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A novel quantum model of mass function for uncertain information fusion

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ABSTRACT

Understanding the uncertainty involved in a mass function is a central issue in Dempster–Shafer evidence theory for uncertain information fusion. Recent advances suggest to interpret the mass function from a view of quantum theory. However, existing studies do not truly implement the quantization of evidence. In order to solve the problem, a usable quantization scheme for mass function is studied in this paper. At first, a novel quantum model of mass function is proposed, which effectively embodies the principle of quantum superposition. Then, a quantum averaging operator is designed to obtain the quantum average of evidence, which not only retains many basic properties, for example idempotency, commutativity, and quasi-associativity, required by a rational approach for uncertain information fusion, but also yields some new characters, namely nonlinearity and globality, caused by the quantization of mass functions. At last, based on the quantum averaging operator, a new rule called quantum average combination rule is developed for the fusion of multiple pieces of evidence, which is compared with other representative average-based combination methods to show its performance. Numerical examples and applications for classification tasks are provided to demonstrate the effectiveness of the proposed quantum model, averaging operator, and combination rule.

1. Introduction

Uncertainty is one of the hardest challenges in information processing [1]. To address this issue, various mathematical theories have been proposed, typically including fuzzy set [2], possibility theory [3,4], Z numbers [5,6], Dempster–Shafer evidence theory (D-S evidence theory) [7,8] and its many derivatives [9–14], as well as other methods [15,16]. These methods have been applied in a variety of tasks, such as medical decision-making [17], expert system [18], complex network [19], and multi-source information or data fusion [20–22]. As one of the most representative theories of uncertainty reasoning, D-S evidence theory has attracted wide attention because it needs weaker condition than the Bayesian probability theory. By allowing to assign basic probabilities among the power set of events, this theory is able to better represent and model uncertainty and imprecision, moreover it provides means to fuse multiple pieces of uncertain information. Thanks to these advantages, D-S evidence theory has been used extensively in many fields to handle the uncertain information [23–27].

Although D-S evidence theory can grasp the ignorance and uncertainty of the events by considering objective lack of complete knowledge or subjective preferences and biases [28,29], it cannot effectively integrate evidence with high conflicts in some cases [30,31]. When evidence conflicts with each others, results of combination by classical Dempster's rule provided by D-S evidence theory may be inconsistent with common sense, or D-S evidence theory simply fails. In order to solve the problem in combining conflicting evidence, many alternative combination rules have been proposed [32–36]. In addition, to help better understand the uncertainty involved in the mass function of D-S evidence theory, various interpretations about mass function have been proposed, typically for example lower & upper probabilities, non-additive probability measure, degree of subjective belief, random set, modal logic, and so on.

All the studies mentioned above are on the basis of the theoretical framework of probability or belief, therefore a mass function is also called basic probability assignment (BPA) or basic belief assignment (BBA). Recently, quantum theory [37], as a mathematical formalism to describe microscopic particles in quantum physics, has been applied in many fields to model the uncertainty [38,39]. The research on the utilization of quantum theory to D-S evidence theory based uncertain information fusion is very scarce. Several attempts have been conducted to study the connection between D-S evidence theory and quantum theory. Gao and Deng [40] proposed a quantum model of mass function. Xiao [41,42] presented a generalized D-S evidence theory based on complex numbers, but this generalization cannot be seen as a real quantum model since the property of quantum superposition is not involved. Resconi and Nikolov [43] provided a composition rule to implement the quantization of BPAs. Deng and Jiang [44] presented a quantum representation of BPA based on mixed quantum states. In

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addition, Vourdas [45,46] proposed that D-S evidence theory can be used in a quantum system. By surveying these innovative attempts, the models given by Gao and Deng [40], Xiao [41,42], are relatively well-defined. But instead of treating their models as quantum models, these models are more like a generalization of D-S evidence theory into complex number field, which provides more powerful ability to express uncertain information. Some typical quantum features, for example quantum superposition, are not well-included in these models. For example, given a mass function $m(A) = 0.8$ and $m(A, B) = 0.2$, the generated model is $Q(|A\rangle) = \sqrt{0.8}e^{j\theta_A}$, $Q(|A\rangle, |B\rangle) = \sqrt{0.2}e^{j\theta_{AB}}$ in terms of Gao and Deng's method, and that is $M(A) = 0.8e^{j\theta_1}$ and $M(A, B) = 0.2e^{j\theta_2}$ satisfying $0.8e^{j\theta_1} + 0.2e^{j\theta_2} = 1$ by means of Xiao's model. Obviously, the quantum superposition is not displayed in these models, hence both of them are not real quantum models. Additionally, existing studies are very theoretical, the established models are hard to really apply to practical information fusion applications since many parameters cannot be determined effectively.

Inspired by the challenge of implementing a usable quantization of mass function, in this paper, a new scheme, on the basis of quantum theory, is presented to provide a new view to model and handle the mass function by a way with more explicit physical meanings. Compared to existing studies, the following contributions are made in the study:

1. A novel quantum model for mass function in D-S evidence theory is proposed, which models the mass function as a quantum pure state, and a group of constraints are appended to express the epistemic and stochastic uncertainty in mass function. The quantum superposition is well-defined in the proposed model.
2. A quantum averaging operator is designed based the above quantization of mass function to obtain the quantum average of multiple mass functions. The proposed operator is not only idempotent, commutative, quasi-associative, but also nonlinear and global caused by the impact of quantum superposition.
3. A new evidence combination method, called quantum average combination rule, is developed based on the proposed quantum averaging operator for the fusion of bodies of evidence in D-S evidence theory.
4. The performance of established models above are verified by illustrative examples and applications for classification in decision-level information fusion.

The rest of paper is organized as follows. In Section 2, the preliminaries about D-S evidence theory and quantum theory are briefly introduced. Then, previous studies on quantum model of mass function are reviewed in Section 3. Section 4 gives a novel quantum model of mass function. Based on that, a quantum averaging operator for evidence is proposed in Section 5, whose mathematical properties are also analyzed and proved. Section 6 provides numerical examples to illustrate the quantum averaging operator proposed. In Section 7, a new evidence combination rule is proposed and applied to classification tasks to demonstrate its effectiveness. Finally, the conclusion is given in Section 8.

2. Preliminaries

In this section, some basic definitions and concepts about D-S evidence theory and quantum theory are introduced.

2.1. D-S evidence theory

D-S evidence theory is a tool to deal with uncertainty reasoning based on "evidence" and "combination", which is very common used in information fusion field. Some basic concepts and definitions related to D-S evidence theory are given as below [8].

Definition 1 (*Frame of Discernment*)**.** Let Ω be a set of mutually exclusive and exhaustive events, defined by $\Omega = \{e_1, e_2, \ldots, e_N\}$, then Ω is called a frame of discernment in D-S evidence theory.

The power set of Ω is composed of 2^N elements, indicated by 2^Ω, namely $2^\Omega = \{\emptyset, \{e_1\}, \{e_2\}, \ldots, \{e_1, e_2\}, \ldots, \Omega\}$, where $\emptyset$ is the empty set. If $A \in 2^\Omega$, A is called a proposition.

Definition 2 (*Mass Function*)**.** A mass function m defined on a frame of discernment Ω, also known as a BPA or BBA, is a mapping from 2^Ω to $[0\ ,\ 1]$ and satisfies:

$$m(\emptyset) = 0 \tag{1}$$

$$\sum_{A \in 2^\Omega} m(A) = 1 \tag{2}$$

In D-S evidence theory, if $m(A)$ is greater than zero, where $A \in 2^\Omega$, A is called a focal element.

Definition 3 (*Belief Function*)**.** Let A be a proposition in a frame of discernment Ω. The belief function of proposition A, denoted as $Bel(A)$, is defined by

$$Bel(A) = \sum_{B \subseteq A} m(B) \tag{3}$$

$Bel(A)$ represents the total belief degree to A. According to the characteristics of BPAs, the following equation holds:

$$Bel(\emptyset) = m(\emptyset) = 0 \tag{4}$$

$$Bel(\Omega) = \sum_{B \subseteq \Omega} m(B) = 1 \tag{5}$$

Definition 4 (*Plausibility Function*)**.** Let A be a proposition in a frame of discernment Ω. The plausibility function of proposition A, denoted as $Pl(A)$, is defined by

$$Pl(A) = 1 - Bel(\overline{A}) = \sum_{B \cap A \neq \emptyset} m(B) \tag{6}$$

where $\overline{A} = \Omega - A$.

The plausibility function represents the belief degree that does not deny A, which has the following features:

$$Pl(\emptyset) = 0 \tag{7}$$

$$Pl(\Omega) = 1 \tag{8}$$

Obviously, $Pl(A) \geq Bel(A)$ for each $A \subseteq \Omega$. Belief function $Bel(A)$ and plausibility function $Pl(A)$ constitute a belief interval $[Bel(A), Pl(A)]$, which represent the lower and upper bounds of support to proposition A, respectively. The width of $[Bel(A)\ ,\ Pl(A)]$ represents the degree of imprecision of proposition A. And there is an one-to-one correspondence between mass function and its belief function, and plausibility function through the fast mobius transformation [47] as follows:

$$m(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} Bel(B) \tag{9}$$

$$m(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|+1} Pl(\bar{B}) \tag{10}$$

Definition 5 (*Dempster's Rule of Combination*)**.** Suppose there are two independent mass functions, indicated by m_1 and m_2, in the frame of discernment Ω. Dempster's rule, denoted as $m = m_1 \oplus m_2$, is used to combine them as follows:

$$m(C) = \begin{cases} \frac{1}{1-K} \sum\limits_{A \cap B = C} m_1(A) m_2(B), & C \neq \emptyset \\ 0, & C = \emptyset \end{cases} \tag{11}$$

with

$$K = \sum_{A \cap B = \emptyset} m_1(A) m_2(B) \tag{12}$$

where $A, B \in 2^{\Omega}$.

2.2. Quantum theory

Quantum theory [37] is a mathematical framework to model the microscopic particles. Within this theory, the state of a quantum system at any time is expressed by a state vector $|\psi\rangle$ defined on a Hilbert space H, which satisfies the following superposition principle.

