HW3 report

1.

(a)

x_i	f_i	$f[x_i, x_{i+1}]$	$f[x_i, \dots, x_{i+2}]$	$f[x_i, \dots, x_{i+3}]$	$f[x_i, \dots, x_{i+4}]$
-0.2	1.23	2.22	-11.883	-103.58	73.6
0.3	2.34	-8.475	-1.525	-81.5	
0.7	-1.05	-7.56	14.775		
-0.3	6.51	-16.425			
0.1	-0.06				

(b) $P(x)$ is the interpolation function for $f(x)$

$\Rightarrow P_2(x) = 1.23 + 2.22(x+0.2) - 11.883(x+0.2)(x-0.3)$

(using first three points)

$P_2(0.4) = 1.23 + 1.332 - 0.71298 = 1.84902$

(c)

I think if we want to interpolate $f(x)$ near $x=0.4$, we should pick three nearest points to $x=0.4$ to minimize error, so I pick $x=0.1, 0.3, 0.7$ to interpolate $f(0.4)$. But we need to reconstruct a different divided-difference table, for convenience, I only use three points

x_i	f_i	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$
0.1	-0.06	12	-34.125
0.3	2.34	-8.475	
0.7	-1.05		

$\Rightarrow P_2(x) = -0.06 + 12(x-0.1) - 34.125(x-0.1)(x-0.3)$

$\Rightarrow P_2(0.4) = -0.06 + 3.6 - 1.02375 = 2.51625$

2. To construct a cubic spline, we can use $HS = Y$ to compute its coefficients

$\Rightarrow$ find a natural spline: $S_0 = S_4 = 0$

$\Rightarrow h_0 = h_1 = h_2 = h_3 = 0.5$, $f[-1,-0.5] = 0$, $f[-0.5,0] = 2$, $f[0,0.5] = -2$, $f[0.5,1] = 0$

$$\Rightarrow \begin{bmatrix} 2 & 0.5 & 0 \\ 0.5 & 2 & 0.5 \\ 0 & 0.5 & 2 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} 12 \\ -24 \\ 12 \end{bmatrix}$$

$\Rightarrow S_1 \doteq 10.29$, $S_2 \doteq -17.14$, $S_3 \doteq 10.29$

$\Rightarrow g_0(x) = 1.715(x+1)^3 - 0.8575(x+1)$, $g_1(x) = -4.572(x+0.5)^3 + 5.145(x+0.5)^2$ $1.713(x+0.5)$, $g_2(x) = 4.572x^3 - 8.57x^2 - 0.0008x + 1$,

