Silver Ratio in Maximum Deng Entropy Triangle

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Abstract

Pascal's triangle is a mathematical triangular form of binomial coefficients, from which Fibonacci number sequence and golden ratio can be obtained. Recently, the relations between Pascal's triangle and maximum Deng entropy are studied and presented. A straightforward idea comes: if we design a triangle based on maximum Deng entropy, what would the generated number sequence and the corresponding ratio be like? Hence, this paper proposes a triangle based on maximum Deng entropy, named as the maximum Deng entropy triangle (MDET). Besides, the number sequences based on MDET are investigated. Next, the general term for the MDET sequence is presented and the ratio in MDET sequence is analyzed. We prove that, under the condition of $n \to \infty$, the ratio in right MDET sequence converges to silver ratio $1 + \sqrt{2}$. Moreover, numerical examples are shown to expound the proposed MDET and the MDET sequence.

Keywords: Pascal's triangle, Deng entropy, Maximum Deng entropy triangle (MDET), Number sequence, Silver ratio

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1. Introduction

Pascal triangle refers to a triangular arrangement of the binomial coefficients and it has many remarkable numerical properties [1]. One of the appealing properties of Pascal's triangle is that its diagonal sum can generate the Fibonacci number sequence [2]. The Fibonacci numbers had occurred in lots of fields from the pattern of gerbera flower heads [3] to the rhythm of architecture [4]. The ratio of two successive terms in Fibonacci numbers converges to 1.618, which shows the relationship between Pascal's triangle and golden ratio [2, 5]. Pascal Triangle, Fibonacci numbers, and golden ratio have been attracted many attentions and have been wildly used in modern sciences, such as set theory [6], architecture [4], and engineering [7].

The connection between Pascal's triangle and entropy is an interesting topic. Based on the probability theory, Tsallis entropy is a generalization of Boltzmann-Gibbs statistics [8], whose relationship to Pascal's triangle is investigated in [9]. In 2016, Deng proposed a novel entropy, called Deng entropy [10], also named as belief entropy, which is an extension of Shannon entropy based on Dempster-Shafer evidence theory [11, 12]. Then, Kang and Deng presented the maximum Deng entropy [10], where the analytical solution of the maximum Deng entropy and its associated distribution are analyzed. Since the form of the maximum Deng entropy contains binomial coefficients, Deng entropy has many relations to Pascal's triangle. In 2019, Gao and Deng studied the relation between Pascal's triangle and the maximum Deng entropy [13], which reveals that pseudo-Pascal's triangle can be obtained based on the distribution of the maximum Deng entropy. In 2021, for exploring the physical meaning of the power set in evidence theory, Song and Deng presented an explanation of power set from the view of Pascal's triangle and entropy [14], in which the relation between

Deng entropy and Pascal's triangle is also discussed.

On the one hand, Pascal's triangle can construct Fibonacci number sequence and further yield golden ratio. On the other hand, Pascal's triangle has much relations to maximum Deng entropy. A straightforward idea comes: if we design a triangle based on maximum Deng entropy, what would the generated number sequence and the corresponding ratio be like?

Inspired by this idea, this paper proposes a triangle based on maximum Deng entropy, named as the maximum Deng entropy triangle (MDET). Besides, the number sequence based on MDET is investigated. MDET can yield two types of sequences, *i.e.*, the left and the right MDET sequence. Next, the general term for the MDET sequence is presented and proved. Based on the general term, the ratio for two successive terms in MDET sequence is analyzed. We prove that, under the condition of $n \to \infty$, the ratio in right MDET sequence converges to silver ratio $1 + \sqrt{2}$. Moreover, numerical examples are shown to expound the proposed MDET and the MDET sequence, where the MDET sequence is compared with several well-known number sequence, namely Fibonacci numbers, Pell numbers, and Jacobsthal numbers.

The rest of this article is as follows. Section 2 reviews the preliminaries. Section 3 proposes the MDET, and then analyzes its corresponding sequence and ratio. Section 4 shows some examples for illustration. Section 5 makes a brief conclusion.

2. Preliminaries

In this section, we briefly review some preliminaries of this paper.

