Introduction to Amplitude Analysis

GlueX Software Tutorial
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Amplitude Analysis Outline

- Idea #1: information about the production of resonances and resonance properties is reflected in the distributions of final state particles that we detect
  - peaks in invariant mass suggest resonances
  - angular distributions provide information about spin and parity of resonances (that are produced or exchanged)

- Idea #2: the method of maximum likelihood estimation provides a mechanism to estimate parameters of a probability density function
  - without having to bin phase space or count events in a bin
  - independent of the number of dimensions spanned by the p.d.f.

- Amplitude analysis connects these two ideas: the parameters of the p.d.f. we fit to the observed events are often connected to fundamental parameters of nature whose values and uncertainties we want to estimate from the observed data

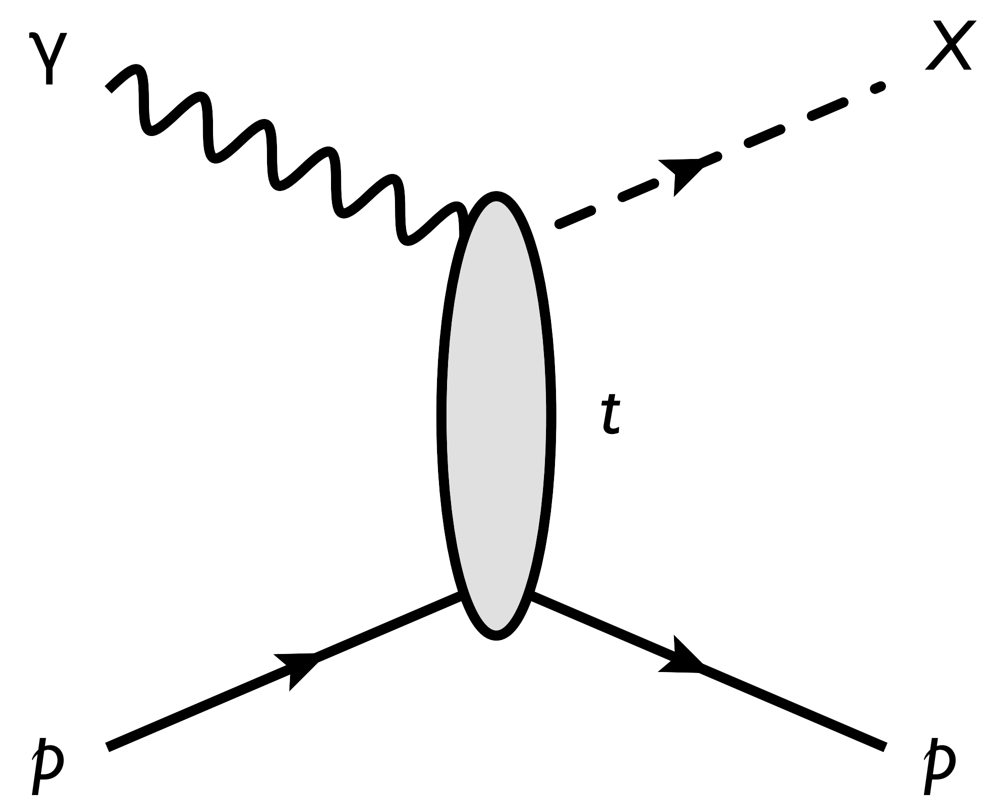
Amplitudes

Amplitude Fundamentals

- Quantum mechanics: amplitudes are complex-valued functions that describe the various indistinguishable paths one can take to go between some distinguishable incoming and outgoing particles
    - amplitudes add coherently to form an intensity (density of observed events per unit phase space)
    - one may need multiple coherent sums to completely describe a system, especially one where incoming and outgoing particles have spin
- Writing equations for the amplitudes generally involves making two types of assumptions:
    - kinematic: assumes conservation of parity, angular momentum, etc. (pretty safe assumptions) -- e.g., using $Y_\ell^m(\theta, \phi)$
    - dynamic: assumes some model for the underlying physics (validity depends on model and application) -- e.g., using a Breit-Wigner function to describe the lineshape and phase of the $a_2(1320)$
- The existence of exotic mesons and their masses and widths is a question about dynamics -- where to build that dynamics into the analysis is a matter of strategy
    - collaboration with theory is essential

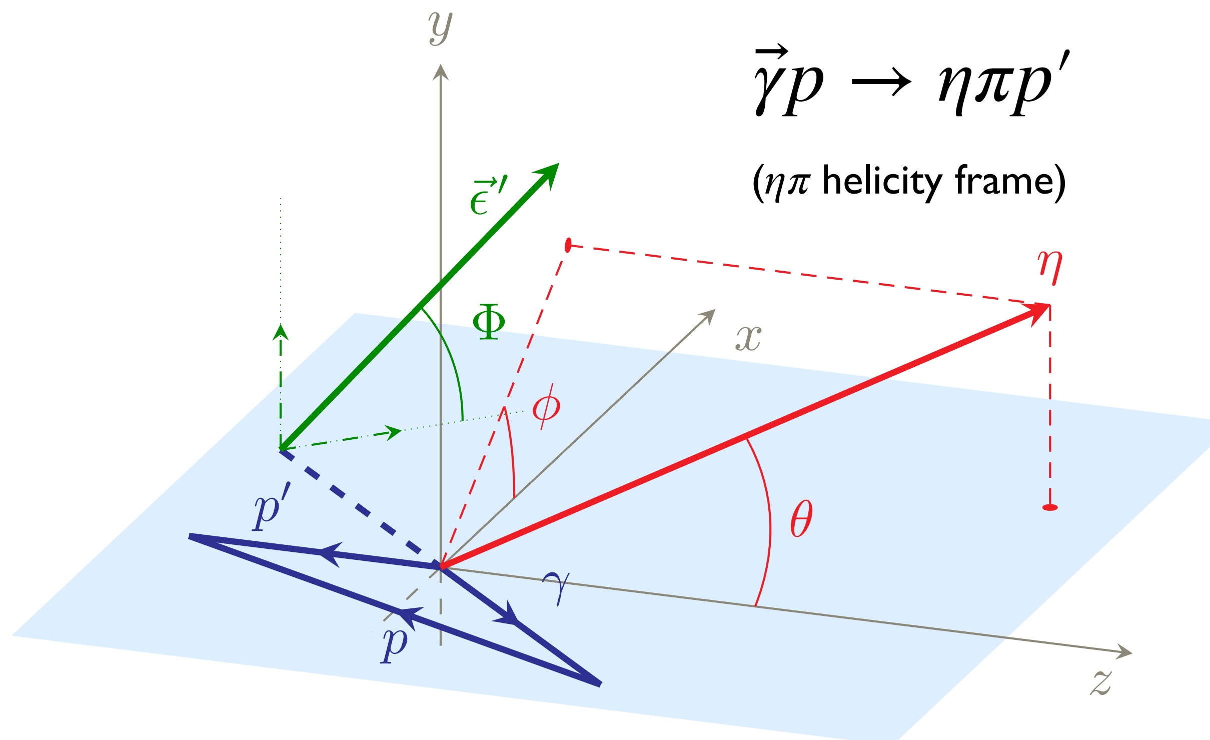
What are we trying to determine?

