

Differential Flatness-Based Trajectory Planning for Autonomous Vehicles

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Abstract—As a core part of autonomous driving systems, motion planning has received extensive attention from academia and industry. However, there is no efficient trajectory planning solution capable of spatial-temporal joint optimization due to nonholonomic dynamics, particularly in the presence of unstructured environments and dynamic obstacles. To bridge the gap, we propose a versatile and real-time trajectory optimization method that can generate a high-quality feasible trajectory using a full vehicle model under arbitrary constraints. By leveraging the differential flatness property of car-like robots, we use flat outputs to analytically formulate all feasibility constraints to simplify the trajectory planning problem. Moreover, obstacle avoidance is achieved with full dimensional polygons to generate less conservative trajectories with safety guarantees, especially in tightly constrained spaces. We present comprehensive benchmarks with cutting-edge methods, demonstrating the significance of the proposed method in terms of efficiency and trajectory quality. Real-world experiments verify the practicality of our algorithm. We will release our codes as open-source packages with the purpose for the reference of the research community.¹

Index Terms—Autonomous Vehicles: Motion Planning, Trajectory Optimization, Collision Avoidance.

I. INTRODUCTION

AUTONOMOUS driving has become one of the hottest research topics in recent years because of its vast potential social benefits. The rapid development of autonomous driving technologies reveals a huge demand for robust and safe motion planning in complex and high-dynamic environments. The goal of motion planning is to generate a comfortable, low-energy, and physically feasible trajectory that makes the ego vehicle reach end states as soon as possible with safety guarantees and expected velocities in constrained environments. Moreover, in real-world applications, light weight and efficiency are strongly demanded to ensure rapid response to dynamic and unknown environments with limited onboard computing power. However, in recent years, few approaches are versatile and efficient enough to generate feasible and high-quality trajectories online in arbitrarily complex scenarios. Existing methods either rely on environmental features

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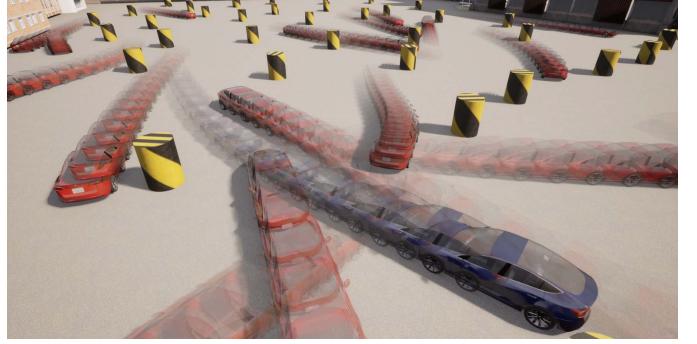
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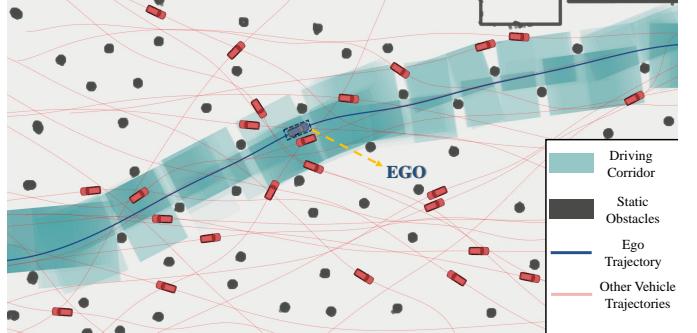
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¹<https://github.com/ZJU-FAST-Lab/Dftpav>



(a) The movement diagram.



(b) The trajectory visualization.

Fig. 1: The two figures show the capability of our planner in a highly dynamic environment. Because of full-dimensional obstacle avoidance, the ego vehicle has the ability to pass through tight areas near other moving objects while ensuring safety.

with predefined rules or oversimplify trajectory generation problems to improve time efficiency. These methods can not apply to general or highly constrained environments. Overall, there is no universal solution for trajectory planning, which makes it one of the Achilles heels hindering the development of autonomous driving. In fact, an ideal motion planning for autonomous driving typically faces five challenging problems.

- 1) Variabilities of Driving Scenarios: In real life, a complete driving trip often involves multiple driving scenarios. For example, a driving route from a parking space to a university includes at least three different scenarios: parking lot, roads, and campus, all of which need to be processed by the same trajectory planner for consistency. Scenario-based approaches can not cope with multiple types of driving environments due to the lack of gen-

- eralization capabilities. For instance, lane-based methods rely heavily on reference lanes which limit their ability to extend to unstructured constrained environments such as parking lots and trails of the campus.
- 2) Nonholonomic Dynamics: Unlike holonomic robots such as omnidirectional mobile robots and quadrotors, nonholonomic dynamic constraints of vehicles must be considered in trajectory planning. However, the strong non-convexity and nonlinearity of nonholonomic dynamics make it difficult to ensure the physical feasibility of states and control inputs in highly constrained environments.
 - 3) Precise Obstacle Avoidance: A common method to simplify safety constraints is to model a vehicle as one or multiple circles with the same radius along the centerline. However, in real-world applications, this rough modeling always compresses the solution space, which introduces conservativeness or even fails to find a collision-free solution in extremely cluttered areas. Nevertheless, accurate modeling of the ego vehicle increases the dimension of the planning problem, resulting in a large computation burden for online vehicle (re)planning. Besides, motion planning in dynamic real-world environments always requires that the distance between the ego vehicle and moving objects is greater than a user-defined safety margin at each moment. In this case, not only the ego vehicle, but also other moving objects need to be modeled accurately in order to ensure full passability.
 - 4) Efficiency and Quality: A trajectory planning problem is conventionally simplified by discretizing the dynamic process. However, there is always a tradeoff between time efficiency and trajectory quality because high-quality motions always require fine discretization which quickly induces unacceptable computing overhead, resulting in expensive time budgets.
 - 5) Joint Optimization on Time and Space: Time information is an inherent attribute of a trajectory. Trajectory generation methods that optimize time and space separately waste a large partition of solutions, especially in tightly coupled spatial-temporal scenarios such as highly dynamic environments. By contrast, spatial-temporal joint optimization can make full use of the solution space to achieve better trajectory optimality. Unfortunately, joint optimization tends to complicate the optimization problem and reduce the real-time performance.

This paper overcomes the above critical issues by proposing a real-time spatial-temporal trajectory planning scheme that generates a trajectory with nonholonomic constraints and achieves obstacle avoidance for full-dimensional objects, as shown in Fig. 1. Crucially, we use the differential flatness nature of car-like robots to encode all feasibility constraints as analytically and continuously differentiable expressions in the flat space, which are integrated into an optimization formulation. Moreover, to ensure traversability in highly constrained environments, we model the ego vehicle as a convex polygon to accomplish accurate collision avoidance. For static obstacles in environments, we construct the geometric free space which is used to constrain the full model of the ego vehicle to ensure

safety. As for dynamic obstacles, we use convex polygons to cover their shapes. Then, we constrain the lower-bound approximations of the signed distances [1] between the ego vehicle and obstacle polygons at each moment to ensure its safety. A minimal control effort trajectory class [2] is used to parameterize trajectories to speed up the optimization process. Furthermore, the originally constrained programming problem is reformulated as an unconstrained one which is further solved by the quasi-Newton method [3] robustly. We conduct benchmark comparisons in simulation with other prevalent trajectory planners in many different cases to demonstrate the significant superiority of the proposed method. Additionally, we validate our planner on a real platform with fully autonomous onboard computing and no external positioning. The main contributions of this paper can be summarized as follows:

- We present a differential flatness-based spatial-temporal optimization formulation where all states and constraints can be analytically derived from flat outputs, thus immensely simplifying the trajectory planning.
- We achieve full-model obstacle avoidance by using a convex polygon to enclose the ego vehicle. Afterward, we formulate static safety constraints based on geometric free spaces extracted from environments. Moreover, we use signed distances between convex polygons to achieve efficient full-size dynamic avoidance.
- We analyze the characteristics of the constraints in the trajectory planning problem and reformulate the original optimization as an efficiently solvable one without sacrificing optimality.
- We open source the trajectory planning code, aiming to facilitate future progress in the field of motion planning for autonomous vehicles.

The remaining content is organized as follows: Sect. II describes related literature work. The spatial-temporal trajectory planning for autonomous vehicles is presented in Sect. III. Instantaneous state constraints and dynamic obstacle avoidance constraints are discussed in Sect. IV and Sect. V, respectively. In Sect. VI, we reformulate the trajectory optimization. Sect. VII presents benchmarks and real-world experiments. This article is concluded in Sect. VIII.

II. RELATED WORK

A. Sampling-Based Trajectory Generation

Sampling-based methods [4]–[14] are widely used for robot motion planning [15] due to the ease of incorporating user-defined objectives. These typical methods, including state lattice approaches and probabilistic planners, always sample the robot state in the configuration space to find a feasible trajectory connecting the starting node and the goal node. Lattice-based planners [4]–[7] discretize the continuous state space into a lattice graph for planning. Then, graph-search algorithms such as Dijkstra are used to obtain the optimal trajectory in the graph. Probabilistic planners [8]–[14] represented by rapidly-exploring random tree (RRT) [8] obtain a feasible path by expanding a state tree rooted at the starting node. Shkolnik et al. [9] propose reachability-guided RRT, which measures a reachable set of any node to reduce useless

expansions. Karaman and Frazzoli [10] propose RRT* that can ensure global optimality with sufficiently dense sampling. Webb and Berg [11] present kinodynamic RRT* which can satisfy the nonlinear dynamics of a robot. Additionally, more RRT-based algorithms [12]–[14] for nonholonomic robots are presented to improve the efficiency or the adaptability to specific cases. Han et al. [12] propose a lightweight approximate distance metric for nonholonomic constraints to accelerate convergence. The combination of graph search and RRT is presented in [13], where sampled state distribution matches the geometric information of the any-angle path. Similarly, to reduce the sample space, the RRT tree in [14] is expanded along routes in different homotopy classes which are generated by the waypoint planner. Despite the ability to avoid local minima in non-convex space, sampling-based methods confront a dilemma between computation consumption and trajectory quality which limits the direct application in realistic settings.

B. Optimization-Based Trajectory Generation

In structured road environments, the longitudinal-lateral decomposition in Frenét frame [16] with the explicit demonstration of driving progress along the reference lanes is usually employed for autonomous driving [16]–[24]. Adopting the Frenét frame for on-lane planning better imitates human-like driving behaviors [17] and easily encodes lane geometrical constraints. By separate planning of each direction, Lim et al. [18] generate a safe and dynamically feasible trajectory by numerically optimizing the longitudinal movement based on the lateral candidate movements obtained with multiple predefined maneuvers, but with limitations when the lateral and longitudinal motions are tightly coupled. Moreover, the quality of the final solution depends on the number of candidates, which limits real-time applications. Ding et al. [19, 20] propose the spatial-temporal safe corridor (SSC) to ensure dynamic safety. Nevertheless, the temporal resolution is fixed before the optimization, which limits the application in general cases. In works [19, 20], the lateral and longitudinal motions are represented as Bernstein piece-wise polynomials whose convexity is used to simplify the constraints and thus transform the original optimization into quadratic programming (QP). However, it introduces a formal conservatism that prevents the ego vehicle from achieving a desired physical performance. Besides, these above Frenét-based methods always do not model the true kinematics of a vehicle. Therefore, the feasibility of nonholonomic dynamics is not always guaranteed, especially on curvy roads. In addition, the Frenét-based methods are not suitable for employment in unstructured environments because of the dependence on lane lines.

Motion planning in a Cartesian coordinate system is often considered to have sufficient generalization capability to apply to different driving scenarios [25]. To simplify the trajectory generation problem, some intuitive approaches [26]–[28] decouple the spatial shape and dynamics profile of the trajectory. Zhu et al. [26] propose convex elastic band smoothing (CES) algorithm which eliminates the non-convexity of the curvature constraint by assuming that the path lengths before and after

optimization are essentially the same at each iteration. Then, the initial non-convex problem is transformed into a convex optimization with quadratic objective and quadratic constraints (QCQP), and is then solved by FORCES Pro [29]. However, as presented in work [28], the length consistency assumption does not always hold, which invalidates the curvature constraint and thus reducing the feasibility of the control. Based on CES framework, Zhou et al. [28] propose the dual-loop iterative anchoring path smoothing (DL-IAPS) algorithm to generate a smooth and safe path, where sequential convex optimization (SCP) is used to relax the curvature constraint. Whereas, the efficiency of this method relies heavily on the initial path obtained by hybridA* [30], which limits the application in complex environments.

Highly adaptable model predictive control (MPC) approaches always formulate a trajectory planning problem as an optimal control problem (OCP) which is further discretized into a nonlinear programming (NLP) problem. These approaches [31]–[39] always directly optimize discrete states and control inputs, which can conveniently integrate dynamic and safety constraints. Kondak and Hommel [31] first model the motion planning in autonomous parking as an OCP, and then it is solved numerically using sequential quadratic programming (SQP) implemented in the software package SNOPT [40]. The safety constraint is described by the artificial potential field at each obstacle point based on the triangle-area criterion. Nonetheless, this method does not guarantee effective convergence in dense environments. A unified OCP formulation for trajectory generation in unstructured environments is presented in work [34] with arbitrarily placed obstacles. Whereas, the collision-free constraint in work [34] is nominally non-differentiable. Zhang et al. [41] assume that obstacles are convex and propose the optimization-based collision avoidance (OBCA) algorithm which removes the integer variables used to model full-dimensional object collision avoidance [42]. They introduce dual variables to reformulate the distance between the robot and obstacles, and transform the safety constraint into a continuous differentiable form. Then, Zhang et al. [37] integrate OBCA into the MPC problem of motion planning and propose H-OBCA, a hierarchical framework for trajectory planning in unstructured environments. Besides, a reasonable method is presented in work [37] for warm-starting dual variables to speed up the optimization. However, the introduction of dual variables in OBCA-based methods [37, 38] increases the dimension of the problem, making it more difficult to solve. Besides, the number of dual variables is positively correlated with the number of obstacles. As obstacles increases, the problem dimension will rise rapidly, leading to unacceptable computational and memory costs. Li et al. [39] cover the ego vehicle with two circles along the centerline, and then the ego vehicle model is simplified to two points by inflating obstacles. In work [39], obstacle avoidance is ensured by constraining these two points within a safe corridor. This approach essentially enjoys the property that the dimension of collision-free constraints is independent of the number of obstacles. Whereas, such a point-based modeling method does not fully utilize the solution space and often generates overly conservative trajectories, especially in complex environments.

