Implementation of the Pion Cross Section Model for E91-003

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1 Introduction.

In order to make data vs. SIMC comparisons for pion electroproduction data, we need a model for the pion production cross section. This is necessary in order to properly model the effect of the spectrometer acceptances on the measurement. With a reasonable starting model, we can adopt an iterative procedure in order to generate a model that gives good agreement with the data. In order to compare pion production from Hydrogen with pion production from heavier targets (D and ^{3}He), we need to modify our hydrogen model to give an extimate for the D and ^{3}He cross section in such a way that we do not introduce a false A-dependence. In the older SIMC model, there were several points where approximations were made assuming that the initial nucleon momentum was negligible compared to the virtual photon (or final pion) momentum. This is not a good approximation in the case of E91-003, where low momentum pions (290 MeV/c) are created. In addition, calculations in the hydrogen lab frame (proton at rest) were being used in the real lab frame (nucleus at rest) for the D and ^{3}He calculations. Once again, this is a smaller effect for large pion momentum, but becomes increasingly important as the pion momentum is reduced.

2 H(e,e$'$ π)n Cross Section Model.

We are currently using the pion production model provided by Henk Blok. This is a parameterization of data by Brauel [1]. It takes data measured at Q^2=0.7 (GeV/c)^2, W=2.19 GeV to provide a cross section, $\frac{d\sigma}{dtd\phi_{CM}}$, as a function of t, θ_{CM}, and ϕ_{CM}.

Experiment	Q^2	W	ϵ	Model/Data
fpi	0.75	1.95	high	0.8
fpi	0.75	1.95	low	1.0
fpi	1.60	1.95	high	1.0
fpi	1.60	1.95	low	1.0
nucpi	0.40	1.60	high	1.1
nucpi	0.40	1.60	low	???
nucpi	0.40	1.16	high	4.2
nucpi	0.40	1.16	low	???

Table 1: Ratio of model(SIMC) to data for several nucpi and fpi hydrogen kinematics. The Model/Data value is good to roughly 10% (due to variation with proton angle, etc...).

It then applies additional terms to introduce a Q^2 and W dependence. The W dependence is designed to be appropriate for large values of W, and is expected to be reasonable for $W = 1.6$ GeV, but not for $W = 1.16$ GeV. Table 1 shows the ratio of model to data for several fpi and nucpi kinematics.

3 Effects of Nucleon Momentum for A>1.

There are three ways in which the initial nucleon momentum can enter into SIMC. Events have to be generated or weighted according to the momentum distribution for the target nucleus. Kinematics have to be adjusted in order to take the initial nucleon momentum into account. And finally, the cross section must be modified to take into account the modified kinematics, and the conversion from the proton rest frame (used in the hydrogen model) to the frame where the entire target nucleus is at rest. In addition, one can apply off-shell effects to the cross section. This has not been attempted yet, and will not be discussed here.

In Deuterium and Helium, the target nucleon has an initial momentum within the nucleus. When generating events, some kind of weight or selection must be applied according to the momentum distribution for the given target. Initial momentum values can be generated according to the momentum distribution, or initial momentum values can be generated uniformly, and a weight applied to each event based on the momentum distribution. SIMC takes the first approach, and generates an initial momentum for each event according to a parameterization of the D or ^{3}He momentum distribution. There were problems with the way SIMC was generating the initial momentum, caused by truncating the distribution at 490 MeV/c, and not properly normalizing the distribution. This will be fixed by taking Harry Lee's momentum distribution all the way to 1.2 GeV/c, and normalizing the momentum distributions to 1.0.

Once the initial nucleon momentum has been chosen, the kinematics of the scattering have to be modified to take into account the motion of the target nucleon. This means taking into account the motion of the nucleon when calculating kinematic variables in the lab frame, but also involves being careful to convert everything to the correct frame when calculating frame dependent quantities. In particular, when using the cross section model for hydrogen, one must remember that the parameterization is taken in terms of γ-p center of mass and proton rest frame, while the SIMC kinematic variables are taken in the frame where the target nucleus at rest.

Finally, one must take the motion of the target nucleon into account when generating the pion production cross section in the lab. This means taking the correct kinematics (using angles and momenta in the frame used in the model), as well as taking the proper conversion factors when transforming from the γ-p center of mass frame to the true lab frame (nucleus at rest).

While the following discussion is applicable for all heavy nuclei (A>1), and for both π^+ and π^- production, I will call always refer to the target nucleon as the proton, and the recoil nucleon as the neutron, even though we will use the same procedure for π^- production from Helium. The spectator nucleons will be neglected throughout, except to the extent that they define the lab frame (entire nucleus at rest). In addition, I will often refer to the target nucleus as a deuteron, but this discussion is general for all A. Variable names are as before, except that the target proton has an initial momentum of p_f in the lab frame. Note that we will refer to the 'real' lab frame (which now has the nucleus at rest) and the proton rest frame, which was the lab frame in the Hydrogen case.