2.1. Dempster-Shafer evidence theory

Dempster-Shafer evidence theory [11, 12] is an efficient tool for uncertainty modeling [10] and data fusion [15].

Definition 2.1 (Frame of discernment). *Let Θ, called the frame of discernment (FOD), denotes a fixed set of N mutually exclusive and exhaustive elements, indicated by $\Theta = \{\theta_1, \theta_2, \ldots, \theta_N\}$. The power set of Θ, denoted as 2^Θ, contains all possible subsets of Θ and has 2^N elements. 2^Θ is indicated by*

$$\begin{aligned} 2^\Theta =& \{A_1, A_2, \cdots, A_{2^N}\} \\ =& \{\emptyset, \{\theta_1\}, \{\theta_2\}, \cdots, \{\theta_N\}, \\ & \{\theta_1, \theta_2\}, \{\theta_1, \theta_3\}, \cdots, \{\theta_1, \theta_N\}, \cdots, \Theta\} \end{aligned} \tag{1}$$

Definition 2.2 (Mass function). *Given a FOD of Θ, a mass function, also called a basic probability assignment (BPA), is a mapping from 2^Θ to [0, 1], formally defined by:*

$$m : 2^\Theta \rightarrow [0, 1] \tag{2}$$

which satisfies $m(\emptyset) = 0$ and $\sum_{A \in 2^\Theta} m(A) = 1$.

2.2. Deng entropy and maximum Deng entropy

Deng entropy is the generalization of Shannon entropy, which is designed to measure the uncertainty in evidence theory [10]. Deng entropy is further developed into information volume [16], which has been applied in many fields, such as fuzzy information volume [17], higher-order information volume [18], pattern recognition [19], and fractal dimension [20].

Definition 2.3 (Deng entropy). *Given a certain mass function distribution defined on FOD Θ, Deng entropy is defined as:*

$$H_{DE}(m) = -\sum_{A \in 2^\Theta} m(A) \log\left(\frac{m(A)}{2^{|A|} - 1}\right) \tag{3}$$

where $|A|$ is the cardinal of a certain focal element A.

The maximum entropy is an important conception in statistic. The maximum Deng entropy and its corresponding distribution are summarized below [10].

Theorem 1. *The analytical solution of maximum Deng entropy is that*

$$H_{\max\text{-}DE} = \log \sum_{A \in 2^\Theta} (2^{|A|} - 1) \tag{4}$$

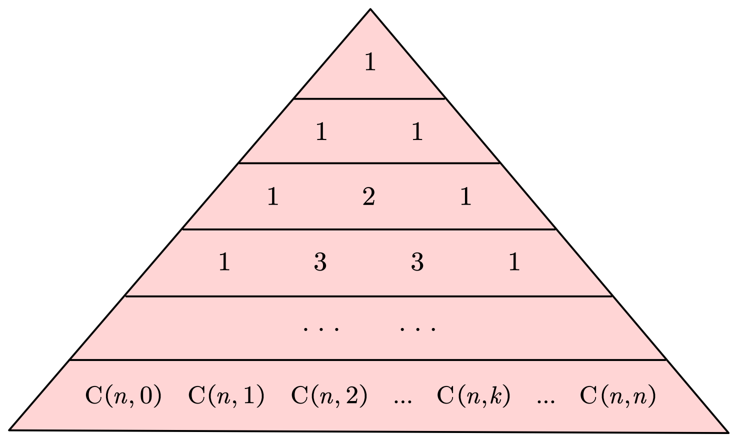

Figure 1: The illustration of Pascal's triangle

Theorem 2. *Let the frame of discernment be Θ. The maximum Deng entropy appears if and only if the mass function distribution satisfies*

$$m(A) = \frac{(2^{|A|}-1)}{\sum_{A\in 2^{\Theta}}(2^{|A|}-1)}, A \in 2^{\Theta} \tag{5}$$

2.3. Pascal's triangle

In mathematics, Pascal's triangle is a triangular array of the binomial coefficients that arises in probability theory [21]. As is illustrated in Figure 1, Pascal's triangle is a geometric arrangement of binomial coefficients in a triangle, which satisfies:

- The k-th number in the n-th row can be expressed as $C(n, x)$, meaning the combinatorial number of drawing x elements from n elements, where $n \geq 0$ and $0 \leq k \leq n$.
- In the $(n-1)$-th row of the triangle, the sum of the $(k-1)$-th and the k-th number equals to the k-th number in the n-th row, namely, $C(n, k) = C(n-1, k-1) + C(n-1, k)$.
- The sum of all the numbers in the n-th row is 2^n.