Definition 6 (*Superposition Principle of States*)**.** Let $\Phi = \{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle\}$ be a set of complete and mutually orthogonal basic states over the complex number field for a quantum system, then any linear superposition of these basic states is a possible state $|\psi\rangle$ of the system, i.e.,

$$|\psi\rangle = \sum_{i=1}^{n} \lambda_i |\psi_i\rangle \tag{13}$$

where $\{\lambda_i\}$ is a set of complex constants, satisfying the normalization condition

$$\sum_{i=1}^{n} |\lambda_i|^2 = 1 \tag{14}$$

in which $|\cdot|$ represents the modulus of a complex number.

In quantum theory, Eq. (13) defines a quantum pure state $|\psi\rangle$ which is the superposition of n basic states $|\psi_i\rangle$, $i = 1, 2, \ldots, n$. When a measurement operation is done on the state $|\psi\rangle$, it will collapse randomly into an explicit basic state $|\psi_i\rangle$ with the probability of $|\lambda_i|^2$.

In terms of quantum mechanics, the coefficient λ_i is a probability amplitude, which represents the wave function of basic state $|\psi_i\rangle$. A superposed state is the superposition of wave functions or probability amplitudes, but not the superposition of probabilities. Therefore, there is

$$\mathrm{Prob}(|\psi_1\rangle + |\psi_2\rangle) \neq \mathrm{Prob}(|\psi_1\rangle) + \mathrm{Prob}(|\psi_2\rangle)$$

since

$$\mathrm{Prob}(|\psi_1\rangle + |\psi_2\rangle) = |\lambda_1 + \lambda_2|^2 = |\lambda_1|^2 + |\lambda_2|^2 + (\lambda_1{}^*\lambda_2 + \lambda_1\lambda_2{}^*)$$

$$\mathrm{Prob}(|\psi_1\rangle) + \mathrm{Prob}(|\psi_2\rangle) = |\lambda_1|^2 + |\lambda_2|^2$$

where $(\lambda_1{}^*\lambda_2 + \lambda_1\lambda_2{}^*)$ represents the interference effect between $|\psi_1\rangle$ and $|\psi_2\rangle$.

Furthermore, a good manner to enhance the understanding of quantum theory is through Feynman's rules, which are originally cast by Feynman [48] and deeply studied by Goyal et al. in recent years [49–52]. Given an individual system, these rules can be used to explain how amplitudes of complex transitions of the system is generated by means of the combination of amplitudes associated with given transitions of the system, and how probabilities are calculated from amplitudes. In terms of Ref. [49], a system can classically take every path from an initial event E_i to a final event E_f, where each such event is the outcome of a measurement performed on the system and is associated with a complex number, Feynman's rules are stated as follows

(1) Amplitude sum rule: If there are $n > 1$ possible paths from E_i to E_f, but it cannot determine which path has been taken, then the total amplitude, denoted as z, for the transition from E_i to E_f is the sum of amplitudes z_k associated with all possible paths, namely $z = \sum_{k=1}^{n} z_k$.
(2) Amplitude product rule: If the transition from E_i to E_f happens via an intermediate event E_m, then the total amplitude for the transition from E_i to E_f is the product of z_{im} and z_{mf}, where z_{im} and z_{mf} are the amplitudes for the transitions $E_i \to E_m$ and $E_m \to E_f$ respectively, namely $z = z_{im} \times z_{mf}$.
(3) Probability - amplitude rule: The probability of the transition from E_i to E_f is the square of the modulus of total amplitude z, i.e., $p = |z|^2$.

As shown in the above Feynman's rules, in quantum theory, the laws of probability combination are replaced by the operations on probability amplitudes. As a result, interference terms appear. For example, if $z = z_1 + z_2$, then the probability of the transition of the system is calculated as $p = |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + (z_1{}^*z_2 + z_1z_2{}^*)$, but not $p = |z_1|^2 + |z_2|^2$ in terms of the probability laws, in which an interference term $(z_1{}^*z_2 + z_1z_2{}^*)$ yields.

3. Previous studies on quantum model of mass function

In this section, some existing schemes on the quantum representation of mass function in previous studies are reviewed briefly.

3.1. Resconi and Nikolov's composition rule

In [43], Resconi and Nikolov expressed a mass function m in quantum model as $m(A) = k|\psi(A)|^2$, where ψ is the amplitude function, and provided the following composition rule to implement the quantization of a mass function:

$$m(S_1 \cup S_2) = |\alpha\psi(S_1) + \beta\psi(S_2)|^2 = m(S_1) + m(S_2) + \Gamma_{12} \tag{15}$$

where S_1 and S_2 are propositions with $m(S_1) = \alpha^2|\psi(S_1)|^2$, $m(S_2) = \beta^2|\psi(S_2)|^2$ and $\Gamma_{12} = \alpha\beta\psi(S_2)^*\psi(S_1) + \alpha\beta\psi(S_1)^*\psi(S_2)$ is the interference term which can be positive, null or negative. By using the composition rule, the authors attempt to explain why $m(A \cup B) \neq m(A) + m(B)$ in a mass function for D-S evidence theory, while there is $\mathrm{Prob}(A \cup B) = \mathrm{Prob}(A) + \mathrm{Prob}(B)$ within a probability distribution. However the authors do not give methods to calculate related parameters, and there were not further studies on this idea.

3.2. Gao and Deng's quantum model

In [40], Gao and Deng proposed a quantum model for mass function m, indicated by Q, as follows:

$$Q(|A\rangle) = \psi e^{\mathrm{j}\theta_A} \tag{16}$$

satisfying

$$Q(\emptyset) = 0 \tag{17}$$

$$\sum_{|A\rangle \subseteq |\Theta\rangle} \|Q(|A\rangle)\|^2 = 1 \tag{18}$$

where $\mathrm{j} = \sqrt{-1}$, $\|Q(|A\rangle)\|^2 = \psi^2 = m(A)$ and Θ is the frame of discernment. By analyzing this model, we find there are some points that need to be further clarified:

(i) What is the set of basic states, Θ or 2^{Θ}?

(ii) Given a focal element $m(a, b) = p$, which is indicated by $Q(|a, b\rangle) = \sqrt{p}e^{\mathrm{j}\theta}$ in this quantum model, how does the quantum superposition between states a and b being embodied?

(iii) How to determine the amplitude angle θ for each focal element? And what is the physical meaning of θ?

3.3. Xiao's complex mass function

In [41,42], Xiao defined a complex mass function M as below

$$\begin{aligned} & M(\emptyset) = 0, \\ & M(A) = m(A)e^{\mathrm{j}\theta(A)}, A \subseteq \Theta \\ & \sum_{A \subseteq \Theta} M(A) = 1 \end{aligned} \tag{19}$$

where $\mathrm{j} = \sqrt{-1}$, $m(A) \in [0, 1]$ representing the magnitude of $M(A)$, $\theta(A) \in [-\pi, \pi]$ denoting a phase term.

Essentially, the model proposed by Xiao is a generalization of complexification to classical mass function, but not a quantum model: (i) The quantum superposition property is not contained in the model of complex mass function; (ii) In the transformation of a classical mass function, the probability of a proposition is defined as the modulus of corresponding complex vector, but not the square of the modulus; (iii) The normalization condition about the probability amplitudes, no matter $\sum_{A\subseteq\Theta}|m(A)| = 1$ or $\sum_{A\subseteq\Theta}|m(A)|^2 = 1$, where $|\cdot|$ represents the absolute value, is not satisfied.

3.4. Deng and Jiang's model based on mixed quantum states

In [44] Deng and Jiang proposed a quantum representation of basic probability assignments based on mixed quantum states. Given a frame of discernment $\Theta = \{h_1, h_2, \dots, h_n\}$, they first quantize it to a quantum frame of discernment $\Theta_q = \{\,|h_1\rangle, |h_2\rangle, \cdots, |h_n\rangle\,\}$. Then, a proposition F in Θ is regarded as a quantum pure state:

$$|F\rangle = \sum_{h_i\in F} z_i\,|h_i\rangle \tag{20}$$

where z_i is a complex number satisfying $\sum_{h_i\in F}|z_i|^2 = 1$. At last, a mass function $m(F_i) = q_i$ with $F_i \subseteq \Theta$, $F_i \neq \emptyset$, $q_i \geq 0$ and $\sum_{i=1}^{2^n-1} q_i = 1$, is represented as a mixed system of quantum pure states:

$$\left\{|F_i\rangle \text{ with probability } q_i,\ \ \text{where } q_i > 0\right\} \tag{21}$$

which can be simply formulated by a density matrix $\rho = \sum_i q_i\rho_i$ with $\rho_i = |F_i\rangle\langle F_i|$.

Deng and Jiang's model provided a solution to quantize non-bayesian mass functions. However, for a bayesian mass function, in which the basic probabilities are only assigned to singletons of the frame of discernment, the model just treats it as a mixture of deterministic events in terms of a probability distribution.

4. Proposed quantum model of mass function

In this article, a novel quantum model of mass function is proposed, which can effectively implement the quantization of mass function in D-S evidence theory.

As well known, one of the most essential features of a quantum system is the principle of superposition. Given a set of basic events $\Phi = \{\phi_1, \phi_2, \dots, \phi_n\}$, the associated quantum state is represented as

$$|\varphi\rangle = \sum_{i=1}^{n} \lambda_i\,|\phi_i\rangle \tag{22}$$

where the probability of each basic state ϕ_i is $|\lambda_i|^2$ and $\sum_{i=1}^{n}|\lambda_i|^2 = 1$.

Meanwhile, in the context of D-S evidence theory, epistemic and stochastic uncertainty can be contained simultaneously in a mass function. According to the lower & upper probabilities interpretation, a mass function can be seen as a cluster of probability distributions. In other words, given a mass function m defined on Φ, its all information is contained in a set $\{\mathrm{Prob}(\phi_i), \phi_i \in \Phi\}$ satisfying the following conditions:

$$\begin{cases} \mathrm{Prob}(\phi_i) \in [0,1], \forall \phi_i \in \Phi \\ \displaystyle\sum_{\phi_i\in\Phi} \mathrm{Prob}(\phi_i) = 1 \\ Bel(A) \leq \displaystyle\sum_{\phi_i\in A} \mathrm{Prob}(\phi_i) \leq Pl(A), \forall A \subseteq \Phi \end{cases} \tag{23}$$

Therefore, the quantization of mass function must take the above information into account. With this consideration, in this paper a novel quantum model of mass function is proposed as below.

