Problem Formulation

Let a robot's trajectory through its environment be represented by a sequence of N poses: $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N$. Each pose lies on a manifold: $\mathbf{p}_i \in \mathcal{M}$. Simple examples of manifolds used in Graph SLAM include 1-D, 2-D, and 3-D space, i.e., $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$. These environments are *rectilinear*, meaning that there is no concept of orientation. By contrast, in $SE(2)$ problem settings a robot's pose consists of its location in $\mathbb{R}^2$ and its orientation θ. Similarly, in $SE(3)$ a robot's pose consists of its location in $\mathbb{R}^3$ and its orientation, which can be represented via Euler angles, quaternions, or $SO(3)$ rotation matrices.

As the robot explores its environment, it collects a set of M measurements $\mathcal{Z} = \{\mathbf{z}_j\}$. Examples of such measurements include odometry, GPS, and IMU data. Given a set of poses $\mathbf{p}_1, \ldots, \mathbf{p}_N$, we can compute the estimated measurement $\hat{\mathbf{z}}_j(\mathbf{p}_1, \ldots, \mathbf{p}_N)$. We can then compute the *residual* $\mathbf{e}_j(\mathbf{z}_j, \hat{\mathbf{z}}_j)$ for measurement j. The formula for the residual depends on the type of measurement. As an example, let $\mathbf{z}_1$ be an odometry measurement that was collected when the robot traveled from $\mathbf{p}_1$ to $\mathbf{p}_2$. The expected measurement and the residual are computed as

$$
\begin{aligned}
\hat{\mathbf{z}}_1(\mathbf{p}_1, \mathbf{p}_2) &= \mathbf{p}_2 \ominus \mathbf{p}_1 \\
\mathbf{e}_1(\mathbf{z}_1, \hat{\mathbf{z}}_1) &= \mathbf{z}_1 \ominus \hat{\mathbf{z}}_1 = \mathbf{z}_1 \ominus (\mathbf{p}_2 \ominus \mathbf{p}_1),
\end{aligned}
$$

where the $\ominus$ operator indicates inverse pose composition. We model measurement $\mathbf{z}_j$ as having independent Gaussian noise with zero mean and covariance matrix Ω_j^{-1}; we refer to Ω_j as the *information matrix* for measurement j. That is,

$$
p(\mathbf{z}_j \mid \mathbf{p}_1, \ldots, \mathbf{p}_N) = \eta_j \exp\big((-\mathbf{e}_j(\mathbf{z}_j, \hat{\mathbf{z}}_j))^T \Omega_j \mathbf{e}_j(\mathbf{z}_j, \hat{\mathbf{z}}_j)\big),
$$

where η_j is the normalization constant.

The objective of Graph SLAM is to find the maximum likelihood set of poses given the measurements $\mathcal{Z} = \{\mathbf{z}_j\}$; in other words, we want to find

$$
\underset{\mathbf{p}_1, \ldots, \mathbf{p}_N}{\arg\max}\, p(\mathbf{p}_1, \ldots, \mathbf{p}_N \mid \mathcal{Z})
$$

Using Bayes' rule, we can write this probability as

$$
\begin{aligned}
p(\mathbf{p}_1, \ldots, \mathbf{p}_N \mid \mathcal{Z}) &= \frac{p(\mathcal{Z} \mid \mathbf{p}_1, \ldots, \mathbf{p}_N) p(\mathbf{p}_1, \ldots, \mathbf{p}_N)}{p(\mathcal{Z})} \\
&\propto p(\mathcal{Z} \mid \mathbf{p}_1, \ldots, \mathbf{p}_N),
\end{aligned}
$$

since $p(\mathcal{Z})$ is a constant (albeit, an unknown constant) and we assume that $p(\mathbf{p}_1, \ldots, \mathbf{p}_N)$ is uniformly distributed. Therefore, we can simplify the Graph SLAM optimization as follows:

