L(A) := \|F_A\|^2 = \int_X \langle F_A \wedge \star F_A \rangle$$

We note that for any gauge transformation $\varphi \in \mathscr{G}(P)$, we have $L(\varphi^* A) = L(A)$, since we have

$$L(\varphi^* A) = \int_X \langle \operatorname{Ad}_{g_\varphi^{-1}} F_A \wedge \star \operatorname{Ad}_{g_\varphi^{-1}} F_A \rangle = \int_X \langle F_A \wedge \star F_A \rangle = L(A)$$

because of this we say that L is ***gauge invariant***.

The Yang-Mills equations are the variational equations for the Yang-Mills functional.

Proposition 2.9 (***The first variation***)**.** *Let A be a local extremum of L. Then we have*

$$d_A \star F_A = 0$$

Proof. Let $\eta \in \Omega^1_X(\mathfrak{g}_P)$. We then compute

$$\begin{aligned}
L(A) &= \int_X \langle F_{A+t\eta} \wedge \star F_{A+t\eta} \\
&= \int_X \langle F_A + \frac{t^2}{2}[\eta \wedge \eta] + t d_A \eta \wedge \star(F_A + \frac{t^2}{2}[\eta \wedge \eta] + t d_A \eta)
\end{aligned}$$

The term linear in t is

$$\int_X \langle F_A \wedge \star d_A \eta + \langle d_A \eta \wedge \star F_A \rangle = 2(F_A, d_A \eta)$$

Then let $d_A^* = (-1)^{2n+1} \star d_A \star$ denote the formal adjoint to d_A. Since A is a local extremum, the term linear in t must vanish, so for every η, we must have

$$(F_A, d_A\eta) = (d_A^* F_A, \eta) = 0$$

Then since up to sign $d_A^* = \star d_A \star$ and $\star$ is an isomorphism, we have $d_A \star F_A = 0$. ■

The first variation gives us what are referred to as the ***Yang-Mills equations***

$$\begin{aligned} d_A F_A &= 0 \\ d_A^* F_A &= 0 \end{aligned}$$

Definition 2.10. A ***Yang-Mills connection*** is a connection $A \in \mathscr{A}(P)$ satisfying the Yang-Mills equations, i.e. a local extremum of L.

Exercise 2.11. In the case that $G = \mathrm{U}_1$, show that the curvature of a connection A can be identified as an element of Ω_X^2. Show that A is a Yang-Mills connection if and only if F_A is a harmonic form, i.e. $\Delta F_A = 0$, where $\Delta = dd^* + d^*d$ is the Hodge Laplacian. Use this to show that the space of Yang-Mills connections on a principal U_1-bundle P is a torsor over the vector space of closed 1-forms on X.

The first equation is simply the ***Bianchi identity*** and the second comes from the first variation.

Proposition 2.12 (***The second variation***)**.** *Let A be a Yang-Mills connection. Then for every $\eta \in \Omega_X^1(\mathfrak{g}_P)$, we have*

$$\left.\frac{d}{dt}\right|_{t=0} d_{A+t\eta}^* F_{A+t\eta} = d_A^* d_A \eta + \star[\eta \wedge \star F_A]$$

The proof of this is similar to the proof of the first variation, and involves expanding out $d_{A+t\eta}^* F_{A+t\eta}$ and then taking the term linear in t. If we think of L as a Morse function on $\mathscr{A}(P)$, for a Yang-Mills connection A, the operator $d_A^* d_A + \star[\cdot \wedge \star F_A]$ can be interpreted as the Hessian of L at the critical point A. In particular, if η is tangent to the critical submanifold of Yang-Mills connections, one can use the Atiyah-Singer index theorem with this operator to compute the dimension of the space of Yang-Mills connections.

We now restrict ourselves to the case where X is a Riemann surface. Let $\Gamma_{\mathbb{R}}$ denote the central extension of $\pi_1(X)$ by $\mathbb{R}$ where if we let J denote a generator for $\mathbb{R}$, we have the relation $\prod_i [a_i, b_i] = J$ where the a_i and b_i are the generators for the usual presentation of a closed surface of genus g. Using this group one can prove the following theorems, though we will omit the proofs.