Definition 7. Let $\Phi = \{\phi_1, \phi_2, \dots, \phi_n\}$ be a set of mutually exclusive and collectively exhaustive events, whose quantum transformation is $\Phi_q = \{|\phi_1\rangle, |\phi_2\rangle, \dots, |\phi_n\rangle\}$ satisfying $\langle\phi_i|\phi_j\rangle = \delta_{ij}$, where δ_{ij} is the Kronecker symbol meeting $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$, $i, j = 1, 2, \dots, n$. As for a mass function m defined on Φ, it can be represented as the following quantum state:

$$\begin{aligned} |\varphi\rangle &= \rho_1 e^{j\theta_1}|\phi_1\rangle + \rho_2 e^{j\theta_2}|\phi_2\rangle + \cdots + \rho_n e^{j\theta_n}|\phi_n\rangle \\ &= \sum_{i=1}^{n} \rho_i e^{j\theta_i}|\phi_i\rangle \\ & s.t. \begin{cases} \rho_i \geq 0, \\ \theta_i \in [-\pi, \pi], \\ \displaystyle\sum_{\phi_i\in\Phi} {\rho_i}^2 = 1, \\ Bel(A) \leq \displaystyle\sum_{\phi_i\in A} {\rho_i}^2 \leq Pl(A), \forall A \subseteq \Phi. \end{cases} \end{aligned} \tag{24}$$

where $j = \sqrt{-1}$ and e is Euler's constant.

More specifically, in Eq. (24), ϕ_i represents the ith basic event in the frame of discernment Φ, and $|\phi_i\rangle$ is the quantum state counterpart of ϕ_i. A mass function m defined on $\Phi = \{\phi_1, \phi_2, \dots, \phi_n\}$ is represented as a quantum superposition state $|\varphi\rangle$ composed of $|\phi_1\rangle, |\phi_2\rangle, \dots, |\phi_n\rangle$, i.e., $|\varphi\rangle = \sum_{i=1}^{n} \rho_i e^{j\theta_i}|\phi_i\rangle$, in which $\rho_i e^{j\theta_i}$ is the probability amplitude associated with basic state $|\phi_i\rangle$, and ρ_i and θ_i are the modulus and phase angle of this probability amplitude, respectively. With respect to the constraints in Eq. (24), $\sum_{\phi_i\in\Phi}{\rho_i}^2 = 1$ is the normalization condition required by the definition of standard quantum state. Meanwhile, the modulus of each probability amplitude, i.e., ρ_i, $i = 1, \dots, n$, has to meet the constraint $Bel(A) \leq \sum_{\phi_i\in A}{\rho_i}^2 \leq Pl(A), \forall A \subseteq \Phi$, required by the definition of mass function, since $|\varphi\rangle$ is the quantum counterpart of mass function m. Thus, an explicit connection between mass function and quantum model is established through Eq. (24) in Definition 7.

From the above definition, the proposed quantum model of mass function has these characteristics as below.

1. A mass function is modeled as a quantum pure state with constraints, and the epistemic uncertainty contained in a mass function is considered in the constraints.
2. The principle of quantum superposition is involved in the model since the quantum state of a mass function, denoted as $|\varphi\rangle$, is a linear superposition of basic states with complex coefficients $\rho_i e^{j\theta_i}$, $i = 1, 2, \dots, n$.
3. The physical meanings of parameters are clear: the square of probability amplitude ρ_i expresses the probability of event ϕ_i, while the amplitude angle θ_i can have different interpretations which will be discussed in following sections.
4. The model is very succinct. A mass function can be easily represented as a quantum state. Conversely, given such a quantum model, the associated mass function can also be generated easily by using following optimization function

$$Bel(A) = \min \sum_{\phi_i\in A}\rho_i^2 \Big/ \sum_{\phi_i\in\Phi}\rho_i^2, \quad \forall A \subseteq \Phi \tag{25}$$

or

$$Pl(A) = \max \sum_{\phi_i\in A}\rho_i^2 \Big/ \sum_{\phi_i\in\Phi}\rho_i^2, \quad \forall A \subseteq \Phi \tag{26}$$

For the optimization problems in Eqs. (25) and (26), since ρ_i^2 is handled as a whole, both of them are essentially linear programming problems with constraints. For the problem of linear programming, classical simplex algorithm has an exponential worst case time complexity $\mathcal{O}(2^n)$ [53], where $n = |\Phi|$ is the size of frame of discernment Φ. In order to reduce the computational complexity, many polynomial time algorithms have been developed for linear programming, such as interior-point methods (IPM). IPM [54,55] is the first practical polynomial time algorithm for linear programming, whose computational complexity is $\mathcal{O}(n^{3.5}L)$ where L is the length of all inputs encoded in binary bits. In this paper, we can use IPM to solve the optimization problems in Eqs. (25) and (26).

5. In the model, the set of basic quantum states are transformed from Φ, but not 2^{Φ}.

For the sake of explaining more clearly how the classical mass function is mapped to a quantum superposition, a numerical example is illustrated as follows.

Example 1. Given $\Phi = \{a, b, c\}$, two mass functions defined on Φ are respectively $m_1(a) = 0.6, m_1(a, b, c) = 0.4$; and $m_2(a) = 0.5, m_2(b) = 0.4, m_2(c) = 0.1$. Herein, m_1 is a non-Bayesian mass function, and m_2 is a Bayesian mass function. In terms of Definition 7, the quantum models of m_1 and m_2 can be derived as below:

$$\left|\varphi_1\right\rangle = \rho_{11}e^{j\theta_{11}}\left|a\right\rangle + \rho_{12}e^{j\theta_{12}}\left|b\right\rangle + \rho_{13}e^{j\theta_{13}}\left|c\right\rangle$$

$$s.t.\begin{cases} 0.6 \le {\rho_{11}}^2 \le 1, \\ 0 \le {\rho_{12}}^2 \le 0.4, \\ 0 \le {\rho_{13}}^2 \le 0.4, \\ 0.6 \le {\rho_{11}}^2 + {\rho_{12}}^2 \le 1, \\ 0.6 \le {\rho_{11}}^2 + {\rho_{13}}^2 \le 1, \\ 0 \le {\rho_{12}}^2 + {\rho_{13}}^2 \le 0.4, \\ {\rho_{11}}^2 + {\rho_{12}}^2 + {\rho_{13}}^2 = 1, \\ \rho_{11} \ge 0, \rho_{12} \ge 0, \rho_{13} \ge 0, \\ \theta_{11}, \theta_{12}, \theta_{13} \in [-\pi, \pi]. \end{cases}$$

$$\left|\varphi_2\right\rangle = \sqrt{0.5}e^{j\theta_{21}}\left|a\right\rangle + \sqrt{0.4}e^{j\theta_{22}}\left|b\right\rangle + \sqrt{0.1}e^{j\theta_{23}}\left|c\right\rangle$$
$$s.t. \quad \theta_{21}, \theta_{22}, \theta_{23} \in [-\pi, \pi].$$

From the results, for non-Bayesian mass function m_1, the obtained quantum state $\left|\varphi_1\right\rangle$ is a superposition state composed of $|a\rangle$, $|b\rangle$ and $|c\rangle$, in which the probability amplitudes associated with different basic states have to satisfy a series of constraints required by m_1. For m_2, its quantum model $\left|\varphi_2\right\rangle$ is also a superposition state composed of $|a\rangle$, $|b\rangle$ and $|c\rangle$. However, different from $\left|\varphi_1\right\rangle$, in $\left|\varphi_2\right\rangle$ the module of each probability amplitude is deterministic since m_2 is a Bayesian mass function. Through this example, the difference between quantum models of Bayesian and non-Bayesian mass functions is presented.

5. Quantum averaging operator for mass functions

In this section, a quantum averaging operator for mass functions is presented based on our proposed quantum model, in which the weights of mass functions are given in advance.

Definition 8. Assuming that m_1, m_2 are two pieces of evidence defined on $\Phi = \{\phi_1, \phi_2, \ldots, \phi_n\}$ with weights w_1 and w_2, let $\left|\varphi_{m_1}\right\rangle$, $\left|\varphi_{m_2}\right\rangle$ be the quantum models of m_1, m_2, respectively, i.e.,

$$\left|\varphi_{m_1}\right\rangle = \sum_{i=1}^{n} \rho_{1i}e^{j\theta_{1i}}\left|\phi_i\right\rangle$$
$$s.t.1\begin{cases} \rho_{1i} \ge 0, \\ \theta_{1i} \in [-\pi, \pi], \\ \sum_{\phi_i \in \Phi} {\rho_{1i}}^2 = 1, \\ Bel(A) \le \sum_{\phi_i \in A} {\rho_{1i}}^2 \le Pl(A), \forall A \subseteq \Phi. \end{cases} \tag{27}$$

$$\left|\varphi_{m_2}\right\rangle = \sum_{i=1}^{n} \rho_{2i}e^{j\theta_{2i}}\left|\phi_i\right\rangle$$
$$s.t.2\begin{cases} \rho_{2i} \ge 0, \\ \theta_{2i} \in [-\pi, \pi], \\ \sum_{\phi_i \in \Phi} {\rho_{2i}}^2 = 1, \\ Bel(A) \le \sum_{\phi_i \in A} {\rho_{2i}}^2 \le Pl(A), \forall A \subseteq \Phi. \end{cases} \tag{28}$$

The quantum average of m_1 and m_2, denoted as $m = w_1 m_1 +_q w_2 m_2$, or $\left|\varphi_m\right\rangle = w_1 \cdot \left|\varphi_{m_1}\right\rangle +_q w_2 \cdot \left|\varphi_{m_2}\right\rangle$, is defined by

$$\left|\varphi_m\right\rangle = \frac{1}{S}\sum_{i=1}^{n} \rho_i e^{j\theta_i}\left|\phi_i\right\rangle$$
$$s.t.\begin{cases} s.t.1, \\ s.t.2, \\ \rho_i \ge 0, S > 0, \\ \theta_i \in [-\pi, \pi], \\ \sum_{i=1}^{n}\left({w_1}^2{\rho_{1i}}^2 + {w_2}^2{\rho_{2i}}^2 + 2w_1w_2\rho_{1i}\rho_{2i}\cos(\theta_{1i} - \theta_{2i})\right) = S^2. \end{cases} \tag{29}$$

with $\forall i \in \{1, 2, \ldots, n\}$,

$${\rho_i}^2 = {w_1}^2{\rho_{1i}}^2 + {w_2}^2{\rho_{2i}}^2 + 2w_1w_2\rho_{1i}\rho_{2i}\cos(\theta_{1i} - \theta_{2i}) \tag{30}$$

$$\theta_i = \arctan\left(\frac{w_1\rho_{1i}\sin\theta_{1i} + w_2\rho_{2i}\sin\theta_{2i}}{w_1\rho_{1i}\cos\theta_{1i} + w_2\rho_{2i}\cos\theta_{2i}}\right) \tag{31}$$

