Linear Algebra

Lecture 11-15

Lecture 11

New vector space M

suppose M: All 3 by 3 matrices. dimension is 9

Subspace of M:

- upper triangulars: dimension is 6
- symmetric matrices: dimension is 6
- diagonal matrices: dimension is 3.

Basis for M = all 3 by 3's
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dots \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rank one matrix

$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \end{bmatrix}$

Rank One Matrix $A = uv^T$

In R^4, $v = \begin{bmatrix} v_1 & v_2 & v3 & v_4 \end{bmatrix}, v_1 + v_2 + v_3 + v_4 = 0$ is a subspace.

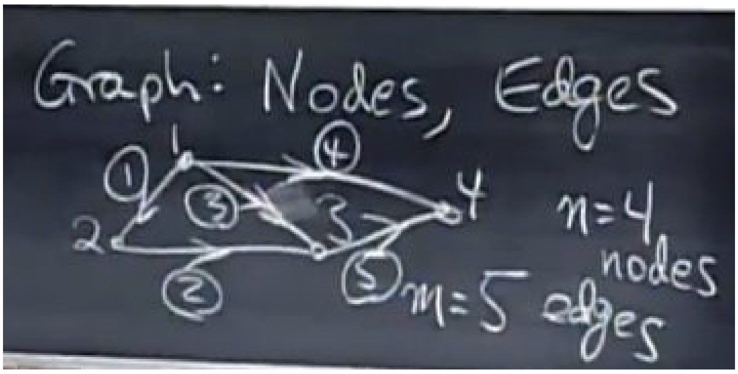
Graph

nodes, edges.
small world graph

Lecture 12

Graph and Network

Graph: Nodes, Edge



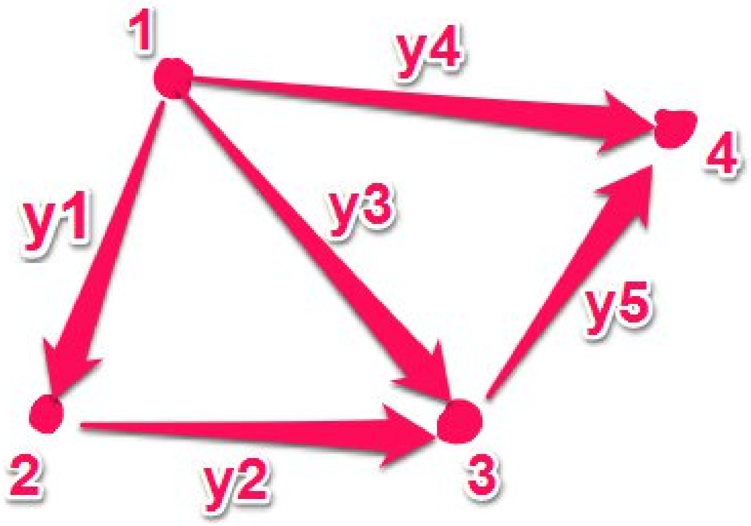
Incidence Matrix

The columns represent nodes, the rows represent edges.

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

> **Note:** Edge 1,2,3 form subgraph or loop.
> $dimN(A) = 1$

1. potentials at nodes: $x = x_1, x_2, x_3, x_4$
   Ax: find potential difference on edges
2. currents y_1, y_2, y_3, y_4, y_5 one edges
3. $A^T y = 0$ — Kirchaffs Current Law
   $dimN(A^T) = 2$



$$\begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

即是:

$-y_1 - y_4 - y - 4 = 0 \Rightarrow$ KCL，节点电流和为0

$\vdots$

> **Note:** Basis for $N(T)$ 可以根据graph中loop来找。
> $dimN(A^T) = m - r$

Tree: graph with no loop.

> **Note:** number of loops = number of edges - (number of nodes - 1)
> That is Euler's formula: **number of nodes - number of edges + number of loops = 1**

Lecture 13

review

Lecture 14

Orthogonal Vectors

$x \cdot y = x^T y = 0 \Rightarrow x \perp y \Rightarrow \| x \|^2 + \| y \|^2 = \| x + y \|^2$

$\| x \|^2 = x^T x$

> **Note:** 利用勾股定理可证明以上formula

Zero vector is orthogonal to any vector.

