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DEEP INELASTIC SCATTERING FROM ARBITRARY SPIN TARGETS*

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We analyze inclusive inelastic lepton scattering from polarized targets of arbitrary spin, J. We express the cross-section and lepton spin asymmetry in terms of Lorentz invariant structure functions. In the Bjorken limit all information about the target is summarized by a set of $2J+1$ quark distribution functions (for each flavor of quark and antiquark) which evolve logarithmically with Q^2 in a manner prescribed by QCD. We present both parton model and operator product expansion analyses. For nuclear targets we present the predictions for these structure functions based on a convolution model in which nuclei are composed only of bound protons and neutrons. Specific examples are worked out in detail.

1. Introduction

Recently [1] we pointed out the possibility of novel effects in deep inelastic scattering from polarized spin-one hadrons★. Experimenters are contemplating deep inelastic scattering from polarized nuclear targets ranging from the deuteron ($J=1$) and ^{3}He ($J=\frac{1}{2}$) to ^{7}Li ($J=\frac{3}{2}$), ^{10}B ($J=3$) and ^{27}Al ($J=\frac{5}{2}$). These polarization-dependent effects arise only from the unpaired nucleons in a nucleus, and are therefore suppressed by a factor $1/A$ relative to the dominant structure function F_1. In order to clarify the effects measurable in such experiments we have extended the analysis of ref. [1] to the case of arbitrary spin targets. At the same time we have unified the treatment of target spin so that our analysis applies as well to $J=0$, $\frac{1}{2}$ and 1 where it reproduces familiar results.

We find that in the Bjorken limit (augmented by Callan–Gross-like relations which hold only approximately in QCD) all information about deep inelastic electron (or muon) scattering from a target of spin-J is summarized by $2J+1$ "parton distribution functions," $q_s^{JH}(x)$, for each flavor of quark and antiquark.

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★ Some of these effects were studied previously for the case of the deuteron by L.I. Frankfurt and M.S. Strickman [2].

$q_s^{JH}(x)$ is defined to be the probability to find a quark with momentum fraction x and spin-component s in a target with spin-component H in an infinite momentum frame. s and H are measured along the photon momentum, $\boldsymbol{q}^{\star}$. H takes values from $-J$ to J, while s is either $\downarrow(-\frac{1}{2})$ or $\uparrow(+\frac{1}{2})$. Parity invariance of the strong interactions requires $q_s^{JH}(x) = q_{-s}^{J-H}(x)$ and leaves $(2J+1)$ independent distributions. (Of course, $q_s^{JH}(x)$ can be as well defined in the laboratory where $\hat{q}$ defines the $\hat{e}_3$-axis, and x becomes the $+$component of momentum, $p^+ = (p^0 + p^3)/\sqrt{2}\mathscr{M}$, where $\mathscr{M}$ is the target mass.) In QCD, the parton distributions $\{q_s^{JH}(x)\}$ develop a logarithmic Q^2-dependence which is calculable, and is described in sect. 4.

There is an additional distribution function $\ell^{JH}(x)$ for each target helicity H if we do not use the Callan–Gross like relations. Parity invariance requires $\ell^{JH}(x) = \ell^{J-H}(x)$, so there are $J+1$ $(J+\frac{1}{2})$ independent functions ℓ^{JH} for integer J (half-integer J). The functions $\ell^{JH}(x)$ have no direct parton model interpretation, just as the quantity $F_2(x) - 2xF_1(x)$ has no simple parton interpretation in the case of a spinless target. To keep the connection with spinless and spin-$\frac{1}{2}$ targets as close as possible, we define**

$$F_1^{JH}(x) \equiv \tfrac{1}{2}\left(q_{\uparrow}^{JH}(x) + q_{\downarrow}^{JH}(x)\right), \tag{1.1}$$

$$F_2^{JH}(x) \equiv x\left(q_{\uparrow}^{JH}(x) + q_{\downarrow}^{JH}(x)\right) + 2x\ell^{JH}(x), \tag{1.2}$$

$$g_1^{JH}(x) \equiv \tfrac{1}{2}\left(q_{\uparrow}^{JH}(x) - q_{\downarrow}^{JH}(x)\right). \tag{1.3}$$

The familiar target-spin-averaged structure functions, $F_1(x)$ and $F_2(x)$, are given by

$$F_k(x) = \frac{1}{2J+1} \sum_{H=-J}^{J} F_k^{JH}(x), \qquad k = 1, 2.$$

The deep inelastic scattering cross-section and polarization asymmetry for a target with helicity H and mass $\mathscr{M}$ have a very simple form in the Bjorken limit,

$$\frac{\mathrm{d}\bar{\sigma}^{JH}}{\mathrm{d}x\,\mathrm{d}y} = \frac{e^4 ME}{2\pi Q^4}\left\{y^2 x F_1^{JH}(x) + (1-y) F_2^{JH}(x)\right\}, \tag{1.4}$$

$$\frac{\mathrm{d}\Delta\sigma^{JH}}{\mathrm{d}x\,\mathrm{d}y} = \frac{e^4 ME}{2\pi Q^4} y(2-y) x g_1^{JH}(x), \tag{1.5}$$

* We use $\boldsymbol{q}$ as the spin quantization axis for both the target and the virtual photon even through it has the minor inconvenience that s and H are *minus* the helicity of the quark and target in the photon-target center-of-mass frame.

** All flavor labels will be suppressed in the following discussion. We will also not explicitly put in the quark charges or the antiquark distributions. Since electroproduction involves charge conjugation even operators, one should add the antiquark distributions to the quark distributions.

where $y \equiv \nu/\mathscr{M}E$, $x = Q^2/2\nu$, and $\nu = P \cdot q$. $\bar{\sigma}$ is the cross-section averaged over the incident lepton helicity, and $\Delta\sigma$ is the asymmetry, $\frac{1}{2}(\sigma_\uparrow - \sigma_\downarrow)$, where the $\uparrow\downarrow$ label refers to the lepton helicity.

If we do not take the Bjorken limit, there are a total of $6J + 2$ $(6J + 1)$ functions for J integer (half-integer). The additional $3J$ $(3J - \frac{1}{2})$ functions are higher twist for J-integer (half-integer) and their effect on the cross section vanishes as $1/\sqrt{Q^2}$ or $1/Q^2$ (relative to leading twist) as $Q^2 \to \infty$. The only such structure function for $J \leqslant \frac{1}{2}$ is $g_2(x)$, which contributes $\sim 1/\sqrt{Q^2}$ in the Bjorken limit. For $J = 1$, there are three such structure functions, $g_2(x)$ as for $J = \frac{1}{2}$, and $b_3(x)$ and $b_4(x)$ defined in ref. [1], which contribute $\sim 1/\sqrt{Q^2}$ and $\sim 1/Q^2$ respectively in the Bjorken limit. These higher twist structure functions are hard to measure, nevertheless for completeness we compute the inelastic scattering cross-section for all Q^2 in terms of a full set of $6J + 2$ (or $6J + 1$) structure functions. The result may be found in sect. 2.

In practice it is useful to replace the structure function $\{q_s^{JH}(x)\}$ by a set of "multipole" structure functions which have better (irreducible) transformation properties under rotations. In sects. 3 and 4 we show that the proper choice is

$$\sum_{H=-J}^{J} (-)^{J-H} (JHJ - H|L0) q_\uparrow^{JH}(x) \equiv \begin{cases} {}^J_L F_1(x), & L \text{ even}, 0 \leqslant L \leqslant 2J, \\ {}^J_L g_1(x), & L \text{ odd}, 1 \leqslant L \leqslant 2J, \end{cases} \tag{1.6}$$

where $(JHJ - H|L0)$ is a Clebsch–Gordan coefficient. ${}^J_L F_1$ contributes only to the cross-section averaged over lepton helicities ($\bar{\sigma}$) whereas ${}^J_L g_1$ contributes only to the lepton spin asymmetry ($\Delta\sigma$). The $\{{}^J_L F_1(x)\}$ and $\{{}^J_L g_1(x)\}$ are a decomposition of the leading twist structure functions into irreducible representations of the rotation group. In sect. 4 we show that the multipole structure functions ${}^J_L F_1$ and ${}^J_L g_1$ are related to matrix elements of twist-2 operators in a simple way. In particular we are able to use the rotation group properties of the structure functions and the operators to derive important new sum rules:

$$\left.\begin{aligned} \int_0^1 x^{n-1}\, {}^J_L F_1(x)\,\mathrm{d}x = 0 \\ \int_0^1 x^{n-2}\, {}^J_L F_2(x)\,\mathrm{d}x = 0 \end{aligned}\right\} L, n \text{ even}, \qquad 0 < n < L, \tag{1.7}$$

$$\int_0^1 x^{n-1}\, {}^J_L g_1(x) = 0 \quad L, n \text{ odd}, \qquad 1 \leqslant n < L, \tag{1.8}$$

which significantly constrain the form of these structure functions. (These sum rules are expected to converge at $x = 0$ on the basis of traditional Regge analysis.)

Other virtues of the $\{{}^{J}_{L}F_1(x)\}$ and $\{{}^{J}_{L}g_1(x)\}$ become apparent when one calculates deep inelastic scattering from a target which is composed of spin-$\frac{1}{2}$ protons and neutrons in (non-relativistic) relative motion. Then ${}^{J}_{L}F_1(x)$ (${}^{J}_{L}g_1(x)$) is given by an integral over the nucleon momentum distribution weighted by $Y_{L0}(\Omega)$ ($Y_{L\pm 1,0}$) for L even (odd). This is discussed further in sect. 5.

For familiar values of J and L, ${}^{J}_{L}F_1$ and ${}^{J}_{L}g_1$ reduce to familiar structure functions. For example,

$$
\begin{aligned}
{}^{J}_{0}F_1(x) &= \frac{1}{\sqrt{2J+1}}\sum_H \left(q_\uparrow^{JH}(x) + q_\downarrow^{JH}(x)\right) = \sqrt{2J+1}\,F_1(x)\,,\\
{}^{1/2}_{1}g_1(x) &= \frac{1}{\sqrt{2}}\left(q_\uparrow^{1/2\,1/2}(x) - q_\downarrow^{1/2\,1/2}(x)\right) = \sqrt{2}\,g_1(x)\,,\\
{}^{1}_{1}g_1(x) &= \frac{1}{\sqrt{2}}\left(q_\uparrow^{11}(x) - q_\downarrow^{11}(x)\right) = \sqrt{2}\,g_1(x)\,,\\
{}^{1}_{2}F_1(x) &= \frac{1}{\sqrt{6}}\left(q_\uparrow^{11}(x) + q_\uparrow^{1\,-1}(x) - 2q_\uparrow^{10}(x)\right) = -\sqrt{\tfrac{2}{3}}\,b_1(x)\,.
\end{aligned}
\tag{1.9}
$$

And for unfamiliar values of J and L, ${}^{J}_{L}F_1(x)$ and ${}^{J}_{L}g_1$ provide the most natural generalization of the structure functions F_1, g_1 and b_1. Thus, the most convenient definitions of the spin asymmetries for a spin-$\frac{3}{2}$ target are

$$
{}^{3/2}_{1}g_1(x) = \frac{1}{\sqrt{20}}\left(3q_\uparrow^{3/2\,3/2} - 3q_\downarrow^{3/2\,3/2} + q_\uparrow^{3/2\,1/2} - q_\downarrow^{3/2\,1/2}\right)
$$

and

$$
{}^{3/2}_{3}g_1(x) = \frac{1}{\sqrt{20}}\left(q_\uparrow^{3/2\,3/2} - q_\downarrow^{3/2\,3/2} - 3g_\uparrow^{3/2\,1/2} + 3q_\downarrow^{3/2\,1/2}\right) \tag{1.10}
$$

as opposed to $q_\uparrow^{3/2\,3/2} - q_\downarrow^{3/2\,3/2}$ and $q_\uparrow^{3/2\,1/2} - q_\downarrow^{3/2\,1/2}$ individually.

The calculations in sect. 5 are applied to some cases of special interest in sect. 6.

2. Kinematics

We are interested in inelastic scattering of an electron (or muon) of initial four-momentum k^μ and final four-momentum $k'^\mu \equiv k^\mu - q^\mu$ from a target of four-momentum p^μ. We will generally work in the rest frame of the target, $p^\mu = (\mathscr{M}, \mathbf{0})$, and orient the virtual photon along the $\hat{e}_3(\hat{z})$-direction, $q^\mu = (q^0, 0, 0, q^3)$, which we also choose to be the quantization axis for the target spin. We can choose the azimuthal angle of the final lepton (with respect to the x-axis) in the xy plane to be φ. We will refer to this frame we have chosen as the photon frame. By contrast, in the laboratory frame, the incident lepton direction defines the z-axis, and the outgoing lepton has polar angles (θ, φ) with respect to the z-axis. In the Bjorken limit, the photon frame and laboratory frame coincide. As usual, we define $Q^2 \equiv -q^2 = (q^3)^2 - (q^0)^2$ and $\nu \equiv p \cdot q = \mathscr{M} q^0$.

