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FOUNDATION, ALGEBRAIC, AND ANALYTICAL METHODS IN SOFT COMPUTING

Maximum entropy of random permutation set

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Abstract
Recently, a new type of set, called random permutation set (RPS), is proposed by considering all the permutations of elements in a certain set. For measuring the uncertainty of RPS, the entropy of RPS is presented. However, the maximum entropy principle of RPS entropy has not been discussed. To address this issue, this paper presents the maximum entropy of RPS. The analytical solution of maximum RPS entropy and its PMF condition are proven and discussed. Besides, numerical examples are used to illustrate the maximum RPS entropy. The results show that the maximum RPS entropy is compatible with the maximum Deng entropy and the maximum Shannon entropy. Moreover, in order to further apply RPS entropy and maximum RPS entropy in practical fields, a comparative analysis of the choice of using Shannon entropy, Deng entropy, and RPS entropy is also carried out.

Keywords Random permutation set · Shannon entropy · Deng entropy · Type-2 Deng entropy · Maximum entropy of random permutation set · Uncertainty

1 Introduction

In order to model and process the information with uncertainty, many theories have been developed, such as probability theory (Lee 1980), fuzzy set theory (Zadeh 1965), Dempster–Shafer evidence theory (evidence theory) (Dempster 1967; Shafer 1976), rough set theory (Pawlak 1982), and Z-numbers (Zadeh 2011), which have been further used in various fields, such as information fusion (Lai et al. 2020; Pan et al. 2020), risk assessment (Wang et al. 2021b, a; Gao et al. 2021; Wang et al. 2022), game theory (Cheong et al. 2019; Babajanyan et al. 2020), fault diagnosis (Chen et al. 2021; Huang et al. 2021; Wang et al. 2020b, a), complex networks (Wen and Cheong 2021), target recognition (Wen et al. 2021; Xiao 2022), and decision making (Xiao 2021; Xie et al. 2021; Cheng and Xiao 2021).

Set theory is a fundamental theory, which provides a basis for most existing theories (Jech 2013). For example, the sample space of probability theory is the set that contains all the possible outcomes of a certain experiment (Lee 1980). In Dempster–Shafer evidence theory, the power set considers all the possible subsets of a frame of discernment (Dempster 1967; Shafer 1976).

Recently, Song and Deng (2021) pointed out that a power set can be viewed as all the possible combination of the elements in a frame of discernment. A straightforward question is: "what a set would be if it not consider all the combinations of elements but all the permutations of elements in the frame of discernment?" To solve this problem, Deng (2022) proposed a new type of set, called random permutation set (RPS), which consists of permutation event space (PES) and permutation mass function (PMF). The PES of a certain set considers all the permutations of that set. PMF describes the chance that a certain element in PES would happen.

As is an efficient tool for deal with uncertainty, lots of information entropy have been proposed, such as Shannon entropy (Shannon 1948), Tsallis entropy (Tsallis 1988), and Deng entropy (Deng 2020). Also, entropy has a wide variety of applications (Balakrishnan et al. 2022), including

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effect algebras (Di Nola et al. 2005), statistics (Balakrishnan et al. 2022a), pattern recognition (Balakrishnan et al. 2022b), data fusion (Xiao and Pedrycz 2022), information dimension (Qiang et al. 2022; Zhou et al. 2022), time-series (Song and Xiao 2022), and uncertainty measurements (Gao et al. 2021; Zhou and Deng 2021). For measuring the uncertainty of RPS, Chen and Deng (2021) presented the entropy of RPS, also known as RPS entropy, which is compatible with Deng entropy and Shannon entropy. RPS entropy can be seen as a type-2 Deng entropy. Compared with the classical Deng entropy, type-2 Deng entropy can measure the uncertainty of ordered information.

The maximum entropy principle states that the distribution with maximal entropy best represents the current state of a system based on given information, which is widely used in many fields, such as negation transformation (Yager 2014; Wu et al. 2022), imprecise side conditions (Buckley 2005), fuzzy cognitive maps (Feng et al. 2019), decision making (Yager 2009), and uncertainty theory (Ma 2021). As for the maximum entropy principle of RPS entropy, two aspects should be handled. The first one would be, "what is the maximum form of RPS entropy?". The second one would be, "what is the PMF condition of the maximum RPS entropy?".

To address these issues, in this paper, the maximum entropy of RPS is presented, also called the maximum RPS entropy. The analytical solution of maximum RPS entropy and its corresponding PMF condition are, respectively, proven and discussed. In addition, numerical examples are used to illustrate the maximum RPS entropy. The results show that the maximum RPS entropy is compatible with the maximum Deng entropy and the maximum Shannon entropy. When the order of the element in permutation event is ignored, the maximum RPS entropy will degenerate into the maximum Deng entropy. When each permutation event is limited to containing just one element, the maximum RPS entropy will degenerate into the maximum Shannon entropy. Furthermore, to illustrate the practical use of RPS entropy and maximum RPS entropy, the selection of using Shannon entropy, Deng entropy, and RPS entropy is compared and analyzed.

The rest of this article is as follows. Section 2 introduces the preliminaries. Section 3 presents the maximum entropy of RPS. Section 4 uses some numerical examples to illustrated the maximum RPS entropy. Section 5 compares and analyzes the choice of using different types of entropy. Section 6 makes a brief conclusion.

2 Preliminaries

This section introduces some preliminaries of this paper, including evidence theory, Deng entropy, random permutation set (RPS), and RPS entropy.

2.1 Dempster–Shafer evidence theory

Dempster–Shafer evidence theory (Dempster 1967; Shafer 1976) is an efficient tool for dealing with uncertainty (Deng 2020), which has a variety of applications, such as classification (Liu et al. 2021a, b), complex networks (Xiong et al. 2021), data fusion (Song et al. 2022), and interference effects prediction(Xiao 2021).

