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The Distance of Random Permutation Set

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ABSTRACT

The Random Permutation Set (RPS) is a newly proposed type of set. An RPS takes the permutation of a certain set into consideration and can be regarded as a generalization of evidence theory. Since distance is an important metric, an immediate question which arises is how to calculate the distance between RPSs. To address this problem, the distance between RPSs is first presented in this paper. Based on a geometrical interpretation of RPS, the distance between RPSs is measured by considering the similarity between their permutation sets, both in the number and order of elements. Numerical examples demonstrate the behavior of the proposed RPS distance. Thereafter, based on the proposed RPS distance, a Data-Driven Reliability Determination (DDRD) algorithm is put forth and applied in threat assessment. The experimental results show that the proposed DDRD algorithm can effectively identify the fusion order of RPSs. The corresponding fusion result is in agreement with theoretical predictions and has the highest accuracy of identification.

1. Introduction

Uncertainty is universal in our daily lives [1,2]. Many theories have been proposed to deal with uncertainty, such as probability theory [3], fuzzy sets [4], intuitionistic fuzzy sets [5] and Dempster-Shafer evidence theory (evidence theory) [6,7]. Amongst them, evidence theory is distinguished from probability theory as providing a weaker condition. Evidence theory distributes belief in a power set space while probability theory does so in a sample space, which is the basis for Basic Probability Assignment (BPA).

An explanation of power set has recently been proposed from the view of Pascal's triangle and combinatorial numbers, in a bid to explore the meaning of power set in evidence theory [8]. Inspired by the idea of replacing combinatorial numbers with permutation numbers, Deng proposed a new type of set called Random Permutation Set (RPS) [9]. An RPS consists of a Permutation Event Space (PES) and a Permutation Mass Function (PMF). The PES of a set considers all permutations of elements in the set. Each object in a PES is called a permutation event. Correspondingly, the PMF describes the chance of a certain permutation event happening. RPS is compatible with evidence theory and probability theory [9,10].

Distance measure, also known as similarity measure, is an important mathematical tool and has been a topic of immense interest to researchers [11]. For instance, Han *et al.* proposed a belief interval-based distance measure between bodies of evidence [12]. In fuzzy sets, Shuvasree *et al.* defined a normalized Minkowski distance of Type-2 intuitionistic fuzzy sets applied to a biogas-plant

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implementation project [13]. Based on the Jensen-Shannon divergence, Xiao introduced a new distance between intuitionistic fuzzy sets [14] and BPAs [15,16]. Various distance measures have been applied in related applications such as group decision-making and performance evaluation [17].

Permutation distance is a specific kind of distance measure for problems which solutions can be represented with permutations [18]. Permutations have corresponding problem-dependent interpretations in different applications, and many different ways of measuring permutation distance have been developed. For example, Levenshtein [19] first introduced edit distance in the context of error-correcting codes, which measured the distance of two binary strings. Sörensen [20] later presented several extensions of edit distance for multiple combinatorial optimization problems, where the representation of solutions is a permutation of items. Ronald [21] reported two permutation distance functions, namely the exact match and deviation distance functions, for order-based encodings. Cicirello [22] introduced a new framework for studying permutation search landscapes, taking into consideration the different characteristics of permutation distance. Unmistakably, the distance of RPS also belongs to the category of permutation distance at a fundamental level.

A question which arises now is how the distance measure in RPS should be defined. One classic distance measure between BPAs is Jousselme's distance [23]. Jousselme *et al.* provides a geometrical interpretation of BPA. Specifically, a power set is considered a linear space, where a BPA is a special case of vectors. The distance between two BPAs is thus defined from the view of vectors. It has been recognized that Jousselme's distance meets all requirements for a metric [24]. Inspired by the idea of Jousselme's distance in evidence theory, the distance between RPSs is first proposed in this paper. The PES is regarded as a linear space, where RPS is a vector with coordinates PMF. Within such a linear space, the distance between RPSs is defined by considering the similarity between sets both in the number and the order of elements. The main contributions of this paper are as follows:

1) The Left Union (LU) and Right Union (RU) are firstly defined for RPS.
2) An Ordered Degree (OD) of two RPSs is presented to measure the similarity of their permutation sets.
3) Based on the definitions of union and ordered degrees, the distance between RPSs is proposed from a geometrical interpretation.
4) Based on the proposed distance, a Data-Driven Reliability Determination (DDRD) algorithm is presented and applied in threat assessment.

The rest of this paper is organised as follows. Section 2 introduces some related preliminaries. Section 3 proposes the distance between RPSs. Section 4 uses several numerical examples to illustrate the presented distance. Section 5 provides an application for threat assessment. Section 6 concludes our paper.

2. Preliminaries

In this section, we briefly review some preliminaries related to this paper.

2.1. Dempster-Shafer evidence theory

As an extension of probability theory, evidence theory has an advantage in directly expressing uncertain and unknown information [25–27]. Furthermore, evidence theory can fuse different pieces of information to reduce uncertainty. Evidence theory has been developed from different aspects, especially in various open issues. Some examples include the decision-making model based on mass function [28,29], complex mass function [30,31], combination of belief function [32], uncertainty measure in evidence theory [33,34] and quantum mass function [35,36]. With the advantage in uncertainty handling and the practicability in engineering, evidence theory has been applied in many fields such as pattern recognition [37,38], fault diagnosis [39–41] and risk assessment [42–44]. Some basic concepts in evidence theory are summarized as follows.

Definition 2.1 *(Frame of discernment).* Let Ω, called the Frame Of Discernment (FOD), denote a fixed set of N mutually exclusive and exhaustive elements, indicated by [45,46]

$$\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\} \tag{1}$$

$PS(\Omega)$ is the power set of Ω, which contains all subsets of Ω and has 2^N elements, indicated by

$$\begin{aligned} PS(\Omega) =& \{A_1, A_2, \cdots, A_{2^N}\} \\ =& \{\emptyset, \{\omega_1\}, \{\omega_2\}, \cdots, \{\omega_N\}, \\ & \{\omega_1, \omega_2\}, \{\omega_1, \omega_3\}, \cdots, \{\omega_1, \omega_N\}, \cdots, \Omega\} \end{aligned} \tag{2}$$

Definition 2.2 *(Mass function).* Given an FOD of Ω, a mass function, also called a Basic Probability Assignment (BPA), is a mapping from 2^Ω to [0, 1], formally defined by [45,46]

$$m : PS(\Omega) \to [0, 1] \tag{3}$$

constrained by

$$\sum_{A \in PS(\Omega)} m(A) = 1, \;\; m(\emptyset) = 0 \tag{4}$$

A is called a focal element if $m(A) > 0$.

2.2. The distance in evidence theory

Distance is an important metric in measuring the difference between BPAs in evidence theory [47]. From a novel geometric perspective, Jousselme *et al.* gave the vector form of BPA and presented a principled distance between BPAs [23].

