LOW-ENERGY THEOREMS FOR SPIN $S \geq 1$†

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Two new results for Compton scattering are noted: (a) a relation for $S \geq 1$ which tests whether the total cross sections for unpolarized photons on a polarized stable target are asymptotically independent of the target polarization, (b) the existence of low-energy theorems for all intrinsic multipole moments. The electric-quadrupole theorem is given explicitly.

It is the purpose of this Letter to state two theorems concerning the Compton scattering on targets with spin $S \geq 1$. Each of these theorems comprises a group of low-energy theorems. The assumptions made are the same as for the known theorems[1-3] for spin 0 and $\frac{1}{2}$. In particular: (1) P- and/or T-noninvariant effects are neglected, (2) the target is assumed to be nondegenerate with any other particle with the same mass and other quantum numbers. The results, then, are exact in the strong interactions. They will also be electromagnetically exact if the infrared limit spelled out by Low[1] exists.[4] In any event, the present arguments for $S \geq 1$ are on the same footing as for $S = 0, \frac{1}{2}$ as long as a consistent treatment of electromagnetic interactions exists at all.[4]

We define $a(S_3, \omega)$ as the elastic forward amplitude for the scattering of a photon with frequency ω and some fixed helicity on a target[5] with polarization S_3.

Theorem I.—For any $S \geq 1$, there exists one and only one nontrivial relation between the $2S+1$ threshold amplitudes $a(S_3, 0)$. If this relation may be converted to a sum rule, there follows a nontrivial homogeneous relation between integrals over total cross sections of unpolarized photons on polarized targets. By "nontrivial" is meant that the relation is not a consequence of rotational and crossing symmetry alone.

Theorem II.—There exist low-energy theorems for all $2S+1$ intrinsic multipole moments of a particle with spin S.

The detailed proofs of the theorems[6] are somewhat lengthy, but quite straightforward. The only complexity of some substance is that one has to answer the following question: Consider the scattering amplitude in the nonforward direction without implementation of the transversality condition for the external photons. The number[7] of independent amplitudes is then equal to $N_\rho(S) = (4S - [S] + 2)(2S + [S] + 2)$. What is an explicit basis for these $N_\rho(S)$ functions which is free of kinematical singularities? Having found this basis, one also has the right basis for the number $N_\gamma(S) = 2(S+1)(2S+1)$ of amplitudes which survive after transversality is imposed. A method of explicit construction for arbitrary S will be given in a paper now in course of preparation, along with other technical details. Here we shall only indicate what are the main steps in the argument.

Details of Theorem I.—The general forward amplitude is written as $A(\omega)$, where A is a matrix function of S. A contains $[3S+1]$ independent amplitudes and can be written as[8]

$$A(\omega) = A_1(\omega)\vec{\epsilon}' \cdot \vec{\epsilon} + iA_2(\omega)\vec{S} \cdot (\vec{\epsilon}' \times \vec{\epsilon}) + A_3(\omega)[\vec{S} \cdot \vec{\epsilon}', \vec{S} \cdot \vec{\epsilon}] + O(\omega^2). \tag{1}$$

For example, for $S = 1$, the term $O(\omega^2)$ is $A_4(\omega) \times \vec{\epsilon}' \cdot \vec{\epsilon}(\vec{S} \cdot \vec{k})^2$. Further analysis shows that $A_4(0)$ exists; and similarly for $O(\omega^2)$ for higher S. Thus $A(0)$ has at most three independent amplitudes, as is obvious from angular-momentum conservation alone.[9] Furthermore, $A_2(\omega) \sim \omega$, as follows essentially from crossing, hence at $\omega = 0$ only A_1 and A_3 survive. But now, in addition, it can be shown by the standard method[1] that for any[10] S,

$$A_3(0) = 0. \tag{2}$$

It is only because of the additional Eq. (2) that the Thomson limit follows (to no one's surprise) for $S \geq 1$. What is more remarkable is that Eq. (2) has still further consequences (apart from the vanishing of double spin flip at $\omega = 0$). Namely, as $A_1(0) = -e^2/M$ (M = target mass),

$$a(S_3, 0) = -e^2/M, \quad \text{for any } S. \tag{3}$$

Thus if and only if all terms in $a(S_3, \omega)$ which do not vanish as $\omega \to \infty$ (and there are such terms!) are independent of S_3, then it follows from Eq. (3) that we have an unsubtracted dispersion relation for $a(S_3, \omega) - a(S_3', \omega)$. Summation over photon polarizations then yields

$$\int [\sigma(S_3, \omega) - \sigma(S_3', \omega)] d\omega = 0, \quad \text{all } S_3, S_3'. \tag{4}$$

$\sigma(S_3,\omega)$ is the total cross section for the scattering of an unpolarized photon on a target with polarization S_3. Because of

$$\sigma(S_3,\omega)=\sigma(-S_3,\omega), \tag{5}$$

there are only $[S]$ nontrivial relations of the type of Eq. (4). Thus for $S=1$ we only have one new relation:

$$\int[\sigma(1,\omega)-\sigma(0,\omega)]d\omega=0, \quad S=1. \tag{6}$$

Equations (4) and (6) may be of interest for experimental study. Indications of failure of these relations would contribute evidence against "asymptotic polarization independence."