Definition 2.1 *(Frame of discernment)* Frame of discernment (FOD) a set of mutually exclusive and exhaustive elements, denoted by $\Theta = \{\theta_1, \theta_2, \ldots, \theta_N\}$. The power set of Θ contains all possible subsets of Θ, which is indicated by $2^\Theta = \{\varnothing, \{\theta_1\}, \ldots, \{\theta_N\}, \{\theta_1, \theta_2\}, \ldots, \{\theta_1, \theta_N\}, \ldots, \Theta\}$.

Definition 2.2 *(Mass function)* Given a certain FOD Θ, a mass function is a mapping function defined by $m : 2^\Theta \to [0, 1]$, which is constrained by $m(\emptyset) = 0$ and $\sum\limits_{A\in 2^\Theta} m(A) = 1$.

2.2 Deng entropy and maximum Deng entropy

In order to measure the uncertainty of mass function in evidence theory, a novel entropy, called Deng entropy, also named as belief entropy, is proposed (Deng 2020), which is further generalized into information volume (Zhou and Deng 2022; Gao et al. 2021). Deng entropy has been applied in many fields, such as time-series analysis (Cui et al. 2022) and eXtropy (Buono and Longobardi 2020; Kazemi et al. 2021; Zhou and Deng 2021).

Definition 2.3 *(Deng entropy)* Given a certain mass function distribution defined on FOD Θ, Deng entropy is defined as:

$$H_{DE}(m) = -\sum_{A\in 2^\Theta} m(A) \log\left(\frac{m(A)}{2^{|A|}-1}\right) \tag{1}$$

where $|A|$ is the cardinal of a certain focal element A.

The maximum entropy is an important conception in statistic. The analytical solution of the maximum Deng entropy and its corresponding condition are given as follows (Deng 2020).

Theorem 1 (The mass function condition of maximum Deng entropy) *Let the frame of discernment be* Θ. *The maximum Deng entropy appears if and only if the mass function distribution satisfies*

$$m(A) = \frac{(2^{|A|}-1)}{\sum_{A\in 2^\Theta}(2^{|A|}-1)}, A \in 2^\Theta. \tag{2}$$

Theorem 2 (The analytic solution of maximum Deng entropy) *The analytical solution of maximum Deng entropy is that*

$$H_{\max\text{-}DE} = \log \sum_{A \in 2^{\Theta}} (2^{|A|} - 1). \tag{3}$$

2.3 Random permutation set

In 2022, Deng (2022) proposed random permutation set (RPS), which is a novel set consisting of permutation event space (PES) and permutation mass function (PMF). Some basic definitions of RPS theory are given as follows.

Definition 2.4 *(Permutation event space)* Given a fixed set $\Theta = \{\theta_1, \theta_2, \ldots, \theta_N\}$, the corresponding **permutation event space** (PES) of Θ is a set containing all possible permutations of Θ:

$$\begin{aligned} PES(\Theta) = &\{ A_{ij} \mid i = 0, ..., N;\ j = 1, ..., P(N,i) \} \\ = &\{ \emptyset, (\theta_1), (\theta_2), ..., (\theta_N), \\ &(\theta_1, \theta_2), (\theta_2, \theta_1), ..., (\theta_{N-1}, \theta_N), \quad (4) \\ &(\theta_N, \theta_{N-1}), ..., (\theta_1, \theta_2, ..., \theta_N), ..., \\ &(\theta_N, \theta_{N-1}, ..., \theta_1) \} \quad (5) \end{aligned}$$

in which $P(N,i) = \frac{N!}{(N-i)!}$ and A_{ij} is called the **permutation event** of PES.

Definition 2.5 *(Random permutation set)* Given a fixed set $\Theta = \{\theta_1, \theta_2, \ldots, \theta_N\}$, its **random permutation set** (RPS) is a set of pairs defined as follows:

$$RPS(\Theta) = \{\langle A, \mathscr{M}(A)\rangle \mid A \in PES(\Theta)\} \tag{6}$$

where $\mathscr{M} : PES(\Theta) \to [0,1]$ is called the **permutation mass function** (PMF), which is constrained by $\mathscr{M}(\emptyset) = 0$ and $\sum_{A \in PES(\Theta)} \mathscr{M}(A) = 1$.

2.4 Entropy of random permutation set

For modeling the uncertainty of the RPS, Chen and Deng (2021) presented the entropy of random permutation set, also called random permutation set entropy or RPS entropy.

Definition 2.6 *(Entropy of random permutation set)* Let a RPS be denoted as $RPS(\Theta) = \left\{\left\langle A_{ij}, \mathscr{M}\left(A_{ij}\right)\right\rangle | A_{ij} \in PES(\Theta)\right\}$, which is defined on $PES(\Theta) = \left\{A_{ij} \,|i = 0, ..., N;\ j = 1, ..., P(N,i)\right\}$. The entropy of this RPS is defined as:

$$H_{RPS}(\mathscr{M}) = -\sum_{i=1}^{N} \sum_{j=1}^{P(N,i)} \mathscr{M}\left(A_{ij}\right) \log \left(\frac{\mathscr{M}\left(A_{ij}\right)}{F(i)-1}\right) \tag{7}$$

where $P(N,i) = \frac{N!}{(N-i)!}$ is the i-permutation of N and $F(i) = \sum_{k=0}^{i} P(i,k) = \sum_{k=0}^{i} \frac{i!}{(i-k)!}$ is the sum from 0-permutation of i to i-permutation of i.

3 Maximum entropy of random permutation set

In this section, the maximum entropy of RPS and its PMF condition are, respectively, presented and proven.

