HW2 report

1.

$$Ax = b,\quad [A\ b] = \begin{bmatrix} 3 & 1 & -4 & 7 \\ -2 & 3 & 1 & -5 \\ 2 & 0 & 5 & 10 \end{bmatrix},\quad x = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ -\frac{2}{3} & 0 & 1 \end{bmatrix},\quad M_1[A\ b] = \begin{bmatrix} 3 & 1 & -4 & 7 \\ 0 & \frac{11}{3} & \frac{-5}{3} & -\frac{1}{3} \\ 0 & -\frac{2}{3} & \frac{23}{3} & \frac{16}{3} \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{2}{11} & 1 \end{bmatrix},\quad M_2M_1[A\ b] = \begin{bmatrix} 3 & 1 & -4 & 7 \\ 0 & \frac{11}{3} & \frac{-5}{3} & \frac{-1}{3} \\ 0 & 0 & \frac{243}{33} & \frac{174}{33} \end{bmatrix}$$

$\Rightarrow$ By backward substitution $\Rightarrow c = \frac{174}{243} = \frac{58}{81}$

$\Rightarrow \frac{11}{3}b = \frac{209}{243} \Rightarrow b = \frac{19}{81}$

$\Rightarrow 3a + \frac{19}{81} - \frac{232}{81} = 7 \Rightarrow 3a = \frac{780}{81} \Rightarrow a = \frac{260}{81}$

$\Rightarrow x = \begin{bmatrix} \frac{260}{81} \\ \frac{19}{81} \\ \frac{58}{81} \end{bmatrix}$, and no row interchanges needed

2.

(a) $Ax = b$, $[A\ b] = \begin{bmatrix} 0.1 & 51.7 & 104 \\ 5.1 & -7.3 & 16 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$

$M_1 = \begin{bmatrix} 1 & 0 \\ -51 & 1 \end{bmatrix}$, $M_1[A\ b] = \begin{bmatrix} 0.1 & 51.7 & 104 \\ 0 & -2650 & -5280 \end{bmatrix}$

$\Rightarrow$ by backward substitution $\Rightarrow y = 1.9924 \sim \doteqdot 1.99$

$\Rightarrow 0.1x = 1$, $\Rightarrow x = 10 \Rightarrow X = \begin{bmatrix} 10 \\ 1.99 \end{bmatrix}$, its x value is very different from correct value

(b) Do partial pivoting $\Rightarrow$ need row interchange

$\Rightarrow [A\ b] = \begin{bmatrix} 5.1 & -7.3 & 16 \\ 0.1 & 51.7 & 104 \end{bmatrix}$, $M_1 = \begin{bmatrix} 1 & 0 \\ \frac{-1}{51} & 1 \end{bmatrix}$

$M_1[A\ b] = \begin{bmatrix} 5.1 & -7.3 & 16 \\ 0 & 51.8 & 104 \end{bmatrix}$ $\Rightarrow$ by backward substitution

$\Rightarrow y = 2.0077 \sim \doteqdot 2.01 \Rightarrow 5.1x = 30.7 \Rightarrow x = 6.0196 \sim \doteqdot 6.02$

$\Rightarrow X = \begin{bmatrix} 6.02 \\ 2.01 \end{bmatrix}$, x 和 y 分別比 correct value 多 0.01 和 0.02

(c) Do scaled partial pivoting $\Rightarrow$ scaling matrix $S = \begin{bmatrix} \frac{1}{5.1} & 0 \\ 0 & \frac{1}{51.7} \end{bmatrix}$

$\Rightarrow$ new $[A\ b] = \begin{bmatrix} 0.00193 & 1 & 2.01 \\ 1 & -1.43 & 3.14 \end{bmatrix}$ $\Rightarrow$ need row interchange,

$[A\ b] = \begin{bmatrix} 1 & -1.43 & 3.14 \\ 0.00193 & 1 & 2.01 \end{bmatrix}$, $M_1 = \begin{bmatrix} 1 & 0 \\ -0.00193 & 1 \end{bmatrix}$, $M_1[A\ b] = \begin{bmatrix} 1 & -1.43 & 3.14 \\ 0 & 1 & 2 \end{bmatrix}$

$\Rightarrow$ by backward substitution $\Rightarrow y = 2$

$\Rightarrow x = 3.14 + 1.43y \Rightarrow x = 6 \Rightarrow X = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

$\Rightarrow$ This matches part (b), but it has better result

3.

