Autocovariance, Autocorrelation, Distance Autocovariance and Distance Autocorrelation

Exercise 1. Note that this theorem implies that whenever $\rho = 0$ also $\mathcal{R}(X,Y) = 0$. However in the lecture we discussed that the reason to use $\mathcal{R}(X,Y)$ is that it can be larger zero, even if $\rho = 0$ and so can detect dependencies that ρ misses. How do we solve this contradiction?

Solution

The reason why $\rho = 0$ for two Gaussian random variables X, Y implies $\mathcal{R}(Y,X) = 0$ is because when (X,Y) are jointly Gaussian, then they are independent if and only if $\rho = 0$. This is true in all examples here, as for white noise, AR(1) and MA(1), (Y_t, Y_{t+h}) will be jointly Gaussian for all t and h. This is however not true in general. For instance, as we saw in the lecture if:

$$Y_t = Y_{t-1}^2 + \varepsilon_t, \varepsilon_t \sim N(0,1),$$

(Y_t, Y_{t+1}) are no longer jointly Gaussian and one can show that the correlation between Y_t, Y_{t+1} is zero, while $\mathcal{R}(Y,X) > 0$.

Exercise 2. Autocovariance and autocorrelation of White Noise

- Simulate a white noise process $Y_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, 5^2)$ of length $n = 500$.
- Plot the generated time series.
- Compute the empirical autocovariance $\hat{\gamma}(h)$ and autocorrelation $\hat{\rho}(h)$ for lags $h = 0, 1, \ldots n-1$ manually and using R function `acf()`.
- Derive the theoretical autocovariance and autocorrelation function of the process.
- Compare the empirical and theoretical autocorrelation and distance autocorrelation (Hint: Use Theorem 7 in Székely, Rizzo, and Bakirov 2007 to compute the theoretical distance autocorrelation).

Solution

$$Y_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma^2) \tag{1}$$

Autocovariance

$$\gamma(h) = \operatorname{cov}(Y_t, Y_{t+h}) = \begin{cases} \sigma^2, & h = 0 \\ 0, & h \neq 0 \end{cases}$$

Autocorrelation

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \begin{cases} 1, & h = 0 \\ 0, & h \neq 0 \end{cases}$$

Distance Autocorrelation

$$\mathcal{R}(Y_t, Y_{t+h}) = f(\rho(h)) = \begin{cases} 1, & h = 0 \\ 0, & h \neq 0 \end{cases}$$

where f is described in Theorem 7 of Székely, Rizzo, and Bakirov 2007:

$$f(\rho) = \left(\frac{\rho \arcsin \rho + \sqrt{1-\rho^2} - \rho \arcsin \rho/2 - \sqrt{4-\rho^2} + 1}{1 + \pi/3 - \sqrt{3}} \right)^{1/2}. \tag{2}$$

Exercise 3. Autocovariance and autocorrelation of an AR(1) process

Consider the AR(1) model:

$$Y_t = 0.7Y_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, 2^2).$$

- Simulate a trajectory of length $n = 500$
- Plot the generated time series.
- Compute the empirical autocovariance $\hat{\gamma}(h)$ and autocorrelation $\hat{\rho}(h)$ for lags $h = 0, 1, \ldots n-1$ manually and using `R` function `acf()`.
- Derive the theoretical autocovariance and autocorrelation function of the process.
- Compare the empirical and theoretical autocorrelation and distance autocorrelation (Hint: Use Theorem 7 in Székely, Rizzo, and Bakirov 2007 to compute the theoretical distance autocorrelation).

Solution

We first derive the variance of Y_t:
With $|\phi| < 1$, the process is (weak) stationary, hence $\mathrm{var}(Y_t) = \mathrm{var}(Y_{t-1}) = \gamma(0)$. Taking variance on both sides:

$$\mathrm{var}(Y_t) = \mathrm{var}(\phi Y_{t-1} + \varepsilon_t)$$

Since Y_{t-1} and ε_t are independent:

$$\mathrm{var}(Y_t) = \phi^2 \, \mathrm{var}(Y_{t-1}) + \mathrm{var}(\varepsilon_t)$$

$$\gamma(0) = \phi^2 \gamma(0) + \sigma^2 \quad \Rightarrow \quad \gamma(0) = \frac{\sigma^2}{1 - \phi^2}.$$

Autocovariance
The autocovariance at lag h is

$$\gamma(h) = \mathrm{cov}(Y_t, Y_{t-h}).$$

Lag 1:

$$\gamma(1) = \mathrm{cov}(Y_t, Y_{t-1}) = \mathrm{cov}(\phi Y_{t-1} + \varepsilon_t, Y_{t-1}) = \phi \, \mathrm{var}(Y_{t-1}) = \phi \gamma(0).$$

