Engineering Tripos Part IIA THIRD YEAR

Module 3A1: Fluid Mechanics I

INCOMPRESSIBLE FLOW

Preparatory Problems

The Incompressible Flow lecture course makes extensive use of complex variables and vector calculus. In addition, the examples paper will be *much* easier if you are able to use Matlab to produce plots of various types. This set of exercises is intended to help you get up to speed. You should be able to complete it independently. If you encounter difficulties, go back to your IA (Complex Variables) and IB (Vector Calculus) maths notes. Your IB Octave course will have given you a grounding in basic Matlab commands and syntax; use the Product Help pages to resolve any problems.

Complex Variables

1. (a) The complex variable $z = x + iy$. What is z^2 in terms of x and y?

   (b) z can also be written as $re^{i\theta}$, where $r\cos\theta = x$ and $r\sin\theta = y$. What is z^2 in terms of r and θ?

   (c) Show that your answers to (a) and (b) are equal to one another.

2. For $z = x + iy$, the modulus $|z|$ is defined as $\sqrt{x^2 + y^2}$, and the complex conjugate z^* as $x - iy$.

   (a) What is $z \times z^*$?

   (b) What is $|z|^2$ in terms of r and θ (as defined in question 1)?

   (c) How could you use your answer to (a) to derive that to (b) more quickly? (Hint: what is the complex conjugate of $e^{i\theta}$?)

3. (a) Express the ratio

$$\frac{x + iy}{a + ib}$$

   as the sum of separate real and imaginary parts.

(b) Simplify

$$\frac{1+re^{i\theta}}{1-re^{i\theta}}$$

(i) by writing the numerator and denominator in terms of their real and imaginary parts, and using your result from (a);

(ii) by multiplying numerator and denominator by $1-re^{-i\theta}$.

4. The complex number z is equal to 3 + 4i. Find, from first principles (i.e. without using any of your calculator's complex number functionality):

(a) e^z; (b) $\ln z$; (c) $\sin z$; (d) $z^{1/3}$ (all three possibilities).

Check your answers using Matlab.

5. The complex variable ζ is defined as a function of z by:

$$\zeta = z + \frac{1}{z}.$$

(a) Show that, as the unit circle $z = e^{i\theta}$ is traversed (i.e. θ varies from 0 to 2π) in the z plane, ζ varies along the real axis from 2 to –2, and back again. (The unit circle is said to be 'mapped' onto this line.)

(b) Form a vector of z values around the unit circle within Matlab as follows:
```
theta = linspace(0,2*pi,101);
z = exp(1i*theta);
```
Then find the corresponding z values and plot them:
```
zeta = z + 1./z;
plot(zeta)
axis('equal')
```
Make sure you understand what you've done here. (Why do you need to use ./ instead of / for the 1/z term? How does Matlab handle a plot request with a single argument when that argument is: (i) real; (ii) complex? What was the point of the axis command?) If you're unsure about something, just try out alternatives and see what happens.

(c) Now find out what happens to circles in the z plane with radii greater than one. For example:
```
z = 2*exp(1i*theta);
zeta = z + 1./z;
hold
plot(zeta)
```

(d) Finally, plot the mapped versions of some radial lines (defined by a set of z values with constant argument and varying modulus).

Vector Calculus, Surfaces and Contour Plotting

6. An unfeasibly smooth hill has height

$$h = h_0 - a(x - x_0)^2 - b(y - y_0)^2.$$

(a) What are the gradients of the hill surface in the x and y directions?

(b) Generate a numerical representation of the hill in Matlab, as follows:

```
h0 = 10; a = 0.25; x0 = 2; b = 4; y0 = -1;
x = linspace(-10,15,26);
y = linspace(-5,5,11);
for ix = 1:length(x)
   for iy = 1:length(y)
      xmat(ix,iy) = x(ix);
      ymat(ix,iy) = y(iy);
   end
end
h = h0 – a*(xmat-x0).^2 – b*(ymat-y0).^2;
```

(N.B. The command `meshgrid` is a much more efficient way to generate `xmat` and `ymat`; the `for` loops are only used to make it easier for you to see what's going on. You've also reached the stage where it's worth grouping a set of commands together in a file, called a 'script'. Call your file (say) `qu6.m`; you can then run it by typing `qu6` in the Matlab command window.) Now try the following:

```
plot(x,hmat(:,6))
plot(y,hmat(11,:))
```

Are the results what you expect? Finally, plot the contours of the surface:

```
contour(xmat,ymat,h)
```

The contour levels in this plot were set automatically. Sometimes you'll want to control these yourself; can you work out how to produce a plot with contours at -50, -10, -1, 0, 2 and 7?