3. Silver ratio in maximum Deng entropy triangle

3.1. Maximum Deng entropy triangle

In this subsection, based on the maximum Deng entropy [10], we propose a novel triangle, called the maximum Deng entropy triangle (MDET). The definition of MDET is as follows:

Definition 3.1 (Maximum Deng entropy triangle). *In the maximum Deng entropy triangle (MDET), the k-th element of the n-th row is defined by:*

$$MDET(n,k) = C(n,k) * (2^k - 1) \tag{6}$$

where $C(n,k) = \frac{n!}{k!(n-k)!}$, $n \geq 0$, *and* $0 \leq k \leq n$.

For better understanding, the illustration of MDET is shown in Fig.2.

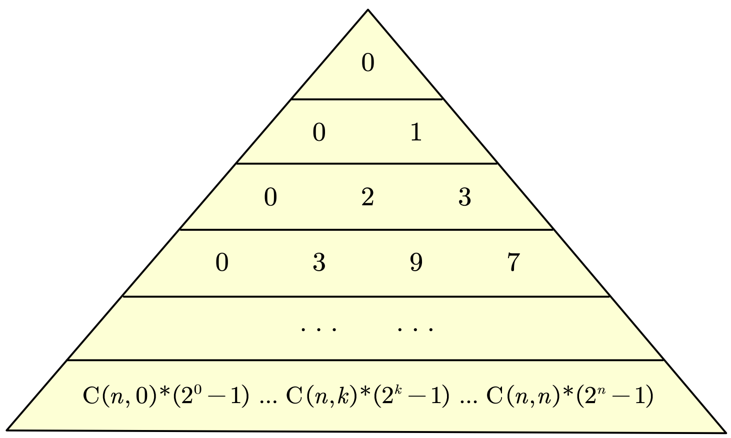

Figure 2: The illustration of MDET

3.2. MDET sequence

According to [2], Fibonacci number sequence can be derived from the diagonal sum of the entries in Pascal's triangle, which is illustrated in Fig. 3 (a). Inspired by this idea, we present the diagonal sum of MDET, so as to obtain number sequences based on MDET. It should be noted that, since MDET is asymmetric, there are two directions for the diagonal sum of MDET, namely the left diagonal sum and the right diagonal sum, which are shown in Fig. 3 (b) and (c).

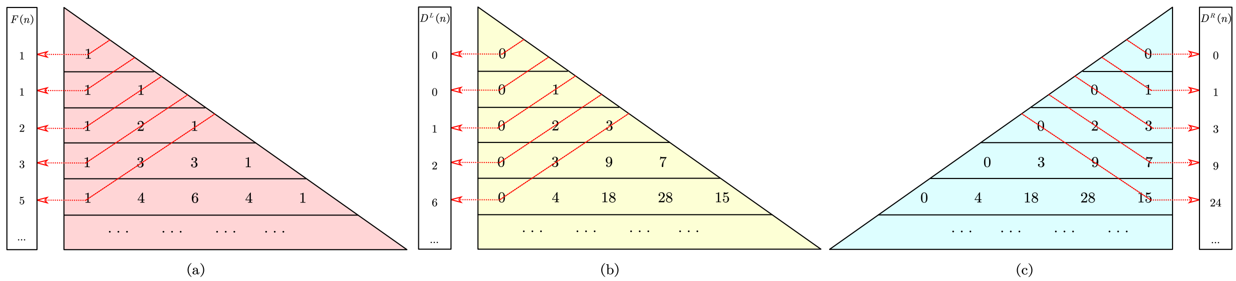

Figure 3: (a) The diagonal sum of Pascal's triangle and its generated Fibonacci numbers $F(n)$. (b) The left diagonal sum of MDET and its generated left MDET sequence $D^L(n)$. (c) The right diagonal sum of MDET and its generated right MDET sequence $D^R(n)$.

