Linear Algebra

Lecture 6-10

Lecture 6

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = b$$

column space $C(A)$

Definition

: all linear combinations of the columns. — subspace of R^m

> **Note:** In the exmaple, the column space is subspace of R^4

Which b's allow this system $Ax = b$ to be solved?

: Can solve $Ax = b$ exactly when b is $C(A)$

Pivot Column

: the columns that are linear independent.

> **Note:** In the example, pivot columns are column 1 and 2. Or column 2 and 3.
> Because $Column3 = Column1 + Column2$

Dimension

: equals to the number of pivot columns.

Nullspace $N(A)$

Definition

all solutions x that $Ax = 0$ — subspace of R^n

> **Note:**
>
> - N(A) is $c\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$
> - Nullspace must contains **ZERO vector**

Row space

Definition

all linear combinations of the rows. — subspace of R^n

Lecture 7

sovle $Ax = 0$, Nullspace

- First step: **Elimination**

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

> **Note:**
>
> - U is called **Echelon Form**
> - Column 1 and 3 are **pivot columns**, Column 2 and 4 are free columns.
> - x_1 andx_3 are pivot variables, x_2 andx_3 are free variables.
> - **Rank**: The number of pivots

- Second step: **Solve** $Ux = 0$
  分别令free variable为0，求得Nullspace的base
  令$x_2 = 1, x_4 = 0$，得$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

令$x_2 = 0, x_4 = 1$，得$\begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

所以x解为：

$$x = c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

也即是A的Nullspace $N(A)$

> **Note:**
>
> - Reduced row echelon form: zeros above and below pivots and pivots equal 1.
>
> $$R = \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
>
> - $Rank(A) = Rank(A^T)$

Lecture 8

sovle $Ax = b$

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

argmented matrix: $\begin{bmatrix} A & b \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix}$$

Solvability Condition on b

$Ax = b$ solvable if when b is in $C(A)$

if a combination of rows of A gives zero row, the same of combination of the enties of b must give 0.

To find complete solutions to $Ax = b$

1. $x_{particular}$
    - set all free variables(x_2 and x_4) to zero
    - solve $Ax = b$ for pivot variables. (we get $x_1 = -2$ and $x_3 = \frac{3}{2}$)
    - $x_p = \begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix}$
2. $x_{nullspace}$
3. $x_{complete} = x_p + x_n$, x_n is any verctor in nullspace.

prove:
$Ax_p = b$ and $Ax_b = 0$, so $A(x_p + x_n) = b$

m by n matrix of rank r ($r \leq m, r \leq n$)

Full column rank means $r = n$

- No free variable
- $N(A) = \vec{0}$
- solutions to $Ax = b$,$x = x_p$, unique solution if it exists. That is, 0 or 1 solution.

Full row rank means $r = m$

- can solve $Ax = b$ for every b
- left with $n - m$ free variables
- $Ax = b$ has 1 or infinite solutions

Full row rank and column rank $r = m = n$

- invertible
- the reduced echelon form is identity matrix
- the nullspace is zero vector only
- $Ax = b$ has 1 solution

Rank $r < n, r < m$
- $Ax = b$ has 0 or infinite solutions

Lecture 9

Suppose A is m by n with m < n. Then there are nonzero solutions to $Ax = 0$. (more unknowns than equations)
Reason: There will be free variables.

Linear independence

Definition

Vectors x_1 , x_2 , ... , x_n are independent if no combination gives zero vector (except the zero combination).

> **Note:**
>
> - Independent vecotors must not contain zero vector.

When v_1, ... v_n are columns of A,

- they are independent if nullspace of A is only the zero vector. **rank=n**
- they are dependent if $Ac = 0$ for some nonzero c. **rank < n**

Spaning a space

Definition

Vectors v_1 , ... , v_l spans a space means: the space consists of all combinations of those vectors.

> **Note:** The column vectors span the column space.

Basis

Definition

Basis for a space is a sequence of vectors v_1 , ... , v_d with two properties
1. They are independent
2. They span the space

In R^n , n vectors give basis if n*n matrix with those columns is invertible.

Given a space, every basis for the space has the same number of vectors. — DIMENSION

> **Note:**
>
> - Rank(A) = number of pivot columns = dimension of C(A)
> - dimesion N(A) = number of free variables = n - r

standard basis: the column of indentity.

Lecture 10

A is m by n.

Four fundamental subspace

1. Column space $C(A)$ in R^m
   dim $C(A)$ = rank
2. Nullspace $N(A)$ in R^n
3. Row space
   all combinations of rows = all combinations of column of A^T = $C(A^T)$ in R^n
4. Nullspace of A^T = $N(A^T)$ — left nullspace of A in R^m

	$C(A)$	**raw space**	$N(A)$	$N(A^T)$
basis	pivot columns	the first r row of R	special solutions	basis is in E, $EA = R$
dimensions	r	r	n-r	m-r

> **Note:** $C(A) \neq C(R)$, R is the echelon form of A