4 Determination of $\frac{d^5\sigma}{dE_{e'}d\Omega_{e'}d\Omega_\pi}$ for Hydrogen.

Henk's model provides the 5-fold cross section in the lab frame for pion production from hydrogen. Below are the steps necessary to determine the cross section from the kinematics for the event (starting in the lab frame). Variables with a '$*$' superscript are center of mass variables. The virtual photon 4-momentum is $q^\mu = (\nu, q)$, the pion momentum is p_π, and the target mass is m_p (mass of the proton).

1. Calculate kinematics in the lab frame $(\nu, t, Q^2, \theta_{\gamma\pi}, \phi_{\gamma\pi}, \epsilon)$.

2. Calculate center of mass values for $p_\pi^*, \theta_{\gamma\pi}^*$, and $\phi_{\gamma\pi}^*$. $\phi_{\gamma\pi}^*$ is taken relative to the electron scattering plane, and is equal to $\phi_{\gamma\pi}$ (lab frame) because the boost from lab to center of mass is along $\hat{q}$.

3. Determine $\frac{d^2\sigma}{dtd\phi^*}$ from parameterization in terms of t, Q^2, θ^*, and ϕ^*. This gives the cross section for W=2.19 GeV. Apply a W^2 dependence of the form $1/(W^2-m_p^2)^2$.

4. Convert $\frac{d^2\sigma}{dtd\phi^*}$ to $\frac{d^2\sigma}{d\Omega}$ by multiplying by $\frac{1}{2\pi}\frac{dt}{d\cos(\theta)}$ (since $\phi = \phi^*$). The $\frac{1}{2\pi}$ comes from Brauel's convention:

$$2\pi\frac{d^2\sigma}{dtd\phi} = \sigma_T + \epsilon\sigma_L + \epsilon\cos(2\phi^*)\sigma_{TT} + \sqrt{2\epsilon(1-\epsilon)}\sigma_{LT}. \tag{1}$$

To get to $\frac{d^2\sigma}{d\Omega^*}$ (center of mass), you need:

$$\frac{dt}{d\cos(\theta^*)} = 2p_\pi^* q^*, \tag{2}$$

while to get to the lab cross section, the factor is (see Appendix H):

$$\frac{dt}{d\cos(\theta)} = \frac{2m_p q p_\pi}{m_p + \nu - \frac{E_\pi}{p_\pi} q\cos(\theta)}. \tag{3}$$

5. Convert to $\frac{d^5\sigma}{dE_{e'}d\Omega_{e'}d\Omega_\pi}$ by multiplying by Γ_T' ($\frac{d^2\sigma}{d\Omega} \equiv \frac{1}{\Gamma_T'}\frac{d^5\sigma}{dE_{e'}d\Omega_{e'}d\Omega_\pi}$) [2], where

$$\Gamma_T' = \frac{\alpha}{2\pi^2}\frac{E_e'}{E_e}\frac{W^2-m_p^2}{2m_p}\frac{1}{Q^2(1-\epsilon)}. \tag{4}$$

5 Determination of $\frac{d^5\sigma}{dE_{e'}d\Omega_{e'}d\Omega_\pi}$ for A>1.

For the most part, the steps are the same as for hydrogen. However, care must be taken that all variable are calculated in the correct frame, and that all transformations are correct for the case where the photon and target proton are not necessarily collinear in the final lab frame (due to the initial proton momentum).

1. Calculate kinematics in the lab frame $(\nu, t, Q^2, \theta_{\gamma\pi}, \phi_{\gamma\phi}, \epsilon, p_f)$. Except for the addition of p_f, this is the same as in the hydrogen case, and these are the SIMC starting variables.

2. Calculate center of mass values for $p^*_\pi, \theta^*_{\gamma\pi}$, and $\phi^*_{\gamma\pi}$. $\phi^*_{\gamma\pi}$ is no longer identical to $\phi_{\gamma\pi}$, as the boost from the lab frame to the center of mass is no longer along the $\hat{q}$ direction.