Given two BPAs m_1 and m_2 under the same FOD, the distance is defined as [23]

$$d_{BPA}(m_1, m_2) = \sqrt{\frac{1}{2}\left(\overrightarrow{m_1} - \overrightarrow{m_2}\right) \underline{D} \left(\overrightarrow{m_1} - \overrightarrow{m_2}\right)^T} \tag{5}$$

where $\overrightarrow{m_1}$ and $\overrightarrow{m_2}$ are two 2^N-dimensional vectors with coordinates m(A). $\underline{D}$ is a $2^N \times 2^N$ matrix whose elements are

$$\underline{D}(A, B) = \frac{|A \cap B|}{|A \cup B|}, \;\; A, B \in PS(\Omega) \tag{6}$$

d_{BPA} has another mathematical form as

$$d_{BPA}\left(m_1, m_2\right) = \sqrt{\frac{1}{2}\left(\left\|\overrightarrow{m_1}\right\|^2 + \left\|\overrightarrow{m_2}\right\|^2 - 2\left\langle \overrightarrow{m_1}, \overrightarrow{m_2} \right\rangle\right)} \tag{7}$$

where $\langle \overrightarrow{m_1}, \overrightarrow{m_2} \rangle$ is the scalar product defined by

$$\langle \overrightarrow{m_1}, \overrightarrow{m_2} \rangle = \sum_{i=1}^{2^N} \sum_{j=1}^{2^N} m_1\left(A_i\right) m_2\left(A_j\right) \frac{\left|A_i \cap A_j\right|}{\left|A_i \cup A_j\right|} \tag{8}$$

where $A_i,\ A_j \in PS(\Omega)$. $\|\overrightarrow{m_1}\|^2$ is the square norm of $\overrightarrow{m_1}$, i.e., $\|\overrightarrow{m_1}\|^2 = \langle \overrightarrow{m_1}, \overrightarrow{m_1} \rangle$.

2.3. Random permutation set

Random Permutation Set (RPS) is a novel set which considers the permutation of elements in a set. Some basic definitions of RPS are given as follows:

Definition 2.3 *(Permutation Event Space)*. Given a fixed set of N mutually exclusive and exhaustive elements $\Theta = \{\theta_1, \theta_2, \ldots, \theta_N\}$, its Permutation Event Space (PES) is a set of all possible permutations of Θ, defined as follows [9]:

$$PES(\Theta) = \{ A_{ij} \mid i = 0, \ldots, N; \; j = 1, \ldots, P(N, i) \} \tag{9}$$

$$\begin{aligned} &= \{ \emptyset, \left(\theta_1\right), \left(\theta_2\right), \ldots, \left(\theta_N\right), \left(\theta_1, \theta_2\right), \left(\theta_2, \theta_1\right), \ldots, \left(\theta_{N-1}, \theta_N\right), \\ &\quad \left(\theta_N, \theta_{N-1}\right), \ldots, \left(\theta_1, \theta_2, \ldots, \theta_N\right), \ldots, \left(\theta_N, \theta_{N-1}, \ldots, \theta_1\right) \; \} \end{aligned} \tag{10}$$

$P(N, i) = \frac{N!}{(N-i)!}$, which is the i-permutation of N. The element A_{ij} in PES is called a Permutation Event (PE), which is a tuple representing a possible permutation of Θ. i indicates the index for the cardinality of A_{ij} and j denotes the index for the possible permutation. The cardinality of $PES(\Theta)$ is $\Delta = \sum_{i=0}^{N} P(N, i)$.

Definition 2.4 *(Random Permutation Set)*. Given a fixed set of N mutually exclusive and exhaustive elements $\Theta = \{\theta_1, \theta_2, \ldots, \theta_N\}$, its RPS is a set of pairs [9]:

$$RPS(\Theta) = \{ \langle A, \mathscr{M}(A) \rangle \mid A \in PES(\Theta) \} \tag{11}$$

where $\mathscr{M}$ is called Permutation Mass Function (PMF), described as:

$$\mathscr{M} : PES(\Theta) \to [0, 1] \tag{12}$$

satisfying

$$\mathscr{M}(\emptyset) = 0, \;\; \sum_{A \in PES(\Theta)} \mathscr{M}(A) = 1 \tag{13}$$

Some important properties of RPSs are discussed in [9]. Specifically, when not considering the permutation of elements in PES, the PES of an RPS is the same as the power set, while the PMF of an RPS is the same as the BPA in evidence theory. When each

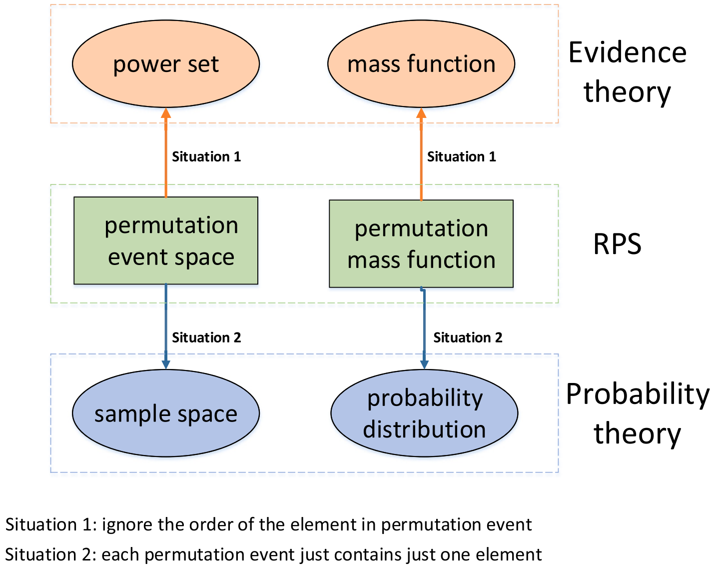

Fig. 1. The relationships between RPS, evidence theory and probability theory.

permutation event contains just one element, the PES of an RPS degenerates into the sample space, while the PMF of an RPS degenerates into the probability distribution. Fig. 1 shows the relationships between RPS, evidence theory and probability theory.

Definition 2.5 *(Intersection of permutation events).* Given two permutation events $A, B \in PES(\Theta)$, the Right Intersection (RI) and Left Intersection (LI) of A and B are defined as follows [9]:

$$A\vec{\cap}B = B \backslash\backslash \bigcup_{\theta\in B, \theta\notin A} \{\theta\} \qquad (RI) \tag{14}$$

$$A\overleftarrow{\cap}B = A \backslash\backslash \bigcup_{\theta\in A, \theta\notin B} \{\theta\} \qquad (LI) \tag{15}$$

where $X\backslash\backslash Y$ denotes removing Y from X under the permutation of elements in X.

