HW4 report

1. By definition, $s = \dfrac{x - x_i}{h}$, $\dfrac{d}{dx} P_n(s) = \dfrac{1}{h}\left[\Delta f_i + \sum_{j=2}^{n}\left\{\sum_{k=0}^{j-1}\prod_{\substack{l=0\\ l\neq k}}^{j-1}(s-l)\right\} \cdot \dfrac{\Delta^j f_i}{j!}\right]$

(a)

Choose $i=1$, $S = \dfrac{0.72-0.5}{0.2} = 1.1$

$$\Rightarrow P_3'(1.1) = \frac{1}{0.2}\left[0.2549 + \frac{(1.1+0.1)\times(-0.0086)}{2} + \frac{(-0.09-0.99+0.11)\times(-0.0018)}{6}\right]$$

$$= 1.250155$$

(b)

Choose $i=4$, $S = \dfrac{1.33-1.1}{0.2} = 1.15$

$$\Rightarrow P_2'(1.15) = \frac{1}{0.2}\left[0.2241 + \frac{(1.15+0.15)\times(-0.0128)}{2}\right] = 1.0789$$

(c)

Choose $i=1$, $S = \dfrac{0.5-0.5}{0.2} = 0$

$$\Rightarrow P_4'(0) = \frac{1}{0.2}\left[0.2549 + \frac{0.0086}{2} + \frac{2\times(-0.0018)}{6} + \frac{(-6)\times(0.0004)}{24}\right]$$

$$= 1.2925$$

2. By definition, $f''(x) = C_{-2}f_{-2} + C_{-1}f_{-1} + C_0 f_0 + C_1 f_1 + C_2 f_2$.

And consider 5 special cases $\Rightarrow P(u) = 1, u, u^2, u^3, u^4$

we get $P''(u) = 0, 0, 2, 6u, 12u^2 \Rightarrow P''(0) = 0, 0, 2, 0, 0$.

By above we get
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2h & -h & 0 & h & 2h \\ 4h^2 & h^2 & 0 & h^2 & 4h^2 \\ -8h^3 & -h^3 & 0 & h^3 & 8h^3 \\ 16h^4 & h^4 & 0 & h^4 & 16h^4 \end{bmatrix} \begin{bmatrix} C_{-2} \\ C_{-1} \\ C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ solve this we get
$$\begin{cases} C_{-2} = \dfrac{-1}{12h^2} \\ C_{-1} = \dfrac{4}{3h^2} \\ C_0 = \dfrac{-5}{2h^2} \\ C_1 = \dfrac{4}{3h^2} \\ C_2 = \dfrac{-1}{12h^2} \end{cases} \Rightarrow f''(x_0) = \frac{-f_{-2} + 16f_{-1} - 30f_0 + 16f_1 - f_2}{12h^2}$$

$\Rightarrow$ Find error term by Taylor expansion

$\Rightarrow f(x+nh) = f(x) + f'(x)(nh) + \frac{f''(x)}{2}(nh)^2 + \cdots$, and $x_0 = x + 0 \cdot h$

$$\Rightarrow f''(x_0) = \frac{-f_{-2} + 16f_{-1} - 30f_0 + 16f_1 - f_2}{12h^2} \approx \frac{15 \cdot 12 f''(x)h^2 - 2h^6 f^{(6)}(x)}{15 \cdot 12h^2}$$

$$= f''(x) - \frac{h^4 f^{(6)}(x)}{90} \Rightarrow \text{error term is } O(h^4)$$

For $f'''(x_0)$, we have $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2h & -h & 0 & h & 2h \\ 4h^2 & h^2 & 0 & h^2 & 4h^2 \\ -8h^3 & -h^3 & 0 & h^3 & 8h^3 \\ 16h^4 & h^4 & 0 & h^4 & 16h^4 \end{bmatrix} \begin{bmatrix} C_{-2} \\ C_{-1} \\ C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 6 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{cases} C_{-2} = \frac{-1}{2h^3} \\ C_{-1} = \frac{1}{h^3} \\ C_0 = 0 \\ C_1 = \frac{-1}{h^3} \\ C_2 = \frac{1}{2h^3} \end{cases} \Rightarrow f'''(x_0) = \frac{-f_{-2} + 2f_{-1} - 2f_1 + f_2}{2h^3}$$

