Lec 1 2025/9/3

Random Experiment, Sample Space, Event, & Probability

Random Experiment: Activity, process, or experiment whose outcome is uncertain

Sample Space: Set of all possible outcomes of experiment:

$\Omega = \{ \omega_1, \omega_2, \cdots \}$ Properties: (finite / infinite, continuous / not continuous, ...)

It can be either finite or infinite kind of outcome, and outcomes can be continuous or not as well.

Event: Collection (subset) of outcomes contained in the sample space Ω

Simple: contains only one outcome

Compound: is consisted of more than one outcome

$A = \{ \cdots \}$ and $A \subseteq \Omega$

Head of event, denoted as capital letters

Probability: the proportion of event A occuring, written as $P(A)$

Can be seen as a function

$P: A \rightarrow [0, 1]$

sample space of Ω

Relative Frequency interpretation of probability

Let A be an event associated with some random experiment.

The proportion of A occurance (under infinite tracking of experiment) is probability of A, written as

$P(A)$

Remark: This is the relative frequency interpretation of probability, which says that the probability of event is equal to its relative frequency in a large number of trials.

Limitations on the relative frequency interpretation of probability:

① One cannot actually repeat an experiment infinity many times

② To define probablity carefully, we need to take a formal, axiomatic, mathematical approach

Lec 2 Preview

Discrete sample space:

We say a sample space is discrete if it is either finite or countably infinite

Countably infinite

A set is countably infinity if the elements of the set can be arranged as a sequence

Remark: and all countably infinite sets can be put in one-to-one corresponding with natural number (as they are sequence)

Assuming sample sets discussed are discrete, which means it can be written in

$\Omega = \{w_1, w_2, \cdots, w_k\}$ or

$\Omega = \{w_1, w_2, \cdots\}$

Properties of probabilities and event operations

Event operators: or, and, not

Set operators: union, intersection, complement

Set terminologies and notations:

Capital letters: set

A set consists of a_1, a_2, a_3: $A = \{a_1, a_2, a_3\}$

S: denotes the universal set which is the set of all possible objects

ϕ: denotes as null or empty set ($\phi = \{\}$), a set without any object

Subset:

For two set A, B,

If every object in A is in B,

Then A is a subset of B, denoted $A \subseteq B$

Hint, ϕ is subset of every set

Union:

The union of A and B is the set that all objects are in A or B.

Denoted: $A \cup B$

Intersection

The intersection of A and B is the set that all objects are in both A and B

Denoted $A \cap B$

Complement:

If A is a subset of S, then compliment of A is the set of all object in S that is not in A.

Denoted: A^c

Hint: $A^c \cup A = S$

$A^c \cap A = \phi$

Disjoint:

Two sets A and B are said to be disjoint or mutually exclusive if they have no element in common:

If $A \cap B = \phi$

Then A and B are disjoint

Independent:

Two events are independent if occurance of one event does not alter the probability of occurance of the other in any way.

Logic as set operators (symbolization from sentence to set)

Important laws for set operators:

Common:

$A \cap B = B \cap A$

$A \cup B = B \cup A$

Associate:

$(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

Distribution:

$A \cap (B \cup C) = (A \cap B) \cup (B \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

DeMorgan's laws

$(A \cup B)^c = A^c \cap B^c$

$(A \cap B)^c = A^c \cup B^c$

De Morgan's laws general:

$$\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c$$

$$\left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} A_i^c$$

Probability Functions

In random experiment with sample space Ω, the probability of an event A, denoted as $P(A)$ is a function assigned to event A a numerical value that chance that event A will occurs.

Three axioms:

① $P(A) \geq 0$

② $P(\Omega) = 1$

③ For a set of disjoint event $A_1, A_2, \cdots, A_n$ in Ω

Probability Function:

Given a random experiment with discrete sample space Ω, a probability function P is a function on Ω with following properties:

- $P(w) \geq 0$, for all $w \in \Omega$ always non-negative
- $\sum_{w = \Omega} P(w) = 1$ sum of all event is 1
- For all events $A \in \Omega$, $P(A) = \sum_{w \in A} P(w)$ the probability of A is sum of all its event probability

Probability on discrete sample space:

For finite sample space $\Omega = \{w_1, \dots, w_k\}$

then $\sum_{w \in \Omega} P(w) = P(w_1) + \dots + P(w_k) = 1$

For countable infinite sample space $\Omega = \{w_1, \dots\}$

$$\sum_{w = \Omega} P(w) = \sum_{i=1}^{\infty} P(w_i) = 1$$

Properties of probability function: additive law

Thm: The additive law of probability: (Inclusive-Exclusive Principle)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Thm: The additive law of probability: general case

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \cdots$$
$$+ (-1)^{r+1} \sum_{i_1 < i_2 < \cdots i_r} P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_r}) +$$
$$(-1)^{n+1} P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_n})$$

Thm: Addition rule for mutually exclusive events

If A and B are mutually events, then

$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B)$

since $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

For mutually exclusive set A, B, $P(A \cup B) = P(A) + P(B)$

then $P(A \cap B) = \emptyset$

Thm: The complement rule:

For an event A

$P(A) = 1 - P(A^c)$

Thm: The hierarchical rule

If A implies B ($A \subseteq B$), then

$P(A) \leq P(B)$

Otherwise

$P(B) < P(A)$, then $A \subseteq B$ is false

Lec 3 Preview

Thm: Probability as Relative Frequency

$$P(A) = \frac{\# \text{ of outcomes in } A}{\text{total } \# \text{ of outcomes in random experiment}} = \frac{n(A)}{n(\Omega)}$$

Thm: Weal Law of Large Numbers (WLLN)

When you take more and more trials, the sample average gets closer and closer to that long-run probability.

(repeat experiments will reduce randomness)

Thm WLLN, basic form:

Let $X_1, X_2 \dots$ be random variables with finite mean $\mu = \mathbb{E}[X_i]$ and finite variance $\sigma^2 = Var(X_1) < \infty$, Define the sample mean

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

Then $\bar{X}_n \xrightarrow{P} \mu$

ie. For $\forall \varepsilon > 0$, $\lim_{n \to \infty} P_r(|\bar{X}_n - \mu| < \varepsilon) = 0$

Thm The multiplication principle (Foundamental principal of counting (FPC))

A n-element sequence $(a_1, \cdots, a_n)$ have $k_1, \cdots k_n$ number of choice.

The total number of possible sequence is

$$\prod_{i=1}^{n} k_i = k_1 \cdot k_2 \cdot \cdots \cdot k_n$$