In the proposed quantum averaging operator for mass functions, Eqs. (30) and (31) are the key to understand the impact of mass functions' quantization on the combination of evidence. Specifically, for a basic state $\left|\phi_i\right\rangle$, the probability amplitudes of $\left|\phi_i\right\rangle$ in $\left|\varphi_{m_1}\right\rangle$ and $\left|\varphi_{m_2}\right\rangle$ are $\rho_{1i}e^{j\theta_{1i}}$ and $\rho_{2i}e^{j\theta_{2i}}$ respectively through the quantization to m_1 and m_2, then the quantum averaging operator makes a weighted average on $\rho_{1i}e^{j\theta_{1i}}$ and $\rho_{2i}e^{j\theta_{2i}}$ to generate the probability amplitude of $\left|\phi_i\right\rangle$ in the combination result, i.e., $\rho_i e^{j\theta_i} = w_1\rho_{1i}e^{j\theta_{1i}} + w_2\rho_{2i}e^{j\theta_{2i}}$, thus ${\rho_i}^2 = {w_1}^2{\rho_{1i}}^2 + {w_2}^2{\rho_{2i}}^2 + 2w_1w_2\rho_{1i}\rho_{2i}\cos(\theta_{1i} - \theta_{2i})$, as shown in Eq. (30). The quantity $2w_1w_2\rho_{1i}\rho_{2i}\cos(\theta_{1i} - \theta_{2i})$ is the interference term between $\left|\varphi_{m_1}\right\rangle$ and $\left|\varphi_{m_2}\right\rangle$ on $\left|\phi_i\right\rangle$. The amplitude angle θ_i in Eq. (31) is also obtained by means of the weighted average of complex vectors $\rho_{1i}e^{j\theta_{1i}}$ and $\rho_{2i}e^{j\theta_{2i}}$ with weights w_1 and w_2. As shown in Eq. (31), θ_i is a function of w_1, w_2, ρ_{1i}, ρ_{2i}, θ_{1i}, θ_{2i}, in which w_1, w_2 are given in advance, and ρ_{1i}, ρ_{2i} can be obtained directly according to mass functions to combine, amplitude angles θ_{1i} and θ_{2i} are defined in terms of specific application scenarios. For example, in a recent work [56] the authors have suggested to define the amplitude angle by means of the reliability of corresponding evidence.

From the above definition, the quantum averaging operator has reflected the impact of quantum superposition since there is interference term $\Gamma_i = 2w_1w_2\rho_{1i}\rho_{2i}\cos(\theta_{1i} - \theta_{2i})$. As shown in Fig. 1, $|\theta_{1i} - \theta_{2i}|$ influences the sign of interference term Γ_i. If $|\theta_{1i} - \theta_{2i}| \in [0, \frac{\pi}{2})$, the interference term is positive; If $|\theta_{1i} - \theta_{2i}| \in (\frac{\pi}{2}, \pi]$, the interference term is negative; Otherwise, the interference term is null if $|\theta_{1i} - \theta_{2i}| = \frac{\pi}{2}$. Based on these, we provide two interpretations for the physical meaning of $|\theta_{1i} - \theta_{2i}|$:

- It reflects the dependence between m_1 and m_2 on event ϕ_i. In D-S evidence theory, the Dempster's rule requires that the bodies of evidence to be combined should be independent, here $|\theta_{1i} - \theta_{2i}|$ could provide potential solutions to solve the issue of combination of dependent mass functions.
- It reflects the inter-support effect between m_1 and m_2 on event ϕ_i. If $|\theta_{1i} - \theta_{2i}| \in [0, \frac{\pi}{2})$, the beliefs of m_1 and m_2 on ϕ_i are mutually reinforcing; If $|\theta_{1i} - \theta_{2i}| \in (\frac{\pi}{2}, \pi]$, the beliefs of m_1 and m_2 on ϕ_i are mutually suppressing; $|\theta_{1i} - \theta_{2i}| = \frac{\pi}{2}$ represents there is not any effect between m_1 and m_2 on ϕ_i.

For the sake of understanding the first interpretation, we compare this idea with an existing approach proposed by Yager [57] for the aggregation of non-independent mass functions. In [57], all mass functions to be fused are ordered in terms of a predefined sequence in the light of the reliability of information sources and the information content contained in belief structures, that is $m_{seq(1)} \rightarrow m_{seq(2)} \rightarrow$

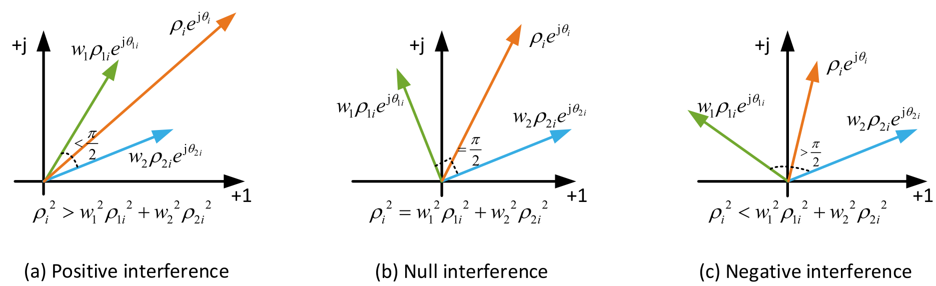

Fig. 1. Interference in quantum averaging operator.

$\cdots\cdots \to m_{seq(q)}$. Then, the fusion of non-independent mass functions is formulated as a sequential aggregation process, i.e.,

$$\begin{aligned} m &= m_{seq(1)} \oplus (\delta_{seq(2)} \otimes m_{seq(2)}) \oplus \cdots \oplus (\delta_{seq(q)} \otimes m_{seq(q)}) \\ &= \bigoplus_{j=1}^{q} (\delta_{seq(j)} \otimes m_{seq(j)}) \end{aligned} \tag{32}$$

where $\delta_{seq(j)}$ is the degree of independence of the evidence $m_{seq(j)}$ from the previous aggregated values, and $\otimes$ represents the discounting operator defined by Shafer [8]. It is found that, for the issue of dependent evidence fusion, Yager's method defined a global degree of independence for every mass function at the level of evidence. However, by means of the idea of quantum model of mass function proposed in this paper we suggest to define a degree of dependence for every basic event ϕ_i between mass functions, which is at the level of proposition, and $|\theta_{1i} - \theta_{2i}|$ could provide useful information to help determine the dependent degree between two pieces of evidence on event ϕ_i.

In this paper, we mainly focus on the latter interpretation. The phase difference $|\theta_{1i} - \theta_{2i}|$ reflects the interaction of different mass functions on a same event, which cannot be obtained directly through the given mass functions and needs to be determined externally. From the perspective of information fusion, classical average between mass functions, i.e., $m(A) = \dfrac{w_1 m_1(A) + w_2 m_2(A)}{w_1 + w_2}$, exactly corresponds to the case of $|\theta_{1i} - \theta_{2i}| = \dfrac{\pi}{2}$. With the decrease of $|\theta_{1i} - \theta_{2i}|$, the mutual support between m_1 and m_2 on ϕ_i, i.e., interference term Γ_i, becomes stronger and stronger. Specially, when $|\theta_{1i} - \theta_{2i}| = 0$, the mutual support Γ_i reaches the biggest. From the sense of fusion, $|\theta_{1i} - \theta_{2i}| = 0$ corresponds to a manner of "enhanced" fusion which is to maximize the mutual support among information from different sources. Having these thinking, we simplify the quantum averaging operator in Definition 8 by setting $|\theta_{1i} - \theta_{2i}| = 0$, then the result of the quantum average of m_1 and m_2 becomes

$$\begin{aligned} &|\varphi_m\rangle = \frac{1}{S}\sum_{i=1}^{n} \rho_i |\phi_i\rangle \\ &s.t. \begin{cases} s.t.1, \\ s.t.2, \\ \rho_i \ge 0, S > 0, \\ \sum_{i=1}^{n} \left(w_1^2 \rho_{1i}^2 + w_2^2 \rho_{2i}^2 + 2 w_1 w_2 \rho_{1i} \rho_{2i}\right) = S^2. \end{cases} \end{aligned} \tag{33}$$

with $\forall i \in \{1, 2, \ldots, n\}$,

$$\rho_i^2 = w_1^2 \rho_{1i}^2 + w_2^2 \rho_{2i}^2 + 2 w_1 w_2 \rho_{1i} \rho_{2i} \tag{34}$$

In the light of Eqs. (33) and (34), the amplitude angles θ_{1i} and θ_{2i} in $\left|\varphi_{m_1}\right\rangle$ and $\left|\varphi_{m_2}\right\rangle$ become insignificant. Moreover, for the results obtained by using Eqs. (33) and (34), the following desirable properties are satisfied.

Property 1. *Idempotency:* $m = w_1 m +_q w_2 m$.

Proof. Given two pieces of evidence m_1, m_2 defined on $\Phi = \{\phi_1, \phi_2, \ldots, \phi_n\}$, and m_1 and m_2 are identical. Let $|\varphi_{m1}\rangle$, $|\varphi_{m2}\rangle$ be the quantum representations of m_1 and m_2, denoted as $|\varphi_{m1}\rangle = \sum_{i=1}^{n} \rho_{1i} e^{j\theta_{1i}} |\phi_i\rangle$, $|\varphi_{m2}\rangle = \sum_{i=1}^{n} \rho_{2i} e^{j\theta_{2i}} |\phi_i\rangle$. According to Eqs. (33) and (34), the obtained belief function using quantum averaging operator of mass functions is denoted as

$$Bel_m(A) = \min \sum_{\phi_i \in A} \rho_i^2 \Big/ \sum_{\phi_i \in \Phi} \rho_i^2$$

with $\rho_i^2 = w_1^2 \rho_{1i}^2 + w_2^2 \rho_{2i}^2 + 2 w_1 w_2 \rho_{1i} \rho_{2i}$.

