CSCC11 Assignment 4

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Question 1.

$$
\begin{aligned}
P(L|X_{1:N}) &= \frac{P(X_{1:N}|L)P(L)}{P(X_{1:N})} && (1)\\
&= \frac{\Pi_{i=1}^{N} P(X_i|L)\frac{1}{M} \cdot I_{[X_{max},M]}(L)}{P(X_{1:N})} && (2)\\
&= \frac{(1/(L))^N(1/(M)) \cdot I_{[X_{max},M]}(L)}{\Sigma_{i=1}^{M}(P(X_{1:N}|L=i) \cdot I_{[X_{max},M]}(i)P(L=i))} && (3)\\
&= \frac{(1/(L))^N \cdot I_{[X_{max},M]}(L)}{\Sigma_{i=1}^{M}(P(X_{1:N}|L=i) \cdot I_{[X_{max},M]}(i))} && \text{Since } P(L=i)=1/(M) \quad (4)\\
&= \frac{(1/(L))^N \cdot I_{[X_{max},M]}(L)}{\Sigma_{i=1}^{M}\Pi_{j=1}^{M} P(X_j|L=i) \cdot I_{[X_j,M]}(i)} && (5)\\
&= \frac{(1/(L))^N \cdot I_{[X_{max},M]}(L)}{\Sigma_{i=1}^{M}(1/(i))^N \cdot I_{[X_{max},M]}(i)} && (6)\\
&= \frac{(1/L)^N \cdot I_{[X_{max},M]}(L)}{\Sigma_{i=X_{max}}^{M}(1/i)^N} && (7)
\end{aligned}
$$

Question 2.

From prior,

$$L \in [1, M]$$

The only way the posterior can be zero is if

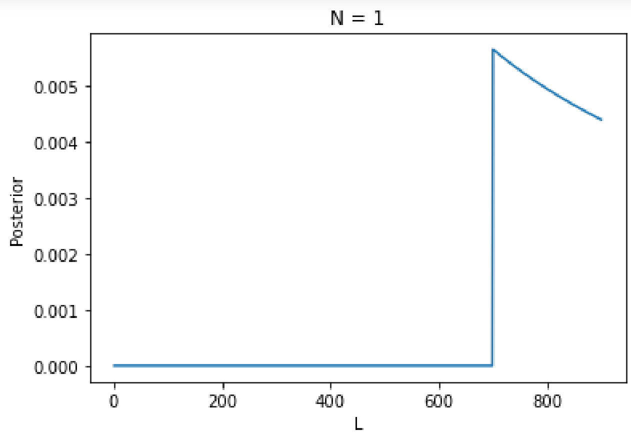
$$I_{[X_{max},M]}(L) = 0 \implies L \notin [X_{max}, M]$$

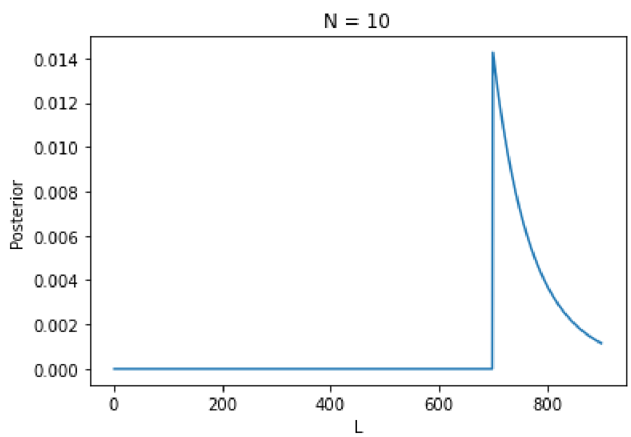
So the range of values of L where posterior will be non zero is $[X_{max}, M]$

