

## CSCC11 Assignment 4

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### Question 1.

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$$P(L|X_{1:N}) = \frac{P(X_{1:N}|L)P(L)}{P(X_{1:N})} \quad (1)$$

$$= \frac{\prod_{i=1}^N P(X_i|L) \frac{1}{M} \cdot I_{[X_{max}, M]}(L)}{P(X_{1:N})} \quad (2)$$

$$= \frac{(1/(L))^N (1/(M)) \cdot I_{[X_{max}, M]}(L)}{\sum_{i=1}^M (P(X_{1:N}|L=i) \cdot I_{[X_{max}, M]}(i) P(L=i))} \quad (3)$$

$$= \frac{(1/(L))^N \cdot I_{[X_{max}, M]}(L)}{\sum_{i=1}^M (P(X_{1:N}|L=i) \cdot I_{[X_{max}, M]}(i))} \quad \text{Since } P(L=i) = 1/(M) \quad (4)$$

$$= \frac{(1/(L))^N \cdot I_{[X_{max}, M]}(L)}{\sum_{i=1}^M \prod_{j=1}^M P(X_j|L=i) \cdot I_{[X_j, M]}(i)} \quad (5)$$

$$= \frac{(1/(L))^N \cdot I_{[X_{max}, M]}(L)}{\sum_{i=1}^M (1/(i))^N \cdot I_{[X_{max}, M]}(i)} \quad (6)$$

$$= \frac{(1/L)^N \cdot I_{[X_{max}, M]}(L)}{\sum_{i=X_{max}}^M (1/i)^N} \quad (7)$$

### Question 2.

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From prior,

$$L \in [1, M]$$

The only way the posterior can be zero is if

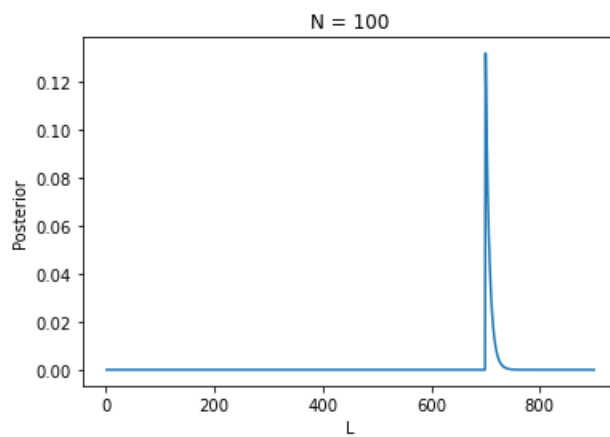
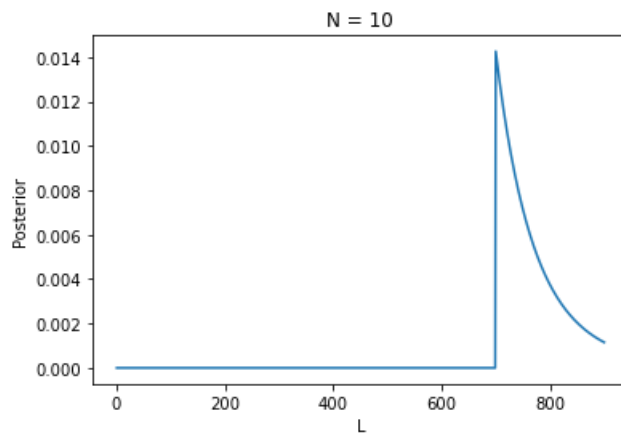
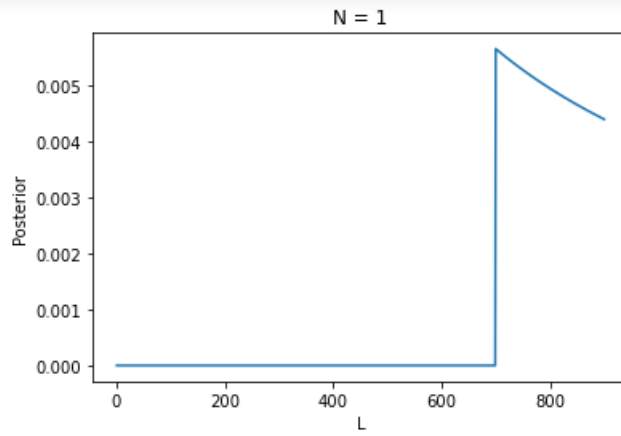
$$I_{[X_{max}, M]}(L) = 0 \implies L \notin [X_{max}, M]$$

So the range of values of L where posterior will be non zero is  $[X_{max}, M]$

### Question 3.

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$$P(L|X_{1:N}) = \frac{1/L^N}{\sum_{i=X_{max}}^M 1/i^N} \quad (8)$$

**Question 4.**

**Question 5.**

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Since the posterior probability is maximized at  $L = M_{max}$ ,

$$L_{MAP} = X_{max}$$

This is perfect intuitive sense, since the probability is spread out uniformly, we want the shortest possible range of possible license plate numbers to maximize the probability of observing each individual one. Therefore,  $L = M_{max}$

**Question 6.**

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Here are the results I calculated for each N

for N = 1  
Posterior mean = 795.7738100301044

for N = 10  
Posterior mean = 760.9800930703402

for N = 100  
Posterior mean = 706.6496652868035

These estimates are more reasonable than the posterior mode (MAP), Since the posterior distribution is heavily skewed, it might be better to estimate using the mean than the mode. Furthermore, It seems that the mean approaches the mode fairly quickly as  $n \rightarrow \infty$ , so for large  $n$ , this distinction should not matter.