$$\begin{aligned}
\underset{\mathbf{p}_1,\cdots,\mathbf{p}_N}{\arg\max}\, p(\mathbf{p}_1,\ldots,\mathbf{p}_N \mid \mathcal{Z}) &= \underset{\mathbf{p}_1,\cdots,\mathbf{p}_N}{\arg\max}\, p(\mathcal{Z} \mid \mathbf{p}_1,\ldots,\mathbf{p}_N) \\
&= \underset{\mathbf{p}_1,\cdots,\mathbf{p}_N}{\arg\max} \prod_{j=1}^{M} p(\mathbf{z}_j \mid \mathbf{p}_1,\ldots,\mathbf{p}_N) \\
&= \underset{\mathbf{p}_1,\cdots,\mathbf{p}_N}{\arg\max} \prod_{j=1}^{M} \exp\big(-(\mathbf{e}_j(\mathbf{z}_j,\hat{\mathbf{z}}_j))^T\Omega_j\mathbf{e}_j(\mathbf{z}_j,\hat{\mathbf{z}}_j)\big) \\
&= \underset{\mathbf{p}_1,\cdots,\mathbf{p}_N}{\arg\min} \sum_{j=1}^{M} (\mathbf{e}_j(\mathbf{z}_j,\hat{\mathbf{z}}_j))^T\Omega_j\mathbf{e}_j(\mathbf{z}_j,\hat{\mathbf{z}}_j).
\end{aligned}$$

We define

$$\chi^2 := \sum_{j=1}^{M} (\mathbf{e}_j(\mathbf{z}_j,\hat{\mathbf{z}}_j))^T\Omega_j\mathbf{e}_j(\mathbf{z}_j,\hat{\mathbf{z}}_j),$$

and this is what we seek to minimize.

Dimensionality and Pose Representation

Before proceeding further, it is helpful to discuss the dimensionality of the problem. We have:

- A set of N poses $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_N$, where each pose lies on the manifold $\mathcal{M}$
  - Each pose $\mathbf{p}_i$ is represented as a vector in (a subset of) $\mathbb{R}^d$. For example:
    - An $SE(2)$ pose is typically represented as (x, y, θ), and thus $d = 3$.
    - An $SE(3)$ pose is typically represented as $(x, y, z, q_x, q_y, q_z, q_w)$, where (x, y, z) is a point in $\mathbb{R}^3$ and (q_x, q_y, q_z, q_w) is a *quaternion*, and so $d = 7$.
  - We also need to be able to represent each pose compactly as a vector in (a subset of) $\mathbb{R}^c$.
    - Since an $SE(2)$ pose has three degrees of freedom, the (x, y, θ) representation is again sufficient and $c = 3$.
    - An $SE(3)$ pose only has six degrees of freedom, and we can represent it compactly as (x, y, z, q_x, q_y, q_z), and thus $c = 6$.
  - We use the $\boxplus$ operator to indicate pose composition when one or both of the poses are represented compactly. The output can be a pose in $\mathcal{M}$ or a vector in $\mathbb{R}^c$, as required by context.
- A set of M measurements $\mathcal{Z} = \{\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_M\}$
  - Each measurement's dimensionality can be unique, and we will use $\bullet$ to denote a "wildcard" variable.
  - Measurement $\mathbf{z}_j \in \mathbb{R}^\bullet$ has an associated information matrix $\Omega_j \in \mathbb{R}^{\bullet\times\bullet}$ and residual function $\mathbf{e}_j(\mathbf{z}_j, \hat{\mathbf{z}}_j) = \mathbf{e}_j(\mathbf{z}_j, \mathbf{p}_1, \ldots, \mathbf{p}_N) \in \mathbb{R}^\bullet$.
  - A measurement could, in theory, constrain anywhere from 1 pose to all N poses. In practice, each measurement usually constrains only 1 or 2 poses.

