University of Pennsylvania

CIS 1600 F23

Theorems, Lemmas & Proofs

by

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Contents

1 Logic & Number Theory

Logic

Theorem 1.1

$$p \to q \equiv \neg p \vee q \equiv \neg q \to \neg p$$

Theorem 1.2 (Basis for Contradiction)

$$p \equiv \neg p \to C \qquad p \to q \equiv p \wedge \neg q \to C$$

Number Theory

Theorem 1.3 *If the sum of two integers is even, their difference is also even.*

Theorem 1.4 *If the product of n integers is odd, then all of the integers are odd.*

Theorem 1.5 *If n is odd, then $n^2 + n + 1$ is odd.*

Theorem 1.6 *For all $x \in \mathbb{Z}^+$, $x^3 + 1$ is composite*

Theorem 1.7 *For all $x, y \in \mathbb{Z}$, if $x + y$ is even, then x and y are both odd or both even.*

Theorem 1.8 *For all $x, y \in \mathbb{R}$, if ab is irrational, then either a or b or both are irrational.*

Theorem 1.9 (Sum of Arithmetic Series) *For all $n \in \mathbb{Z}^+$, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$*

Theorem 1.10 (Sum of Geometric Series) *For all $n \in \mathbb{Z}^+$, $\sum_{i=1}^{n} r^i = \frac{r^{n+1}-1}{r-1}$*

Theorem 1.11 (Sum of Infinite Geometric Series) *For all $|r| < 1$, $\sum_{i=1}^{\infty} r^i = \frac{1}{1-r}$*

Theorem 1.12 (Unique Factorization Theorem) *For all $n \in \mathbb{Z}^+$, n can be written as a product of primes in a unique way.*

Theorem 1.13 *There exists infinitely many primes.*

Theorem 1.14 (Binomial Theorem) *For all $n \in \mathbb{Z}^+$, $(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$*

Theorem 1.15 (Generalized Pigeonhole Principle) *For all $n \in \mathbb{Z}^+$, to place n objects into k boxes, at least one box must contain $\lceil \frac{n}{k} \rceil$ objects.*

Theorem 1.16 *For a sequence of n positive integers, there exists a consecutive subsequence whos sum is devisable by n*

2 Sets

Theorem 2.1

$$|P(A)| = 2^{|A|}$$

Theorem 2.2 $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$

Theorem 2.3 $(T \cap S) \cup (F \cap G) \subseteq (T \cup F) \cap (S \cup G)$

Theorem 2.4 $A \times B = B \times A \leftrightarrow A = B$

Theorem 2.5 (De'Morgan's Laws)

$$A - (B \cup C) = (A - B) \cap (A - C)$$
$$A - (B \cap C) = (A - B) \cup (A - C)$$

Theorem 2.6 (Principal of Inclusion-Exclusion)

$$|A_i \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

3 Counting & Probability

Theorem 3.1 (Multiplicatoin Rule) *For all $n \in \mathbb{Z}^+$, if a task consists of n steps, where the ith step can be done in n_i ways, then the task can be done in $\prod_{i=1}^{n} n_i$ ways.*

Theorem 3.2 (Permutation of Multisets) *If there ar n_1 T_1 objects, n_2 T_2 objects, $\ldots$, n_k T_k objects, then there are*

$$\frac{(n_1 + n_2 + \ldots + n_k)!}{n_1! n_2! \ldots n_k!}$$

ways to arrange them.

Theorem 3.3 (r-Combinations with Repetition) *For all $n, r \in \mathbb{Z}^+$, there are*

$$\binom{n + r - 1}{r}$$

ways to choose r objects from n types of objects with repetition.

