三次螺旋曲线

Hermite_Curves

三阶曲线方程的标准形式

$$
\begin{aligned}
x(t) &= a_x t^3 + b_x t^2 + c_x t + d_x \\
y(t) &= a_y t^3 + b_y t^2 + c_y t + d_y \\
z(t) &= a_z t^3 + b_z t^2 + c_z t + d_z
\end{aligned} \tag{1}
$$

单独就x变量来考虑

$$
\begin{aligned}
P(t)_x &= a_x t^3 + b_x t^2 + c_x t + d_x \\
\frac{d}{dt}P(t)_x &= 3a_x t^2 + 2b_x t + c_x
\end{aligned} \tag{2}
$$

起始点 $(P(0)_X, \frac{d}{dt}P(0)_X)$，终点$(P(1)_X, \frac{d}{dt}P(1)_X)$

带入上述方程可得到

$$
\begin{vmatrix} P(0)_X \\ P(1)_X \\ \frac{d}{dt}P(0)_X \\ \frac{d}{dt}P(1)_X \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{vmatrix} \begin{vmatrix} a_x \\ b_x \\ c_x \\ d_x \end{vmatrix} => \begin{vmatrix} a_x \\ b_x \\ c_x \\ d_x \end{vmatrix} = \begin{vmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} P(0)_X \\ P(1)_X \\ \frac{d}{dt}P(0)_X \\ \frac{d}{dt}P(1)_X \end{vmatrix} \tag{3}
$$

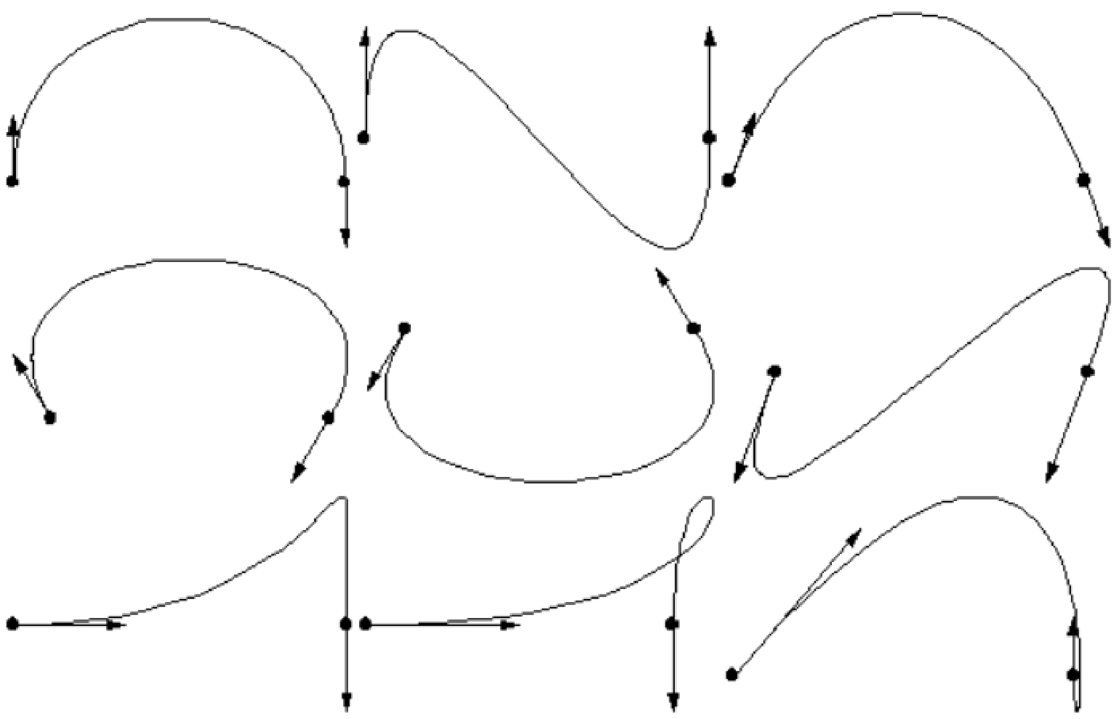
故

$$
P(t)_x = [t^3, t^2, t, 1] \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P(0)_X \\ P(1)_X \\ \frac{d}{dt}P(0)_X \\ \frac{d}{dt}P(1)_X \end{bmatrix} \tag{4}
$$