Remark.—It is curious that, for $S\geq 1$, we now meet a homogeneous "consistency condition" (a superconvergence relation) which is not implied by rotational invariance, while recently it was found[11] that, for isospin ≥ 1, one also meets homogeneous consistency conditions not implied by isospin invariance, and likewise dictated by low-energy theorems combined with asymptotic behavior. It may be of interest to combine these two phenomena!

Details of Theorem II.—For a 2^L-pole moment one will need a low-energy theorem accurate to $O(\omega^L)$. However, this does not mean that we need the full amplitude to this order. [In fact, it is well known[1,2] that to $O(\omega^2)$ a fully structure-independent description is impossible.] Indeed, more specifically, for an electric 2^L pole it suffices[12] to have a theorem for the terms $\sim\omega^L Y_{LM}(\theta,\varphi)$ (L even), while for 2^L magnetic we need $\sim\omega^L Y_{L-1,M}(\theta,\varphi)$ (L odd). Hence for $L\geq 2$ no multipole theorems can be found from the forward direction only.

Thus we expand the full amplitude $A(\omega,\theta,\varphi)$ as follows:

$$A=\epsilon_m{}'A_{mn}\epsilon_n$$
$$=\frac{1}{M}\sum_{NLM}\left(\frac{\omega}{M}\right)^N\epsilon_m{}'A_{mn}{}^{(NLM)}\epsilon_n Y_{LM}(\theta,\varphi). \tag{7}$$

The $A_{mn}{}^{(NLM)}$ still are matrices in S. In this language,

$$A_{mn}{}^{(000)}=-e^2\delta_{mn}(4\pi)^{1/2} \text{ ("Thomson")}, \tag{8}$$

$$A_{mn}{}^{(100)}=-\tfrac{1}{4}ie^2(g-2)^2\epsilon_{mnl}S_l(4\pi)^{1/2}. \tag{9}$$

g is the gyromagnetic ratio. Thus low-energy Compton scattering includes a "$g-2$ experiment" for any S, independently of the high-energy behavior of amplitudes.[13]

For the electric quadrupole moment eQ (in units of $1/M^2$) one can show that[12]

$$(A_{11}+A_{22})^{(220)}=-\tfrac{1}{3}(4\pi/5)^{1/2}e^2Q(\vec{S}^2-S_3{}^2). \tag{10}$$

The proofs of this result as well as the ones for higher moments makes use of the following basic ingredients.

(a) Put $A=\epsilon_\mu{}'A_{\mu\nu}\epsilon_\nu$ and $A_{\mu\nu}=U_{\mu\nu}+E_{\mu\nu}$, where U and E refer to the "unexcited" (or s, u-channel target pole) and excited contributions, respectively. We have[1]

$$k_m{}'(U_{mn}+E_{mn})k_n=-\omega\omega'(U_{44}+E_{44}). \tag{11}$$

(b) Crucial to the present argument is Singh's lemma which states that[14]

$$E_{44}=k_m{}'k_n[\Lambda_{mn}(\omega',\vec{k}',\omega,\vec{k},)$$
$$+\Lambda_{nm}(-\omega,-\vec{k};-\omega',-\vec{k})], \tag{12}$$

where Λ_{mn} is a three-tensor of second rank, free of kinematical singularities. Expand Λ_{mn} just like A_{mn} in Eq. (7) and insert in Eq. (11). We exemplify what happens for the Q calculation where we need Eq. (11) to the accuracy $O(\omega^4)$. Here we need Λ_{mn} to $O(\omega^0)$, hence the most general form of E_{44} to our order is $a\vec{k}\cdot\vec{k}'+b\{\vec{S}\cdot\vec{k},\vec{S}\cdot\vec{k}'\}$, with a and b unknown numerical constants. On the other hand, e^2Q appears in Eq. (11) through a term $\sim\vec{k}\cdot\vec{k}'[(\vec{S}\cdot\vec{k})^2+(\vec{S}\cdot\vec{k}')^2]$. Hence to the order considered, E_{44} contains "the wrong tensor" and cannot affect the determination of e^2Q. As is well known,[1] one does have a low-energy theorem as soon as one has proved that E_{44} cannot affect the answer.

The procedure for a general 2^L-pole moment follows similar lines. Expand Λ_{mn} in the base of $N_\rho(S)$ functions mentioned earlier, and likewise for E_{mn}. By considering simultaneously (a) the power dependence in ω, (b) the LM dependence, and (c) the dependence on irreducible tensors in spin space,[15] one consistently finds that, for multipole moment terms, E_{44} drops behind in a power counting in both ω and $\cos\theta$, as compared to the unexcited terms in Eq. (11).