Theorem 3.4 (Pascals Formula) *For all $n, r \in \mathbb{Z}^+$, $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$*

Theorem 3.5 $\sum_{i=0}^{n} \binom{n}{i} = 2^n$

Probability

Theorem 3.6 (Conditional Probability) *For all events A, B,*

$$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}$$

Corollary 3.6.1 *For all events A, B,*

$$Pr[A \cap B] = Pr[A|B]Pr[B]$$

Theorem 3.7 (Total Probability Theorem) *For all events $A, B_1, \ldots, B_n$,*

$$Pr[A] = \sum_{i=1}^{n} Pr[A|B_i]Pr[B_i]$$

Theorem 3.8 *Event A and B are independent $\leftrightarrow Pr[A \cap B] = Pr[A]Pr[B]$*

Theorem 3.9 (Expected Value) *For all events $A_1, \ldots, A_n$,*

$$E[X] = \sum_{i=1}^{n} Pr[A_i]X_i$$

Theorem 3.10 (Linearity of Expectation) *For all events $A_1, \ldots, A_n$,*

$$E[X] = \sum_{i=1}^{n} E[X_i]$$

Theorem 3.11 (Markov's Inequality) *For all random variables X and $a > 0$,*

$$Pr[X \geq a] \leq \frac{E[X]}{a}$$

Theorem 3.12 (Varience) *] For all random variables X,*

$$Var[X] = E[X^2] - E[X]^2$$

Theorem 3.13 *If X and Y are independent random variables, then $E[XY] = E[X]E[Y]$ and $Var[X+Y] = Var[X] + Var[Y]$*

Theorem 3.14 $E[Y] = \sum_{k=0}^{\infty} Pr[Y > k]$

Theorem 3.15 (De'Morgan's Laws)

$$Pr[\overline{A \cup B}] = Pr[\overline{A} \cap \overline{B}]$$
$$Pr[\overline{A \cap B}] = Pr[\overline{A} \cup \overline{B}]$$

4 Graph Theory

General Graphs

Lemma 4.1 (Handshaking Lemma) *For any graph G, the sum of the degrees of all vertices is twice the number of edges. That is:*

$$\sum_{v \in V} \deg(v) = 2|E|$$

Theorem 4.2 *There is an even amount of vertices with odd degree in any graph G.*

Theorem 4.3 *Graph with n vertices and m edges must have $\geq n - m$ connected components*

Corollary 4.3.1 *Each connected graph must have at least $n - 1$ edges*

Theorem 4.4 *Graph $G = (V, E)$ has $|V| \geq 1$ and $|E| \geq 1$, then $|V| \leq |V|^2 - 2|E|$*

Theorem 4.5 *If v is a cut vertex in G, v is not a cut vertex in $\overline{G}$ and $\overline{G}$ is connected.*

Theorem 4.6 *Either G or $\overline{G}$ is connected. Where $\overline{G}$ is the complement of G.*

Trees

Theorem 4.7 *The following are equivalent for a graph G:*

1. *G is a tree*
2. *G is connected and has $|V| - 1$ edges*
3. *G is minimally connected, $G - e$ is disconnected for any $e \in E$*
4. *G is acyclic and $G + \{x, y\}$ for non-adjacent $x, y \in V$ creates a unique cycle*
5. *G There is a unique path between any two vertices*

Theorem 4.8 *Every connected graph contains a spanning subtree.*

Theorem 4.9 *A full binary tree with k internal vertices has $2k+1$ vertices and $k+1$ leaves.*

Theorem 4.10 *A binary tree of height h has at most 2^h leaves.*

Theorem 4.11 *For any graph with $\delta(G) \geq 2$, G contains a cycle*

Theorem 4.12 *For any graph with $\delta(G) \geq \frac{n}{2}$, G is connected*

Theorem 4.13 *A tree T such that the degree of all vertices adjacent to a leaf is ≥ 3 has leaf $x \neq y$ wherh $N(x) = N(y)$.*

Theorem 4.14 *Every three tree with ≥ 4 leaves has some internal vertex thats adjacent to two leaves.*