3. Determine $\frac{d^2\sigma}{dtd\phi^*}$ from parameterization of cross section. Apply W^2 dependent term.

4. Convert $\frac{d^2\sigma}{dtd\phi^*}$ to $\frac{d^2\sigma}{d\Omega}$. To get $\frac{d^2\sigma}{d\Omega^*}$ (center of mass of photon-pion system), the factor is still:

$$\frac{dt}{d\cos(\theta^*)} = 2q^*p^*_\pi. \tag{5}$$

To convert to the cross section in the proton rest frame, the formula is the same as for the hydrogen case, but with all variables taken in the proton rest frame. However, in order to convert to the real lab frame (nucleus at rest), we need the full jacobian (Appendix J):

$$\begin{vmatrix} \frac{\partial t}{\partial \cos(\theta)} & \frac{\partial t}{\partial \phi} \\ \frac{\partial \phi^*}{\partial \cos(\theta)} & \frac{\partial \phi^*}{\partial \phi} \end{vmatrix}. \tag{6}$$

For hydrogen, this reduced to $\frac{dt}{d\cos(\theta_{lab})}$, since $\frac{\partial\phi^*}{\partial\phi} = 1$ and $\frac{\partial t}{\partial\phi} = \frac{\partial t}{\partial\phi*} = 0$ for the hydrogen case, where the photon and proton are collinear in both the center of mass and lab frames. For A>1, the photon and proton are not collinear in the lab frame, meaning that we need to use the full jacobian (and making the $\frac{dt}{d\cos(\theta_{lab})}$ much more complicated).

5. Convert to $\frac{d^5\sigma}{dE_{e'}d\Omega_{e'}d\Omega_\pi}$ by multiplying by Γ'_T, where Γ'_T is modified from the hydrogen case due to the proton momentum (see Appendix Γ):

$$\Gamma'_T = \frac{\alpha}{2\pi^2}\frac{E'_e}{E_e}\frac{W^2 - m_p^2}{2(E_p - p_p\cos(\theta_{pq}))}\frac{1}{Q^2(1-\epsilon)}. \tag{7}$$

6 Errors in Initial SIMC Pion Model.

There were several things in the original SIMC pion model that were either incorrect in general, or were approximations that do not work well for low pion momentum. The errors are listed as they apply to the steps above.

1. No problem in getting the lab variables. However, in later steps, variables in the lab frame were used when the parameterization expected variables in the proton rest frame.

2. We can write the velocity of the γ-p system as seen in the lab system as $\vec{\beta}_{CM} = \sum \vec{p} / \sum E$. This is the boost we need to apply in order to transform the kinematic variables into the γ-p center of mass frame. The initial version of SIMC applied a boost with the correct magnitude, but applied it along the $\hat{q}$ direction (*i.e.* $\vec{\beta} = \mid \vec{\beta}_{CM} \mid \hat{q}$).

3. $\frac{d^2\sigma}{dt d\phi^*}$ was taken from the parameterization in terms of t, Q^2, θ_{CM}, and ϕ_{CM}. However, due to the incorrect boost direction, the center of mass angles were not quite correct. Because t, Q^2, and W^2 are invariants, they can be calculated in any frame.

4. The cross section was being converted to $\frac{d^2\sigma}{d\Omega}$ using the formula which is takes you to the proton rest frame. However, it uses values of momenta in the real lab frame. Therefore, after applying this conversion, the cross section isn't exactly correct for either the proton rest frame or the nucleus rest frame. In addition, because it was being assumed that this factor took the cross section to the proton rest frame, there was an additional transformation being applied to go from the proton rest frame to deuteron rest frame. It is not clear if the factor being applied was correct or not, but by using the correct $\frac{dt}{d\cos(\theta)}$ and $\frac{d\phi^*}{d\phi}$, we will convert the cross section to the true lab frame, and this additional transformation will not be necessary.

5. The cross section was being converted to $\frac{d^5\sigma}{dE_{e'} d\Omega_{e'} d\Omega_\pi}$ using the same definition of Γ'_T as was used in the hydrogen case. This factor needs to be modified as shown in section 5 (see Appendix Γ).

7 Updated SIMC Model.

The current version of SIMC uses the same pion cross section model, but replaces the approximations above with exact calculations.

1. Particle momenta and angles are calculated in both the lab frame, and the proton rest frame so that both are available.

2. $\vec{\beta}_{CM}$ is now calculated and applied correctly. All variables that were expected to be center of mass for the γ-p system are now center of mass for the proton-photon system, and all 'lab' variables from the hydrogen model are now taken in the proton rest frame.

3. The value of $\frac{d^2\sigma}{dt d\phi^*}$ is now determined from the parameterization in terms of center of mass angles, and W^2 in the proton rest frame, as assumed in the parameterization of the hydrogen data.

4. The transformation is applied with $\frac{dt}{d\cos(\theta)}$ and $\frac{d\phi^2}{d\phi}$, with the proper lab frame θ, and without the assumption that the target proton is at rest. As this now takes us to the real lab frame, there is no additional transformation from the proton rest frame to the nucleus rest frame.

5. SIMC is now using the modified definition of Γ'_T, as shown in section 5.

8 Conclusion and Remaining Work.

At this point, it looks as thought the calculation that converts the hydrogen cross section to a cross section for deuterium and helium is correct. We still are not generating a binding energy for the target nucleon, just using the average binding energy per nucleon (WHEN CONSERVING ENERGY IN GENERATING THE OUTGOING NEUTRON. IS SIMC ALWAYS CONSISTANT IN THE TREATMENT OF BINDING ENERGY AND ENERGY CONSERVATION?), and have not applied any off-shell prescription to the cross section. This is left as an exercise to the reader (we hate readers).