$\Rightarrow$ Error term $\Rightarrow f'''(x_0) = \frac{-f_{-2} + 2f_{-1} - 2f_1 + f_2}{2h^3}$

$$= \frac{-(f(x) - 2hf'(x) + 2h^2f''(x) - \frac{4}{3}h^3f'''(x) + \cdots) + 2(f(x) - hf'(x) + \frac{1}{2}h^2f''(x) - \frac{1}{6}h^3f'''(x) + \cdots)}{2h^3}$$

$$\frac{-2(f(x) + hf'(x) + \frac{1}{2}h^2f''(x) + \frac{1}{6}h^3f'''(x) + \cdots) + (f(x) + 2hf'(x) + 2h^2f''(x) + \frac{4}{3}h^3f'''(x) + \cdots)}{2h^3}$$

$$\approx \frac{2h^3f'''(x) + \frac{1}{2}h^5f^{(5)}(x)}{2h^3} = f'''(x) + \frac{1}{4}h^2f^{(5)}(x)$$

$\Rightarrow$ error term is $O(h^2)$

For $f^{(4)}(x_0)$, we have $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2h & -h & 0 & h & 2h \\ 4h^2 & h^2 & 0 & h^2 & 4h^2 \\ -8h^3 & -h^3 & 0 & h^3 & 8h^3 \\ 16h^4 & h^4 & 0 & h^4 & 16h^4 \end{bmatrix} \begin{bmatrix} C_{-2} \\ C_{-1} \\ C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 24 \end{bmatrix}$

$$\Rightarrow \begin{cases} C_{-2} = \frac{1}{h^4} \\ C_{-1} = -\frac{4}{h^4} \\ C_0 = \frac{6}{h^4} \\ C_1 = -\frac{4}{h^4} \\ C_2 = \frac{1}{h^4} \end{cases} \Rightarrow f^{(4)}(x_0) = \frac{f_{-2} - 4f_{-1} + 6f_0 - 4f_1 + f_2}{h^4}$$

$$\Rightarrow \text{Error term} \Rightarrow f^{(4)}(x_0) = \frac{f_{-2} - 4f_{-1} + 6f_0 - 4f_1 + f_2}{h^4}$$

$$= \frac{(f(x) - 2hf'(x) + 2h^2f''(x) + \dots) - 4(f(x) - hf'(x) + \frac{1}{2}h^2f''(x) + \dots) +}{h^4}$$

$$\frac{6(f(x_0)) - 4(f(x) + hf'(x) + \frac{1}{2}h^2f''(x) + \dots) + (f(x) + 2hf'(x) + 2h^2f''(x) + \dots)}{h^4}$$

$$\approx \frac{h^4f^{(4)}(x) + \frac{1}{6}h^6f^{(6)}(x)}{h^4} = f^{(4)}(x) + \frac{1}{6}h^2f^{(6)}(x)$$

$\Rightarrow$ error term is $O(h^2)$

3. We first set $x_0 = a$, $x_1 = \frac{a+b}{2}$, $x_2 = b$, so $h = \frac{b-a}{2}$

and suppose a cubic: $f(x) = cx^3 + dx^2 + ex + f$

The area under $f(x)$ between $[a, b]$ $\Rightarrow$

$$A_f = \int_a^b cx^3 + dx^2 + ex + f \, dx = \frac{1}{4}cx^4 + \frac{1}{3}dx^3 + \frac{1}{2}ex^2 + fx \Big|_a^b$$

$$= \frac{1}{4}cb^4 + \frac{1}{3}db^3 + \frac{1}{2}eb^2 + fb - \left(\frac{1}{4}ca^4 + \frac{1}{3}da^3 + \frac{1}{2}ea^2 + fa\right)$$

And suppose a parabola: $g(x)$ matches the cubic at $x = a$, $x = b$, and $x = \frac{a+b}{2}$, so the area of $g(x)$ between $[a, b]$ $\Rightarrow$

$$A_g = \int_a^b g(x) \, dx = \frac{h}{3}\left(g(a) + 4g\left(\frac{a+b}{2}\right) + g(b)\right)$$