推广到三维，可得到

$$
Q(t)_x = [t^3, t^2, t, 1] \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \\ dx_0 & dy_0 & dz_0 \\ dx_1 & dy_1 & dz_1 \end{bmatrix} \tag{5}
$$

Example

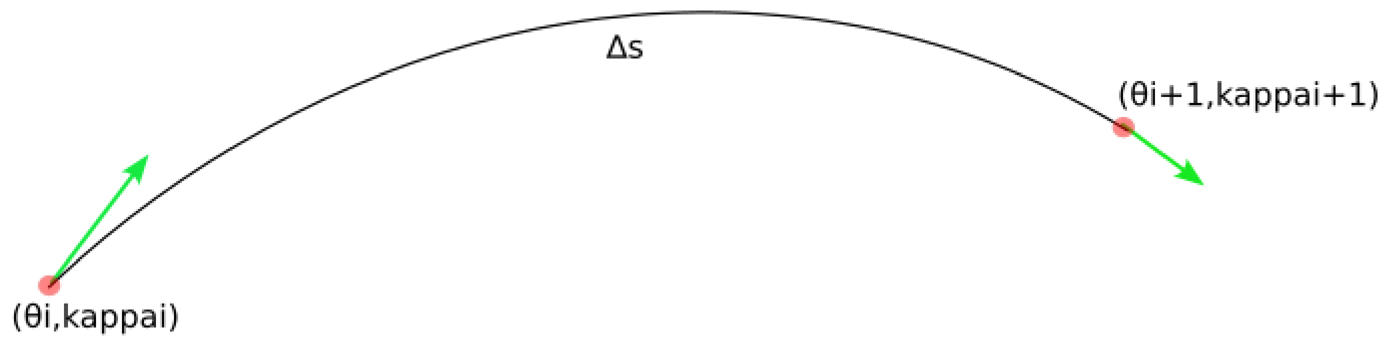


构建三次螺旋曲线

三次螺旋曲线表达式推导

$$
\begin{aligned}
\theta(s) &= as^3 + bs^2 + cs + d \\
\kappa(s) &= 3as^2 + 2bs + c
\end{aligned} \tag{6}
$$

在单位区间$(0,1)$，给定**起点** $(\theta(0), kappa(0))$，**终点** $(\theta(1), kappa(1))$



得到系数矩阵为:

$$\begin{vmatrix} a \\ b \\ c \\ d \end{vmatrix} = \begin{vmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} \theta(0) \\ \theta(1) \\ kappa(0) \\ kappa(1) \end{vmatrix} \tag{7}$$

推广到任意区间 (s_i, s_{i+1})，将后者变量映射到 $(0,1)$，则得到三次螺旋曲线表达式为:

$$\theta(s) = [t^3, t^2, t, 1] \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_i \\ \theta_{i+1} \\ (s_{i+1} - s_i) kappa_i \\ (s_{i+1} - s_i) kappa_{i+1} \end{bmatrix}, \quad t = \frac{s - s_i}{s_{i+1} - s_i}, s \in [s_i, s_{i+1}] \tag{8}$$

