Lec 2

$\frac{dx}{dt} = f(x, t)$ differential equation

Different kinds of differential equation

Linear :

- linear in x :

  $\frac{dx}{dt} = a(t) \cdot x + b(t)$

Standard form

$p(t)x'(t) + q(t)x(t) = r(t)$

Reduced standard form

$x'(t) + p(t)x(t) = q(t)$

Example :

A cup of coffee cooling down

$C(t)$: temperature of coffee after t times

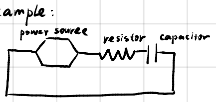
A : temperature of surrounding

$\frac{dc}{dt} = -k(C - A)$ $\quad$ k: certain constant

if temperature is higher than surrounding, negative, Otherwise positive (to indicate cooling)

$\frac{1}{k} \cdot \frac{dc}{dt} + C(t) = A$

Example :



$U(t) = V_R(t) + V_C(t)$

voltage of resistor $\quad$ voltage of capacitor

$V_R(t) = RI(t)$

current in t

$V_C'(t) = \frac{1}{C} \cdot I(t)$

capacitance

$U'(t) = V_R'(t) + V_C'(t)$

$V'(t) = R \cdot I'(t) + \frac{1}{C} \cdot I(t)$

Integrating Factor :

$\frac{dy}{dx} + P(x) \cdot y = q(x)$

Recall: $\frac{d}{dx}(u(x) \cdot v(x)) = u'(x)v(x) + u(x)v'(x)$

let : $\phi(x) = e^{\int p(x)\,dx}$

$\phi(x) \cdot y'(x) + \underbrace{\phi'(x)}_{= \phi(x) \cdot p(x)} \cdot y = \phi(x) \cdot q(x)$

$y(x) \cdot \phi(x) = \int \phi(x) \cdot q(x)\,dx$

$y(x) = \frac{1}{\phi(x)} \int \phi(x) \cdot q(x)\,dx$

$y(x) = e^{-\int p(x)dx} \cdot \int q(x) \cdot e^{\int p(x)dx} dx$

Example :

$\frac{dc}{dt} + kC(t) = kA$

$\phi(t) = e^{\int k\,dt}$

$= e^{kt+c}$

$\phi(t) \frac{dc}{dt} + \phi(t) \cdot k \cdot C(t) = k \cdot A \cdot \phi(t)$

$\phi(t) \frac{dc}{dt} + \phi'(t) C(t) = k \cdot A \cdot \phi(t)$

$\phi(t) \cdot c(t) = \int k \cdot A \cdot \phi(t)\,dt$

$c(t) = \frac{\int k \cdot A \cdot e^{kt+c}\,dt}{e^{kt+c}}$

$= [A \cdot e^{kt+c} + B] \cdot e^{-kt}$

$C(t) = A + B \cdot e^{-kt}$

$t = 0$, coffee $= 80\,k$, room temp $= 15\,k$

$C(0) = 15 + B \cdot e^{-k \cdot 0}$

$80 = 15 + B$

$B = 65$

$y' = y - x^2$

$y' - y = -x^2$

$\phi(x) = e^{\int -1 dx} = e^{-x}$

$\phi(x) \cdot y'(x) - \phi(x) \cdot y = \phi(x) \cdot -x^2$

$\phi(x) \cdot y'(x) + \phi'(x) y = \phi(x) \cdot -x^2$

$y \cdot \phi(x) = \int x^2 e^{-x} dx$

Unit 1 First-Order Differential Equations

Sec 1.1 Differential Equations and Mathematical Models

Differential equation: an equation relating an unknown function and one or more of its derivative.

EX 1: $\frac{dx}{dt} = x^2 + t^2$ is a differential equation as it contains an unknown function $x(t)$, and its derivative $\frac{dx}{dt} = x'(t)$

Three principal goals to study differential equations:

1. To discover the differential equation that describes a specified physical situation
2. To find the approciate solution of that equations
3. To interpret solution is found

In differential equation, we are expected to find out a function that satisfies the equations.