**Question 7.**

$$P(X_{n+1}|X_{1:N}) = \frac{P(X_{N+1}|L, X_{1:N})P(L|X_{1:N})}{P(L|X_{N+1}, X_{1:N})}$$

$$\text{Since } P(X|Y, Z) = \frac{P(Y|X, Z)P(X|Z)}{P(Y|Z)}$$

(9)

$$= \frac{P(X_{N+1}|L)P(L|X_{1:N})}{P(L|X_{N+1}, X_{1:N})} \quad (10)$$

$$= \frac{\frac{1}{L} \cdot I_{[X_{N+1}, M]}(L) \cdot P(L|X_{1:N})}{P(L|X_{N+1}, X_{1:N})} \quad (11)$$

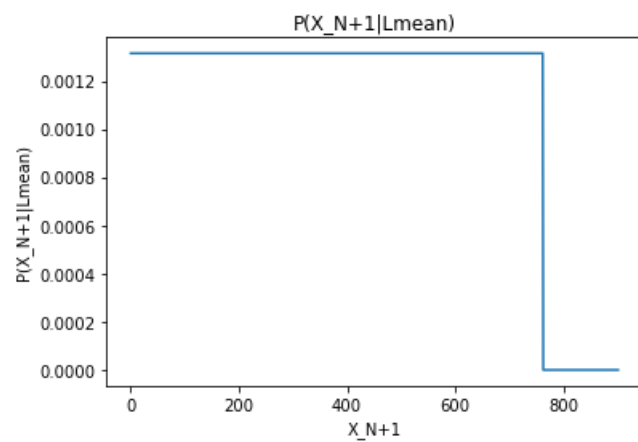
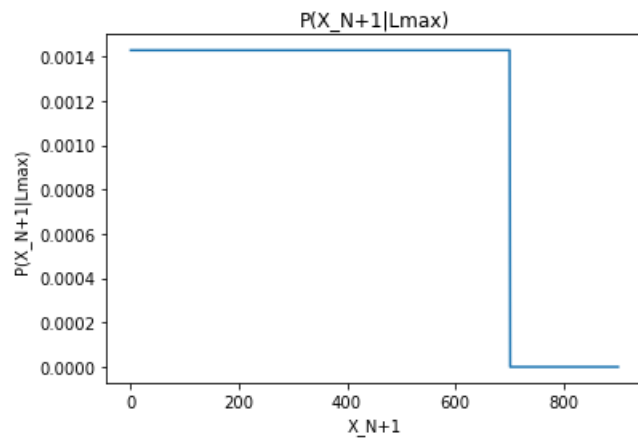
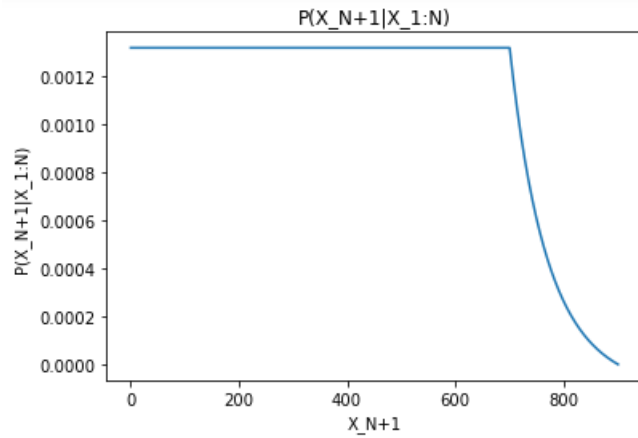
$$= \frac{\frac{1}{L} \cdot I_{[X_{N+1}, M]}(L) \cdot P(L|X_{1:N})}{\frac{1/L^{N+1} \cdot I_{[X_{max}(X_{max}, X_{N+1}), M]}(L)}{\sum_{i=max(X_{max}, X_{N+1})}^M 1/i^{N+1}}} \quad (12)$$

$$= \frac{\frac{1}{L} \cdot I_{[X_{N+1}, M]}(L) \cdot \frac{1/L^N \cdot I_{[X_{max}, M]}(L)}{\sum_{i=X_{max}}^M 1/i^N}}{\frac{1/L^{N+1} \cdot I_{[max(X_{max}, X_{N+1}), M]}(L)}{\sum_{i=max(X_{max}, X_{N+1})}^M 1/i^{N+1}}} \quad (13)$$

$$= \frac{\frac{1}{L} \cdot I_{[X_{N+1}, M]}(L) \cdot I_{[X_{max}, M]}(L)}{\frac{1}{L} I_{[max(X_{max}, X_{N+1}), M]}(L)} \cdot \frac{\sum_{i=max(X_{max}, X_{N+1})}^M 1/i^{N+1}}{\sum_{i=X_{max}}^M 1/i^N} \quad (14)$$

$$= \frac{I_{[max(X_{max}, X_{N+1}), M]}(L)}{I_{[max(X_{max}, X_{N+1}), M]}(L)} \cdot \frac{\sum_{i=max(X_{max}, X_{N+1})}^M 1/i^{N+1}}{\sum_{i=X_{max}}^M 1/i^N} \quad (15)$$

$$= \frac{\sum_{i=max(X_{max}, X_{N+1})}^M 1/i^{N+1}}{\sum_{i=X_{max}}^M 1/i^N} \quad (16)$$

**Question 8.**

**Question 9.**

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The  $P(X_{N+1}|X_{1:N})$  is the most consistent with my intuition. Based on my derivation of  $P(X_{N+1}|X_{1:N})$ , there is a 3.078209% chance that the next car has a license plate greater than 750. Therefore I would not take the bet as it's very likely I would lose.