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 \end{bmatrix}, \quad M_1 A = \begin{bmatrix} 2 & -1 & 3 & 2 \\ 0 & 3 & -3 & 2 \\ 0 & \frac{3}{2} & -\frac{7}{2} & 1 \\ 0 & \frac{7}{2} & \frac{5}{2} & -2 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{7}{6} & 0 & 1 \end{bmatrix}, \quad M_2 M_1 A = \begin{bmatrix} 2 & -1 & 3 & 2 \\ 0 & 3 & -3 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 6 & -\frac{13}{3} \end{bmatrix}$$

$$M_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}, \quad M_3 M_2 M_1 A = \begin{bmatrix} 2 & -1 & 3 & 2 \\ 0 & 3 & -3 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -\frac{13}{3} \end{bmatrix} = U$$

$$L = M_1^{-1} M_2^{-1} M_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & \frac{7}{6} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{7}{6} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{7}{6} & -3 & 1 \end{bmatrix},$$ but each

diagonal position of L has 2's rather than 1's

$$\Rightarrow \underset{A}{\begin{bmatrix} 2 & -1 & 3 & 2 \\ 2 & 2 & 0 & 4 \\ 1 & 1 & -2 & 2 \\ 1 & 3 & 4 & -1 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & \frac{7}{3} & -6 & 2 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 1 \\ 0 & \frac{3}{2} & -\frac{3}{2} & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\frac{13}{6} \end{bmatrix}}$$

4.

$$[A\ \ b] = \begin{bmatrix} 7 & -3 & 4 & 6 \\ -3 & 2 & 6 & 2 \\ 2 & 5 & 3 & -5 \end{bmatrix},\quad \text{so } A = \begin{bmatrix} 7 & -3 & 4 \\ -3 & 2 & 6 \\ 2 & 5 & 3 \end{bmatrix}$$

rearrang A to $\begin{bmatrix} 7 & -3 & 4 \\ 2 & 5 & 3 \\ -3 & 2 & 6 \end{bmatrix}$ (swap row2 and row3) to be diagonally dominant

So new $[A\ \ b] = \begin{bmatrix} 7 & -3 & 4 & 6 \\ 2 & 5 & 3 & -5 \\ -3 & 2 & 6 & 2 \end{bmatrix}$, 我用python來實現

Jacobi method $\Rightarrow X^{(k+1)} = D^{-1}(b-(L+U)X^{(k)})$,

$$L = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -3 & 2 & 0 \end{bmatrix},\quad D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix},\quad U = \begin{bmatrix} 0 & -3 & 4 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix},\quad b = \begin{bmatrix} 6 \\ 5 \\ 2 \end{bmatrix}$$

直到 $X^{(k+1)}$ 和 $X^{(k)}$ 足夠接近。

若代[0,0,0]進去 ⇒ 需要32次iterations得到

solution = [-0.14332, -1.3746, 0.71987]

```
[-0.14331712 -1.37459193  0.71987013]
Number of iterations : 32
```

Result for Problem 4

5. 承上題，我用python來實現 Gauss-Seidel method

$\Rightarrow X^{(k+1)} = (L+D)^{-1}(b - Ux^{(k)})$，直到 $X^{(k+1)}$ 和 $X^{(k)}$ 足夠接近

若代入[0,0,0]進去 $\Rightarrow$ 只需要14次 iterations

即可得到 solution = [-0.14332, -1.3746, 0.71987]

```
[-0.14332299 -1.37459376  0.71986976]
Number of iterations : 14
```

Result for Problem 5

6. $A=\begin{bmatrix}2&-2\\-2&2\end{bmatrix}$, $L=\begin{bmatrix}0&0\\-2&0\end{bmatrix}$, $D=\begin{bmatrix}2&0\\0&2\end{bmatrix}$, $U=\begin{bmatrix}0&-2\\0&0\end{bmatrix}$