3.1 The PMF condition for maximum entropy of RPS

Let the PES be $PES(\Theta) = \{A_{ij} \,|i = 0, ..., N;\ j = 1, ..., P(N,i)\}$, $P(N,i)$ be as follows $P(N,i) = \frac{N!}{(N-i)!}$, and $F(i)$ be defined as $F(i) = \sum_{k=0}^{i} P(i,k) = \sum_{k=0}^{i} \frac{i!}{(i-k)!}$. The PMF condition for maximum entropy of RPS is presented as follows.

Theorem 3 (The PMF condition for maximum entropy of RPS) *The maximum entropy of RPS happens if and only if the PMF satisfies the following condition*

$$\mathscr{M}\left(A_{ij}\right) = \frac{F(i)-1}{\sum_{i=1}^{N}\left[P(N,i)(F(i)-1)\right]}. \tag{8}$$

Proof 3.1 (Proof for Theorem 3) Let the entropy of RPS be denoted as

$$H(\mathscr{M}) = -\sum_{i=1}^{N} \sum_{j=1}^{P(N,i)} \mathscr{M}\left(A_{ij}\right) \log_b \left(\frac{\mathscr{M}\left(A_{ij}\right)}{F(i)-1}\right) \tag{9}$$

which is constrained by

$$\sum_{A_{ij} \in PES(\Theta)} \mathscr{M}(A_{ij}) = 1 \tag{10}$$

where b is the base of logarithmic function. Then, the Lagrange function with Lagrangian multiplier λ can be defined as follows:

$$\begin{aligned} H_0(\mathscr{M}) = &-\sum_{i=1}^{N} \sum_{j=1}^{P(N,i)} \mathscr{M}\left(A_{ij}\right) \log_b \left(\frac{\mathscr{M}\left(A_{ij}\right)}{F(i)-1}\right) \\ &+\lambda \left(\sum_{A_{ij} \in PES(\Theta)} \mathscr{M}(A_{ij}) - 1\right). \end{aligned} \tag{11}$$

In order to get the maximum of $H(\mathscr{M})$, the gradient of $H_0(\mathscr{M})$ should be equal to 0. Hence, the gradient can be deduced as follows:

$$\frac{\partial H_0(\mathscr{M})}{\partial \mathscr{M}(A_{ij})} = -\log_b \left(\frac{\mathscr{M}(A_{ij})}{F(i)-1}\right) - \frac{1}{\ln b} + \lambda = 0. \tag{12}$$

Based on Eq. 12), it can be concluded that, with respect to different $i \in \{1, 2, ..., N\}$ and $j \in \{1, 2, ..., P(N, i)\}$, $\frac{\mathscr{M}(A_{ij})}{F(i)-1}$ is a constant denoted as C:

$$C \triangleq \frac{\mathscr{M}(A_{ij})}{F(i) - 1}. \tag{13}$$

According to Eq. (13) and Eq.(10), we can get:

$$\begin{aligned}\sum_{A_{ij}\in PES(\Theta)} \mathscr{M}(A_{ij}) &= \sum_{i=1}^{N}\sum_{j=1}^{P(N,i)} \mathscr{M}\left(A_{ij}\right) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{P(N,i)} C * [F(i) - 1] \\ &= \sum_{i=1}^{N} [P(N, i) * C * (F(i) - 1)] = 1\end{aligned} \tag{14}$$

so that C can be calculated as

$$C = \frac{1}{\sum_{i=1}^{N} [P(N, i)(F(i) - 1)]}. \tag{15}$$

Based on Eq. (13) and Eq. (15), we can get this equation:

$$\frac{\mathscr{M}(A_{ij})}{F(i) - 1} = \frac{1}{\sum_{i=1}^{N} [P(N, i)(F(i) - 1)]}. \tag{16}$$

Therefore, the condition for maximum entropy of RPS can be obtained:

$$\mathscr{M}\left(A_{ij}\right) = \frac{F(i) - 1}{\sum_{i=1}^{N} [P(N, i)(F(i) - 1)]}. \tag{17}$$

□

3.2 The analytic solution for maximum entropy of RPS

Let $P(N, i)$ be as follows $P(N, i) = \frac{N!}{(N-i)!}$, and $F(i)$ be defined as $F(i) = \sum_{k=0}^{i} P(i, k) = \sum_{k=0}^{i} \frac{i!}{(i-k)!}$. The analytical solution for the maximum entropy of RPS is detailed as follows.

Theorem 4 (The analytic solution for maximum entropy of RPS) *The analytical solution for maximum entropy of RPS is that*

$$H_{\max\text{-}RPS} = \log\left(\sum_{i=1}^{N} [P(N, i)(F(i) - 1)]\right). \tag{18}$$

Proof 3.2 (Proof for Theorem 4) According to **Theorem 3**, the maximum entropy of RPS happens when PMF satisfies Eq. (17). Hence, the analytic solution for maximum entropy of RPS can be calculated by substituting Eq. (17) into Eq. (7):

$$H_{\max\text{-}RPS} = -\sum_{i=1}^{N}\sum_{j=1}^{P(N,i)} \left\{ \frac{F(i) - 1}{\sum_{i=1}^{N} [P(N, i)(F(i) - 1)]} \log\left(\frac{1}{\sum_{i=1}^{N} [P(N, i)(F(i) - 1)]}\right)\right\}. \tag{19}$$

For a certain PES, N is a constant, so that $\sum_{i=1}^{N} [P(N, i)(F(i) - 1)]$ is also a constant. Therefore, Eq. (19) can be calculated as:

$$H_{\max\text{-}RPS} = \frac{-\log\left(\frac{1}{\sum_{i=1}^{N}[P(N,i)(F(i)-1)]}\right)}{\sum_{i=1}^{N} [P(N, i)(F(i) - 1)]} \sum_{i=1}^{N}\sum_{j=1}^{P(N,i)} (F(i) - 1) \tag{20}$$

$$= \frac{\log\left(\sum_{i=1}^{N} [P(N, i)(F(i) - 1)]\right)}{\sum_{i=1}^{N} [P(N, i)(F(i) - 1)]} \sum_{i=1}^{N} [P(N, i)(F(i) - 1)] \tag{21}$$

$$= \log\left(\sum_{i=1}^{N} [P(N, i)(F(i) - 1)]\right). \tag{22}$$

As a result, the analytic solution for maximum entropy of RPS is obtained. □

4 Numerical examples and discussion

This section shows some numerical examples to illustrate the presented maximum RPS entropy and its PMF condition.

