HW4 report

1. The modified Euler method: $y_{n+1} = y_n + \frac{h}{2}\left(f(x_n, y_n) + f(x_{n+1}, \tilde{y}_{n+1})\right)$,

where $\tilde{y}_{n+1} = y_n + h f(x_n, y_n)$, and $f(x, y) = \sin x + y$.

I set $h = 0.01$ and use python to implement iterations.

$\Rightarrow y(0.1) \fallingdotseq 2.21550$, $y(0.5) \fallingdotseq 3.44326$

2. $\frac{dx}{dt} = f(x, t) \Rightarrow \int_{t_n}^{t_{n+1}} f(x, t)\, dt = X(t_{n+1}) - X(t_n) = X_{n+1} - X_n$

let $\int_{t_n}^{t_{n+1}} f(x, t)\, dt \approx C_0 f(x_{n-1}, t_{n-1}) + C_1 f(x_n, t_n)$

$\Rightarrow \int_{t_n}^{t_{n+1}} f(t)\, dt \approx C_0 f_{n-1} + C_1 f_n$, interpolate 2 points

$(0, f_n), (-h, f_{n-1}) \Rightarrow f(t) = 1, \int_{t_n}^{t_{n+1}} 1 \cdot dt = h = C_0 + C_1$

$f(t) = t, \int_{t_n}^{t_{n+1}} t\, dt = \frac{h^2}{2} = C_0(-h) + C_1(0)$

$\Rightarrow C_0 = -\frac{h}{2}, C_1 = \frac{3}{2}h \Rightarrow X_{n+1} - X_n \approx \frac{h}{2}\left[-f_{n-1} + 3f_n\right]$

$\Rightarrow X_{n+1} \approx X_n + \frac{h}{2}\left[-f_{n-1} + 3f_n\right]$

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2.215502979492508
3.443257441278822
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Result of Problem 1

3. (a) First let $x_1 = y,\ x_2 = y',\ x_3 = y'' \Rightarrow dx = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ t + 2x_1 - x_2 t \end{bmatrix}$

And I use matlab ode45 to implement RKF $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$\Rightarrow y(0.2) \fallingdotseq 0.200133,\ y(0.4) \fallingdotseq 0.402132,\ y(0.6) \fallingdotseq 0.610778$

(b) Set $h = 0.2$.

$$X_{0.8} = X_{0.6} + \frac{0.2}{24}(55f_{0.6} - 59f_{0.4} + 37f_{0.2} - 9f_0) = \begin{bmatrix} 0.8340 \\ 1.1694 \\ 0.6296 \end{bmatrix}$$

$\Rightarrow$ we get $f_{0.8}$ by $dx = \begin{bmatrix} 1.1694 \\ 0.6296 \\ 1.5325 \end{bmatrix}$

$$\Rightarrow \tilde{X}_{0.8} = X_{0.8} + \frac{0.2}{24}(9f_{0.8} + 19f_{0.6} - 5f_{0.4} + f_{0.2}) = \begin{bmatrix} 0.8340 \\ 1.1692 \\ 0.6291 \end{bmatrix}$$

$$X_{1.0} = X_{0.8} + \frac{0.2}{24}(55f_{0.8} - 59f_{0.6} + 37f_{0.4} - 9f_{0.2}) = \begin{bmatrix} 1.0827 \\ 1.3284 \\ 0.9679 \end{bmatrix}$$

$\Rightarrow$ we get $f_{1.0}$ by $dx = \begin{bmatrix} 1.3284 \\ 0.9679 \\ 1.8370 \end{bmatrix}$

$$\Rightarrow \hat{X}_{1.0} = X_{0.8} + \frac{0.2}{24}(9f_{1.0} + 19f_{0.8} - 5f_{0.6} + f_{0.4}) = \begin{bmatrix} 1.0826 \\ 1.3281 \\ 0.9675 \end{bmatrix}$$

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y(0.2) = 0.200133
y(0.4) = 0.402132
y(0.6) = 0.610778
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Result of Problem 3 (a)

4. $[0, \frac{\pi}{2}]$ is divided into four subintervals.

$$\Rightarrow \quad \begin{array}{ccccc} x_0 & x_1 & x_2 & x_3 & x_4 \\ 0 & \frac{\pi}{8} & \frac{\pi}{4} & \frac{3\pi}{8} & \frac{\pi}{2} \end{array}, \quad h = \frac{\pi}{8}$$

$\Rightarrow y''(x_i) \approx \dfrac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$ and by $y'' + y = 0$ we get

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + y_i = 0 \Rightarrow y_{i+1} - (2-h^2)y_i + y_{i-1} = 0$$

$\Rightarrow$ To deal with boundary condition:

$$x = 0 \Rightarrow \frac{y_1 - y_{-1}}{2h} + y_0 = 2 \Rightarrow y_{-1} = (2-h^2)y_0 - y_1$$

$$x = \frac{\pi}{2} \Rightarrow \frac{y_5 - y_3}{2h} + y_4 = -1 \Rightarrow y_5 = (2-h^2)y_4 - y_3$$

$$\Rightarrow \begin{bmatrix} -\frac{(2+2h-h^2)}{2h} & \frac{1}{h} & 0 & 0 & 0 \\ 1 & -(2-h^2) & 1 & 0 & 0 \\ 0 & 1 & -(2-h^2) & 1 & 0 \\ 0 & 0 & 1 & -(2-h^2) & 1 \\ 0 & 0 & 0 & -\frac{1}{h} & \frac{(2+2h-h^2)}{2h} \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y(0) \\ y(\frac{\pi}{8}) \\ y(\frac{\pi}{4}) \\ y(\frac{3\pi}{8}) \\ y(\frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} -140.9353 \\ -184.6194 \\ -199.8721 \\ -184.3445 \\ -140.4278 \end{bmatrix}$$