(c) Click the 'data cursor' icon at the top of your figure window, and work out the change in height in going from (3,0) to (4,1). How does this compare with the approximate value obtained using the surface gradients at (3,0)?

7. A surface has height $h(x,y)$. A particle is on the surface.

(a) Show that, as the particle moves a small distance $\delta\mathbf{r} = (\delta x, \delta y)$, it experiences a height change $\delta h = \delta\mathbf{r} \cdot \nabla h$.

(b) Hence show that:

(i) the surface contours are perpendicular to ∇h;

(ii) the maximum δh for a given $|\delta\mathbf{r}|$ is when $\delta\mathbf{r}$ is parallel to ∇h.

(c) The particle velocity (in the x–y plane) is $\mathbf{v}$. What is its rate of change of height?

8. You will sometimes need to produce a contour plot of a function defined in polar, rather than cartesian, coordinates. Consider, for example,

$$\psi(r,\theta) = re^{-r}\cos^2\theta.$$

(a) Produce a contour plot of ψ via the following script:

```
r = linspace(0,5,51);
theta = linspace(0,2*pi,101);
for ir = 1:length(r)
   for ith = 1:length(theta)
      xmat(ir,ith) = r(ir) * cos(theta(ith));
      ymat(ir,ith) = r(ir) * sin(theta(ith));
      psi(ir,ith) = r(ir) * exp(-r(ir)) * (cos(theta(ith)))^2;
   end
end
contour(xmat,ymat,psi)
```

(You should obtain a nice bow-tie pattern.)

(b) What is the maximum value you can find with the data cursor?

(c) Prove, analytically, that you have located the maximum correctly.

(d) Try another way to produce the contour plot: generate x and y matrices (c.f. qu. 6), then matrices of the corresponding r and θ values (familiarise yourself with the `atan2` function), then ψ.

Functions of the Complex Variable $z = x + iy$

9. Consider the (complex) function $F = z^2$.

(a) Find the real functions $\phi(x,y) = \mathrm{Re}(F)$ and $\psi(x,y) = \mathrm{Im}(F)$.

(b) How are the partial derivatives of ϕ and ψ related to one another?

(c) Plot contours of ϕ and ψ on the same axes (remember the `hold` command used in question 5). N.B. Try to take advantage of Matlab's ability to handle complex numbers, by computing $F(z)$ and then extracting its real and imaginary parts, rather than going through two separate processes for ϕ and ψ.

(d) At what angle do you think the contours of ϕ and ψ intersect?

10. Can you prove:

(a) that the relations you found in 9(b) are true in general for any function $F(z)$ that can be differentiated with respect to z?

(b) that your estimate of the intersection angle in 9(d) is correct?

Don't worry if you can't; these are results which will be derived in the lectures.

Answers

1 (a) $x^2 - y^2 + i2xy$ (b) $r^2 e^{i2\theta}$

2 (a) $|z|^2$ (b) r^2

3 (a) $\dfrac{xa + yb}{a^2 + b^2} + i\dfrac{ya - xb}{a^2 + b^2}$ (b) $\dfrac{1 - r^2 + 2ir\sin\theta}{1 + r^2 - 2r\cos\theta}$

6 (a) $-2a(x - x_0)$, $-2b(y - y_0)$ (c) $-12.75, -8.5$

7 (c) $\mathbf{v} \cdot \nabla h$

8 (b) 0.368, at $(\pm 1, 0)$

9 (a) $\phi = x^2 - y^2$, $\psi = 2xy$ (b) $\dfrac{\partial \phi}{\partial x} = \dfrac{\partial \psi}{\partial y}$, $\dfrac{\partial \phi}{\partial y} = -\dfrac{\partial \psi}{\partial x}$

W R Graham, October 2015