Definition 2.6 *(Left orthogonal sum of permutation mass functions).* Given two RPSs defined on Θ: $RPS_1 = \{\langle A, \mathscr{M}_1(A)\rangle\}$ and $RPS_2 = \{\langle A, \mathscr{M}_2(A)\rangle\}$, the left orthogonal sum (LOS), represented by $\mathscr{M}^L = \mathscr{M}_1 \overleftarrow{\oplus} \mathscr{M}_2$, is defined by

$$\mathscr{M}^L(A) = \begin{cases} \frac{1}{1-\overleftarrow{K}} \sum\limits_{B\overleftarrow{\cap}C=A} \mathscr{M}_1(B)\mathscr{M}_2(C), \text{ A} \neq \emptyset \\ 0, \text{ A} = \emptyset \end{cases} \tag{16}$$

where $A, B, C \in PES(\Theta)$, $\overleftarrow{\cap}$ denotes the left intersection, and $\overleftarrow{K}$ is defined as

$$\overleftarrow{K} = \sum_{B\overleftarrow{\cap}C=\emptyset} \mathscr{M}_1(B)\mathscr{M}_2(C) \tag{17}$$

Definition 2.7 *(Right orthogonal sum of permutation mass functions).* Given two RPSs defined on Θ: $RPS_1 = \{\langle A, \mathscr{M}_1(A)\rangle\}$ and $RPS_2 = \{\langle A, \mathscr{M}_2(A)\rangle\}$, the right orthogonal sum (ROS), represented by $\mathscr{M}^R = \mathscr{M}_1 \vec{\oplus} \mathscr{M}_2$, is defined by

$$\mathscr{M}^R(A) = \begin{cases} \frac{1}{1-\vec{K}} \sum\limits_{B\vec{\cap}C=A} \mathscr{M}_1(B)\mathscr{M}_2(C), \text{ A} \neq \emptyset \\ 0, \text{ A} = \emptyset \end{cases} \tag{18}$$

where $A, B, C \in PES(\Theta)$, $\vec{\cap}$ denotes the right intersection, and $\vec{K}$ is defined as

$$\vec{K} = \sum_{B\vec{\cap}C=\emptyset} \mathscr{M}_1(B)\mathscr{M}_2(C) \tag{19}$$

Table 1
RU and LU of permutation events in RPS_1 and RPS_2.

RPS_1	RPS_2	LU	RU	RPS_1	RPS_2	LU	RU	RPS_1	RPS_2	LU	RU
(R)	(G)	(R,G)	(G,R)	(R,B)	(G)	(R,B,G)	(G,R,B)	(B,R)	(G)	(B,R,G)	(G,B,R,)
	(R,B)	(R,B)	(R,B)		(R,B)	(R,B)	(R,B)		(R,B)	(B,R)	(R,B)
	(R,G,B)	(R,G,B)	(R,G,B)		(R,G,B)	(R,B,G)	(R,G,B)		(R,G,B)	(B,R,G)	(R,G,B)
	(G,B,R)	(R,G,B)	(G,B,R)		(G,B,R)	(R,B,G)	(G,B,R)		(G,B,R)	(B,R,G)	(G,B,R)

3. Distance of Random permutation set

In this section, the distance between two RPSs is proposed, by considering the similarity between sets both in the number and order of elements.

3.1. The union of RPS

Definition 3.1 *(Union of permutation events).* Given two permutation events $A, B \in PES(\Theta)$, the Right Union (RU) and Left Union (LU) of A and B are defined as follows:

$$A\vec{\cup}B = B// \bigcup_{\theta\in A,\theta\notin B} \{\theta\} \qquad (RU) \tag{20}$$

$$A\overleftarrow{\cup}B = A// \bigcup_{\theta\in B,\theta\notin A} \{\theta\} \qquad (LU) \tag{21}$$

where $X//Y$ denotes adding Y to the end of X keeping the permutation of elements in Y.

Combined with the intersection of permutation events in Definition 2.5, some properties about the intersection and union of permutation events are discussed as follows.

Proposition 3.1. $|A\vec{\cap}B| = |A\overleftarrow{\cap}B| = |A\cap B|$

Proposition 3.2. $|A\vec{\cup}B| = |A\overleftarrow{\cup}B| = |A\cup B|$

For better understanding of RU and LU, let us consider the following example.

Example 3.1. Let two RPSs defined on $\Theta = \{R, B, G\}$ be as follows:

$$RPS_1(\Theta) = \{\langle (R), 0.4\rangle, \langle (R,B), 0.3\rangle, \langle (B,R), 0.3\rangle\}$$

$$RPS_2(\Theta) = \{\langle (G), 0.4\rangle, \langle (R,B), 0.1\rangle, \langle (R,G,B), 0.15\rangle, \langle (G,B,R), 0.35\rangle\}$$

Take two permutation events as an instance, i.e. (R, B) in RPS_1 and (G, B, R) in RPS_2, then the RU and LU of these two permutation events can be respectively obtained as:

$$(R,B)\vec{\cup}(G,B,R) = (G,B,R)//\emptyset = (G,B,R)$$

$$(R,B)\overleftarrow{\cup}(G,B,R) = (R,B)//\{G\} = (R,B,G)$$

Similarly, all union results of RPS_1 and RPS_2 are shown in Table 1.

Example 3.1 shows that the result of RU of two permutation events is different from that of LU, since permutation events consider the order of the elements in $PES(\Theta)$.

3.2. The ordered degree of two RPSs

Definition 3.2 *(Ordered degree of permutation events).* Given two permutation events $A, B \in PES(\Theta)$, the ordered degree between A and B is defined as follows:

$$OD(A,B) = \exp\left(-\frac{\sum_{\theta\in A\cap B} \left|rank_A(\theta) - rank_B(\theta)\right|}{|A\cup B|}\right) \tag{22}$$

where $rank_A(\theta)$ and $rank_B(\theta)$ are the order of θ in A and B, respectively. The term inside the exponential of Eq. (22), i.e., $-\frac{\sum_{\theta\in A\cap B}|rank_A(\theta) - rank_B(\theta)|}{|A\cup B|}$ is called the pseudo-deviation distance. Obviously, the range of $OD(A,B)$ is $(0,1]$. $OD(A,B)$ reflects the difference of A and B resulting from the order of elements. The greater the number of elements with different orders in A and B, the smaller the ordered degree, and the greater the difference between A and B. If A and B are the same, $OD(A,B) = 1$.

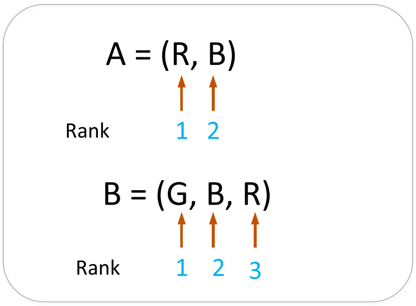

Fig. 2. The rank of elements in Example 3.2.

It should be noted that the core of the proposed pseudo-deviation distance is the same as the deviation distance [21], both of which consider the sum of positional deviations of permutation elements. However, there are differences in the operation and form between them. Deviation distance is designed to measure the difference between two genotypes for genetic algorithm. Hence, deviation distance is only applicable for two permutation sets with the same domain and length. However, in the context of RPS, the proposed pseudo-deviation function can measure the difference between two permutation sets with different domains and lengths. Furthermore, the normalization method of two functions also differs. For two permutation sets with the same length s in the same domain, the pseudo-deviation distance uses s while deviation distance adopts $s-1$ to normalize the sum of positional deviations. Therefore, deviation distance has a restriction of $s \geq 2$. The pseudo-deviation distance is also valid when $s = 1$, which refers to the difference between single elements. To summarize, the pseudo-deviation distance is a more flexible operation than deviation distance.