Example 4.1 Suppose a box contains three balls, namely the red, blue, and yellow ball, which can be denoted by $\{R, B, G\}$. Then, we take the following actions:

Action 1: Randomly take out **one ball** from the box.
Action 2: Randomly take out **a number of balls** from the box **without replacement**.
Action 3: Randomly take out **a number of balls** from the box **in sequence without replacement**.

All the possible results of the three actions can be, respectively, represented by sample space Θ, power set $2^{\{R,B,G\}}$,

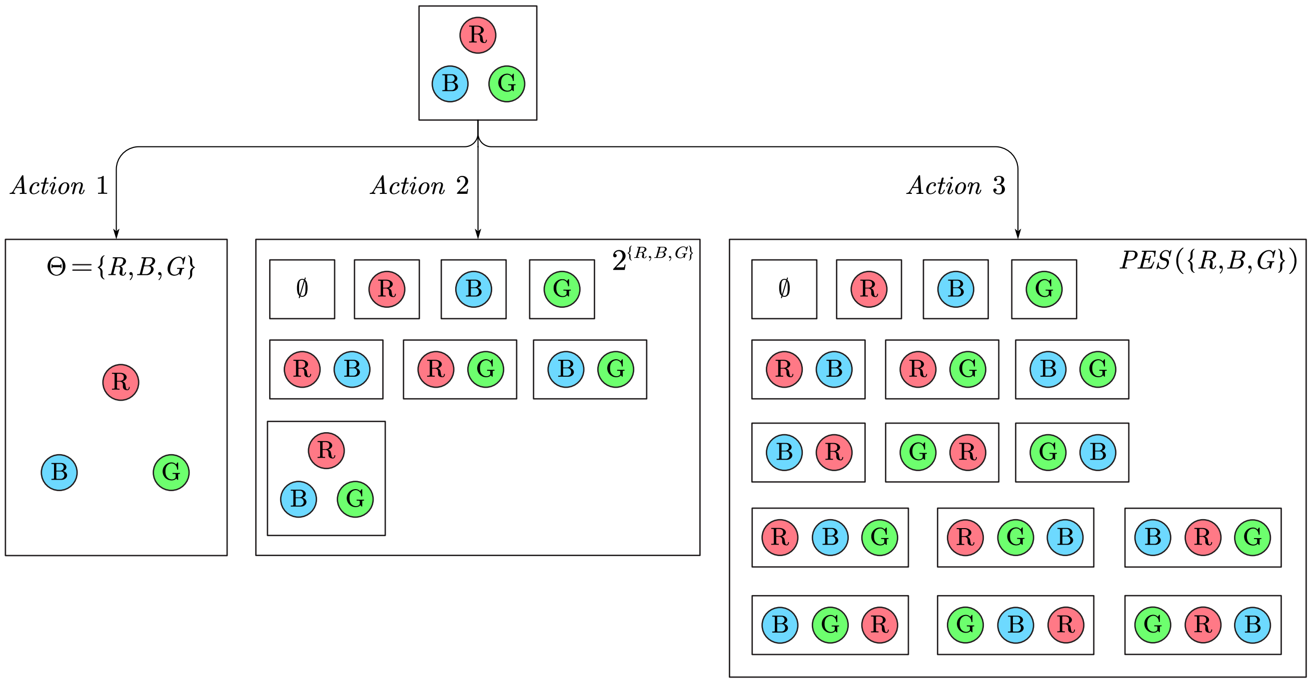

Fig. 1 The illustration for all the possible results based on Actions 1 to 3

and permutation event space $PES(\{R, B, G\})$, which are illustrated in Figure 1 and detailed as follows:

$$\Theta = \{R, B, G\} \tag{23}$$

$$2^{\{R,B,G\}} = \{\emptyset, \{R\}, \{B\}, \{G\}, \{R, B\}, \{R, G\}, \{B, G\}, \{R, B, G\}\} \tag{24}$$

$$\begin{aligned} PES(\{R, B, G\}) = \{&\emptyset, (R), (B), (G), \\ &(R, B), (R, G), (B, G), \\ &(B, R), (G, R), (G, B), \\ &(R, B, G), (R, G, B), (B, R, G), \\ &(B, G, R), (G, B, R), (G, R, B)\}. \end{aligned} \tag{25}$$