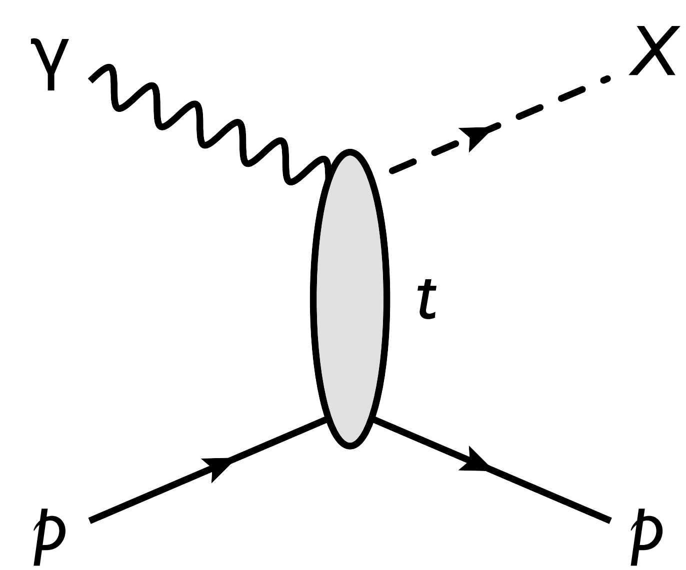
- The external parts of the reaction look like:
  $\vec{\gamma} p \to \eta\pi^0 p$
- Let's assume that the picture on the right is valid: production of a resonance $X \to \eta\pi^0$, where X has spin ℓ and spin-projection m
- Kinematics tells us the distribution of final state particles given a certain polarization of γ and spin and parity of the exchange particle
- We often want to fit for the dynamics, e.g., how does the magnitude and phase of this process depend on $M(\eta\pi)$
  - extract in bins of $M(\eta\pi)$, or
  - assume some shape like Breit-Wigner and extract the overall strength
  - referred to in my notes/talks as "production coefficients": the complex-valued coupling of the input particles to the output particles according to some amplitude that may be indistinguishable from other amplitudes



$\eta\pi$ Polarized Photoproduction Kinematics

- Write amplitudes with definite "reflectivity": $\epsilon = \pm 1$, which have a non-trivial distribution in Φ
- Definine naturality: $P(-1)^J$
    - natural parity: $P(-1)^J = +1$; $J^P = 0^+, 1^-, 2^+, \ldots$
    - unnatural parity: $P(-1)^J = -1$; $J^P = 0^-, 1^+, 2^-, \ldots$
- High energy *t*-channel picture: the reflectivity fixes the product of the naturalities of the exchange particle and the produced resonance
- Make use of this additional dimension (Φ) in the amplitude analysis to gain insight into production mechanisms