Since $m_1 = m_2$, then $Bel_{m_1}(A) = Bel_{m_2}(A), \forall A \subseteq \Phi$, and

$$Bel_{m_1}(A) = \min \sum_{\phi_i \in A} \rho_{1i}^2 \Big/ \sum_{\phi_i \in \Phi} \rho_{1i}^2$$

$$Bel_{m_2}(A) = \min \sum_{\phi_i \in A} \rho_{2i}^2 \Big/ \sum_{\phi_i \in \Phi} \rho_{2i}^2$$

For every $A \subseteq \Phi$ in m, $Bel_m(A)$ gets the minimum value while $\rho_{1i} = \rho_{2i}$, $i \in \{1, 2, \ldots, n\}$. Therefore,

$$\begin{aligned} Bel_m(A) &= \min \sum_{\phi_i \in A} \rho_i^2 \Big/ \sum_{\phi_i \in \Phi} \rho_i^2 \\ &= \min \frac{\sum_{\phi_i \in A} (w_1 \rho_{1i} + w_2 \rho_{2i})^2}{\sum_{\phi_i \in \Phi} (w_1 \rho_{1i} + w_2 \rho_{2i})^2} \\ &= \min \frac{\sum_{\phi_i \in A} (w_1 + w_2)^2 \rho_{1i}^2}{\sum_{\phi_i \in \Phi} (w_1 + w_2)^2 \rho_{1i}^2} \\ &= \min \frac{\sum_{\phi_i \in A} \rho_{1i}^2}{\sum_{\phi_i \in \Phi} \rho_{1i}^2} \end{aligned}$$

Namely, $Bel_m(A) = Bel_{m_1}(A) = Bel_{m_2}(A), \forall A \subseteq \Phi$. The property of idempotency is proved. □

Property 2. *Commutativity:* $w_1 m_1 +_q w_2 m_2 = w_2 m_2 +_q w_1 m_1$.

Proof. The property of commutativity is evident. □

Property 3. *Quasi-Associativity:* $(+_q)_{i=1}^{n} w_i m_i$.

Proof. As explained in [33], quasi-associativity implies that a combination method of evidence is able to be extended to be directly applied on several mass functions. Evidently, the proposed quantum averaging operator defined for the combination of two mass functions can be easily used to the case of multiple mass functions. □

It is noted that the mass functions are combined simultaneously, not in a way of one by one. For a combination rule satisfying the quasi-associativity but not associativity, on practical and real fusion systems we must identify the information fusion schemes to which the question belongs for the first of all, mainly including aggregating scheme and updating scheme [58]. In the case of aggregating scheme where evidences express different opinions about the same event, we must combine all evidences at one time, while the way to combine evidences one by one may lead to wrong conclusions for a quasi-associative rule. In the case of updating scheme where evidences express sequential opinions about a dynamic event, the associativity is not required, a quasi-associative combination rule can directly be used to combine the evidences one by one in terms of the sequencing of evidences. Therefore, the quasi-associativity of the combination rule makes a constraint on the way people use it for practical and real fusion systems.

Example 2. A simple example is given to show how to implement the average of multiple mass functions by using the proposed quantum averaging operator.

Assuming that m_1, m_2 and m_3 are three pieces of evidence defined on $\Phi = \{a, b, c\}$ with weights w_1, w_2 and w_3, respectively. At first, the quantum models of these mass functions are generated as below

$$|\psi_1\rangle = \rho_{11}e^{j\theta_{11}}|a\rangle + \rho_{12}e^{j\theta_{12}}|b\rangle + \rho_{13}e^{j\theta_{13}}|c\rangle$$

$$s.t.1 \begin{cases} Bel(a) \le {\rho_{11}}^2 \le Pl(a), \\ Bel(b) \le {\rho_{12}}^2 \le Pl(b), \\ Bel(c) \le {\rho_{13}}^2 \le Pl(c), \\ Bel(ab) \le {\rho_{11}}^2 + {\rho_{12}}^2 \le Pl(ab), \\ Bel(ac) \le {\rho_{11}}^2 + {\rho_{13}}^2 \le Pl(ac), \\ Bel(bc) \le {\rho_{12}}^2 + {\rho_{13}}^2 \le Pl(bc), \\ {\rho_{11}}^2 + {\rho_{12}}^2 + {\rho_{13}}^2 = 1. \end{cases}$$

$$|\psi_2\rangle = \rho_{21}e^{j\theta_{21}}|a\rangle + \rho_{22}e^{j\theta_{22}}|b\rangle + \rho_{23}e^{j\theta_{23}}|c\rangle$$

$$s.t.2 \begin{cases} Bel(a) \le {\rho_{21}}^2 \le Pl(a), \\ Bel(b) \le {\rho_{22}}^2 \le Pl(b), \\ Bel(c) \le {\rho_{23}}^2 \le Pl(c), \\ Bel(ab) \le {\rho_{21}}^2 + {\rho_{22}}^2 \le Pl(ab), \\ Bel(ac) \le {\rho_{21}}^2 + {\rho_{23}}^2 \le Pl(ac), \\ Bel(bc) \le {\rho_{22}}^2 + {\rho_{23}}^2 \le Pl(bc), \\ {\rho_{21}}^2 + {\rho_{22}}^2 + {\rho_{23}}^2 = 1. \end{cases}$$

$$|\psi_3\rangle = \rho_{31}e^{j\theta_{31}}|a\rangle + \rho_{32}e^{j\theta_{32}}|b\rangle + \rho_{33}e^{j\theta_{33}}|c\rangle$$

$$s.t.3 \begin{cases} Bel(a) \le {\rho_{31}}^2 \le Pl(a), \\ Bel(b) \le {\rho_{32}}^2 \le Pl(b), \\ Bel(c) \le {\rho_{33}}^2 \le Pl(c), \\ Bel(ab) \le {\rho_{31}}^2 + {\rho_{32}}^2 \le Pl(ab), \\ Bel(ac) \le {\rho_{31}}^2 + {\rho_{33}}^2 \le Pl(ac), \\ Bel(bc) \le {\rho_{32}}^2 + {\rho_{33}}^2 \le Pl(bc), \\ {\rho_{31}}^2 + {\rho_{32}}^2 + {\rho_{33}}^2 = 1. \end{cases}$$

Then, the quantum average of m_1, m_2 and m_3 is obtained

$$\begin{aligned} |\psi\rangle &= w_1 \cdot |\psi_1\rangle +_q w_2 \cdot |\psi_2\rangle +_q w_3 \cdot |\psi_3\rangle \\ &= \frac{1}{S}(\rho_1 e^{j\theta_1}|a\rangle + \rho_2 e^{j\theta_2}|b\rangle + \rho_3 e^{j\theta_3}|c\rangle) \end{aligned}$$

$$s.t. \begin{cases} s.t.1, \\ s.t.2, \\ s.t.3, \\ \rho_1 \ge 0, \rho_2 \ge 0, \rho_3 \ge 0, S > 0, \\ \theta_1, \theta_2, \theta_3 \in [-\pi, \pi], \\ \sum_{i=1}^{3} \big({w_1}^2{\rho_{1i}}^2 + {w_2}^2{\rho_{2i}}^2 + {w_3}^2{\rho_{3i}}^2 + 2w_1w_2\rho_{1i}\rho_{2i} + 2w_1w_3\rho_{1i}\rho_{3i} \\ +2w_2w_3\rho_{2i}\rho_{3i}\big) = S^2. \end{cases}$$

with $\forall i \in \{1, 2, 3\}$,

$$\begin{aligned} \rho_i^2 &= {w_1}^2{\rho_{1i}}^2 + {w_2}^2{\rho_{2i}}^2 + {w_3}^2{\rho_{3i}}^2 \\ &\quad + 2w_1w_2\rho_{1i}\rho_{2i} + 2w_1w_3\rho_{1i}\rho_{3i} + 2w_2w_3\rho_{2i}\rho_{3i} \\ \theta_i &= \arctan\Big(\frac{w_1\rho_{1i}\sin\theta_{1i} + w_2\rho_{2i}\sin\theta_{2i} + w_3\rho_{3i}\sin\theta_{3i}}{w_1\rho_{1i}\cos\theta_{1i} + w_2\rho_{2i}\cos\theta_{2i} + w_3\rho_{3i}\cos\theta_{3i}}\Big) \end{aligned}$$

Property 4. *Nonlinearity: the quantum averaging operator "$+_q$" is a nonlinear combination of multiple mass functions.*

Proof. Traditional average of evidence, i.e., $m(A) = \frac{w_1m_1(A) + w_2m_2(A)}{w_1 + w_2}$, is a linear combination of mass function. However, as shown in Eqs. (33) and (34), since the existence of interference item $2w_1w_2\rho_{1i}\rho_{2i}$, the quantum averaging operator "$+_q$" is a nonlinear combination of multiple mass functions. This is a new important property caused by the quantization, i.e., quantum superposition. □

Property 5. *Globality: the quantum averaging operator "$+_q$" is a global combination of mass functions.*

Proof. Given two mass functions m_1 and m_2 to combine, globality means that in the combination result of quantum averaging operator, denoted as m, the mass $m(A)$ is not only related to $m_1(A)$ and $m_2(A)$, but also related to the other focal elements in m_1 and m_2. This is a property different from classical averaging combination of evidence $m(A) = \frac{w_1m_1(A) + w_2m_2(A)}{w_1 + w_2}$. Here, we give an illustrative example to show the property.