Although the above MPC-based methods have the ability to easily accommodate the motion model of robots, they also have some disadvantages: 1) Since the trajectory is represented by discrete states, ensuring trajectory constraints in highly constrained environments requires increasing the density of states, which significantly reduces real-time performance. 2) The planned trajectory does not completely conform to the real physical motion model due to the discretization of the problem. 3) These methods are unable to analytically obtain the full state of the system at any time.

C. Geometry-Based Trajectory Generation

Geometry-based methods [43]–[50] have been used for autonomous parking due to their high efficiency. Such approaches typically use a series of special curves with limited curvature to compose the path. Then, the path planning is modeled as a two-dimensional Euclidean geometry problem by incorporating the spatial features of the parking spot. Some methods [43]–[46] are presented in one maneuver where circular arcs and line segments are used to construct the path. However, the curvature of such a path in works [43]–[46] is not completely continuous. The abrupt changes in curvature at the junction of different curves always cause discomfort and undesired motion pauses. Two-step methods that guarantee curvature continuity are proposed in works [47]–[50]. These methods firstly use a lower-level geometric planner without consideration of kinematic restrictions for generating a collision-free path. Then, the path is divided and replaced by continuous-curvature components such as β -splines [47], clothoid [48, 49] and Bézier curves [50]. Although the above geometry-based methods are lightweight and intuitive, they can only handle specific tasks with predefined rules and priori information, and lack the versatility to extend to general environments with irregular obstacles.

Instead of decomposing the trajectory planning, discretizing the motion model, or over-relying on the spatial characteristics of environments, we use the differential flatness property of car-like robots to achieve spatial-temporal joint trajectory planning with guaranteed full-dimensional obstacle avoidance. Our method enjoys low computational cost and high robustness against unstructured environments. A brief qualitative comparison with SSC planner [19], DL-IAPS + Piecewise-Jerk Speed Optimization (PJSO) [28] and H-OBCA [37] is shown in Fig. 2, while more detailed quantitative comparisons are presented in Sect. VII.

D. Differential Flatness Vehicles

A differentially flat system, with its natural advantages of expressing state and inputs by flat outputs, has been widely studied and utilized for robot control, tracking, and planning [51]. Different flat outputs tend to represent properties of different flat systems. Especially for trajectory planning, the physical meaning of flat outputs is crucial to generating a trajectory. Intuitively, flat outputs are usually selected as an easy-to-use state related to a point associated with the vehicle. The well-known simplified bicycle model [52] which describes the dynamics of car-like robots without side slip, is widely used

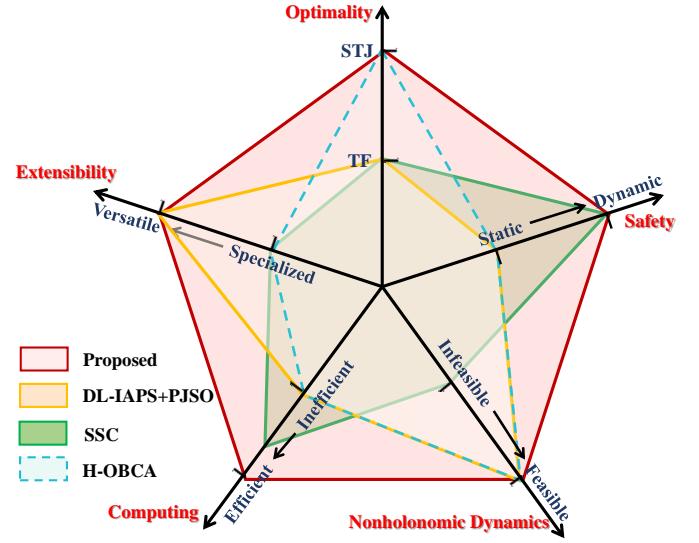


Fig. 2: Illustration of qualitative comparisons with SSC [19], DL-IAPS+PJSO [28] and H-OBCA [37]. In **Optimality** axis: *TF* is Time-Fixed trajectory optimization and *STJ* is Spatial-Temporal Joint trajectory optimization. In **Safety** axis: *Static* means that the planner can only handle static environments, while *Dynamic* represents the ability to ensure dynamic safety. In **Nonholonomic Dynamics** axis: *Infeasible* means nonholonomic constraints are not considered in the planning, while *Feasible* signifies dynamic feasibility guarantees. In **Extensibility** axis: *Specialized* means the requirement for additional assumptions such as the convexity of obstacles or the existence of lane lines, while *Versatile* represents the generalizability to all driving environments. In **Computing** axis, the efficiency increases from the center to the outward.

in trajectory planning for autonomous vehicles. A nonlinear planar holonomic model which involves the lateral tire forces is outlined in work [53], and its flat output is the combination of lateral and longitudinal velocities of a distinct point determined by vehicle parameters. Another representation [54] maps the dynamics of a half-car dynamical model with wheel slip on front and rear Huygens centers of oscillation (CO) and additional yaw dynamics. Because the acceleration of CO is independent of lateral tire friction forces, the flatness model associated with CO has been extended for tracking and planning [55, 56]. The dynamic feasibility constraints on the above flat description for vehicles that introduce flip and friction model are the same as the common bicycle model [52], while model identification and analysis of stability increase the complexity of applications. In this paper, we employ the simplified bicycle model [52] with a flat output as the rear-wheel position, which concisely encodes dynamic feasibility and is extensively applied in trajectory parameterization [57].

III. SPATIAL-TEMPORAL OPTIMAL TRAJECTORY PLANNING

In this section, we present the spatial-temporal joint optimization formulation for trajectory planning. To construct

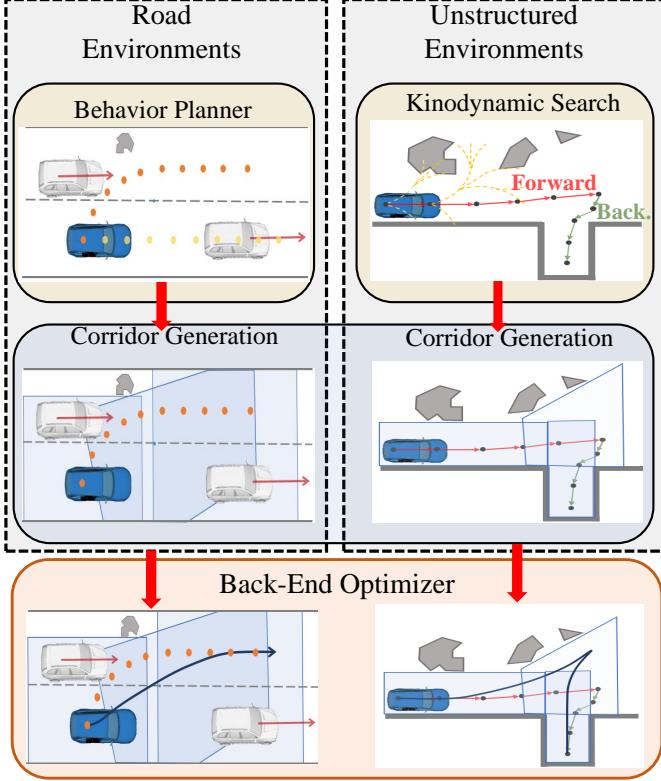


Fig. 3: The trajectory planning framework.

the optimization formulation, we discuss the complete motion planning pipeline and introduce the differential flat model of car-like robots. Then, we show the formulation of the trajectory optimization problem in flat-output space considering human comfort, execution time, and feasibility constraints. Last but not least, we analyze the gradient propagation chain of the problem for subsequent numerical optimization.

A. Planning Pipeline

The whole pipeline follows a hierarchical structure, as shown in Fig. 3. In practical applications, the proposed planner cooperates with a front-end planner whose main role is to provide an initial guess. In structured road environments, our planner is integrated into a multi-layer planning framework [19], where the behavior policy planner is used to provide front-end outputs. In unstructured environments without reference lines, we adopt the lightweight hybridA* algorithm to find a collision-free path that is further optimized by the proposed planner. Moreover, for each node to be expanded, we try using Reed Shepp Curve [58] to shoot the end state for the earlier termination of the search process. For complex driving tasks such as autonomous parking, the front-end output often contain both forward and back vehicle movements. By reasonably assuming the vehicle always reaches a complete stop at the gear shifting position, we parameterize the forward and backward segments of the trajectory as piecewise polynomials, respectively, whose specific formulations are presented in Sect. III-C. Besides, we define an additional variable $\eta \in \{-1, 1\}$ to characterize the motion direction

of the vehicle, where $\eta = -1$ and $\eta = 1$ represent the backward and forward movements, respectively. Additionally, η is determined by the front-end output and is prefixed before the back-end optimization process.

B. Differentially Flat Vehicle Model

We use the simplified kinematic bicycle model in the Cartesian coordinate frame to describe a four-wheel vehicle. Assuming that the car is front-wheel driven and steered with perfect rolling and no slipping, the model can be described as Fig. 4. The state vector is $\mathbf{x} = (p_x, p_y, \theta, v, a_t, a_n, \phi, \kappa)^T$, where $\mathbf{p} = (p_x, p_y)^T$ denotes the position at the center of the rear wheels, v is the longitudinal velocity w.r.t vehicle's body frame, a_t represents the longitude acceleration, a_n is the latitude acceleration, ϕ is the steering angle of the front wheels and κ is the curvature. The dynamics are given by :

$$\left\{ \begin{array}{l} \dot{p}_x = v \cos \theta, \\ \dot{p}_y = v \sin \theta, \\ \dot{\theta} = \frac{1}{L} v \tan \phi, \\ \dot{v} = a_t, \\ a_n = v^2 \kappa, \\ \kappa = \tan \phi / L, \end{array} \right. \quad (1)$$

where L is the wheelbase length of the car and CoG is the abbreviation of center of gravity. Thanks to the thorough study of the differentially flat car model [51], we choose the flat output as $\boldsymbol{\sigma} := (\sigma_x, \sigma_y)^T$ with a physical meaning that $\boldsymbol{\sigma} = \mathbf{p}$ is the position centered on the rear wheel of the car. Other variable transformations except p_x, p_y can be expressed as:

$$v = \eta \sqrt{\dot{\sigma}_x^2 + \dot{\sigma}_y^2}, \quad (2a)$$

$$\theta = \arctan 2(\eta \dot{\sigma}_y, \eta \dot{\sigma}_x), \quad (2b)$$

$$a_t = \eta(\dot{\sigma}_x \ddot{\sigma}_x + \dot{\sigma}_y \ddot{\sigma}_y) / \sqrt{\dot{\sigma}_x^2 + \dot{\sigma}_y^2}, \quad (2c)$$

$$a_n = \eta(\dot{\sigma}_x \ddot{\sigma}_y - \dot{\sigma}_y \ddot{\sigma}_x) / \sqrt{\dot{\sigma}_x^2 + \dot{\sigma}_y^2}, \quad (2d)$$

$$\phi = \arctan \left(\eta(\dot{\sigma}_x \ddot{\sigma}_y - \dot{\sigma}_y \ddot{\sigma}_x) L / (\dot{\sigma}_x^2 + \dot{\sigma}_y^2)^{\frac{3}{2}} \right), \quad (2e)$$

$$\kappa = \eta(\dot{\sigma}_x \ddot{\sigma}_y - \dot{\sigma}_y \ddot{\sigma}_x) / (\dot{\sigma}_x^2 + \dot{\sigma}_y^2)^{\frac{3}{2}}. \quad (2f)$$

Consequently, with the natural differential flatness property, we can use the flat outputs and their finite derivatives to characterize arbitrary state quantities of the vehicle, which simplifies the trajectory planning and facilitates optimization.

C. Optimization Formulation with Vehicle Flat Outputs

The i -th segment of the trajectory is formulated as a 2-dimensional and time-uniform M_i -piece polynomial with degree $N = 2s - 1$, which is parameterized by the intermediate waypoints $\mathbf{q}_i = (q_{i,1}, \dots, q_{i,M_i-1}) \in \mathbb{R}^{2 \times (M_i-1)}$, the time interval for each piece $\delta T_i \in \mathbb{R}^+$, and the coefficient matrix

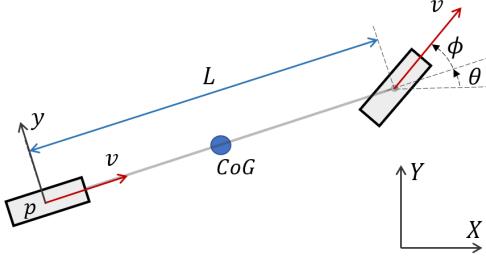


Fig. 4: The kinematic bicycle model.

