Discrete Structures, CSCI 246, Fall 2018
Exam 2, Oct. 31

Indicate the option that most accurately completes the sentence.

1. A contingency is a compound proposition (4 pts.)
   a. that is always true
   b. that is always false
   c. that is neither always true nor always false
   d. where p holds but q does not hold
   e. where q holds but p does not hold

2. The following holds for a function which is an injection (4 pts.)
   a. it is invertible
   b. it is a bijection
   c. it is a function f where $\forall a \forall b\ (f(a)=f(b) \rightarrow a=b)$
   d. it is a function f where $\forall y \exists x\ (f(x)=y)$
   e. all of the above

3. The function f(x) = 2x-3 from R to R is (4 pts.)
   a. is a bijection
   b. is 1-1 but is not onto
   c. is onto but is not 1-1
   d. is not 1-1 and is not onto
   e. is not a function

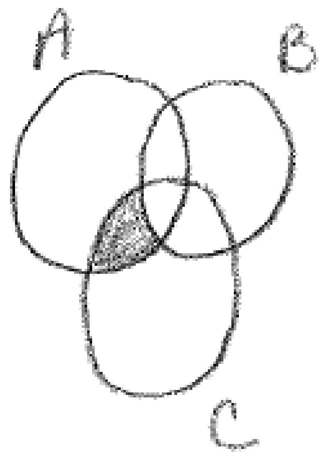
4. An arithmetic sequence with the initial term 5 and the common difference 3 can be written (4 pts.)
   a. 3, 8, 13, 18, …
   b. 5, 8, 11, 14, …
   c. 3, 15, 75, 375, …
   d. 5, 15, 45, 135, …
   e. There is insufficient information to write the series

5. The symbol $\equiv$ (4pts.)
   a. is a logical connective (also called a logical operator)
   b. can be used in compound propositions
   c. says that compound propositions have the same truth values for all possible variable values
   d. is another way of writing implication
   e. all of the above

6. Draw the Venn diagram for the following combination of the sets A, B and C,

$(A\text{-}B) \cap C$ (5pts.)

Answer:



7. Using summation notation, express the sum of the first 100 terms of the arithmetic sequence with initial term 12 and the common difference of 13. (5 pts.)

Arithmetic sequence with initial term 12 and common difference of 13.

12, 12+13*1=25, 12 + 13*2 = 38, 12 + 13*3 = 51, …, 12 + 13*99

Want 12 + 25 + 38 + 51 + … + a big number

Using summation notation:

$$\sum_{i=0}^{99} (12 + 13i)$$

8. Determine whether the following function is a bijection from R to R. If the function is not a bijection, tell if it is 1-1 and tell if it is onto. If the function is a bijection, give its inverse.

$$f(x) = -2x^3 - 4 \qquad \text{(5 pts.)}$$

Is a bijection

$y = -2x^3 - 4$

$y + 4 = -2x^3$

$-\frac{y+4}{2} = x^3$

$\sqrt[3]{-\frac{y+4}{2}} = x$

$f^{-1}(x) = \sqrt[3]{-\frac{y+4}{2}}$

9. Give the first four terms of the following sequence. (5 pts.)

$a_1 = 1$, $a_2 = 9$, and $a_n = 3*a_{n-1} + 2*a_{n-2}$ for $n>2$.

1, 9, 3*9 + 2*1 = 29, 3*29 + 2* 9 = 105

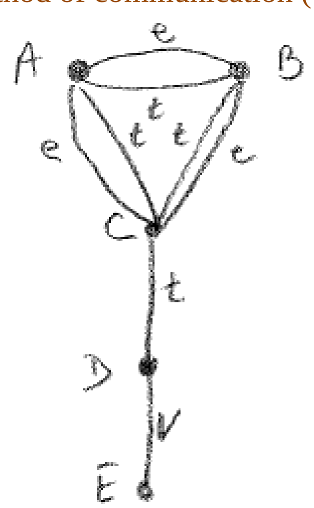
First 4 terms are 1, 9, 29, 105

10. Tell the number of vertices and edges in the graph $K_{n,m}$. (5 pts.)

Vertices in $K_{n,m}$	Edges in $K_{n,m}$
n+m	n*m

11. Describe a discrete structure based on a graph that can be used to represent all forms of electronic communication between two people. Say that people A, B and C communicate via email and text. C and D communicate with each other only via text, while D and E only communicate by talking on the telephone. Describe the graph model which you will use. Follow this by drawing the graph. (5 pts.)

Use an undirected graph where the vertices are people. Place labels on the edges to show the method of communication (e for email, t for text and v for voice).



12. Translate the following English statement into logical expressions. The domain of discourse is the set of all real numbers. (5 pts.)

The ratio of every two positive numbers is also positive.

$\forall x \ \forall y \ ((x>0 \wedge y>0) \rightarrow x/y > 0)$

13. Is the following collection of subsets a partitions on the set of bit strings of length 5? Explain why or why not.

The set of bit strings that end with 111, the set of bit strings that end with 011, and the set of bit strings that end with 001. (5 pts.)

No. The property $A = A_1 \cup A_2 \cup ... \cup A_n$ doesn't hold. For instance, the strings that end with 010 aren't included.

14. Show that $\neg p \rightarrow (p \rightarrow q)$ is a tautology not using truth tables. (In other words, prove that the statement holds using the rules given at the end of the exam.) (10 pts.)

