Problem 4

Suppose $AB = BA$. Prove that

(a) $(A+B)^2 = A^2 + 2AB + B^2$, (b) $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$.

How do these formulas change if $AB \neq BA$?

Solution

We use associativity and distributivity of operator/matrix multiplication. Commutativity *is not* assumed unless explicitly stated.

(a) Square. Expand without assuming commutativity:

$$\begin{aligned}(A+B)^2 &= (A+B)(A+B)\\ &= A(A+B) + B(A+B)\\ &= A^2 + AB + BA + B^2.\end{aligned}$$

If $AB = BA$, then AB and BA are equal and can be combined:

$$(A+B)^2 = A^2 + 2AB + B^2.$$

If $AB \neq BA$, the correct form is

$$\boxed{(A+B)^2 = A^2 + AB + BA + B^2}.$$

(b) Cube. Start from the general square and expand:

$$\begin{aligned}(A+B)^3 &= (A+B)^2(A+B)\\ &= (A^2 + AB + BA + B^2)(A+B)\\ &= A^3 + A^2B + ABA + AB^2 + BA^2 + BAB + B^2A + B^3.\end{aligned}$$

If $AB = BA$, then we may reorder factors to group like terms:

$$ABA = A(BA) = A(AB) = A^2B, \qquad BA^2 = A^2B,$$

and similarly

$$BAB = B(AB) = B(BA) = B^2A, \qquad AB^2 = B^2A.$$

Hence,

$$(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3.$$

If $AB \neq BA$, there is no further simplification in general and the correct expansion is

$$\boxed{(A+B)^3 = A^3 + A^2B + ABA + AB^2 + BA^2 + BAB + B^2A + B^3}.$$

Summary.

$$\begin{aligned}&\textbf{Always:} && (A+B)^2 = A^2 + AB + BA + B^2,\\ & && (A+B)^3 = A^3 + A^2B + ABA + AB^2 + BA^2 + BAB + B^2A + B^3.\\ &\textbf{If } AB = BA: && (A+B)^2 = A^2 + 2AB + B^2, \quad (A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3.\end{aligned}$$

Problem 5 (Raising and Lowering Operators)

Let $L = L_-L_+$ and suppose the commutation relation $[L_-, L_+] = \mathbb{I}$ holds, where

$$[A, B] := AB - BA, \qquad \mathbb{I} \text{ is the identity operator.}$$

If $L\,|x\rangle = \lambda\,|x\rangle$, prove that:

(a) $|y\rangle := L_+|x\rangle$ satisfies $L|y\rangle = (\lambda{+}1)|y\rangle$, (b) $|z\rangle := L_-|x\rangle$ satisfies $L|z\rangle = (\lambda{-}1)|z\rangle$,

provided the respective vectors are nonzero. (This is why L_+/L_- are called raising/lowering operators.)

Tools (from Volumen UNO: Linear Operators)

- **Commutator:** $[A, B] = AB - BA$.
- **Product rule:** $[AB, C] = A[B, C] + [A, C]B$.
- **Rearrangement:** from $[A, B] = AB - BA$ we get $AB = BA + [A, B]$.
- **Identity:** $\mathbb{I}$ with $A\mathbb{I} = \mathbb{I}A = A$.
- From $[L_-, L_+] = \mathbb{I}$ it follows that $[L_+, L_-] = -\mathbb{I}$.
- Eigenpair notation: $L|x\rangle = \lambda|x\rangle$.

(a) Raising: L_+ increases the eigenvalue by $+1$

Define $|y\rangle := L_+|x\rangle$. We aim to show $L|y\rangle = (\lambda + 1)|y\rangle$.

Step 1: Composition. By definition of $|y\rangle$,

$$L|y\rangle = L\big(L_+|x\rangle\big) = (LL_+)|x\rangle.$$

Step 2: Use the commutator.

$$[L, L_+] = LL_+ - L_+L \quad \Rightarrow \quad LL_+ = L_+L + [L, L_+].$$

Step 3: Compute $[L, L_+]$ using $L = L_-L_+$ and the product rule.

$$[L, L_+] = [L_-L_+, L_+] = L_-[L_+, L_+] + [L_-, L_+]\,L_+ = 0 + \mathbb{I}\,L_+ = L_+.$$

Step 4: Substitute and left-factor.

$$LL_+ = L_+L + [L, L_+] = L_+L + L_+ = L_+(L + \mathbb{I}).$$

Step 5: Act on $|x\rangle$ and use $L|x\rangle = \lambda|x\rangle$.

$$\begin{aligned} L|y\rangle = (LL_+)|x\rangle &= L_+(L + \mathbb{I})|x\rangle = L_+\big(L|x\rangle + \mathbb{I}|x\rangle\big) \\ &= L_+\big(\lambda|x\rangle + |x\rangle\big) = (\lambda + 1)\,L_+|x\rangle = (\lambda + 1)|y\rangle. \end{aligned}$$

(b) Lowering: L_- decreases the eigenvalue by -1

Define $|z\rangle := L_-|x\rangle$. We show $L|z\rangle = (\lambda - 1)|z\rangle$.

Step 1: Composition.

$$L|z\rangle = L\big(L_-|x\rangle\big) = (LL_-)|x\rangle.$$

Step 2: Use the commutator.

$$[L, L_-] = LL_- - L_-L \quad \Rightarrow \quad LL_- = L_-L + [L, L_-].$$

Step 3: Compute $[L, L_-]$ using $L = L_-L_+$ and the product rule.

$$[L, L_-] = [L_-L_+, L_-] = L_-[L_+, L_-] + [L_-, L_-]\, L_+ = L_-(-\mathbb{I}) + 0 = -\, L_-.$$

Step 4: Substitute and left-factor.

$$LL_- = L_-L + [L, L_-] = L_-L - L_- = L_-(L - \mathbb{I}).$$

Step 5: Act on $|x\rangle$ and use $L|x\rangle = \lambda|x\rangle$.

$$\begin{aligned} L|z\rangle = (LL_-)|x\rangle &= L_-(L - \mathbb{I})|x\rangle = L_-\big(L|x\rangle - \mathbb{I}|x\rangle\big) \\ &= L_-\big(\lambda|x\rangle - |x\rangle\big) = (\lambda - 1)\, L_-|x\rangle = (\lambda - 1)|z\rangle. \end{aligned}$$

Remarks (boundary cases and ladder structure)

- If $L_+|x\rangle = 0$, $|x\rangle$ is a *top state* (cannot raise further).
- If $L_-|x\rangle = 0$, $|x\rangle$ is a *bottom state* (cannot lower further).
- Otherwise, $L_\pm$ generate an eigenvalue ladder

$$\cdots \xrightarrow{L_-} \lambda - 1 \xrightarrow{L_-} \lambda \xrightarrow{L_+} \lambda + 1 \xrightarrow{L_+} \lambda + 2 \xrightarrow{\cdots} \ .$$