Eight Parameter Gauge Compression & Twisted Boundary Conditions

Abstract Eight parameter gauge compression destroys the twisted boundary conditions. One has to apply the phase after reconstructing.

First we have to define some Gell-Mann matrices in order generate some non-trivial SU(3) matrix.

```
In[161]:= gm1 = {{0, 1, 0}, {1, 0, 0}, {0, 0, 0}};
          gm5 = {{0, 0, -I}, {0, 0, 0}, {I, 0, 0}};
```

Then it is helpful to let *Mathematica* assume that ϕ is a real number and generate an arbitrary comples phase.

```
In[163]:= ? Assuming
```

Assuming[*assum*, *expr*] evaluates *expr* with *assum* appended to $Assumptions, so that *assum* is included in the default assumptions used by functions such as Refine, Simplify, and Integrate. »

```
In[164]:= $Assumptions

Out[164]= ϕ ∈ Reals

In[165]:= $Assumptions = Element[ϕ, Reals];
          phase = Exp[I ϕ];
```

An SU(3) matrix is obtained via the exponential function. One could choose a more complicated linear combination of the generators, but that is sufficient for a counter-example. We multiply all nine matrix elements with the chosen complex phase.

```
In[167]:= u = MatrixExp[I (gm1 + gm5)] phase;
          MatrixForm[FullSimplify[u]]
```

Out[168]//MatrixForm=

$$\begin{pmatrix} e^{i\phi}\cos\left[\sqrt{2}\right] & \frac{i\, e^{i\phi}\sin\left[\sqrt{2}\right]}{\sqrt{2}} & \frac{e^{i\phi}\sin\left[\sqrt{2}\right]}{\sqrt{2}} \\ \frac{i\, e^{i\phi}\sin\left[\sqrt{2}\right]}{\sqrt{2}} & e^{i\phi}\cos\left[\frac{1}{\sqrt{2}}\right]^2 & i\, e^{i\phi}\sin\left[\frac{1}{\sqrt{2}}\right]^2 \\ -\frac{e^{i\phi}\sin\left[\sqrt{2}\right]}{\sqrt{2}} & \sin\left[\frac{1}{\sqrt{2}}\right]^2\left(-i\cos[\phi]+\sin[\phi]\right) & e^{i\phi}\cos\left[\frac{1}{\sqrt{2}}\right]^2 \end{pmatrix}$$

This matrix is not unitary any more:

```
In[169]:= UnitaryMatrixQ[u]

Out[169]= False
```

The gauge compression works as follows: Take the first and second row of the matrix only. Just discard the third row.

```
a = u[[1, ;;]];
b = u[[2, ;;]];
```

The gauge reconstruction then works by taking the complex conjugate of the cross product of the first and second row.

```
c = FullSimplify[Conjugate[Cross[a, b]]];
ur = {a, b, c};
```

We can do a quick check and see whether everything is normalized to unity:

```
In[179]:= {Norm[a], Norm[b], Norm[c]} // FullSimplify
Out[179]= {1, 1, 1}
```

Let's have a look at the original and the reconstructed matrix:

```
In[175]:= MatrixForm[FullSimplify[u]]
          MatrixForm[FullSimplify[ur]]
```

Out[175]//MatrixForm=

$$\begin{pmatrix} e^{i\phi}\,\text{Cos}[\sqrt{2}] & \frac{i\,e^{i\phi}\,\text{Sin}[\sqrt{2}]}{\sqrt{2}} & \frac{e^{i\phi}\,\text{Sin}[\sqrt{2}]}{\sqrt{2}} \\ \frac{i\,e^{i\phi}\,\text{Sin}[\sqrt{2}]}{\sqrt{2}} & e^{i\phi}\,\text{Cos}\left[\frac{1}{\sqrt{2}}\right]^2 & i\,e^{i\phi}\,\text{Sin}\left[\frac{1}{\sqrt{2}}\right]^2 \\ -\frac{e^{i\phi}\,\text{Sin}[\sqrt{2}]}{\sqrt{2}} & \text{Sin}\left[\frac{1}{\sqrt{2}}\right]^2 \left(-i\,\text{Cos}[\phi]+\text{Sin}[\phi]\right) & e^{i\phi}\,\text{Cos}\left[\frac{1}{\sqrt{2}}\right]^2 \end{pmatrix}$$

Out[176]//MatrixForm=

$$\begin{pmatrix} e^{i\phi}\,\text{Cos}[\sqrt{2}] & \frac{i\,e^{i\phi}\,\text{Sin}[\sqrt{2}]}{\sqrt{2}} & \frac{e^{i\phi}\,\text{Sin}[\sqrt{2}]}{\sqrt{2}} \\ \frac{i\,e^{i\phi}\,\text{Sin}[\sqrt{2}]}{\sqrt{2}} & e^{i\phi}\,\text{Cos}\left[\frac{1}{\sqrt{2}}\right]^2 & i\,e^{i\phi}\,\text{Sin}\left[\frac{1}{\sqrt{2}}\right]^2 \\ -\frac{e^{-2i\phi}\,\text{Sin}[\sqrt{2}]}{\sqrt{2}} & -i\,e^{-2i\phi}\,\text{Sin}\left[\frac{1}{\sqrt{2}}\right]^2 & e^{-2i\phi}\,\text{Cos}\left[\frac{1}{\sqrt{2}}\right]^2 \end{pmatrix}$$

As one can see, the complex phase is not preserved properly:

```
In[184]:= FullSimplify[u / ur]
```

Out[184]= $\{\{1, 1, 1\}, \{1, 1, 1\}, \{e^{3 i \phi}, e^{3 i \phi}, e^{3 i \phi}\}\}$

The simple reason is that the reconstruction process always creates a unitary matrix. Since the original matrix is not a unitary matrix, the last row will absorb all the non-unitariness. We can check that the reconstructed matrix is unitary by helping *Mathematica* by giving ϕ some arbitrary but real value:

```
In[192]:= UnitaryMatrixQ[ur /. ϕ → 1.345]
Out[192]= True
```

We can fix the phase by multiplying the last row with the desired phase cubed in the reconstruction process:

In[188]:=
```
ur2 = {a, b, c phase^3};
MatrixForm[FullSimplify[ur]]
```

Out[189]//MatrixForm=

$$\begin{pmatrix} e^{i\phi}\,\text{Cos}\left[\sqrt{2}\right] & \frac{i\,e^{i\phi}\,\text{Sin}\left[\sqrt{2}\right]}{\sqrt{2}} & \frac{e^{i\phi}\,\text{Sin}\left[\sqrt{2}\right]}{\sqrt{2}} \\ \frac{i\,e^{i\phi}\,\text{Sin}\left[\sqrt{2}\right]}{\sqrt{2}} & e^{i\phi}\,\text{Cos}\left[\frac{1}{\sqrt{2}}\right]^2 & i\,e^{i\phi}\,\text{Sin}\left[\frac{1}{\sqrt{2}}\right]^2 \\ -\frac{e^{-2i\phi}\,\text{Sin}\left[\sqrt{2}\right]}{\sqrt{2}} & -i\,e^{-2i\phi}\,\text{Sin}\left[\frac{1}{\sqrt{2}}\right]^2 & e^{-2i\phi}\,\text{Cos}\left[\frac{1}{\sqrt{2}}\right]^2 \end{pmatrix}$$

In[187]:=
```
FullSimplify[u == ur2]
```

Out[187]= True