The cross-section for inelastic lepton scattering is proportional to the imaginary part of the amplitude for forward, virtual Compton scattering from the polarized target,

$$W_{\mu\nu}^{JH'H}(q,p) \equiv \frac{1}{4\pi}\int \mathrm{d}^4x\, \mathrm{e}^{iq\cdot x}\left\langle p, JH'\right|\left[J_\mu(x), J_\nu(0)\right]\left|p, JH\right\rangle = \frac{1}{2\pi}\,\mathrm{Im}\, T_{\mu\nu}^{JH'H},$$

$$T_{\mu\nu}^{JH'H} \equiv i\int \mathrm{d}^4x\, \mathrm{e}^{iq\cdot x}\left\langle p, JH'\right|T\left(J_\mu(x) J_\nu(0)\right)\left|p, JH\right\rangle. \tag{2.1}$$

If we were only interested in targets polarized along $\hat{e}_3$ we could set $H = H'$; to obtain the cross-section for arbitrary target polarizations it is necessary to keep $H \neq H'$. Since the forward Compton amplitude $T_{\mu\nu}^{JH'H}$ can be expressed in terms of helicity amplitudes, $W_{\mu\nu}^{JH'H}$ can be written using the imaginary parts of the helicity amplitudes. The number of independent Lorentz invariant structure functions in $W_{\mu\nu}^{JH'H}$ is the same as the number of independent helicity amplitudes. Let $A_{hH,h'H'}$ denote the imaginary part of the forward Compton helicity amplitude for $\gamma_h + \mathrm{target}_H \to \gamma_{h'} + \mathrm{target}_{H'}$,

$$A^J_{hH,h'H'} \equiv \epsilon_{h'}^{*\mu} W_{\mu\nu}^{JH'H} \epsilon_h^\nu, \tag{2.2}$$

where $\{\epsilon^\mu(\lambda)\}$ are photon polarization vectors

$$\epsilon_\pm^\mu = \mp\frac{1}{\sqrt{2}}(0,1,\pm i,0), \qquad \epsilon_0^\mu = \frac{1}{\sqrt{Q^2}}\left(q^3,0,0,q^0\right). \tag{2.3}$$

The $\{A_{hH,h'H'}\}$ are easily enumerated. Angular momentum conservation requires that the total helicity is conserved, $h + H = h' + H'$, which leaves $18J + 1$ independent helicity amplitudes. Time reversal invariance requires $A^J_{hH,h'H'} = A^J_{h'H',hH}$ which leaves $12J + 2$ independent amplitudes. Parity invariance requires $A^J_{hH,h'H'} = A^J_{-h-H,-h'-H'}$ leaving $2J + 1$ diagonal transverse amplitudes $\{A^J_{\pm H,\pm H}$ for $H > 0$ and $A^J_{+0,+0}$ if $H = 0\}$; $J + 1$ $(J + \frac{1}{2})$ diagonal longitudinal amplitudes for J-integer (half-integer) $\{A^J_{0H,0H}$ for $H \geqslant 0\}$; $2J$ helicity-flip amplitudes $\{A^J_{+H,0H+1}$ for $H \geqslant 0$ and $A^J_{-H,0H-1}$ for $H > 0\}$; and J $(J - \frac{1}{2})$ double helicity-flip amplitudes for J-integer (half-integer) $\{A^J_{+H-1,-H+1}$ for $H > 0\}$, for a grand total of $6J + 2$ $(6J + 1)$ for J-integer (half-integer). In going from a spin J-$\frac{1}{2}$ target to a spin J target, there are two additional amplitudes if $2J$ is odd, and four additional amplitudes if $2J$ is even.

The cross-section for inelastic lepton scattering is obtained by contracting $W_{\mu\nu}^{JH'H}$ with the lepton tensor $\ell_{\mu\nu}$ for the (possibly) polarized lepton, which may be calculated to lowest order in QED (ignoring lepton masses).

$$\ell_{\mu\nu}^{\pm}(q,k) \equiv \tfrac{1}{2}\operatorname{Tr} \not{k}\gamma_\mu(\not{k}-\not{q})\gamma_\nu(1\pm\gamma_5)$$
$$= 4k_\mu k_\nu - 2k_\mu q_\nu - 2k_\nu q_\mu + 2k\cdot q g_{\mu\nu} \pm 2i\epsilon_{\mu\nu\alpha\beta}k^\alpha q^\beta ,$$

where $\epsilon_{0123} = +1$ and + or − correspond to right- or left-handed initial leptons respectively. We define

$$\bar{\ell}_{\mu\nu} \equiv \tfrac{1}{2}\left(\ell_{\mu\nu}^{+} + \ell_{\mu\nu}^{-}\right) = 4k_\mu k_\nu + 2k\cdot q g_{\mu\nu} - 2k_\mu q_\nu - 2k_\nu q_\mu ,$$

which controls lepton spin averaged inelastic scattering; and

$$\Delta\ell_{\mu\nu} \equiv \tfrac{1}{2}\left(\ell_{\mu\nu}^{+} - \ell_{\mu\nu}^{-}\right) = 2i\epsilon_{\mu\nu\alpha\beta}k^\alpha q^\beta ,$$

which determines the lepton spin asymmetry. $\bar{\ell}_{\mu\nu}$ and $\Delta\ell_{\mu\nu}$ can be expanded in the basis of virtual photon helicity eigenstates,

$$\bar{\ell}^{\mu\nu} = \frac{2Q^2}{\kappa y^2}\Big\{\lambda^2\epsilon_0^\mu\epsilon_0^\nu + \tfrac{1}{2}(\lambda^2+\kappa y^2)\left(\epsilon_+^{\mu *}\epsilon_+^\nu + \epsilon_-^{\mu *}\epsilon_-^\nu\right) - (\lambda^2/2)\left(\epsilon_+^{\mu *}\epsilon_-^\nu e^{2i\varphi} + \epsilon_-^{\mu *}\epsilon_+^\nu e^{-2i\varphi}\right)$$
$$+\lambda(1-y/2)\left(\epsilon_-^{\mu *}\epsilon_0^\nu e^{-i\varphi} + \epsilon_0^\mu\epsilon_-^\nu e^{i\varphi} - \epsilon_+^{\mu *}\epsilon_0^\nu e^{i\varphi} - \epsilon_0^\mu\epsilon_+^\nu e^{-i\varphi}\right)\Big\}, \quad (2.4)$$

$$\Delta\ell^{\mu\nu} = \frac{2Q^2}{y\sqrt{\kappa}}\Big\{(1-y/2)\left(\epsilon_+^{\mu *}\epsilon_+^\nu - \epsilon_-^{\mu *}\epsilon_-^\nu\right)$$
$$-(\lambda/2)\left(\epsilon_-^{\mu *}\epsilon_0^\nu e^{-i\varphi} + \epsilon_0^\mu\epsilon_-^\nu e^{i\varphi} + \epsilon_+^{\mu *}\epsilon_0^\nu e^{i\varphi} + \epsilon_0^\mu\epsilon_+^\nu e^{-i\varphi}\right)\Big\}, \quad (2.5)$$

where $\lambda^2 = 2(1-y) - (y^2/2)(\kappa-1)$,

$$\kappa = 1 + \frac{4x^2\mathcal{M}^2}{Q^2} = 1 + \frac{\mathcal{M}^2 Q^2}{\nu^2},$$

and φ is the azimuthal angle (measured with respect to the x-axis in the xy-plane) of the final lepton.

The cross-section (or asymmetry) for deep inelastic scattering can be obtained by contracting the leptonic tensor $\bar{\ell}_{\mu\nu}$ (or $\Delta\ell_{\mu\nu}$) with $W_{\mu\nu}^{JH'H}$. $W_{\mu\nu}^{JH'H}$ can be expressed in terms of the imaginary parts of the Compton helicity amplitudes by using eq. (2.2).

The resulting expressions for the cross-section (denoted by Σ) in terms of the helicity amplitudes are

$$\frac{\mathrm{d}\bar{\Sigma}^{JH'H}}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\varphi}=\frac{e^4MEx}{4\pi^2Q^4\kappa}\Big\{\lambda^2\big(A^J_{0H,0H'}-\tfrac{1}{2}A^J_{+H,-H'}\mathrm{e}^{-2i\varphi}-\tfrac{1}{2}A^J_{-H,+H'}\mathrm{e}^{2i\varphi}\big)$$
$$+\tfrac{1}{2}(\lambda^2+\kappa y^2)\big(A^J_{+H,+H'}+A^J_{-H,-H'}\big)$$
$$+(1-y/2)\lambda\big(A^J_{-H,0H'}\mathrm{e}^{i\varphi}+A^J_{0H,-H'}\mathrm{e}^{-i\varphi}-A^J_{+H,0H'}\mathrm{e}^{-i\varphi}-A^J_{0H,+H'}\mathrm{e}^{i\varphi}\big)\Big\}\,, \tag{2.6}$$

$$\frac{\mathrm{d}\Delta\Sigma^{JH'H}}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\varphi}=\frac{e^4MEx}{4\pi^2Q^4\sqrt{\kappa}}\Big\{y\Big(1-\frac{y}{2}\Big)\big(A^J_{+H,+H'}-A^J_{-H,-H'}\big)$$
$$-\tfrac{1}{2}y\lambda\big(A^J_{-H,0H'}\mathrm{e}^{i\varphi}+A^J_{0H,-H'}\mathrm{e}^{-i\varphi}+A^J_{+H,0H'}\mathrm{e}^{-i\varphi}+A^J_{0H,+H'}\mathrm{e}^{i\varphi}\big)\Big\}\,. \tag{2.7}$$

In eqs. (2.6) and (2.7) we have used an abbreviated notation: overall helicity is conserved, so for a specific choice of H and H' a given term is present only if it conserves overall helicity. The cross-sections (2.6) and (2.7) are not in a useful form, because the helicities H and H' are defined with respect to the direction $\boldsymbol{q}$, which changes on an event by event basis according to the scattering kinematics. It is better to define cross-sections for targets with definite helicities λ and λ' with respect to a fixed direction in the laboratory frame, such as the incident beam direction. To transform from the laboratory coordinate frame to the photon frame, first perform a rotation about the laboratory z-axis by angle φ, then about the new y-axis by angle β, and finally, about the new z-axis by angle $-\varphi$. β is the angle between $\boldsymbol{k}$ and $\boldsymbol{q}$, and is given by

$$\cos\beta=\frac{y+Q^2/2E^2}{\left(y^2+Q^2/E^2\right)^{1/2}}=\frac{1}{\sqrt{\kappa}}\Big(1+\frac{y}{2}(\kappa-1)\Big)\,. \tag{2.8}$$

Thus the state $|J\lambda\rangle$ with helicity λ along the beam direction can be written in terms of the states $|JH\rangle$ with helicity H along the photon direction,

$$|J\lambda\rangle=\sum_H \mathrm{e}^{i(\lambda-H)\varphi}d^J_{H\lambda}(\beta)|JH\rangle\,, \tag{2.9}$$

where $d^J_{H\lambda}(\beta)$ is the usual Wigner rotation matrix. The cross-sections for targets with definite polarizations in the laboratory frame can then be obtained from eqs. (2.6), (2.7) and (2.9) as

$$\frac{\mathrm{d}\bar{\sigma}^{J\lambda'\lambda}}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\varphi}=\sum_{H,H'}\frac{\mathrm{d}\bar{\Sigma}^{JH'H}}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\varphi}d^J_{H\lambda}(\beta)\,d^J_{H'\lambda'}(\beta)\,\mathrm{e}^{i(\lambda-\lambda'+H'-H)\varphi},\tag{2.10}$$

$$\frac{\mathrm{d}\Delta\sigma^{J\lambda'\lambda}}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\varphi}=\sum_{H,H'}\frac{\mathrm{d}\Delta\Sigma^{JH'H}}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\varphi}d^J_{H\lambda}(\beta)\,d^J_{H'\lambda'}(\beta)\,\mathrm{e}^{i(\lambda-\lambda'+H'-H)\varphi}.\tag{2.11}$$

The cross-sections of eqs. (2.10) and (2.11) appear to be complex numbers. It must be remembered, however, that we are interested only in physical cross-sections, i.e. in the case in which the initial and final target states in the Compton process are described by the *same* spin density matrix. In this case the cross-sections are real and positive. Consider, for example, an initial state which is a helicity eigenstate with respect to the beam direction, so that $\lambda=\lambda'$. Then the relevant cross-sections $\bar{\sigma}^{J\lambda\lambda}$ and $\Delta\sigma^{J\lambda\lambda}$ are real and $\bar{\sigma}^{J\lambda\lambda}\geqslant|\Delta\sigma^{J\lambda\lambda}|$. $\bar{\sigma}^{J\lambda\lambda}$ can be expressed in terms of two structure functions,

$$F_1^{JH}(x,Q^2)=\tfrac{1}{2}\left(A^J_{+H,+H}+A^J_{-H,-H}\right),$$
$$F_2^{JH}(x,Q^2)=(x/\kappa)\left(2A^J_{0H,0H}+A^J_{+H,+H}+A^J_{-H,-H}\right).\tag{2.12}$$

$\Delta\sigma^{J\lambda\lambda}$ depends only on $A^J_{+H,+H}-A^J_{-H,-H}$ which may in turn be written as a linear combination of g_1^{JH} and g_2^{JH} (see sect. 4 for the definition of g_2^{JH}) though the result is not very illuminating.

