Small World Networks

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Adapted from the research paper: "Collective dynamics of 'small-world' networks", by Duncan J. Watts and Steven H. Strogatz and the presentation of Daniel Lazarev

Structural Properties

- Path Length, L(p): Typical separation between 2 vertices in the graph
  -Global property

- Clustering Coefficient, C(p): "cliquishness" of a typical neighborhood
  -Local property

Path Length, L(p)

- *L* is the number of edges in the shortest path between two vertices, averaged over all pairs of vertices

Clustering Coefficient, C(p)

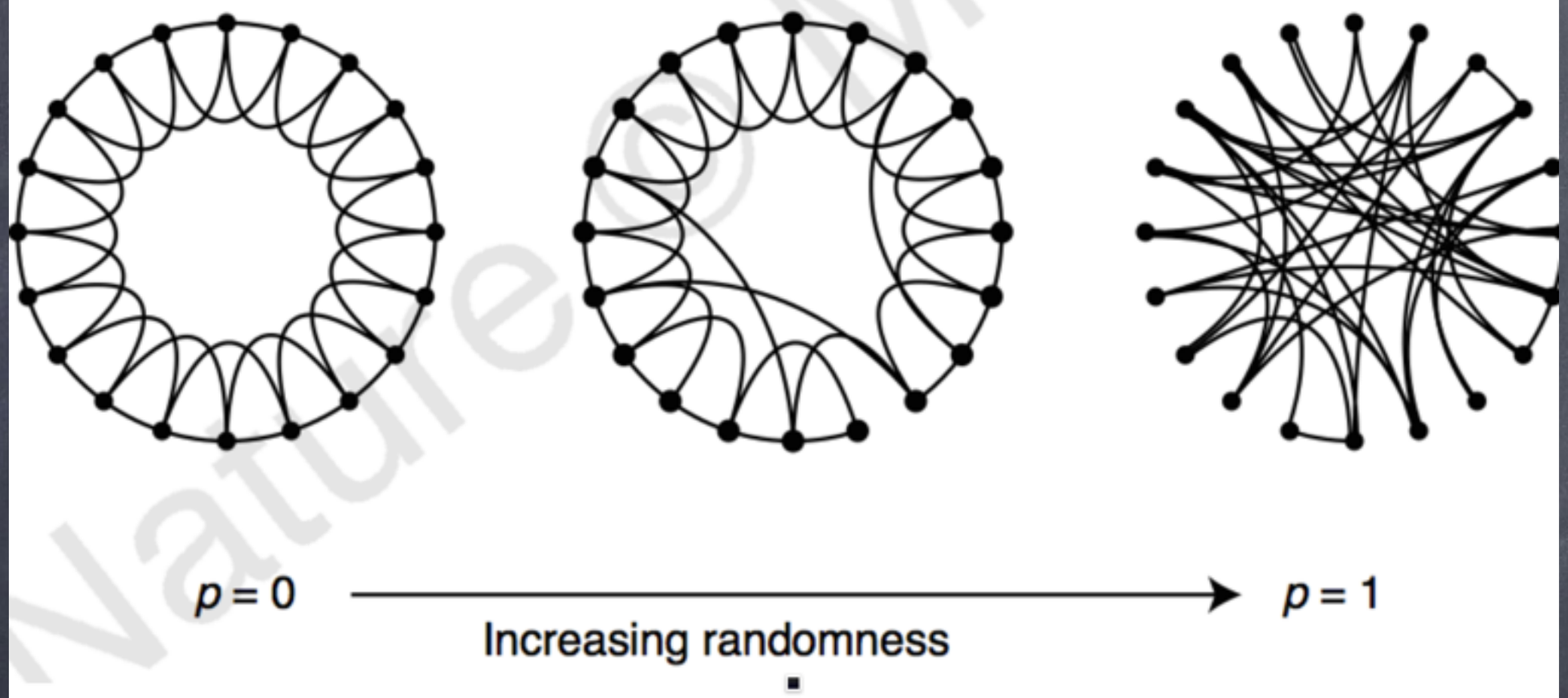
- Suppose a vertex has k_v neighbors
- Maximum number of edges—when every neighbor is connected with every other—is $k_v(k_v - 1)/2$
- Let C_v be the fraction of these that actually exist for an arbitrary vertex v
- The average C_v over all v is the clustering coefficient, C
- Mathematically, this is $C_v = \frac{2E_v}{k_v(k_v - 1)}$
  Where E is the fraction of edges that exist between the neighbors of a vertex, v
- Then C for a system of n vertices is

$$C = \frac{1}{n}\sum_{v=1}^{n} C_v$$

Watts-Strogatz (WS) Model

- Consider a ring lattice with n vertices and k edges/vertex
- Rewire each edge at random with probability p
- We can tune the graph between $p = 0$ (regularity) and $p = 1$ (disorder)
- So what happens in $0 < p < 1$?