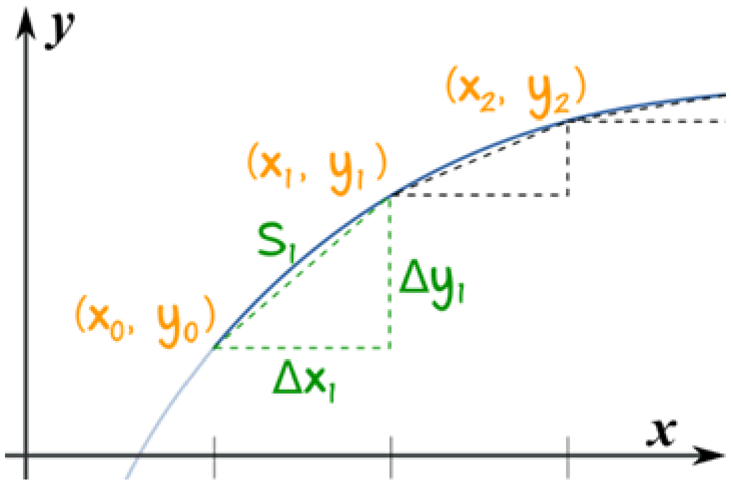
建立数学模型

优化变量

$$\vec{q} = [\vec{\theta}, \dot{\vec{\theta}}, \vec{x}, \vec{y}] \tag{9}$$

建立约束方程

$$costfunction = \frac{1}{2} \| (x_{i+1} - x_i - \int_0^s \cos(\theta(s)) ds) + (y_{i+1} - y_i - \int_0^s \sin(\theta(s)) ds) \|^2 \tag{10}$$



设置边界条件

$$\begin{cases} x_{ref_i} - r_i \le x_i \le x_{ref_i} + r_i \\ y_{ref_i} - r_i \le y_i \le y_{ref_i} + r_i \end{cases} \tag{11}$$

求解约束方程中的积分问题

$$\begin{aligned} x_{i+1} = x_i + \int_0^s \cos(\theta(s)) ds \approx x_i + \sum_{i=1}^{7} w_i \cdot cos(\theta(\Delta s_i)) \\ y_{i+1} = y_i + \int_0^s \sin(\theta(s)) ds \approx y_i + \sum_{i=1}^{7} w_i \cdot sin(\theta(\Delta s_i)) \end{aligned} \tag{12}$$

将区间$s \in [s_i, s_{i+1}]$等分为7份，通过高斯勒让得求积来获得在笛卡尔坐标系下的位置

Δs_i(采样点)	w_i(权重)
0.025446s	0.0647425s
0.129234s	0.139853s
0.297077s	0.190915s
0.5s	0.20898s
0.702923s	0.190915s
0.870766s	0.139853s
0.974554s	0.0647425s

最终该问题可表述为:

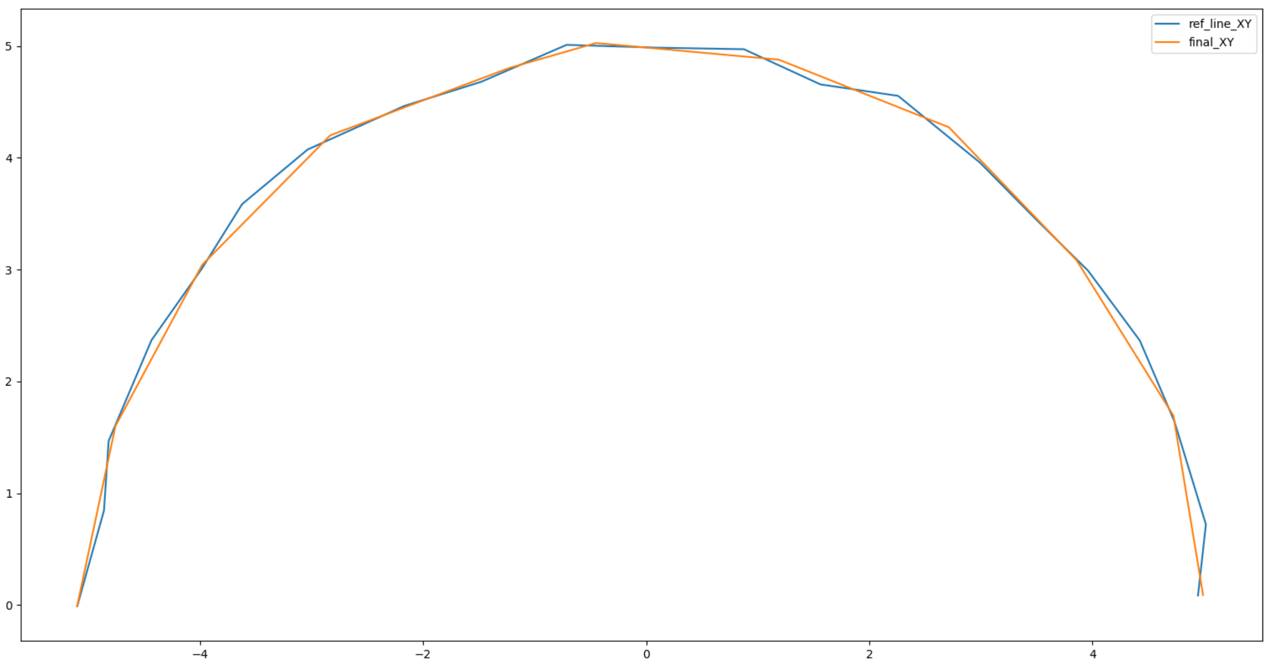
$$\min_x \sum_i \frac{1}{2} \| (x_{i+1} - x_i - \int_0^s \cos(\theta(s)) ds) + (y_{i+1} - y_i - \int_0^s \sin(\theta(s)) ds) \|^2$$

$$s.t.\begin{cases}x_{ref_i} - r_i \le x_i \le x_{ref_i} + r_i \\ y_{ref_i} - r_i \le y_i \le y_{ref_i} + r_i\end{cases} \tag{13}$$

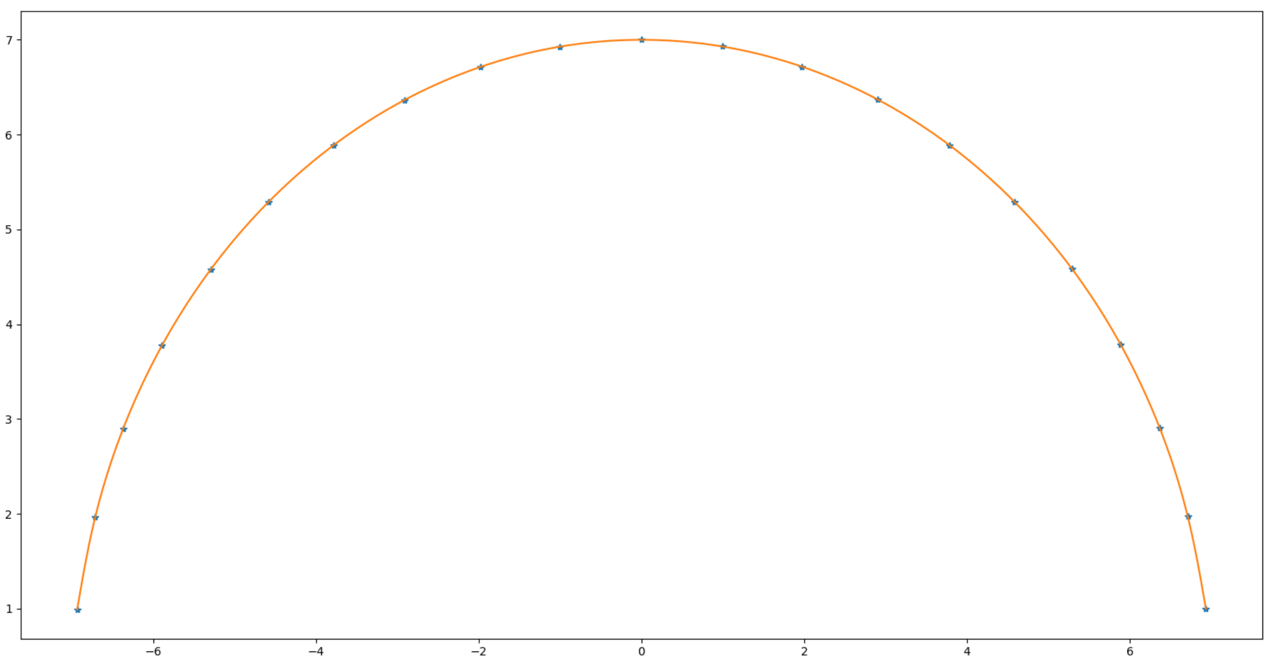
利用ceres库求解上述问题

```
自动微分耗时:
Time for solve time = 320.6 msec.
手动微分耗时:
Time for solve time = 0.773201 msec.
```