Based on Θ, $2^{\{R,B,G\}}$, and $PES(\{R, B, G\})$, the probability condition for maximum Shannon entropy (Shannon 1948), the mass function condition for maximum Deng entropy (Deng 2020), and the PMF condition for maximum entropy of RPS can be, respectively, obtained:

$$P : P(\{R\}) = P(\{B\}) = P(\{G\}) = 0.3333. \tag{26}$$

$$\begin{aligned} m : \; & m(\{R\}) = m(\{B\}) = m(\{G\}) = 0.0526; \\ & m(\{R, B\}) = m(\{R, G\}) = m(\{B, G\}) = 0.1579; \\ & m(\{R, B, G\}) = 0.3684. \end{aligned} \tag{27}$$

$$\begin{aligned} \mathscr{M} : \; & \mathscr{M}(R) = \mathscr{M}(B) = \mathscr{M}(G) = 0.0085; \\ & \mathscr{M}(R, B) = \mathscr{M}(R, G) = \mathscr{M}(B, G) \\ & = \mathscr{M}(B, R) = \mathscr{M}(G, R) = \mathscr{M}(G, B) = 0.0342; \\ & \mathscr{M}(R, B, G) = \mathscr{M}(R, G, B) = \mathscr{M}(B, R, G) \\ & = \mathscr{M}(B, G, R) = \mathscr{M}(G, B, R) \\ & = \mathscr{M}(G, R, B) = 0.1282. \end{aligned} \tag{28}$$

This example illustrates the PMF condition for maximum entropy of RPS. It can be seen in Eq. (28) that, with respect to the same cardinality of permutation event, the PMFs for that permutation events have the same value. For example, the permutation events (R), (B), (G) are of the same value of cardinality. Their corresponding PMFs are the same, which is 0.0085. In addition, according to Eq. (28), the larger the cardinality of a certain permutation event, the larger the corresponding PMF is.

Example 4.2 Let a fixed set of N elements be denoted by $\Theta = \{\theta_1, \theta_2, \ldots, \theta_N\}$. Based on Θ, the sample space of maximum Shannon entropy, the power set of maximum Deng entropy, and the PES of maximum entropy of RPS can be, respectively, indicated by Θ, 2^Θ, and $PES(\Theta)$. Hence, their corresponding maximum Shannon entropy $H_{\max\text{-}SE}$, maximum Deng entropy $H_{\max\text{-}DE}$, and maximum entropy of RPS $H_{\max\text{-}RPS}$ can be obtained:

$$H_{\max\text{-}SE} = \log_2(N), \tag{29}$$

$$H_{\max\text{-}DE} = \log_2\left(\sum_{A \in 2^\Theta} (2^{|A|} - 1)\right), \tag{30}$$

$$H_{\max\text{-}RPS} = \log_2\left(\sum_{i=1}^{N} [P(N, i)(F(i) - 1)]\right). \tag{31}$$

Table 1 Maximum Shannon entropy, maximum Deng entropy, and maximum entropy of RPS with different values of N

N	$H_{\max\text{-}SE}$	$H_{\max\text{-}DE}$	$H_{\max\text{-}RPS}$
1	0.0000	0.0000	0.0000
2	1.0000	2.3219	3.3219
3	1.5850	4.2479	6.8704
4	2.0000	6.0224	10.9278
5	2.3219	7.7211	15.5406
6	2.5850	9.3772	20.6691
7	2.8074	11.0077	26.2495
8	3.0000	12.6223	32.2231
9	3.1699	14.2266	38.5424
10	3.3219	15.8244	45.1699

The logarithm is based on 2

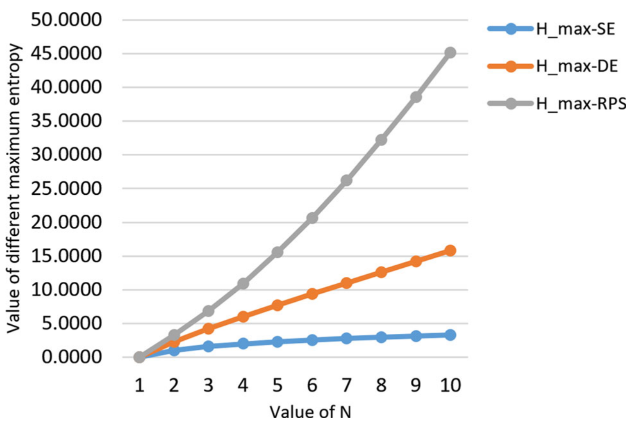

Fig. 2 The trend of maximum Shannon entropy, maximum Deng entropy, and maximum entropy of RPS with different values of N

With different values of N, the values of $H_{\max\text{-}SE}$, $H_{\max\text{-}DE}$, and $H_{\max\text{-}RPS}$ can be calculated, which are shown in Table 1 and Figure 2.

This example shows that the trend of the maximum entropy of RPS changing with different values of N. In addition, according to Figure 2, with respect to the same value of N, the value of maximum entropy of RPS is larger than that of maximum Shannon entropy or maximum Deng entropy. This means that, with respect to the same number of samples of a certain set, the uncertainty of a RPS is larger than that of a power set or a sample space, since RPS takes permutation of a certain set into account while sample space and power set do not consider that.

Example 4.3 Given a fixed set $\Theta = \{X, Y\}$, its corresponding PES is that

$$PES(\Theta) = \{\emptyset, (X), (Y), (X, Y), (Y, X)\}. \tag{32}$$

Based on this PES, the maximum entropy of RPS is calculated by:

$$\begin{aligned} H_{\max\text{-}RPS} &= \log_2 \left(\sum_{i=1}^{2} [P(2, i)(F(i) - 1)] \right) \\ &= \log_2 [2 * (1 + 1 - 1) \\ &\quad +2 * (1 + 2 + 2 - 1)] = 3.3219 \end{aligned} \tag{33}$$

in which $F(i)$ is defined as $F(i) = \sum_{k=0}^{i} P(i, k)$.