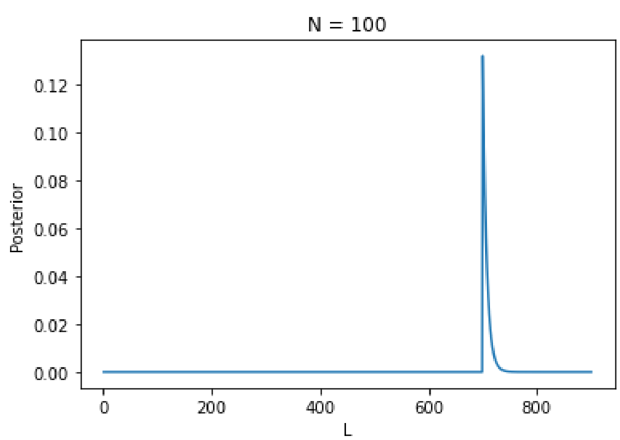
Question 3.

$$P(L|X_{1:N}) = \frac{1/L^N}{\Sigma_{i=X_{max}}^{M} 1/i^N} \tag{8}$$

Question 4.


N = 1
Posterior
0.005
0.004
0.003
0.002
0.001
0.000
0
200
400
600
800
L


N = 10
Posterior
0.014
0.012
0.010
0.008
0.006
0.004
0.002
0.000
0
200
400
600
800
L


N = 100
Posterior
0.12
0.10
0.08
0.06
0.04
0.02
0.00
0
200
400
600
800
L

Question 5.

Since the posterior probability is maximized at $L = M_{max}$,

$$L_{MAP} = X_{max}$$

This is perfect intuitive sense, since the probability is spread out uniformly, we want the shortest possible range of possible license plate numbers to maximize the probability of observing each individual one. Therefor, $L = M_{max}$

Question 6.

Here are the results I calculated for each N

```
for N = 1
Posterior mean = 795.7738100301044

for N = 10
Posterior mean = 760.9800930703402

for N = 100
Posterior mean = 706.6496652868035
```

These estimates are more reasonable than the posterior mode (MAP), Since the posterior distribution is heavily skewed, it might be better to estimate using the mean than the mode.
Furthermore, It seems that the mean approaches the mode fairly quickly as $n \longrightarrow \infty$, so for large n, this distinction should not matter.

Question 7.

$$P(X_{n+1}|X_{1:N}) = \frac{P(X_{N+1}|L, X_{1:N})P(L|X_{1:N})}{P(L|X_{N+1}, X_{1:N})} \qquad \text{Since } P(X|Y,Z) = \frac{P(Y|X,Z)P(X|Z)}{P(Y|Z)} \tag{9}$$

$$= \frac{P(X_{N+1}|L)P(L|X_{1:N})}{P(L|X_{N+1}, X_{1:N})} \tag{10}$$

$$= \frac{\frac{1}{L} \cdot I_{[X_{N+1},M](L)} \cdot P(L|X_{1:N})}{P(L|X_{N+1}, X_{1:N})} \tag{11}$$

$$= \frac{\frac{1}{L} \cdot I_{[X_{N+1},M](L)} \cdot P(L|X_{1:N})}{\frac{1/L^{N+1} \cdot I_{[X_{max(X_{max},X_{N+1})},M]}(L)}{\Sigma_{i=max(X_{max},X_{N+1})}^{M} 1/i^{N+1}}} \tag{12}$$

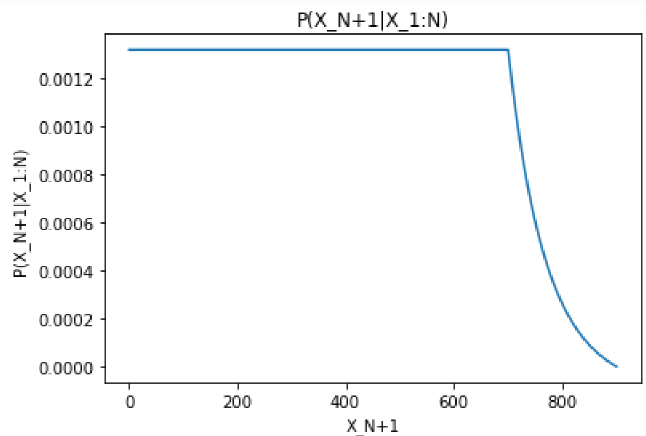
$$= \frac{\frac{1}{L} \cdot I_{[X_{N+1},M](L)} \cdot \frac{1/L^{N} \cdot I_{[X_{max},M]}(L)}{\Sigma_{i=X_{max}}^{M} 1/i^{N}}}{\frac{1/L^{N+1} \cdot I_{[max(X_{max},X_{N+1}),M]}(L)}{\Sigma_{i=max(X_{max},X_{N+1})}^{M} 1/i^{N+1}}} \tag{13}$$