$\mathbf{c}_i = (\mathbf{c}_{i,1}^T, \dots, \mathbf{c}_{i,M_i}^T)^T \in \mathbb{R}^{2M_i s \times 2}$. Then, the j -th piece of the i -th segment $\boldsymbol{\sigma}_{i,j}$ is written as:

$$\begin{aligned}\boldsymbol{\sigma}_{i,j}(t) &:= \mathbf{c}_{i,j}^T \boldsymbol{\beta}(t), \\ \boldsymbol{\beta}(t) &:= (1, t, t^2, \dots, t^N)^T,\end{aligned}\quad (3)$$

$$\forall t \in [0, \delta T_i], \forall i \in \{1, 2, \dots, n\}, \forall j \in \{1, 2, 3, \dots, M_i\},$$

where n is the number of trajectory segments and $\boldsymbol{\beta}(t)$ is a natural basis. The M_i -piece polynomial trajectory $\boldsymbol{\sigma}_i : [0, T_i]$ is obtained:

$$\begin{aligned}\boldsymbol{\sigma}_i(t) &= \boldsymbol{\sigma}_{i,j}(t - (j-1) * \delta T_i), \\ \forall j &\in \{1, 2, \dots, M_i\}, t \in [(j-1) * \delta T_i, j * \delta T_i].\end{aligned}\quad (4)$$

Here, the total duration of the i -th segment of the trajectory is $T_i = M_i * \delta T_i$. Then, the complete trajectory representation $\boldsymbol{\sigma}(t) : [0, T_s]$ is formulated:

$$\begin{aligned}\boldsymbol{\sigma}(t) &= \boldsymbol{\sigma}_i(t - \hat{T}_i), \\ \forall i &\in \{1, 2, \dots, n\}, t \in [\hat{T}_i, \hat{T}_{i+1}],\end{aligned}\quad (5)$$

where $T_s = \sum_{i=1}^n T_i$ is the duration of the whole trajectory, $\hat{T}_i = \sum_{i=1}^{i-1} T_i$ is the timestamp of the starting point of the i -th segment and \hat{T}_1 is set as 0. Moreover, we define a coefficient set $\mathbf{c} = (\mathbf{c}_1^T, \mathbf{c}_2^T, \dots, \mathbf{c}_n^T)^T \in \mathbb{R}^{(\sum_{i=1}^n 2M_i s) \times 2}$ and a time set $\mathbf{T} = (T_1, T_2, \dots, T_n)^T \in \mathbb{R}^n$ for the subsequent derivation. With constraints for obstacles avoidance and dynamic feasibility, the minimal control effort problem involving time regularization can be expressed as a nonlinear constrained optimization:

$$\min_{\mathbf{c}, \mathbf{T}} J(\mathbf{c}, \mathbf{T}) = \int_0^{T_s} \boldsymbol{\mu}(t)^T \mathbf{W} \boldsymbol{\mu}(t) dt + w_T T_s \quad (6a)$$

$$\text{s.t. } \boldsymbol{\mu}(t) = \boldsymbol{\sigma}^{[s]}(t), \forall t \in [0, T_s], \quad (6b)$$

$$\boldsymbol{\sigma}_0^{[s-1]}(0) = \bar{\boldsymbol{\sigma}}_0, \boldsymbol{\sigma}_n^{[s-1]}(T_n) = \bar{\boldsymbol{\sigma}}_f, \quad (6c)$$

$$\boldsymbol{\sigma}_i^{[s-1]}(T_i) = \boldsymbol{\sigma}_{i+1}^{[s-1]}(0) = \tilde{\boldsymbol{\sigma}}_i, 1 \leq i < n, \quad (6d)$$

$$\boldsymbol{\sigma}_{i,j}^{[\tilde{d}]}(\delta T_i) = \boldsymbol{\sigma}_{i,j+1}^{[\tilde{d}]}(0), 1 \leq i \leq n, 1 \leq j < M_i, \quad (6e)$$

$$T_i > 0, 1 \leq i \leq n, \quad (6f)$$

$$\mathcal{G}_d(\boldsymbol{\sigma}(t), \dots, \boldsymbol{\sigma}^{(s)}(t), t) \preceq 0, \quad \forall d \in \mathcal{D}, \forall t \in [0, T_s], \quad (6g)$$

where $\mathbf{W} \in \mathbb{R}^{2 \times 2}$ is a diagonal matrix to penalize control efforts. Eq. (6c) is the boundary condition, where $\bar{\boldsymbol{\sigma}}_0, \bar{\boldsymbol{\sigma}}_f \in \mathbb{R}^{2 \times s}$ are the user-specified initial and final states. $\tilde{\boldsymbol{\sigma}}_i \in \mathbb{R}^{2 \times s}$ is the switching state between forward and reverse gears in which the position and the tangential direction of the motion

curve are optimized. Moreover, the specific constraint Eq. (6d) will be described in Sect. VI. Eq. (6e) is the continuity constraint up to degree \tilde{d} . The second term $w_T T_s$ in the objective function is the time regularization term to restrict the total duration T_s , with a weight $w_T \in \mathbb{R}^+$. The constraint function at d is defined as \mathcal{G}_d . In our formulation, the set $\mathcal{D} = \{d : d = v, a_t, a_n, \kappa, \zeta, \Theta\}$ includes dynamic feasibility (v, a_t, a_n, κ) , static and dynamic obstacle avoidance (ζ, Θ) . Besides, s is chosen as 3, which means that the integration of jerk is minimized to ensure human comfort [59].

D. Gradient Derivation

Without loss of completeness, we first derive the analytic gradients of the objective function J w.r.t $\mathbf{c}_{i,j}$ and T_i :

$$\frac{\partial J}{\partial \mathbf{c}_{i,j}} = 2 \left(\int_0^{\delta T_i} \boldsymbol{\beta}^{(s)}(t) \boldsymbol{\beta}^{(s)}(t)^T dt \right) \mathbf{c}_{i,j}, \quad (7)$$

$$\frac{\partial J}{\partial T_i} = \frac{1}{M_i} \sum_{j=1}^{M_i} \mathbf{c}_{i,j}^T \boldsymbol{\beta}^{(s)}(\delta T_i) \boldsymbol{\beta}^{(s)}(\delta T_i)^T \mathbf{c}_{i,j} + w_T. \quad (8)$$

The feasibility constraints Eq.(6g) imposed on the entire trajectory are equivalent to each piece of any piece-wise polynomial trajectory segment complying with these constraints:

$$\begin{aligned}\mathcal{G}_d(\boldsymbol{\sigma}(t), \dots, \boldsymbol{\sigma}^{(s)}(t), t) &\preceq 0, \forall d \in \mathcal{D}, \forall t \in [0, T_s] \iff \\ \mathcal{G}_d(\boldsymbol{\sigma}_{i,j}(\bar{t}), \dots, \boldsymbol{\sigma}_{i,j}^{(s)}(\bar{t}), \bar{t}) &\preceq 0, \forall d \in \mathcal{D}, \forall \bar{t} \in [0, \delta T_i], \\ \forall i &\in \{1, 2, \dots, n\}, \forall j \in \{1, 2, 3, \dots, M_i\},\end{aligned}\quad (9)$$

where \bar{t} is the relative timestamp and $\hat{t} = \hat{T}_i + \delta T_i * (j-1) + \bar{t}$ is the absolute timestamp. Therefore, to approximate the continuous-time formula Eq. (9), we uniformly discretize each piece of the piece-wise polynomial into $\lambda \in \mathbb{N}_{>0}$ constraint points. Moreover, we ensure trajectory feasibility by imposing constraints on these constraint points. Then, the continuous-time formula Eq. (9) is transformed into a discrete form:

$$\begin{aligned}\mathcal{G}_{d,i,j,k}(\mathbf{c}_{i,j}, \mathbf{T}) &\preceq 0, \\ \mathcal{G}_{d,i,j,k}(\mathbf{c}_{i,j}, \mathbf{T}) &:= \mathcal{G}_d(\boldsymbol{\sigma}_{i,j,k}, \dots, \boldsymbol{\sigma}_{i,j,k}^{(s)}, \hat{t}), \\ \boldsymbol{\sigma}_{i,j,k}^{(\bar{d})} &:= \boldsymbol{\sigma}_{i,j}^{(\bar{d})}(\bar{t}), \quad \forall \bar{d} \in \{0, 1, \dots, s\}, \\ \bar{t} &= \frac{k T_i}{\lambda M_i}, \quad \hat{t} = \hat{T}_i + \left(\frac{j-1}{M_i} + \frac{k}{\lambda M_i} \right) T_i, \\ \forall k &\in \{0, 1, 2, \dots, \lambda\}, \quad \forall d \in \mathcal{D}.\end{aligned}\quad (10)$$

Without loss of generality, we derive the gradient propagation at any constraint point based on the chain rule:

$$\frac{\partial \mathcal{G}_{d,i,j,k}}{\partial \mathbf{c}_{i,j}} = \sum_{\bar{d}=0}^s \beta^{(\bar{d})}(\bar{t}) \left(\frac{\partial \mathcal{G}_{d,i,j,k}}{\partial \boldsymbol{\sigma}_{i,j,k}^{(\bar{d})}} \right)^T, \quad (11)$$

$$\frac{\partial \mathcal{G}_{d,i,j,k}}{\partial \mathbf{T}} = \frac{\partial \mathcal{G}_{d,i,j,k}}{\partial \bar{t}} \frac{\partial \bar{t}}{\partial \mathbf{T}} + \frac{\partial \mathcal{G}_{d,i,j,k}}{\partial \hat{t}} \frac{\partial \hat{t}}{\partial \mathbf{T}}, \quad (12)$$

We further derive time-related gradient terms inside Eq. (12):

$$\frac{\partial \mathcal{G}_{d,i,j,k}}{\partial \bar{t}} = \sum_{\bar{d}=0}^s \left(\boldsymbol{\sigma}_{i,j,k}^{(\bar{d}+1)} \right)^T \frac{\partial \mathcal{G}_{d,i,j,k}}{\partial \boldsymbol{\sigma}_{i,j,k}^{(\bar{d})}}, \quad (13)$$

$$\frac{\partial \bar{t}}{\partial \mathbf{T}} = \left(\mathbf{0}_{i-1}^T, \frac{k}{\lambda M_i}, \mathbf{0}_{n-i}^T \right)^T, \quad (14)$$

$$\frac{\partial \hat{t}}{\partial \mathbf{T}} = \left(\mathbf{1}_{i-1}^T, \frac{k}{\lambda M_i} + \frac{j-1}{M_i}, \mathbf{0}_{n-i}^T \right)^T, \quad (15)$$

where all the above gradient calculations w.r.t vectors follow the denominator layout. As a result, by substituting Eq.(13)-(15) to Eq.(12), we can obtain the gradients of $\mathcal{G}_{d,i,j,k}$ w.r.t $\boldsymbol{\sigma}_{i,j,k}$ and \mathbf{T} once $\partial \mathcal{G}_{d,i,j,k} / \partial \boldsymbol{\sigma}_{i,j,k}^{(\bar{d})}$ and $\partial \mathcal{G}_{d,i,j,k} / \partial \hat{t}$ are specified. Besides, it is worth mentioning that constraint functions $\mathcal{G}_{d,i,j,k}$ can be precisely expressed by some of the quantities in \hat{t} and $\boldsymbol{\sigma}_{i,j,k}^{(\bar{d})}$, where $\bar{d} \in \{0, 1, 2, \dots, s\}$. Therefore, the gradients w.r.t irrelevant variables are 0 without derivation. In subsequent sections, we present the specific formulation of the constraint functions $\mathcal{G}_{d \in \mathcal{D}}$ and derive gradients. For simplification, i, j, k and relative timestamp \hat{t} are omitted in Sect. IV and Sect. V.

IV. INSTANTANEOUS STATE CONSTRAINTS

In this section, we introduce the instantaneous state constraints for trajectory optimization, where these constraint functions are only related to the instantaneous states of the vehicle.

A. Dynamic Feasibility

1) *Longitude Velocity Limit*: For autonomous driving, the longitude velocity always needs to be limited within a reasonable range because of practical factors such as traffic rules, physical vehicle performance, and environmental uncertainty. Then, the constraint function of longitude velocity at a constraint point is defined as follows:

$$\mathcal{G}_v(\dot{\boldsymbol{\sigma}}) = \dot{\boldsymbol{\sigma}}^T \dot{\boldsymbol{\sigma}} - v_m^2. \quad (16)$$

where v_m is the magnitude of maximal longitude velocity. The gradient of $\mathcal{G}_v(\dot{\boldsymbol{\sigma}})$ is written as:

$$\frac{\partial \mathcal{G}_v}{\partial \dot{\boldsymbol{\sigma}}} = 2\dot{\boldsymbol{\sigma}}. \quad (17)$$

Accordingly, we can obtain the gradients of \mathcal{G}_v by combining Eq.(17) and Eq.(11)-(15).

2) *Acceleration Limit*: The acceleration is always required to be limited to prevent skidding due to friction limits between the tire and the ground. From Eq. (2c)(2d), we define the constraint functions of longitude and latitude acceleration at a constraint point:

$$\mathcal{G}_{a_t}(\dot{\boldsymbol{\sigma}}, \ddot{\boldsymbol{\sigma}}) = \frac{(\ddot{\boldsymbol{\sigma}}^T \dot{\boldsymbol{\sigma}})^2}{\dot{\boldsymbol{\sigma}}^T \dot{\boldsymbol{\sigma}}} - a_{tm}^2, \quad (18)$$

$$\mathcal{G}_{a_n}(\dot{\boldsymbol{\sigma}}, \ddot{\boldsymbol{\sigma}}) = \frac{(\ddot{\boldsymbol{\sigma}}^T \mathbf{B} \dot{\boldsymbol{\sigma}})^2}{\dot{\boldsymbol{\sigma}}^T \dot{\boldsymbol{\sigma}}} - a_{nm}^2, \quad (19)$$

where a_{tm} and a_{nm} are the maximal longitude and latitude acceleration and $\mathbf{B} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is an auxiliary antisymmetric

matrix. The gradients of $\mathcal{G}_{a_t}(\dot{\boldsymbol{\sigma}}, \ddot{\boldsymbol{\sigma}})$ and $\mathcal{G}_{a_n}(\dot{\boldsymbol{\sigma}}, \ddot{\boldsymbol{\sigma}})$ are derived as :

$$\frac{\partial \mathcal{G}_{a_t}}{\partial \dot{\boldsymbol{\sigma}}} = 2 \frac{\ddot{\boldsymbol{\sigma}}^T \dot{\boldsymbol{\sigma}}}{\|\dot{\boldsymbol{\sigma}}\|_2^2} \ddot{\boldsymbol{\sigma}} - 2 \left(\frac{\ddot{\boldsymbol{\sigma}}^T \dot{\boldsymbol{\sigma}}}{\|\dot{\boldsymbol{\sigma}}\|_2^2} \right)^2 \dot{\boldsymbol{\sigma}}, \quad (20)$$

$$\frac{\partial \mathcal{G}_{a_t}}{\partial \ddot{\boldsymbol{\sigma}}} = 2 \frac{\ddot{\boldsymbol{\sigma}}^T \dot{\boldsymbol{\sigma}}}{\|\dot{\boldsymbol{\sigma}}\|_2^2} \dot{\boldsymbol{\sigma}}, \quad (21)$$

$$\frac{\partial \mathcal{G}_{a_n}}{\partial \dot{\boldsymbol{\sigma}}} = 2 \frac{\ddot{\boldsymbol{\sigma}}^T \mathbf{B} \dot{\boldsymbol{\sigma}}}{\|\dot{\boldsymbol{\sigma}}\|_2^2} \mathbf{B}^T \dot{\boldsymbol{\sigma}} - 2 \left(\frac{\ddot{\boldsymbol{\sigma}}^T \mathbf{B} \dot{\boldsymbol{\sigma}}}{\|\dot{\boldsymbol{\sigma}}\|_2^2} \right)^2 \dot{\boldsymbol{\sigma}}, \quad (22)$$

$$\frac{\partial \mathcal{G}_{a_n}}{\partial \ddot{\boldsymbol{\sigma}}} = 2 \frac{\ddot{\boldsymbol{\sigma}}^T \mathbf{B} \dot{\boldsymbol{\sigma}}}{\|\dot{\boldsymbol{\sigma}}\|_2^2} \mathbf{B} \dot{\boldsymbol{\sigma}}. \quad (23)$$