(by Simpson's $\frac{1}{3}$ rule)

$$= \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$ (by $g(a) = f(a)$, $g(b) = f(b)$, $g\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right)$)

$$= \frac{b-a}{6}\left(ca^3 + da^2 + ea + f + cb^3 + db^2 + eb + f + \frac{ca^3 + ca^2b + cab^2 + cb^3}{2} + da^2 + 2dab + db^2 + 2ea + 2eb + 4f\right) = \frac{b-a}{6}\left(\frac{3}{2}ca^3 + \left(d + \frac{cb}{2} + d\right)a^2 + \left(e + \frac{cb^2}{2} + db + 2e\right)a + \frac{3}{2}cb^3 + \left(d + \frac{ca}{2} + d\right)b^2 + \left(e + \frac{ca^2}{2} + da + 2e\right)b + bf\right)$$

$$= \frac{1}{4}cb^4 + \frac{1}{3}db^3 + \frac{1}{2}eb^2 + fb - \left(\frac{1}{4}ca^4 + \frac{1}{3}da^3 + \frac{1}{2}ea^2 + fa\right)$$

$\therefore$ $A_f = A_g$. Proof completes.

4.

With $h=0.5$, $\int_0^1 f(x)\,dx = \frac{0.5}{3}\left(f(0)+4f(0.5)+f(1)\right)$

$\fallingdotseq 0.946146$

With $h=0.25$, $\int_0^1 f(x)\,dx = \frac{0.25}{3}\left(f(0)+4f(0.25)+2f(0.5)+4f(0.75)+f(1)\right)$

$\fallingdotseq 0.946087$

Extrapolate to get abetter result

$\Rightarrow 0.946087 + \frac{1}{2^4-1}(0.946087-0.946146) \fallingdotseq 0.94608307$

We know original Simpson's $\frac{1}{3}$ rule approximation is 4^{th} order error $\Rightarrow O(h^4)$, and by extrapolation, it can achieve 5^{th} order error $\Rightarrow O(h^5)$

True value of $\int_0^1 f(x)\,dx \fallingdotseq 0.9460831$, we can notice that better result got by extrapolation indeed reduce the error.

5.

(a)
$$\int_{-0.2}^{1.4}\int_{0.4}^{2.6} e^{x}\sin(2y)\,dy\,dx = \left(\int_{-0.2}^{1.4} e^{x}\,dx\right)\left(\int_{0.4}^{2.6}\sin(2y)\,dy\right)$$

$$= \frac{0.1}{2}\left(e^{-0.2}+2e^{-0.1}+\cdots+e^{1.4}\right)\times\frac{0.1}{2}\left(\sin(0.8)+2\sin(1)+\cdots+\sin(5.2)\right)$$

$$\doteqdot 0.36834$$

(b)
$$\left(\int_{-0.2}^{1.4} e^{x}\,dx\right)\left(\int_{0.4}^{2.6}\sin(2y)\,dy\right)$$

$$= \frac{0.1}{3}\left(e^{-0.2}+4e^{-0.1}+2e^{0}+\cdots+e^{1.4}\right)\times\frac{0.1}{3}\left(\sin(0.8)+4\sin(1)+2\sin(1.2)+\cdots+\sin(5.2)\right)$$

$$\doteqdot 0.369269$$

(c) We first change variable

$$\Rightarrow x=\frac{1.6t+1.2}{2}=0.8t+0.6,\quad dx=0.8dt$$

$$\Rightarrow y=\frac{2.2u+3}{2}=1.1u+1.5,\quad dy=1.1du$$

$$\Rightarrow I = 0.8\times1.1\times\int_{-1}^{1}\int_{-1}^{1} e^{0.8t+0.6}\sin(2.2u+3)\,du\,dt$$

$$= 0.88\times\left(\frac{5}{9}\cdot e^{-0.01968}+\frac{8}{9}e^{0.6}+\frac{5}{9}e^{1.21968}\right)\times\left(\frac{5}{9}\sin(1.29588)+\right.$$

$$\left.\frac{8}{9}\cdot\sin(3)+\frac{5}{9}\sin(4.70412)\right)\doteqdot 0.372374$$

6.

$$\int_{-1}^{2}\int_{-2}^{3} f(x,y)\,dx\,dy \approx (-3)(-5)\cdot\frac{1}{N}\sum_{i=1}^{N} f(x_i,y_i),$$

I use python to implement Monte Carlo Intergration, and choose $N = 10^6$, so the result $\Rightarrow$

$I \fallingdotseq 35.937564$

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35.93756390631453
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Result for Problem 6