1 Parametric equations

1.1 Mapping curves

Do get a curve:

$x = f(t),\ y = g(t)$

where $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ is parametric, t is a parameter

EG: ① $x = t^2 - 3$

$y = t + 2$

For $-3 \leq t \leq 3$

$t = y - 2$

$x = y^2 - 4y + 4 - 3$

$x = y^2 - 4y + 1$

② $x = \cos(t)$

$y = \sin(t)$, $0 \leq t \leq 2\pi$

$x^2 + y^2 = \cos^2(t) + \sin^2(t) = 1$

$\cos^{-1}(x) = t$

$y = \sin(\cos^{-1}(x))$

Hint compare it with

$x = \cos(2t)$

$y = \sin(2t)$

1.2 Derivative:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

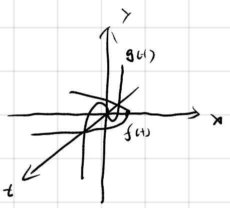
$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

1.3 Areas

Find the area enclosed:

$x = 1 - t^2 = f(t)$

$y = t - t^3 = g(t)$



$$\text{Area} = \int_{a_1}^{a_2} y\, dx = \int_{a_1}^{a_2} y \cdot \frac{dx}{dt} \cdot dt$$

$$= \int_{a_1}^{a_2} y \cdot f'(t)\, dt$$

$$= \int_{a_1}^{a_2} g(t) \cdot f'(t)\, dt \text{, for same reason, } = \int_{a_1}^{a_2} f(t) \cdot g'(t)\, dt$$

1.4 Arc length

$$\text{Arc} = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

From Arc-formula in 2D

Chp 1 Parametric equations

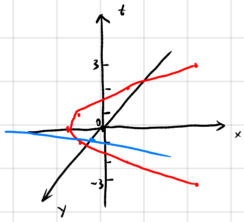
1.1 Mapping Curves

To make a curve :

$x = f(t)$ $\quad f: \mathbb{R} \to \mathbb{R},\ g: \mathbb{R} \to \mathbb{R}$

$y = g(t)$, where t is parameter

t	x	y
-3	6	-1
-2	1	0
-1	-2	1
0	-3	2
1	-2	3
2	1	4
3	6	5



EX 1:

$x = t^2 - 3$

$y = t + 2$

To check : make parameter equation to cartesian form

$t = y - 2$

$x = (y-2)^2 - 3$

$x = y^2 - 4y + 1$

EX 2

$x = \cos(t)$

$y = \sin(t)$

$t = \cos^{-1}(x)$

$y = \sin(\cos^{-1}(x))$

$$\boxed{\begin{aligned} y^2 &= \sin^2(\cos^{-1}(x)) \\ y^2 &= (1 - \cos^2(\cos^{-1}(x)) \\ y^2 &= 1 - x^2 \\ y^2 + x^2 &= 1 \end{aligned}}$$

Formula of circle:

$x = x_0 + r \cdot \cos(t)$

$y = y_0 + r \cdot \sin(t)$

center : (x_0, y_0)

radium : r

$$\frac{x - x_0}{r} = \cos(t)$$

$$t = \cos^{-1}\left(\frac{x - x_0}{r}\right)$$

$$\frac{y - y_0}{r} = \sin(t)$$

$$\left(\frac{y - y_0}{r}\right)^2 = 1 - \cos^2\left(\cos^{-1}\left(\frac{x - x_0}{r}\right)\right)$$

$$\left(\frac{y - y_0}{r}\right)^2 = 1 - \left(\frac{x - x_0}{r}\right)^2$$

$$y^2 + y_0^2 - 2yy_0 = r^2 - x^2 + 2xx_0 - x_0^2$$

1.2 Calculus with Parametric

Suppose two functions:

$x = f(t),\ y = g(t)$

If $\frac{dx}{dt} \neq 0$ and $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

Then $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Horizontal tangent:

$\frac{dy}{dt} = 0$, and $\frac{dx}{dt} \neq 0$

Vertical tangent:

$\frac{dx}{dt} = 0$, and $\frac{dy}{dt} \neq 0$

Second Derivative:

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{\frac{d\left(\frac{dy}{dx}\right)}{dt}}{\frac{dx}{dt}}$$

EX: $x = t^2,\ y = t^3 - 3t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d\left(\frac{dy}{dx}\right)}{dt}}{\frac{dx}{dt}} = \frac{\left(\frac{3t^2-3}{2t}\right)}{2t} = \frac{3t^2 - 3}{4t^2}$$

$t = 2,\ \frac{dy}{dt} = \frac{9}{4}$

1.3 Area:

Area under $f(x) = y$ between a and b: $\int_a^b f(x)\,dx = \int_a^b y\,dx$

Consider $x = f(t),\ y = g(t)$

$A = \int_a^b y\,dx = \int g(t)\,d(f(t)) = \int_\alpha^\beta g(t) f'(t)\,dt = \int_a^b y(t) \cdot x'(t)\,dt$ $\qquad$ Hint $a = f(\alpha),\ b = f(\beta)$

Ex Find area under loop

Lec 1

Foundamental Theorem of Calculus (FTC)

$$F(x) = \int_a^x f(t)\,dt$$

IF f is continuous

THEN $F(x)$ is differentiable and $F'(x) = f(x)$ } Part I

$$\int_a^b f(t)\,dt = F(b) - F(a)$$ } Part II

$\partial[a,b] = \{a,b\}$ boundary sign: ∂

A sample form of integral

$$\underbrace{\int_{D}}_{\text{Domain } [a,b]} \underbrace{f(x)\,dx}_{\text{Differential of Function}} = \underbrace{f}_{\text{Function}}\Big|_{\underbrace{\partial[a,b]=\{a,b\}}_{\text{Boundary}}}$$

Interplay analytic & geometric

Course outline:

FALL: Differentiation
- Different kind of Multivariable Function
- Partial Derivative
- Optimization
- Combination of Diff from 135

WINTER:
- Integral of multivariable funcs
- New types of integral
- Combination of 136

12.1 Functions on two variables:

$f: \mathbb{R} \to \mathbb{R}$ Single variable function
(Domain, Range)

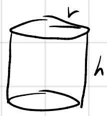
with some descriptions (differentiable, continuous, ...)

Representation: algebraically, verbally, geometrically, numerically

Recall: $f: \underbrace{\mathbb{R} \times \mathbb{R}}_{\text{independent var}} \to \underbrace{\mathbb{R}}_{\text{dependent var}}$

Algebraically: $f(x,y)$ is a formula which gives the value of the function f with input x, and y

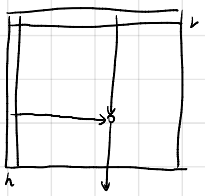
Example: A cylinder with closed ends has radius r and height of h. If its volumne is V and its surface area is A, find the formula of function $V = f(r,h)$ and $A = g(r,h)$



$f(r,h) = \pi r^2 \cdot h$ = Base area $\times$ height

$g(r,h) = 2\pi r \cdot h + 2\pi r^2$ = $2\times$ Base + Side Area

Numerically: Table of values



Verbally:

Let $T = f(d,t)$, where T is temperature, and t is minutes after a heater is turned on, at distance d from the heater

a) Is T an increasing or decreasing function of t? Explain

Increasing

You should firstly fix the other variable

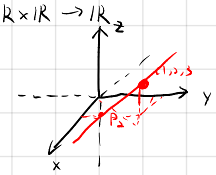
b) Is T an increasing or decreasing function of d? Explain

Decreasing.

Graphically : Visualizing a function

- Graphs
- shape of various levels or sections of graph

$f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$



P_1 (1, 2, 3)
P_2 (0, 0, -1)

3D

12.1 – 12.2 , P698. 699, 703, 704 & Videos