3) *Front Steer Angle Limit*: The front steer angle needs to be limited to ensure the nonholonomic dynamic feasibility of the vehicle. Due to the monotonicity of the tangent function, we restrict the front steer angle by limiting the curvature $\kappa = \tan \phi / L$ in $[-\tan \phi_m / L, \tan \phi_m / L] := [-\kappa_m, \kappa_m]$, where ϕ_m is the preset maximum steer angle and κ_m is the corresponding maximum curvature. Then, the nonholonomic dynamic constraint function $\mathcal{G}_\kappa = (\mathcal{G}_{\kappa_l}, \mathcal{G}_{\kappa_r})^T \in \mathbb{R}^2$ is defined as two linear penalties on the curvature, expressed as:

$$\mathcal{G}_{\kappa_l}(\dot{\boldsymbol{\sigma}}, \ddot{\boldsymbol{\sigma}}) = \frac{\ddot{\boldsymbol{\sigma}}^T \mathbf{B} \dot{\boldsymbol{\sigma}}}{\|\dot{\boldsymbol{\sigma}}\|_2^3} - \kappa_m, \quad (24)$$

$$\mathcal{G}_{\kappa_r}(\dot{\boldsymbol{\sigma}}, \ddot{\boldsymbol{\sigma}}) = -\kappa_m - \frac{\ddot{\boldsymbol{\sigma}}^T \mathbf{B} \dot{\boldsymbol{\sigma}}}{\|\dot{\boldsymbol{\sigma}}\|_2^3}. \quad (25)$$

Furthermore, we derive the gradients w.r.t $\dot{\boldsymbol{\sigma}}_{i,j}$ and $\ddot{\boldsymbol{\sigma}}_{i,j}$:

$$\frac{\partial \mathcal{G}_{\kappa_l}}{\partial \dot{\boldsymbol{\sigma}}} = \frac{\mathbf{B}^T \dot{\boldsymbol{\sigma}}}{\|\dot{\boldsymbol{\sigma}}\|_2^3} - 3 \frac{\ddot{\boldsymbol{\sigma}}^T \mathbf{B} \dot{\boldsymbol{\sigma}}}{\|\dot{\boldsymbol{\sigma}}\|_2^5} \dot{\boldsymbol{\sigma}}, \quad (26)$$

$$\frac{\partial \mathcal{G}_{\kappa_l}}{\partial \ddot{\boldsymbol{\sigma}}} = \frac{\mathbf{B} \dot{\boldsymbol{\sigma}}}{\|\dot{\boldsymbol{\sigma}}\|_2^3}, \quad (27)$$

$$\frac{\partial \mathcal{G}_{\kappa_r}}{\partial \dot{\boldsymbol{\sigma}}} = -\frac{\partial \mathcal{G}_{\kappa_l}}{\partial \dot{\boldsymbol{\sigma}}}, \quad (28)$$

$$\frac{\partial \mathcal{G}_{\kappa_r}}{\partial \ddot{\boldsymbol{\sigma}}} = -\frac{\partial \mathcal{G}_{\kappa_l}}{\partial \ddot{\boldsymbol{\sigma}}}. \quad (29)$$

B. Static Obstacle Avoidance

In this subsection, we analytically present static safety constraints that are efficiently computable based on the geometric representation of the free space in the environment. We first decompose the semantic environment and extract the safe space to construct a driving corridor consisting of a series of convex polygons. Then, we derive the necessary and sufficient condition for enforcing the full vehicle model in the driving corridor, which is used to construct static no-collision constraints.

Before specific derivation, we introduce the pipeline of the constraint modeling. We first discretize the collision-free path generated by the front end into sampling points whose number is the same as the number of constraint points in the back-end optimization. Then, combined with the environmental information, we generate a free convex polygon based on the sampling point by the method [60] or directly expanding in each defined direction. As a result, the entire trajectory is guaranteed to be safe by confining the full vehicle model at

each constraint point to the corresponding convex polygon, as shown in Fig. 5. We use a convex polygon to enclose the full shape of the ego vehicle which is defined as \mathbb{E} . Moreover, we define the vertex set \mathcal{E} of the convex polygon as:

$$\mathcal{E} = \{\mathbf{v}_e \in \mathbb{R}^2 : \mathbf{v}_e = \boldsymbol{\sigma} + \mathbf{R}\mathbf{l}_e, e = 1, 2, \dots, n_e\}, \quad (30)$$

where \mathbf{R} is the rotation matrix from the body to the world frame, transformed into flat outputs as:

$$\mathbf{R} = \frac{\eta}{\|\dot{\boldsymbol{\sigma}}\|_2} (\dot{\boldsymbol{\sigma}}, \mathbf{B}\dot{\boldsymbol{\sigma}}). \quad (31)$$

Here, n_e is the number of vertexes, and \mathbf{l}_e is the coordinate of the e -th vertex in the body frame. $\eta \in \{-1, 1\}$ is a prefixed auxiliary variable to indicate the motion direction of the segment of the trajectory. Note n_e and \mathbf{l}_e are also constant once the vehicle model is identified. The H-representation [61] of each convex polygon \mathcal{P}^H in the driving corridor is obtained:

$$\begin{aligned} \mathcal{P}^H &= \{\mathbf{q} \in \mathbb{R}^2 : \mathbf{A}\mathbf{q} \leq \mathbf{b}\}, \\ \mathbf{A} &= (\mathbf{A}_1, \dots, \mathbf{A}_z, \dots, \mathbf{A}_{n_z})^T \in \mathbb{R}^{n_z \times 2}, \\ \mathbf{b} &= (b_1, \dots, b_z, \dots, b_{n_z})^T \in \mathbb{R}^{n_z}, \end{aligned} \quad (32)$$

where n_z is the number of hyperplanes, $\mathbf{A}_z \in \mathbb{R}^2$ and $b_z \in \mathbb{R}$ are the descriptors of a hyperplane, which can be determined by a point on the hyperplane and the normal vector. Additionally, once the driving corridor is generated, the hyperplane descriptors \mathbf{A}_z and b_z are also completely determined. A sufficient and necessary condition for containing the full model of the vehicle in a convex polygon is that each vertex of the vehicle model is contained in the convex polygon:

$$\begin{aligned} \mathbb{E} \subseteq \mathcal{P}^H &\iff \\ \boldsymbol{\sigma} + \mathbf{R}\mathbf{l}_e &\subseteq \mathcal{P}^H \quad \forall e \in \{1, 2, \dots, n_e\}. \end{aligned} \quad (33)$$

A sufficient and necessary condition for containing a vertex in a convex polygon is that the vertex is on the inner side of each hyperplane:

$$\begin{aligned} \boldsymbol{\sigma} + \mathbf{R}\mathbf{l}_e &\subseteq \mathcal{P}^H \iff \\ \mathbf{A}_z^T (\boldsymbol{\sigma} + \mathbf{R}\mathbf{l}_e) &\leq b_z \quad \forall z \in \{1, 2, \dots, n_z\}. \end{aligned} \quad (34)$$

Therefore, the spatial constraint function at a constraint point is $\mathcal{G}_\zeta = (\mathcal{G}_{\zeta_{1,1}}, \dots, \mathcal{G}_{\zeta_{e,z}}, \dots, \mathcal{G}_{\zeta_{n_e,n_z}})^T \in \mathbb{R}^{n_e n_z}$, with $n_e n_z$ linear constraint penalty about vertices of the ego vehicle, which is defined in the flat-output space as:

$$\mathcal{G}_{\zeta_{e,z}}(\boldsymbol{\sigma}, \dot{\boldsymbol{\sigma}}) = \mathbf{A}_z^T (\boldsymbol{\sigma} + \mathbf{R}\mathbf{l}_e) - b_z. \quad (35)$$

Before further derivation, we define an auxiliary expression $\mathcal{F}(\mathbf{l}) : \mathbb{R}^2 \rightarrow \mathbb{R}^{2 \times 2}$ to simplify the form:

$$\mathcal{F}(\mathbf{l}) = \frac{\eta(\mathbf{l}, \mathbf{B}\mathbf{l})^T}{\|\dot{\boldsymbol{\sigma}}\|_2} - \frac{\dot{\boldsymbol{\sigma}}(\mathbf{R}\mathbf{l})^T}{\|\dot{\boldsymbol{\sigma}}\|_2^2}. \quad (36)$$

The gradients of the constraint function w.r.t $\boldsymbol{\sigma}$ and $\dot{\boldsymbol{\sigma}}$ are:

$$\frac{\partial \mathcal{G}_{\zeta_{e,z}}}{\partial \boldsymbol{\sigma}} = \mathbf{A}_z, \quad (37)$$

$$\frac{\partial \mathcal{G}_{\zeta_{e,z}}}{\partial \dot{\boldsymbol{\sigma}}} = \mathcal{F}(\mathbf{l}_e)\mathbf{A}_z. \quad (38)$$

Then, the gradients w.r.t the polynomial coefficients and durations can also be calculated by propagating equations Eq.(11)-

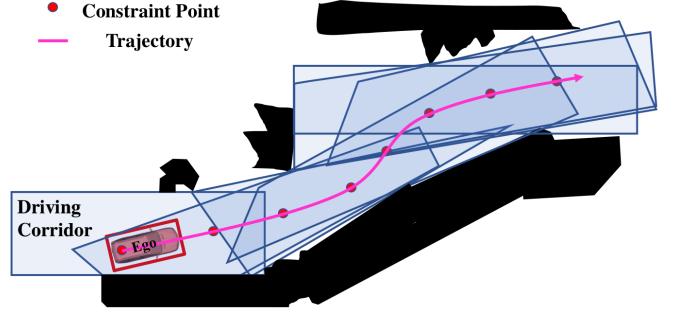


Fig. 5: Visualization of the safe driving corridor. The convex polygon contains the full model of the ego vehicle at any constraint point. With a proper trajectory resolution, we can practically guarantee the static no-collision property.

(15). Moreover, the proposed planner can also be combined with other common map representations to guarantee static obstacle avoidance, such as Euclidean Sign Distance Field (ESDF). Intuitively, we can obtain gradient information from the ESDF map to push the full rigid body of the vehicle away from obstacles.

V. DYNAMIC OBSTACLE AVOIDANCE

Dynamic safety is guaranteed by ensuring that the minimum distance between the ego vehicle and obstacle convex polygons at each moment of the trajectory is greater than the safety threshold. To increase readability, we introduce helpful priors for evaluating the signed distance between convex polygons. Then, we elaborate on the dynamic avoidance constraint function which is further relaxed into a continuously differentiable form.

A. Distance Representations

1) *Signed Distances for Rigid Objects*: We consider two convex polytopes \mathbb{E}, \mathbb{O} bounded by the intersection of half-spaces, as $\mathbb{E} = \cap_{e=1}^{K_e} \mathcal{P}_e$, $\mathbb{O} = \cap_{o=1}^{K_o} \mathcal{P}_o$. The usual convex collision avoidance method penalizes the signed distance between two sets. The distance is defined with the minimal translation \mathcal{T} as:

$$\text{dist}(\mathbb{E}, \mathbb{O}) = \min_{\mathcal{T}} \{\|\mathcal{T}\| : (\mathbb{E} + \mathcal{T}) \cap \mathbb{O} \neq \emptyset\}. \quad (39)$$

When overlapping, $\text{dist} = 0$ holds, which is insufficient to obtain gradient directions to separate them. The penetration depth can be combined to solve this issue:

$$\text{pen}(\mathbb{E}, \mathbb{O}) = \min_{\mathcal{T}} \{\|\mathcal{T}\| : (\mathbb{E} + \mathcal{T}) \cap \mathbb{O} = \emptyset\}. \quad (40)$$

Hence, we can get the signed distance:

$$\text{sd}(\mathbb{E}, \mathbb{O}) := \text{dist}(\mathbb{E}, \mathbb{O}) - \text{pen}(\mathbb{E}, \mathbb{O}). \quad (41)$$

Computing the signed distance requires solving minimum optimization problems Eq. (39, 40), which is unsuitable for embedding into our trajectory optimization problem. We will discuss a Minsky difference-based algorithm for the approximate efficient computation of signed distances in the next section.

2) *Approximation Distances*: We follow the definition of Minkowski difference used in GJK algorithm [62]. Considering the general case where $A, B \in \mathbb{R}^n$ are two sets, the Minkowski Difference is defined by:

$$A - B = \{a - b \in \mathbb{R}^n : a \in A, b \in B\}. \quad (42)$$

Based on the core property [63] extensively used for collision checking:

$$\text{sd}(\mathbb{E}, \mathbb{O}) = \text{sd}(\mathbf{0}, \mathbb{O} - \mathbb{E}). \quad (43)$$

The problem of computing signed distances between two sets can be reduced to the distance between the origin point $\mathbf{0}$ to the set $\mathbb{O} - \mathbb{E}$. A concise formulation to bound signed distances is proposed in [1], defined as the maximum signed distance from the origin to the Minkowski difference between a polygon and each hyperplane:

$$\max_{\mathcal{P}_e, \mathcal{P}_o} \{\text{sd}(\mathbf{0}, \mathbb{O} - \mathcal{P}_e), \text{sd}(\mathbf{0}, \mathcal{P}_o - \mathbb{E})\} \leq \text{sd}(\mathbb{E}, \mathbb{O}),$$

$$\forall e = \{1, \dots, K_e\}, \forall o = \{1, \dots, K_o\}. \quad (44)$$

By extending the Minkowski difference to the case between a polygon and each hyperplane, we have

$$\begin{aligned} \mathbb{O} - \mathcal{P}_e &= \{\mathbf{p} \in \mathbb{R}^n : \mathbf{p} + \mathbf{y} \in \mathbb{O}, \mathbf{y} \in \mathcal{P}_e\}, \\ &= \{\mathbf{p} \in \mathbb{R}^n : (\mathbf{H}^e)^T \mathbf{p} \geq -h^e + (\mathbf{H}^e)^T \mathbf{u}, \mathbf{u} \in \mathbb{O}\}. \end{aligned} \quad (45)$$

with $\mathcal{P}_e = \{\mathbf{y} \in \mathbb{R}^n : (\mathbf{H}^e)^T \mathbf{y} \leq h^e\}$ as a hyperplane of the set \mathbb{E} . Similarly, we obtain

$$\mathcal{P}_o - \mathbb{E} = \{\mathbf{p} \in \mathbb{R}^n : (\mathbf{G}^o)^T \mathbf{p} \leq g^o - (\mathbf{G}^o)^T \mathbf{y}, \mathbf{y} \in \mathbb{E}\}. \quad (46)$$

The signed distance can be computed as:

$$\text{sd}(\mathbf{0}, \mathbb{O} - \mathcal{P}_e) = \frac{1}{\|\mathbf{H}^e\|_2} (-h^e + \min_{\mathbf{u}} (\mathbf{H}^e)^T \mathbf{u}), \quad (47)$$

$$\text{sd}(\mathbf{0}, \mathcal{P}_o - \mathbb{E}) = \frac{1}{\|\mathbf{G}^o\|_2} (-g^o + \min_{\mathbf{y}} (\mathbf{G}^o)^T \mathbf{y}). \quad (48)$$

The physical meaning of this formulation is that the signed distance of two sets is bounded by a maximum signed distance of one set to each hyperplane and vice versa. For convex polytopes, we only need to check each vertex to get the minimum value and then the lower bound can be analytically calculated for optimization, as shown in Fig. 6. In practical applications, we can constrain the lower bound to the user-defined minimal safety distance to ensure that convex objects will not collide with each other.