Theorem 4.15 *Every three tree with l has $l - 2$ vertices of degree 3.*

Theorem 4.16 *A tree T must have at least $\Delta(T)$ leaves.*

Eularian & Hamiltonian Graphs

Corollary 4.16.1 *For any graph with $\delta(G) \geq \frac{n}{2}$ and $n \geq 3$, G is Hamiltonian*

Theorem 4.17 *If a graph contains a Eularian circuit $\rightarrow$ the line graph of G is Hamiltonian*

Planar Graphs

Theorem 4.18 (Euler's Formula) *$n - m + f = 2$ for any connected graph*

Theorem 4.19 *Sum of all degrees of faces in a crossing is $2|E|$*

Corollary 4.19.1 *$|E| \leq 3|V| - 6$ for any planar graph with $|V| \geq 2$*

Theorem 4.20 *All planr graph have $\delta(G) \leq 5$*

Theorem 4.21 *A graph is planar $\leftrightarrow$ it contains no subdivision of K_5 or $K_{3,3}$*

Theorem 4.22 *If a graph is planar, $\chi(G) \leq 6$*

Graph Coloring & Matchings

Theorem 4.23 *A graph with $\Delta(G) \leq k$, G is $k+1$-colorable*

Theorem 4.24 *Graph G is bipartite $\leftrightarrow$ G contains no odd length cycle*

Theorem 4.25 *Matching M is maximum $\leftrightarrow$ contains no M-augmenting path*

Theorem 4.26 (Hall's Theorem) *Bipartite graph $G = (X, Y, E)$ has a X-saturating matching $\leftrightarrow$ $|N(S)| \geq |S|$ for all $S \subseteq X$*

Theorem 4.27 *A n-vertex tournament graph has $\frac{n!}{2^{n-1}}$ Hamiltonian paths*

Theorem 4.28 *For graph $G = (V, E)$ and $d = \frac{2m}{n}$ is the average degree, $\alpha(G) \geq \frac{n}{2d}$*

Theorem 4.29 *For graph G where $|V| \geq 2$ and $\delta(G) = \delta$ contains a dominating set of at most $\frac{n \cdot (1+\ln(1+\delta))}{1+\delta}$*

Theorem 4.30 *In graph G, For $k \in \mathbb{Z}^+$, if n is larnge enough, there exists a k-dominated tournament on n vertices.*

Theorem 4.31 *For a bipartite graph $G = (X, Y, E)$, $\delta(X) \geq \Delta(Y)$ there exists a X-saturating matching.*

Theorem 4.32 *For any graph $G = (V, E)$, there exists a partition of $V_1 \cup V_2 = V$ such that there are $\geq \frac{|E|}{2}$ edges between them.*

Theorem 4.33 *For $G = (V, E)$, if $\delta(G) \geq 2k$, then there exists a matching of size at least k.*

Theorem 4.34 *If every vertex in a bipartite graph $G = (X, Y, E)$ has the same degree $d \geq 0$, then $|X| = |Y|$*

Theorem 4.35 *If*

$$\binom{n}{k} 2^{1-\binom{k}{2}} < 1, \text{ then } R(k,k) > n$$

5 Relations & Functions

Theorem 5.1 *f is a function*

1. *f is injective $\leftrightarrow$ for all $a, b \in A$, $f(a) = f(b) \rightarrow a = b$*
2. *f is surjective $\leftrightarrow$ for all $b \in B$, there exists $a \in A$ such that $f(a) = b$*
3. *f is bijective $\leftrightarrow$ f is injective and surjective (one-to-one correspondence)*

Theorem 5.2 *There are $2^{n(n-1)}$ relations on a set of size n.*

Theorem 5.3 *R is a relation on set A. R is transitive $\leftrightarrow R^n \subseteq R$ for all $n \in \mathbb{Z}^+$*

Theorem 5.4 *f and g are surjective/injective/bijective $\leftrightarrow f \circ g$ is surjective/injective/bijective*

A Appendix: Proofs