Theorem 2.13. *Every principal G-bundle $P \to X$ admits a Yang-Mills connection.*

Theorem 2.14. *There is a bijective correspondence*

$$\mathrm{Hom}(\Gamma_{\mathbb{R}}, G)/G \longleftrightarrow \{\textit{Principal } G\textit{-bundles } P \to X \textit{with a Yang-Mills connection}\} \;/\sim$$

where the action of G is conjugation and the equivalence relation is gauge equivalence.

The second theorem should be thought of an analogue of the classical Riemann-Hilbert correspondence.

Exercise 2.15. The classical Riemann-Hilbert correspondence gives a bijection

$$\mathrm{Hom}(\pi_1(X), G)/G \leftrightarrow \{\text{Principal } G\text{-bundles } P \to X \text{ equipped with a flat connection } A\} \,/ \sim$$

where the action of G on $\mathrm{Hom}(\pi_1(X), G)$ is by conjugation, and the equivalence relation is gauge equivalence of conenctions. The correspondence assigns to $\rho \in \mathrm{Hom}(\pi_1(X), G)$ the associated bundle

$$\widetilde{X} \times_{\pi_1(X)} G$$

where $\widetilde{X}$ is the universal cover of X, and the connection is the one induced by descending the trivial connection on $\widetilde{X} \times G$ to the quotient. In the other direction, the holonomy of a flat connection defines (up to conjugation by G) a homomorphism $\pi_1(X) \to G$.

A principal U_1-bundle $P \to X$ corresponds to a Hermitian line bundle $L \to X$ by taking the associated bundle $P \times_{\mathrm{U}_1} \mathbb{C}$ with the standard action of U_1 on $\mathbb{C}$. Using this correspondence and the classical Riemann-Hilbert correspondence, show that there is a bijection

$$\mathrm{Hom}(\Gamma_{\mathbb{R}}, \mathrm{U}_1)/\mathrm{U}_1 \longleftrightarrow \{\text{Principal } \mathrm{U}_1\text{-bundles } P \to X \text{with a Yang-Mills connection}\} \,/ \sim$$

For the rest of our discussion, we will restrict to case where $G = \mathrm{U}_n$. We first make a remark involving the proofs of the two preceding theorems. As with the U_1 case, the data of a principal U_n-bundle $P \to X$ is equivalent to a rank n complex vector bundle $E \to X$ equipped with a Hermitian metric. In the proofs, one shows that a Yang-Mills connection A is equivalent to the choice of a Lie algebra element $X \in \mathfrak{u}_n$. Writing $X = -2\pi i \Lambda$ for a Hermitian matrix Λ, the Yang-Mills condition implies that the trace of Λ is equal to the first Chern class of E, thought of as an integer by integrating over X. If we let λ_i denote the i^{th}-eigenvalue (arranged in ascending order) and n_i the multiplicity of λ_i, one can show that $n_i \lambda_i$ must also be integral. These observations will be useful when we relate Yang-Mills connections with holomorphic vector bundles.

3. Holomorphic Vector Bundles and Yang-Mills Connections

A good references for this material would be [4] and [6]

Let X denote a complex manifold.

Definition 3.1. A ***holomorphic vector bundle*** is a complex vector bundle $\pi : E \to X$ such that the total space E is a complex manifold and π is holomorphic.

Given a holomorphic vector bundle $E \to X$, we can find a trivialization of E such that the transition functions are holomorphic. In a neighborhood $U \subset X$ such that $E|_U$ is trivial, the smooth sections can be identified with functions $U \to \mathbb{C}^n$, and the holomorphic sections can be identified with the holomorphic functions $U \to \mathbb{C}^n$. We have a local operator $\overline{\partial}$, which we can apply componentwise to a local section to get an operator on smooth sections over U. Furthermore, since $\overline{\partial}$ annihilates holomorphic functions and the transition functions are holomorphic, we have that $\overline{\partial}$ glues to a well defined operator $\overline{\partial}_E : \mathcal{A}^0_X(E) \to \mathcal{A}^{0,1}_X(E)$. The holomorphic sections of E are then exactly the sections annihilated by $\overline{\partial}_E$. Furthermore, the operator $\overline{\partial}_E$ extends to operators $\overline{\partial}_E : \mathcal{A}^k_X(E) \to \mathcal{A}^{k+1}_X(E)$, and satifies the condition $\overline{\partial}_E^2 = 0$, since $\overline{\partial}^2 = 0$. The punchline is that the holomorphic structure on E is entirely determined by this operator.