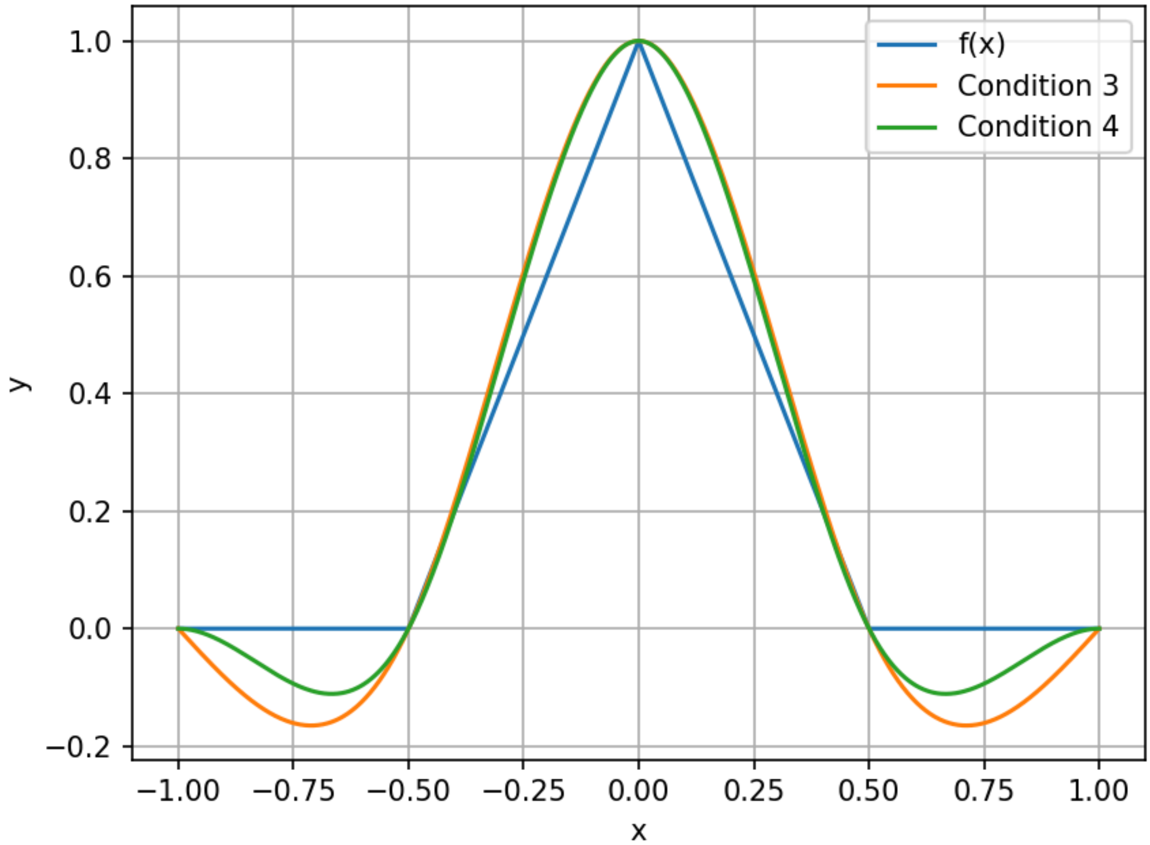
$g_3(x) = -1.715(x-0.5)^3 + 5.145(x-0.5)^2 - 1.715(x-0.5)$.

I use python to implement condition 3, 4 spline

$\Rightarrow$ condition 3: $S_0 = S_1$, $S_4 = S_3$

condition 4: S_0, S_4 are linear extrapolations

$\Rightarrow$ I find I can choose spline type = "natural" to achieve condition 3. And spline type = "clamped" to achieve condition 4. I find that end condition 4 gives the best fit



Result for Problem 2

3. (a) $P_i(u) = u^T M p = [u^3\ u^2\ u\ 1] \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{bmatrix} = \begin{bmatrix} (1-u)^3 \\ 3u(1-u)^2 \\ 3u^2(1-u) \\ u^3 \end{bmatrix}^T \begin{bmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{bmatrix}$

$\Rightarrow$ use two interior points as controls

$$\Rightarrow P(u) = \begin{bmatrix} (1-u)^3 \\ 3u(1-u)^2 \\ 3u^2(1-u) \\ u^3 \end{bmatrix}^T \begin{bmatrix} (0,0) \\ (1,0.3) \\ (2,1.7) \\ (3,1.5) \end{bmatrix} \Rightarrow \begin{array}{l} x(u) = 3u \quad (u \in [0,1]) \\ y(u) = -2.7u^3 + 3.3u^2 + 0.9u \end{array}$$

I use python to plot zigzag and Bezier curve

(b)