Then, based on these two directions of diagonal sum of MDET, two types of number sequences are derived, called left MDET sequence and right MDET sequence. In specific, as is shown in Fig. 3 (b) and (c), the left (right) MDET sequence are respectively generated based on the left (right) diagonal sum of MDET. These two types of sequences are mathematically defined as below.

Definition 3.2 (MDET sequence). *Based on the left and right diagonal sum of MDET, the left MDET sequence $D^L(n)$ and the right MDET sequence $D^R(n)$ are defined by*

$$D^L(n) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor + 1} MDET(n-k, k) \tag{7}$$

$$D^R(n) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor + 1} MDET(n-k, n-k) \tag{8}$$

where $\lfloor \cdot \rfloor$ is the floor function, which takes a real number as an input and returns the greatest integer less than or equal to that number.

3.3. General term of MDET sequence

In this subsection, the general terms for the two types of MDET sequences are investigated. If Eq. (6) is substituted into Eqs. (7) and (8), then the recurrence relations of left MDET sequence and that of right MDET sequence can be obtained:

$$\begin{aligned} D^L(n) =& 2D^L(n-1) + 2D^L(n-2) - 3D^L(n-3) - 2D^L(n-4) \qquad (9) \\ & \textit{with } D^L(0) = D^L(1) = 0, D^L(2) = 1, D^L(3) = 2 \\ D^R(n) =& 3D^R(n-1) - 3D^R(n-3) - D^R(n-4) \qquad (10) \\ & \textit{with } D^R(0) = 0, D^R(1) = 1, D^R(2) = 3, D^R(3) = 9 \end{aligned}$$

Based on the two recurrence relations, the general terms for the two types of MDET sequences can be derived:

Theorem 3. *The general term of left MDET sequence is as follows:*

$$D^L(n) = \frac{1}{3}\left[2^{n+1} - (-1)^{n+1}\right] - \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right] \qquad (11)$$

Theorem 4. *The general term of right MDET sequence is as follows:*

$$D^R(n) = \frac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)^{n+1} - \left(1-\sqrt{2}\right)^{n+1}\right] - \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right] \qquad (12)$$

Proof 3.1 (Proof for Theorem 3).

The corresponding characteristic equation of Eq. (9) is that

$$\varphi^4 - 2\varphi^3 - 2\varphi + 3\varphi + 2 = 0 \qquad (13)$$

from which the characteristic roots can be solved:

$$\varphi_1 = 2, \varphi_2 = -1, \varphi_3 = \frac{1+\sqrt{5}}{2}, \varphi_4 = \frac{1-\sqrt{5}}{2} \qquad (14)$$

Based on the roots, the general term of left MDET sequence can be constructed:

$$D^L(n) = c_1 2^n + c_2(-1)^n + c_3\left(\frac{1+\sqrt{5}}{2}\right)^n + c_4\left(\frac{1-\sqrt{5}}{2}\right)^n \tag{15}$$

Substitute the initial values $D^L(0) = D^L(1) = 0, D^L(2) = 1, D^L(3) = 2$, and then calculate the constants:

$$c_1 = \frac{1}{3},\ c_2 = -\frac{1}{3},\ c_3 = -\frac{1}{\sqrt{5}},\ c_4 = \frac{1}{\sqrt{5}} \tag{16}$$

Hence, the general term of left MDET sequence can be obtained.

□

Proof 3.2 (Proof for Theorem 4).