Theorem 3.2. *Let $\pi : E \to X$ be a C^∞ complex vector bundle, and let $D : \mathcal{A}^0_X(E) \to \mathcal{A}^{0,1}(E)$ be an operator satifying $D^2 = 0$. Then there exists a unique complex structure on E such that π is holomorphic and D coincides with the operator $\overline{\partial}_E$.*

This can be seen as a linearized version of the Newlander-Nirenberg theorem. In particular, a holomorphic vector bundle $E \to X$ can be thought of as a smooth vector bundle along with a choice of operator $\overline{\partial}_E$. Since the operator $\overline{\partial}$ satisfies a Leibniz rule, the operator $\overline{\partial}_E$ behaves like a connection. In a *smooth* local trivialization, we can write

$$\overline{\partial}_E = \overline{\partial} + B$$

where B is a smooth $\mathrm{M}_n\mathbb{C}$-valued $(0, 1)$-form. Indeed, we have that the space of holomorphic structures on a smooth vector bundle $E \to X$ is an affine space over $\mathcal{A}^{1,0}(\operatorname{End} E)$. We let $\mathscr{C}(E)$ denote the space of holomorphic structures on E.

We now restrict to the case where X is a Riemann surface.

Definition 3.3. The ***slope*** of a holomorphic vector bundle $E \to M$ is

$$\mu(E) := \frac{c_1(E)}{\operatorname{rank}(E)}$$

where we think of $c_1(E) \in H^2(X, \mathbb{Z})$ as an integer via integration over X.

Sometimes the integer $c_1(E)$ is also referred to as the ***degree*** of E. One thing to note is that the slope of a holomorphic vector bundle is independent of the holomorphic structure – both the degree and rank are topological invariants, and only depend on the underlying C^∞ complex vector bundle.

Definition 3.4. A holomorphic vector bundle $E \to X$ is

1. ***Stable*** if for every holomorphic subbundle $F \subset E$, we have $\mu(F) < \mu(E)$.
2. ***Semistable*** if for every holomorphic subbundle $F \subset E$, we have $\mu(F) \leq \mu(E)$.
3. ***Unstable*** if E is not semistable.

While the slope is a topological invariant, stability is not, since we only consider holomorphic subbundles – which depend on the holomorphic structure. The terminology comes from Geometric Invariant Theory (GIT). The main result will use is:

Theorem 3.5 (***The Harder-Narasimhan Filtration***)**.** *Let $E \to X$ be a holomorphic vector bundle. Then E admits a canonical filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

by holomorphic subbundles E_i such that E_i/E_{i-1} is semistable and

$$\mu(E_1/E_0) > \mu(E_2/E_1) > \cdots > \mu(E_n/E_{n-1})$$

The proof of the above theorem is not extremely difficult, but we omit it. The main idea is that any holomorphic vector bundle has a unique maximal semistable subbundle, which we take to be E_1. We then take E_2 to be the preimage of the maximal semistable bundle of E_1/E_0 under the quotient map, and continue inductively. The slopes $\mu_i := \mu(E_i/E_{i-1})$ gives us n rational numbers. If k denotes the rank of E, then we construct an element of $\mathbb{Q}^k$ by arranging the μ_i in order, and repeating the entry μ_i a total of $\operatorname{rank}(E_i/E_{i-1})$ times.

We call this vector the ***Harder-Narasimhan type*** of E.