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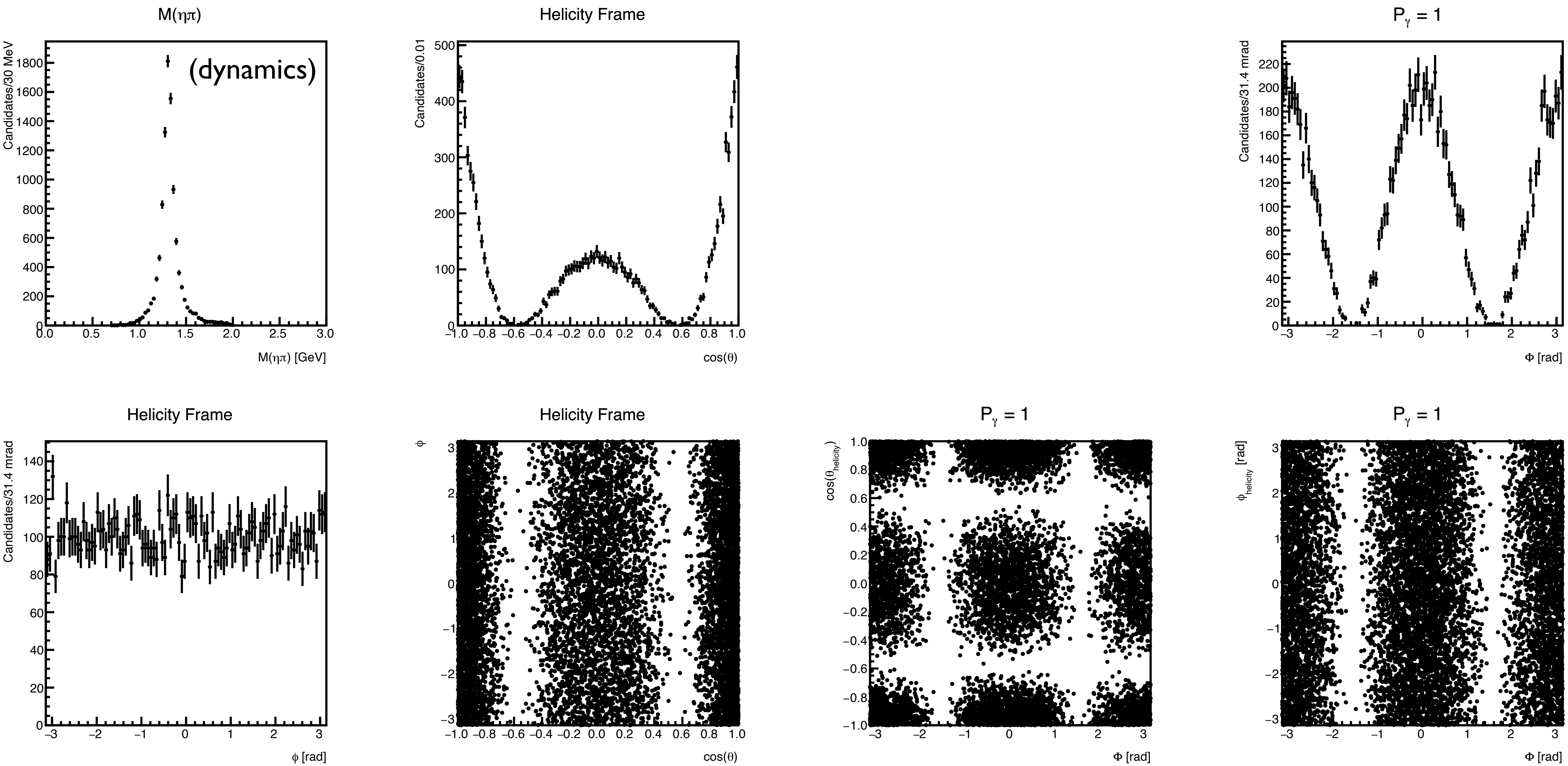
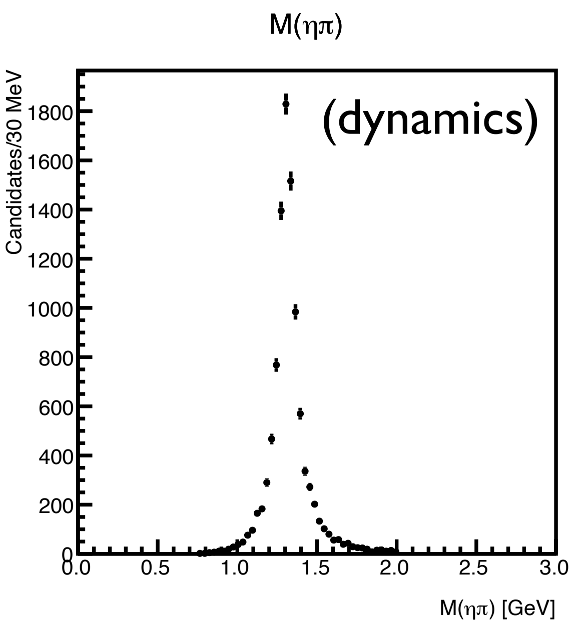
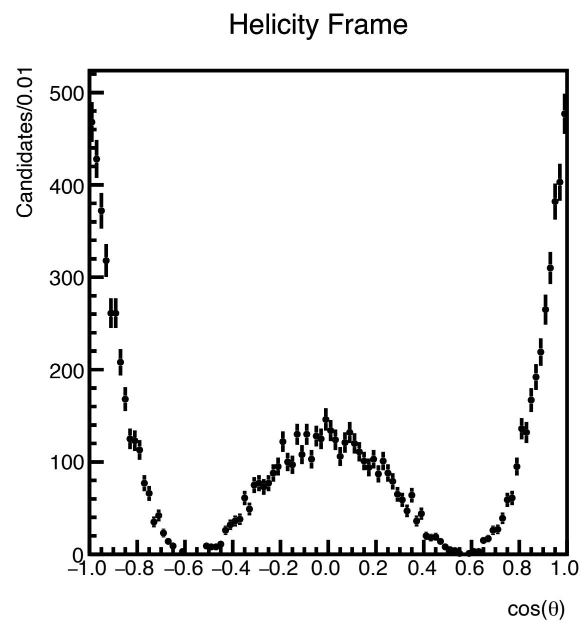
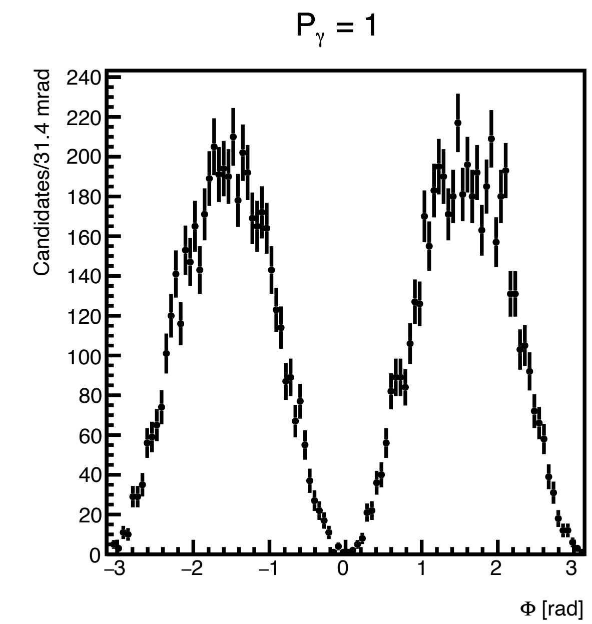
$a_2(1320) \to \eta\pi$ in the $\ell^{\epsilon}_{m} = D^{+}_{0}$ amplitude

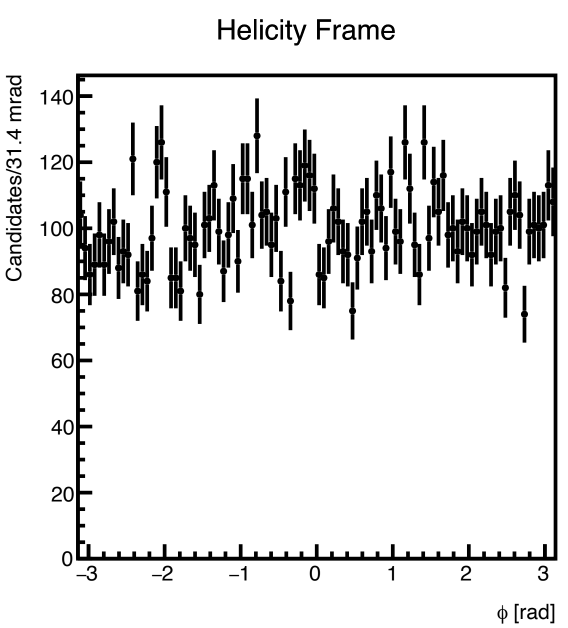


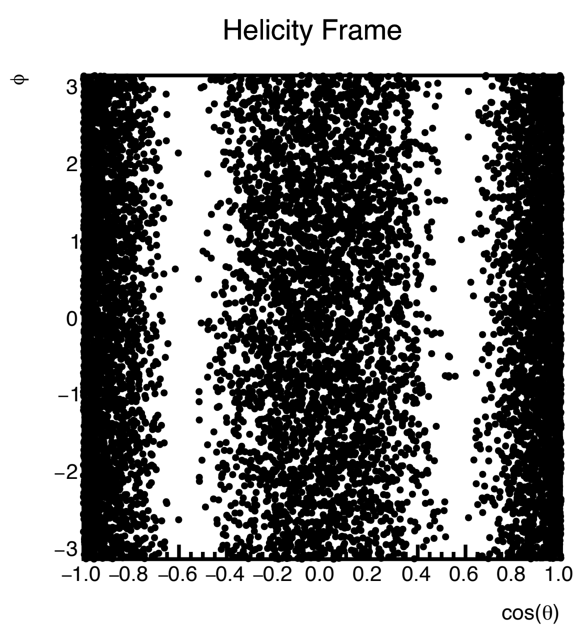
$a_2(1320) \to \eta\pi$ in the $\ell_m^\epsilon = D_0^-$ amplitude

