

Introduction to Statistical Data Science

October 23, 2018

1 Lecture 1

1.1 Normal Distribution

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-u)^2}{2\sigma^2}\right)$$

$$F(x) \equiv P(X \leq x)$$

Central Limit Theorem (for normal distribution)

The more random variables we average over, the closer the resulting distribution will be to the Normal distribution

Parameter: u The mean is the location parameter.

Parameter: σ^2 The variance is the scale parameter

1.2 Uniform Distribution

$$X \sim U[0, 1] \tag{1}$$

$$0 \leq x \leq 1 \tag{2}$$

$$p(x) = 1 \tag{3}$$

$$F(x) = P(X \leq x) = x \tag{4}$$

$$\tag{5}$$

Use uniform distribution to construct normal distribution

$$\mathbf{X} = \begin{bmatrix} X^{(1)} \\ X^{(2)} \\ \vdots \\ X^{(n)} \end{bmatrix} \quad (6)$$
$$X^{(i)} \sim U[0, 1]$$
$$Y = \frac{1}{n} \sum_{i=1}^n X^{(i)}$$

subsampling \mathbf{X} vector to construct Y . The distribution of $Y_j \sim ?$, $j = 1, 2, 3, \dots, p$ would close to normal distribution.

1.3 Poisson distribution

$$P(X = k) = \frac{e^{-\lambda} \lambda^k}{k!}$$
$$E(X) = V(X) = \lambda$$

1.4 empirical CDF

$$F_n(x) = \frac{1}{n} \sum_{i=1}^n I(x^{(i)} \leq x)$$

quantile is simply the inverse of the CDF: -The 0.9 quantile is the value of x such that $F(x) = 0.9$
i.e. $x = F^{-1}(0.9)$

2 Hypothesis

null hypothesis

In inferential statistics, the null hypothesis is a general statement or default position that there is no relationship between two measured phenomena, or no association among groups.

p-value

The probability of obtaining results as or more extreme than that observed, assuming H_0 is true, is the p-value, under the assumption that the null hypothesis.

$$p \equiv P(X \leq 15; H_0)$$

$$p = \sum_{x=0}^{15} \binom{40}{x} 0.5^x (1 - 0.5)^{(40-x)}$$

The p-value is most certainly not the probability of H_0 being true

When used in practice with a threshold of 0.05 this is an informal method of reasoning and can be easily criticized

Statistical power

The power of a hypothesis test is the probability of avoiding a false negative

P-value distribution (CDF)

$$P(F(X) \leq z) = P(F^{-1}(F(x)) \leq F^{-1}(z)) = P(X \leq F^{-1}(z)) = F(F^{-1}(z)) = z$$

Level

The threshold probability of 0.05 was the level of the test.

The choice of a particular level may be guided by the need to trade off Type 1 and Type 2 errors.

Type 1 errors

Type 1 error occurs when we reject the null hypothesis H_0 , when it is true

Type 2 errors

Type 2 error occurs when we fail to reject H_0 when it is false.

power of the test

The probability of avoiding a Type 2 error is the power of the test.

That is the probability that we reject H_0 given that it is false.

The power of a test varies with sample size

The power of a test also varies with the level of the test.

The ways to increase the power of our test: collect more data, allow for a higher Type 1 error, use a better test statistic, make stronger assumption.

Testing procedure

Specify a null and alternative hypothesis.

Specify the level of the test.

Specify a suitable test statistic.

critical region

The set of all test statistic values which would cause us to reject H_0 .

t-test

For the small sample, and iid normal distribution:

$$T = \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{S_n} \sim \tau(n - 1)$$

This is called a t-distribution with $n - 1$ degrees of freedom

Wald test

Sample $X^{(i)}$ can be from any distribution.

Sample sizes must be assumed to be ‘big enough’ that CLT applies.

Hence, the distribution of test statistic is $N(0, 1)$. That is the case of t-distribution which $n \rightarrow \infty$

Goodness-of-fit tests (chi-square test)

still needs to be discussed.

Paired tests

Bonferroni correction

While testing n independent hypothesis in the same data set, the p value should be $1/n$

Confidence Intervals

$$X \sim N(\mu, \sigma^2)$$

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$$

\bar{X} is the sample average. If $X_i \sim N(\mu, \frac{\sigma^2}{n})$, then

$$\bar{X} \sim N$$

$$Var(\bar{X}) = \frac{(\frac{\sum_i x_i}{n} - \mu)^2}{1} = (\frac{\sum_i (x_i - \mu)}{n})^2 = \sigma^2/n$$

Then

$$\frac{\bar{X} - \mu}{\sqrt{\sigma^2/n}} \sim N(0, 1)$$

This a pivot, a function of parameter of interest which has a known distribution.

$$P\left(\frac{\bar{X} - \mu}{\sqrt{\sigma^2/n}} \leq 1.5\right) = p(Z \leq 1.5) = p\left(\mu \geq \bar{X} - \frac{1.5\sigma}{\sqrt{n}}\right) = 0.93$$

Coverage

The interval $[\bar{X} - 1.5\sigma/\sqrt{n}, \infty)$ will contain μ 93% of the time

two side interval

It is hard to calculate the confidence interval, therefore a CLT could help to solve it with large sample.

Bootstrap

A foundation of bootstrap is sampling with replacement

1. Randomly select a data point.
2. Add it to the 're-sample'
3. Put it back in the box.

Using Bootstrap to calculate SE

1. Draw $X^{(1)*}, \dots, X^{(n)*} \sim \hat{F}_n$
2. Compute \hat{X}_n^* by averaging X_1^*, \dots, X_n^*
3. Repeat steps 1 and 2, B times, to get $\bar{X}_{n,1}^*, \dots, \bar{X}_{n,B}^*$
4. Let

$$s.e._{boot} = \sqrt{\frac{1}{B} \sum_{b=1}^B \left(\bar{X}_{n,b}^* - \frac{1}{B} \sum_{r=1}^B \bar{X}_{n,r}^* \right)^2}$$

Bootstrap pivotal interval

Define $H(r) = P(\hat{\theta} - \theta \leq r)$

Define quantiles such that we get coverage $1 - \alpha$:

$$\begin{aligned}P(a(\hat{\theta}_n) \leq \theta \leq b(\hat{\theta}_n)) &= 1 - \alpha \\a(\hat{\theta}_n) &= \hat{\theta} - H^{-1}(1 - \alpha/2) \\b(\hat{\theta}_n) &= \hat{\theta} - H^{-1}(\alpha/2)\end{aligned}$$