B. Constraint for Dynamic Avoidance

We apply discrete obstacle avoidance constraints on the constraint points along the trajectory regarding the trajectories of other obstacles at the same time stamp. Therefore, the dynamic safety constraint is defined as $\mathcal{G}_{\Theta}(\boldsymbol{\sigma}, \dot{\boldsymbol{\sigma}}, \hat{t}) = \{\mathcal{G}_{\Theta_1}, \dots, \mathcal{G}_{\Theta_u}, \dots, \mathcal{G}_{\Theta_{N_u}}\}^T \in \mathbb{R}^{N_u}$ where N_u is the number of dynamic obstacles. The dynamic avoidance constraint function with the u -th moving object at a constraint point is defined as:

$$\mathcal{G}_{\Theta_u}(\boldsymbol{\sigma}, \dot{\boldsymbol{\sigma}}, \hat{t}) = d_m - U(\mathbb{E}(\boldsymbol{\sigma}, \dot{\boldsymbol{\sigma}}), \mathbb{O}_u(\hat{t})), \quad (49)$$

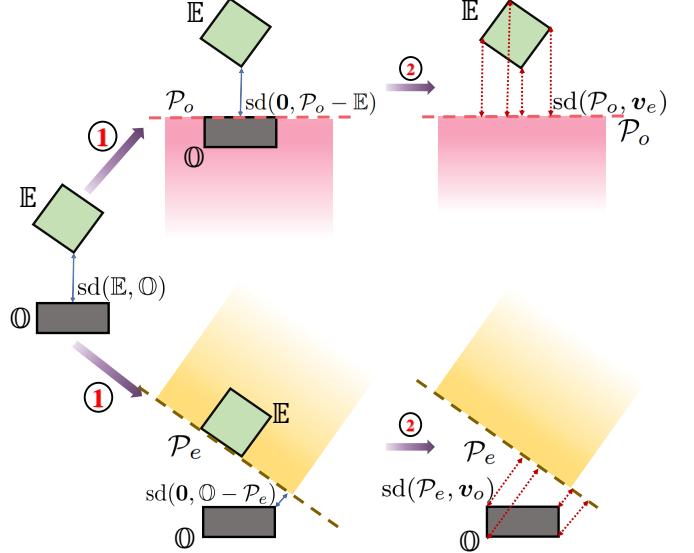


Fig. 6: Illustration of computing the lower bound of the signed distance between convex sets \mathbb{E} and \mathbb{O} , where \mathcal{P}_e and \mathcal{P}_o refer to any hyperplane of \mathbb{E} and \mathbb{O} . Besides, we define v_e and v_o as any vertex of \mathbb{E} and \mathbb{O} , respectively. The computation process follows the above two-stage structure. We first use the maximum of the signed distances between convex sets and hyperplanes to approximate $\text{sd}(\mathbb{E}, \mathbb{O})$. Then, due to the convexity of \mathbb{E} and \mathbb{O} , $\text{sd}(\mathbf{0}, \mathcal{P}_o - \mathbb{E})$ and $\text{sd}(\mathbf{0}, \mathbb{O} - \mathcal{P}_e)$ are converted to point-to-hyperplane distances $\text{sd}(\mathcal{P}_o, v_e)$ and $\text{sd}(\mathcal{P}_e, v_o)$ which can be analytically calculated.

where d_m is the minimum safe distance (safety margin) and $U(\mathbb{E}(t), \mathbb{O}_u(t))$ is the distance between the ego vehicle and the moving obstacle. With the lower approximation of the signed distance between two convex objects (Sect. V-A2), we have:

$$\text{sd}(\mathbb{E}(\boldsymbol{\sigma}, \dot{\boldsymbol{\sigma}}), \mathbb{O}_u(\hat{t})) \geq \text{lb}_{\text{sd}}(\mathbb{E}(\boldsymbol{\sigma}, \dot{\boldsymbol{\sigma}}), \mathbb{O}_u(\hat{t})). \quad (50)$$

The lower bound lb_{sd} is paraphrased by Eq. (44) as:

$$\begin{aligned} \text{lb}_{\text{sd}}(\mathbb{E}(\boldsymbol{\sigma}, \dot{\boldsymbol{\sigma}}), \mathbb{O}_u(\hat{t})) &= \\ \max_{\mathcal{P}_e, \mathcal{P}_o} \{\text{sd}(\mathbf{0}, \mathbb{O}_u(\hat{t}) - \mathcal{P}_e(\boldsymbol{\sigma}, \dot{\boldsymbol{\sigma}})), \text{sd}(\mathbf{0}, \mathcal{P}_o^u(\hat{t}) - \mathbb{E}(\boldsymbol{\sigma}, \dot{\boldsymbol{\sigma}}))\}, \\ e \in \{1, \dots, n_e\}, o \in \{1, \dots, n_u\}, \end{aligned} \quad (51)$$

where n_u is the number of hyperplanes of the u -th moving obstacle and \mathcal{P}_o^u is the hyperplane. Since the motion planning is on a two-dimensional plane, a hyperplane of a convex set degenerates into a straight line determined by two vertices. Then, the hyperplane descriptors of the ego vehicle can be determined as follows:

$$\mathbf{H}^e = \frac{\mathbf{B}(\mathbf{v}_{e+1} - \mathbf{v}_e)}{\|\mathbf{v}_{e+1} - \mathbf{v}_e\|_2}, \quad (52)$$

$$h^e = (\mathbf{H}^e)^T \mathbf{v}_e, e \in \{1, \dots, n_e\}, \quad (53)$$

where $\{\mathbf{v}_1, \dots, \mathbf{v}_e, \dots, \mathbf{v}_{n_e+1}\}$ are the vertices arranged clockwise, defined in Sec.IV-B, and $\mathbf{v}_{n_e+1} = \mathbf{v}_1$. Due to the convexity of the model, the distance between the moving obstacle and the hyperplane is converted into the minimum distance between vertices and the hyperplane, thus simplifying

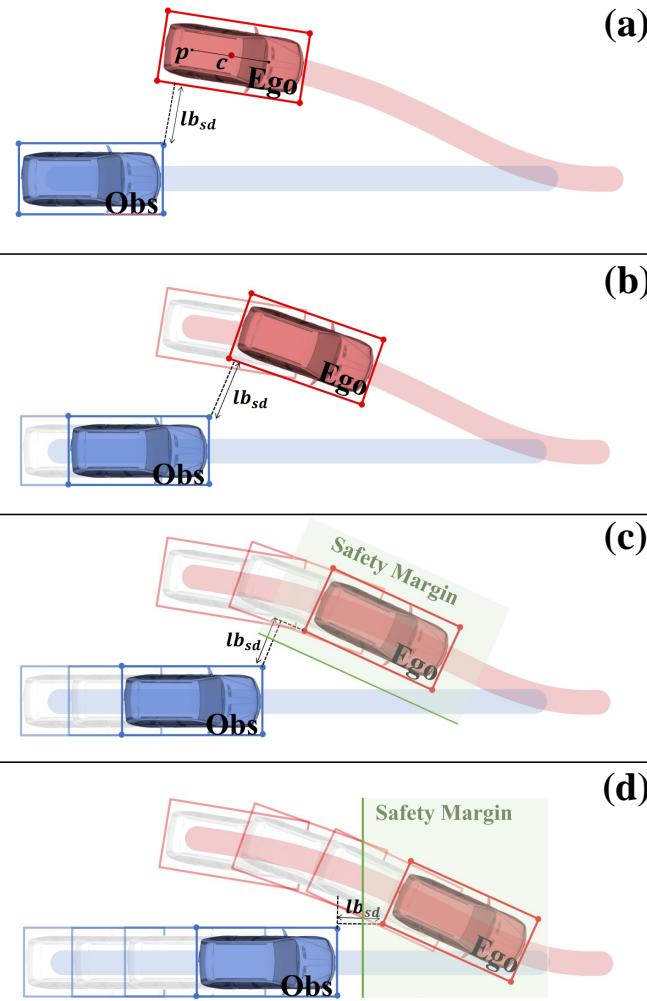


Fig. 7: The illustration of the lower bound of signed distance. When the ego vehicle is passing another moving object, the dynamic obstacle avoidance distance is evaluated over a short time stamp, as shown in (a)-(d).

Eq.(47):

$$\begin{aligned} \text{sd}(\mathbf{0}, \mathbb{O}_u - \mathcal{P}_e) &= \frac{1}{\|\mathbf{H}^e\|_2} (-\mathbf{h}^e + \min_o (\mathbf{H}^e)^T \mathbf{v}_o^u), \\ \mathbf{v}_o^u &= \mathbf{w}_u + \mathbf{R}_u \mathbf{l}_o^u, \quad o \in \{1, \dots, n_e\}, \end{aligned} \quad (54)$$

where \mathbf{v}_o^u is any vertex of the moving obstacle \mathbb{O}_u . \mathbf{w}_u and \mathbf{R}_u are the origin and rotation matrix of the obstacle body coordinate system, respectively. \mathbf{l}_o^u is the translation vector determined in advance. Before further derivation, we define an auxiliary expression $\mathcal{H}(\tilde{\mathbf{R}}, \Delta l) : (\mathbb{R}^{2 \times 2}, \mathbb{R}^2) \rightarrow \mathbb{R}^2$:

$$\mathcal{H}(\tilde{\mathbf{R}}, \Delta l) = \frac{\tilde{\mathbf{R}} \Delta l}{\|\Delta l\|_2}. \quad (55)$$

By combining Eq.(30)(52-54), the signed distance can be analytically in flat output space:

$$\text{sd}(\mathbf{0}, \mathbb{O}_u - \mathcal{P}_e) = \min_o \mathcal{H}(\mathbf{R}, \Delta l_e)^T (\mathbf{v}_o^u - \mathbf{v}_e), \quad (56)$$

where $\Delta l_e = \mathbf{l}_{e+1} - \mathbf{l}_e$, and the physical meaning of $\mathcal{H}(\mathbf{R}, \Delta l_e)$ is the normal vector outwards of the plane \mathcal{P}_e . Similarly, the signed distance between \mathcal{P}_o^u and \mathbb{E} can also be obtained:

$$\text{sd}(\mathbf{0}, \mathcal{P}_o^u - \mathbb{E}) = \min_e \mathcal{H}(\mathbf{R}_u, \Delta l_o^u)^T (\mathbf{v}_e - \mathbf{v}_o^u). \quad (57)$$

Finally, we substitute Eq.(56)(57) into Eq.(51) to obtain the analytical expression of lb_{sd} :

$$\begin{aligned} lb_{sd}(\mathbb{E}(\sigma, \dot{\sigma}), \mathbb{O}_u(\hat{t})) &= \\ \max\{\max_e \min_o \mathcal{H}(\mathbf{R}, \Delta l_e)^T (\mathbf{v}_o^u(\hat{t}) - \mathbf{v}_e), \\ \max_o \min_e \mathcal{H}(\mathbf{R}_u(\hat{t}), \Delta l_o^u)^T (\mathbf{v}_e - \mathbf{v}_o^u(\hat{t}))\}, \\ e \in \{1, \dots, n_e\}, o \in \{1, \dots, n_u\}. \end{aligned} \quad (58)$$

The calculation of the lower bound of the signed distance between two vehicles is shown in Fig. 7. To smooth the maximum and minimum operations, a widely adopted log-sum-exp function is applied, defined as follows to approximate the vector-max(min) function:

$$\text{lse}_\alpha(\gamma) = \alpha^{-1} \log \left(\sum_{\omega=1}^{\Omega} \exp(\alpha r_\omega) \right), \quad (59)$$

where r_ω is an element of the vector $\gamma = \{r_1, \dots, r_\omega, \dots, r_\Omega\}^T \in \mathbb{R}^{\Omega > 0}$. if $\alpha > 0$ then it approximately gets the maximum value in γ , or $\alpha < 0$ selects the minimum term. We smooth the discrete function by the log-sum-exp function with the advantage that the gradient of the log-sum-exp function is exactly the softmax(min) function. Moreover, the approximation error of the log-sum-exp is lower bounded by:

$$\text{lse}_{\alpha>0}(\gamma) \geq \max\{\gamma\} \geq \text{lse}_{\alpha>0}(\gamma) - \frac{\log(\Omega)}{\alpha}. \quad (60)$$