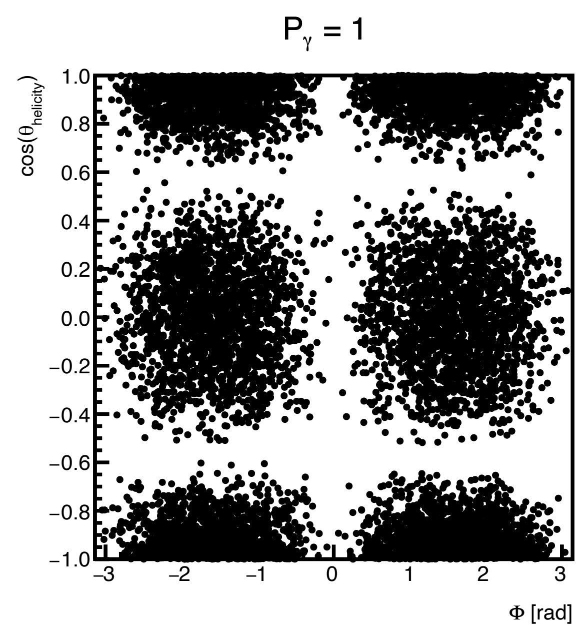


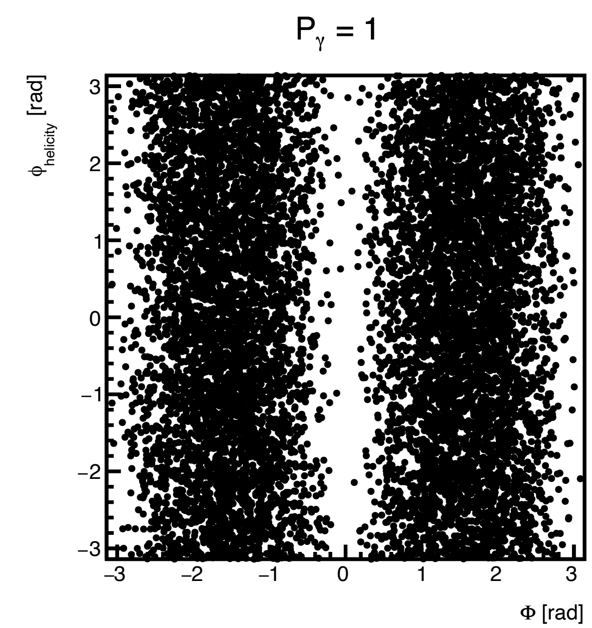




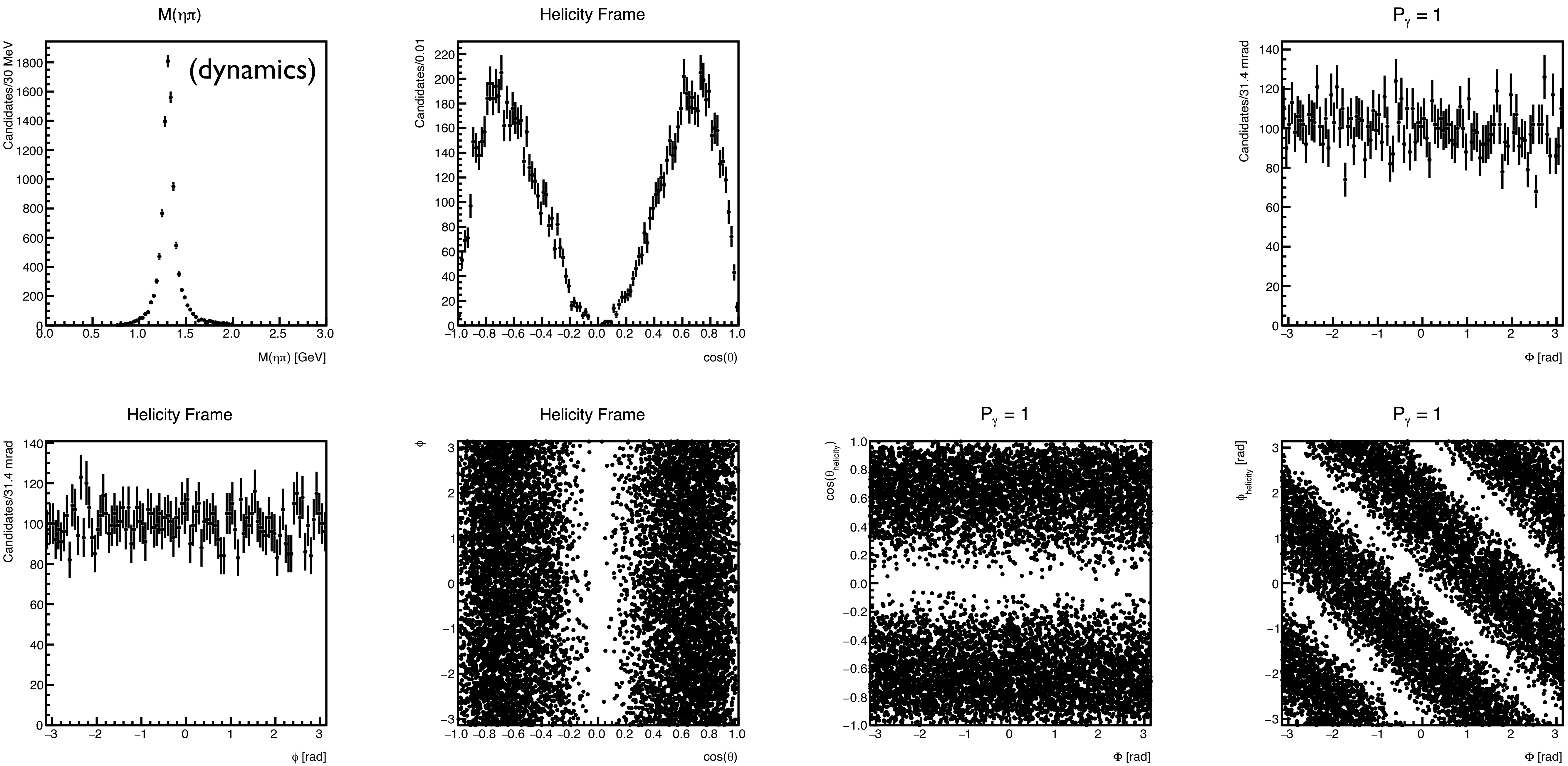




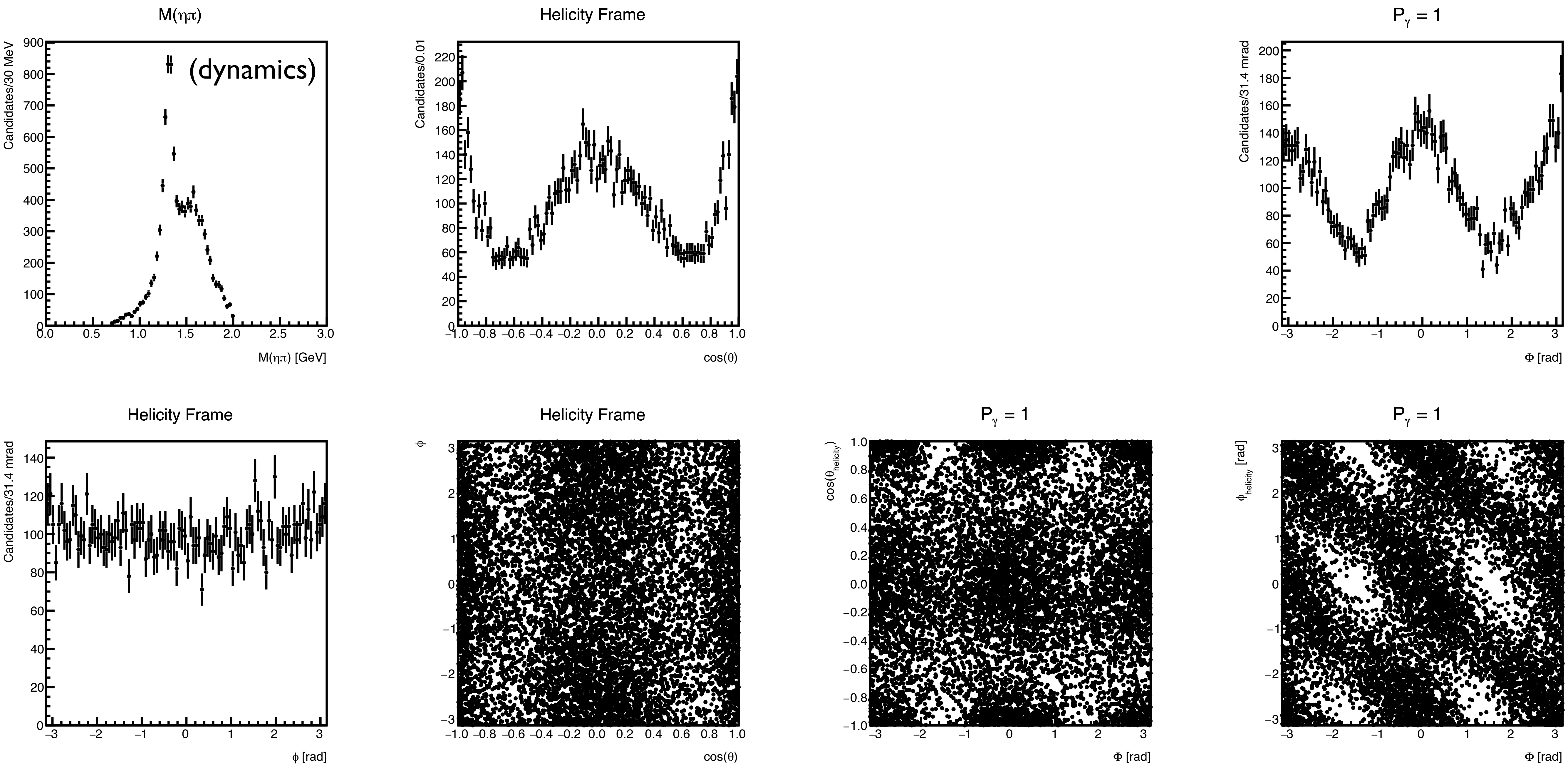




$a_2(1320) \to \eta\pi$ in the $\ell_m^\epsilon = D_1^-$ amplitude



Interference between resonances in the D_0^+ and P_1^+ amplitudes



Describing the Intensity for $\eta\pi$ Polarized Photoproduction

- The previous plots rely on deriving and expression for the intensity as a function of the key angles (see GlueX-doc 4094):

$$I(\Omega,\Phi) = 2\kappa \sum_k \left\{ (1-P_\gamma)\left|\sum_{\ell,m}[\ell]^{(-)}_{m;k}\mathrm{Re}[Z^m_\ell(\Omega,\Phi)]\right|^2 + (1-P_\gamma)\left|\sum_{\ell,m}[\ell]^{(+)}_{m;k}\mathrm{Im}[Z^m_\ell(\Omega,\Phi)]\right|^2 + \right.$$
$$\left. (1+P_\gamma)\left|\sum_{\ell,m}[\ell]^{(+)}_{m;k}\mathrm{Re}[Z^m_\ell(\Omega,\Phi)]\right|^2 + (1+P_\gamma)\left|\sum_{\ell,m}[\ell]^{(-)}_{m;k}\mathrm{Im}[Z^m_\ell(\Omega,\Phi)]\right|^2 \right\}$$

- In this expression: $Z^m_\ell(\Omega,\Phi) \equiv Y^m_\ell(\theta,\phi)e^{-i\Phi}$
- Comments:
  - k indexes nucleon spin-flip or non-flip amplitudes -- these are experimentally indistinguishable so we effectively only consider one of them
  - The dynamics is in $[\ell]^{(\epsilon)}_m$, which describes the production of a resonance with spin ℓ and spin-projection m in an amplitude with reflectivity ϵ
  - In all generality, for each resonance with spin ℓ there are $2(2\ell+1)$ amplitudes, each with a unique angular signature
    - example: the $a_2(1320)$ can populate up to ten amplitudes, but models predict that some combinations of m, ϵ should be insignificant