More broadly, a generalized representation of ordered degree can be defined as

$$OD(A, B) = \exp(-d_p) \tag{23}$$

where d_p denotes a distance function on permutations, just like the pseudo-deviation distance in Eq. (22). Besides the proposed pseudo-deviation distance, other permutation distance functions can be considered, for example, exact match distance [21] and Kendall Tau Distance [48]. A review on permutation distance can be found in Ref. [49]. It is worth noting that a valid d_p not only depends on specific problems that consider the interpretation of permutation, but also considers the characteristics of RPS.

Example 3.2. Following Example 3.1, according to Fig. 2, for A = (R, B) and B = (G, B, R), OD(A, B) is computed by

$$\begin{aligned} OD(A, B) &= \exp\left(-\frac{|rank_A(R) - rank_B(R)| + |rank_A(B) - rank_B(B)|}{3}\right) \\ &= \exp\left(-\frac{|1-3| + |2-2|}{3}\right) \\ &= \exp\left(-\frac{2}{3}\right) \\ &= 0.5134 \end{aligned}$$

3.3. The proposed distance of RPS

Before introducing our proposed method, a geometrical interpretation for RPS will first be given. Specifically, a body of RPS can be regarded as a discrete random variable whose values are in $PES(\Theta)$ with a probability distribution in the form of PMF. Let $\Phi_{PES(\Theta)}$ be the space generated by elements of $PES(\Theta)$. $\Phi_{PES(\Theta)}$ is a vector space if any linear combination of the objects of $\Phi_{PES(\Theta)}$ is in $\Phi_{PES(\Theta)}$. This means

$$\vec{V} = \sum_{i=1}^{\Delta} \alpha_i A_i \in \Phi_{PES(\Theta)} \tag{24}$$

where Δ is the total number of elements in $PES(\Theta)$, A_i is the element of $PES(\Theta)$ and α_i is a real number. $A_1, A_2, \ldots, A_\Delta$ form a base for $\Phi_{PES(\Theta)}$.

Therefore, an RPS can be defined as a special case of vectors in $\Phi_{PES(\Theta)}$, i.e. a vector $\overrightarrow{RPS}$ in $\Phi_{PES(\Theta)}$ with coordinates $\mathscr{M}(\theta)$ such that

$$\sum_{i=1}^{\Delta} \mathscr{M}(\theta) = 1 \ and \ \mathscr{M}(\theta) \geq 0, \ \theta \in PES(\Theta) \tag{25}$$

Definition 3.3 *(Distance of RPS)*. Let RPS_1 and RPS_2 be two RPSs on the same event set Θ. The distance between RPS_1 and RPS_2 is defined as:

$$d_{RPS}(\overrightarrow{RPS_1}, \overrightarrow{RPS_2}) = \sqrt{\frac{1}{2}\left(\overrightarrow{RPS_1} - \overrightarrow{RPS_2}\right) \underline{RD} \left(\overrightarrow{RPS_1} - \overrightarrow{RPS_2}\right)^T} \tag{26}$$

where $\overrightarrow{RPS_1}$ and $\overrightarrow{RPS_2}$ are two vectors in the permutation event space with coordinates $\mathscr{M}(A)$. $\underline{RD}$ is a $\Delta \times \Delta$ matrix whose elements are

$$\underline{RD}(A, B) = \frac{\left|A \vec{\cap} B\right|}{\left|A \vec{\cup} B\right|} \times OD(A, B) \tag{27}$$

From Propositions 3.1 and 3.2, we can then obtain

$$\frac{\left|A \vec{\cap} B\right|}{\left|A \vec{\cup} B\right|} = \frac{\left|A \overleftarrow{\cap} B\right|}{\left|A \overleftarrow{\cup} B\right|} = \frac{|A \cap B|}{|A \cup B|} \tag{28}$$

Therefore, $\underline{RD}(A, B)$ can also be computed by

$$\underline{RD}(A, B) = \frac{\left|A \vec{\cap} B\right|}{\left|A \vec{\cup} B\right|} \times OD(A, B) = \frac{|A \cap B|}{|A \cup B|} \times OD(A, B) \tag{29}$$

Another way to write d_{RPS} is

$$d_{RPS}(\overrightarrow{RPS_1}, \overrightarrow{RPS_2}) = \sqrt{\frac{1}{2}\left(\left\|\overrightarrow{RPS_1}\right\|^2 + \left\|\overrightarrow{RPS_2}\right\|^2 - 2\left\langle \overrightarrow{RPS_1}, \overrightarrow{RPS_2} \right\rangle\right)} \tag{30}$$

where $\left\langle \overrightarrow{RPS_1}, \overrightarrow{RPS_2} \right\rangle$ is the scalar product defined by

$$\left\langle \overrightarrow{RPS_1}, \overrightarrow{RPS_2} \right\rangle = \sum_{i=1}^{\Delta} \sum_{j=1}^{\Delta} RPS_1\left(A_i\right) RPS_2\left(A_j\right) \frac{\left|A_i \cap A_j\right|}{\left|A_i \cup A_j\right|} \times OD(A_i, A_j) \tag{31}$$

where $A_i,\ A_j \in PES(\Theta)$. $\left\|\overrightarrow{RPS_1}\right\|^2$ is the square norm of $\overrightarrow{RPS_1}$, i.e., $\left\|\overrightarrow{RPS_1}\right\|^2 = \left\langle \overrightarrow{RPS_1}, \overrightarrow{RPS_1} \right\rangle$.

Proposition 3.3. $d_{RPS}(\overrightarrow{RPS_1}, \overrightarrow{RPS_2}) \geq 0$. *If two RPSs are the same, then the distance is 0.*

Proposition 3.4. $d(\overrightarrow{RPS_1}, \overrightarrow{RPS_2}) = d(\overrightarrow{RPS_2}, \overrightarrow{RPS_1})$

Example 3.3. Following Example 3.1, the distance of RPS_1 and RPS_2 is calculated as follows: First, the vector form of two RPSs is given as

$$\overrightarrow{RPS_1} = [0.4 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.3 \quad 0.3 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$$
$$\overrightarrow{RPS_2} = [\ 0 \quad 0.4 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.35 \quad 0 \quad 0 \quad 0.15]$$

Then the matrix $\underline{RD}$ is calculated based on Eq. (27). For A = (R, B) and B = (G, B, R),

$$\begin{aligned} RD(A, B) &= \frac{\left|(R, B) \vec{\cap} (G, B, R)\right|}{\left|(R, B) \vec{\cup} (G, B, R)\right|} \times OD(A, B) \\ &= \frac{|(B, R)|}{|(G, B, R)|} \times OD_{dd}(A, B) \\ &= \frac{2}{3} \times 0.5134 \\ &= 0.3423 \end{aligned}$$

Finally, based on Eq. (26), we can obtain $d_{RPS}(\overrightarrow{RPS_1}, \overrightarrow{RPS_2}) = 0.5844$.