Our ultimate goal will be to relate moduli spaces of holomorphic vector bundles over X to Yang-Mills connections. To see this, let $E \to X$ be a C^∞ complex vector bundle of rank n, and fix a Hermitian metric on E. Then let $P \to X$ denote the principal U_n-bundle of frames for E. We abbreviate the gauge group $\mathscr{G}(P)$ as $\mathscr{G}$.

Proposition 3.6. *There is a bijection $\mathscr{A}(P) \leftrightarrow \mathscr{C}(E)$.*

Proof. We provide maps in both directions. Suppose we have a connection $A \in \mathscr{A}(P)$. Then $\mathcal{A}$ induces a covariant derivative $d_A : \mathcal{A}^0_X(E) \to \mathcal{A}^1_X(E)$. The $(0,1)$ part of d_A automatically satisfies $(d_A^{0,1})^2 = 0$, since $\mathcal{A}^2_X = 0$ by dimension reasons. Therefore, $d_A^{0,1}$ defines a holomorphic structure on E.

In the other direction, given a holomorphic structure $\overline{\partial}_E$, there exists a unique Hermitian connection A such that $d_A^{1,0} = \overline{\partial}_E$ called the ***Chern connection***, which is a sort of analogue to the Levi-Civita connection in Riemannian geometry. ■

Let $\mathscr{G}_\mathbb{C}$ denote the group of smooth bundle automorphisms of E. Though both $\mathscr{G}_\mathbb{C}$ and $\mathscr{G}$ are both infinite dimensional, the former can be seen as the complexification of the latter. The space $\mathscr{C}(E)$ has a natural action by $\mathscr{G}_\mathbb{C}$ by conjugation. Furthermore, the orbits under this action are exactly the isomorphism classes of holomorphic structures on E. This is most easily seen by characterizing an isomorphism $\varphi : E \to F$ of holomorphic vector bundles as a smooth bundle isomorphism intertwining $\overline{\partial}_E$ and $\overline{\partial}_F$. However, the naïve quotient $\mathscr{C}(E)/\mathscr{G}_\mathbb{C}$ is poorly behaved (for example, it is not Hausdorff). To remedy this, we restrict our attention to semistable bundles.

The relationship between $\mathscr{G}_\mathbb{C}$ and $\mathscr{G}$ as well as the identification of $\mathscr{A}(P)$ and $\mathscr{C}(E)$ suggests that isomorphism classes of holomorphic bundles should have something to do with gauge equivalence classes of connections on P. This is turns out to be true, and is an infinite dimensional version of the relationship between a GIT quotient and a symplectic quotient. To investigate further, we make a short digression regarding this relationship.

Let G be a reductive complex group, and X a Kähler manifold with Kähler metric ω, equipped with a "nice" action of G. In the usual setting, X is a smooth projective variety with a fixed embedding $X \hookrightarrow \mathbb{CP}^N$, the Kähler metric ω is the restriction of the Fubini-Study form, and the G-action is induced by a homomorphism $G \to \mathrm{GL}_{N+1}(\mathbb{C})$. In general, the naïve quotient X/G is not well behaved, and one restricts the action to a subset X_{ss} consisting of ***semistable points*** to construct the ***GIT quotient*** X_{ss}/G.

Then let $K \subset G$ denote the maximal compact subgroup, which has the property that its complexification is isomorphic to G. Suppose that the action of K on X is symplectic, i.e. the action of any $k \in K$ preserves the Kähler metric on X. Let $\mathfrak{k}$ denote the Lie algebra of K. Then the infinitesimal action of K is given by the Lie algebra homomorphism $\mathfrak{k} \to \mathfrak{X}(X)$ (where $\mathfrak{X}(X)$ denotes the space of vector fields on X) defined by $\xi \mapsto X_\xi$ where

$$(X_\xi)_p := \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi) \cdot p$$

Definition 3.7. A symplectic action of K on X is ***Hamiltonian*** if for each $\xi \in \mathfrak{k}$, there exists a function $H_\xi : X \to \mathbb{R}$ such that for all $p \in X$ and $v \in T_pX$, we have

$$\omega_p((X_\xi)_p, v) = (dH_\xi)_p(v)$$

and the mapping $\xi \mapsto H_\xi$ is K-equivariant with respect to the right actions of K on $\mathfrak{k}$ by the Adjoint action and precomposition with left translation L_k on $C^\infty(X)$. The functions H_ξ are called ***Hamiltonian functions***.