Given a frame of discernment $\Phi = \{a, b, c\}$, there are two pieces of evidence m_1, m_2, whose quantum models are $|\psi_1\rangle$ and $|\psi_2\rangle$:

$$|\psi_1\rangle = \alpha_1 e^{j\theta_a}|a\rangle + \beta_1 e^{j\theta_b}|b\rangle + \gamma_1 e^{j\theta_c}|c\rangle$$

$$s.t.1 \begin{cases} Bel(a) \le {\alpha_1}^2 \le Pl(a), \\ Bel(b) \le {\beta_1}^2 \le Pl(b), \\ Bel(c) \le {\gamma_1}^2 \le Pl(c), \\ Bel(ab) \le {\alpha_1}^2 + {\beta_1}^2 \le Pl(ab), \\ Bel(ac) \le {\alpha_1}^2 + {\gamma_1}^2 \le Pl(ac), \\ Bel(bc) \le {\beta_1}^2 + {\gamma_1}^2 \le Pl(bc), \\ {\alpha_1}^2 + {\beta_1}^2 + {\gamma_1}^2 = 1. \end{cases}$$

$$|\psi_2\rangle = \alpha_2 e^{j\theta_a}|a\rangle + \beta_2 e^{j\theta_b}|b\rangle + \gamma_2 e^{j\theta_c}|c\rangle$$

$$s.t.2 \begin{cases} Bel(a) \le {\alpha_2}^2 \le Pl(a), \\ Bel(b) \le {\beta_2}^2 \le Pl(b), \\ Bel(c) \le {\gamma_2}^2 \le Pl(c), \\ Bel(ab) \le {\alpha_2}^2 + {\beta_2}^2 \le Pl(ab), \\ Bel(ac) \le {\alpha_2}^2 + {\gamma_2}^2 \le Pl(ac), \\ Bel(bc) \le {\beta_2}^2 + {\gamma_2}^2 \le Pl(bc), \\ {\alpha_2}^2 + {\beta_2}^2 + {\gamma_2}^2 = 1. \end{cases}$$

The result obtained by the quantum average of m_1 and m_2 is

$$|\varphi\rangle = \alpha e^{j\theta_a}|a\rangle + \beta e^{j\theta_b}|b\rangle + \gamma e^{j\theta_c}|c\rangle$$

$$s.t. \begin{cases} s.t.1, \\ s.t.2, \\ \alpha \ge 0, \beta \ge 0, \gamma \ge 0, \\ \theta_a, \theta_b, \theta_c \in [-\pi, \pi], \end{cases}$$

with

$$\alpha^2 = (w_1\alpha_1 + w_2\alpha_2)^2 \Big/ \sum_{(\lambda_1,\lambda_2)\in\{(\alpha_1,\alpha_2),(\beta_1,\beta_2),(\gamma_1,\gamma_2)\}} (w_1\lambda_1 + w_2\lambda_2)^2$$

$$\beta^2 = (w_1\beta_1 + w_2\beta_2)^2 \Big/ \sum_{(\lambda_1,\lambda_2)\in\{(\alpha_1,\alpha_2),(\beta_1,\beta_2),(\gamma_1,\gamma_2)\}} (w_1\lambda_1 + w_2\lambda_2)^2$$

$$\gamma^2 = (w_1\gamma_1 + w_2\gamma_2)^2 \Bigg/ \sum_{(\lambda_1,\lambda_2)\in\{(\alpha_1,\alpha_2),(\beta_1,\beta_2),(\gamma_1,\gamma_2)\}} (w_1\lambda_1 + w_2\lambda_2)^2$$

Therefore, the belief function $Bel(a)$ associated with $|\varphi\rangle$ is generated by

$$\begin{aligned}Bel(a) &= \min\ \alpha^2 \\ &= \min\ (w_1\alpha_1 + w_2\alpha_2)^2 \Bigg/ \sum\nolimits_{(\lambda_1,\lambda_2)\in\{(\alpha_1,\alpha_2),(\beta_1,\beta_2),(\gamma_1,\gamma_2)\}} (w_1\lambda_1 + w_2\lambda_2)^2 \\ &= \min\ (w_1\alpha_1 + w_2\alpha_2)^2 \big/ \left(w_1{}^2 + w_2{}^2 + 2w_1w_2(\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2)\right)\end{aligned}$$

Since $m(a) = Bel(a)$, we can find that $m(a)$ is related to all parameters α_1, α_2, β_1, β_2, γ_1, γ_2, but not only related to α_1 and α_2, where $m_1(a) = \min\ \alpha_1{}^2$ and $m_2(a) = \min\ \alpha_2{}^2$. □

6. Numerical examples

In this section, a few of numerical examples are given to show the performance of proposed quantum averaging operator, in comparison of other averaging methods.

Here, the referenced methods are two classical averaging operators for mass functions. One is evidential average, which is defined by the following formula:

$$m_{e-avg}(A) = \frac{w_1 m_1(A) + w_2 m_2(A)}{w_1 + w_2} \tag{35}$$

where m_1 and m_2 are mass functions defined on a frame of discernment Φ to combine, and w_1, w_2 are the weights of these two mass functions, respectively.

The other one is probabilistic average which is defined as below.

Let P_1 and P_2 are the probability representations for mass functions m_1 and m_2 defined on Φ, satisfying

$$\begin{cases} P_1(\phi_i) \in [0,1], \forall \phi_i \in \Phi \\ \sum\limits_{\phi_i\in\Phi} P_1(\phi_i) = 1 \\ Bel_{m_1}(A) \le \sum\limits_{\phi_i\in A} P_1(\phi_i) \le Pl_{m_1}(A), \forall A \subseteq \Phi \end{cases} \tag{36}$$

$$\begin{cases} P_2(\phi_i) \in [0,1], \forall \phi_i \in \Phi \\ \sum\limits_{\phi_i\in\Phi} P_2(\phi_i) = 1 \\ Bel_{m_2}(A) \le \sum\limits_{\phi_i\in A} P_2(\phi_i) \le Pl_{m_2}(A), \forall A \subseteq \Phi \end{cases} \tag{37}$$

The probabilistic average of m_1 and m_2, denoted as m_{p-avg}, is generated from $\left\{ P_{p-avg}(\phi_i) = \dfrac{w_1 P_1(\phi_i) + w_2 P_2(\phi_i)}{w_1 + w_2},\ \phi_i \in \Phi \right\}$ which satisfies

$$\begin{cases} P_{p-avg}(\phi_i) \in [0,1], \forall \phi_i \in \Phi \\ \sum\limits_{\phi_i\in\Phi} P_{p-avg}(\phi_i) = 1 \\ Bel_{m_{p-avg}}(A) \le \sum\limits_{\phi_i\in A} P_{p-avg}(\phi_i) \le Pl_{m_{p-avg}}(A), \forall A \subseteq \Phi \end{cases} \tag{38}$$

The result generated by the quantum averaging operator is denoted as m_{q-avg}.

Example 3. Given $\Phi = \{a, b, c\}$, there are two identical mass functions defined on Φ:

$m_1(a) = 0.5,\ m_1(b) = 0.2,\ m_1(c) = 0.1,\ m_1(a, b, c) = 0.2$
$m_2(a) = 0.5,\ m_2(b) = 0.2,\ m_2(c) = 0.1,\ m_2(a, b, c) = 0.2$

Table 1 shows the results obtained by evidential average, probabilistic average, and quantum average. Since $m_1 = m_2$ in this example, all the three kinds of averaging operators are idempotent, which accord with the intuition.

Table 1
Results of Example 3.

	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$
m_{e-avg}	0.5	0.2	0.1	0.0	0.0	0.0	0.2
m_{p-avg}	0.5	0.2	0.1	0.0	0.0	0.0	0.2
m_{q-avg}	0.5	0.2	0.1	0.0	0.0	0.0	0.2

Example 4. Herein, Zadeh's example [30] is examined. Suppose there are two mass functions defined on $\Phi = \{a, b, c\}$:

$m_1(a) = 0.9,\ m_1(b) = 0.1,\ m_1(c) = 0.0$
$m_2(a) = 0.0,\ m_2(b) = 0.1,\ m_2(c) = 0.9$

If using the Dempster's rule to fuse them, the result is $m(b) = 1$, which is counterintuitive.

Previous studies show that the operation of average is an effective way to overcome the problem resulting from the high conflict between mass functions. In this example, in order to compare the performance of different averaging operators, we set the weights w_1 and w_2 change from 1 to 5, respectively. The results are displayed in Fig. 2. From the figure, the results obtained by evidential average and probabilistic average, i.e., m_{e-avg} and m_{p-avg}, are identical: (i) Despite of the values of w_1 and w_2, the mass of b is always 0.1 in m_{e-avg} and m_{p-avg}; (ii) While $w_1 = w_2$, there is $m(a) = m(b) = 0.45$ in the combination result; (iii) If $w_1 > w_2$, the mass of a is bigger than that of c, on the contrary $m(a) < m(c)$ if $w_1 < w_2$.

By using the quantum averaging operator, the masses of a and c in m_{q-avg} display similar changes as m_{e-avg} and m_{p-avg}. However, the result of mass of b in m_{q-avg} shows different features. It is found that $m_{q-avg}(b)$ is not constant, but changed with w_1 and w_2. Moreover, a phenomenon like wave interference appears in $m_{q-avg}(b)$: proposition b gets the largest mass 0.1818 (meanwhile $m_{q-avg}(a) = m_{q-avg}(c) = 0.4091$) when $w_1 = w_2$; while $w_1 \neq w_2$, $m_{q-avg}(b) < 0.1818$. Theoretically, this is caused by the quantum superposition involved in the proposed quantum model of mass function.

From the results, on one hand, the proposed quantum averaging operator has overcome the counterintuitive phenomenon appeared in Dempster's rule. On the other hand, compared to other averaging methods, i.e., evidential average and probabilistic average, the proposed quantum averaging operator has better sensitivity for the change of weights of mass functions, as the mass of b in the combination result is sensitive to the change of weights w_1 and w_2 since the import of quantum interference.

Example 5. Assuming there are two mass functions defined on $\Phi = \{a, b, c\}$:

$m_1(a) = 0.7,\ m_1(b) = 0.1,\ m_1(a, b, c) = 0.2;$
$m_2(a) = 0.5,\ m_2(b, c) = 0.5.$

Tables 2–5 give the results for the cases of $w_1 : w_2 = 0.1$, $w_1 : w_2 = 0.5$, $w_1 : w_2 = 1.0$, $w_1 : w_2 = 1.5$. From these results, m_{e-avg} and m_{p-avg} obtained by evidential average and probabilistic average, are identical. And, the focal elements in m_{e-avg} and m_{p-avg} are entirely from the set of focal elements of m_1 and m_2. In contrast, by using the quantum average, new focal elements $\{c\}$ and $\{a, b\}$ appear in m_{q-avg}, which is caused by the quantum interference effect. In addition, as the rise of the value of $w_1 : w_2$, m_{q-avg} exhibits reasonable change which is gradually close to m_1 and away from m_2.

From the results as shown in Tables 2–5, new propositions with non-zero masses, which are different from focal elements in original mass functions, appear in the combination result obtained by using the quantum averaging operator. This is a new characteristic caused by the quantum model of mass function, which is not possessed by traditional evidential average and probabilistic average.