4. Numerical examples and discussion

In this section, some numerical examples are given to illustrate the presented distance of two RPSs.

Table 2
The distance between RPS and BPA when X changes in Example 4.3.

(X) and {X}	d_{BPA}	d_{RPS}	(X) and {X}	d_{BPA}	d_{RPS}
1	0.5802	0.5802	3,2,1	0.2944	0.7646
2	0.7326	0.8018	3,1,2	0.2944	0.7557
3	0.8287	0.8762	2,3,1	0.2944	0.7834
1,2	0.3464	0.7150	2,1,3	0.2944	0.6680
2,1	0.3464	0.3464	1,3,2	0.2944	0.6218
1,3	0.5657	0.7736	1,2,3	0.2944	0.2944
3,1	0.5657	0.6438			
2,3	0.6481	0.7630			
3,2	0.6481	0.7814			

Example 4.1. Given the fixed set of $\Theta = \{1, 2, 3\}$, two RPSs defined on Θ are given as follows:

$$RPS_1 = \{< (1), 1 >\}$$
$$RPS_2 = \{< (2,3), 1 >\}$$

The associated distance is $d_{RPS}(\overrightarrow{RPS_1}, \overrightarrow{RPS_2}) = d_{RPS}(\overrightarrow{RPS_2}, \overrightarrow{RPS_1}) = 1$.

Example 4.2. Given the fixed set of $\Theta = \{1, 2, 3\}$, three RPSs defined on Θ are given as follows:

$$RPS_1 = \{< (1,3), 1 >\}$$
$$RPS_2 = \{< (3,1), 1 >\}$$
$$RPS_3 = \{< (1,3,2), 1 >\}$$

The associated distances between RPSs are calculated as

$$d_{RPS}(\overrightarrow{RPS_1}, \overrightarrow{RPS_3}) = 0.5774$$
$$d_{RPS}(\overrightarrow{RPS_2}, \overrightarrow{RPS_3}) = 0.8110$$
$$d_{RPS}(\overrightarrow{RPS_1}, \overrightarrow{RPS_2}) = 0.7951$$

Evidently, the difference between RPS_1 and RPS_2 is the reverse order of elements. The order of events in RPS_1 is the same as the order of corresponding elements in RPS_3, while RPS_3 has the reverse order as compared with RPS_2. Therefore, the distance of RPS_3 to RPS_1 should be smaller than the distance of RPS_3 to RPS_2. It is obvious that the distance result, i.e. $d_{RPS}(\overrightarrow{RPS_1}, \overrightarrow{RPS_3}) \leq d_{RPS}(\overrightarrow{RPS_2}, \overrightarrow{RPS_3})$, is consistent with the intuitive result. Besides, it can be found that $d_{RPS}(\overrightarrow{RPS_1}, \overrightarrow{RPS_3}) + d_{RPS}(\overrightarrow{RPS_1}, \overrightarrow{RPS_2}) = 0.5774 + 0.7951 = 1.3725 \geqslant d_{RPS}(\overrightarrow{RPS_2}, \overrightarrow{RPS_3}) = 0.8110$, which shows that the proposed distance satisfies the triangle inequality.

Example 4.3. Given a fixed set of $\Theta = \{1, 2, 3\}$, two RPSs defined on Θ are shown as follows.

$$RPS_1 = \{< (1), 0.2 >, < (1,2), 0.3 >, < (1,2,3), 0.5 >\}$$
$$RPS_2 = \{(X), 1\}$$

If the order in permutation events is not considered, the two RPSs are actually mass functions in evidence theory.

$$m_1(\{1\}) = 0.2, m_1(\{1,2\}) = 0.3, m_1(\{1,2,3\}) = 0.5$$
$$m_2(\{X\}) = 1$$

As shown in the first column of Table 2, 15 different kinds of situations are being considered regarding the set X. The distance between RPSs and mass functions are then calculated, shown in Table 2 and Fig. 3. It can be found that for the curve of RPS distance, when $X = (3)$, the RPS distance reaches the maximum point with 0.8762. This is because in such a situation the RPS_2 has minimal similarity with RPS_1. When $X = (1,2,3)$, the RPS distance reaches the minimal point with 0.2944, since in this situation RPS_2 has the largest similarity with RPS_1. Obviously, the curve of BPA distance has the same rule, i.e. obtaining the maximum value when $X = \{3\}$ and obtaining the minimal value when $X = \{1,2,3\}$. However, the two curves have different changing trends. For the sets with identical elements like $\{1,2\}$ and $\{2,1\}$, the RPS distance has different values if the sets have different element orders, while the BPA distance obtains the same value. Therefore, the curve of BPA distance is a straight line while the curve of RPS distance has a fluctuation in these situations. To summarize, with regard to the sets which have the same elements but different orders, the RPS distance can effectively measure the difference while the BPA distance cannot.

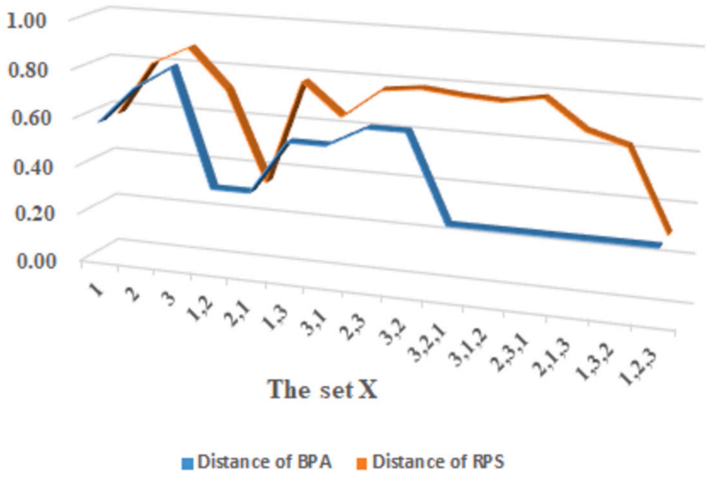

Fig. 3. The distance between RPS and BPA when X changes in Example 4.3.

Table 3
The distance between RPS and mass function when X changes.

X	d_{BPA}	d_{RPS}
1	0.8780	0.8644
1, 2	0.8119	0.7801
1, 2, 3	0.7690	0.6856
1, 2, 3, 4	0.7456	0.5757
1, 2, 3, 4, 5	0.7364	0.4205
1, 2, 3, 4, 5, 6	0.7545	0.5455
1, 2, 3, 4, 5, 6, 7	0.7715	0.6176

Example 4.4. Given a fixed set of $\Theta = \{1, 2, 3, \ldots, 7\}$. Two RPSs defined on Θ is shown as follows.

$$RPS_1 = \{< (2,3,4), 0.05 >, < (6), 0.05 >, < (X), 0.7 >, < (1,2,3,\ldots,7), 0.1 >\}$$

$$RPS_2 = \{(2,3,4,5), 1\}$$

If the order in permutation events is ignored, these two RPSs change to mass functions in evidence theory.

$$m_1(\{2,3,4\}) = 0.05, m_1(\{6\}) = 0.05, m_1(\{X\}) = 0.7, m_1(\Theta) = 0.1$$

$$m_2(\{2,3,4,5\}) = 1$$

When X in RPS_1 and m_1 change from $\{1\}$ to $\{1, 2, 3, \ldots, 7\}$, the distances between RPS and mass function are calculated, which are shown in Table 3 and Fig. 4.