In the Bjorken limit, Q^2, $\nu\to\infty$, $Q^2/2\nu\equiv x$ fixed, eqs. (2.10) and (2.11) simplify. From eqs. (2.8) and (2.9), we see that $\beta\to 0$, and that helicity eigenstates with respect to the beam direction are also helicity eigenstates with respect to the photon direction. We will use H to denote these helicity eigenstates in the scaling limit. In the next section we will see that only helicity non-flip amplitudes, $A^J_{hH,hH}$, are non-zero in the scaling limit, giving

$$\begin{aligned}\frac{\mathrm{d}\bar{\sigma}^{JHH}}{\mathrm{d}x\,\mathrm{d}y}&=\frac{e^4ME}{2\pi Q^4}\Big\{\left[1+(1-y)^2\right]\left[\tfrac{1}{2}xA^J_{+H,+H}(x)+\tfrac{1}{2}xA^J_{-H,-H}(x)\right]\\&\qquad\qquad+2(1-y)xA^J_{0H,0H}(x)\Big\}\\&=\frac{e^4ME}{2\pi Q^4}\left\{y^2xF_1^{JH}(x)+(1-y)F_2^{JH}(x)\right\},\end{aligned}\tag{2.13}$$

and

$$\frac{\mathrm{d}\Delta\sigma^{JHH}}{\mathrm{d}x\,\mathrm{d}y} = \frac{e^4ME}{2\pi Q^4}\left[1-(1-y)^2\right]\left[\tfrac{1}{2}xA^J_{-H,-H}(x) - \tfrac{1}{2}xA^J_{-H,-H}(x)\right]$$

$$= \frac{e^4ME}{2\pi Q^4} y(2-y)xg_1^{JH}(x), \tag{2.14}$$

and

$$\mathrm{d}\bar{\sigma}^{JH'H}/\mathrm{d}x\,\mathrm{d}y = \mathrm{d}\Delta\sigma^{JH'H}/\mathrm{d}x\,\mathrm{d}y \to 0, \qquad H \neq H'.$$

We leave further interpretation to the next section.

3. Scaling limit and structure functions

The scaling limit of the helicity amplitudes, $A^J_{hH,h'H'}$, can be obtained most easily from the parton model. The same results will be obtained from the operator product expansion in the next section. In the Bjorken limit, Compton scattering from quarks is incoherent and elastic and the quark helicity is conserved. With $q_s^{JH}(x)$ defined as in sect. 1, an elementary calculation yields (suppressing flavor labels which may be reintroduced at leisure),

$$\lim_{B_j} W^{JH'H}_{\mu\nu} = \left\{\left(-\tfrac{1}{2}g_{\mu\nu} + \frac{x}{\nu}p_\mu p_\nu\right)\left(q^{JH}_\uparrow + q^{JH}_\downarrow\right) + i\epsilon_{\mu\nu\alpha\beta}\frac{q^\alpha p^\beta}{2\nu}\left(q^{JH}_\uparrow - q^{JH}_\downarrow\right)\right\}\delta_{HH'}. \tag{3.1}$$

If this is compared with eqs. (2.13) and (2.14) we find

$$\lim_{B_j} A^J_{+H,+H} = q^{JH}_\uparrow(x), \tag{3.2a}$$

$$\lim_{B_j} A^J_{-H,-H} = q^{JH}_\downarrow(x), \tag{3.2b}$$

$$\lim_{B_j} A^J_{0H,0H} = 0, \tag{3.2c}$$

$$\lim_{B_j} A^J_{hH,h'H'} = 0, \qquad \text{for } h \neq h', \quad H \neq H'. \tag{3.2d}$$

Eqs. (3.2a, b) reflect the fact that a helicity + photon can only be absorbed by a helicity + quark and vice-versa. Eq. (3.2c) is a generalization of the Callan–Gross relation to arbitrary H. Eq. (3.2d) indicates that helicity flip amplitudes vanish in the parton model, and are higher twist in QCD.

The spin structure of eq. (3.1) may be rendered more transparent if we relate the kinematic factors to photon polarization vectors,

$$\lim_{B_j} W_{\mu\nu}^{JH'H} = \left\{ \epsilon_{\mu+}\epsilon_{\nu+}^* q_\uparrow^{JH}(x) + \epsilon_{\mu-}\epsilon_{\nu-}^* q_\downarrow^{JH}(d) \right\} \delta_{HH'} . \tag{3.3}$$

This form suggests the correct generalization to allow for failure of the Callan–Gross relations and for logarithmic scaling violation generated by QCD radiative corrections,

$$W_{\mu\nu}^{JH'H} \underset{Q^2,\nu\to\infty}{\sim} \left\{ \epsilon_{\mu+}\epsilon_{\nu+}^* q_\uparrow^{JH}(x,Q^2) + \epsilon_{\mu-}\epsilon_{\nu-}^* q_\downarrow^{JH}(x,Q^2) + \epsilon_{\mu0}\epsilon_{\nu0}\ell^{JH}(x,Q^2) \right\} \delta_{HH'} , \tag{3.4}$$

where $\ell^{JH}(x,Q^2) \equiv A_{0H,0H}^J$. q_s^{JH} and ℓ^{JH} are determined by unusual (helicity dependent) matrix elements of the usual twist-two operators in QCD. Therefore the q_s^{JH}, ℓ^{JH} and F_1^{JH}, F_2^{JH} and g_1^{JH} defined in eqs. (1.1–1.3) evolve with $\log Q^2$ exactly as do the familiar F_1, F_2 and g_1.

Consider a target characterized by a spin density matrix, ρ^J, ($\rho^{J\dagger} = \rho^J$, $\mathrm{Tr}\,\rho^J = 1$) in the basis in which J_3 is diagonal. ($\rho_{H_1H_2}^J = \delta_{H_1H}\delta_{H_2H}$ describes a target with helicity H.) It is easy to see that in the Bjorken limit,

$$\frac{\mathrm{d}\bar{\sigma}^J(\rho)}{\mathrm{d}x\,\mathrm{d}y} = \frac{e^4ME}{2\pi Q^4}\left\{ y^2x\,\mathrm{Tr}\left[\rho^J F_1^J\right] + (1-y)\mathrm{Tr}\left[\rho^J F_2^J\right] \right\} ,$$

and

$$\frac{\mathrm{d}\Delta\sigma^J(\rho)}{\mathrm{d}x\,\mathrm{d}y} = \frac{e^4ME}{2\pi Q^4} y(2-y)x\,\mathrm{Tr}\left[\rho^J g_1^J\right] , \tag{3.5}$$

where F_1^J, F_2^J are $(2J+1)\times(2J+1)$ diagonal matrices with elements $(F_1^J)_{HH} = F_1^{JH}$, etc. For the remainder of this section we shall assume the Callan–Gross-like relations and set $2xF_1^J = F_2^J$.

It is convenient to introduce multipole structure functions which will simplify the study of complex (nuclear) targets composed of spin-$\frac{1}{2}$ protons and neutrons. The simplest approach is to insert into the traces in eq. (3.5), a sum over matrices which project out the desired multipoles. Let M_L^J be a matrix defined by

$$\begin{aligned}\left(M_L^J\right)_{HH'} &= (-1)^{J-H}(JHJ-H|L0)\delta_{HH'} ,\\ &= (-1)^{J-H}\sqrt{2L+1}\begin{pmatrix} J & J & L \\ H & -H & 0 \end{pmatrix}\delta_{HH'} ,\end{aligned} \tag{3.6}$$

in the basis in which J_3 is diagonal. The matrix M_L^J obeys

$$\mathrm{Tr}\, M_L^J M_{L'}^J = \delta_{LL'}, \tag{3.7a}$$

$$\sum_L \left(M_L^J\right)^2 = 1. \tag{3.7b}$$

If we insert eq. (3.7b) into eq. (3.5) (and use $2xF_1^J = F_2^J$) we obtain

$$\frac{\mathrm{d}\bar{\sigma}^J(\rho)}{\mathrm{d}x\,\mathrm{d}y} = \frac{e^4 ME}{2\pi Q^4}\left[1+(1-y)^2\right] x \sum_{L\ \mathrm{even}} {}_L^J F_1(x) \rho_L^J,$$

$$\frac{\mathrm{d}\Delta\sigma^J(\rho)}{\mathrm{d}x\,\mathrm{d}y} = \frac{e^4 ME}{2\pi Q^4}\left[1-(1-y)^2\right] x \sum_{L\ \mathrm{odd}} {}_L^J g_1(x) \rho_L^J, \tag{3.8}$$

where

$$\rho_L^J = \mathrm{Tr}\, M_L^J \rho^J,$$

$${}_L^J F_1(x) = \mathrm{Tr}\, M_L^J F_1^J(x),$$

$${}_L^J g_1(x) = \mathrm{Tr}\, M_L^J g_1^J(x). \tag{3.9}$$

Using $F_1^{JH} = F_1^{J-H}$ and $g_1^{JH} = -g_1^{J-H}$ it is easy to show that ${}_L^J F_1(x) = 0$ for L odd and ${}_L^J g_1(x) = 0$ for L even, and

$$\sum_{H=-J}^{J} (-1)^{J-H} (JHJ-H|L0)\, q_\uparrow^{JH}(x) = \begin{cases} {}_L^J F_1(x) & \text{for } L \text{ even}, \\ {}_L^J g_1(x) & \text{for } L \text{ odd}. \end{cases} \tag{3.10}$$

The virtues of this decomposition will become apparent in sects. 4 and 5. At present, it is enough to note that ${}_L^J F_1$ are proportional to the familiar structure functions F_1 and b_1 (for $L=0$ and 2, respectively) on targets with $J \leqslant 1$, and ${}_L^J g_1$ is proportional to g_1 (for $L=1$) on targets with $J \leqslant 1$. Table 1 gives $\{{}_L^J F_1, {}_L^J g_1\}$ for all $L \leqslant 6$ and $J \leqslant 3$ in terms of $q_\uparrow^{JH}$ (remember $q_\downarrow^{JH} = q_\uparrow^{J-H}$). ${}_L^J F_1$ contributes only to lepton-spin averaged scattering; ${}_L^J g_1$ contributes only to the spin asymmetry. Consistent with this, as the table indicates, ${}_L^J F_1$ is independent of quark spin (it weighs $q_\uparrow^{JH}$ and $q_\downarrow^{JH}$ equally) and ${}_L^J g_1$ vanishes if $q_\uparrow^{JH} = q_\downarrow^{JH}$.