Hence, we can formulate the distance function:

$$\begin{aligned} U(\mathbb{E}(\sigma, \dot{\sigma}), \mathbb{O}_u(\hat{t})) &= \text{lse}_{\alpha>0}(\mathbf{d}) - \frac{\log(n_e + n_u)}{\alpha}, \\ \mathbf{d} &= (\mathbf{d}_U^T, \mathbf{d}_E^T)^T \in \mathbb{R}^{n_e + n_u}, \\ \mathbf{d}_U &= (d_U^1, \dots, d_U^e, \dots, d_U^{n_e})^T \in \mathbb{R}^{n_e}, \\ \mathbf{d}_E &= (d_E^1, \dots, d_E^o, \dots, d_E^{n_u})^T \in \mathbb{R}^{n_u}, \end{aligned} \quad (61)$$

where $d_U^e = \text{sd}(\mathbf{0}, \mathbb{O}_u(\hat{t}) - \mathcal{P}_e(\sigma, \dot{\sigma}))$ and $d_E^o = \text{sd}(\mathbf{0}, \mathcal{P}_o^u(\hat{t}) - \mathbb{E}(\sigma, \dot{\sigma}))$ are defined to simplify the formulation. Similarly, the minimum operation in d_U^e and d_E^o can be approximated by the log-sum-exp function with $\alpha < 0$. Combined with Eq.(56)(57) We transform the distance into the flat-output space as:

$$\begin{aligned} d_U^e &= \text{lse}_{\alpha<0} \left(\left(d_{U_1}^e, \dots, d_{U_o}^e, \dots, d_{U_{n_e}}^e \right)^T \right) + \tilde{d}_U^e, \\ d_E^o &= \text{lse}_{\alpha<0} \left(\left(d_{E_1}^o, \dots, d_{E_e}^o, \dots, d_{E_{n_u}}^o \right)^T \right) + \tilde{d}_E^o, \\ d_{U_o}^e &= \mathcal{H}(\mathbf{R}, \Delta l_e)^T \mathbf{R}_u(\hat{t}) \mathbf{l}_o^u, \\ d_{E_e}^o &= \mathcal{H}(\mathbf{R}_u(\hat{t}), \Delta l_o^u)^T \mathbf{R} \mathbf{l}_e, \\ \tilde{d}_U^e &= \mathcal{H}(\mathbf{R}, \Delta l_e)^T (\mathbf{w}_u(\hat{t}) - \mathbf{v}_e), \\ \tilde{d}_E^o &= \mathcal{H}(\mathbf{R}_u(\hat{t}), \Delta l_o^u)^T (\sigma - \mathbf{v}_o^u(\hat{t})). \end{aligned} \quad (62)$$

By substituting Eq.(62) into Eq.(61) and then into Eq.(49), $\mathcal{G}_{\Theta_u}(\boldsymbol{\sigma}, \dot{\boldsymbol{\sigma}}, \hat{t})$ is transformed into a continuously differentiable function that can be analytically expressed by flat outputs. Based on the chain rule, we calculate the gradients of $\mathcal{G}_{\Theta_u}(\boldsymbol{\sigma}, \dot{\boldsymbol{\sigma}}, \hat{t})$ w.r.t $\boldsymbol{\sigma}$:

$$\begin{aligned}\frac{\partial \mathcal{G}_{\Theta_u}}{\partial \boldsymbol{\sigma}} &= -\sum_{e=1}^{n_e} \text{lse}'_{\alpha>0}(d_U^e) \frac{\partial \tilde{d}_U^e}{\partial \boldsymbol{\sigma}} - \sum_{o=1}^{n_o} \text{lse}'_{\alpha>0}(d_E^o) \frac{\partial \tilde{d}_E^o}{\partial \boldsymbol{\sigma}}, \\ \frac{\partial \tilde{d}_U^e}{\partial \boldsymbol{\sigma}} &= -\mathcal{H}(\mathbf{R}, \Delta \mathbf{l}_e), \quad \frac{\partial \tilde{d}_E^o}{\partial \boldsymbol{\sigma}} = \mathcal{H}(\mathbf{R}_u(\hat{t}), \Delta \mathbf{l}_o^u),\end{aligned}\quad (63)$$

where $\text{lse}' : \mathbb{R} \rightarrow \mathbb{R}$ is the gradient of the log-sum-exp function. In a similar way, we derive the gradients w.r.t \hat{t} and $\dot{\boldsymbol{\sigma}}$:

$$\begin{aligned}\frac{\partial \mathcal{G}_{\Theta_u}}{\partial \hat{t}} &= -\sum_{e=1}^{n_e} \text{lse}'_{\alpha>0}(d_U^e) \left(\sum_{o=1}^{n_o} \text{lse}'_{\alpha<0}(d_{U,o}^e) \frac{\partial d_{U,o}^e}{\partial \hat{t}} + \frac{\partial \tilde{d}_U^e}{\partial \hat{t}} \right) \\ &\quad - \sum_{o=1}^{n_o} \text{lse}'_{\alpha>0}(d_E^o) \left(\sum_{e=1}^{n_e} \text{lse}'_{\alpha<0}(d_{E,e}^o) \frac{\partial d_{E,e}^o}{\partial \hat{t}} + \frac{\partial \tilde{d}_E^o}{\partial \hat{t}} \right), \\ \frac{\partial \mathcal{G}_{\Theta_u}}{\partial \dot{\boldsymbol{\sigma}}} &= -\sum_{e=1}^{n_e} \text{lse}'_{\alpha>0}(d_U^e) \left(\sum_{o=1}^{n_o} \text{lse}'_{\alpha<0}(d_{U,o}^e) \frac{\partial d_{U,o}^e}{\partial \dot{\boldsymbol{\sigma}}} + \frac{\partial \tilde{d}_U^e}{\partial \dot{\boldsymbol{\sigma}}} \right) \\ &\quad - \sum_{o=1}^{n_o} \text{lse}'_{\alpha>0}(d_E^o) \sum_{e=1}^{n_e} \text{lse}'_{\alpha<0}(d_{E,e}^o) \frac{\partial d_{E,e}^o}{\partial \dot{\boldsymbol{\sigma}}}.\end{aligned}\quad (64)$$

From Eq.(64), we can obtain the gradients of the dynamic safety constraint once the gradients of auxiliary distances are determined. Next, we derive the gradients w.r.t $\dot{\boldsymbol{\sigma}}$:

$$\begin{aligned}\frac{\partial d_{U,o}^e}{\partial \dot{\boldsymbol{\sigma}}} &= \frac{\mathcal{F}(\Delta \mathbf{l}_e) \mathbf{B}^T \mathbf{R}_u(\hat{t}) \mathbf{l}_o^u}{\|\Delta \mathbf{l}_e\|_2}, \\ \frac{\partial \tilde{d}_U^e}{\partial \dot{\boldsymbol{\sigma}}} &= \frac{\mathcal{F}(\Delta \mathbf{l}_e) \mathbf{B}^T (\mathbf{w}_u(\hat{t}) - \mathbf{v}_e) + \mathcal{F}(\mathbf{l}_e) \mathbf{B}^T \mathbf{R} \Delta \mathbf{l}_e}{\|\Delta \mathbf{l}_e\|_2}, \\ \frac{\partial d_{E,e}^o}{\partial \dot{\boldsymbol{\sigma}}} &= \mathcal{F}(\mathbf{l}_e) \mathcal{H}(\mathbf{R}_u(\hat{t}), \Delta \mathbf{l}_o^u).\end{aligned}\quad (65)$$

Then, the gradients w.r.t abstract timestamp \hat{t} are also derived as follows:

$$\begin{aligned}\frac{\partial d_{U,o}^e}{\partial \hat{t}} &= \left(\dot{\mathbf{R}}_u(\hat{t}) \mathbf{l}_o^u \right)^T \mathcal{H}(\mathbf{R}, \Delta \mathbf{l}_e), \\ \frac{\partial \tilde{d}_U^e}{\partial \hat{t}} &= (\dot{\mathbf{w}}_u(\hat{t}))^T \mathcal{H}(\mathbf{R}, \Delta \mathbf{l}_e), \\ \frac{\partial d_{E,e}^o}{\partial \hat{t}} &= (\mathbf{R} \mathbf{l}_e)^T \mathcal{H}(\dot{\mathbf{R}}_u(\hat{t}), \Delta \mathbf{l}_o^u), \\ \frac{\partial \tilde{d}_E^o}{\partial \hat{t}} &= \mathcal{H}(\dot{\mathbf{R}}_u(\hat{t}), \Delta \mathbf{l}_o^u)^T (\boldsymbol{\sigma} - \mathbf{w}_u(\hat{t}) - \mathbf{R}_u(\hat{t}) \mathbf{l}_o^u) \\ &\quad - \mathcal{H}(\mathbf{R}_u(\hat{t}), \Delta \mathbf{l}_o^u)^T (\dot{\mathbf{w}}_u(\hat{t}) + \dot{\mathbf{R}}_u(\hat{t}) \mathbf{l}_o^u),\end{aligned}\quad (66)$$

where $\dot{\mathbf{R}}_u(\hat{t}) \in \mathbb{R}^{2 \times 2}$ and $\dot{\mathbf{w}}_u(\hat{t}) \in \mathbb{R}^2$ are the gradients w.r.t \hat{t} . In practice, trajectories of other obstacles are fitted by piece-wise polynomials. Therefore, the position $\mathbf{w}_u(\hat{t})$, the rotation matrix $\mathbf{R}_u(\hat{t})$ and their gradients can also be calculated analytically. Besides, we set α as 100 to approximate the maximum function and -100 to smooth the minimum function. Here, we have derived all gradients of the dynamic obstacle avoidance constraint function in the flat output space for further optimization. The expressions can be complicated, but due to the good property of convexity and monotonicity

of the log-sum-exp function, the smoothed constraint function can work well with our trajectory construction.

VI. REFORMULATION OF TRAJECTORY OPTIMIZATION

In this section, we analyze the characteristics of the constraints Eq.(6c-6f)(10) in trajectory planning and use targeted approaches to eliminate them, respectively. Then, the original optimization problem is reformulated into an unconstrained program that can be further solved efficiently.

A. Feasibility Constraints

We adopt the discrete-time summation-type penalty term S_Σ to relax the feasibility constraints Eq. (10):

$$\begin{aligned}S_\Sigma(\mathbf{c}, \mathbf{T}) &= \sum_{d \in \mathcal{D}} w_d \sum_{i=1}^n \sum_{j=1}^{M_i} \sum_{k=0}^\lambda P_{d,i,j,k}(\mathbf{c}_{i,j}, \mathbf{T}), \\ P_{d,i,j,k}(\mathbf{c}_{i,j}, \mathbf{T}) &= \frac{\delta T_i}{\lambda} \bar{\omega}_k L_1(\mathcal{G}_{d,i,j,k}),\end{aligned}\quad (67)$$

where w_d is the penalty weight corresponding to different kinds of constraints. $[\bar{\omega}_0, \bar{\omega}_1, \dots, \bar{\omega}_{\lambda-1}, \bar{\omega}_\lambda] = [1/2, 1, \dots, 1, 1/2]$ are the quadrature coefficients from the trapezoidal rule [64] and $P_{d,i,j,k}$ is the violation penalty for a constraint point. Moreover, we define a first-order relaxation function $L_1(\cdot)$ to guarantee the continuous differentiability and non-negativity of penalty terms:

$$L_1(x) = \begin{cases} 0 & x \leq 0, \\ -\frac{1}{2a_0^3}x^4 + \frac{1}{a_0^2}x^3 & 0 < x \leq a_0, \\ x - \frac{a_0}{2} & a_0 < x. \end{cases}\quad (68)$$

$$L'_1(x) = \begin{cases} 0 & x \leq 0, \\ -\frac{2}{a_0^3}x^3 + \frac{3}{a_0^2}x^2 & 0 < x \leq a_0, \\ 1 & a_0 < x. \end{cases}\quad (69)$$

Here $a_0 = 10^{-4}$ is the demarcation point. Such discrete penalty formulation ensures that continuous-time constraints Eq.(6g) are satisfied within an acceptable tolerance. Then, the trajectory optimization for vehicles is reformatted as follows:

$$\min_{\mathbf{c}, \mathbf{T}} \mathcal{J}(\mathbf{c}, \mathbf{T}) = J(\mathbf{c}, \mathbf{T}) + S_\Sigma(\mathbf{c}, \mathbf{T})\quad (70a)$$

$$\text{s.t. } \boldsymbol{\sigma}_0^{[s-1]}(0) = \bar{\boldsymbol{\sigma}}_0, \quad \boldsymbol{\sigma}_n^{[s-1]}(T_n) = \bar{\boldsymbol{\sigma}}_f,\quad (70b)$$

$$\boldsymbol{\sigma}_i^{[s-1]}(T_i) = \boldsymbol{\sigma}_{i+1}^{[s-1]}(0) = \tilde{\boldsymbol{\sigma}}_i, \quad 1 \leq i < n,\quad (70c)$$

$$\boldsymbol{\sigma}_{i,j}^{[\tilde{d}]}(\delta T_i) = \boldsymbol{\sigma}_{i,j+1}^{[\tilde{d}]}(0), \quad 1 \leq i \leq n, 1 \leq j < M_i,\quad (70d)$$

$$T_i > 0, \quad 1 \leq i \leq n.\quad (70e)$$

Without loss of generality, the gradients of the violation penalty at each constraint point on the trajectory are derived:

$$\frac{\partial P_{d,i,j,k}}{\partial \mathbf{c}_{i,j}} = \frac{\partial P_{d,i,j,k}}{\partial \mathcal{G}_{d,i,j,k}} \frac{\partial \mathcal{G}_{d,i,j,k}}{\partial \mathbf{c}_{i,j}},$$

$$\begin{aligned}\frac{\partial P_{d,i,j,k}}{\partial \mathbf{T}} &= \left(\mathbf{0}_{i-1}^T, \frac{P_{d,i,j,k}}{T_i}, \mathbf{0}_{n-i}^T \right)^T + \frac{\partial P_{d,i,j,k}}{\partial \mathcal{G}_{d,i,j,k}} \frac{\partial \mathcal{G}_{d,i,j,k}}{\partial \mathbf{T}}, \\ \frac{\partial P_{d,i,j,k}}{\partial \mathcal{G}_{d,i,j,k}} &= \frac{\delta T_i}{\lambda} \bar{\omega}_k \mathbf{L}'_1(\mathcal{G}_{d,i,j,k}).\end{aligned}\quad (71)$$

Since the gradients of the constraints $\mathcal{G}_{d,i,j,k}$ have been calculated in Sect. IV and Sect. V, we can also analytically obtain the gradients of the violation penalty $\partial P_{d,i,j,k}$ by the above propagation chain Eq.(71). Then, the gradients of the newly defined objective function $\mathcal{J}(\mathbf{c}, \mathbf{T})$ can be easily obtained in a similar way.