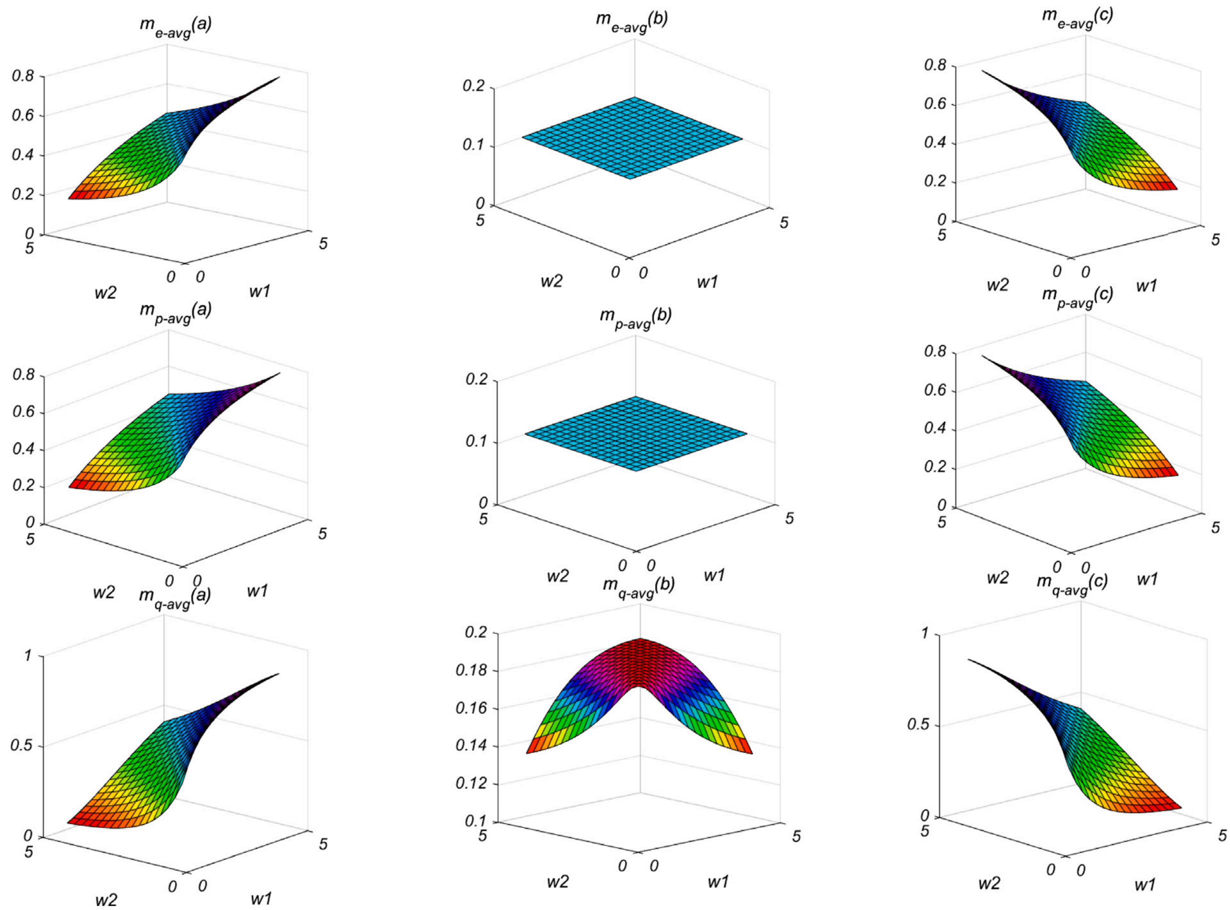

Fig. 2. Results of Example 4.

Table 2
Results of Example 5 for the case $w_1 : w_2 = 0.1$.

	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$
m_{e-avg}	0.5182	0.0091	0.0	0.0	0.0	0.4545	0.0182
m_{p-avg}	0.5182	0.0091	0.0	0.0	0.0	0.4545	0.0182
m_{q-avg}	0.5186	0.0012	0.0176	0.0058	0.0	0.4247	0.0321

Table 3
Results of Example 5 for the case $w_1 : w_2 = 0.5$.

	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$
m_{e-avg}	0.5667	0.0333	0.0	0.0	0.0	0.3333	0.0667
m_{p-avg}	0.5667	0.0333	0.0	0.0	0.0	0.3333	0.0667
m_{q-avg}	0.5682	0.0116	0.0	0.0187	0.0	0.2998	0.1017

Table 4
Results of Example 5 for the case $w_1 : w_2 = 1.0$.

	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$
m_{e-avg}	0.6000	0.0500	0.0	0.0	0.0	0.2500	0.1000
m_{p-avg}	0.6000	0.0500	0.0	0.0	0.0	0.2500	0.1000
m_{q-avg}	0.6020	0.1420	0.0	0.0730	0.0	0.1220	0.0610

Table 5
Results of Example 5 for the case $w_1 : w_2 = 1.5$.

	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$
m_{e-avg}	0.6200	0.0600	0.0	0.0	0.0	0.2000	0.1200
m_{p-avg}	0.6200	0.0600	0.0	0.0	0.0	0.2000	0.1200
m_{q-avg}	0.6223	0.0377	0.0	0.1539	0.0	0.1000	0.0861

7. Algorithm and application

In this section, based on the defined quantum averaging operator, a new evidence fusion rule, called quantum average combination (QAC) rule, is proposed to implement the fusion of mass functions in D-S evidence theory. And two applications are provided to verify the proposed QAC rule.

7.1. Quantum average combination rule

Given a group of evidence $m_1, m_2, \ldots, m_K$ defined over Φ with weights $w_1, w_2, \ldots, w_K$ respectively, by integrating the quantum averaging operator and ideas of Murphy's [32] and Deng's [59] average combination rules, a quantum average combination (QAC) rule is proposed in this paper, as shown in Algorithm 1. Through the QAC rule, it is shown that the proposed quantization scheme of mass function is really usable for uncertain information fusion in the framework of D-S evidence theory.

7.2. Application to tree species classification from multi-source remotely sensed data

In this subsection, the proposed QAC rule is applied to tree species classification from multi-source remotely sensed data. Data of three scenarios from literature [60] are adopted. In these scenarios, the tree species examined include Norway maple (MN), honey locust (LH), Austrian pine (PA), blue spruce (SB), and white spruce (SW), which have constituted a frame of discernment $\Phi = \{\mathrm{MN}, \mathrm{LH}, \mathrm{PA}, \mathrm{SB}, \mathrm{SW}\}$. The classification information comes from three information sources,

Algorithm 1 Quantum average combination (QAC) rule

Input: K mass functions $m_1, m_2, \cdots, m_K$ defined on $\Phi = \{\phi_1, \phi_2, \cdots, \phi_n\}$, with weights $w_1, w_2, \cdots, w_n$ respectively.

Output: Combination result m.

1: For each mass function m_k, generate corresponding quantum model:

$$|\varphi_k\rangle = \sum_{i=1}^{n} \rho_{ki} \, |\phi_i\rangle, \quad k = 1, 2, \cdots, K$$

with constraint $s.t.k \begin{cases} \rho_{ki} \geq 0, \quad \sum_{i=1}^{n} {\rho_{ki}}^2 = 1 \\ Bel_{m_k}(A) \leq \sum_{\phi_i \in A} {\rho_{ki}}^2 \leq Pl_{m_k}(A), \quad \forall A \subseteq \Phi \end{cases}.$

2: Calculate the quantum average of $|\varphi_1\rangle, |\varphi_2\rangle, \cdots, |\varphi_K\rangle$:

$$|\varphi\rangle = \sum_{i=1}^{n} \left(\left(\sum_{k=1}^{K} w_k \rho_{ki} \right) |\phi_i\rangle \right)$$

satisfying a series of constraints $s.t.1, s.t.2, \cdots, s.t.K$.

3: Generate associated belief function according to $|\varphi\rangle$:

$$Bel(A) = \min \sum_{\phi_i \in A} \left(\sum_{k=1}^{K} w_k \rho_{ki} \right)^2 \Bigg/ \sum_{\phi_i \in \Phi} \left(\sum_{k=1}^{K} w_k \rho_{ki} \right)^2, A \subseteq \Phi$$

4: Obtain the quantum average evidence based on fast mobius transformation:

$$m_{q-avg}(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} Bel(B), \quad \forall A \subseteq \Phi$$

5: Derive the final combination result in terms of Dempster's rule:

$$m = \underbrace{m_{q-avg} \oplus \cdots \oplus m_{q-avg}}_{n-1 \text{ times}}$$

including spectral, textural, structural features derived from multi-spectral and panchromatic imagery and Light Detection And Ranging (LiDAR) data.

Scenario A: There are not substantive conflicts among spectral, textural, structural features.

The mass functions given by the spectral, textural, structural features are shown in Table 6. It is found that the three mass functions to be combined all support the tree class MN with the largest beliefs, therefore they are not conflicting. Then, by using different combination rules, including Dempster's, Murphy's [32], Deng's [59], and proposed QAC, we have obtained a same result $m(\text{MN}) = 1$. In this scenario, all the considered combination rules obtained reasonable result. In this paper, Murphy's and Deng's rules are selected for comparison because they are two representative combination rules in which the average of evidence plays a central role in the combination process, which makes the evaluation to proposed QAC rule more comparable and equitable. Specifically, Murphy's rule corresponds to the equally weighted QAC rule, Deng's rule corresponds to the QAC rule with different weights, the only difference between the QAC rule and Murphy's and Deng's rules is that the evidential average is replaced by the quantum average.

Scenario B: Two types of features are highly conflicting.

In this scenario, as shown in Table 7, the mass functions from spectral and structural features give very high beliefs to LH and PA, respectively. These two types of features are highly conflicting. The third feature, i.e., textural, exhibits a high uncertainty between LH and PA, but the support to PA is relatively higher than the support to LH. Therefore, intuitively, the tree class should belong to PA. For the scenario, Dempster's rule produces wrong combination result which supports LH with the highest mass 0.803. In contrast, all average-based rules yield correct results. Specifically, Murphy's rule and QAC rule

Table 6
Tree species classification for Scenario A.