The associated characteristic equation of Eq. (10) is that

$$\delta^4 - 3\delta^3 + 3\delta + 1 = 0 \tag{17}$$

from which the characteristic roots can be solved:

$$\delta_1 = 1+\sqrt{2}, \delta_2 = 1-\sqrt{2}, \delta_3 = \frac{1+\sqrt{5}}{2}, \delta_4 = \frac{1-\sqrt{5}}{2} \tag{18}$$

Based on the roots, the general term of right MDET sequence can be constructed:

$$D^R(n) = c_1\left(1+\sqrt{2}\right)^n + c_2\left(1-\sqrt{2}\right)^n + c_3\left(\frac{1+\sqrt{5}}{2}\right)^n + c_4\left(\frac{1-\sqrt{5}}{2}\right)^n \tag{19}$$

Substitute the initial values $D^R(0) = 0, D^R(1) = 1, D^R(2) = 3, D^R(3) = 9$, and then get the constants:

$$c_1 = \frac{1}{2\sqrt{2}},\ c_2 = -\frac{1}{2\sqrt{2}},\ c_3 = -\frac{1}{\sqrt{5}},\ c_4 = \frac{1}{\sqrt{5}} \tag{20}$$

As a result, the general term of right MDET sequence can be obtained.

□

Remark 1: Based on Theorem 3 and Eqs. (**??**) (**??**), the left MDET sequence can be calculated by Jacobsthal numbers $J(n)$ and Fibonacci numbers $F(n)$:

$$D^L(n) = J(n+1) - F(n+1),\ n \geq 0. \tag{21}$$

Remark 2: According to Theorem 4 and Eqs. (**??**) (**??**), the right MDET sequence can be calculated based on Pell numbers $P(n)$ and Fibonacci numbers $F(n)$:

$$D^R(n) = P(n+1) - F(n+1),\ n \geq 0. \tag{22}$$

3.4. Silver ratio in MDET

In this subsection, we attempt to investigate the silver ratio in MDET. Based on Theorems 3 and 4, the ratio for two successive terms in the left MDET sequence and that in the right MDET sequence satisfy the following theorems:

Theorem 5. *When $n \to \infty$, the ratio for two successive terms in the left MDET sequence converges to:*

$$\lim_{n\to\infty} \frac{D^L(n)}{D^L(n-1)} = 2 \tag{23}$$

Theorem 6. *When $n \to \infty$, the ratio for two successive terms in the right MDET sequence converges to:*

$$\lim_{n\to\infty} \frac{D^R(n)}{D^R(n-1)} = 1 + \sqrt{2} \tag{24}$$

Proof 3.3 (Proof for Theorem 5).
Based on Theorem 3, the ratio can be calculated as

$$\lim_{n\to\infty} \frac{D^L(n)}{D^L(n-1)} = \lim_{n\to\infty} \frac{\frac{1}{3}\left[2^{n+1} - (-1)^{n+1}\right] - \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]}{\frac{1}{3}\left[2^{n} - (-1)^{n}\right] - \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n}\right]} \tag{25}$$

Since $\varphi_1 = 2$ is the largest characteristic root among the four roots, the ratio can be written as:

$$\lim_{n\to\infty} \frac{D^L(n)}{D^L(n-1)} \to \lim_{n\to\infty} \frac{\frac{1}{3}\left(2^{n+1}\right)}{\frac{1}{3}\left(2^{n}\right)} = 2 \tag{26}$$

□

Proof 3.4 (Proof for Theorem 6).

Based on Theorem 4, the ratio can be calculated as

$$\lim_{n\to\infty}\frac{D^R(n)}{D^R(n-1)} = \lim_{n\to\infty}\frac{\frac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)^{n+1}-\left(1-\sqrt{2}\right)^{n+1}\right]-\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]}{\frac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)^{n}-\left(1-\sqrt{2}\right)^{n}\right]-\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n}-\left(\frac{1-\sqrt{5}}{2}\right)^{n}\right]} \tag{27}$$

Since $\delta_1 = 1+\sqrt{2}$ is the largest characteristic root among the four roots, the ratio can be written as:

$$\lim_{n\to\infty}\frac{D^R(n)}{D^R(n-1)} \to \lim_{n\to\infty}\frac{\frac{1}{2\sqrt{2}}\left(1+\sqrt{2}\right)^{n+1}}{\frac{1}{2\sqrt{2}}\left(1+\sqrt{2}\right)^{n}} = 1+\sqrt{2} \tag{28}$$

□

Remark 3: Based on Theorem 5, under the condition of $n \to \infty$, the ratio in the left MDET sequence is the same as the ratio in Jacobsthal numbers, which is 2.