As one example we work out the case $J=2$ ($J \leqslant 1$ are familiar, most results for $J=\frac{3}{2}$ were quoted in the Introduction). There are five non-trivial structure func-

TABLE 1
Coefficients of $q_{\uparrow}^{JH}(x)$ in the expansion of ${}^{J}_{H}F_1(x)$ for $L \leqslant 6$ and $J \leqslant 3$. The columns for L even give ${}^{J}_{L}F_1$ and for L odd give ${}^{J}_{L}g_1$

$J \backslash L$		0	1	2	3	4	5	6
$J=\frac{1}{2}$	$H=\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$					
	$H=-\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$					
$J=1$	$H=1$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{6}}$				
	$H=0$	$\frac{1}{\sqrt{3}}$	0	$-\frac{2}{\sqrt{6}}$				
	$H=-1$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{6}}$				
$J=\frac{3}{2}$	$H=\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{\sqrt{20}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{20}}$			
	$H=\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{20}}$	$-\frac{1}{2}$	$-\frac{3}{\sqrt{20}}$			
	$H=-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{\sqrt{20}}$	$-\frac{1}{2}$	$\frac{3}{\sqrt{20}}$			
	$H=-\frac{3}{2}$	$\frac{1}{2}$	$-\frac{3}{\sqrt{20}}$	$\frac{1}{2}$	$-\frac{1}{\sqrt{20}}$			
$J=2$	$H=2$	$\frac{1}{\sqrt{5}}$	$\frac{2}{\sqrt{10}}$	$\frac{2}{\sqrt{14}}$	$\frac{1}{\sqrt{10}}$	$\frac{1}{\sqrt{70}}$		
	$H=1$	$\frac{1}{\sqrt{5}}$	$\frac{1}{\sqrt{10}}$	$-\frac{1}{\sqrt{14}}$	$-\frac{2}{\sqrt{10}}$	$-\frac{4}{\sqrt{70}}$		
	$H=0$	$\frac{1}{\sqrt{5}}$	0	$-\frac{2}{\sqrt{14}}$	0	$\frac{6}{\sqrt{70}}$		
	$H=-1$	$\frac{1}{\sqrt{5}}$	$-\frac{1}{\sqrt{10}}$	$-\frac{1}{\sqrt{14}}$	$\frac{2}{\sqrt{10}}$	$-\frac{4}{\sqrt{70}}$		
	$H=-2$	$\frac{1}{\sqrt{5}}$	$-\frac{2}{\sqrt{10}}$	$\frac{2}{\sqrt{14}}$	$-\frac{1}{\sqrt{10}}$	$\frac{1}{\sqrt{70}}$		
$J=\frac{5}{2}$	$H=\frac{5}{2}$	$\frac{1}{\sqrt{6}}$	$\frac{5}{\sqrt{70}}$	$\frac{5}{\sqrt{84}}$	$\frac{5}{\sqrt{180}}$	$\frac{1}{\sqrt{28}}$	$\frac{1}{\sqrt{252}}$	
	$H=\frac{3}{2}$	$\frac{1}{\sqrt{6}}$	$\frac{3}{\sqrt{70}}$	$-\frac{1}{\sqrt{84}}$	$-\frac{7}{\sqrt{180}}$	$-\frac{3}{\sqrt{28}}$	$-\frac{5}{\sqrt{252}}$	
	$H=\frac{1}{2}$	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{70}}$	$-\frac{4}{\sqrt{84}}$	$-\frac{4}{\sqrt{180}}$	$\frac{2}{\sqrt{28}}$	$\frac{10}{\sqrt{252}}$	
	$H=-\frac{1}{2}$	$\frac{1}{\sqrt{6}}$	$-\frac{1}{\sqrt{70}}$	$-\frac{4}{\sqrt{84}}$	$\frac{4}{\sqrt{180}}$	$\frac{2}{\sqrt{28}}$	$-\frac{10}{\sqrt{252}}$	
	$H=-\frac{3}{2}$	$\frac{1}{\sqrt{6}}$	$-\frac{3}{\sqrt{70}}$	$-\frac{1}{\sqrt{84}}$	$\frac{7}{\sqrt{180}}$	$-\frac{3}{\sqrt{28}}$	$\frac{5}{\sqrt{252}}$	
	$H=-\frac{5}{2}$	$\frac{1}{\sqrt{6}}$	$-\frac{5}{\sqrt{70}}$	$\frac{5}{\sqrt{84}}$	$-\frac{5}{\sqrt{180}}$	$\frac{1}{\sqrt{28}}$	$-\frac{1}{\sqrt{252}}$	

TABLE 1 (continued)

$J\backslash L$		0	1	2	3	4	5	6
$J=3$	$H=3$	$\frac{1}{\sqrt{7}}$	$\frac{3}{\sqrt{28}}$	$\frac{5}{\sqrt{84}}$	$\frac{1}{\sqrt{6}}$	$\frac{3}{\sqrt{154}}$	$\frac{1}{\sqrt{84}}$	$\frac{1}{\sqrt{924}}$
	$H=2$	$\frac{1}{\sqrt{7}}$	$\frac{2}{\sqrt{28}}$	0	$-\frac{1}{\sqrt{6}}$	$-\frac{7}{\sqrt{154}}$	$-\frac{4}{\sqrt{84}}$	$-\frac{6}{\sqrt{924}}$
	$H=1$	$\frac{1}{\sqrt{7}}$	$\frac{1}{\sqrt{28}}$	$-\frac{3}{\sqrt{84}}$	$-\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{154}}$	$\frac{5}{\sqrt{84}}$	$\frac{15}{\sqrt{924}}$
	$H=0$	$\frac{1}{\sqrt{7}}$	0	$-\frac{4}{\sqrt{84}}$	0	$\frac{6}{\sqrt{154}}$	0	$-\frac{20}{\sqrt{924}}$
	$H=-1$	$\frac{1}{\sqrt{7}}$	$-\frac{1}{\sqrt{28}}$	$-\frac{3}{\sqrt{84}}$	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{154}}$	$-\frac{5}{\sqrt{84}}$	$\frac{15}{\sqrt{924}}$
	$H=-2$	$\frac{1}{\sqrt{7}}$	$-\frac{2}{\sqrt{28}}$	0	$\frac{1}{\sqrt{6}}$	$-\frac{7}{\sqrt{154}}$	$\frac{4}{\sqrt{84}}$	$-\frac{6}{\sqrt{924}}$
	$H=-2$	$\frac{1}{\sqrt{7}}$	$-\frac{3}{\sqrt{28}}$	$\frac{5}{\sqrt{84}}$	$-\frac{1}{\sqrt{6}}$	$\frac{3}{\sqrt{154}}$	$-\frac{1}{\sqrt{84}}$	$\frac{1}{\sqrt{924}}$

tions in the scaling limit (assuming Callan–Gross relations):

$$ {}^{2}_{0}F_{1} = \left(1/\sqrt{5}\right)\left(q^{22}_{\uparrow} + q^{22}_{\downarrow} + q^{21}_{\uparrow} + q^{21}_{\downarrow} + q^{20}_{\uparrow}\right), $$

$$ {}^{2}_{1}g_{1} = \left(1/\sqrt{10}\right)\left(2q^{22}_{\uparrow} - 2q^{22}_{\downarrow} + q^{21}_{\uparrow} - q^{21}_{\downarrow}\right) $$

$$ {}^{2}_{2}F_{1} = \left(1/\sqrt{14}\right)\left(2q^{22}_{\uparrow} + 2q^{22}_{\downarrow} - q^{21}_{\uparrow} - q^{21}_{\downarrow} - 2q^{20}_{\uparrow}\right), $$

$$ {}^{2}_{3}g_{1} = \left(1/\sqrt{10}\right)\left(q^{22}_{\uparrow} - q^{22}_{\downarrow} - 2q^{21}_{\uparrow} + 2q^{21}_{\downarrow}\right), $$

$$ {}^{2}_{4}F_{1} = \left(1/\sqrt{70}\right)\left(q^{22}_{\uparrow} + q^{22}_{\downarrow} - 4q^{21}_{\uparrow} - 4q^{21}_{\downarrow} + 6q^{20}_{\uparrow}\right). $$

The ${}^{2}_{0}F_{1}$, ${}^{2}_{2}F_{1}$ and ${}^{2}_{1}g_{1}$ are analogous to F_1, b_1 and g_1, respectively, for $J=1$. ${}^{J}_{3}g_{1}$ first appears at $J=\frac{3}{2}$, see eq. (1.9). ${}^{J}_{4}F_{1}$ first appears for $J=2$. ${}^{2}_{0}F_{1}$ is measured with unpolarized beam and target. ${}^{2}_{2}F_{1}$ and ${}^{2}_{4}F_{1}$ can be measured with unpolarized beam by comparing the cross-section for targets polarized in different directions with the cross-section for an unpolarized target. ${}^{2}_{1}g_{1}$ and ${}^{2}_{3}g_{1}$ can be separated by measuring the lepton spin-asymmetry, $\Delta\sigma$, for two different target orientations. One must keep in mind, however, that there are higher twist structure functions (similar to $g_2(x)$ in ep scattering) which give small ($O(\mathcal{M}^2/Q^2)$) beam and target polarization dependent effects not included in eq. (3.8).

4. QCD analysis of structure functions

To write the general form of $W_{\mu\nu}^{JH'H}$, it is necessary to introduce objects which generalize the concept of the Pauli–Lubanski spin vector s^μ for a spin-$\frac{1}{2}$ particle. Let us define irreducible tensor operators, $\mathscr{S}_{LM}$, which transform as $|LM\rangle$ under the rotation group. They are defined in the rest frame of the target by the Wigner–Eckart theorem,

$$\langle JH',\, \boldsymbol{p}=0|\mathscr{S}_{LM}\,|JH,\, \boldsymbol{p}=0\rangle = \mathscr{M}^L(-1)^{J-H}\big(JHJ-H'|L-M\big), \quad (4.1)$$

where $\mathscr{M}$ is the target mass.

This specifies all matrix elements of the operators $\{\mathscr{S}_{LM}\}$ between target states at rest. These matrix elements are non-zero for $0 \leqslant L \leqslant 2J$. Since the completely symmetric, traceless tensor product of L spin-one states transforms like spin L, one can equivalently represent $\mathscr{S}_{LM}$ in a cartesian basis by $\mathscr{S}^{i_1 \cdots i_L}$ where $i_k = 1,2,3$, and $\mathscr{S}^{i_1\cdots i_L}$ is completely symmetric and traceless in all index pairs. For example, for $L=1$,

$$\mathscr{S}_{L=1,\, M=1} = -\frac{1}{\sqrt{2}}\mathscr{S}^1 - \frac{i}{\sqrt{2}}\mathscr{S}^2\,,$$

$$\mathscr{S}_{L=1,\, M=0} = \mathscr{S}^3\,,$$

$$\mathscr{S}_{L=1,\, M=-1} = \frac{1}{\sqrt{2}}\mathscr{S}^1 - \frac{i}{\sqrt{2}}\mathscr{S}^2\,.$$

We define Lorentz generalizations of the operators $\mathscr{S}^{i_1\cdots i_L}$ by

$$\big\langle JH,\, \boldsymbol{p}=0\big|\mathscr{S}^{\mu_1\cdots\mu_L}\,\big|JH,\, \boldsymbol{p}=0\big\rangle = \begin{cases} \langle JH,\, \boldsymbol{p}=0|\mathscr{S}^{i_1\cdots i_L}\,|JH,\, \boldsymbol{p}=0\rangle & \text{for } \mu_r = i_r = 1,2,3, \\ 0 & \text{if any } \mu_r = 0. \end{cases} \quad (4.2)$$

We obtain the matrix elements of $\mathscr{S}^{\mu_1\cdots\mu_L}$ in an arbitrary frame by boosting (boost matrix $\Lambda^\mu_\nu(p)$) from the rest frame.

$$\langle p,\, JH'|\mathscr{S}^{\mu_1\cdots\mu_L}\,|p,\, JH\rangle \equiv \theta_{H'H}^{J\mu_1\cdots\mu_L}(p)$$
$$= \Lambda^{\mu_1}_{\nu_1}(p)\Lambda^{\mu_2}_{\nu_2}(p)\ldots\Lambda^{\mu_L}_{\nu_L}(p)\langle JH',\, \boldsymbol{p}=0|\mathscr{S}^{\nu_1\cdots\nu_L}\,|JH,\, \boldsymbol{p}=0\rangle,$$

A few examples are in order: θ^μ corresponds to s^μ for a spin-$\frac{1}{2}$ target and to

$S^\mu = (-i/\mathcal{M}^2)\epsilon^{\mu\alpha\beta\sigma}E^*_\alpha E_\beta p_\sigma$ for a spin-1 target; $\theta^{\mu\nu}$ vanishes for spin-$\frac{1}{2}$; for spin-1

$$\theta^{\mu\nu} \propto \left(E^{*\mu}E^\nu + E^{*\nu}E^\mu + \tfrac{2}{3}\left(g^{\mu\nu}\mathcal{M}^2 - p^\mu p^\nu\right)\right).$$