Example 2:

$\frac{dy}{dx} = 2xy$

If c is a constant, and

$y(x) = Ce^{x^2}$

then

$\frac{dy}{dx} = C \cdot 2x \cdot e^{x^2} = 2xy$

then $y(x) = Ce^{x^2}$ is one of solutions

Example 3:

Consider the equation (Newton's law of cooling)

$\frac{dT}{dt} = -k(T - A)$

k: a constant

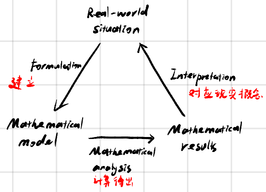
T: temperature of a matter at time t

A: surrounding temperature.

We may need to find out formula of $T(t)$.

Process of mathematical modeling involves:

1. A formulation of real-world problem in mathematical terms
2. The analysis of resulting mathematical problems
3. The interpretation of mathematical result under real-world context



Example 8:

Verify $y(x) = 2x^{\frac{1}{2}} - x^{\frac{1}{2}} \cdot \ln x$ satisfy

$4x^2y'' + y = 0$ (to verify is a solution of 0)

$y'(x) = x^{-\frac{1}{2}} - (\frac{1}{2}x^{-\frac{1}{2}} \cdot \ln x + x^{\frac{1}{2}} \cdot \frac{1}{x})$

$= x^{-\frac{1}{2}} - x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}} \cdot \ln x$

$= -\frac{1}{2}x^{-\frac{1}{2}} \cdot \ln x$

$y''(x) = -\frac{1}{2}(-\frac{1}{2}x^{-\frac{3}{2}} \cdot \ln x + x^{-\frac{1}{2}} \cdot \frac{1}{x})$

$= \frac{x^{-\frac{3}{2}} \cdot \ln x}{4} - \frac{1}{2}x^{-\frac{3}{2}}$

$4x^2(y''(x)) + y' = x^{\frac{1}{2}} \cdot \ln x - 2x^{\frac{1}{2}} + 2x^{\frac{1}{2}} - x^{\frac{1}{2}} \ln x = 0$ as required

However, differential equation may not have a solution

have infinite / two more solutions

The order of differential equation is order of the highest derivative that appears in it.

Examples:

$y' + y = 0$, ord: 1

$y'' + 3y' + 2y = 0$, ord = 2

$y''' + (y')^2 = \sin(x)$, ord = 3

General form of n-th order of differential equation:

$F(x, y, y', \dots, y^{n}) = 0$, which includes $n+2$ variables

Solution of differential equation $F(x, u, \dots, u^{n}) = 0$ on Interval I is a continuous function $u = u(x)$ s.t. $u', \dots, u^{(n)}$ exist on I and

ensure derivative exists

$F(x, u, u', \dots, u^{(n)}) = 0$

Sec 1.2 Integrals as General and Particular Solutions

For a general form of differential equation:

$$\frac{dy}{dx} = f(x)$$

If $G(x)$ has an antiderivative $G'(x)$, and $G'(x) \equiv f(x)$.

then
$$\int \frac{dy}{dx} = \int f(x)\,dx$$
$$y = G(x) + C$$ will gain a series of solutions of y

And we need more information to determine what the arbitrary constant "C" is.

Example 1:

$\frac{dy}{dx} = 2x + 3$. $y(1) = 2$.

$\int \frac{dy}{dx} = \int 2x+3\,dx$

$y = x^2 + 3x + C$

take $y(1) = 2$

$2 = 4 + C$, $C = -2$

$y = x^2 + 3x - 2$

Example. a lunar lander is falling toward to moon at 450 m/s
Its retrorocket provides a constant decleration of $2.5\,m/s^2$,
At what height the lunar lander can land the surface with $v=0$?

$a(t) = 2.5$

$v(t) = \int a(t)\,dt = 2.5t + C_1$

$d(t) = \int v(t) = \frac{5}{4}t^2 + C_1 t + C_2$

we need to gain $d(0)$

since initial velocity is 450m/s, then

$v(0) = 450$, $C_1 = 450$. and

$d(t) = \frac{5}{4}t^2 + 450t + C_2$

since we know $t = \frac{v}{a} = \frac{450}{2.5} = 180s$

then $d(180) = 40,500\,m$

Physical unit may meet:

mks unit (metre, kelogram, second)