From Table 3 and Fig. 4, the results show that both BPA distance and RPS distance go in tandem consistently with the change of X. When $X = \{1\}$, both d_{RPS} and d_{BPA} obtain the maximum distance value; and when $X = \{1,2,3,4,5\}$, both d_{RPS} and d_{BPA} obtain the minimum value. However, the d_{RPS} values are always larger than the corresponding d_{BPA} values under the same X. The reason is that d_{RPS} not only considers the difference in each set and the corresponding belief value (just like d_{BPA}), but also considers the order in each set which d_{BPA} does not.

5. Application in threat assessment

In this section, the proposed distance is applied for reliability ranking in threat assessment. In Ref. [9], an RPS-based Data Fusion (RPSDF) algorithm is presented for threat assessment, as shown in Algorithm 1. It designed a scenario of enemy aircraft detection, where several radars provided detection reports represented by RPS. Assuming that there are three types of aircraft (potential threatening targets) represented by $\Theta = \{A, B, C\}$, the permutation event space $PES(\Theta)$ denotes all possible threat events. The order in permutation events denotes the threat ranking in threat events. PMF defined on $PES(\Theta)$ represents the threat degree of threat events. For example, a report of $< (B, A, C), 0.6 >$ means that the enemy may send all three types of aircraft, amongst which B is the most threatening, A is the next, while the threat of C is the lowest threat. The chance of this threat event occurring is relatively high at 0.6. Given three reports by three radars, denoted as RPS_1, RPS_2 and RPS_3, RPSDF is used to fuse these reports to make a comprehensive assessment.

A critical step in the RPSDF algorithm [9] is in determining the data fusion sequence according to the reliability ranking of RPSs. Different fusion sequence may produce different results. However, Ref. [9] suggested a hypothetical expert-based reliability

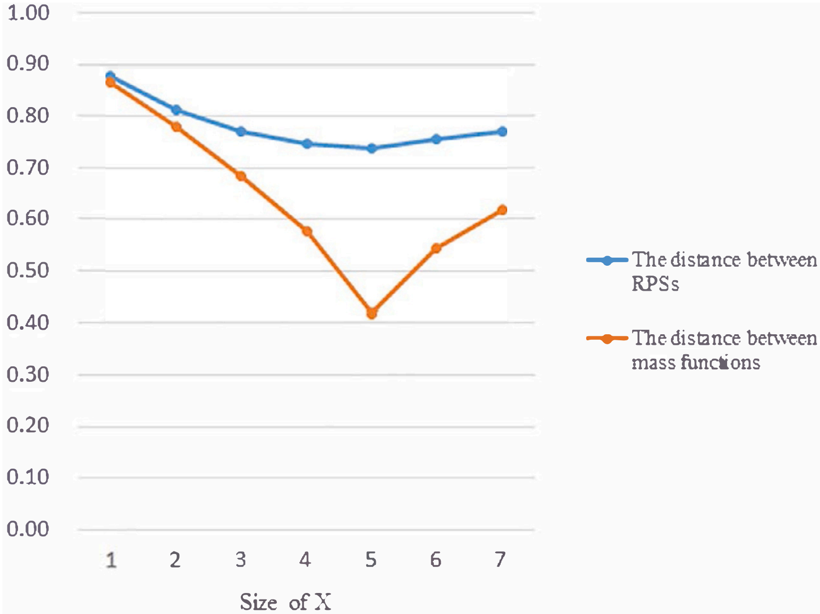

Fig. 4. The distance between RPS and BPA when X changes.

Algorithm 1: The RPS-based Data Fusion (RPSDF) algorithm for threat assessment [9].

Input: Threat assessment reports represented by H RPSs: $\{RPS_i, i = 1, 2, \ldots, H\}$;
Output: The Final Threat Assessment (FTA) for different types;

1 Based on the reliability ranking of RPSs, combine M RPSs based on LOS for $H - 1$ times:

$$\mathscr{M}^L = \mathscr{M}_1 \overleftarrow{\oplus} \mathscr{M}_2 \overleftarrow{\oplus} \ldots \overleftarrow{\oplus} \mathscr{M}_H$$

2 Calculate the relative threat degree (RTD) for the $j - th$ threat event A_{ij} with the $i - th$ type of aircraft:

$$RTD(A_{ij}) = \frac{\mathscr{M}^L(A_{ij})}{\sum_{j=1} P(N,i)\mathscr{M}^L(A_{ij})}$$

where $i = 1, \ldots, N$ is the number of types for aircraft, and $j = 1 \ldots, P(N,i) = \frac{N!}{(N-i)!}$ is the index for the i-type threat event;

3 Determine the Final Threat Assessment (FTA) for different types of threat events:

$$FTA_i = argmax_{1 \le j \le P(N,i)}[RTD(A_{ij})]$$

4 return FTA_i;

ranking, which may be subjective. A specific method to determine the reliability ranking is thus required. To address this problem, a Data-Driven Reliability Determination (DDRD) algorithm is proposed based on the proposed distance of RPS.

5.1. The data-driven reliability determination algorithm

Given a fixed set $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$, along with M RPSs based on $PES(\Theta)$, the proposed DDRD algorithm is described as follows:
Step 1: Calculate the distance between RPS_i and RPS_j, denoted as $d_{RPS}(RPS_i, RPS_j)$, i, $j \in [1, M]$.
Step 2: Calculate the similarity between RPS_i and RPS_j:

$$Sim(RPS_i, RPS_j) = 1 - d_{RPS}(RPS_i, RPS_j) \tag{32}$$

Step 3: Calculate the support degree of RPS_i:

$$Sup(RPS_i) = \sum_{j=1, j \ne i}^{M} Sim(RPS_i, RPS_j) \tag{33}$$

Table 4
The RPS-based threat assessment reports by three radars.

	(A)	(B)	(C)	(B,A)	(A,B)	(A,C)	(C,A,B)	(B,C,A)	(B,A,C)	(A,C,B)	(A,B,C)
RPS_1	0.2	0.08	0	0.05	0.12	0.03	0	0.05	0.1	0.25	0.12
RPS_2	0.07	0.13	0.02	0.2	0.07	0.1	0.08	0	0.2	0	0.13
RPS_3	0.14	0.09	0	0.08	0.12	0	0.05	0	0.1	0.3	0.12

Table 5
The distance and similarity between three RPSs.