9 Appendix H: $\frac{dt}{d\cos(\theta)}$ for Hydrogen.

The following is from Henk's notes (with slight modifications), and describes the calculation he uses to convert from $\frac{d^2\sigma}{dt d\phi^*}$ to $\frac{d^2\sigma}{d\Omega}$ for the hydrogen case:

For Hydrogen, the photon and proton are collinear in both the center of mass and lab frame. Therefore, $\phi^* = \phi$. This means that

$$\frac{dt d\phi^*}{d\Omega} = \frac{dt}{d\cos\theta}. \tag{8}$$

We now use the fact that

$$t = -Q^2 + m_\pi^2 - 2\nu E_\pi + 2p_\pi q \cos\theta \tag{9}$$

is an invariant and differentiate t with respect to $\cos\theta$. Since in the cm system all momenta and energies do not depend on θ,

$$\frac{dt}{d\cos\theta^*} = 2p_\pi^* q^*. \tag{10}$$

In the lab system one needs to take into account the dependence of E_π and p on $\cos\theta$. Using the triangle relation ($\vec{p_\pi}^2 = (\vec{q} + \vec{p_n})^2$), differentiating with respect to $y = \cos\theta$, and using energy convservation $(m_p + \nu = E_n + E_\pi)$ gives:

$$(m_p + \nu)\frac{dE_\pi}{dy} = \frac{d(p_\pi q \cos\theta)}{dy}. \tag{11}$$

Evaluating the derivative on the right hand side (and replacing p_π by $\sqrt{E_\pi^2 - m_\pi^2}$) gives:

$$(m_p + \nu)\frac{dE_\pi}{dy} = p_\pi q + \frac{E_\pi}{p_\pi} q \cos\theta \frac{dE_\pi}{dy}, \tag{12}$$

and thus

$$\frac{dE_\pi}{dy} = \frac{p_\pi q}{m_p + \nu - \frac{E_\pi}{p_\pi} q \cos\theta}. \tag{13}$$

Now differentiate t to get

$$\frac{dt}{dy} = -2\nu \frac{dE_\pi}{dy} + 2\frac{d(p_\pi q \cos\theta)}{dy}, \tag{14}$$

and use 12 to get

$$\frac{dt}{dy} = 2m_p \frac{dE_\pi}{dy} = \frac{2m_p p_\pi q}{m_p + \nu - \frac{E_\pi}{p_\pi} q \cos(\theta)}. \tag{15}$$

10 Appendix Γ: Γ_T'.

We start by taking the definition of Γ_T', following the convention of Close[3]. Γ_T' is defined in terms of the photon flux factor, K:

$$\Gamma_T' = \frac{K\alpha}{2\pi^2 Q^2}\frac{E'}{E}\frac{1}{1-\epsilon} \tag{16}$$

where $\epsilon^{-1} = \left[1 + 2\frac{Q^2+\nu^2}{Q^2}\tan^2(\frac{\theta}{2})\right]$. For real photons, the photon flux is $K = \nu$. For virtual photons, we take the Hand convention[4] and choose K to be the energy necessary for a real photon to produce a final hadron state of the same mass (the 'equivilent photon energy'). The final hadron state mass (assuming a target proton at rest) is

$$W^2 = m_p^2 + 2m_p\nu - Q^2. \tag{17}$$

For a real photon of energy (and momentum) K,

$$W_\gamma^2 = m_p^2 + 2m_pK. \tag{18}$$

Equating these gives:

$$K = \frac{2m_p\nu - Q^2}{2m_p} = \frac{W^2 - m_p^2}{2m_p}. \tag{19}$$

Using this value of K in equation (16) gives the expression for Γ_T' for pion production from hydrogen (eqn. 4). For A>1 nuclei, we must consider the proton momentum when determining the photon flux. For a real photon with $\nu = q = K$, striking a proton with momentum p_p,

$$W_\gamma^2 = (E_p + \nu)^2 - (\vec{p_p} + \vec{q})^2 = m_p^2 + 2E_pK - 2p_pK\cos(\theta_{pq}). \tag{20}$$

Setting this equal to W^2 for the virtual photon scattering gives

$$K = \frac{W^2 - m_p^2}{2(E_p - p_p\cos(\theta_{pq}))}. \tag{21}$$

and thus

$$\Gamma_T' = \frac{\alpha}{2\pi^2}\frac{E_e'}{E_e}\frac{W^2 - m_p^2}{2(E_p - p_p\cos(\theta_{pq}))}\frac{1}{Q^2(1-\epsilon)}. \tag{22}$$

11 Appendix J: THE Jacobian.

References

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