Moved vertically $\Rightarrow (1,0.3) \to (1,a)$, $(2,1.7) \to (2,b)$

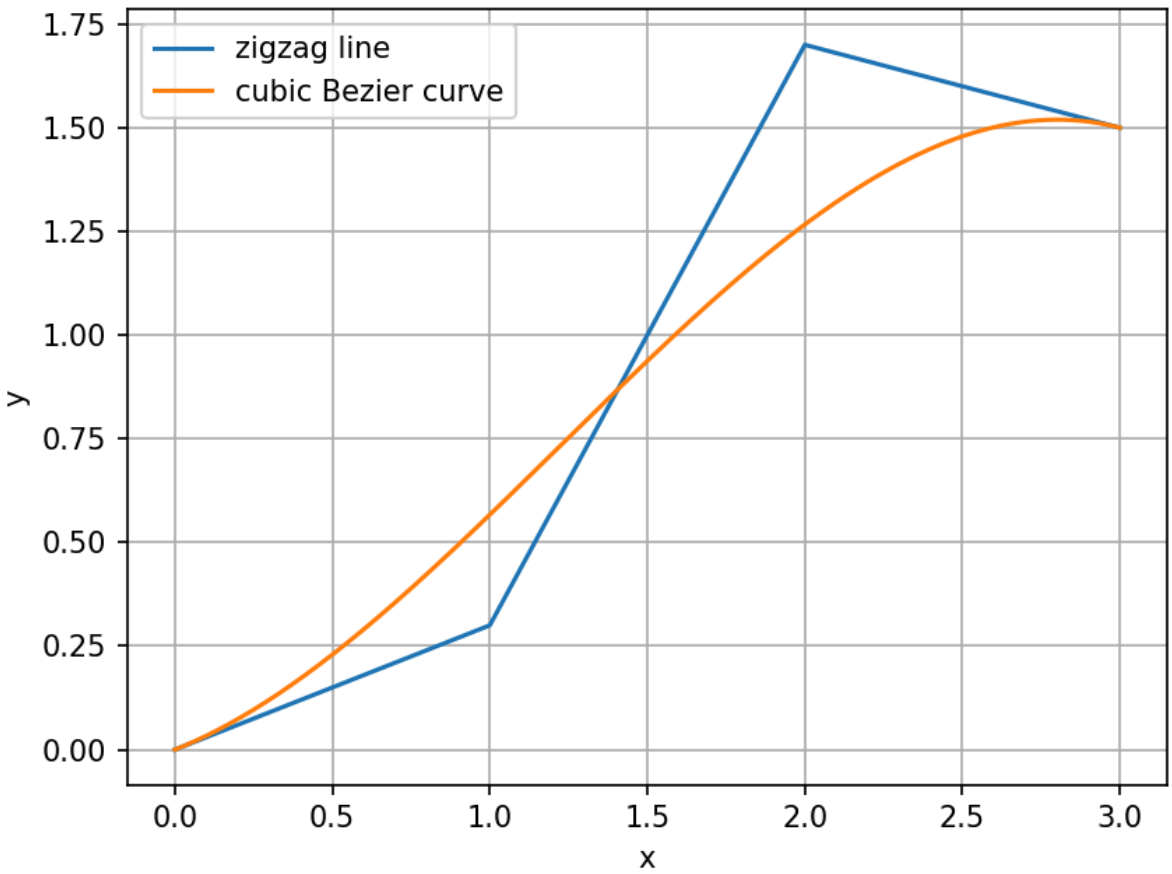
$$\Rightarrow \tilde{P}(u) = \begin{bmatrix} (1-u)^3 \\ 3u(1-u)^2 \\ 3u^2(1-u) \\ u^3 \end{bmatrix}^T \begin{bmatrix} (0,0) \\ (1,a) \\ (2,b) \\ (3,1.5) \end{bmatrix} \Rightarrow \begin{array}{l} \tilde{x}(u) = 3u \\ \tilde{y}(u) = (3a-3b+1.5)u^3 + (3b-6a)u^2 \\ \quad + 3au \end{array}$$

$\Rightarrow$ To pass through all of the original four points

$\Rightarrow$ check $u = 0, \frac{1}{3}, \frac{2}{3}, 1$

$$\Rightarrow \begin{cases} \dfrac{3a-3b+1.5}{27} + \dfrac{3b-6a}{9} + a = 0.3 \\ \dfrac{8(3a-3b+1.5)}{27} + \dfrac{4(3b-6a)}{9} + 2a = 1.7 \end{cases} \Rightarrow \begin{cases} a = -\dfrac{59}{60} \\ b = \dfrac{199}{60} \end{cases}$$

$\Rightarrow$ So $(1,0.3) \to (1, -\frac{59}{60})$, $(2,1.7) \to (2, \frac{199}{60})$



Result for Problem 3(a)