Remark 4: According to Theorem 6, when $n \to \infty$, the ratio in the right MDET sequence is the same as the ratio in Pell numbers, which is actually the silver ratio $1+\sqrt{2}$.

4. Examples and discussions

In this section, some numerical examples are shown to illustrate the presented maximum Deng entropy triangle (MDET) as well as the left and the right MDET sequences.

Example 4.1. *Assume that $n = 0, 1, 2, 3, ..., 9$. Then, the associated Pascal's triangle and MDET are shown in Tables 1 and 2. It can be seen from the tables that Pascal's triangle has only one type of diagonal sum since it is symmetric. By contrast, because MDET is asymmetric, there exists two types of diagonal sum of MDET. Based on the diagonal sums, Fibonacci numbers and two MDET sequences can be obtained, which are shown in Tables 1 and 2. As shown in the tables, there are two different kinds of MDET sequences because of the two types of diagonal sum of MDET.*

Table 1: Pascal's triangle and Fibonacci numbers $F(n)$

n	$F(n)$	Pascal's triangle									
0	1	1									
1	1	1	1								
2	2	1	2	1							
3	3	1	3	3	1						
4	5	1	4	6	4	1					
5	8	1	5	10	10	5	1				
6	13	1	6	15	20	15	6	1			
7	21	1	7	21	35	35	21	7	1		
8	34	1	8	28	56	70	56	28	8	1	
9	55	1	9	36	84	126	126	84	36	9	1

Table 2: MDET, left MDET sequence $D^L(n)$, and right MDET sequence $D^R(n)$

n	$D^L(n)$	$D^R(n)$	Maximum Deng entropy triangle									
0	0	0	0									
1	0	1	0	1								
2	1	3	0	2	3							
3	2	9	0	3	9	7						
4	6	24	0	4	18	28	15					
5	13	62	0	5	30	70	75	31				
6	30	156	0	6	45	140	225	186	63			
7	64	387	0	7	63	245	525	651	441	127		
8	137	951	0	8	84	392	1050	1736	1764	1016	255	
9	286	2323	0	9	108	588	1890	3906	5292	4572	2295	511

Example 4.2. *Let n be from 0 to 10. Then, the illustrations of the trend for Pell numbers $P(n)$, Fibonacci numbers $F(n)$, Jacobsthal numbers $J(n)$, left MDET sequence $D^L(n)$, and right MDET sequence $D^R(n)$ are shown in Fig. 4. This example shows that Pell number sequence has the fastest growth rate among the five sequences because it has the largest characteristic root $\varphi_P = 1 + \sqrt{2}$. The growth rate of Fibonacci number sequence is the lowest, since the biggest characteristic root for Fibonacci numbers is $\varphi_F = \frac{1+\sqrt{5}}{2}$, which is much lower than the roots of other sequence.*

Example 4.3. *Given n from 0 to 100, the values of the ratio for two successive terms in Pell numbers $P(n)$, Fibonacci numbers $F(n)$, Jacobsthal numbers $J(n)$, left MDET sequence $D^L(n)$, and right MDET sequence $D^R(n)$ are shown in Fig. 5. It can be seen from the figure that, when n becomes more and more larger, the ratio for each sequence finally converges to a certain value. The ratio of $P(n)$ and $D^R(n)$ are the same. Both of them are silver ratio. The*