The Method of Maximum Likelihood for Parameter Estimation

Maximum Likelihood

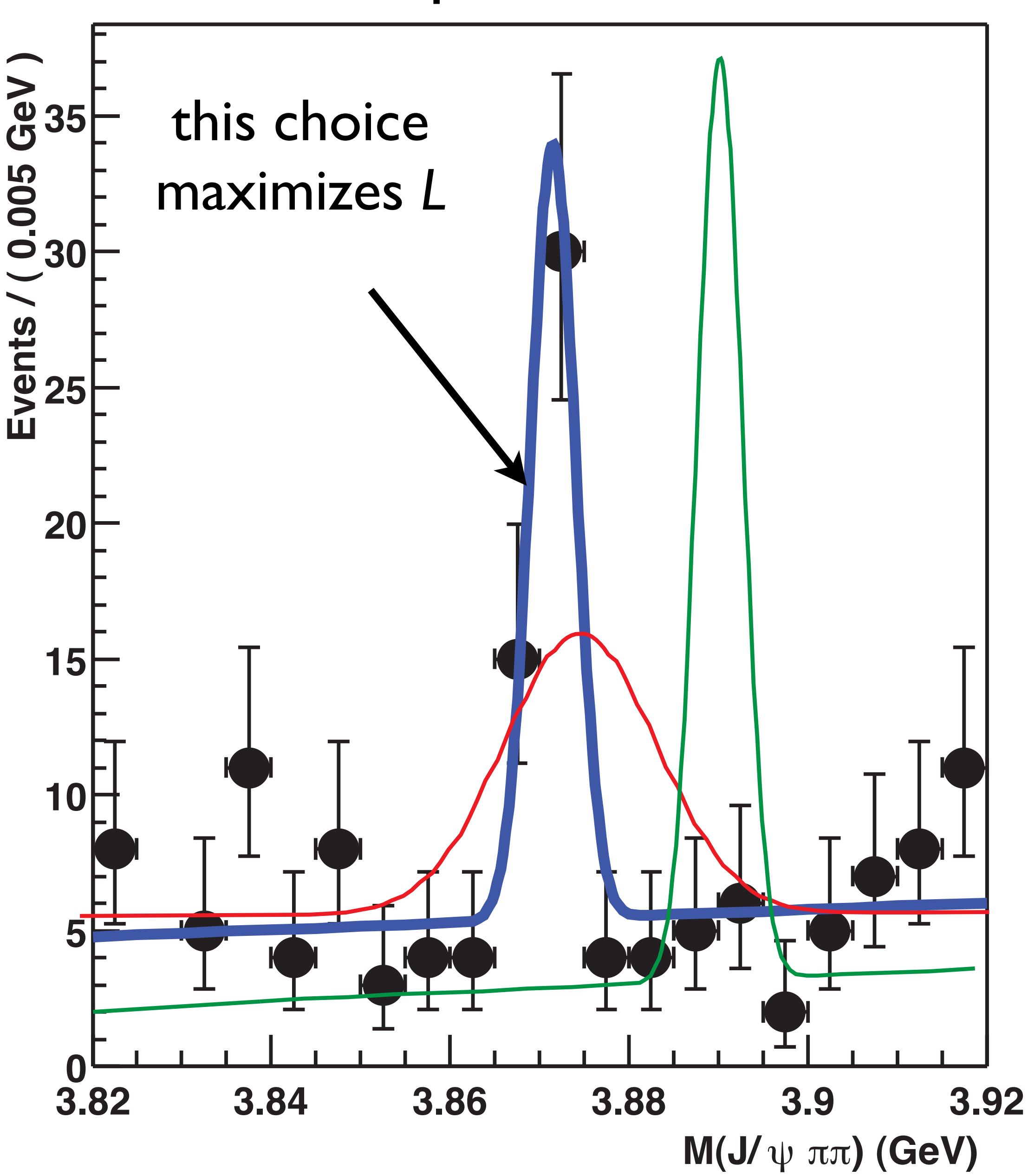
- Amplitude analysis is built around the (extended) maximum likelihood method
- Start with a model that contains free parameters (θ) and predicts the probability of having an event with a particular set of kinematic variables x (angles, invariant mass, etc.)

$$\mathcal{P}(\vec{x};\vec{\theta})$$

- Vary the free parameters to maximize the probability of observing one's data set

$$\mathcal{L} = \prod_{i=1}^{N_{\mathrm{events}}} \mathcal{P}(\vec{x}_i;\vec{\theta})$$