	(RPS_1, RPS_2)	(RPS_1, RPS_3)	(RPS_2, RPS_3)
d_{RPS}	0.2152	0.0687	0.2132
Sim	0.7848	0.9313	0.7868

Table 6
The support degree and weight for each RPS.

	RPS_1	RPS_2	RPS_3
Sup	1.7161	1.5716	1.7180
W	0.3428	0.3140	0.3432

Table 7
The fusion results of RPSs in threat assessment.

	Fusion	(A)	(B)	(C)	(A,B)	(B,A)	(A,C)	(C,A)	(A,B,C)	(A,C,B)	(B,A,C)	(B,C,A)	(C,A,B)
Case 1	$RPS_1\overleftarrow{\oplus}RRS_3\overleftarrow{\oplus}RPS_2$	0.349	0.196	0.007	0.190	0.078	0.040	0.003	0.031	0.065	0.026	0.013	0.000
Case 2	$RPS_1\overleftarrow{\oplus}RRS_2\overleftarrow{\oplus}RPS_3$	0.349	0.196	0.007	0.190	0.078	0.040	0.003	0.031	0.065	0.026	0.013	0.000
Case 3	$RPS_2\overleftarrow{\oplus}RRS_1\overleftarrow{\oplus}RPS_3$	0.349	0.196	0.007	0.097	0.172	0.041	0.002	0.043	0.000	0.066	0.000	0.027
Case 4	$RPS_2\overleftarrow{\oplus}RRS_3\overleftarrow{\oplus}RPS_1$	0.349	0.196	0.007	0.097	0.172	0.041	0.002	0.043	0.000	0.066	0.000	0.027
Case 5	$RPS_3\overleftarrow{\oplus}RRS_1\overleftarrow{\oplus}RPS_2$	0.349	0.196	0.007	0.198	0.071	0.039	0.004	0.029	0.072	0.024	0.000	0.012
Case 6	$RPS_3\overleftarrow{\oplus}RRS_2\overleftarrow{\oplus}RPS_1$	0.349	0.196	0.007	0.198	0.071	0.039	0.004	0.029	0.072	0.024	0.000	0.012

Table 8
The relative threat degree for different permutation events in threat assessment.

	Fusion	(A)	(B)	(C)	(A,B)	(B,A)	(A,C)	(C,A)	(A,B,C)	(A,C,B)	(B,A,C)	(B,C,A)	(C,A,B)
Case 1	$RPS_1\overleftarrow{\oplus}RRS_3\overleftarrow{\oplus}RPS_2$	**0.632**	0.355	0.013	**0.612**	0.250	0.128	0.010	0.231	**0.481**	0.192	0.096	0.000
Case 2	$RPS_1\overleftarrow{\oplus}RRS_2\overleftarrow{\oplus}RPS_3$	**0.632**	0.355	0.013	**0.612**	0.250	0.128	0.010	0.231	**0.481**	0.192	0.096	0.000
Case 3	$RPS_2\overleftarrow{\oplus}RRS_1\overleftarrow{\oplus}RPS_3$	**0.632**	0.355	0.013	0.311	**0.551**	0.133	0.005	0.317	0.000	**0.488**	0.000	0.195
Case 4	$RPS_2\overleftarrow{\oplus}RRS_3\overleftarrow{\oplus}RPS_1$	**0.632**	0.355	0.013	0.311	**0.551**	0.133	0.005	0.317	0.000	**0.488**	0.000	0.195
Case 5	$RPS_3\overleftarrow{\oplus}RRS_1\overleftarrow{\oplus}RPS_2$	**0.632**	0.355	0.013	**0.635**	0.227	0.126	0.012	0.211	**0.526**	0.175	0.000	0.088
Case 6	$RPS_3\overleftarrow{\oplus}RRS_2\overleftarrow{\oplus}RPS_1$	**0.632**	0.355	0.013	**0.635**	0.227	0.126	0.012	0.211	**0.526**	0.175	0.000	0.088

Step 4: The reliability degree of RPS_i is defined as:

$$W(RPS_i) = \frac{Sup(RPS_i)}{\sum_{j=1}^{M} Sup(RPS_j)} \tag{34}$$

5.2. Experimental results

Table 4 shows a data example of detection reports for threat assessment. As shown in Table 4, RPS_1 and RPS_3 support each other and RPS_2 is relatively a'bad' report. Obviously, the targets for different types of threat events in agreement with theoretical predictions should be (A), (A,B) and (A,C,B). According to the proposed DDRD algorithm, the distance and similarity between three RPSs are calculated based on Eq. (32), as shown in Table 5. Based on Eqs. (33)-(34), the support degree and reliability degree for each RPS are further computed, as shown in Table 6. Therefore, the reliability ranking of three radars is $RPS_3 \succ RPS_1 \succ RPS_2$.

Based on the reliability ranking, the fusion sequence is $RPS_3\overleftarrow{\oplus}RRS_1\overleftarrow{\oplus}RPS_2$ by LOS. Note that the fusion sequence can also be represented as $RPS_2\overrightarrow{\oplus}RRS_1\ \overrightarrow{\oplus}RPS_3$ by ROS. Since the RPSDF algorithm (Algorithm 1) adopts LOS to fuse RPSs, we also use LOS operation for consistency in the paper. In addition, the other five possible fusion sequences are also considered for comparison. Based on the RPSDF algorithm [9], the combination results and the relative threat degrees for different permutation events are given in Table 7 and Table 8, respectively. The final threat assessment for different types of threat events are reflected in Figs. 5–7.

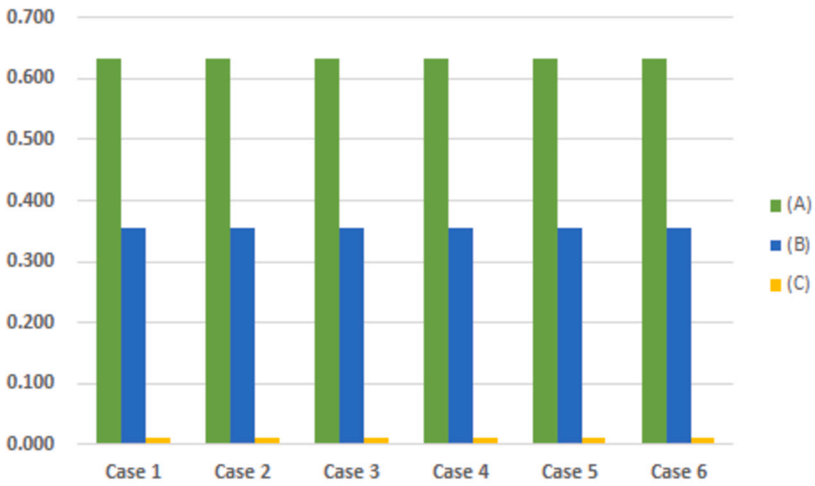

Fig. 5. The final threat assessment for one type of aircraft.

