AR2 model

we have:

$$Y_t = \phi Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t \qquad , \varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$$

Full likelihood:

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$$L(\theta) = \frac{1}{2\pi \det(\Sigma)^{1/2}} \exp\left(-\frac{1}{2}[Y_1, Y_2]^T \Sigma^{-1} [Y_1, Y_2]^T\right) \prod_{t=3}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \left(-\frac{(Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2})^2}{2\sigma^2}\right) \qquad \text{with } \Sigma = \begin{bmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{bmatrix}$$

$$\ell(\theta) = -\log(2\pi) - \frac{1}{2}\log(\det(\Sigma)) - \frac{1}{2}[Y_1, Y_2]^T \Sigma^{-1} [Y_1, Y_2]^T - \frac{n-2}{2}$$

consider $|\phi_2| < 1$, $|\phi_1 + \phi_2| < 1$, $|\phi_2 - \phi_1| < 1$, and therefore Y_t is stationary.

$$\begin{aligned} Cov(Y_t, Y_{t+h}) &= cov(Y_t, Y_{t-h}) \\ &= \mathbb{E}[Y_t Y_{t-h}] \quad \text{as} \quad \mathbb{E}[Y_t] = 0 \\ &= \mathbb{E}[\{\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t\} Y_{t-h}] \\ &= \phi_1 \mathbb{E}[Y_{t-1} Y_{t-h}] + \phi_2 \mathbb{E}[Y_{t-2} Y_{t-h}] + \mathbb{E}[\varepsilon_t Y_{t-h}] \end{aligned}$$

hence we obtain:

$$\text{I} \quad \gamma(0) = \phi_1 \gamma(1) + \phi_2 \gamma(2) + \sigma^2$$

$$\text{II} \quad \gamma(1) = \phi_1 \gamma(0) + \phi_2 \gamma(1)$$

$$\text{III} \quad \gamma(2) = \phi_1 \gamma(1) + \phi_2 \gamma(0)$$

also called Yules Walker equation

1) find $\gamma(1)$ in terms of $\gamma(0)$ from II

$$\begin{aligned} \gamma(1) &= \phi_1 \gamma(0) + \phi_2 \gamma(1) \\ \gamma(1) - \phi_2 \gamma(1) &= \phi_1 \gamma(0) \\ (1 - \phi_2)\gamma(1) &= \phi_1 \gamma(0) \end{aligned}$$

$$\boxed{\gamma(1) = \frac{\phi_1 \gamma(0)}{1 - \phi_2}}$$

2) find $\gamma(2)$ in terms of $\gamma(0)$

$$\begin{aligned} \gamma(2) &= \phi_1 \gamma(1) + \phi_2 \gamma(0) \\ &= \phi_1 \left\{\frac{\phi_1 \gamma(0)}{1 - \phi_2}\right\} + \phi_2 \gamma(0) \\ &= \frac{\phi_1^2 \gamma(0)}{1 - \phi_2} + \phi_2 \gamma(0) = \frac{\phi_1^2 \gamma(0)}{1 - \phi_2} + \frac{(1 - \phi_2)\phi_2 \gamma(0)}{(1 - \phi_2)} = \boxed{\frac{\gamma(0)\{\phi_1^2 + (1 - \phi_2)\phi_2\}}{1 - \phi_2}} \end{aligned}$$

Now, we plug in $\gamma(1)$ and $\gamma(2)$ in I

$$\begin{aligned} \gamma(0) &= \phi_1 \gamma(1) + \phi_2 \gamma(2) + \sigma^2 \\ &= \phi_1 \frac{\phi_1 \gamma(0)}{1 - \phi_2} + \phi_2 \frac{\gamma(0)\{\phi_1^2 + (1 - \phi_2)\phi_2\}}{1 - \phi_2} + \sigma^2 \\ &= \frac{\gamma(0)}{1 - \phi_2}\left[\phi_1^2 + \phi_2\{\phi_1^2 + (1 - \phi_2)\phi_2\}\right] + \sigma^2 \\ &= \frac{\gamma(0)}{1 - \phi_2}\left[\phi_1^2 + \phi_2 \phi_1^2 + \phi_2(1 - \phi_2)\phi_2\right] + \sigma^2 \\ &= \frac{\gamma(0)}{1 - \phi_2}\left[\phi_1^2 + \phi_2 \phi_1^2 + (\phi_2 - \phi_2^2)\phi_2\right] + \sigma^2 \\ &= \frac{\gamma(0)}{1 - \phi_2}\left[\phi_1^2 + \phi_2 \phi_1^2 + \phi_2^2 - \phi_2^3\right] + \sigma^2 \\ &= \frac{\gamma(0)}{1 - \phi_2}\left[\phi_1^2(1 + \phi_2) + \phi_2^2(1 - \phi_2)\right] + \sigma^2 \end{aligned}$$

Solve for $\gamma(0)$

$$\gamma(0) = \frac{\gamma(0)}{1 - \phi_2}\left[\phi_1^2(1 + \phi_2) + \phi_2^2(1 - \phi_2)\right] + \sigma^2$$

$$\gamma(0) - \frac{\gamma(0)}{1 - \phi_2}\left[\phi_1^2(1 + \phi_2) + \phi_2^2(1 - \phi_2)\right] = \sigma^2$$

$$\gamma(0)\left[1 - \frac{\phi_1^2(1 + \phi_2) + \phi_2^2(1 - \phi_2)}{1 - \phi_2}\right] = \sigma^2$$

so, $$\gamma(0) = \frac{\sigma^2}{1 - \frac{\phi_1^2(1 + \phi_2) + \phi_2^2(1 - \phi_2)}{1 - \phi_2}} = \frac{\sigma^2(1 - \phi_2)}{1 - \phi_2 - \phi_1^2(1 + \phi_2) - \phi_2^2(1 - \phi_2)} = \frac{\sigma^2(1 - \phi_2)}{1 - \phi_2 - \phi_1^2(1 + \phi_2) - \phi_2^2 + \phi_2^3}$$

from there, we can obtain $\gamma(1)$:

$$\gamma(1) = \frac{\phi_1 \gamma(0)}{1-\phi_2} = \frac{\phi_1}{1-\phi_2}\left[\frac{\sigma^2(1-\phi_2)}{1-\phi_2-\phi_1^2(1+\phi_2)-\phi_2^2+\phi_2^3}\right]$$

$$\gamma(2) = \frac{\gamma(0)\left\{\phi_1^2 + (1-\phi_2)\phi_2\right\}}{1-\phi_2} = \frac{\sigma^2(1-\phi_2)}{1-\phi_2-\phi_1^2(1+\phi_2)-\phi_2^2+\phi_2^3}\left\{\frac{\phi_1^2+(1-\phi_2)\phi_2}{1-\phi_2}\right\}$$