Example: 1D in *x*



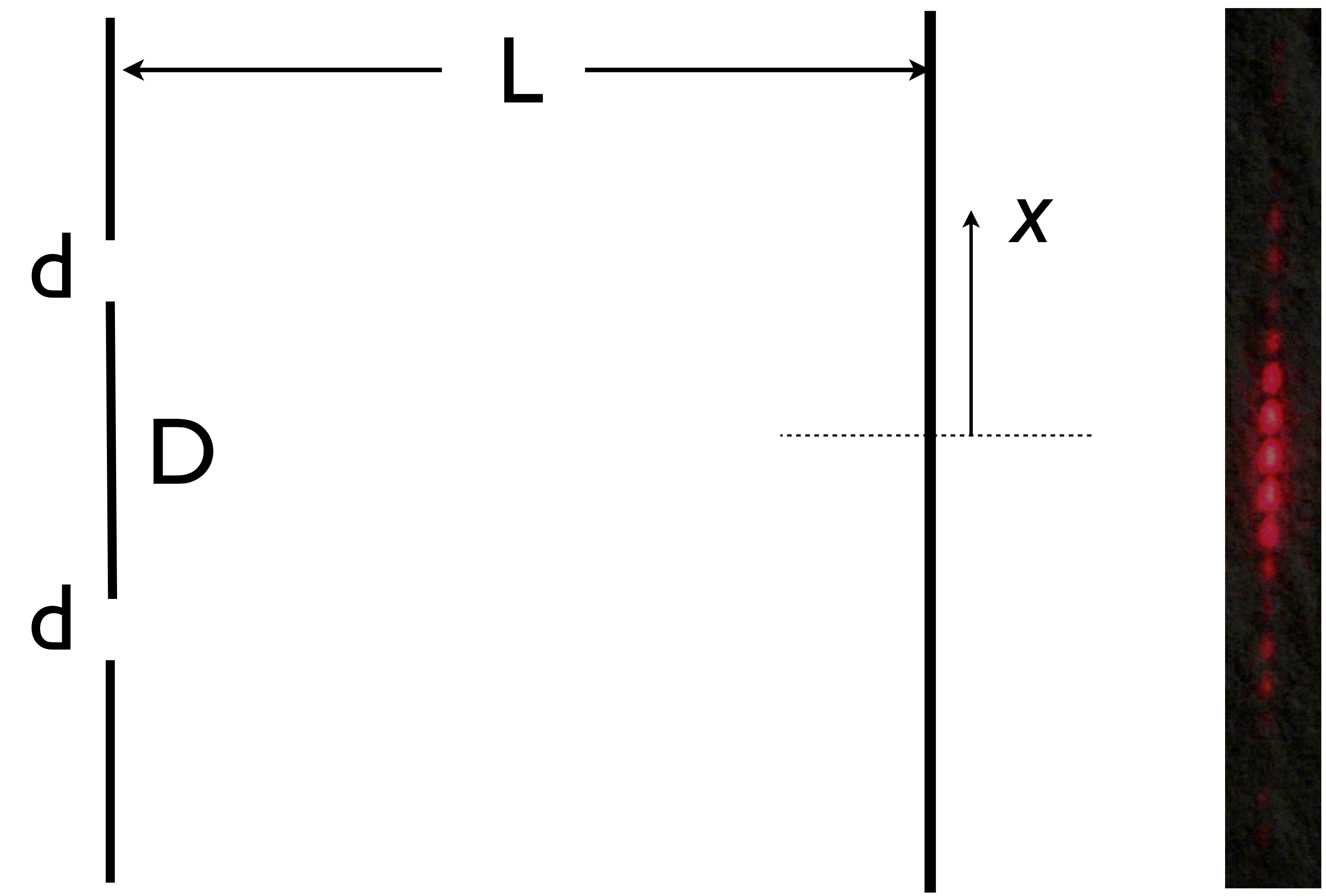
Experiment Application

Step 1: Shoot
particles at slits

<u>Probe</u>
Beam of Particles
wavelength λ

<u>Goal</u>: determine
from data the best
estimates
for *d* and *D*

Step 2: For each particle record
location *x* where it was detected



<u>Physical System Under Study</u>
Two Slits: width d, separation D

<u>Detector</u>
Measures location x_i
for each arriving particle

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The Fit Procedure

- Our "theoretical model" that parametrizes the intensity of the particles in the detector is given by

$$I(x) = I_0 \left(\frac{\sin(d\pi x/\lambda L)}{d\pi x/\lambda L} \right)^2 \cos^2(2D\pi x/\lambda L)$$

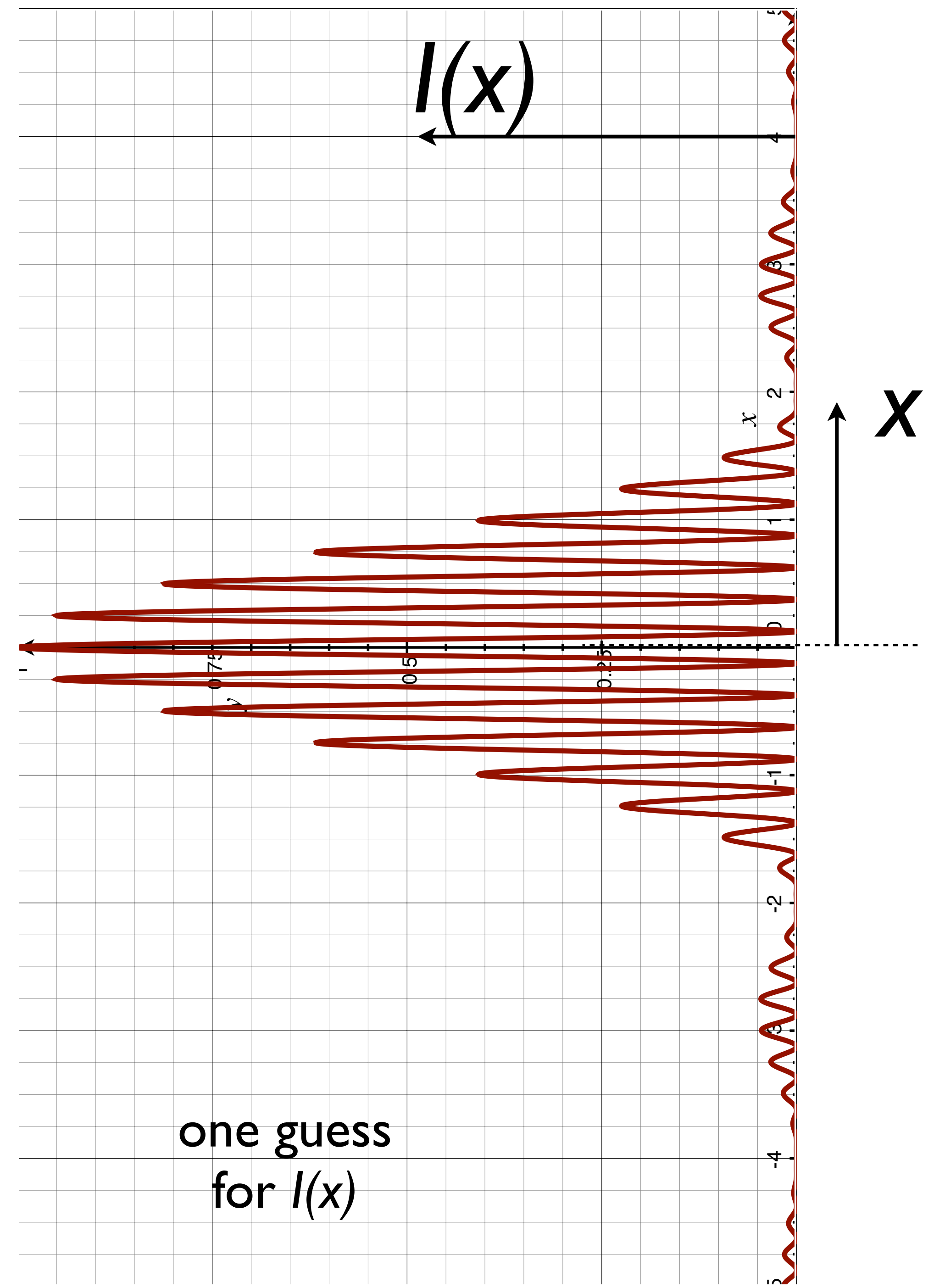
- Start with a guess for values for *d* and *D*
- Convert *I*(*x*) into a properly normalized PDF -- multiple techniques are available for evaluating the integral

$$\mathcal{P}(x) = \frac{I(x)}{\int_{x_{\min}}^{x_{\max}} I(x)dx}$$

- Compute the likelihood by taking the product over all detected events

$$\mathcal{L} = \prod_{i=1}^{N} \mathcal{P}(x_i)$$

- Iterate with a new choice of *d* and *D* until the likelihood is maximized



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Ingredients for Maximum Likelihood