Consider the following two scenarios:

- According to Deng (2022), if the order of the element in $PES(\Theta)$ is ignored, $PES(\Theta)$ will degenerate into the power set: $2^{\Theta} = \{\emptyset, \{X\}, \{Y\}, \{X, Y\}\}$. Based on this power set, calculate the maximum Deng entropy

$$H_{\max\text{-}DE} = \log_2 \left(\sum_{A \in 2^{\Theta}} (2^{|A|} - 1) \right) \tag{34}$$

$$= \log_2 \left[2 * (2^1 - 1) + 1 * (2^2 - 1)\right] = 2.3219. \tag{35}$$

Besides, since the order of the element is ignored, the permutation number $P(N, i)$ degenerates into combinatorial number $C(N, i)$, and $F(i)$ should be calculated as $F(i) = \sum_{k=0}^{i} C(i, k)$. Hence, the corresponding maximum entropy of RPS in this scenario can be detailed by:

$$\begin{aligned} H_{\max\text{-}RPS} &= \log_2 [C(2, 1)(F(1) - 1) \\ &\quad +C(2, 2)(F(2) - 1)] \\ &= \log_2 [2 * (1 + 1 - 1) \\ &\quad +1 * (1 + 2 + 1 - 1)] = 2.3219 \end{aligned} \tag{36}$$

which is the same as $H_{\max\text{-}DE}$. As a result, if the order of the element in PES is ignored, the maximum entropy of RPS will degenerate into the maximum Deng entropy.

- According to Deng (2022), if each permutation event is limited to containing just one element, $PES(\Theta)$ will degenerate into the sample space in probability theory: $\Omega = \{\{X\}, \{Y\}\}$. Based on this sample space, calculate the maximum Shannon entropy:

$$H_{\max\text{-}SE} = \log_2 (2) = 1. \tag{37}$$

Additionally, since permutation event is limited to containing just one element, the permutation number $P(N, i)$

degenerates into

$$\widetilde{P}(N,i) \triangleq \begin{cases} 1 & (i=0, N \geq 1) \\ N & (i=1, N \geq 1) \\ 0 & (otherwise) \end{cases} \tag{38}$$

so that $F(i)$ should be calculated as $F(i) = \sum_{k=0}^{i} \widetilde{P}(i,k) = i+1$. Hence, the corresponding maximum entropy of RPS in this scenario is that:

$$\begin{aligned} H_{\max\text{-}RPS} &= \log_2 \left[\widetilde{P}(2,1) * (F(1)-1) + \widetilde{P}(2,2) * \right. \\ &\quad \times (F(2)-1)] = \log_2 [2 * (1+1-1) \\ &\quad +0 * (2+1-1)] = 1 \end{aligned} \tag{39}$$

which is the same as $H_{\max\text{-}SE}$. As a result, if each permutation event is limited to containing just one element, the maximum entropy of RPS degenerates into the maximum Shannon entropy.

This example shows that the maximum entropy RPS is compatible with the maximum Deng entropy and the maximum Shannon entropy.

5 Comparison and analysis

For the practical applications of RPS entropy and maximum RPS entropy, this section compares and analyzes the choice of using Shannon entropy, Deng entropy, and RPS entropy. It should be noted that in the following calculations, the logarithm is based on 2.

Let us consider the scenario of product sales. Recently, a company developed four products denoted by θ_1, θ_2, θ_3, and θ_4, respectively. The profit and importance of each product is different. The product manager needs to formulate the most profitable sales plan for these four products according to the company's strategy and the profit of the product. Consider the following three strategies for the four products.

Strategy 1: Select only one of the four products as the final product for sale. For example, the company has developed four prototypes of a mobile phone, and only one prototype can be used as the final product on the market.
Strategy 2: Select several of these products and then combine them into product suites for sale. For example, the company has developed four different products, such as mobile phones, computers, watches, and wireless headphones. Mobile phone can be paired with wireless headphone to form a product suite for a higher profit.
Strategy 3: Firstly, select several of these products. Then, rank the selected products by their importance. Finally, sell the selected products in order of importance, so as to avoid the impact of conflicts between these products on overall profits. For example, the company has developed four mobile phones (denoted by θ_1, θ_2, θ_3, and θ_4). In order to avoid the mutual influence between these mobile phones, the manager can choose two of them (such as θ_2 and θ_3) and sell them in a certain order, such as selling mobile phone θ_3 in the first half of the year and selling mobile phone θ_2 in the second half of the year.

For modeling the uncertainty of these three strategies, different theories and different types of entropy should be used.

- Since Strategy 1 takes only one prototype of the four mobile phones in to consideration, the decision space of Strategy 1 should be modeled as $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$, where the events are **mutually exclusive**. In this case, it is better to use **probability theory** to deal with the uncertain information of Strategy 1 and make further decisions based on this information. For example, based on Strategy 1, two product managers provide their sales plans for the products in the form of **probability distributions**:

$$\begin{aligned} P_1 : & \; p_1(\theta_1) = 0.20,\; p_1(\theta_2) = 0.30, \\ & \; p_1(\theta_3) = 0.30,\; p_1(\theta_4) = 0.20, \end{aligned} \tag{40}$$

$$\begin{aligned} P_2 : & \; p_2(\theta_1) = 0.20,\; p_2(\theta_2) = 0.30, \\ & \; p_2(\theta_3) = 0.45,\; p_2(\theta_4) = 0.05. \end{aligned} \tag{41}$$