Note, $\theta^{\mu_1 \cdots \mu_L}$ is a (pseudo) tensor for (odd L) even L. For brevity we shall suppress the labels J, H, H', and p on θ and $W_{\mu\nu}$ in the following. Then the most general expression for $W_{\mu\nu}^{JH'H}$ is

$$\begin{aligned} W_{\mu\nu} = {} & \sum_{L=1,3,\dots}^{2J} i\frac{{}^J_Lg_1}{\nu^L}\epsilon_{\mu\nu\alpha}{}^{\mu_1}\theta_{\mu_1\dots\mu_L}q^\alpha q^{\mu_2}\dots q^{\mu_L} + \sum_{L=1,3,\dots}^{2J} i\frac{{}^J_Lg_2}{\nu^{L+1}}\epsilon_{\mu\nu\alpha\beta}p_{[\mu_1}\theta_{\beta]\mu_2\dots\mu_L}q^\alpha q^{\mu_1}\dots q^{\mu_L} \\ & + \sum_{L=0,2,\dots}^{2J} -g_{\mu\nu}\frac{{}^J_LF_1}{\nu^L}\theta_{\mu_1\dots\mu_L}q^{\mu_1}\dots q^{\mu_L} + \sum_{L=0,2,\dots}^{2J} \frac{{}^J_LF_2}{\nu^{L+1}}p_{(\mu}p_{\nu}\theta_{\mu_1\dots\mu_L)}q^{\mu_1}\dots q^{\mu_L} \\ & + \sum_{L=2,4,\dots}^{2J} \frac{{}^J_Lb_3}{\nu^{L+1}}\left[\theta_{\mu_1\dots\mu_L}p_\mu p_\nu - \theta_{\mu\nu\mu_3\dots\mu_L}p_{\mu_1}p_{\mu_2}\right]q^{\mu_1}\dots q^{\mu_L} \\ & + \sum_{L=2,4,\dots}^{2J} \frac{{}^J_Lb_4}{\nu^{L+1}}\left[2\theta_{\mu_1\dots\mu_L}p_\mu p_\nu - \theta_{\mu\mu_2\dots\mu_L}p_{\mu_1}p_\nu - \theta_{\nu\mu_2\dots\mu_L}p_{\mu_1}p_\mu\right]q^{\mu_1}\dots q^{\mu_L}, \end{aligned} \tag{4.3}$$

where we have omitted terms proportional to q^μ and/or q^ν. The coefficients b, g, F have been normalized so that they tend to finite functions of x in the Bjorken limit. In going from a spin $J - \frac{1}{2}$ target to a spin J target, there will be either two new structure functions ${}^J_{2J}g_1$ and ${}^J_{2J}g_2$ if $2J$ is odd, or four new structure functions ${}^J_{2J}F_1, {}^J_{2J}F_2, {}^J_{2J}b_3, {}^J_{2J}b_4$ if $2J$ is even. ${}^J_Lg_1$, ${}^J_Lg_2$, ${}^J_LF_1$, ${}^J_LF_2$, ${}^J_Lb_3$ and ${}^J_Lb_4$ are the generalizations of g_1, g_2, F_1, F_2, b_3 and b_4 to an arbitrary spin target*. Note that ${}^J_LF_i$ and ${}^J_Lb_i$ are only defined for even L, $L \leqslant 2J$, and ${}^J_Lg_i$ are only defined for odd L, $L \leqslant 2J$. The ${}^J_Lg_i$ contribute to the antisymmetric part of $W_{\mu\nu}$, and can be measured using the lepton asymmetry. ${}^J_LF_i$ and ${}^J_Lb_i$ contribute to the symmetric part of $W_{\mu\nu}$ and can be measured using an unpolarized beam. ${}^J_Lg_1$, ${}^J_LF_1$ and ${}^J_LF_2$ get contributions at leading twist (twist two) in QCD. ${}^J_Lg_2$ first get contributions at twist three, and ${}^J_Lb_i$ at twist four. To lowest order in QCD, there is also the generalization of the Callan–Gross relation,

$${}^J_LF_2^J = 2x\,{}^J_LF_1^J. \tag{4.4}$$

* b_1 and b_2 for a spin-one target are now ${}^1_2F_1$ and ${}^1_2F_2$, respectively.

The structure functions are calculated in QCD by performing an operator product expansion on the product of two currents,

$$
\begin{aligned}
& i\int \mathrm{d}^4x\, \mathrm{e}^{iq\cdot x} T\big(J_\mu(x) J_\nu(x)\big) \\
&= \sum_{n=2,4,\dots}^{\infty} 2C_n^{(1)}\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)\frac{2^n q_{\mu_1}\cdots q_{\mu_n}}{\left(-q^2\right)^n} O_{\mathrm{V}}^{\mu_1\cdots\mu_n} \\
&+ \sum_{n=2,4,\dots}^{\infty} 2C_n^{(2)}\left(g_{\mu\mu_1}-\frac{q_\mu q_{\mu_1}}{q^2}\right)\left(g_{\nu\mu_2}-\frac{q_\nu q_{\mu_2}}{q^2}\right)\frac{2^n q_{\mu_3}\cdots q_{\mu_n}}{\left(-q^2\right)^{n-1}} O_{\mathrm{V}}^{\mu_1\cdots\mu_n} \\
&+ \sum_{n=1,3,\dots}^{\infty} 2C_n^{(3)} i\epsilon_{\mu\nu\lambda\mu_1} q^\lambda \frac{2^n q_{\mu_2}\cdots q_{\mu_n}}{\left(-q^2\right)^n} O_{\mathrm{A}}^{\mu_1\cdots\mu_n},
\end{aligned}
\tag{4.5}
$$

where*

$$
O_{\mathrm{V}}^{\mu_1\cdots\mu_n}=\frac{1}{2}\left(\frac{i}{2}\right)^{n-1} S\left\{\bar{\psi}\gamma^{\mu_1}\overleftrightarrow{\mathrm{D}}^{\mu_2}\dots\overleftrightarrow{\mathrm{D}}^{\mu_n}\psi\right\}, \tag{4.6}
$$

$$
O_{\mathrm{A}}^{\mu_1\cdots\mu_n}=\frac{1}{2}\left(\frac{i}{2}\right)^{n-1} S\left\{\bar{\psi}\gamma^{\mu_1}\overleftrightarrow{\mathrm{D}}^{\mu_2}\dots\overleftrightarrow{\mathrm{D}}^{\mu_n}\gamma_5\psi\right\}, \tag{4.7}
$$

(S denotes symmetrizing with respect to $\mu_1,\dots,\mu_n$ and subtracting traces). The matrix elements of $O_{\mathrm{V}}^{\mu_1\cdots\mu_n}$ and $O_{\mathrm{A}}^{\mu_1\cdots\mu_n}$ between target states can be written as

$$
\left\langle JH'\right| O_{\mathrm{V}}^{\mu_1\cdots\mu_n}\left|JH\right\rangle = S\left\{\sum_{L=0,2,\dots}^{n} {}^J_L a_n \theta^{\mu_1\cdots\mu_L} p^{\mu_{L+1}}\dots p^{\mu_n}\right\}, \tag{4.8}
$$

$$
\left\langle JH'\right| O_{\mathrm{A}}^{\mu_1\cdots\mu_n}\left|JH\right\rangle = S\left\{\sum_{L=1,3,\dots}^{n} {}^J_L a_n \theta^{\mu_1\cdots\mu_L} p^{\mu_{L+1}}\dots p^{\mu_n}\right\}. \tag{4.9}
$$

Eqs. (4.8) and (4.9) are the most general tensor structures allowed by charge conjugation and parity.

If we substitute eqs. (4.8) and (4.9) into the matrix element of the current product as defined by eq. (4.5), we obtain a Taylor expansion of the Compton amplitude in the variable $\omega \equiv 1/x = 2\nu/Q^2$, which converges for $|\omega|<1$. Using standard methods of dispersion theory we obtain another Taylor expansion from eq. (4.3) in terms of the moments of the structure functions ${}^J_L F_1$, ${}^J_L F_2$, ${}^J_L g_1$, ${}^J_L g_2$, ${}^J_L b_3$ and ${}^J_L b_4$.

* Once again we suppress flavor indices. We also ignore gluonic operators. Restoring these complications does not change our results significantly.

Comparing the two we find ${}^J_L g_2 = {}^J_L b_3 = {}^J_L b_4 = 0$ to this order (they are higher twist), and a series of sum rules for ${}^J_L g_1$, ${}^J_L F_1$ and ${}^J_L F_2$. If we define

$$M_n(f) \equiv \int_0^1 x^{n-1} f(x)\,\mathrm{d}x\,, \tag{4.10}$$

then

$$\left.\begin{aligned} M_n\left({}^J_L F_1\right) &= 0 \\ M_{n-1}\left({}^J_L F_2\right) &= 0 \end{aligned}\right\} L, n \text{ even}, \qquad 0 < n < L\,,$$

$$M_n\left({}^J_L g_1\right) = 0, \quad L, n \text{ odd}, \qquad 1 \leqslant n < L\,, \tag{4.11}$$

and

$$\left.\begin{aligned} 2M_n\left({}^J_L F_1\right) &= C_n^{(1)}\,{}^J_L a_n \\ M_{n-1}\left({}^J_L F_2\right) &= C_n^{(2)}\,{}^J_L a_n \end{aligned}\right\} L, n \text{ even}, \qquad n \geqslant L,\ n \neq 0\,,$$

$$2M_n\left({}^J_L g_1\right) = C_n^{(3)}\,{}^J_L a_n, \quad L, n \text{ odd}, \qquad n \geqslant L\,, \tag{4.12}$$

where $C_n^{(1)}$, $C_n^{(2)}$ and $C_n^{(3)}$ are coefficients in the operator product expansion (4.5). To lowest order in QCD $C_n^{(i)} = 1$. The moments of ${}^J_L F_1$ and ${}^J_L F_2$ are given by the matrix elements of the operators $O_V^{\mu_1 \cdots \mu_n}$. Therefore the anomalous dimensions of $M_n({}^J_L F_1)$ and $M_{n-1}({}^J_L F_2)$ are the same as the familiar anomalous dimensions of $M_n(F_1)$ for ep scattering. Similarly, the anomalous dimensions of $M_n({}^J_L g_1)$ are equal to those of $M_n(g_1)$ in ep scattering. This establishes the claims about the Q^2-dependence of ${}^J_L F_1$ and ${}^J_L g_1$ made earlier.

To make contact with the results of the previous sections, let us evaluate the leading twist piece of eq. (4.3) for a target with helicity H in the Bjorken limit. Recall that in the Bjorken limit, the photon direction and beam direction coincide, so that target helicity can be measured with respect to the photon direction $\boldsymbol{q}$. Then (in the target rest frame)

$$\begin{aligned} q_{\mu_1} \cdots q_{\mu_L} \left\langle JH \middle| \mathcal{S}^{\mu_1 \cdots \mu_L} \middle| JH \right\rangle &= (q_3)^L \left\langle JH \middle| \mathcal{S}^{33 \ldots 3} \middle| JH \right\rangle \\ &= (q_3)^L \mathcal{M}^L (JHJ - H | L0)(-1)^{J-H}\,, \end{aligned}$$

using eq. (4.1). $\mathscr{M} q_3 = -\mathscr{M} q^3 \simeq -\mathscr{M} q^0 = -\nu$ in the Bjorken limit. Therefore,

$$W_{\mu\nu}^{HH} = \sum_{L=1,3}^{2J} i\frac{{}^J_L g_1}{\nu}\epsilon_{\mu\nu\alpha\beta}q^\alpha\eta^\beta(-)^{J-H}(JHJ-H|L0)$$

$$+\sum_{L=0,2}^{2J}\left(-g_{\mu\nu}{}^J_L F_1^J + \frac{{}^J_L F_2}{\nu}p_\mu p_\nu\right)(-)^{J-H}(JHJ-H|L0),$$

where $\eta^\beta = (0,0,0,\mathscr{M})$. Comparing this equation with eq. (3.1) we see that

$${}^J_L F_2 = 2x\,{}^J_L F_1,$$

$$\sum_{L=0,2,\ldots}^{2J}(-)^{J-H}\,{}^J_L F_1(JHJ-H|L0) = \tfrac{1}{2}\left(q_\uparrow^{JH}+q_\downarrow^{JH}\right),$$

$$\sum_{L=1,3,\ldots}^{2J}(-)^{J-H}\,{}^J_L g_1(JHJ-H|L0) = \tfrac{1}{2}\left(q_\uparrow^{JH}-q_\downarrow^{JH}\right),$$

which can be written using eq. (3.6) as

$$\tfrac{1}{2}\left(q_\uparrow^{JH}+q_\downarrow^{JH}\right) = \sum_{L\text{ even}}\left(M_L^J\right)_{HH}{}^J_L F_1,$$

$$\tfrac{1}{2}\left(q_\uparrow^{JH}-q_\downarrow^{JH}\right) = \sum_{L\text{ odd}}\left(M_L^J\right)_{HH}{}^J_L g_1.$$

Inverting these using eq. (3.7) we obtain the same relations (eqs. (3.9) and (3.10)) we obtained from the parton model in the previous section.

5. Nucleons moving non-relativistically in a potential

The only available targets with $J \geqslant 1$ are nuclei. To a good approximation nuclei may be described as quasi-non-relativistic bound states of neutrons and protons. It is interesting to know the contribution to the structure functions ${}^J_L F_1$ and ${}^J_L g_1$ from a nucleon bound in a potential. This calculation is analogous to the "Fermi smearing" which must be computed to relate the spin-independent structure function (F_1) of a bound nucleon to that of a free nucleon. We use a refinement of the convolution formalism developed for that problem [3,4].

The convolution model is based on several ad hoc assumptions – for example, that deep inelastic scattering is incoherent off nucleons in the nucleus – which we are not trying to justify here. Instead, we view this calculation as a zeroth order estimate of ${}^J_L F_1$ and ${}^J_L g_1$ with which to compare data. Departures from these

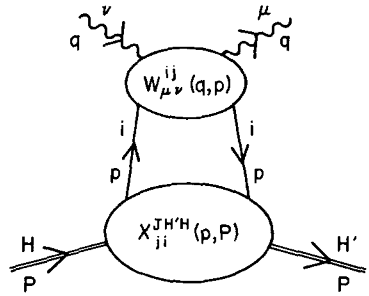

Fig. 1. Forward virtual Compton scattering in the convolution approximation.

estimates may point to straightforward modifications of the convolution model (e.g., different binding energy prescriptions) or may signal more significant new physics.