$b=\begin{bmatrix}0\\0\end{bmatrix}$

(a) Jacobi methon $\Rightarrow X^{(k+1)}=D^{-1}(b-(L+U)X^{(k)})$

代 [1, 1] ⇒ converges to [1, 1]

代 [1, -1] ⇒ 發現不會收斂，檢查後發現會在[1, -1]

[-1, 1] 之間反復交換

代 [-1, 1] ⇒ 同 [1, -1]

代 [2, 5] ⇒ 不會收斂，檢查後發現會在 [2, 5], [5, 2]

之間反復交換

代 [5, 2] ⇒ 同 [2, 5]

(b) Gauss-Seidel method $\Rightarrow X^{(k+1)}=(L+D)^{-1}(b-UX^{(k)})$

代 [1, 1] ⇒ converges to [1, 1]

代 [1, -1] ⇒ converges to [-1, -1]

代 [-1, 1] ⇒ converges to [1, 1]

代 [2, 5] ⇒ converges to [5, 5]

代 [5, 2] ⇒ converges to [2, 2]

```
[1. 1.]
Number of iterations : 1

[ 1. -1.]
Number of iterations : 100000
[array([ 1., -1.]), array([-1.,  1.]), array([ 1., -1.]), array([-1.,  1.]), array([ 1., -1.]), array([-1.,  1.]), array([ 1., -1.]), array([-1.,  1.
]), array([ 1., -1.]), array([-1.,  1.])]


[-1.  1.]
Number of iterations : 100000
[array([-1.,  1.]), array([ 1., -1.]), array([-1.,  1.]), array([ 1., -1.]), array([-1.,  1.]), array([ 1., -1.]), array([-1.,  1.]), array([ 1., -1.
]), array([-1.,  1.]), array([ 1., -1.])]


[2. 5.]
Number of iterations : 100000
[array([2., 5.]), array([5., 2.]), array([2., 5.]), array([5., 2.]), array([2., 5.]), array([5., 2.]), array([2., 5.]), array([5., 2.]), array([2., 5
.]), array([5., 2.])]


[5. 2.]
Number of iterations : 100000
[array([5., 2.]), array([2., 5.]), array([5., 2.]), array([2., 5.]), array([5., 2.]), array([2., 5.]), array([5., 2.]), array([2., 5.]), array([5., 2
.]), array([2., 5.])]
```

Result for Problem 6 (a)

```
[1. 1.]
Number of iterations : 1

[-1. -1.]
Number of iterations : 2

[1. 1.]
Number of iterations : 2

[5. 5.]
Number of iterations : 2

[2. 2.]
Number of iterations : 2
```

Result for Problem 6 (b)

(c) $A=\begin{bmatrix}2 & -1.99\\ -1.99 & 2\end{bmatrix}$, $L=\begin{bmatrix}0 & 0\\ -1.99 & 0\end{bmatrix}$, $D=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$, $U=\begin{bmatrix}0 & -1.99\\ 0 & 0\end{bmatrix}$

$b=\begin{bmatrix}0\\ 0\end{bmatrix}$

用 Jacobi method 和 Gauss-Seidel method 測試過後會發現不論是哪個 starting vectors 和 method，最終都會趨近 [0, 0]，但 Gauss-Seidel 依舊收斂的比 Jacobi 快不少

```
[0.00198812 0.00198812]
Number of iterations : 1241

[ 4.9776539e-06 -4.9776539e-06]
Number of iterations : 2436

[-4.9776539e-06  4.9776539e-06]
Number of iterations : 2436

[6.60005133e-06 1.65001283e-05]
Number of iterations : 2518

[1.65001283e-05 6.60005133e-06]
Number of iterations : 2518

[0.00098554 0.00098061]
Number of iterations : 691

[-0.00098554 -0.00098061]
Number of iterations : 691

[0.00098554 0.00098061]
Number of iterations : 691

[0.0009909  0.00098595]
Number of iterations : 851

[0.00098694 0.000982  ]
Number of iterations : 760
```

Resutl for Problem 6 (c)