Graph SLAM Algorithm

The "Graph" in Graph SLAM refers to the fact that we view the problem as a graph. The graph has a set $\mathcal{V}$ of N vertices, where each vertex v_i has an associated pose $\mathbf{p}_i$. Similarly, the graph has a set $\mathcal{E}$ of M edges, where each edge e_j has an associated measurement $\mathbf{z}_j$. In practice, the edges in this graph are either unary (i.e., a loop) or binary. (Note: e_j refers to the edge in the graph associated with measurement $\mathbf{z}_j$, whereas $\mathbf{e}_j$ refers to the residual function associated with $\mathbf{z}_j$.)

We want to optimize

$$\chi^2 = \sum_{e_j \in \mathcal{E}} \mathbf{e}_j^T \Omega_j \mathbf{e}_j.$$

Let $\mathbf{x}_i \in \mathbb{R}^c$ be the compact representation of pose $\mathbf{p}_i \in \mathcal{M}$, and let

$$\mathbf{x} := \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_N \end{bmatrix} \in \mathbb{R}^{cN}$$

We will solve this optimization problem iteratively. Let

$$\mathbf{x}^{k+1} := \mathbf{x}^k \boxplus \Delta\mathbf{x}^k = \begin{bmatrix} \mathbf{x}_1 \boxplus \Delta\mathbf{x}_1 \\ \mathbf{x}_2 \boxplus \Delta\mathbf{x}_2 \\ \vdots \\ \mathbf{x}_N \boxplus \Delta\mathbf{x}_2 \end{bmatrix}$$

The χ^2 error at iteration $k+1$ is

$$\chi_{k+1} = \sum_{e_j \in \mathcal{E}} \underbrace{\left[\mathbf{e}_j(\mathbf{x}^{k+1})\right]^T}_{1\times\bullet} \underbrace{\Omega_j}_{\bullet\times\bullet} \underbrace{\mathbf{e}_j(\mathbf{x}^{k+1})}_{\bullet\times 1}.$$

We will linearize the errors as:

$$\begin{aligned} \mathbf{e}_j(\mathbf{x}^{k+1}) &= \mathbf{e}_j(\mathbf{x}^k \boxplus \Delta\mathbf{x}^k) \\ &\approx \mathbf{e}_j(\mathbf{x}^k) + \frac{\partial}{\partial \Delta\mathbf{x}^k}\left[\mathbf{e}_j(\mathbf{x}^k \boxplus \Delta\mathbf{x}^k)\right] \Delta\mathbf{x}^k \\ &= \mathbf{e}_j(\mathbf{x}^k) + \left(\left. \frac{\partial \mathbf{e}_j(\mathbf{x}^k \boxplus \Delta\mathbf{x}^k)}{\partial(\mathbf{x}^k \boxplus \Delta\mathbf{x}^k)} \right|_{\Delta\mathbf{x}^k = \mathbf{0}} \right) \frac{\partial(\mathbf{x}^k \boxplus \Delta\mathbf{x}^k)}{\partial \Delta\mathbf{x}^k} \Delta\mathbf{x}^k. \end{aligned}$$

Plugging this into the formula for χ^2, we get:

$$
\begin{aligned}
\chi^2_{k+1} \approx \quad & \sum_{e_j \in \mathcal{E}} \underbrace{[\mathbf{e}_j(\mathbf{x}^k)]^T}_{1\times\bullet} \underbrace{\Omega_j}_{\bullet\times\bullet} \underbrace{\mathbf{e}_j(\mathbf{x}^k)}_{\bullet\times 1} \\
& + \sum_{e_j \in \mathcal{E}} \underbrace{[\mathbf{e}_j(\mathbf{x^k})]^T}_{1\times\bullet} \underbrace{\Omega_j}_{\bullet\times\bullet} \underbrace{\left(\left.\frac{\partial \mathbf{e}_j(\mathbf{x}^k \boxplus \Delta\mathbf{x}^k)}{\partial(\mathbf{x}^k \boxplus \Delta\mathbf{x}^k)}\right|_{\Delta\mathbf{x}^k=\mathbf{0}} \right)}_{\bullet\times dN} \underbrace{\frac{\partial(\mathbf{x}^k \boxplus \Delta\mathbf{x}^k)}{\partial \Delta\mathbf{x}^k}}_{dN\times cN} \underbrace{\Delta\mathbf{x}^k}_{cN\times 1} \\
& + \sum_{e_j \in \mathcal{E}} \underbrace{(\Delta\mathbf{x}^k)^T}_{1\times cN} \underbrace{\left(\frac{\partial(\mathbf{x}^k \boxplus \Delta\mathbf{x}^k)}{\partial \Delta\mathbf{x}^k} \right)^T}_{cN\times dN} \underbrace{\left(\left.\frac{\partial \mathbf{e}_j(\mathbf{x}^k \boxplus \Delta\mathbf{x}^k)}{\partial(\mathbf{x}^k \boxplus \Delta\mathbf{x}^k)}\right|_{\Delta\mathbf{x}^k=\mathbf{0}} \right)^T}_{dN\times\bullet} \underbrace{\Omega_j}_{\bullet\times\bullet} \underbrace{\left(\left.\frac{\partial \mathbf{e}_j(\mathbf{x}^k \boxplus \Delta\mathbf{x}^k)}{\partial(\mathbf{x}^k \boxplus \Delta\mathbf{x}^k)}\right|_{\Delta\mathbf{x}^k=\mathbf{0}}}_{\bullet\times dN} \\
= \chi^2_k + 2\mathbf{b}^T \Delta\mathbf{x}^k + (\Delta\mathbf{x}^k)^T H \Delta\mathbf{x}^k,
\end{aligned}
$$

where

$$
\mathbf{b}^T = \sum_{e_j \in \mathcal{E}} \underbrace{[\mathbf{e}_j(\mathbf{x^k})]^T}_{1\times\bullet} \underbrace{\Omega_j}_{\bullet\times\bullet} \underbrace{\left(\left.\frac{\partial \mathbf{e}_j(\mathbf{x}^k \boxplus \Delta\mathbf{x}^k)}{\partial(\mathbf{x}^k \boxplus \Delta\mathbf{x}^k)}\right|_{\Delta\mathbf{x}^k=\mathbf{0}} \right)}_{\bullet\times dN} \underbrace{\frac{\partial(\mathbf{x}^k \boxplus \Delta\mathbf{x}^k)}{\partial \Delta\mathbf{x}^k}}_{dN\times cN}
$$

$$
H = \sum_{e_j \in \mathcal{E}} \underbrace{\left(\frac{\partial(\mathbf{x}^k \boxplus \Delta\mathbf{x}^k)}{\partial \Delta\mathbf{x}^k} \right)^T}_{cN\times dN} \underbrace{\left(\left.\frac{\partial \mathbf{e}_j(\mathbf{x}^k \boxplus \Delta\mathbf{x}^k)}{\partial(\mathbf{x}^k \boxplus \Delta\mathbf{x}^k)}\right|_{\Delta\mathbf{x}^k=\mathbf{0}} \right)^T}_{dN\times\bullet} \underbrace{\Omega_j}_{\bullet\times\bullet} \underbrace{\left(\left.\frac{\partial \mathbf{e}_j(\mathbf{x}^k \boxplus \Delta\mathbf{x}^k)}{\partial(\mathbf{x}^k \boxplus \Delta\mathbf{x}^k)}\right|_{\Delta\mathbf{x}^k=\mathbf{0}} \right)}_{\bullet\times dN} \underbrace{\frac{\partial(\mathbf{x}^k \boxplus \Delta\mathbf{x}^k)}{\partial \Delta\mathbf{x}^k}}_{dN\times cN}
$$

Using this notation, we obtain the optimal update as

$$
\Delta\mathbf{x}^k = -H^{-1}\mathbf{b}.
$$