Section 1: Effective Rates of Interest and Discount

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ACT240: Mathematics of Investment and Credit

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Introduction

- In this section, we introduce the concept of interest
- Interest is
  - The cost of borrowing money
  - The return on investment for saving money
- By the end of this section, you should be able to:
  - Understand how to move money from time 1 to time 2
  - Understand the difference between simple and compound interest
  - Understand the difference between present and future value
  - Understand the concepts of accumulating and discounting

Interest

- Interest: cost of borrowing money or the return on investment for saving money
- Amount of interest depends on:
  - Principal (initial amount)
  - Interest rate
  - Time period
- Ways of quoting interest rates:
  - As a percentage: 10%
  - As a decimal: 0.1
- Conventions:
  - Rates are quoted for specific periods of time (e.g., per year, per month)
  - Standard practice: annual interest rate

Interest

Simple setup:

- **Principal**: the initial amount of money
- **Interest rate**: the percentage charged on the principal
- **Time**: the duration for which the money is borrowed or invested

Here is a breakdown of the amount repaid:

- **Principal** (amount borrowed)
- **Amount of interest charged**
  - Amount borrowed $\times$ interest rate for time period

Effective rate of interest

Definition (Effective rate of interest)

The effective rate of interest i is the interest earned at the end of a period by an investment of one unit made at the beginning of the period.

Interest and time

- Quoting separate interest rates for every period (e.g. 1 month, 1 quarter, 1 year, 5 years) is impractical
- Recording all possible time frames would be cumbersome
- Standard practice:
  1. Quote a single **annual effective interest rate**.
  2. Use an **interest rule** to derive rates for other periods
- Two most common interest rules: simple interest and compound interest

Simple interest

- Suppose the annual simple interest rate is i
- An investment of 1 unit for t years will earn interest given by:

$$\text{Interest} = i \times t$$

- The total amount owed after t years will be:

$$\text{Total Amount} = \text{Principal} + \text{Interest} = 1 + i \times t$$

Time conventions

- Simple interest requires a measure of time t
- Common conventions:
  1. Ordinary simple interest:

$$t = \frac{m}{12}, \quad m = \text{number of months}$$

  2. Exact simple interest:

$$t = \frac{d}{365}, \quad d = \text{exact number of days}$$

  3. Banker's rule (variation):

$$t = \frac{d}{360}$$

- In practice, simple interest is usually limited to periods shorter than one year

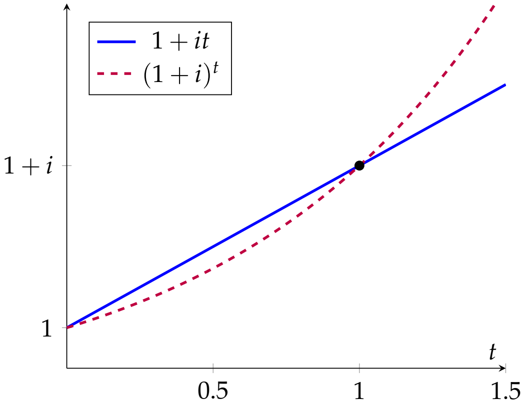
Compound interest

- A more common scenario involves continuous compounding
- At the end of each period, automatically reinvest the interest

Compound interest

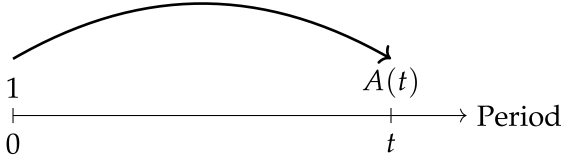
- At annual effective compound interest rate i, an initial investment of 1 accumulates to $(1+i)^t$ at time t
- Usually t positive integer, but fractional t possible (called true or exact compound interest)
- Effective interest rate typically refers to **annual** compounding, it can also be defined for other periods such as monthly or quarterly
- In such cases t is measured in the appropriate unit of time (months, quarters, etc); see nominal rates of interest in Section 2

Comparison of simple vs compound accumulation



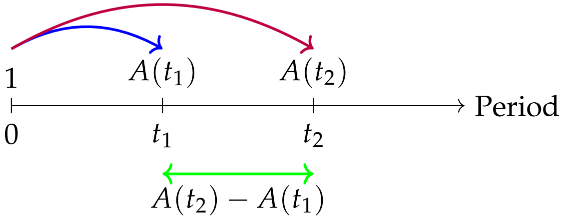
Accumulation function

- When we are interested in how an initial investment grows over time, we are interested in the future value
- The **accumulation function** determines the future value of a 1 unit investment at time t
- Also called the **future value** function



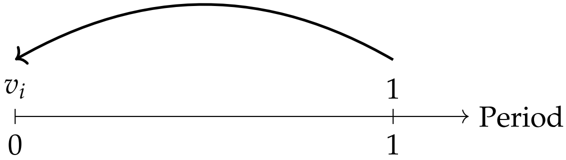
Accumulation function

- We have $A(0) = 1$
- The amount of interest earned at time t is $A(t) - 1$
- The amount of interest earned between t_1 and t_2 is $A(t_2) - A(t_1)$