Small-world!



- Watts-Strogatz (WS)

$$C_{WS} = \frac{1}{n}\sum_{v=1}^{n}\frac{2E_v}{k_v(k_v-1)}$$

Defining a Small-world

- Definition 1: A network N is said to be small-world if

$$L(p) \geq L_{random}$$

but

$$C(p) \gg C_{random}$$

- Or S > 1 where

$$\gamma = \frac{C(p)}{C_{random}}$$

and

$$\lambda = \frac{L(p)}{L_{random}}$$

$$S = \frac{\gamma}{\lambda}$$

Table 1. Table of small-world-ness values and other topological properties of real networks.

class	#	network	n	m	$\langle k \rangle$	ξ	ι	c^{Δ}	C^{ws}	S^{Δ}	S^{ws}	p (WS)	Reference
Social	1	Dolphins†	62	159	5.13	0.084	3.36	0.31	0.26	2.8	2.35	0.64	[41]
	2	film actors	449913	25516482	113.43	2.5×10^{-4}	3.48	0.2	0.78	627	2446	0.95	[1,3]
	3	company directors	7673	55392	14.44	0.002	4.6	0.59	0.88	228	341	0.77	[1,23]
	4	math coauthorship	253339	496489	3.92	1.6×10^{-5}	7.57	0.15	0.34	11666	26443	0.7	[1,45]
	5	physics coauthorship	52909	245300	9.27	1.8×10^{-4}	6.19	0.45	0.56	2026	2521	0.73	[1,46]
	6	biology coauthorship	1520251	11803064	15.53	1×10^{-5}	4.92	0.088	0.6	9089	61967	0.88	[1,46]
	7	email messages	59912	86300	1.44	4.8×10^{-5}	4.95	-	0.16	-	40524	n/a	[1,47]
	8	email address books	16881	57029	3.38	4×10^{-4}	5.22	0.17	0.13	1301	995	0.64	[1,48]
	9	student relationships	573	477	1.67	0.0029	16.01	0.005	0.001	1.34	0.27	n/a	[1,20]
	10	newspaper article co-occurence	459	1422	6.2	0.0135	2.98	-	0.02	-	1.67	n/a	[49]
	11	US directors	11057	74414	13.46	0.0012	5.19	0.56	0.87	315	494	0.77	[50]
	12	UK directors	8850	39741	8.98	0.001	6.46	0.61	0.89	386	561	0.71	[50]
	13	German directors	4185	30438	14.55	0.0035	6.4	0.72	0.93	100.71	129.7	0.79	[50]
Information	14	WWW nd.edu	269504	1497135	5.56	4×10^{-5}	11.27	0.11	0.29	3453	9104	0.81	[1,51]
	15	Roget's Thesaurus	1022	5103	4.99	0.0098	4.87	0.13	0.15	23.54	27.17	0.76	[1,52]
	16	word adjacency†	112	425	7.59	0.0684	2.54	0.16	0.17	2.13	2.34	0.74	[26]
	17	book purchases†	105	441	8.4	0.081	3.08	0.35	0.49	3.09	4.33	0.71	V.Kreb, unpublished (www.orgnet.com)
Technological	18	Internet	10697	31992	5.98	5.6×10^{-4}	3.31	0.035	0.39	98.09	1093	0.83	[1,53]
	19	power grid	4941	6594	2.67	5.4×10^{-4}	18.99	0.1	0.08	84.45	67.56	0.8	[1,3]
	20	train routes	587	19603	66.79	0.114	2.16	-	0.69	-	4.26	n/a	[1,54]
	21	software packages	1439	1723	1.2	0.0017	2.42	0.07	0.082	1403	1644	n/a	[1,25]
	22	software classes	1377	2213	1.61	0.0023	1.51	0.033	0.012	285.26	103.73	n/a	[1,55]
	23	electronic circuits	24097	53248	4.42	1.8×10^{-4}	11.05	0.01	0.03	33.5	100.5	0.91	[1,56]
	24	peer-to-peer network	880	1296	2.95	0.0034	4.28	0.012	0.011	5.26	4.82	0.85	[1,57]
Biological	25	metabolic network	765	3686	9.65	0.0126	2.56	0.09	0.67	8.18	60.89	0.82	[1,58]
	26	yeast protein interactions	2115	2240	0.001	2.12	6.8	0.072	0.071	107.85	106.35	0.73	[1,59]
	27	marine food web	135	598	4.43	0.0661	2.05	0.16	0.23	7.84	11.27	0.64	[1,60]