Show that $\neg p \rightarrow (p \rightarrow q) \equiv T$

Proof:

Statement	Reason
1. $\neg p \rightarrow (p \rightarrow q) \equiv \neg\neg p \vee (p \rightarrow q)$	Conditional identity
2. " $\equiv \neg\neg p \vee (\neg p \vee q)$	Conditional identity
3. " $\equiv p \vee (\neg p \vee q)$	Double negation law
4. " $\equiv (p \vee \neg p) \vee q$	Associative law
5. " $\equiv T \vee q$	Negation law
6. " $\equiv T$	Domination law

■ Thus, $\neg p \rightarrow (p \rightarrow q)$ is a tautology.

15. Prove that if n is an integer and 3n+2 is odd, then n is odd. (10 pts.)

Restate the problem mathematically:
In the domain of integers,
odd(3n+2) → odd(n)

Proof: This will be proven by contrapositive
Assume: ¬ odd(n)
Show: ¬ odd(3n+2)

Statement	Reason
1. ¬ odd(n)	Assumption
2. even(n)	Every integer is either odd or even
3. n=2k for some integer k	Definition of even
4. 3n+2 = 3(2k) + 2 for some integer k	Substituting 2k for n
5. 3n+2 = 6k + 2 for some integer k	Algebra
6. 3n+2 = 2(3k+1) for some integer k	Algebra
7. 3n+2 = 2j for some integer j	For j=3k+1
8. even(3n+2)	Definition of even
9. ¬ odd(3n+2)	Every integer is either odd or even

It has been shown that if n is an integer and 3n+2 is odd, then n is odd.

■

16. Use mathematical induction to prove that $\sum_{j=0}^{n}(-\frac{1}{2})^j = \frac{2^{n+1}+(-1)^n}{3*2^n}$ whenever n is a nonnegative integer.

(20 pts.)

This theorem can be written mathematically:

Let S(n): $\sum_{j=0}^{n}(-\frac{1}{2})^j = \frac{2^{n+1}+(-1)^n}{3*2^n}$

Prove S(n) for n ≥ 0

This will be proven using mathematical induction

Proof:

Basis:
Show that S(0) holds.

Statement	Reason
1. $\sum_{j=0}^{0}(-\frac{1}{2})^j = (-\frac{1}{2})^0$	Algebra
2. " $= 1$	Algebra
3. " $=$ [2+1]/[3*1]	Algebra
4. " $= \frac{2^{0+1}+(-1)^0}{3*2^0}$	Algebra

■

Induction:

Assume that S(n) holds for n ≥ 0
Show that S(n+1) holds

Statement	Reason
1. $\sum_{j=0}^{n+1}(-\frac{1}{2})^j = \sum_{j=0}^{n}(-\frac{1}{2})^j + (-\frac{1}{2})^{n+1}$	Definition of $\sum^1$
2. " $= \frac{2^{n+1}+(-1)^n}{3*2^n} + (-\frac{1}{2})^{n+1}$	Inductive hypothesis
3. " $= \frac{2^{n+2}+2(-1)^n}{3*2^{n+1}} + \frac{3*(-1)^{n+1}}{3*2^{n+1}}$	Multiply first term by 2/2 and second term by 3/3
4. " $= \frac{2^{n+2}+2(-1)^n+3(-1)^n*-1}{3*2^{n+1}}$	Algebra
5. " $= \frac{2^{n+2}+2(-1)^n-3(-1)^n}{3*2^{n+1}}$	Algebra
6. " $= \frac{2^{n+2}+-1*(-1)^n}{3*2^{n+1}}$	Algebra
7. " $= \frac{2^{n+2}+(-1)^{n+1}}{3*2^{n+1}}$	Algebra

■

Thus by the Principle of Mathematical induction, S(n) holds for n ≥ 0.

Extra Credit:

Let S(x,y) be that statement "x can scare y" where the domain of discourse consists of all people in the world. Use quantifiers to express the statement

There is someone who can scare exactly one person besides himself or herself. (5 pts.)

$$\exists x\ \exists y\ (x \neq y \wedge S(x,y) \wedge \forall z\ ((z \neq x \wedge z \neq y) \rightarrow \neg S(x,z)))$$

Table of Logical Equivalences

TABLE of Logical Equivalences	
Equivalence	*Name*
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$	Negation laws
$p \rightarrow q \equiv \neg p \vee q$ $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$	Conditional identities

Tables of Logical Inferences

TABLE of Rules of Inference		
Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$[p \wedge (p \to q)] \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$[\neg q \wedge (p \to q)] \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$[(p \vee q) \wedge \neg p] \to q$	Disjunctive syllogism
p $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p q $\therefore p \wedge q$	$[(p) \wedge (q)] \to (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \to (q \vee r)$	Resolution

TABLE of Rules of Inference for Quantified Statements	
Rule of Inference	*Name*
$\forall x P(x)$ $\therefore P(c)$	Universal instantiation
$P(c)$ for an arbitrary c $\therefore \forall x P(x)$	Universal generalization
$\exists x P(x)$ $\therefore P(c)$ for some element c	Existential instantiation
$P(c)$ for some element c $\therefore \exists x P(x)$	Existential generalization

TABLE of Universal Modus Ponens and Modus Tollens	
Rule of Inference	*Name*
$\forall x(P(x) \rightarrow Q(x))$ P(a), where a is a particular element in the domain $\therefore Q(a)$	Universal modus ponens
$\forall x(P(x) \rightarrow Q(x))$ $\neg Q(a)$, were a is a particular element in the domain $\therefore \neg P(a)$	Universal modus tollens

DeMorgan's Laws for quantifiers	
Rule of Inference	*Name*
$\neg \forall x\, P(x) \equiv \exists x \neg P(x)$ $\neg \exists x\, Q(x) \equiv \forall x \neg Q(x)$	De Morgan's laws for quantifiers