B. Equality Constraints

Before discussing the elimination of the equality constraints Eq.(70b)-(70d), we first decompose and reformulate Eq.(70c):

$$\boldsymbol{\sigma}_i(T_i) = \boldsymbol{\sigma}_{i+1}(0) = \mathbf{p}_i^g, \quad (72a)$$

$$\boldsymbol{\sigma}_i^{(1)}(T_i) = -\boldsymbol{\sigma}_{i+1}^{(1)}(0) = \mathbf{v}_i^g, \quad (72b)$$

$$\boldsymbol{\sigma}_i^{(\bar{d})}(T_i) = \boldsymbol{\sigma}_{i+1}^{(\bar{d})}(0) = \mathbf{0}_2, \quad (72c)$$

$$\|\mathbf{v}_i^g\|_2 = \bar{v}, \quad (72d)$$

$$\forall i \in \{1, 2, 3, \dots, n-1\}, \forall \bar{d} \in \{2, \dots, s-1\}. \quad (72e)$$

p_i^g is the gear shifting position and v_i^g is the final velocity before the shift. It is worth noting that the velocity direction is reversed before and after the shift and its magnitude is set to a small non-zero value \bar{v} to prevent singularities during optimization. For instance, \bar{v} is set to 0.05 in the actual implementation. Based on the optimality condition proved in [2], the minimum control effort piece-wise polynomial coefficients \mathbf{c}_i are uniquely determined by the intermediate waypoints \mathbf{q}_i , the time interval of each piece, and the head and tail states:

$$\mathbf{M}_i(T_i)\mathbf{c}_i = \mathbf{b}_i, 1 \leq i \leq n, \quad (73)$$

where $\mathbf{M}_i(T_i) \in \mathbb{R}^{2M_i s \times 2M_i s}$ is an invertible banded matrix whose specific form can be found in [2]. $\mathbf{b}_{i \in \{1 \dots n\}} \in \mathbb{R}^{2M_i s \times 2}$ is defined as follows:

$$\begin{aligned}\mathbf{b}_1 &= (\bar{\boldsymbol{\sigma}}_0, \mathbf{q}_{1,1}, \mathbf{0}_{2 \times \tilde{d}}, \dots, \mathbf{q}_{1,M_1-1}, \mathbf{0}_{2 \times \tilde{d}}, p_1^g, v_1^g, \mathbf{0}_{2 \times (s-2)})^T, \\ \mathbf{b}_n &= (p_{n-1}^g, v_{n-1}^g, \mathbf{0}_{2 \times (s-2)}, \mathbf{q}_{n,1}, \mathbf{0}_{2 \times \tilde{d}}, \dots, \mathbf{q}_{n,M_n-1}, \mathbf{0}_{2 \times \tilde{d}}, \bar{\boldsymbol{\sigma}}_f)^T. \\ \mathbf{b}_i &= (p_{i-1}^g, v_{i-1}^g, \mathbf{0}_{2 \times (s-2)}, \mathbf{q}_{i,1}, \mathbf{0}_{2 \times \tilde{d}}, \dots, \\ \mathbf{q}_{i,M_i-1}, \mathbf{0}_{2 \times \tilde{d}}, p_i^g, v_i^g, \mathbf{0}_{2 \times (s-2)})^T, 1 < i < n,\end{aligned}\quad (74)$$

where the degree of continuity \tilde{d} is set to $2s-1$ to satisfy the optimality condition. Based on Eq.(73), we use the waypoints $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_n) \in \mathbb{R}^{2 \times \sum_{i=1}^n (M_i-1)}$, the time set \mathbf{T} , the gear shifting position $\mathbf{p}^g = (p_1^g, \dots, p_{n-1}^g) \in \mathbb{R}^{2 \times (n-1)}$ and the velocity $\mathbf{v}^g = (v_1^g, \dots, v_{n-1}^g) \in \mathbb{R}^{2 \times (n-1)}$ as decision variables in the optimization problem without sacrificing optimality, then the constraints Eq.(70b)-(70d)-(72a)-(72c) naturally satisfy:

$$\min_{\mathbf{q}, \mathbf{T}, \mathbf{p}^g, \mathbf{v}^g} \mathcal{J}(\mathbf{q}, \mathbf{T}, \mathbf{p}^g, \mathbf{v}^g) = \mathcal{J}(\mathbf{c}(\mathbf{q}, \mathbf{T}, \mathbf{p}^g, \mathbf{v}^g), \mathbf{T}) \quad (75a)$$

$$\text{s.t. } \|\mathbf{v}_i^g\|_2 = \bar{v}, 1 \leq i \leq n, \quad (75b)$$

$$T_i > 0, 1 \leq i \leq n. \quad (75c)$$

Then, the gradients of the objective function w.r.t waypoints and time are derived as follows:

$$\frac{\partial \mathcal{J}}{\partial \mathbf{q}_{i,j}} = \left(\mathbf{M}_i^{-T} \frac{\partial \mathcal{J}}{\mathbf{c}_i} \right)^T e_{(2j-1)s+1}, \quad (76)$$

$$\frac{\partial \mathcal{J}}{\partial T_i} = \frac{\partial \mathcal{J}}{\partial T_i} - \text{Tr} \left\{ \left(\mathbf{M}_i^{-T} \frac{\partial \mathcal{J}}{\mathbf{c}_i} \right)^T \frac{\partial \mathbf{M}_i}{\partial T_i} \mathbf{c}_i \right\}, \quad (77)$$

$$\forall i \in \{1, 2, 3, \dots, n\}, \forall j \in \{1, 2, 3, \dots, M_i-1\}.$$

e_k is a column vector of proper dimension, where the element of the k -th row is 1 and all others are 0. $\text{Tr}(\cdot)$ is the trace operation and $\partial \mathbf{M}_i / \partial T_i$ is also analytically computable [2]. Similarly, the gradients w.r.t the gear shifting position and the velocity are as follows:

$$\frac{\partial \mathcal{J}}{\partial \mathbf{p}_i^g} = \left(\mathbf{M}_i^{-T} \frac{\partial \mathcal{J}}{\mathbf{c}_i} \right)^T e_{(2M_i-1)s+1} + \left(\mathbf{M}_{i+1}^{-T} \frac{\partial \mathcal{J}}{\mathbf{c}_{i+1}} \right)^T e_1, \quad (78)$$

$$\frac{\partial \mathcal{J}}{\partial \mathbf{v}_i^g} = \left(\mathbf{M}_i^{-T} \frac{\partial \mathcal{J}}{\mathbf{c}_i} \right)^T e_{(2M_i-1)s+2} + \left(\mathbf{M}_{i+1}^{-T} \frac{\partial \mathcal{J}}{\mathbf{c}_{i+1}} \right)^T e_2, \quad (79)$$

$$\forall i \in \{1, 2, 3, \dots, n-1\}.$$

Furthermore, to satisfy the constraint Eq.(75b), we express $\mathbf{v}_i^g = (\bar{v} \cos \theta_i^g, \bar{v} \sin \theta_i^g)^T$ in polar coordinates, where the physical meaning of θ_i^g is the direction of velocity before gear switching. Besides, we define the angle set $\boldsymbol{\theta}^g = (\theta_1^g, \theta_2^g, \dots, \theta_{n-1}^g)^T \in \mathbb{R}^{n-1}$ for subsequent derivation and the cost function is transformed to $\mathcal{K}(\mathbf{q}, \mathbf{T}, \mathbf{p}^g, \boldsymbol{\theta}^g) = \mathcal{J}(\mathbf{q}, \mathbf{T}, \mathbf{p}^g, \mathbf{v}^g(\boldsymbol{\theta}^g))$. The gradient of the cost function \mathcal{K} w.r.t $\boldsymbol{\theta}^g$ can be obtained as follows:

$$\frac{\partial \mathcal{K}}{\partial \theta_i^g} = (-\bar{v} \sin \theta_i^g, \bar{v} \cos \theta_i^g) \frac{\partial \mathcal{J}}{\partial \mathbf{v}_i^g}, 1 \leq i < n. \quad (80)$$

We eliminate all equation constraints in the original planning which is the basis for subsequent efficient optimization.

C. Positiveness Condition

To remove the strict positiveness condition Eq.(75c), we define an unconstrained virtual time $\boldsymbol{\tau} = [\tau_1, \dots, \tau_i, \dots, \tau_n]^T \in \mathbb{R}^n$ and employ a diffeomorphism map [65] from $\tau_i \in \mathbb{R}$ to the real duration $T_i \in \mathbb{R}^+$:

$$T_i = \begin{cases} \frac{1}{2} \tau_i^2 + \tau_i + 1 & \tau_i > 0 \\ \frac{2}{\tau_i^2 - 2\tau_i + 2} & \tau_i \leq 0 \end{cases} \quad (81)$$

$$\forall i \in \{1, 2, 3, \dots, n\}.$$

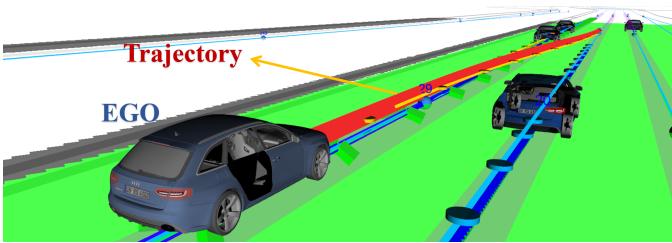
By deriving Eq.(81) backward, the surjection from T_i to τ_i is as follows:

$$\tau_i = \begin{cases} \sqrt{2T_i - 1} - 1 & T_i > 1 \\ 1 - \sqrt{\frac{2}{T_i} - 1} & T_i \leq 1 \end{cases} \quad (82)$$

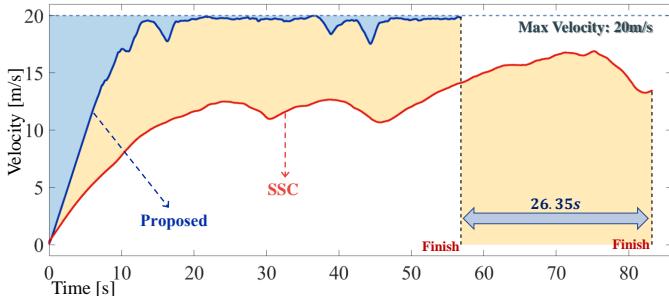
$$\forall i \in \{1, 2, 3, \dots, n\}.$$

Then, we use $\boldsymbol{\tau}$ to replace the real time set \mathbf{T} so that the strict positiveness constraints Eq.(75c) are naturally satisfied. The corresponding reformulated optimization problem is as follows:

$$\min_{\mathbf{q}, \boldsymbol{\tau}, \mathbf{p}^g, \boldsymbol{\theta}^g} \mathcal{W}(\mathbf{q}, \boldsymbol{\tau}, \mathbf{p}^g, \boldsymbol{\theta}^g) = \mathcal{K}(\mathbf{q}, \boldsymbol{\tau}, \mathbf{p}^g, \boldsymbol{\theta}^g). \quad (83)$$



(a) A snapshot when overtaking a moving vehicle.



(b) Comparison of velocity curves of trajectories. The area of the yellowish region on the left and the right is equal.

Fig. 8: Illustration of the comparison on a road track.

The gradient can be propagated from T_i to τ_i :

$$\frac{\partial \mathcal{W}}{\partial \tau_i} = \begin{cases} (\tau_i + 1) \frac{\partial \mathcal{K}}{\partial T_i} & \tau_i > 0 \\ \frac{4(1 - \tau_i)}{(\tau_i^2 - 2\tau_i + 2)^2} \frac{\partial \mathcal{K}}{\partial T_i} & \tau_i \leq 0 \end{cases} \quad (84)$$

$\forall i \in \{1, 2, 3, \dots, n\}$.

The gradients w.r.t τ are efficiently calculated once we obtain the gradients w.r.t \mathbf{T} .

In summary, we have removed all constraints in the optimization and analytically derived the gradients. Finally, we robustly solve the transformed unconstrained optimization Eq.(83) via the quasi-Newton method [3].

VII. EVALUATIONS

The proposed method is evaluated both in simulation and in the real world. Benchmarks show that our planner outperforms state-of-the-art methods in terms of time efficiency and trajectory quality.

A. Simulations

Simulation scenarios include structured roads and unstructured areas. All simulation experiments are conducted on a desktop computer running Ubuntu 18.04 with an Intel Core i7-10700 CPU and a GeForce RTX 2060 GPU.

1) Dynamical Urban Road Environments: In an urban road, a trajectory planner is combined with a behavior planner for policy elections as lane choosing. We adopt the multi-agent simulation with a high-level decision-making implementation in work [19] for our urban traffic simulation. Since the predictions and potential interactions are out of the scope of this paper, we follow the pipelines in work [19] and pedestrians

TABLE I: Comparison Of Trajectory Generation With SSC

Procedure	Ours	SSC [19]
Trajectory Optimization	8.14 ms	9.22 ms
Improved Rate	11.7%	-
State Transform	-	2.82 ms
Total Time	8.14 ms	12.04 ms

are ignored. Besides, the ego vehicle only has a limited range for the semantic maps and the interactions with other vehicles. The replanning frequency is set to 20 Hz.

The comparison experiment with SSC [19] planner is conducted on a structured road with a length of 1000m and 21 autonomous moving vehicles, as shown in Fig. 8a. Additionally, the maximum velocity and longitude acceleration are set to 20m/s and 2m/s², respectively. We plot the dynamic profile of the planned trajectories for comparison, as shown in Fig. 8b. The result shows that our trajectory planner can better utilize the maneuverability of the vehicle to finish the track about 26.35s earlier than SSC [19]. That is because SSC [19] uses Bézier curves to parameterize the trajectory. Consequently, SSC [19] inevitably suffers from the conservative nature of the Bézier curve and is unable to approach the constraint limit sufficiently. Moreover, we quantitatively compare the time efficiency of trajectory planning with SSC [19]. Indicated by Table. I, our planner demands less optimization time on average to generate the trajectory and improves 11.7% in time efficiency. Furthermore, since SSC [19] is implemented based on the Frenét coordinate system, additional and non-negligible state transform time is required to convert the information in the Cartesian coordinate system, such as the state and prediction of ego and surrounding vehicles, and the environmental point cloud to the Frenét frame. In conclusion, compared to SSC [19], we can directly generate a trajectory that more closely matches the user's desired physical performance without any state coordinate system transformation while enjoying higher optimization efficiency.