Definition 3.8. Suppose we have a Hamiltonian action of K on X. A ***moment map*** for the action is a K-equivariant map $X \to \mathfrak{k}^*$ (where the action on $\mathfrak{k}$ is the coadjoint action) such that for any $p \in X$, $v \in T_pX$, and $\xi \in \mathfrak{k}$, we have

$$d\mu_p(v)(\xi) = \omega_p((X_\xi)_p, v)$$

One things to note is that the Hamiltonian functions can be recovered by the moment maps. If a Hamiltonian action admits a moment map, then

$$H_\xi(p) = \mu(p)(\xi)$$

The let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathfrak{k}^*$ that is invariant under the coadjoint action, and $\|\cdot\|$ the induced norm. Since X is compact, the map $\|\mu\|^2 : X \to \mathbb{R}$ attains its minimum, and WLOG we assume that the minimum value is 0.

Definition 3.9. The ***symplectic quotient*** of X by K is the quotient space

$$\mu^{-1}(0)/K$$

The symplectic quotient can also be referred to as the ***symplectic reduction*** or the ***Marsden-Weinstein quotient***.

Theorem 3.10. *The symplectic quotient of X by K admits a unique Kähler structure such that the Kähler metric on $\mu^{-1}(0)/K$ is induced by the Kähler metric on X.*

The relationship between the GIT quotient and the symplectic quotient is given by the Kempf-Ness theorem.

Theorem 3.11 (***Kempf-Ness***)**.** *Suppose a complex reductive group G acts on a Kähler manifold X such that the action of the maximal compact subgroup $K \subset G$ is Hamiltonian and admits a moment map $\mu : X \to \mathfrak{k}^*$. Then the G-orbit of any semistable point contains a unique K-orbit minimizing $\|\mu\|^2$. This establishes a homeomorphism*

$$X_{ss}/G \longleftrightarrow \mu^{-1}(0)/K$$

We now want to relate the previous discussion to our situation. Using the identification of $\mathscr{A}(P)$ and $\mathscr{C}(E)$, we want the action of $\mathscr{G}_\mathbb{C}$ to play the role of the complex reductive group G and the gauge group $\mathscr{G}$ to play the role of the maximal compact subgroup. Since the space $\mathscr{A}(P)$ is infinite dimensional, along with the groups $\mathscr{G}_\mathbb{C}$ and $\mathscr{G}$, we are working in an infinite dimensional setting, but we will gloss over the analytic details and work with them formally.

Our first task is to realize $\mathscr{A}(P)$ as a "Kähler manifold." Since X is a surface, the Hodge star maps $\mathcal{A}^1_X(\mathfrak{g}_P)$ to itself and squares to -1, so it defines a "complex structure" on $\mathscr{A}(P)$, where we use the fact that $\mathscr{A}(P)$ is affine over the vector space $\mathcal{A}^1_X(\mathfrak{g}_P)$ to identify the "tangent space" of $\mathscr{A}(P)$ at a connection A with $\mathcal{A}^1_X(\mathfrak{g}_P)$. Furthermore, the fact that for 1-forms $\omega, \eta \in \mathcal{A}^1_X(\mathfrak{g}_P)$ the pairing $\langle \omega \wedge \eta \rangle$ is skew-symmetric, we can identify the pairing

$$\omega \otimes \eta \mapsto \int_X \langle \omega \wedge \eta \rangle$$

as a "symplectic form" on $\mathscr{A}(P)$. Together, these give $\mathscr{A}(P)$ the structure of a "Kähler manifold."

Our next task is to show that the action of $\mathscr{G}$ on $\mathscr{A}(P)$ is "Hamiltonian" with respect to this Kähler structure. One can identify the "Lie algebra" of $\mathscr{G}$ with the space of sections $\Gamma(X, \mathfrak{g}_P)$.