Present value

- When we are interested in how much a **future payment** is worth today, we are interested in the present value
- The present value of 1 due in one year is the amount that, if invested today at the given interest rate, will accumulate to 1 in one year
- Using compound interest at rate i, an amount of X invested today will accumulate to $X(1+i)$ in one year
- Solving for $X(1+i) = 1$, we find $X = (1+i)^{-1}$
- In actuarial notation, $v_i = (1+i)^{-1}$ is called the present value factor or discount factor (we sometimes omit the subscript when it is clear from context)



Present value

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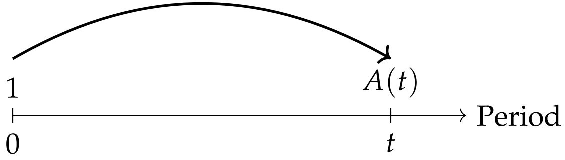
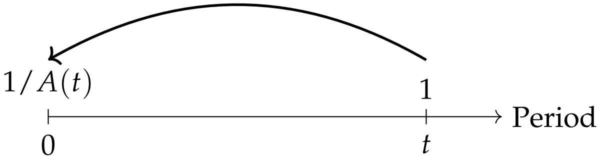
- The present value of 1 due in t years is the amount that, if invested today at the given interest rate, will accumulate to 1 in t years
- Using compound interest at rate i, the present value of 1 due in t years is $v_i^t = (1+i)^{-t}$
- Using simple interest at rate i, the present value of 1 due in t years is $1/(1+it)$

Present value vs future value

- Future value: invest today and **accumulate** over time



- Present value: **discount** future value to today



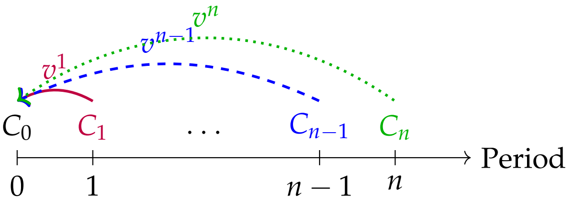
Equation of value

- An investment usually consists of cash outflows and inflows over time
- To compare different investment options, we need to compare the cash flows associated with each option
- In general, to compare different cash flows occurring at different times, we need to compute the present value or future value of all cash flows
- To do so, we must pick a **comparison date**, a date to which all cash flows will be discounted or accumulated
- The **equation of value** accumulates or discounts all past and future cash flows to the comparison date

Net present value

- The **net present value** (NPV) is the present value of all cash flows associated with an investment
- The NPV is an equation of value where the comparison date is period 0
- Let C_t be the net cash flow at time t (negative for outflow, positive for inflow)
- If the target rate of return per period is i, then the net present value is

$$\text{NPV} = \sum_{k=0}^{n} C_k v^k$$



Example

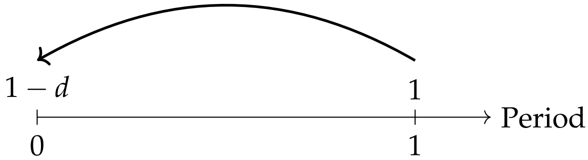
At an annual effective interest rate of $i, i > 0$, each of the following two sets of payments has present value K:

- Payment of 121 immediately and another payment of 121 at the end of one year
- Payment of 144 at the end or two years and another payment of 144 at the end of three years

Calculate K.

Rate of discount

- The **rate of discount** is an alternative way to measure interest
- The **annual effective rate of discount** d is defined as the the value such that 1 - d is the present value of 1 due in one year
- The rate of discount is useful for discounting (computing present values)



Difference between rate of interest and rate of discount

- Rate of interest

$$i = \frac{i}{1} = \frac{A(1) - A(0)}{A(0)} = \frac{\text{Amount of interest for the time period}}{\text{Amount borrowed at the beginning of the period}}$$

- Rate of discount

$$d = \frac{A(1) - A(0)}{A(1)} = \frac{\text{Amount of interest for the time period}}{\text{Amount owing at the end of the period}}$$

Equivalent rates

- We have seen two different rates:
  - The annual effective interest rate i
  - The annual effective rate of discount d
- The rates of interest and discount are **equivalent** if they describe the same accumulation over a given time period
- If i and d are equivalent annual rates, then we have the relationship

$$1 - d = v_i = \frac{1}{1+i}$$

- We can also find

$$d = \frac{i}{1+i} \quad i = \frac{d}{1-d}$$

- We can also show that $id = i - d$ and $1/d = 1/i + 1$
- Note that $d < i$ for all positive i, but different numbers can represent the same accumulation or discount

Average rate of interest

- In some cases, the interest rate may vary over time
- The **average rate of interest** over a period is the constant rate that would yield the same accumulated value as a series of varying rates
- When using compound interest, the arithmetic average is not appropriate
- In general, for n years with yearly effective rates of interest $i_1, i_2, \ldots, i_n$, we have

$$(1+i)^n = (1+i_1)(1+i_2)\cdots(1+i_n)$$