Lag 2:

$$\begin{aligned}\gamma(2) &= \mathrm{cov}(Y_t, Y_{t-2}) = \mathrm{cov}(\phi Y_{t-1} + \varepsilon_t, Y_{t-2}) \\ &= \phi \, \mathrm{cov}(Y_{t-1}, Y_{t-2}) + \mathrm{cov}(\varepsilon_t, Y_{t-2}) = \phi \gamma(1) = \phi^2 \gamma(0),\end{aligned}$$

where we used that ε_t is independent of Y_{t-2}.
Recursion for general lag h:

$$\gamma(h) = \phi \gamma(h-1) = \phi^h \gamma(0).$$

Final formula:

$$\gamma(h) = \phi^{|h|} \gamma(0) = \frac{\sigma^2}{1 - \phi^2} \phi^{|h|}.$$

Autocorrelation

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \phi^{|h|}$$

Distance Autocorrelation

$$\mathcal{R}(Y_t, Y_{t+h}) = f(\phi^{|h|})$$

where f is given in (2).

Exercise 4. Autocovariance and autocorrelation of an MA(1) process

Consider the MA(1) model:

$$Y_t = \varepsilon_t + 0.5\varepsilon_{t-1}, \quad \varepsilon_t \sim \mathcal{N}(0, 2^2).$$

- Simulate a trajectory of length $n = 500$.
- Plot the generated time series.
- Compute the empirical autocovariance $\hat{\gamma}(h)$ and autocorrelation $\hat{\rho}(h)$ for lags $h = 0, 1, \dots n-1$ manually and using `R` function `acf()`.
- Derive the theoretical autocovariance and autocorrelation function of the process.
- Compare the empirical and theoretical autocorrelation and distance autocorrelation (Hint: Use Theorem 7 in Székely, Rizzo, and Bakirov 2007 to compute the theoretical distance autocorrelation).

Solution

Variance
We compute the variance of Y_t:

$$\operatorname{var}(Y_t) = \operatorname{var}(\varepsilon_t + \theta\varepsilon_{t-1}).$$

Since ε_t and ε_{t-1} are independent:

$$\operatorname{var}(Y_t) = \operatorname{var}(\varepsilon_t) + \theta^2 \operatorname{var}(\varepsilon_{t-1}) + 2\theta \operatorname{cov}(\varepsilon_t, \varepsilon_{t-1})$$

Because $\operatorname{cov}(\varepsilon_t, \varepsilon_{t-1}) = 0$ (independence):

$$\operatorname{var}(Y_t) = \sigma^2 + \theta^2\sigma^2 = \sigma^2(1 + \theta^2)$$

Hence,

$$\gamma(0) = \operatorname{var}(Y_t) = \sigma^2(1 + \theta^2).$$

Autocovariance The autocovariance at lag h is defined as:

$$\gamma(h) = \operatorname{cov}(Y_t, Y_{t-h})$$

$$\gamma(1) = \operatorname{cov}(Y_t, Y_{t-1}) = \operatorname{cov}(\varepsilon_t + \theta\varepsilon_{t-1}, \varepsilon_{t-1} + \theta\varepsilon_{t-2})$$

Expanding with bilinearity of covariance:

$$\gamma(1) = \operatorname{cov}(\varepsilon_t, \varepsilon_{t-1}) + \theta \operatorname{cov}(\varepsilon_{t-1}, \varepsilon_{t-1}) + \theta \operatorname{cov}(\varepsilon_t, \varepsilon_{t-2}) + \theta^2 \operatorname{cov}(\varepsilon_{t-1}, \varepsilon_{t-2})$$

Using independence of white noise (all covariances between different ε's are 0):

$$\gamma(1) = \theta \operatorname{var}(\varepsilon_{t-1}) = \theta\sigma^2$$

For $h > 1$, any covariance between ε_t and ε_{t-h} or between ε_{t-1} and ε_{t-h-1} is zero because of independence. Thus:

$$\gamma(h) = 0 \quad \text{for } h > 1$$

Autocorrelation
The autocorrelation function is

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)}$$

Lag 0:

$$\rho(0) = \frac{\gamma(0)}{\gamma(0)} = 1$$

Lag 1:

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{\theta\sigma^2}{\sigma^2(1 + \theta^2)} = \frac{\theta}{1 + \theta^2}$$

Lag $h > 1$:

$$\rho(h) = 0 \quad \text{for } h > 1$$

Thus, the MA(1) autocorrelation function is

$$\rho(0) = 1, \quad \rho(1) = \frac{\theta}{1 + \theta^2}, \quad \rho(h) = 0 \text{ for } h > 1.$$

Distance Autocorrelation Accoring to Theorem 7 of Székely, Rizzo, and Bakirov 2007, this leads to

$$\mathcal{R}(Y_t, Y_{t+h}) = \begin{cases} 1, & \text{if } h = 0 \\ f(\theta/(1 + \theta^2)), & \text{if } h = 1 \\ 0, & \text{if } h > 1, \end{cases}$$

with f given in (2).

References

Székely, Gábor J, Maria L Rizzo, and Nail K Bakirov (2007). "Measuring and Testing Dependence by Correlation of Distances". In: *The Annals of Statistics* 35.6, pp. 2769–2794.