2) Unstructured Environments: We perform more simulation experiments based on the open-source physical simulator CARLA [66]. The proposed method is compared with four impressive methods specifically designed for motion planning of car-like robots in static unstructured environments, including OBTPAP [39], DL-IAPS+PJSO [28], H-OBCA [37] and Timed Elastic Bands (TEB) [67]. All methods are implemented in C++14 without parallel acceleration. OBTPAP [39] and H-OBCA [37] are solved by primal-dual interior-point method IPOPT [68]. DL-IAPS+PJSO [28] is implemented using OSQP [69]. The graph optimization solver G²o [70] is used for TEB [67]. Moreover, for fair comparisons, all planners use hybridA* [30] algorithm as their front-end to provide rough initial guesses for the subsequent optimization. We conduct comparison experiments in two scenes built by Unreal Engine²: one is a parking lot with vehicles, and the other is an abandoned farm with many irregular obstacles, as shown

²<https://www.unrealengine.com/>

TABLE II: Comparison of Dynamic Statistics in Different Cases

Environments	Problem Scale	Small-Scale ($30m \sim 60m$)			Medium-Scale ($60m \sim 120m$)			Large-Scale ($120m \sim 180m$)		
		Method	M.A (m/s^2)	M.JK (m/s^3)	M.TE (m)	M.A (m/s^2)	M.JK (m/s^3)	M.TE (m)	M.A (m/s^2)	M.JK (m/s^3)
Low-Complexity (6 ~ 10 Obstacles)	Proposed	6.96	16.61	0.142	6.76	18.81	0.131	6.03	15.30	0.146
	OBTPAP	30.11	101.99	0.223	51.32	157.19	0.231	36.10	105.56	0.250
	DL-IAPS+PJSO	10.02	66.44	0.208	9.40	77.79	0.240	10.14	86.09	0.252
	H-OBCA	7.42	20.67	0.194	8.23	24.11	0.217	6.40	19.62	0.258
	TEB	16.27	380.52	0.348	15.74	427.35	0.338	14.96	363.07	0.321
Medium-Complexity (11 ~ 20 Obstacles)	Proposed	7.50	20.69	0.142	7.60	18.66	0.135	7.17	17.68	0.115
	OBTPAP	42.27	145.89	0.239	58.01	197.21	0.236	49.94	158.09	0.251
	DL-IAPS+PJSO	11.53	83.92	0.223	13.01	119.75	0.257	14.26	140.55	0.254
	H-OBCA	7.85	21.69	0.216	8.88	26.25	0.213	7.05	20.57	0.231
	TEB	18.72	466.93	0.348	21.94	614.42	0.343	21.32	707.91	0.357
High-Complexity (21 ~ 30 Obstacles)	Proposed	7.53	17.95	0.127	7.97	19.49	0.120	6.97	17.56	0.126
	OBTPAP	36.54	128.29	0.230	43.90	157.32	0.247	57.03	193.75	0.251
	DL-IAPS+PJSO	12.25	94.43	0.210	16.24	155.38	0.259	13.92	134.74	0.268
	H-OBCA	8.09	23.58	0.195	8.17	25.77	0.230	7.05	21.49	0.248
	TEB	19.67	493.70	0.352	23.42	607.32	0.416	19.89	584.01	0.365

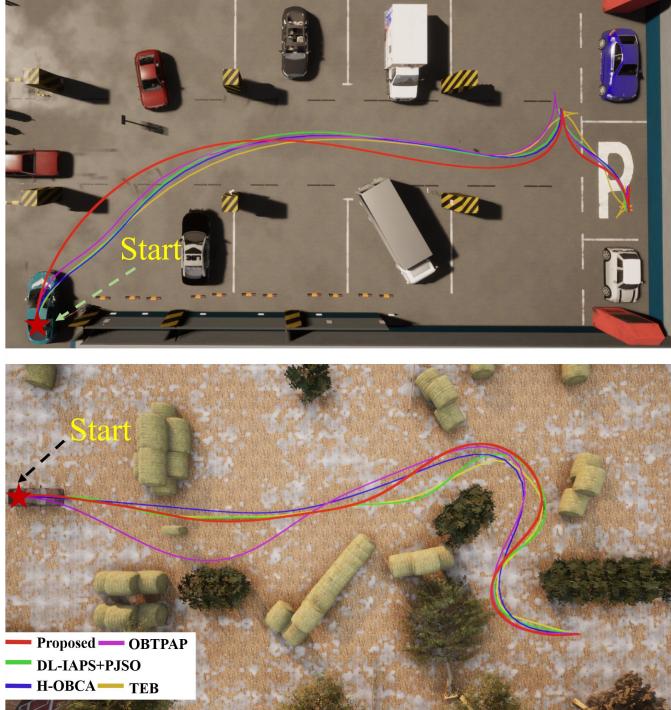


Fig. 9: The trajectory visualization in the parking lot (top) and abandoned farm (bottom).

in Fig. 9. Additionally, the proposed planner, OBTPAP [39], DL-IAPS+PJSO [28] and TEB [67] use known global point clouds in the environment to construct safety constraints. H-OBCA [37] uses known convex polygons manually extracted from obstacles to construct its optimization formulation. All common parameters, including convergence conditions, dynamics constraints, and the ego vehicle dimensions, are set to the same for fairness. We use an MPC controller [71] that minimizes position and velocity errors to follow planned

trajectories to measure the executable performance. We draw the tracking error of planned trajectories in Fig. 11, which visually demonstrates the superiority of the proposed method in terms of physical feasibility.

We conduct extensively quantitative assessments in many cases with different numbers of environmental obstacles and start-end distances (denote problem scale). Besides, the convergence tolerance of the optimization is set to 10^{-4} . Lots of comparison tests are performed in each case with random starting and ending states, and physical dynamics statistics including the mean of acceleration (abbreviated as M.A), jerk (M.JK) and tracking error (M.TE) are shown in Tab. II. The results show that our planner has the low control effort and the best human comfort in all cases. Moreover, the parametric form of the trajectory for the flat system inherently guarantees the continuity of the state and its finite-dimensional derivatives, which makes our trajectory easier to track than the above methods [28, 37, 39, 67] of discrete motion processes. Additionally, to evaluate the real-time performance of planners, the average planning time under each test case is visualized in Fig. 12. The histogram shows that our planner has an order-of-magnitude efficiency advantage over other algorithms, especially for large-scale trajectory generation. Furthermore, the results demonstrate the robustness against the problem scale and the number of environmental obstacles, thus allowing it to adapt to different scenarios.

We also validate the proposed planner in a $210m \times 50m$ dynamic unstructured environment where the ego vehicle is required to avoid static obstacles and the traffic flow consisting of eight other vehicles, as shown in Fig. 10. Since perception is not the focus of this paper, the environment and the trajectories of other vehicles are known to the planner. The maximum velocity of all vehicles is set to $10m/s$. The colors of vehicles and trajectories in Fig. 10 represent motion timestamps, which indicate the absence of spatial-temporal intersections between the ego vehicle and other moving objects, demonstrating dy-

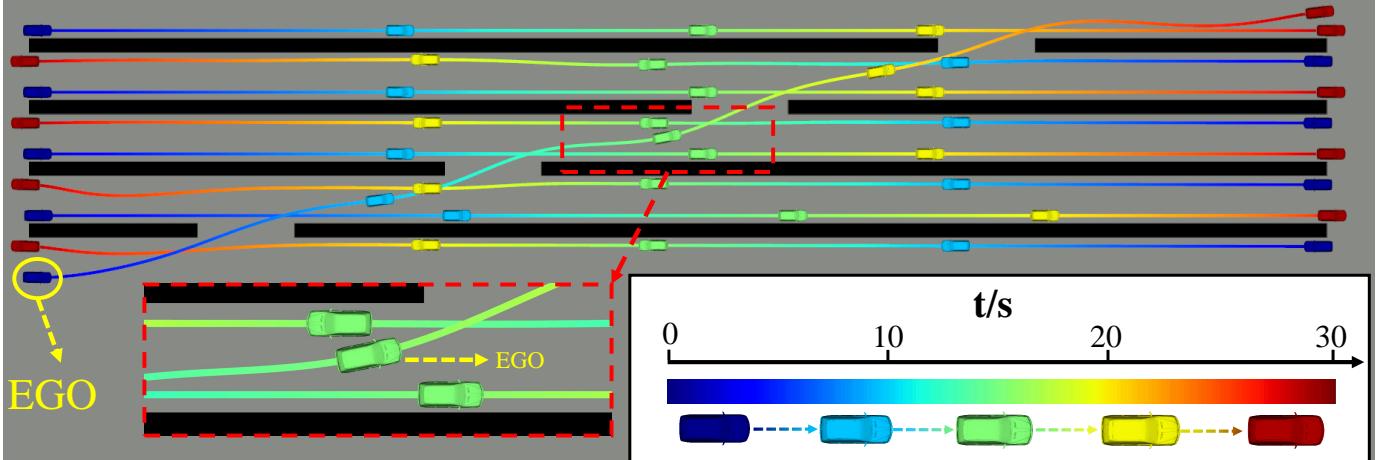


Fig. 10: Motion visualization in the dynamic environment, where the colors represent timestamps.

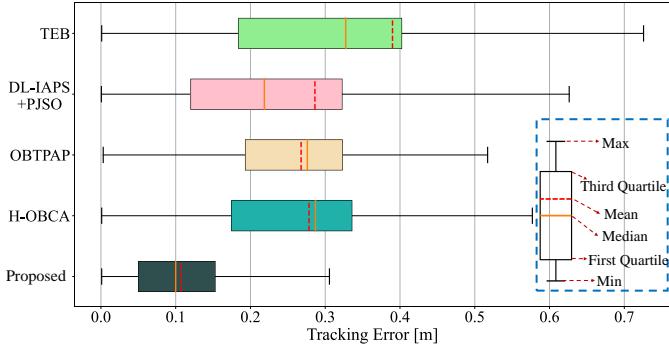


Fig. 11: Comparison of the tracking error profiles for the planned trajectories in the abandoned farm shown in Fig. 9.

namic safety. Thanks to full-dimensional modeling of objects, the ego vehicle has the ability to fully utilize the safe space to get through narrow areas, as the close-up shows. The trajectory length of the ego vehicle is 260m, while the computation time is only 0.18s, which demonstrates the efficiency of our planner in complex dynamic environments, especially for long-distance global trajectory generation.

B. Real-World Experiments

In addition to simulations, we also conduct real-world experiments to verify the feasibility of our planner on a real physical platform. The experimental site is a dense $40m \times 20m$ outdoor unstructured parking lot, as shown in Fig. 13. The moving robot is required to start from an initial state, go around obstacles, and eventually reverse into a parking space at a human-defined heading angle. The length of the whole track is about 42 meters. During the trajectory planning, the vehicle model was expanded by $0.25m$ to ensure safety during actual execution in the presence of unavoidable control and positioning errors. The maximum forward and backward speeds are set to $2m/s$ and $0.5m/s$, respectively. The time weight w_T is set to 50 to ensure the aggressiveness of the trajectory. The motion of the ego vehicle is visualized in Fig. 15, which demonstrates that the ego vehicle can follow the

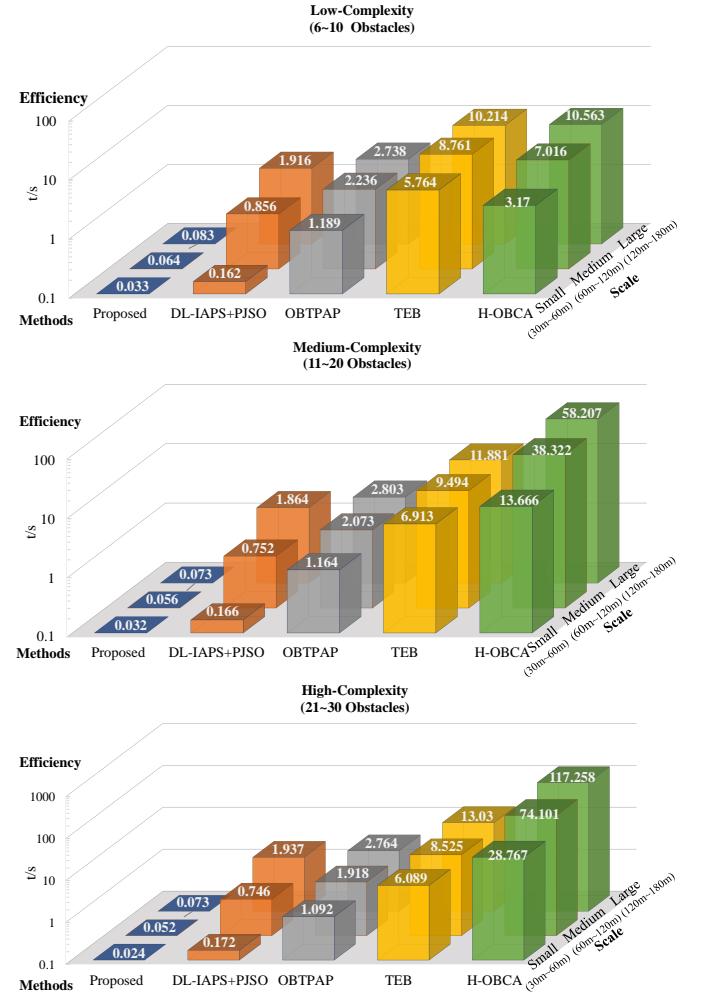


Fig. 12: Illustration of time efficiency comparisons.

planned trajectory to avoid obstacles and eventually reverse into a parking space with the user-given heading angle. Furthermore, dynamic evaluation metrics are quantified in Tab. III. As we can see, the ego vehicle maintains a relatively high speed throughout to reach the target state without exceeding



Fig. 13: The real-world experiment site.



Fig. 14: The experimental platform.

the dynamic limits while keeping a low jerk to ensure the comfort of passengers. Finally, more demonstrations can be found in the supplementary material.

VIII. CONCLUSION

This paper proposes an efficient differential flatness-based trajectory planning for car-like robots. Unlike traditional motion planning in state space for autonomous vehicles, we translate the optimization to a flat output space so that fewer decision variables characterize a continuous trajectory with higher-order information. We decompose ambient point clouds and free space to construct a safe driving corridor, thus modeling geometric constraints used to ensure static obstacle avoidance. To cope with dynamic environments, we constrain continuously derivable analytical lower bounds of the signed distances between the ego vehicle and moving objects to achieve full-dimensional obstacle avoidance. Benchmark results with state-of-the-art methods demonstrate the superiority of our method in terms of time efficiency and trajectory quality. Real-world experiments are also conducted to validate the effectiveness of the proposed method on a real platform.

In the future, we will explore the application of the method to swarm and extend the principle to robots with other motion models. In addition, we will focus on motion planning for rugged terrain in the field, such as going up and down hills.

TABLE III: Dynamic Statistics in Real-World Experiments

Statistics	Mean	Max	STD.
Forward.Vel. (m/s)	1.53	1.95	0.52
Backward.Vel. (m/s)	0.45	0.50	0.10
Forward.Jerk. (m/s^3)	0.25	6.95	0.50
Backward.Jerk (m/s^3)	0.18	7.07	0.55

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Fig. 15: The motion diagram of the ego vehicle in the real-world experiment, where the red curve is the execution trajectory.

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