The convolution formalism is summarized by fig. 1 where it is *assumed* that the structure function of the nucleus, $W^{JH'H}_{\mu\nu}(q, P)$ factorizes (in the t-channel) into a product of the structure function of the nucleon, $w^{ij}_{\mu\nu}(q, p)$, times the (connected) amplitude $\chi^{JH'H}_{ji}(p, P)$, for the nucleon to forward scatter from the target nucleus,

$$\chi^{JH'H}_{ji}(p, P) = \int \mathrm{d}^4\xi\, \mathrm{e}^{-ip\cdot\xi} \left\langle JH', P \middle| \bar{\psi}_i(\xi)\psi_j(0) \middle| JH, P \right\rangle, \tag{5.1}$$

$$w^{ij}_{\mu\nu}(q, p) = -\frac{1}{\nu} g_{\mu\nu} F_1 \not{q}^{ij} + \frac{i}{\nu}\epsilon_{\mu\nu\alpha\beta} q^{\alpha} g_1 \left(\gamma^{\beta}\gamma^{5}\right)^{ij}. \tag{5.2}$$

i and j are Dirac indices. In writing $w^{ij}_{\mu\nu}$ we have displayed only those terms necessary to extract ${}^{J}_{L}F_1$ and ${}^{J}_{L}g_1$ in the scaling limit. Gauge terms and terms subdominant in the Bjorken limit have been omitted. Terms proportional to $p_\mu p_\nu$ which yield F_2 have been dropped because Callan–Gross like relations fix ${}^{J}_{L}F_2 = 2x{}^{J}_{L}F_1$. F_1 and g_1 are (Lorentz invariant) functions of $q\cdot p$, q^2 and p^2 only. From fig. 1 we read off the convolution model for $W^{JH'H}_{\mu\nu}$ (in the Bjorken limit, obtained by letting $q^- \to \infty$ in the target rest frame)

$$\lim_{q^-\to\infty} W^{JH'H}_{\mu\nu}(q, P) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{1}{p^+}$$
$$\times\left\{-F_1 g_{\mu\nu}\,\mathrm{Tr}\left[\gamma^+\chi^{JH'H}\right] + ig_1\epsilon_{\mu\nu-\beta}\,\mathrm{Tr}\left[\gamma^\beta\gamma^5\xi^{JH'H}\right]\right\}. \tag{5.3}$$

We are interested only in the target helicity-dependent structure functions, F_1^{JH} and g_1^{JH}, which survive the Bjorken limit. They may be projected out from $W^{JH'H}_{\mu\nu}$

by contraction with $\epsilon^{\mu}_{\pm}$ etc. The results are

$$F_1^{JH}(x,Q^2)=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{p^+}F_1(q,p)\operatorname{Tr}\gamma^+\chi^{JHH}(p,P),$$

$$g_1^{JH}(x,Q^2)=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{p^+}g_1(q,p)\operatorname{Tr}\gamma^+\gamma^5\chi^{JHH}(p,P). \tag{5.4}$$

These may be converted to a more familiar form by postponing the integrals over $p^2\equiv\mu^2$ and $p^+\equiv yP^+$ until the end,

$$F_1^{JH}(x,Q^2)=\int\mathrm{d}\mu^2\,\mathrm{d}y\,\mathrm{d}z\,\delta(yz-x)F_1(z,\mu^2,Q^2)f^{JH}(y,\mu^2),$$

$$g_1^{JH}(x,Q^2)=\int\mathrm{d}\mu^2\,\mathrm{d}y\,\mathrm{d}z\,\delta(yz-x)g_1(z,\mu^2,Q^2)g^{JH}(y,\mu^2), \tag{5.5}$$

where

$$f^{JH}(y,\mu^2)=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\delta(p^2-\mu^2)\delta(p^+-yP^+)\operatorname{Tr}\left[\gamma^+\chi^{JH}(p,P)\right] \tag{5.6}$$

and likewise for g^{JH} (with $\gamma^+\gamma^5$ replacing γ^+). μ^2 is the invariant square mass of the struck (virtual) nucleon. In reality F_1 and g_1 are likely to depend on μ^2. In practice, that dependence is ignored (see ref. [3] for further discussion), and we obtain

$$F_1^{JH}(x,Q^2)=\int_0^{M_A/M}\mathrm{d}y\int_0^1\mathrm{d}z\,\delta(yz-x)F_1(z,Q^2)f^{JH}(y),$$

$$g_1^{JH}(x,Q^2)=\int_0^{M_A/M}\mathrm{d}y\int_0^1\mathrm{d}z\,\delta(yz-x)g_1(z,Q^2)g^{JH}(y), \tag{5.7}$$

where M_A is the mass of the nucleus, M is the nucleon mass and we have restored the proper limits on the y and z integrations. Combining eqs. (5.1) and (5.6) we obtain simple expressions for f^{JH} and g^{JH},

$$\left\{\begin{matrix} f^{JH}(y)\\ g^{JH}(y)\end{matrix}\right\}=\frac{1}{2\pi}\int\mathrm{d}\xi^-\,\mathrm{e}^{-iyM\xi^-/\sqrt{2}}\langle JH|\bar{\psi}(\xi^-)\left\{\begin{matrix}\gamma^+\\ \gamma^+\gamma^5\end{matrix}\right\}\psi(0)|JH\rangle\Big|_{\xi^+=\xi_\perp=0}, \tag{5.8}$$

in the target rest frame. In eq. (5.7) a sum over different nuclear constituents (protons, neutrons, etc.) has been suppressed.

Next, we replace the matrix element in eq. (5.8) with single particle wavefunctions, leaving the connection between energy, E, and momentum $\boldsymbol{p}$, unspecified:

$$\left\{\begin{matrix} f^{JH}(y) \\ g^{JH}(y) \end{matrix}\right\} = \sqrt{2}\int \mathrm{d}^3p\,\delta\left(y - \frac{E(p)+p^3}{M}\right)\bar{\psi}^{JH}(\boldsymbol{p})\left\{\begin{matrix} \gamma^+ \\ \gamma^+\gamma^5 \end{matrix}\right\}\psi^{JH}(p). \quad (5.9)$$

$\psi^{JH}(\boldsymbol{p})$ is the (Dirac) momentum space wavefunction of a nucleon. A sum over single nucleon levels is understood. It is tempting to make the "non-relativistic" replacement $\sqrt{2}\,\gamma^+ \to 1$ and $\sqrt{2}\,\gamma^+\gamma^5 \to \sigma^3$. However, the error incurred, $O(p^2/M^2)$, is the same order as the effect one is keeping [4,5]. To proceed, we decompose $\psi^{JH}(\boldsymbol{p})$ into upper and lower (Dirac) components, $\psi^{JH}(\boldsymbol{p}) = \begin{pmatrix} \varphi^{JH}(\boldsymbol{p}) \\ \eta^{JH}(\boldsymbol{p}) \end{pmatrix}$. The Dirac equation gives

$$\eta^{JH}(\boldsymbol{p}) = \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{2M}\varphi^{JH}(\boldsymbol{p}), \quad (5.10)$$

up to corrections of order interaction energies divided by the nucleon mass, which we ignore. With the aid of eq. (5.10) we obtain

$$\bar{\psi}^{JH}(\boldsymbol{p})\sqrt{2}\,\gamma^+\psi^{JH}(\boldsymbol{p}) = \varphi^{\dagger JH}(\boldsymbol{p})\varphi^{JH}(\boldsymbol{p})\left(1 + \frac{p^3}{M} + \frac{\boldsymbol{p}^2}{4M^2}\right),$$

$$\bar{\psi}^{JH}(\boldsymbol{p})\sqrt{2}\,\gamma^+\gamma^5\psi^{JH}(\boldsymbol{p}) = \varphi^{\dagger JH}(\boldsymbol{p})\sigma^3\varphi^{JH}(\boldsymbol{p})\left(1 + \frac{p^3}{M} - \frac{\boldsymbol{p}^2 - 2(p^3)^2}{4M^2}\right), \quad (5.11)$$

from which we obtain the nucleon light-cone distribution functions $f^{JH}(y)$ and $g^{JH}(y)$,

$$f^{JH}(y) = \int \mathrm{d}^3p\,|\varphi^{JH}(p)|^2\left(1 + \frac{p^3}{M} + \frac{\boldsymbol{p}^2}{4M^2}\right)\delta\left(y - \frac{E+p^3}{M}\right),$$

$$g^{JH}(y) = \int \mathrm{d}^3p\left[|\varphi_\uparrow^{JH}(p)|^2 - |\varphi_\downarrow^{JH}(p)|^2\right]\left(1 + \frac{p^3}{M} - \frac{\boldsymbol{p}^2 - 2(p^3)^2}{4M^2}\right)\delta\left(y - \frac{E+p^3}{M}\right), \quad (5.12)$$

where $\varphi_s^{JH}(\boldsymbol{p})$ is obtained by decomposing the target nucleus state $|JH\rangle$ according to whether the constituent nucleon's spin is along or opposite to the axis defined by the virtual photon*,

$$\langle \boldsymbol{p}s|JH\rangle = \varphi_s^{JH}(\boldsymbol{p}). \tag{5.13}$$

The multipole structure functions defined in eq. (1.6) are convolutions over multipole projections of $f^{JH}(y)$ and $g^{JH}(y)$,

$$
\begin{aligned}
{}^J_L F_1(x) &= \int \mathrm{d}y\,\mathrm{d}z\,\delta(yz-x)F_1(z)f^J_L(y),\\
{}^J_L g_1(x) &= \int \mathrm{d}y\,\mathrm{d}z\,\delta(yz-x)g_1(z)g^J_L(z),
\end{aligned} \tag{5.14}
$$

with

$$
\begin{aligned}
f^J_L(y) &\equiv \sum_{H=-J}^{J} \left(M^J_L\right)_{HH} f^{JH}(y),\\
g^J_L(y) &\equiv \sum_{H=-J}^{J} \left(M^J_L\right)_{HH} g^{JH}(y).
\end{aligned} \tag{5.15}
$$

The convolution method we have developed can be applied under many circumstances. For definiteness we compute the contribution from a nucleon with orbital angular momentum l_1, spin-$\frac{1}{2}$ and total angular momentum j_1, coupled to a residual "core" of nucleons with angular momentum j_2 to give total angular momentum J. Thus we label our target state $|(l_1\frac{1}{2})j_1j_2, JH\rangle$. This may be a good approximation to a nucleus involving a few valence nucleons outside a core coupled to $J=0$. However, the general method can easily be adapted to the deuteron (where the center of mass must be factored out), a deformed nucleus or a pion distribution within a nucleus.

Returning to eq. (5.15), the sum over H has the effect of projecting out a specific multipole moment of the nucleon momentum distribution. To establish this connection we require a multipole expansion of the operator $|\boldsymbol{p}s\rangle\langle \boldsymbol{p}s'|$ which appears in the definition of $|\varphi_s^{JH}(\boldsymbol{p})|^2 \equiv \langle JH|\boldsymbol{p}s\rangle\langle \boldsymbol{p}s|JH\rangle$.

$$|\boldsymbol{p}s\rangle\langle \boldsymbol{p}s'| = \sum_{lm}\hat{\theta}_{lm}(p)Y^*_{lm}(\Omega_p)\delta_{ss'} + \sum_{jlm}\hat{\theta}_{l1jm}(p)\left(\boldsymbol{Y}_{l1jm}(\Omega_p)\cdot\sigma_{ss'}\right)^*. \tag{5.16}$$

Here $p\equiv|\boldsymbol{p}|$, Y_{lm} and $\boldsymbol{Y}_{l1jm}$ are ordinary and vector-spherical harmonics, respectively. $\hat{\theta}_{lm}$ and $\hat{\theta}_{l1jm}$ are irreducible tensor operators of rank l and j, respectively. It

* The state $|\boldsymbol{p}s\rangle$ is defined with the nucleon spin quantized along $\hat{k}$ in the rest frame of the nucleon.