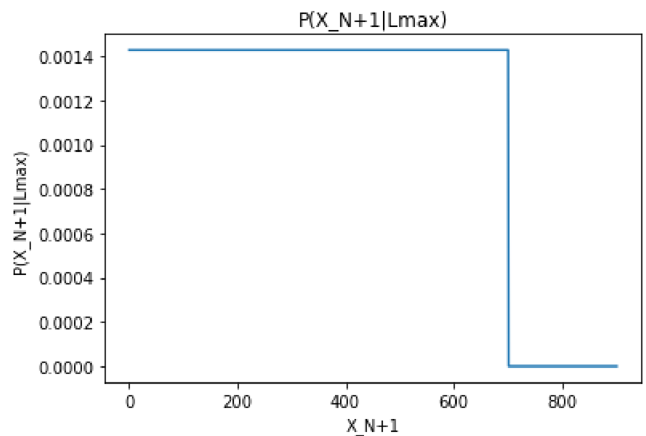
$$= \frac{\frac{1}{L} \cdot I_{[X_{N+1},M](L)} \cdot I_{[X_{max},M]}(L)}{\frac{1}{L} I_{[max(X_{max},X_{N+1}),M]}(L)} \cdot \frac{\Sigma_{i=max(X_{max},X_{N+1})}^{M} 1/i^{N+1}}{\Sigma_{i=X_{max}}^{M} 1/i^{N}} \tag{14}$$

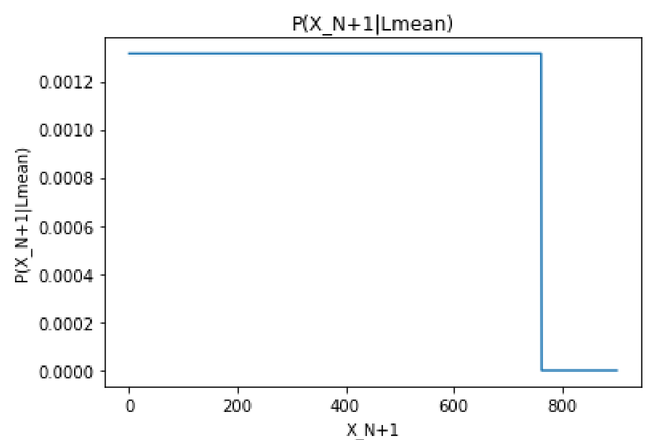
$$= \frac{I_{[max(X_{max},X_{N+1}),M]}(L)}{I_{[max(X_{max},X_{N+1}),M]}(L)} \cdot \frac{\Sigma_{i=max(X_{max},X_{N+1})}^{M} 1/i^{N+1}}{\Sigma_{i=X_{max}}^{M} 1/i^{N}} \tag{15}$$

$$= \frac{\Sigma_{i=max(X_{max},X_{N+1})}^{M} 1/i^{N+1}}{\Sigma_{i=X_{max}}^{M} 1/i^{N}} \tag{16}$$

Question 8.


P(X_N+1|X_1:N)
P(X_N+1|X_1:N)
0.0000
0.0002
0.0004
0.0006
0.0008
0.0010
0.0012
0
200
400
600
800
X_N+1


P(X_N+1|Lmax)
P(X_N+1|Lmax)
0.0000
0.0002
0.0004
0.0006
0.0008
0.0010
0.0012
0.0014
0
200
400
600
800
X_N+1


P(X_N+1|Lmean)
P(X_N+1|Lmean)
0.0000
0.0002
0.0004
0.0006
0.0008
0.0010
0.0012
0
200
400
600
800
X_N+1

Question 9.

The $P(X_{N+1}|X_{1:N})$ is the most consistent with my intuition. Based on my derivation of $P(X_{N+1}|X_{1:N})$, there is a 0.059% that the next car has a license plate greater than 750. Therefor i would not take the bet as its very likily i would lose.