Proposition 3.12. *The infinitesimal action of $\phi \in \Gamma(X, \mathfrak{g}_P)$ on $\mathscr{A}(P)$ is given by the mapping $A \mapsto d_A\phi$.*

Proof. We compute the vector field at a connection $A \in \mathscr{A}(P)$ to be

$$\begin{aligned}
&\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp(t\phi)^{-1}} A + \exp(t\phi)^*\theta = -[\phi, A] + \left.\frac{d}{dt}\right|_{t=0} (dL_{\exp(-t\phi)} d(\exp(t\phi))) \\
&= [A, \phi] + \left(\left.\frac{d}{dt}\right|_{t=0} dL_{\exp(-t\phi)} \right) d(\exp(0)) + dL_{\exp(0)} \left(\left.\frac{d}{dt}\right|_{t=0} d(\exp(t\phi)) \right) \\
&= [A, \phi] + d\phi \\
&= d_A\phi
\end{aligned}$$

where for the third equality we use the product rule, and in the fourth equality we use the fact that $\exp(0) = \mathrm{id}$ and that the derivative of $\exp(t\phi)$ as $t \to 0$ is ϕ. ∎

Proposition 3.13. *Let $\phi \in \Gamma(X, \mathfrak{g}_P)$. Then the function*

$$\begin{aligned}
H_\phi : \mathscr{A}(P) &\to \mathbb{R} \\
A &\mapsto \int_X \langle F_A \wedge \phi \rangle
\end{aligned}$$

is a Hamiltonian function for ϕ.

Exercise 3.14. Prove the previous proposition.

Since $\langle \cdot, \cdot \rangle$ is invariant under the adjoint action, the mapping $\phi \mapsto H_\phi$ is clearly $\mathscr{G}$ equivariant, so this tells us that the action is Hamiltonian. Furthermore, the computation we made identifies the mapping $A \mapsto F_A$ as the moment map for this action. To summarize, we have the following analogies

$$\begin{aligned}
\text{Kähler manifold } X &\longleftrightarrow \mathscr{A}(P) \\
\text{Complex reductive group } G &\longleftrightarrow \mathscr{G}_{\mathbb{C}} \\
\text{Maximal compact subgroup } K \subset G &\longleftrightarrow \mathscr{G} \\
\text{Moment map } \mu &\longleftrightarrow A \mapsto F_A \\
\text{Norm square of the moment map } \|\mu\|^2 &\longleftrightarrow L
\end{aligned}$$

The last missing piece is something analogous to the Kempf-Ness theorem.

Theorem 3.15 (***Narasimhan-Seshadri***)**.** *Let $\mathscr{A}_s(P) \subset \mathscr{A}(P)$ denote the subspace of connections that are absolute minimal for the Yang-Mills functional, and correspond to irreducible representations $\Gamma_{\mathbb{R}} \to \mathrm{U}_n$. Let $\mathscr{C}_s(E)$ denote the subspace of stable holomorphic structures on E. The isomorphism classes of holomorphic bundles in $\mathscr{C}_s(E)$ admit unique Yang-Mills connections* (*up to gauge equivalence*) *minimizing the Yang-Mills functional. In other words, there is a homeomorphism*

$$\mathscr{A}_s(P)/\mathscr{G} \longleftrightarrow \mathscr{C}_s(E)/\mathscr{G}_{\mathbb{C}}$$

Remark. The original proof is more algebraic in flavor. A proof more in the spirit of the Atiyah-Bott paper was given by Donaldson in [2]. The spirit of this proof is carried on by the proof of Hermitian-Yang-Mills and the nonabelian Hodge theorem, which were both grew out of the developments from the Atiyah-Bott paper.

One issue is that the Narasimhan-Seshadri theorem only works for stable bundles. However, in the case that the rank and degree of E are coprime, stability and semistability coincide for numerical reasons. For that reason, we will continue onwards with the assumption that the rank and degree of E are coprime.

4. Equivariant Cohomology

References for this material include [8] and [3]

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