In conclusion, while Compton scattering is hardly the best way to measure multipole moments, it is nevertheless of some interest that as a matter of principle such a measurement is contained in the Compton effect.

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[1]F. E. Low, Phys. Rev. 96, 1428 (1954).

[2]M. Gell-Mann and M. L. Goldberger, Phys. Rev. 96, 1433 (1954).

[3]V. Singh, to be published.

[4]See Ref. 1, Sec. I. The limit in question is $\lambda \to 0$, where λ is an infinitesimal photon mass. λ is introduced to make the target rigorously nondegenerate with target plus soft photons. It should be noted that $\lambda \to 0$ certainly causes no problems for all graphs denoted by class A in Ref. 2, as long as the electromagnetic vertex exists. The insistence on existence is no mere luxury. For $S \geq 1$, not only are there well-known ambiguities in the Lagrangian formulation of electromagnetic interactions. But, furthermore, the power-series expansion in e^2 is most likely not valid [T. D. Lee and C. N. Yang, Phys. Rev. 128, 885 (1962)]. It should be stressed that we are only concerned about the existence of the theory, not about the e^2 expansion. In particular we can use exact electromagnetic-vertex functions in class A graphs. We also bypass the ambiguities noted especially by T. D. Lee, Phys. Rev. 140, B967 (1965), concerning minimality for $S \geq 1$. In any event, it is hard to imagine complications for a target particle like the deuteron; while even in the nonperturbation approach, anomalous multipole moments may be expected to have an electromagnetic contribution small compared with the strong interaction contribution—cf. T. D. Lee, Phys. Rev. 128, 899 (1962).

[5]Throughout this Letter, we consider the spin to be quantized in the direction of the incoming beam. Wherever specification is necessary, all amplitudes shall refer to the laboratory system. We work in the transverse radiation gauge.

[6]Upon communication of Theorem II to Dr. V. Singh, he informed me that he had conjectured that this theorem might be ture.

[7]Effectively, N_ρ is the number of independent amplitudes for the scattering of a massive neutral vector meson on a target with spin S. $[S]$ denotes the integer part of S.

[8]$(\vec{\epsilon}', \vec{k}')$ and $(\vec{\epsilon}, \vec{k})$ are (polarization, momentum) of the final and initial photon, respectively. $\{a,b\} = ab + ba$.

[9]From here on, we always understand that $S \geq 1$.

[10]For $S = 1$, Eq. (2) is equivalent to the relation $c = 0$ in L. I. Lapidus and Chou Kuang-Chao, Zh. Eksperim. i Teor. Fiz. 39, 1286 (1960) [translation: Soviet Phys. —JETP 12, 898 (1961)].

[11]A. Pais, Phys. Rev. Letters 18, 17 (1967).

[12]θ, φ are the polar angles of $\vec{k}'$ relative to $\vec{k}$. $\vec{k}$ is taken in the 3 direction.

[13]Starting from the algebra of electric dipole moments, sum rules for $g-2$ have been derived by M. Hosoda and K. Yamamoto, Progr. Theoret. Phys. (Kyoto) 36, 425, 426 (1966), and G. Konisi and K. Yamamoto, Progr. Theoret. Phys. (Kyoto) 37, 538 (1967). Note that such sum rules, especially for $S \geq 1$, are on a less firm footing than the low-energy theorem (9). See the discussion by H. Pagels, Phys. Rev. 158, 1566 (1967), Appendix, where the theorem (9) is stated for $S = 1$.

[14]See Ref. 3, Eq. (13).

[15]Of course these dependences are correlated by the conservation of angular momentum. The identification of the 2^L-pole terms is facilitated by noting that we need a totally symmetric tensor in S_m of rank L. To be quite precise, the 3×3 spin-matrix relation, Eq. (10), is fully determined by the low-energy theorem apart from the transitions $S_3 = 1 \to 1$ and $-1 \to -1$. Of course these two diagonal elements are equal. The matrices $A_{mn}^{(22M)}$ for $M = 1$ and 2 are fully determined.

NONRESONANT PRODUCTION AMPLITUDES OR THE DECK EFFECT*

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We wish to generalize the class of production processes which can be handled by "single-scattering"[1] methods to cases where more than two particles or resonances are produced. The object exchanged may be a particle or its trajectory but we are especially interested in vacuon, or Pomeranchukon exchange processes, also called diffraction dissociation,[2] because this leads to the largest cross sections. In the present Letter, we give a prescription only for the background amplitude: the amplitude which would be present in a certain channel if there were no resonances decaying into that channel. If there is a decaying resonance, where the same partial wave occurs in the background, there will be a reaction on the background[3] which can be significant in addition to the interference of resonance and modified background