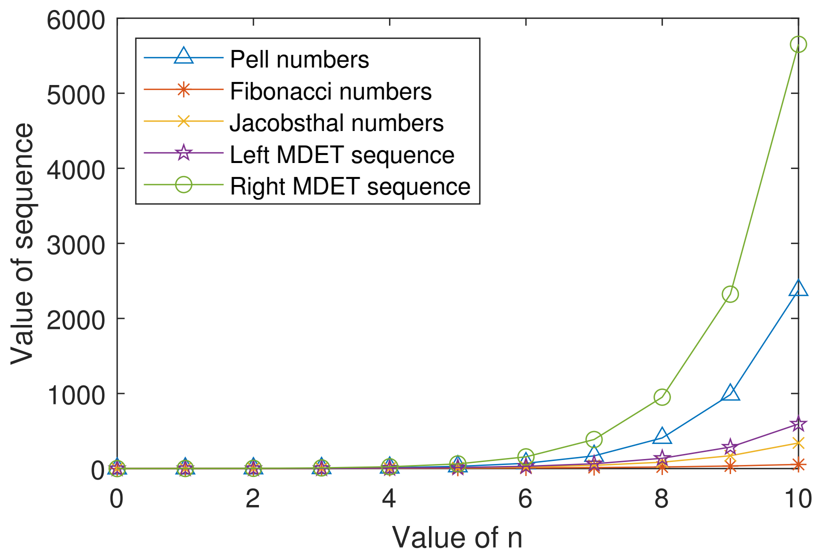

Figure 4: The trends for different types of number sequences

ratio of $J(n)$ equals to that of $D^L(n)$, which is 2. The ratio of $F(n)$ is the well-known golden ratio.

5. Conclusion

Pascal's triangle is a mathematical triangular form of binomial coefficients, from which Fibonacci number sequence and golden ratio can be obtained. Recently, the relations between Pascal's triangle and maximum Deng entropy are studied and presented. A straightforward idea comes: if we design a triangle based on maximum Deng entropy, what would the generated number sequence and the corresponding ratio be like? Hence, this paper proposes a triangle based on maximum Deng entropy, named as the maximum Deng entropy triangle (MDET). Besides, the number sequences based on MDET are investigated. Next, the general term for the MDET sequence is presented and the ratio in MDET sequence is analyzed. We prove that, under the condition of $n \to \infty$, the ratio in right MDET sequence converges to silver

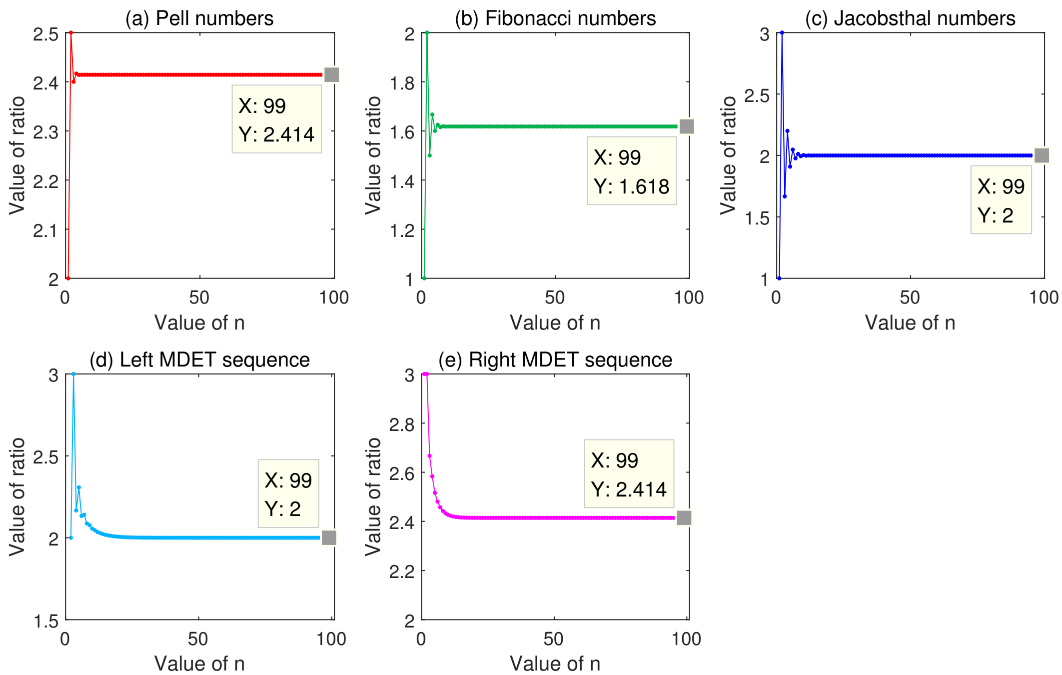

Figure 5: The ratio in different types of number sequences

ratio $1 + \sqrt{2}$. Moreover, numerical examples are shown to expound the proposed MDET and the MDET sequence.

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