Force	N
Mass	kg
Distance	m
Time	s
Gravity	$9.8\,m/s^2$

Example 3

(a) A ball is thorwn straight upward from the ground ($y_0 = 0$) with velocity $v_0 = 96\,ft/s$, and it reaches maximum height when $v=0$
($g \approx 32\,ft/s$)

$h(t) = \int v(t)\,dt = \iint a(t)\,dt$

$a(t) = -32\,ft/s^2$ for opposite direction

$v(t) = -32t + C$. since $v_0 = 96\,m/s$, $C = 96\,ft/s$. then

$v(t) = -32t + 96$

for $v(t) = 0$. $t = 3$

and $d(t) = -16t^2 + 96t + C$

since we thrown at height of 0m, then $d(0) = 0$, $C = 0$

$d(3) = 144$

(b) an arrow shot upward from $h=0$ with velocity $=49$, when will it returns ?
(use $g = 9.8\,m/s^2$)

$d(t) = \int v(t)\,dt = \iint a(t)\,dt$

$v(t) = -9.8t + C$, $a(t) = 9.8$

$v(0) = 49$, $v(t) = -9.8t + 49$

$d(t) = -4.9t^2 + 49t + C$

$d(0) = 0$, $d(t) = -4.9t^2 + 49$, for $d(t) = 0$, $t = 10$, or 0

take $t = 10$, as 0 is starting point

Sec 1.3 Slope Fields and Solution Curves (may need more exercise)

Consider differential equation below:

$$\frac{dy}{dx} = f(x, y)$$

You may want to solve it as shown:

$$\int \frac{dy}{dx} = \int f(x, y)\, dx$$

$$y = \int f(x, y)\, dx$$

But it does not make sense as $f(x,y)$ includes a unknown function y

Slope Fields and Graphical Solutions

There is a geometric way to think about solutions of a given differential equation $y' = f(x, y)$:

At each point of x, y, determine $m = f(x, y)$, then we can generate y' by differentiating m / find out its tangent in the diagram

And derivative of these lines are solution curve of $y' = f(x, y)$

Important definition:

Solution Curve: actual graph satisfies solution of differential equations

Direction field / slope field: (when we can't compute the solution), visualize the $y(x)$ by mapping $y' = f(x, y)$ each $f(x,y)$ points, to understand the shape of y.

It is like a map of arrow to guide how the function is shaped

Existence of unique solution

Before solving, we need to know whether differential equation:

1. have a solution
2. have a unique solution

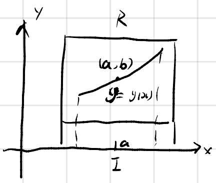
$f(x, y, \dots)$

Partial derivative to x:

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y, \dots) - f(x, y, \dots)}{0}$$

Thm 1 Existence and Uniqueness of Solutions ? (Picard-Lindelöf Theorem)

Suppose that both the function $f(x, y)$ and its partial derivative $D_y f(x, y)$ are Continuous on some rectangular R in x-y plane that contain point (a, b) in its interior. Then for some open interval I containing point a, the initial problem



$$\frac{dy}{dx} = f(x, y), \quad y(a) = b$$

has one and only one solution that defined in I.

Picard-Lindelöf Theorem (A theorem to deal with an initial value problem [IVP])

For a differential equation $f(x, y)$, with initial value $y(x_0) = y_0$

If $f(x, y)$ is continuous in some rectangle around (x_0, y_0) and $\frac{\partial f}{\partial y}$ is also continuous in this region

Then there exists at least one solution $y(x)$ to the IVP and solution is unique

Example 6:

$$x \frac{dy}{dx} = 2y, \quad y(0) = 0$$

Sol: $\frac{dy}{dx} = \frac{2y}{x}$

$$\frac{\partial f}{\partial y} = \frac{2}{x}, \quad y(0) = 0$$

since $\frac{2}{x}$ is not continuous at $x = 0$, then solution of $f(x, y)$ don't have unique solution