To measure the uncertainty of the given information for further decision making, information entropy should be utilized. **Shannon entropy** is a commonly used entropy for probability, which measures the average degree of information inherent in a probability. Shannon entropy of these two probabilities can be calculated as:

$$H_{SE}(P_1) = 1.9710,\; H_{SE}(P_2) = 1.7200. \tag{42}$$

Based on these results, we can conclude that the decision of the second product manager is clearer and the tendency is more obvious than that of the first manager, which means that Shannon entropy can distinguish the difference between P_1 and P_2. Hence, in the case of mutually exclusive events, Shannon entropy is sufficient to measure the corresponding uncertainty.
- In Strategy 2, a product suite can be viewed as a **combination** of several products. Different product suites may affect each other. For instance, product suite $\{\theta_1, \theta_4\}$ may affect the sales of product suite $\{\theta_3, \theta_4\}$, because they all contain product θ_4. It is true that the event can be handled based on joint probability distribution, but the prior probabilities are often difficult to obtain. In addition, conditional probability requires integration of the joint probability, which sometimes is intractable. Fortu-

nately, **evidence theory** does not necessarily require prior information, and its assumptions are weaker than those of probability theory (Dempster 1967; Shafer 1976). Therefore, it is recommended to choose evidence theory to deal with the situation of Strategy 2, especially when facing the problems of subjective reasoning or highly uncertain information processing. The decision space of Strategy 2 can be represented by the **power set** of evidence theory:

$$2^{\Theta} = \{\emptyset, \{\theta_1\}, ..., \{\theta_4\}, \{\theta_1, \theta_2\}, ..., \{\theta_3, \theta_4\}, ..., \Theta\} \tag{43}$$

in which the events are mutually **non-exclusive**. Now, based on Strategy 2, the sales plans are presented in the form of **mass function**:

$$\begin{aligned} m_1 &: m_1(\{\theta_1, \theta_3\}) = 0.2, m_1(\{\theta_3, \theta_4\}) = 0.7, \\ &\quad m_1(\{\theta_1, \theta_2, \theta_4\}) = 0.1, \end{aligned} \tag{44}$$

$$\begin{aligned} m_2 &: m_2(\{\theta_1, \theta_3\}) = 0.2, \\ &\quad m_2(\{\theta_3, \theta_4\}) = 0.7, m_2(\{\theta_1, \theta_2\}) = 0.1. \end{aligned} \tag{45}$$

Different from probability, the uncertainty of a mass function comes is from two parts, namely discord and non-specificity. Discord describes the uncertainty caused by the value of mass function. Non-specificity models the uncertainty caused by the sets with multiple elements (e.g., $\{\theta_2, \theta_3\}$, also called multi-set), which Shannon entropy cannot handle. The reason is that Shannon can only deal with the uncertainty derived from singletons (e.g., $\{\theta_4\}$). To measure discord and non-specificity of mass function, Deng (2020) proposed **Deng entropy**, which provides a useful tool for systematically measuring both types of uncertainty for a given mass function. For comparison, Deng entropy and Shannon entropy are applied to measure uncertainty of the two mass functions:

$$H_{DE}(m_1) = 2.8640, \ H_{DE}(m_2) = 2.7417, \tag{46}$$

$$H_{SE}(\widetilde{m}_1^P) = 1.1568, \ H_{SE}(\widetilde{m}_2^P) = 1.1568. \tag{47}$$

where $\widetilde{m}_1^P$ and $\widetilde{m}_2^P$ are the degenerated mass functions (without the information about multi-sets), which are actually probabilities:

$$\widetilde{m}_1^P : \widetilde{m}_1^P(x_1) = 0.2, \widetilde{m}_1^P(x_2) = 0.7, \widetilde{m}_1^P(x_3) = 0.1 \tag{48}$$

$$\widetilde{m}_2^P : \widetilde{m}_2^P(y_1) = 0.2, \widetilde{m}_2^P(y_2) = 0.7, \widetilde{m}_2^P(y_3) = 0.1. \tag{49}$$

It can be seen from the results that Shannon entropy failed to distinguish m_1 from m_2 because it does not takes into account the uncertainty of multi-set (non-specificity). In contrast, Deng entropy successfully differentiated between m_1 and m_2 by measuring both discord and non-specificity. As a result, when it comes to the non-exclusive case, Deng entropy is more recommended than Shannon entropy.

- Strategy 3 considers the sales plans in order of the product importance, where the relative order of products can be viewed as a **permutation** of several products. Different relative orders will cause different interactions of the products. For example, it is more profitable to sell mobile phones first and then earphones, but selling a phone first and then selling the same type of phone again can lose profits. Such sales plans can be modeled based on Markov chains and state-transition matrices. However, state-transition probabilities may be hard to construct, and prior information may be even more difficult to obtain. These issues can be partially solved by **random permutation set (RPS) theory** (Deng 2022). The reasons are as follows. On the one hand, RPS theory does not necessarily require prior information, which is just like evidence theory. On the other hand, RPS theory considers not only the chance of events happening, but also the inherent order of events. Hence, under the highly uncertain circumstances that Markov chains cannot be easily implemented, we recommend to use RPS theory. The sense of order within Strategy 3 can be indicated by the permutation event of RPS such as "(θ_3, θ_1)." All possible permutation events form the decision space for Strategy 3, namely the **permutation event space** (PES):

$$\begin{aligned} PES(\Theta) = \{ & \emptyset, (\theta_1), (\theta_2), ..., (\theta_4), (\theta_1, \theta_2), \\ & (\theta_2, \theta_1), ..., (\theta_3, \theta_4), \\ & (\theta_4, \theta_3), ..., (\theta_1, \theta_2, ..., \theta_4), ..., \\ & (\theta_4, \theta_3, ..., \theta_1) \} \end{aligned} \tag{50}$$

where the events are **non-exclusive**, and each has an internally **ordered structure**. According to Strategy 3, two sales plans for the product suites are offered by the managers based on **permutation mass function** (PMF):

$$\begin{aligned} \mathscr{M}_1 &: \mathscr{M}_1(\theta_2, \theta_3) = 0.4, \mathscr{M}_1(\theta_3, \theta_2) = 0.3, \\ &\quad \mathscr{M}_1(\theta_2, \theta_4, \theta_3) = 0.3, \end{aligned} \tag{51}$$

$$\begin{aligned} \mathscr{M}_2 &: \mathscr{M}_2(\theta_2, \theta_3) = 0.6, \mathscr{M}_2(\theta_3, \theta_2) = 0.1, \\ &\quad \mathscr{M}_2(\theta_2, \theta_4, \theta_3) = 0.3. \end{aligned} \tag{52}$$