can easily be shown that their (Wigner) reduced matrix elements are

$$\left(l_1\tfrac{1}{2}j_1\middle\|\hat{\theta}_l(p)\middle\|l_1\tfrac{1}{2}j_1\right)=\tfrac{1}{2}|\phi(p)|^2\left(l_1\tfrac{1}{2}j_1\|Y_l\|l_1\tfrac{1}{2}j_1\right),$$

$$\left(l_1\tfrac{1}{2}j_1\middle\|\hat{\theta}_{l1j}(p)\middle\|l_1\tfrac{1}{2}j_1\right)=\tfrac{1}{2}|\phi(p)|^2\left(l_1\tfrac{1}{2}j_1\|\boldsymbol{Y}_{l1j}\cdot\boldsymbol{\sigma}\|l_1\tfrac{1}{2}j_1\right),\tag{5.17}$$

where $\phi(p)$ is the radial wavefunction (in momentum space) of the single nucleon state (normalized so $\int p^2|\phi(p)|^2(1+p^2/4M^2)\,\mathrm{d}p=1$), and the reduced matrix elements of Y_{lm} and $\boldsymbol{Y}_{l1jm}\cdot\boldsymbol{\sigma}$ are given by

$$\left(l_1\tfrac{1}{2}j_1\|Y_l\|l_1\tfrac{1}{2}j_1\right)=\frac{(-1)^{1/2+j_1+l}}{\sqrt{8\pi}}\begin{pmatrix}l_1 & l & l_1\\ 0 & 0 & 0\end{pmatrix}\begin{bmatrix}l_1 & j_1 & \tfrac{1}{2}\\ j_1 & l_1 & l\end{bmatrix},$$

$$\left(l_1\tfrac{1}{2}j_1\|\boldsymbol{Y}_{l1j}\cdot\boldsymbol{\sigma}\|l_1\tfrac{1}{2}j_1\right)=\frac{(-1)^{l_1}}{\sqrt{8\pi}}\begin{bmatrix}l_1 & l_1 & l\\ \tfrac{1}{2} & \tfrac{1}{2} & 1\\ j_1 & j_1 & j\end{bmatrix}\begin{pmatrix}l_1 & l & l_1\\ 0 & 0 & 0\end{pmatrix},\tag{5.18}$$

where

$$\begin{bmatrix}a & b & c\\ d & e & f\end{bmatrix}=\sqrt{(2a+1)(2b+1)\dots}\begin{Bmatrix}a & b & c\\ d & e & f\end{Bmatrix},$$

and $\begin{Bmatrix}a & b & c\\ d & e & f\end{Bmatrix}$ is a standard six-j symbol. Likewise for the nine-j symbol.

We now insert this representation of $|\boldsymbol{p}s\rangle\langle\boldsymbol{p}s|$ into the definition of $|\varphi_s^{JH}(\boldsymbol{p})|^2$, construct $f_L^J(y)$ and perform the sum on H in eq. (5.15) with the aid of orthogonality properties of Clebsch–Gordan coefficients. The results are:
for L even:

$$f_L^J(y)=\frac{2J+1}{\sqrt{2L+1}}(-1)^{j_1+j_2+J}\begin{Bmatrix}j_1 & J & j_2\\ J & j_1 & L\end{Bmatrix}\left(l_1\tfrac{1}{2}j_1\|Y_L\|l_1\tfrac{1}{2}j_1\right)$$
$$\times\int\mathrm{d}^3p\,|\phi(p)|^2Y_{L0}(\cos\theta)\left(1+\frac{p}{M}\cos\theta+\frac{p^2}{4M^2}\right)\delta\left(y-\frac{p\cos\theta+E}{M}\right)\tag{5.19a}$$

and for L odd:

$$g_L^J(y)=-\frac{2J+1}{\sqrt{2L+1}}(-1)^{j_1+j_2+J}\begin{Bmatrix}j_1 & J & j_2\\ J & j_1 & L\end{Bmatrix}\sum_{l=L\pm1}(l010|L0)$$
$$\times\left(l_1\tfrac{1}{2}j_1\|\boldsymbol{Y}_{l1L}\cdot\boldsymbol{\sigma}\|l_1\tfrac{1}{2}j_1\right)\int\mathrm{d}^3p\,|\phi(p)|^2Y_{l0}(\cos\theta)$$
$$\times\left(1+\frac{p}{M}\cos\theta-\frac{p^2}{4M^2}(1-2\cos^2\theta)\right)\delta\left(y-\frac{p\cos\theta+E}{M}\right).\tag{5.19b}$$

The properties of f_L^J and g_L^J are best described by regarding them as distributions at $y=1$. This approach is justified by the fact that in nuclei $\phi(p)$ falls rapidly to zero as p^2/M^2 becomes appreciable. Typically $\langle p^2/M^2\rangle \approx 0.04$. Also $E=M-\epsilon$ where ϵ is a small fraction of the nucleon rest mass. Clearly, the δ-functions in eq. (5.19) force $y\approx 1$ under these circumstances. In the extreme limit, $p=\epsilon=0$ (i.e. $\phi(p)=0$ for $p\neq 0$) only $f_0^J(y)$ and $g_1^J(y)$ survive and are proportional to $\delta(y-1)$. The others, $\{f_L^J(y),\ L=2,4\}$ and $\{g_L^J(y),\ L=3,5\dots\}$ vanish at least as fast as $\langle p^2/M^2\rangle$ (or $\langle \epsilon/M\rangle$) and are proportional to $\delta'(y-1)$ on higher derivations. Thus the nucleon contributions to higher structure functions ($L\geqslant 2$) arise entirely from the motion of the nucleons in the nucleus and are small because the motion is non-relativistic. It is easy to see, using these methods, that the nuclear structure functions obtained by convoluting $f_L^J(y)$ or $g_L^J(y)$ with $F_1(x)$ or $g_1(x)$ (see eq. (5.14)) obey the sum rules, eqs. (1.7) and (1.8). In fact, they obey more restrictive sum rules

$$\begin{aligned} &M_n\left({}^J_LF_1\right)=0, \qquad 1<n<L, \\ &M_n\left({}^J_Lg_1\right)=0, \qquad 0<n<L, \end{aligned} \tag{5.20}$$

which, however, are consequences of their special form and are not of general validity. Note, in particular, the lowest convergent moment of $b_1(x)$ $\left(=-\sqrt{\frac{3}{2}}\,\frac{1}{2}F_1(x)\right)$ need not vanish contrary to the most naive generalization of eq. (1.7). Should these structure functions ever be measured with sufficient sensitivity, the sum rules of eqs. (5.20) which go beyond eqs. (1.7) and (1.8) would provide particularly clear tests of the convolution model.

6. Examples

To illustrate the application of the relatively complex formalism we have developed, we consider a few concrete examples.

(1) ${}^J_0F_1(x)$

According to eq. (1.9), we expect ${}^J_0F_1(x)$ to be proportional to the familiar spin-average structure function, $F_1(x)$. For J, j_1 and j_2 arbitrary (except $j_1\otimes j_2\supset J$) and $L=0$, eq. (5.19) reduces to

$$f_0^J(y)=\sqrt{2J+1}\int\frac{\mathrm{d}^3p}{4\pi}|\phi(p)|^2\left(1+\frac{p}{M}\cos\theta+\frac{p^2}{4M^2}\right)\delta\left(y-\frac{p\cos\theta+E}{M}\right). \tag{6.1}$$

This is the standard Fermi smearing function for $F_1(x)$ (given our treatment of the lower Dirac components). In the extreme non-relativistic limit it reduces to

$$f_0^J(y)=\sqrt{2J+1}\,\delta(y-1),$$

which, via eqs. (5.14) and (1.9), gives the expected result that the nucleus' $F_1(x)$ is merely the sum of the $F_1(x)$ for its constituent neutrons and protons. For realistic nucleon wavefunctions, eq. (6.1) gives Fermi smearing and binding energy corrections to $F_1(x)$ which are thought to account for some of the EMC effect.

(2) ${}^{1/2}_{1}g_1(x)$

A slightly less trivial example and an important check on our algebra is provided by the case of a $j_1 = \frac{1}{2}$ nucleon coupled to an inert core ($j_2 = 0$) to give total $J = \frac{1}{2}$. There are two distinct cases:

(a) $l_1 = 0$. For an $s_{1/2}$ nucleon, the nucleon's angular momentum and spin are parallel. From eq. (5.19) we find

$$g_1^{1/2}(y) = \sqrt{2}\int \frac{\mathrm{d}^3p}{4\pi}|\phi(p)|^2\left(1 + \frac{p\cos\theta}{M} - \frac{p^2(1-2\cos^2\theta)}{4M^2}\right)\delta\left(y - \frac{p\cos\theta + E}{M}\right). \tag{6.2}$$

In the extreme non-relativistic limit this reduces to

$$g_1^{1/2}(y) = \sqrt{2}\,\delta(y-1),$$

which, via eqs. (5.14) and (1.9) gives the expected result that the nucleus' $g_1(x)$ is merely the sum of $g_1(x)$ for its constituent protons and neutrons. For realistic $\phi(p)$, eq. (6.2) provides Fermi smearing and binding energy corrections to $g_1(x)$.

(b) $l_1 = 1$. The spin and the total angular momenta of a $p_{1/2}$ nucleon are not parallel, so the g_1-structure function of the nucleus is related to that of the nucleon by a "polarization factor" in addition to Fermi motion and binding energy effects. From eq. (5.19) we find

$$\begin{aligned} g_1^{1/2}(y) = &-\frac{\sqrt{2}}{3}\int \frac{\mathrm{d}^3p}{4\pi}|\phi(p)|^2\left(1 - 2(3\cos^2\theta - 1)\right) \\ &\times\left(1 + \frac{p\cos\theta}{M} - \frac{p^2(1-2\cos^2\theta)}{4M^2}\right)\delta\left(y - \frac{p\cos\theta + E}{M}\right). \end{aligned} \tag{6.3}$$

In the extreme non-relativistic limit this reduces to

$$g_1^{1/2}(y) = -(\sqrt{2}/3)\,\delta(y-1).$$

The factor $-\frac{1}{3}$ reflects that, on average, the nucleon's spin in a $p_{1/2}$ state is twice as likely to be found opposite its total angular momentum as along it. The factor can, of course, be checked by an elementary angular momentum coupling calculation.

For realistic $\phi(p)$, eq. (6.3) gives rise to both Fermi smearing/binding energy corrections and a momentum dependence of the "polarization factor" embodied in the second term. Attempts to extract $g_1(x)$ for the neutron from asymmetry measurements on real nuclei will have to include these effects in the extraction process.

(3) $J = \frac{3}{2}$ target

As a final, non-trivial, example we consider the general problem of extracting and interpreting the scaling structure functions of a $J = \frac{3}{2}$ target. There are six non-vanishing structure functions in the scaling limit ${}^{3/2}_{\ 0}F_1$, ${}^{3/2}_{\ 0}F_2$, ${}^{3/2}_{\ 2}F_1$, ${}^{3/2}_{\ 2}F_2$, ${}^{3/2}_{\ 1}g_1$ and ${}^{3/2}_{\ 3}g_1$. The first two are proportional to the spin average structure functions F_1 and F_2 and are measured by separating the x and y dependence in the spin and helicity average cross section (see, for example, eq. (1.4)). The second two are analogous to the quadrupole structure functions discussed in ref. [1] (called b_1 and b_2 there) for a spin-1 target. The last two are sensitive to quark spin asymmetries. ${}^{3/2}_{\ 1}g_1$ is the analog to g_1 for a spin-$\frac{1}{2}$ target. ${}^{3/2}_{\ 3}g_1$ is a new asymmetry which first arises for a $J = \frac{3}{2}$ target.

${}^{3/2}_{\ 2}F_1$ and ${}^{3/2}_{\ 2}F_2$ can be extracted by measuring the deep inelastic cross-section $(\mathrm{d}\bar{\sigma}/\mathrm{d}x\,\mathrm{d}y)$ with the target polarized along the beam axis, so that the initial state is a coherent or incoherent superposition only of states with $J_z = \pm\frac{3}{2}$ along the beam axis. In the Bjorken limit the beam axis is parallel to $\hat{q}$ so the target has $H = \pm\frac{3}{2}$, and the experiment measures $F_1^{3/2\,3/2}$ and $F_2^{3/2\,3/2}$ in the notation of eq. (1.4). Then an elementary exercise with Clebsch–Gordan coefficients gives

$$ {}^{3/2}_{\ 2}F_1 = 2\left(F_1^{3/2\,3/2} - F_1\right), \tag{6.4}$$

and likewise for F_2. (Here F_1 is the unpolarized structure function, $2F_1 = {}^{3/2}_{\ 0}F_1$.)