- A set of statistically independent events or observations, each of which lands in a single place in some multi-dimensional phase space
- A properly normalized probability density function, written in terms of parameters (which we want to know) that describes the probability for observing an event at a particular place in phase space
  - hint... an problem is coming: this function depends on both physics and detector/analysis acceptance
- An algorithm for maximizing the likelihood $\mathscr{L}$
  - in practice we minimize $-2\ln\mathscr{L}$ -- note the correspondence with χ^2 in the case that $\mathscr{P} \propto e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
  - computation is non-trivial because the calculation of $\ln\mathscr{L}$ involves (at least) computing the intensity function for each event in the data sample for a given choice of fit parameters

Using Maximum Likelihood for Amplitude Analysis

Formulating the Likelihood

- Generally use the "extended" maximum likelihood written in terms of some parameters θ
- Need the probability of observing an event $\mathbf{x}_i$ at a point in the phase space which depends on
  - how mother nature populates phase space, which depends on fit parameters, and
  - the probability of the event being detected by the GlueX detector and passing all the analyst's selection criteria $\eta(\mathbf{x})$
- The total number of observed events μ is the product of these two functions integrated over all phase space
  - μ can be used to normalize the p.d.f that is needed for the likelihood

$$\mathcal{L}(\boldsymbol{\theta}) = \frac{e^{-\mu}\mu^N}{N!}\prod_{i=1}^{N}\mathcal{P}(\mathbf{x}_i;\boldsymbol{\theta})$$

$$\mu = \int \mathcal{I}(\mathbf{x};\boldsymbol{\theta})\eta(\mathbf{x})\ d\mathbf{x}$$

$$\mathcal{P}(\mathbf{x};\boldsymbol{\theta}) = \frac{1}{\mu}\mathcal{I}(\mathbf{x};\boldsymbol{\theta})\eta(\mathbf{x})$$

Experimental Acceptance

- The integral of any function over some domain can be written in terms of its average value on that domain
- The acceptance function $\eta(\mathbf{x})$ cannot, and does not need to, be written analytically -- one only needs to be able integrate the product of it and the intensity
  - do this by finding the average value using a sample of M_a accepted signal Monte Carlo events from a sample of M_g generated events
- The average value alone is acceptable for use in the log-likelihood
  - R absorbed in parameters that scale the intensity and contribute a constant offset to $-2\ln\mathscr{L}$
- The accepted MC should contain all the "physics" that is not in the intensity expression to ensure proper integration of the intensity
- It is only necessary to have true signal MC events: introducing and subtracting backgrounds just creates statistical noise in the integration procedure

$$\int_R f(x)\ dx = R\langle f(x)\rangle$$

$$\langle \mathcal{I}(\mathbf{x};\boldsymbol{\theta})\eta(\mathbf{x})\rangle = \frac{1}{M_g}\sum_{i=1}^{M_a}\mathcal{I}(\mathbf{x}_i;\boldsymbol{\theta})$$

insert RHS term here

$$-2\ln\mathcal{L}(\boldsymbol{\theta}) = -2\left(\sum_{i=1}^{N}\ln\mathcal{I}(\mathbf{x}_i;\boldsymbol{\theta}) - \int\mathcal{I}(\mathbf{x};\boldsymbol{\theta})\eta(\mathbf{x})\ d\mathbf{x}\right) + c_1$$

Technical Comments

$$-2\ln\mathcal{L}(\boldsymbol{\theta}) = -2\left(\sum_{i=1}^{N}\ln\mathcal{I}(\mathbf{x}_i;\boldsymbol{\theta}) - \int\mathcal{I}(\mathbf{x};\boldsymbol{\theta})\eta(\mathbf{x})\ d\mathbf{x}\right) + c_1$$

sum over data   sum over accepted MC

- Performing a fit involves hundreds to tens of thousands of evaluations of $-2\ln\mathscr{L}$ to find the set of θ parameters that minimize $-2\ln\mathscr{L}$
  - each involves computing the intensity for every data event
  - and a sum over MC may be necessary depending on how fit parameters appear in the intensity function
- Relevant for GlueX: data from RF or other sidebands is used to correct the sum over signal region data so that the first sum is consistent with only signal, which is described by the intensity function
- Large, independent sums lend themselves well to parallel processing
  - partial sums over partial data sets computed on individual processes
  - sums performed in parallel using concurrent compute threads on a graphical processing unit
- Fit speed is dominated by two things:
  - speed of computing $-2\ln\mathscr{L}$: many technical tricks to try to do this as fast as possible
  - reducing the number of computations of $-2\ln\mathscr{L}$ by choice of algorithm used find the minimum

The Typical Procedure

- Devise an event selection method that maximizes purity and statistics of some reaction of interest that one wants to describe as a sum of interfering amplitudes
- Configure the intensity model with free parameters: some implementations may limit which types of values are allowed to freely float in the fit
    - typical parameters: complex production coefficients for each amplitude, mass and width of a resonance, relative normalization of data sets, polarization angle, ...
- Specify the data sample and MC sample used for intensity integration
- Run the fit code to find parameter values and uncertainties (ideally full covariance matrix... or even better)
- Convert parameter values into quantities of interest: number of events associated with production of a particular resonance
    - full example in the afternoon: extract a_2 cross section using Amplitude Analysis

Summary

- Using amplitude analysis to determine resonance properties brings together some key ideas:
  - the properties of resonances affect of how their decay products populate the detector
  - maximum likelihood estimation is an effective technique for determining model parameters in a sparsely populated, high dimensional phase space
- Practical implementation of the technique with a complex model and a large data set is computationally challenging
  - finding the best estimates for model parameters is a starting point... one also wants uncertainties, which depend on the shape of the likelihood in parameter space around the maximum
  - different amplitude analysis tools or algorithms share common physics ideas: most (all?) are using maximum likelihood estimation based on some intensity constructed from amplitudes
    - differences: how to get to maximum and how the user specifies the intensity
- More technical discussion of likelihood construction in the context of AmpTools is available in the AmpTools User Guide.