	MN	LH	PA	SB	SW
Spectral feature	0.787	0.116	0.010	0.015	0.072
Structural feature	0.967	0.024	0.009	0.000	0.000
Textural feature	0.946	0.005	0.021	0.011	0.017
Dempster's rule	1.000	0.000	0.000	0.000	0.000
Murphy's rule	1.000	0.000	0.000	0.000	0.000
Deng's rule	1.000	0.000	0.000	0.000	0.000
QAC rule (weighted equally)	1.000	0.000	0.000	0.000	0.000
QAC rule (different weights)	1.000	0.000	0.000	0.000	0.000

Table 7
Tree species classification for Scenario B.

	MN	LH	PA	SB	SW
Spectral feature	0.021	0.964	0.001	0.006	0.008
Structural feature	0.000	0.005	0.946	0.024	0.025
Textural feature	0.004	0.449	0.539	0.001	0.007
Dempster's rule	0.000	0.803	0.196	0.000	0.001
Murphy's rule	0.000	0.465	0.535	0.000	0.000
Deng's rule	0.000	0.377	0.623	0.000	0.000
QAC rule (weighted equally)	0.000	0.486	0.514	0.000	0.000
QAC rule (different weights)	0.000	0.354	0.646	0.000	0.000

Table 8
Tree species classification for Scenario C.

	MN	LH	PA	SB	SW
Spectral feature	0.006	0.234	0.055	0.151	0.554
Structural feature	0.000	0.816	0.002	0.094	0.088
Textural feature	0.000	0.001	0.000	0.609	0.390
Dempster's rule	0.000	0.006	0.000	0.310	0.684
Murphy's rule	0.000	0.403	0.000	0.216	0.381
Deng's rule	0.000	0.300	0.000	0.219	0.481
QAC rule (weighted equally)	0.000	0.205	0.000	0.260	0.535
QAC rule (different weights)	0.000	0.135	0.000	0.238	0.627

(weighted equally) give 0.535 and 0.514 masses, respectively, to the class of PA. Deng's rule and QAC rule (different weights) give 0.623 and 0.646 masses to PA, respectively. These results show that the QAC rule is still effective for the case of highly conflicting scenario.

Scenario C: Three different species classes are supported by the three types of features.

In this scenario, each feature supports different tree class. As shown in Table 8, the spectral, structural, and textural features support SW, LH, and SB, respectively. By using Dempster's and Deng's rules, SW is finally supported with masses 0.684 and 0.481, respectively. While LH is finally supported by Murphy's rule with a mass of 0.403. Here, we belief in the result of Deng's rule, since it has fully considered the difference of weights of evidence, which improved Murphy's rule where the weights of all mass functions are simply assumed to be the same. In contrast, by using the QAC rule, no matter in the case of "weighted equally" or the case of "different weights", both cases get the correct result, where they give 0.535 and 0.627 masses to SW respectively. Therefore, from the distribution of masses in the combination results, in this scenario the QAC rule performs better than Murphy's and Deng's rules.

7.3. Application to Iris dataset classification based on the fusion of multiple features

In this subsection, the proposed QAC rule is used to the classification task of Iris dataset based on the fusion of multiple features. The Iris dataset [61] contains 150 samples from three classes including Setosa, Versicolour and Virginica, which constitutes $\Phi = \{\text{Setosa}, \text{Versicolour}, \text{Virginica}\}$ as the considered frame of discernment. Each sample has four numeric features which are sepal length (SL), sepal width (SW), petal length (PL), and petal width (PW).

Table 9
Classification performance of different combination rules for the Iris dataset.

	NMC	Acc.	CE	d_J	k_r
Dempster's rule	21	86.00%	977.2772	49.1140	21.4627
Murphy's rule	21	86.00%	941.3383	50.0109	21.6260
Deng's rule	21	86.00%	932.3153	50.6087	21.8118
QAC rule (weighted equally)	20	86.67%	937.5986	50.5141	21.6952
QAC rule (different weights)	20	86.67%	928.6057	51.0496	21.8399

In this paper, at first, a clustering-based construction method of mass functions introduced in literature [62] is adopted to generate mass function for every sample on each feature, as a result 150 × 4 mass functions have been generated. Then, for every sample, use an evidence combination rule to fuse its four mass functions on the four features, the combination result is also a mass function, denoted as m_{comb}. Next, according to m_{comb}, the plausibility transformation proposed by Cobb and Shenoy [63] is utilized to derive a distribution of probabilities belonging to different classes for every sample. Based on the distribution, the classification result of each sample can be obtained. For the sake of comparing the performance of different combination rules, three indexes belonging to different information levels are considered, including

- Numbers of mis-classification (NMC) in all classification results.
- Accuracy (Acc.) in classification results.
- Cross entropy (CE) between predicted distribution of classes and real distribution of class labels.
- Disagreement degree between obtained combination result m_{comb} and real distribution of class labels. Here, two kinds of disagreement of evidence, evidence distance d_J [64] and conflict coefficient K_r [65], are selected.

Table 9 shows the classification performance of different combination rules for the Iris dataset. From the table, as for NMC, the proposed QAC rule gets 20 in the case "weighted equally" or the case of "different weights", which is better than the results obtained by Dempster's, Murphy's and Deng's rules. As for CE, the QAC rule (different weights) obtains the minimum loss, next is Deng's rule. And in the same setting of weights, the QAC rule (weighted equally) is also better than Murphy's rule according to CE loss. In addition, in terms of the index of disagreement degree, d_J and K_r, these rules exhibit similar performance. Overall speaking, the proposed QAC rule is better than other average-based combination rules, the effectiveness of proposed rule is verified.

7.4. Discussion

From the above applications, the proposed QAC method is effective in the fusion of evidence with the cases of no substantive conflict, part conflict and complete conflict, and in the Iris data classification task. By summarizing these given numerical examples and applications, compared with other combination rules, the proposed method has some theoretical advantages. At first, benefiting from the import of quantum model, the proposed method provides a nonlinear and global information fusion for uncertain information. Second, it has better sensitivity for the change of weights of mass functions, as shown in the results for Zadeh's example. Third, new focal elements can yield in the combination process of pieces of evidence, as shown in Example 5. However, compared to traditional average combination of evidence, the proposed QAC method has obviously increased the computation cost, which should be paid much attention in practical applications.

Specifically, let us analyze the time complexity of the proposed QAC rule. In terms of Algorithm 1, the computational complexity of each step is analyzed as shown in Table 10. According to the process in Algorithm 1, the first step is to generate the quantum model for each mass function to combine, the time complexity is $\mathcal{O}(K)$. The second step is to calculate the quantum average $|\varphi\rangle$ of all quantum models of mass functions, the complexity is $\mathcal{O}(1)$ because it only needs to calculate one time. In the third step, the belief function Bel associated with $|\varphi\rangle$ is derived by solving the optimization problem in Eq. (25), which has a $\mathcal{O}(n^{3.5}L)$ computational complexity with the use of interior-point method, with 2^n times corresponding to 2^n propositions, so the total complexity is $\mathcal{O}(n^{3.5}L)\times\mathcal{O}(2^n)$. The fourth step is to generate the quantum average evidence by means of fast mobius transformation, the complexity of this step is $\mathcal{O}(2^n)$ since there are 2^n propositions within a frame of discernment. The last step is to combine the average evidence $n-1$ times by using Dempster's rule, which has a $\mathcal{O}(2^N)$ computational complexity, where $N = K \times q$.

Table 10
Time complexity of each step in the QAC rule.

Steps	Time complexity	Description
Step 1	$\mathcal{O}(K)$	Generation of quantum models for K mass functions
Step 2	$\mathcal{O}(1)$	Calculation of the quantum average
Step 3	$\mathcal{O}(n^{3.5}L)\times\mathcal{O}(2^n)$	Generation of associated belief function
Step 4	$\mathcal{O}(2^n)$	Obtain the quantum average evidence
Step 5	$\mathcal{O}(2^N)$ where $N = K\times q$	Combination of quantum average evidence using Dempster's rule

Explanation of notations.
n: the size of frame of discernment Φ;
K: the number of mass functions to combine;
L: the length of inputs for the optimization problem encoded in bits;
q: the number of focal elements in the quantum average evidence.

Since D-S evidence theory natively works on the powerset space of the frame of discernment Φ, the QAC rule has inherited the high computational complexity of the theory, especially in steps 3, 4, 5 the computational complexity is exponentially growing with the size of Φ. In practical applications, we need to cautiously use the QAC rule: (i) If the size of the considered frame of discernment and the number of mass functions to combine are small, i.e., n and K are small, the direct application of QAC rule is acceptable in the computational expense; (ii) For a big data environment, the implementation of QAC rule should be optimized with some speedup strategies. At first, it suggests to use parallel computing in step 3 to accelerate the calculation of the belief of each proposition, i.e., $Bel(A)$ for $A \subseteq \Phi$. Then, the matrix calculus to belief functions, developed in [66], can be used in step 4 to help obtain the average evidence with less time cost. At last, for the computational expense in step 5 caused by the use of Dempster's rule, some parallel computing approaches, for example a method given in [67] based on the concept of conquer and divide algorithms, can be used to enhance the computing performance on CPU or GPU.

8. Conclusion

D-S evidence theory is an effective methodology to handle uncertain and imprecise information, but there are also some limitations in this theory, especially how to explain and deal with the uncertainty in mass functions. Quantum theory provides a new view to help understand the uncertainty involved in D-S evidence theory. In this paper, motivated by studying a usable quantization for D-S evidence theory, a novel quantum model of mass function at first has been proposed, which effectively embodied the principle of quantum superposition. Then, based on the proposed quantum model, a quantum averaging operator has been designed to calculate the quantum average of mass functions, in which many desirable properties, including idempotency, commutativity, quasi-associativity, nonlinearity, and globality, are satisfied. Furthermore, a new combination rule, called QAC rule, has been developed to implement the fusion of multiple pieces of evidence. These proposed model, operator, and rule in this paper have constituted a basic framework for the representation and fusion of evidence in the context of quantum theory. Finally, the effectiveness of these models have been demonstrated through numerical examples and applications. In this paper, the interference term has been understood as the inter-support effect between mass functions, in the future research, the other interpretation, i.e., dependence, will be further studied for the fusion of dependent mass functions. And because of the involvement of quantum computing, the time complexity rises, approaches to reduce the computational expense in big data applications should be studied in the future research.

CRediT authorship contribution statement

Xinyang Deng: Conceptualization, Methodology, Writing – review & editing. **Siyu Xue:** Data curation, Software, Writing – original draft. **Wen Jiang:** Validation, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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