4. Construct the rectangle array of 16 points nearest to $(2.8, 0.54) \Rightarrow$

$$\Rightarrow \begin{bmatrix} (1.3, 0.2, 2.521) & (1.3, 0.4, 2.792) & (1.3, 0.5, 2.949) & (1.3, 0.7, 3.314) \\ (2.5, 0.2, 3.721) & (2.5, 0.4, 3.992) & (2.5, 0.5, 4.149) & (2.5, 0.7, 4.514) \\ (3.1, 0.2, 4.321) & (3.1, 0.4, 4.592) & (3.1, 0.5, 4.749) & (3.1, 0.7, 5.114) \\ (4.7, 0.2, 5.921) & (4.7, 0.4, 6.192) & (4.7, 0.5, 6.349) & (4.7, 0.7, 6.714) \end{bmatrix}$$

$\Rightarrow$ Construct B-spline surface $\Rightarrow X_{ij}(u,v) = \frac{1}{36} u^T M X_{ij} M^T v$,

and $Y_{ij}(u,v)$, $Z_{ij}(u,v)$ can be computed similarly

$$\Rightarrow u = \begin{bmatrix} u^3 \\ u^2 \\ u \\ 1 \end{bmatrix}, \quad M = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}, \quad M^T = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 0 & 4 \\ -3 & 3 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} v^3 \\ v^2 \\ v \\ 1 \end{bmatrix}$$

$$\Rightarrow X_{ij}(u,v) = u^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1.3 & 1.3 & 1.3 & 1.3 \\ 2.5 & 2.5 & 2.5 & 2.5 \\ 3.1 & 3.1 & 3.1 & 3.1 \\ 4.7 & 4.7 & 4.7 & 4.7 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 0 & 4 \\ -3 & 3 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} V \times \frac{1}{36}$$

$$= u^T \begin{bmatrix} 0 & 0 & 0 & 9.6 \\ 0 & 0 & 0 & -10.8 \\ 0 & 0 & 0 & 32.4 \\ 0 & 0 & 0 & 86.4 \end{bmatrix} V \times \frac{1}{36} = \frac{9.6u^3 - 10.8u^2 + 32.4u + 86.4}{36} = 2.8$$

$$\Rightarrow u \approx 0.4896 \Rightarrow Y_{ij}(u,v) = u^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.2 & 0.4 & 0.5 & 0.7 \\ 0.2 & 0.4 & 0.5 & 0.7 \\ 0.2 & 0.4 & 0.5 & 0.7 \\ 0.2 & 0.4 & 0.5 & 0.7 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 0 & 4 \\ -3 & 3 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} V \times \frac{1}{36} = u^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1.2 & -1.8 & 5.4 & 13.8 \end{bmatrix} V \times \frac{1}{36} = \frac{1.2v^3 - 1.8v^2 + 5.4v + 13.8}{36} = 0.54$$

$$\Rightarrow v \approx 1.1476$$

$$Z_{ij}(u,v) = u^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2.521 & 2.792 & 2.949 & 3.314 \\ 3.721 & 3.992 & 4.149 & 4.514 \\ 4.321 & 4.592 & 4.749 & 5.114 \\ 5.921 & 6.192 & 6.349 & 6.714 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 0 & 4 \\ -3 & 3 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} V \times \frac{1}{36}$$

$$= u^T \begin{bmatrix} 0 & 0 & 0 & 9.6 \\ 0 & 0 & 0 & -10.8 \\ 0 & 0 & 0 & 32.4 \\ 1.932 & -2.052 & 7.104 & 139.428 \end{bmatrix} V \times \frac{1}{36}$$ ，代入 $u = 0.4896$, $V = 1.1476$

$$\Rightarrow Z(2.8, 0.54) = 4.5247$$

5\. (a)

$$A = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \\ x_5 & y_5 & 1 \\ x_6 & y_6 & 1 \\ x_7 & y_7 & 1 \end{bmatrix} \quad x = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad B = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{bmatrix}$$