Orthogonal Subspaces

Subspace S is orthogonal to subspace T
means: every vector in S is orthogonal every vector in T.

若两个子空间互相垂直且相交，肯定不会相交于一个非零向量。

1. row space is orthogonal to nullspace
   prove:
   for x in nullspace, $Ax = 0$
   that is:

$$\begin{bmatrix} row_1 \\ row_2 \\ \cdots \\ row_m \end{bmatrix} x = 0$$
$$\Rightarrow row_1 \cdot x = 0, \cdots$$

   Then the basis of row space is orghogonal to all x in nullspace, so row space is orghogonal to nullspace.

2. column space is orghogonal to nullspace of A^T

Complements

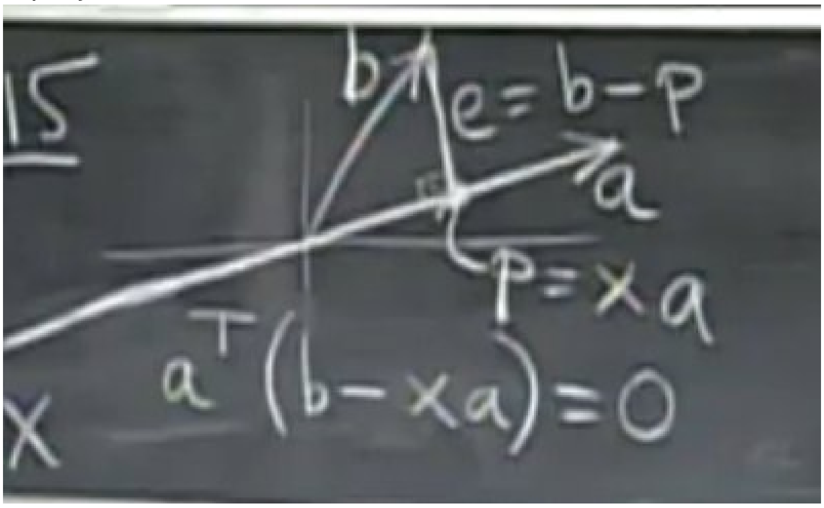
nullspace and row space are orthogonal complements in R^n
Nullspace contains **all** vectors $\perp$ row space.

Lecture 15

$N(A^TA) = N(A)$ and $rank\ of\ AA^T = rank\ of\ A$
AA^T is invertible exactly if A has independent columns.

Two dimension projection

b projection on a:



$a^T(b - xa) = 0 \Rightarrow xa^Ta = a^Tb \Rightarrow x = \frac{a^Tb}{a^Ta}$

Projection $p = a\,\frac{a^Tb}{a^Ta}$

Projection Matrix — P: projection $p = Pb$
$Projectjon\ Matrix\ P = \frac{aa^T}{a^Ta}$

Property of Projection Matrix:

1. $C(P)$ is line through a
2. $rank(P) = 1$
3. $P^T = P$
4. $p^2 = P$

WHY project?

Because $Ax = b$ may have no solution.
Solve $A\hat{x} = p$ instead. p is projection of b onto column space.

Three(or more) dimension projection.

$p = A\hat{x}$ Find $\hat{x}$
Key: $e = b - A\hat{x}$ is perpendicular(垂直) to the plane.

a_1, a_2 are the basis of plane.

$$\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix}(b - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow A^T(b - A\hat{x}) = 0$$
$\Rightarrow e$ in $N(A^T) \Rightarrow e \perp C(A)$
$\Rightarrow A^T A\hat{x} = A^T b$
$\Rightarrow \hat{x} = (A^T A)^{-1} A^T b$
$\Rightarrow p = A\hat{x} = A(A^T A)^{-1} A^T b$
$\Rightarrow P = A(A^T A)^{-1} A^T$

Property of Projection Matrix P:

1. $P^T = P$
2. $P^2 = P$

Least Squares — Fitting by a line
