Linear Algebra

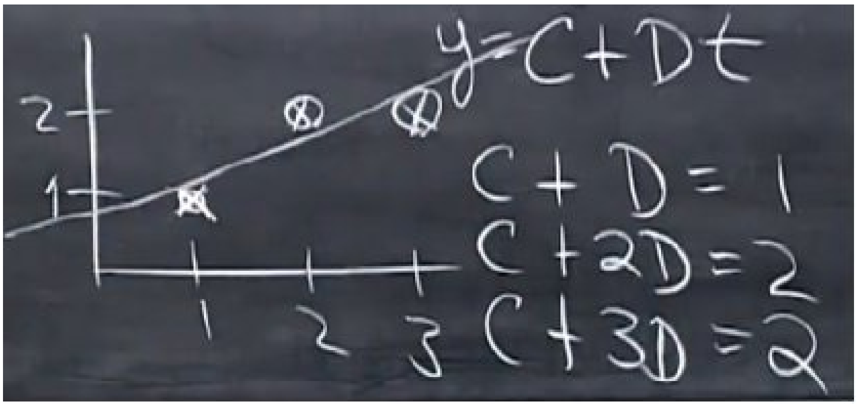
Lecture 16-20

Lecture 16

Projection Matrix P
Pb project b onto a space, then $(I - P)b$ project b onto a $\perp$ space.

Least Square

Find the best line that error is smallest.



$$Ax = b \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

minimize $\| Ax - b \|^2 = \| e \|^2 = \| e_1 \|^2 + \| e_2 \|^2 + \| e_3 \|^2$

Find $\hat{x} = \begin{bmatrix} C \\ D \end{bmatrix}$
$A^T A \hat{x} = A^T b$ — normal equations. we get $C = 1/2, D = 2/3$
so, the best line is $y = \frac{1}{2} + \frac{2}{3} t$

property of e:

1. $p + e = b$

2. $p \cdot e = 0$

If A has independent columns then $A^T A$ is invertible.

Suppose $A^T A x = 0$,
then $x^T A^T A x = 0 \Rightarrow (Ax)^T A x = 0 \Rightarrow Ax = 0 \Rightarrow x = 0$
$\Rightarrow A^T A$ is invertible

Orthonormal vectors

Columns definitely independent if they are **perpendicular unit vectors**, that is orthonormal vectors.

Lecture 17

Orthonomal vectors

$$q_i^T q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$Q = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}$ then $Q^T Q = I$

> **Note:** Q is orthonormal matrix.

If Q is square then $Q^T Q = I$ tells us $Q^T = Q^{-1}$.

Project onto its column space
$P = Q(Q^T Q)^{-1} Q^T = QQ^T$

1. symmetric
2. $PP = QQ^T QQ^T = I$

> **Note:** If Q is square, $P = I$

The component of projection:
$x_j = q_j^T * b$

Gram-Schmidt

independent vectors a, b

1. orthogonal vectors A, B, C
   $A = a,\ B = b - \frac{AA^T}{A^TA} b,\ C = c - \frac{AA^T}{A^TA} c - \frac{BB^T}{B^TB} c$
2. orthonormal vectors q_1, q_2, q_3
   $q_1 = \frac{A}{\| A \|}, q_2 = \frac{B}{\| B \|}, q_3 = \frac{C}{\| C \|}$

Lecture 18

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Determinants det A

Property 1: det I = 1.
Property 2: Exchange rows reverse sign of determinant.
Property 3: LENEAR EACH ROW

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

> **Note:** $det(A+B) \neq det(A) + det(B)$

Property 4: two equal rows -> det = 0
Property 5: Subtract l × row i from row k, **DETERMINANT DOESN'T CHANGE**.
Property 6: Row of zeros -> det A = 0
Property 7: Matrix U is upper triangular, then $det(U) = d_1 * d_2 * ... d_n$, product of pivots(diagonal elements).
Property 8: $det(A) = 0$ when A is singular(noninvertible), $det(A) \neq 0$ when A is invertible.
Property 9: $det(AB) = det(A)det(B)$

- $det(A^{-1}) = \frac{1}{det(A)}$
- $det(A^2) = (det(A))^2$
- $det(2A) = 2^m det(A)$

Property 10: $det(A^T) = det(A)$

- Exchange columns reverse sign
- Column of zeros -> det A = 0

proof:

$|A^T| = |A|$

$\Rightarrow |U^T L^T| = |LU|$

$\Rightarrow |U^T||L^T| = |L||U|$