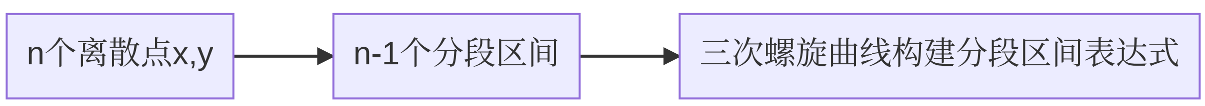
初始点不平滑的拟合结果:

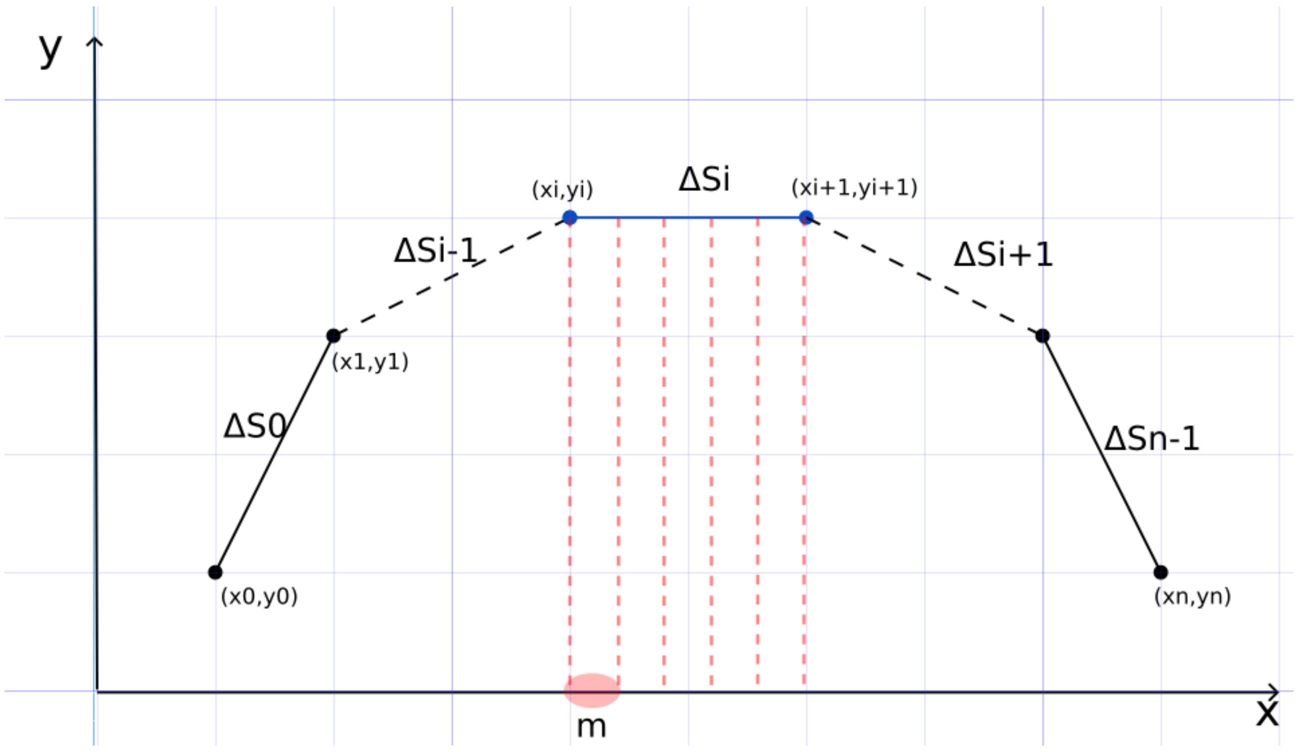


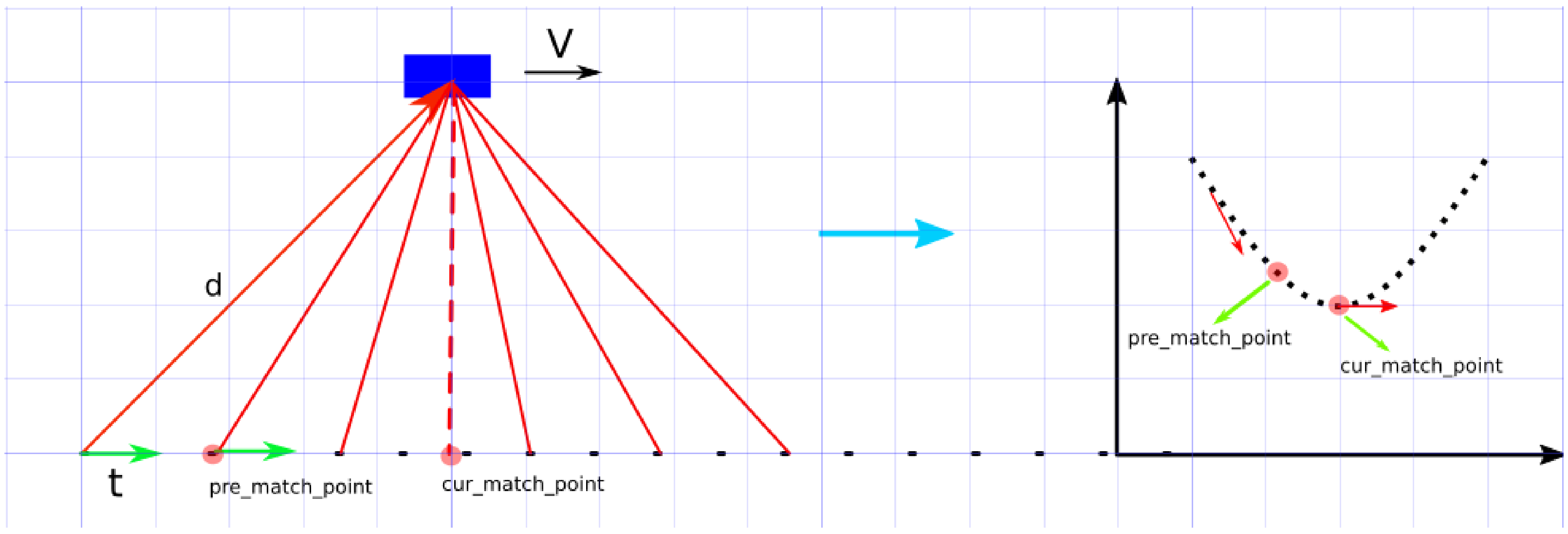
初始点平滑的拟合结果:



整体逻辑




y
(xi,yi)
ΔSi
(xi+1,yi+1)
ΔSi-1
ΔSi+1
(x1,y1)
ΔS0
ΔSn-1
(x0,y0)
(xn,yn)
m
x


V
d
t
pre_match_point
cur_match_point
pre_match_point
cur_match_point