Autocovariance, Autocorrelation, Distance Autocovariance and Distance Autocorrelation

Recall. Let $\{X_t\}_{t\in\mathbb{Z}}$ be a stochastic process with mean $\mu = \mathbb{E}[X_t]$ and variance $\sigma^2 = \mathbb{V}[X_t]$ (assumed finite).

- **Covariance between two time points:** For any two times i, j, the covariance is

$$\mathrm{Cov}(X_i, X_j) = \mathbb{E}\big[(X_i - \mu)(X_j - \mu)\big].$$

  This measures the linear dependence between the values of the process at times i and j.

- **Autocovariance as a function of lag:** If the process is (weakly) stationary, the covariance depends only on the *lag* $h = j - i$, not on the absolute time:

$$\gamma(h) = \mathrm{Cov}(X_t, X_{t+h}) = \mathbb{E}\big[(X_t - \mu)(X_{t+h} - \mu)\big], \quad t \in \mathbb{Z}.$$

  In this case, we simply refer to it as the *autocovariance at lag* h.

- The *autocorrelation function* at lag h is

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)}.$$

- Given observations $X_1, \ldots, X_n$, the *plug-in estimator* of the autocovariance is

$$\hat\gamma(h) = \frac{1}{n}\sum_{t=1}^{n-h}(X_t - \bar X)(X_{t+h} - \bar X), \quad \bar X = \frac{1}{n}\sum_{t=1}^{n} X_t,$$

  and the empirical autocorrelation is

$$\hat\rho(h) = \frac{\hat\gamma(h)}{\hat\gamma(0)}.$$

Exercise 1. Autocovariance and autocorrelation of White Noise

- Simulate a white noise process $X_t \sim \mathcal{N}(0, 5^2)$ of length $n = 500$.
- Plot the generated time series.
- Compute the empirical autocovariance $\hat\gamma(h)$ and autocorrelation $\hat\rho(h)$ for lags $h = 0, 1, \ldots n-1$ manually and using `R` function `acf()`.
- Derive the theoretical autocovariance and autocorrelation function of the process.
- Compare the empirical and theoretical obtained values.

Exercise 2. Autocovariance and autocorrelation of an AR(1) process

Consider the AR(1) model:

$$X_t = 0.7X_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, 2^2).$$

- Simulate a trajectory of length $n = 500$
- Plot the generated time series.
- Compute the empirical autocovariance $\hat{\gamma}(h)$ and autocorrelation $\hat{\rho}(h)$ for lags $h = 0, 1, \dots n-1$ manually and using R function `acf()`.
- Derive the theoretical autocovariance and autocorrelation function of the process.
- Compare the empirical and theoretical obtained values.

Exercise 3. Autocovariance and autocorrelation of an MA(1) process

Consider the MA(1) model:

$$X_t = \varepsilon_t + 0.5\varepsilon_{t-1}, \quad \varepsilon_t \sim \mathcal{N}(0, 2^2).$$

- Simulate a trajectory of length $n = 500$.
- Plot the generated time series.
- Compute the empirical autocovariance $\hat{\gamma}(h)$ and autocorrelation $\hat{\rho}(h)$ for lags $h = 0, 1, \dots n-1$ manually and using R function `acf()`.
- Derive the theoretical autocovariance and autocorrelation function of the process.
- Compare the empirical and theoretical obtained values.