1.3 Application: Computer-Generated Slope fields and solution curves

Sec 1.4 Separate Equations and Applications

For differential equation:

$$\frac{dy}{dx} = H(x, y)$$

Is called separable if it can be written as function of x and y

$$H(x, y) = g(x) \cdot h(y) = \frac{dy}{dx}$$

Then we can compute

$$\frac{1}{h(y)} \cdot dy = g(x) \cdot dx$$

$$\ln(y) = \int g(x)\, dx$$

$$= \quad f(y) \cdot dy = g(x)\, dx$$

$$\int f(y(x)) \cdot \frac{dy(x)}{dx} \cdot dx = \int g(x)\, dx + C$$

$$\int f(y) \cdot dy = \int g(x)\, dx + C$$

take $F(x) = \int f(y)\, dy$, $G(x) = \int g(x)\, dx$

$$D_x[f(y(x))] = F'(y(x)) \cdot y'(x) = f(y) \cdot \frac{dy}{dx} = g(x) = D_x[G(x)]$$

$$F(y(x)) = G(x) + C$$

by using partial derivative

Example 1

$$\frac{dy}{dx} = -6xy \qquad y(0) = 7$$

$$\frac{1}{y} \cdot dy = -6x\, dx$$

$$\ln|y| = -3x^2 + C$$

$$\ln 7 = C$$

$$\ln|y| = -3x^2 + \ln 7$$

$$y = e^{-3x^2 + \ln 7}$$

$$y = 7 \cdot e^{-3x^2}$$

Example 2

$$\frac{dy}{dx} = \frac{4-2x}{3y^2-5} \qquad y(1) = 3$$

$$3y^2 - 5\, dy = 4 - 2x\, dx$$

$$y^3 - 5y = 4x - x^2 + C$$

$$12 = 3 + C$$

$$C = 9$$

$$y^3 - 5y = 4x - x^2 + 9$$

$$H(x, y) = 4x - x^2 + 9 - y^3 + 5y$$

We can solve y for specific value of x

Implicit, General, and Singular Solutions:

The equation for $K(x, y) = 0$ is commonly called an implicit solution of a differential equation if it satisfy some solution for $y = y(x)$

A solution of a differential equation that contains an "arbitrary constant" (c) is commonly called a general solution of the differential equation

Singular solution: a differential equation solution that is not gained by choose C in general solutions.

Ex: $$y = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right)$$

?

Sec 1.5 Linear First-Order Function

Consider $H(x,y) = \frac{dy}{dx}$

(first order) Linear form of differential equation:

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \left[\frac{dy}{dx} = -P(x)y + Q(x) \right]$$ more like linear but not good for computing

If $H(x,y)$ can be written in linear form, then we can take

$$\phi(x) = e^{\int P(x)\,dx}$$ and

$$\phi(x)\cdot\frac{dy}{dx} + \phi(x)P(x)\,y = \phi(x)\cdot Q(x)$$

Imagine $\frac{d\,\phi(x)\,y}{dx} = \phi'(x)\cdot y + \phi(x)\cdot y'$

$= P(x)\cdot\phi(x)\cdot y + \phi(x)\cdot y'$

Make integral on both side: [it is not formed, should be done by partial derivative)

$$\int \phi(x)\frac{dy}{dx} + \phi(x)P(x)\cdot y\,dx = \int \phi(x)\cdot Q(x)\,dx$$

We finally get

$$\phi(x)\cdot y = \int \phi(x)\cdot Q(x)\,dx + C$$

$$y = e^{-\int P(x)\,dx}\cdot\int e^{\int P(x)\,dx}\cdot Q(x)\,dx + C$$

Summary: how to do solution with first-order linear differential equation:

1. Begin with $\phi(x) = e^{\int P(x)\,dx}$
2. Multiply $\phi(x)$ on both sides of equation
3. Recognize left side is:

   $D_x[\phi(x)\,y(x)] = \phi(x)\cdot Q(x)$

4. Integral both side

   $\phi(x)\,y(x) = \int \phi(x)\,Q(x)\,dx + C$

Unit 6 Numerical Methods

6.1 Numberical Approximation: Euler's Method

It is the exception rather than the rule when a differential equation of general form:

$$\frac{dy}{dx} = f(x, y)$$

can be solved exactly and explicitly by elementary method (discussed in chp 1)

elementary function:

built using:

Alegbraic: polynomials, rational func, roots

Exponential / logarithmic: e^x, $\ln x$

Trigonometric and inverse trigonometric:

Any finite combination of using add, substraction, multiplication, divison and composition

Euler's method is a kind of approximation / plotter method to deal with equation with nonelementary function

Eg. error function: $\operatorname{erf}(x) = e^{x^2}$

why nonelementary function:

1. Limit of symbolic integration

   integral cannot be expressed in elementary form

2. Clarity in solutions of differential equation

3. Special usage in physics / real-world application

4. Some theorem may consider it

How to use Euler's Method: (similar to linear approximation)

1. take a step size $h = a$

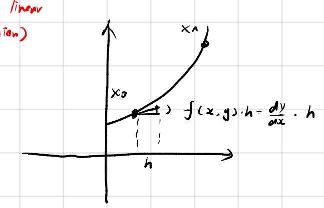
2. compute target x_n from x_0 by:

$x_1 = x_0 + h$, $y_1 = y_0 + h \cdot f(x_0, y_0)$

$x_2 = x_1 + h$, $y_2 = y_1 + h\, f(x_1, y_1)$

...

$x_n = x_{n-1} + h$, $y_n = y_{n-1} + f(x_{n-1}, y_{n-1})$



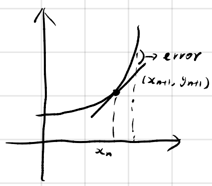
Local and cumulative errors:

Euler's method may have large error when trying to compute y_n at very large n.

The error in the linear approximation:

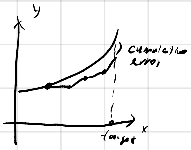
$$y(x_{n+1}) \approx y(x_n) + h \cdot f(x_n, y_n) = y_{n+1}$$

is amount by which the tangent line by which the tangent at (x_n, y_n) departs from solution curve through (x_n, y_n).



For each step's error, it is called local error

Cumulative error:



6.2 A closer Look at the Euler Method

Thm 1 The Error in the Euler Method

Suppose:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

has a unique solution $y(x)$ on the closed interval $[a, b]$ with $a = x_0$,

and assume that $y(x)$ has a continuous second derivative on $[a, b]$.

[to ensure f_x and f_y are continuous for $a \le x \le b$, and $c \le y \le d$. where $c \le y(x) \le d$ for all $x \in [a, b]$)

need more research on how to compute/compare level of error of approximation

Then there exists a constant C such that the following is true:

If the approximation $y_1, \cdots, y_k$ to actual value $y(x_1), \cdots, y(x_k)$ at point of $[a, b]$ are computed using Euler's Method with step size $h > 0$, then

$$|y_n - y(x_n)| \leqq C \cdot h$$

for each $n = 1, \cdots, k$

An Improvement in Euler's Method

Improved Euler's Method:

Given:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

Suppose, carry n steps and step size h

we want to compute $x_n = x_0 + nh$, $f(x_n)$

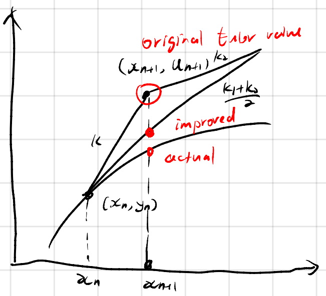
① compute $u_{n+1} = y(x_{n+1})$ in Euler's Method

$$u_{n+1} = y_n + h \cdot f(x_n, y_n) = y_n + h \cdot k_1$$

② now $u_{n+1} \approx y(x_{n+1})$ has been compute, we can take

$$k_2 = f(x_{n+1}, u_{n+1})$$ [by Euler's Method]

③ take average between k_1 and k_2



Algorithm description:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

Step size $= h$

$$k_1 = f(x_n, y_n)$$

$$u_{n+1} = y_n + h \cdot k_1$$

$$k_2 = f(x_{n+1}, u_{n+1})$$

$$y_{n+1} = y_n + h \cdot \frac{1}{2}(k_1 + k_2)$$

summarize:

$$u_{n+1} = y_n + h \cdot f(x_n, y_n)$$

$$y_{n+1} = y_n + h \cdot \frac{1}{2}[f(x_n, y_n) + f(x_{n+1}, u_{n+1})]$$