$$\Rightarrow A^T A x = A^T B \Rightarrow \begin{bmatrix} 0.4 & 1.2 & 3.4 & 4.1 & 5.7 & 7.2 & 9.3 \\ 0.7 & 2.1 & 4 & 4.9 & 6.3 & 8.1 & 8.9 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.4 & 0.7 & 1 \\ 1.2 & 2.1 & 1 \\ 3.4 & 4 & 1 \\ 4.1 & 4.9 & 1 \\ 5.7 & 6.3 & 1 \\ 7.2 & 8.1 & 1 \\ 9.3 & 8.9 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 & 1.2 & 3.4 & 4.1 & 5.7 & 7.2 & 9.3 \\ 0.7 & 2.1 & 4 & 4.9 & 6.3 & 8.1 & 8.9 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.031 \\ 0.933 \\ 3.058 \\ 3.349 \\ 4.87 \\ 5.757 \\ 8.921 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 200.79 & 213.49 & 31.3 \\ 213.49 & 229.42 & 35 \\ 31.3 & 35 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 177.4348 \\ 187.3327 \\ 26.919 \end{bmatrix}$$

$\Rightarrow$ normal equations $\Rightarrow$ $\begin{cases} 200.79a + 213.49b + 31.3c = 177.4348 \\ 213.49a + 229.42b + 35c = 187.3327 \\ 31.3a + 35b + 7c = 26.919 \end{cases}$

(b) Solve $\begin{cases} 200.79a + 213.49b + 31.3c = 177.4348 \\ 213.49a + 229.42b + 35c = 187.3327 \\ 31.3a + 35b + 7c = 26.919 \end{cases}$

$\Rightarrow$ get $\begin{cases} a \doteqdot 1.5961 \\ b \doteqdot -0.7024 \\ c \doteqdot 0.2207 \end{cases}$ $\Rightarrow$ The fitting plane is

$\Rightarrow z = 1.5961x - 0.7024y + 0.2207$

(c) $SSD = \sum_{i=1}^{7} (z_i - \tilde{z}_i)^2$, where $\tilde{z}_i$ is estimated by x_i, y_i, and the plane $z = 1.5961x - 0.7024y + 0.2207$

$\Rightarrow SSD \doteqdot 0.3194$

6. $\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots$

$\approx 1 - \frac{x^2}{2} + \frac{x^4}{24}$ (power series)

Chebyshev series $\Rightarrow 1 - \frac{1}{2} \cdot \frac{1}{2}(T_2 + T_0) + \frac{1}{24} \cdot \frac{1}{8}(T_4 + 4T_2 + 3T_0)$

$- \frac{1}{23040} \cdot (10T_0 + 15T_2 + 6T_4 + T_6) + \cdots$

$\approx 1 - 0.25T_2 - 0.25T_0 + 0.0052T_4 + 0.0208T_2 + 0.015625T_0$

$- 0.000434T_0 - 0.00651T_2 - 0.00026T_4 - 0.0000434T_6$

$\doteq 1 - 0.2348T_0 - 0.2357T_2 + 0.005T_4$

$\Rightarrow$ convert into power series in x

$\Rightarrow 1 - 0.2348 - 0.2357(2x^2 - 1) + 0.005(8x^4 - 8x^2 + 1)$

$= 0.04x^4 - 0.5114x^2 + 1.0059$

I compare the error of both Chebyshev series and power series from $x = -1, -0.8, -0.6, \cdots, 0.8, 1$

X	Chebyshev error	power error
-1	0.005802305868139679	0.0013643607985268646
-0.8	0.001718709347165448	0.0003599573195012251
-0.6	0.0016443850903216095	6.43850903216947e-05
-0.4	0.004039005997114931	5.672663781486342e-06
-0.2	0.005441422158758313	8.882542501531532e-08
0	0.005900000000000016	0.0
0.2	0.005441422158758313	8.882542501531532e-08
0.4	0.004039005997114931	5.672663781486342e-06
0.6	0.0016443850903216095	6.43850903216947e-05
0.8	0.001718709347165448	0.0003599573195012251
1	0.005802305868139679	0.0013643607985268646