Unlike probability and mass function, a PMF consists of three aspects of information: **(a)** the chance of the event happening, **(b)** the cardinality of the permutation event, and **(c)** the intrinsic order of the event. In fact, Shannon entropy can only measure the uncertainty of aspect **(a)**, and Deng entropy is able to measure the uncertainty of aspects **(a)** and **(b)**. In order to deal with uncertainty derived from the three aspects of information, Chen and Deng (2021) proposed a novel entropy for RPS, called **RPS entropy**, which can comprehensively handle the uncertainty of PMF in RPS. To illustrate, we calculate the uncertainty of the two PMFs based on RPS entropy, Deng entropy, and Shannon entropy:

$$H_{RPS}(\mathscr{M}_1) = 4.1430,\ H_{RPS}(\mathscr{M}_2) = 3.8675, \tag{53}$$

$$H_{DE}(\widetilde{\mathscr{M}}_1^m) = 2.8330,\ H_{DE}(\widetilde{\mathscr{M}}_2^m) = 2.8330, \tag{54}$$

$$H_{SE}(\widetilde{\mathscr{M}}_1^P) = 0.8813,\ H_{SE}(\widetilde{\mathscr{M}}_2^P) = 0.8813. \tag{55}$$

where $\widetilde{\mathscr{M}}_1^m$ and $\widetilde{\mathscr{M}}_2^m$ are the first-order degenerated PMFs (without aspect **(c)**), which are actually mass functions:

$$\widetilde{\mathscr{M}}_1^m : \widetilde{\mathscr{M}}_1^m(\{\theta_2, \theta_3\}) = 0.7, \widetilde{\mathscr{M}}_1^m(\{\theta_2, \theta_3, \theta_4\}) = 0.3, \tag{56}$$

$$\widetilde{\mathscr{M}}_2^m : \widetilde{\mathscr{M}}_2^m(\{\theta_2, \theta_3\}) = 0.7, \widetilde{\mathscr{M}}_2^m(\{\theta_2, \theta_3, \theta_4\}) = 0.3; \tag{57}$$

$\widetilde{\mathscr{M}}_1^P$ and $\widetilde{\mathscr{M}}_2^P$ are the second-order degenerated PMFs (without aspects **(b)** and **(c)**), which are actually probabilities:

$$\widetilde{\mathscr{M}}_1^P : \widetilde{\mathscr{M}}_1^P(x_1) = 0.7, \widetilde{\mathscr{M}}_1^P(x_2) = 0.3, \tag{58}$$

$$\widetilde{\mathscr{M}}_2^P : \widetilde{\mathscr{M}}_1^P(y_1) = 0.7, \widetilde{\mathscr{M}}_2^P(y_2) = 0.3. \tag{59}$$

According to the results, it can be seen that neither Shannon entropy nor Deng entropy can discriminate $\mathscr{M}_1$ from $\mathscr{M}_2$. The reason is that both of these two entropies more or less ignore a part of the information in PMF, especially aspect **(c)**. Conversely, RPS entropy is able to distinguish $\mathscr{M}_1$ and $\mathscr{M}_2$ by systemically modeling the uncertainty in the three aspects of PMF. Therefore, in the non-exclusive case with a sense of order, we recommend using RPS entropy instead of Deng entropy or Shannon entropy.

In summary, different types of entropy should be considered when faced with different circumstances. Table 2 shows the choice of using Shannon entropy, Deng entropy, and RPS entropy with respect to different situations.

Table 2 Choice of using Shannon entropy, Deng entropy, and RPS entropy

	Recommended theory	Information representation	Recommended entropy
Exclusive	Probability theory	Probability	Shannon entropy
Non-exclusive	Evidence theory	Mass function	Deng entropy
Non-exclusive + order	RPS theory	PMF	RPS entropy

6 Conclusion

Recently, random permutation set (RPS) is proposed, which is a new kind of set considering all the permutation of elements in a certain set. The entropy of RPS is further presented for modeling the uncertainty of RPS. However, the maximum entropy principle of RPS entropy has not been discussed. To address this issue, in this paper, the maximum entropy of RPS is presented, also known as the maximum RPS entropy. The main contributions of this paper and some remarks on the maximum RPS entropy are summarized as follows:

- The analytical solution of maximum RPS entropy and its corresponding PMF condition are presented and proven.
- Numerical examples are used to illustrate the maximum RPS entropy. The results show that the maximum RPS entropy is compatible with the maximum Deng entropy and the maximum Shannon entropy.
- When the order of the element in permutation event is ignored, the maximum RPS entropy will degenerate into the maximum Deng entropy. When each permutation event is limited to containing just one element, the maximum RPS entropy will degenerate into the maximum Shannon entropy.
- Comparison and analysis are carried out on the selection of using Shannon entropy, Deng entropy, and RPS entropy, which is beneficial to the further application of RPS entropy and maximum RPS entropy in practical fields.

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Data availability All data and materials generated or analyzed during this study are included in this article.

Code availability The code of the current study is available from the corresponding author on reasonable request.

Declarations

Conflict of interest All the authors certify that there is no conflict of interest with any individual or organization for this work.

Ethics approval This article does not contain any studies with human participants or animals performed by any of the authors.

Consent to participate Informed consent was obtained from all individual participants included in the study.

Consent for publication The participant has consented to the submission of the case report to the journal.

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