${}^{3/2}_{\ 1}g_1$ and ${}^{3/2}_{\ 3}g_1$ are more difficult to extract. It is necessary to measure the lepton polarization asymmetry $(\mathrm{d}\Delta\sigma/\mathrm{d}x\,\mathrm{d}y)$ for at least two different target orientations. As an example we take the target to be polarized along the axis-$\hat{n}$ ($\boldsymbol{J}\cdot\hat{n} = \frac{3}{2}$) which lies in the xz plane at an angle β to the $\hat{z} = (\equiv \hat{q})$ axis. Denote this state $|\frac{3}{2}\,\frac{3}{2}\rangle_\beta$ and states quantized along $\hat{k}$ without the β-subscript. Then a textbook exercise with Wigner functions yields

$$\begin{aligned} \left|\tfrac{3}{2}\,\tfrac{2}{2}\right\rangle_\beta = {} & \frac{1+\cos\beta}{2}\left(\cos\tfrac{1}{2}\beta\left|\tfrac{3}{2}\,\tfrac{3}{2}\right\rangle - \sqrt{3}\,\sin\tfrac{1}{2}\beta\left|\tfrac{3}{2}\,\tfrac{1}{2}\right\rangle\right) \\ & - \frac{1-\cos\beta}{2}\left(\sin\tfrac{1}{2}\beta\left|\tfrac{3}{2}\,-\tfrac{3}{2}\right\rangle - \sqrt{3}\,\cos\tfrac{1}{2}\beta\left|\tfrac{3}{2}\,-\tfrac{1}{2}\right\rangle\right). \end{aligned} \tag{6.5}$$

From eq. (6.5) we construct a spin density matrix and from eq. (3.5) we find that the asymmetry measures a linear combination of $g_1^{3/2\,2/2}$ and $g_1^{3/2\,1/2}$:

$$\frac{\mathrm{d}\Delta\sigma(\beta)}{\mathrm{d}x\,\mathrm{d}y} = \frac{e^4 ME}{2\pi Q^4}\,y(2-y)\,x\,\frac{\cos\beta}{4}\left\{\left(3+\cos^2\beta\right)g_1^{3/2\,3/2} + 3\sin^2\beta\, g_1^{3/2\,1/2}\right\}. \tag{6.6}$$

By taking linear combinations of asymmetries at two different values of β one can extract ${}^{3/2}_{1}g_1$ and ${}^{3/2}_{3}g_1$ (see eqs. (1.3) and (1.10)):

$$\begin{aligned} {}^{3/2}_{1}g_1 &= \sqrt{\tfrac{9}{5}}\, g_1^{3/2\,3/2} + \sqrt{\tfrac{1}{5}}\, g_1^{3/1\,1/2} \\ {}^{3/2}_{3}g_1 &= \sqrt{\tfrac{1}{5}}\, g^{3/2\,3/2} - \sqrt{\tfrac{9}{5}}\, g_1^{3/1\,1/2}. \end{aligned} \tag{6.7}$$

Note that transverse polarization ($\beta = \pi/2$) is of no use in extracting ${}^{J}_{L}g_1$ because $\Delta\sigma(\beta)$ vanishes at $\pi/2$ in the Bjorken limit (only $H \neq H'$ is allowed at $\beta = \pi/2$). If the target density matrix is more complex, e.g. if it is not a pure state, or if $\boldsymbol{J}\cdot\hat{n} = \pm\frac{1}{2}$ components are present, nevertheless the same formalism and general experimental approach apply to the extraction of the multipole structure functions.

Once the four scaling structure functions have been extracted what can we say about them? Only ${}^{3/2}_{3}g_1(x)$ obeys a sum rule constraint,

$$\int_0^1 \mathrm{d}x\, {}^{3/2}_{3}g_1(x) = 0.$$

For each multipole function there will be a convolution model prediction. Experiment may well disagree with these predictions (as was the case for the original EMC effect) and we may then learn something about the quark substructure of the nucleus. For definiteness we imagine the target to be ^{7}Li consisting of valence $p_{3/2}$ neutron ($l_1 = 1$, $j_1 = 3/2$) coupled to a closed core ($j_2 = 0$, $J = \frac{3}{2}$). A straightforward calculation then yields the following smearing functions which are to be convoluted with the nucleon's structure functions $F_1(z)$ and $g_1(z)$ to obtain the nucleus' multipole structure functions:

$$\begin{aligned} f_0^{3/2}(y) &= 2a_0(y), \qquad & g_1^{3/2}(y) &= \frac{2\sqrt{5}}{3} b_0(y) - \frac{4}{3\sqrt{5}} b_2(y), \\ f_2^{3/2}(y) &= -2a_2(y), \qquad & g_3^{3/2}(y) &= -\frac{6}{\sqrt{5}} b_2(y), \end{aligned} \tag{6.8}$$

where

$$\begin{aligned} a_L(y) &\equiv \int \frac{\mathrm{d}^3p}{4\pi} |\phi(p)|^2 \left(1 + \frac{p\cos\theta}{M} + \frac{p^2}{4M^2}\right) P_L(\cos\theta)\, \delta\!\left(y - \frac{p\cos\theta + E}{M}\right), \\ b_L(y) &\equiv \int \frac{\mathrm{d}^3p}{4\pi} |\phi(p)|^2 \left(1 + \frac{p\cos\theta}{M} - \frac{p^2(1-2\cos^2\theta)}{4M^2}\right) P_L(\cos\theta)\, \delta\!\left(y - \frac{p\cos\theta + E}{M}\right). \end{aligned} \tag{6.9}$$

For the purpose of convolution, it suffices to expand $a_L(y)$ and $b_L(y)$ as generalized functions about $y=1$. This is especially convenient if we make use of the fact that $p^2 \ll M^2$ and $E-M \ll M$ for nuclear wavefunctions. Then, including all terms through $\mathrm{O}(p^2/M^2)$ and $\mathrm{O}(\epsilon/M)$ (where $E \equiv M-\epsilon$) we find

$$a_0(y) = \delta(y-1+\eta) + \frac{1}{6}\left\langle \frac{p^2}{M^2}\right\rangle \delta''(y-1) + \mathrm{O}\left(\frac{p^4}{M^4}\right),$$

$$b_0(y) = \left(1-\frac{1}{3}\left\langle \frac{p^2}{M^2}\right\rangle\right)\delta(y-1+\eta) + \frac{1}{6}\left\langle \frac{p^2}{M^2}\right\rangle \delta''(y-1) + \mathrm{O}\left(\frac{p^4}{M^4}\right),$$

$$a_2(y) = \frac{1}{15}\left\langle \frac{p^2}{M^2}\right\rangle\{-2\delta'(y-1)+\delta''(y-1)\} + \mathrm{O}\left(\frac{p^4}{M^4}\right),$$

$$b_2(y) = \frac{1}{15}\left\langle \frac{p^2}{M^2}\right\rangle\{\delta(y-1)-2\delta'(y-1)+\delta''(y-1)\} + \mathrm{O}\left(\frac{p^4}{M^4}\right), \tag{6.10}$$

where

$$\eta = \left\langle \frac{\epsilon}{M}\right\rangle - \frac{1}{3}\left\langle \frac{p^2}{M^2}\right\rangle,$$

$$\left\langle \frac{\epsilon}{M}\right\rangle \equiv \int_0^\infty \mathrm{d}p\, p^2 |\phi(p)|^2 \frac{\epsilon(p)}{M},$$

$$\left\langle \frac{p^2}{M^2}\right\rangle = \int_0^\infty \mathrm{d}p\, \frac{p^4}{M^2}|\phi(p)|^2. \tag{6.11}$$

Although eqs. (6.10) are formally correct up to (small) terms of order p^4/M^4 one must be cautious using them to compute ${}^J_L f(x)$ and ${}^J_L g(x)$ too close to $x=1$. The reason is that $F_1(z)$ and $g_1(z)$ vanish quickly as $z \to 1$ (e.g. $\sim (1-z)^3$) and the $\mathrm{O}(p^4/M^3)$ corrections to $a_L(y)$ and $b_L(y)$ multiply higher derivatives of $\delta(y-1)$ (e.g. $\delta'''(y-1)$). These derivatives enhance $\mathrm{O}(p^4/M^4)$ effects. For $x>1$ the generalized function method does not converge at all. A rough measure of the validity of eq. (6.10) is

$$\frac{\langle p^4/M^4\rangle}{(1-x)\langle p^2/M^2\rangle} \ll 1.$$

Taking $\langle p^2/M^2\rangle \sim 1/25$ and $\langle p^4/M^4\rangle \cong \langle p^2/M^2\rangle^2$ we estimate that it is all right to use eq. (6.10) for $x \lesssim 0.85$.

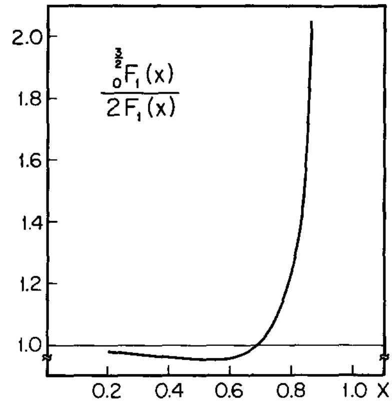

Fig. 2. ${}_0^{3/2}F_1(x)/2F_1(x)$ for a $p_{3/2}$ nucleon with $\langle p^2/M^2\rangle = 0.04$ and $\langle\epsilon/M\rangle = 0.03$.

Finally, then, to illustrate the magnitude of the novel structure functions we take $\langle p^2/M^2\rangle = 0.04$, $\langle\epsilon/M\rangle = 0.03$, $F_1(z) \propto (1-z)^3/z$ and $g_1(z) \propto (1-z)^3$. First we calculate the spin averaged, Fermi-smeared structure function ${}_0^{3/2}F_1(x)$ which we compare with the nucleon structure function $(2F_1(x))$ in fig. 2. The ratio shown in fig. 2 is the convolution model of the EMC effect. The depletion of intermediate x comes from $\eta \neq 0$, the enhancement at large x comes from the $\delta''(y-1)$ in eq. (6.10). The magnitude of the effects agrees well with more detailed (but no more accurate!) calculations. Notice that the depletion at intermediate x is $\lesssim 5\%$ – not enough to account for the EMC effect on heavy nuclei. The excess as $x \to 1$ has been observed experimentally. In fig. 3 we plot ${}_2^{3/2}F_1(x)$, the quadrupole structure function, compared to the spin averaged nuclear structure function $F_1(x)$. It is

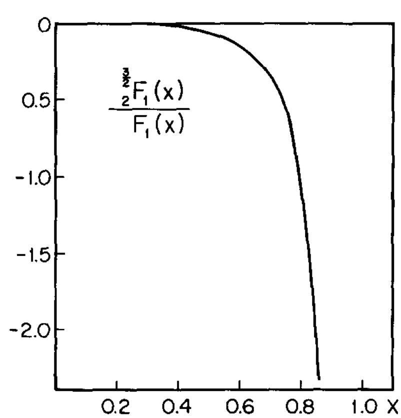

Fig. 3. ${}_2^{3/2}F_1(x)/F_1(x)$ for a $p_{3/2}$ nucleon with $\langle p^2/M^2\rangle = 0.04$ and $\langle\epsilon/M\rangle = 0.03$.

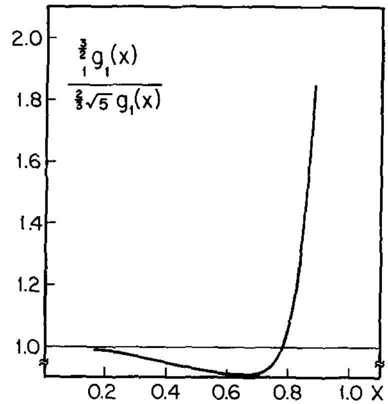

Fig. 4. ${}^{3/2}_{1}g_1(x)/\frac{2}{3}\sqrt{5}\,g_1(x)$ for a $p_{3/2}$ nucleon with $\langle p^2/M^2\rangle = 0.04$ and $\langle \epsilon/M\rangle = 0.03$.

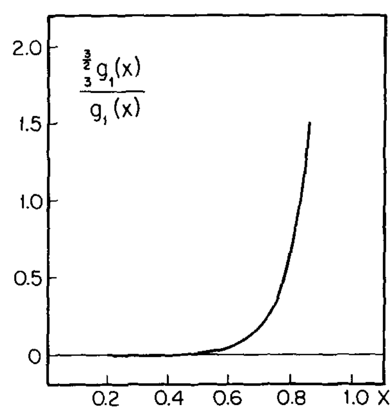

Fig. 5. ${}^{3/2}_{3}g_1(x)/g_1(x)$ for a $p_{3/2}$ nucleon with $\langle p^2/M^2\rangle = 0.04$ and $\langle \epsilon/M\rangle = 0.03$.

significant for $x \gtrsim 0.6$, where Fermi motion affects ${}^{3/2}_{0}F_1(x)$. In figs. 4 and 5 we plot the spin-dependent structure functions ${}^{3/2}_{1}g_1$ and ${}^{3/2}_{3}g_1$ normalized by the free nucleon spin-dependent structure function $g_1(x)$. In graphing ${}^{3/2}_{1}g_1(x)$ we have divided out the "polarization factor" $\frac{2}{3}\sqrt{5}$ so the plotted ratio would be unity in the absence of Fermi smearing. It is important to keep in mind that all nucleons contribute to the spin averaged structure function ${}^{3/2}_{0}F_2$ but that only "valence" nucleons, not coupled to $J = 0$, contribute to the other multipole structure functions.

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