Result for Problem 6

7. $f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{2n\pi x}{p}\right) + B_n \sin\left(\frac{2n\pi x}{P}\right)\right]$, $P = 3$

$$\Rightarrow \int_{-1}^{2} f(x)\,dx = \int_{-1}^{2} \frac{A_0}{2}\,dx \Rightarrow \frac{x^3}{3} - x \Big|_{-1}^{2} = \frac{3}{2}A_0$$

$$\Rightarrow A_0 = \frac{2}{3}\cdot\left(\frac{2}{3} - \frac{2}{3}\right) = 0$$

$$\Rightarrow A_n = \frac{2}{3}\int_{-1}^{2} (x^2-1)\cos\left(\frac{2n\pi x}{3}\right) dx$$

$$= \frac{2}{3}\cdot\left((x^2-1)\cdot\sin\left(\frac{2n\pi x}{3}\right)\cdot\frac{3}{2n\pi}\Big|_{-1}^{2} - \int_{-1}^{2} 2x\cdot\sin\left(\frac{2n\pi x}{3}\right) dx\right)$$

$$= \frac{2}{3}\cdot\left((x^2-1)\cdot\sin\left(\frac{2n\pi x}{3}\right)\cdot\frac{3}{2n\pi}\Big|_{-1}^{2} - \left(-2x\cdot\cos\left(\frac{2n\pi x}{3}\right)\cdot\frac{3}{2n\pi}\Big|_{-1}^{2} - \int_{-1}^{2} 2\cdot\cos\left(\frac{2n\pi x}{3}\right)dx\right)\right)$$

$$= \frac{2}{3}\left(\frac{9}{2n\pi}\sin\left(\frac{4n\pi}{3}\right) - \left(\left(-4\cos\left(\frac{4n\pi}{3}\right) - 2\cos\left(\frac{2n\pi}{3}\right)\right)\times\frac{3}{2n\pi} - \frac{3}{n\pi}\left(\sin\frac{4n\pi}{3} + \sin\frac{2n\pi}{3}\right)\right)\right)$$

$$= \frac{2}{3}\left(\frac{9}{2n\pi}\cdot\sin\left(\frac{4n\pi}{3}\right) + \frac{3}{n\pi}\left(2\cos\left(\frac{4n\pi}{3}\right) + \cos\left(\frac{2n\pi}{3}\right) + \sin\left(\frac{4n\pi}{3}\right) + \sin\left(\frac{2n\pi}{3}\right)\right)\right)$$

$$= \frac{2}{n\pi}\left(2\cos\left(\frac{4n\pi}{3}\right) + \cos\left(\frac{2n\pi}{3}\right) + \frac{5}{2}\sin\left(\frac{4n\pi}{3}\right) + \sin\left(\frac{2n\pi}{3}\right)\right) \text{ for } n \geq 1$$

$$\Rightarrow B_n = \frac{2}{3}\int_{-1}^{2}(x^2-1)\sin\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left(-(x^2-1)\cdot\cos\left(\frac{2n\pi x}{3}\right)\cdot\frac{3}{2n\pi}\Big|_{-1}^{2} - \left(2x\cdot\sin\left(\frac{2n\pi x}{3}\right)\cdot\frac{3}{2n\pi}\Big|_{-1}^{2} - \int_{-1}^{2} 2\sin\left(\frac{2n\pi x}{3}\right)dx\right)\right)$$

$$= \frac{2}{3}\left(\frac{-9}{2n\pi}\cos\left(\frac{4n\pi}{3}\right) - \left(\left(4\sin\left(\frac{4n\pi}{3}\right) + 2\sin\left(\frac{2n\pi}{3}\right)\right)\cdot\frac{3}{2n\pi} - \frac{3}{n\pi}\left(-\cos\left(\frac{4\pi n}{3}\right) + \cos\left(\frac{2n\pi}{3}\right)\right)\right)\right)$$

$$= \frac{2}{3}\left(\frac{-9}{2n\pi}\cos\left(\frac{4n\pi}{3}\right) + \frac{3}{n\pi}\left(\cos\left(\frac{2n\pi}{3}\right) - \cos\left(\frac{4n\pi}{3}\right) - 2\sin\left(\frac{4n\pi}{3}\right) - \sin\left(\frac{2n\pi}{3}\right)\right)\right)$$

$$= \frac{2}{n\pi}\left(\cos\left(\frac{2n\pi}{3}\right) - \frac{5}{2}\cos\left(\frac{4n\pi}{3}\right) - 2\sin\left(\frac{4n\pi}{3}\right) - \sin\left(\frac{2n\pi}{3}\right)\right) \text{ for } n \geq 1$$