As shown in Table 8 and Figs. 5–7, a comparison between the different fusion sequences reveals two key findings. It can be observed that Case 1 and Case 2, Case 3 and Case 4, Case 5 and Case 6 have the same fusion results. This shows that results by the different fusion sequences are mainly determined by the first-fused RPS. For example, for Case 1 and Case 2, the fusion sequences are $RPS_1 \overline{\oplus} RRS_3 \overline{\oplus} RPS_2$ and $RPS_1 \overline{\oplus} RRS_2 \overline{\oplus} RPS_3$ respectively. Although RPS_2 and RPS_3 are fused in different orders in the two cases, RPS_1 is always the first one to fuse. Since the first-fused RPS plays a decisive role in the final fusion result, Case 1 and Case 2 have the same results. In conclusion, in the fusion process of RPS_1, RPS_2 and RPS_3, the final result mainly depends on the RPS which is the first to fuse. On the other hand, it can be observed that Case 1 and Case 2 have the most satisfactory results compared to the other four cases, therefore revealing the validity of the proposed DDRD algorithm:

- From Fig. 5, for threat events with one type of aircraft, i.e. (A), (B) and (C), six cases have the same results, suggesting that A has the highest chance of occurring. This means that if there is only one type of aircraft posing threat, (A) is the most threatening aircraft in the fleet.
- From Fig. 6, for threat events with two types of aircraft, i.e., (A, B), (B, A), (A, C) and (C, A), both Case 1 (Case 2) and Case 5 (Case 6) suggest that the event (A, B) has the highest chance, while Case 2 and Case 3 suggest that the event (F_2, F_1) has the highest chance, which is against intuition. In addition, though Case 1 and Case 5 obtain the same result, Case 1 obtains a higher chance than Case 5 (0.635 v.s. 0.612), which means that Case 1, in which the proposed DDRD algorithm is implemented, has a higher accuracy of recognition.
- From Fig. 7, the desired choice (A, C, B) is found correctly under Case 1 (Case 2) and Case 5 (Case 6), in contrast to the 'bad' result (B, A, C) under Case 2 (Case 3). Additionally, Case 1 (Case 2) also has a higher accuracy of recognition than Case 5 (Case 6) with 0.526 v.s. 0.488.

Overall, different fusion sequences will affect the final fusion results, and the first-fused RPS plays a decisive role in the final fusion results. The proposed DDRD algorithm can effectively determine the fusion order of RPSs with the corresponding fusion result in full agreement with theoretical predictions and has the highest accuracy of recognition in threat assessment.

Considering that the size of permutation event space for a certain set is factorial, the complexity of the proposed distance will be high with the increase of Θ. Therefore, the proposed distance will face the challenge of large-scale matrix operation. Investigating how to overcome the large-scale matrix operation motivates future work. Besides, more applications based on the proposed distance will be investigated, such as conflict management between different RPSs.

6. Conclusion

The Random Permutation Set (RPS) takes the permutation of a certain set into consideration, which can be taken as a generalization of evidence theory. From the perspective of vector space, the permutation event space is regarded as a linear space, where RPS is a vector with coordinates PMF. Under such a linear space, the distance of RPS is measured. The union of RPS, including Left Union (LU) and Right Union (RU), has been defined in this paper. Compared to the traditional union operation, LU and RU consider the order of elements. Regarding any two sets, the cardinality of LU is the same as that of RU. An Ordered Degree (OD) of two RPSs has also been presented to measure the similarity between permutation sets, in which a pseudo-deviation distance is defined by considering positional deviations of permutation elements. The more elements with different orders there are, the smaller the OD value. The distance of RPS is proposed using a permutation distance function, which is defined by a quantification of the similarity between sets both in the number and the order of elements. Based on the proposed distance, a Data-Driven Reliability Determination (DDRD) algorithm is then presented and applied in the context of threat assessment examples. Experimental results

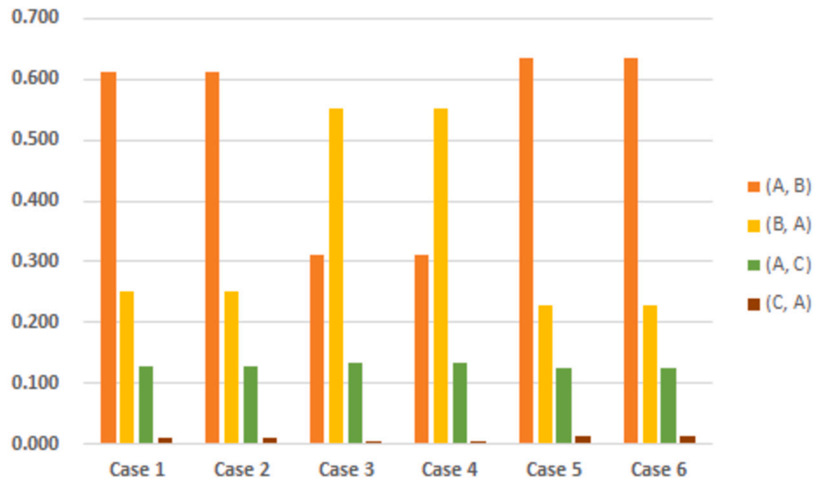

Fig. 6. The final threat assessment for two types of aircraft.

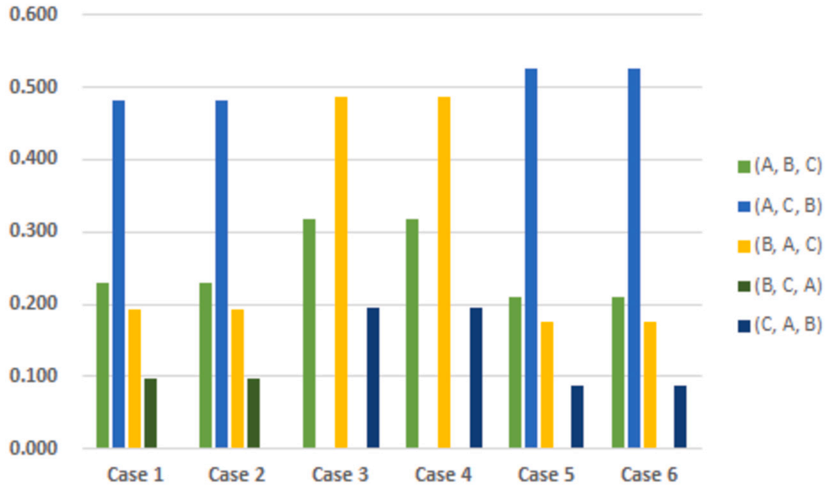

Fig. 7. The final threat assessment for three types of aircraft.

reveal that the threat assessment based on our proposed DDRD algorithm is in agreement with theoretical predictions and has the highest accuracy of identification.

CRediT authorship contribution statement

Luyuan Chen: Conceptualization, Methodology, Writing – original draft, Writing – review & editing. **Yong Deng:** Conceptualization, Funding acquisition, Methodology, Resources, Supervision, Validation, Writing – review & editing. **Kang Hao Cheong:** Conceptualization, Methodology, Resources, Supervision, Validation, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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