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The Role of the DS/AHP in Identifying Inter-Group Alliances and Majority Rule Within Group Decision Making

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Abstract

DS/AHP is a nascent method of multi-criteria decision-making, based on the Dempster-Shafer theory of evidence and indirectly the Analytic Hierarchy Process. It is concerned with the identification of the levels of preference that decision makers have towards certain decision alternatives (DAs), through preference judgements made over a number of different criteria. The working result from a DS/AHP analysis is the body of evidence (BOE), which includes a series of mass values that represent the exact beliefs in the best DA(s) existing within certain subsets of DAs. This paper considers the role of DS/AHP as an aid to group decision-making, through the utilisation of a distance measure (between BOEs). Here, the distance measure enables the identification of the members of the decision-making group who are in most agreement, with respect to the judgements they have individually made. The utilisation of a single linkage dendrite approach to clustering elucidates an appropriate order to the aggregation of the judgements of the group members. This develops the DS/AHP method as a tool to identify inter-group alliances, as well as introduce a 'majority rule' approach to decision-making through consensus building.

Key words: consensus building, Dempster-Shafer theory, group decision-making, ignorance, majority rule

1. Introduction

The basis for making a decision is the ability of a decision maker (DM) to undertake preference judgements on a number of different decision alternatives (DAs). Within multi-criteria decision-making (MCDM), the problem is compounded with judgements necessary over different criteria, where the respective preferential orders of the DAs may be different. As a research problem, the elucidation of MCDM and in particular the creation of effective tools to aid in its action is an often-investigated issue (Bana E. Costa et al. 1997; Mehrez 1997). A further level of complication occurs when MCDM is incumbent within a group decision-making environment (GDM). That is, while an individual DM makes their individual judgements, a process to aggregate the judgements from a number of individuals into a group decision is often enforced on the group (Hinsz 1999).

When included in a GDM environment, the problem of MCDM has similarly been investigated (Forman and Peniwati 1998; Bryson and Joseph 1999). At the theoretical level, this includes how the group dynamics and philosophy are realistically modelled (Belton and Pictet 1997). For example, if final decisions are based on 'majority rule' consensus building and the identification of possible inter-group alliances (Hinsz 1999). Also, whether the group should be interactive or non-interactive in their decision-making (Hollingshead 1996; Armacost et al. 1999). At the technical level, tools in the form of group decision support systems are regularly developed (GDSS), which may adhere to specific group dynamics and philosophies (Matsatsinis and Samaras 2001). The understanding and inclusion of concomitant uncertainty in MCDM and GDM has also been investigated (Roy 1989; Fox and Tversky 1998; Ozdemir and Saaty 2005).

A well-known method of MCDM is the Analytic Hierarchy Process – AHP (Saaty 1977, 1980), which has developed into a versatile decision support system for DMs (see Bard 1992; Wasil and Golden 2003). It is then no surprise that the AHP itself has been developed as a GDSS (Armacost et al. 1999). However, there has grown no definitive approach for its use in MCDM and especially with respect to GDM, including the method of aggregation of the judgements of the individual members of the group (Ramanathan and Ganesh 1994; Van Den Honert and Lootsma 1996). SMART is another MCDM technique derived from the general multi-attribute utility theory, which constructs a ranking of DAs through the simple weighted aggregation of preferences given by a DM (Von Winterfeldt and Edwards 1986).

DS/AHP is a nascent method of MCDM (Beynon et al. 2000; Beynon 2002), whose approach to problem structuring is inspired by the AHP, but its inherent analytic processes are based around the Dempster-Shafer theory of evidence – DST (Dempster 1968; Shafer 1976). It is associated with MCDM in the presence of ignorance and non-specificity (Beynon 2005a), encompassing the notions of incompleteness, imprecision and uncertainty (Bonissone and Tong 1985; Smets 1991). DS/AHP allows a DM to identify subsets of DAs (over different criteria), based on their preference when compared with all the DAs available to be considered (the frame of discernment). For each criterion, the identified subsets of DAs and frame of discernment together with the respective exact belief (mass) values in their preference make up a body of evidence (BOE). It follows, a series of criterion BOEs are constructed from the judgements made by a DM over the different criteria. Since its introduction DS/AHP has been compared with AHP and SMART (see Beynon et al. 2000; Beynon 2005a).

As a tool in a GDM environment, DS/AHP offers a number of advantages (over AHP). This includes the homogeneity of the aggregation of the judgements made over the different criteria for a single group member and the aggregation of the judgements of group members into a group decision. That is, the series of criterion BOEs for a single DM can be combined into an individual BOE (using a combination rule). Then the individual BOEs from the group members can be combined into a group BOE (using the same combination rule), which contains the evidence for a final group decision to be made. While a specific iterative order of aggregation of BOEs is unnecessary (the same individual or group BOE obtained irrespectively), a distance measure between two BOEs is utilised to elucidate an order (Jousselme et al. 2001). Its inclusion allows an understanding to a consensus building approach to GDM using DS/AHP, as well as the elucidation of inter-group alliances.

The desire to explicitly define the mechanism (order) to achieve a final decision highlights the possible existence of conflict within a group. That is, conflict between group members may be due to different ideological beliefs, interpersonal differences and goal incongruities (Noori 1995). A reaction to conflict within a decision making group has often been with a dictatorship to achieve a final decision (Lei and Youmin 1996; Hinsz 1999). DeSanctis and Gallupe (1987) highlight the possible existence of; dominance by one or more individuals in a group, extreme influence of high-status members, lack of acknowledgement of opinions of low-status members and low tolerance of minority or controversial opinions. Cartwright and Zander (1968) identify, as group membership increases so consensus becomes harder to achieve. With respect to the AHP, Lai et al. (2002) found it conducive to consensus building in GDM.

In DS/AHP, a distance measure between criterion BOEs or individual BOEs represents a degree of similarity/consensus in the judgements made on the different criteria by a DM or between DMs, respectively. The notion of a distance measure has been previously considered within MCDM, spearheaded by Kemeny and Snell (1962). Carlsson et al. (1992) constructs a topological measure of the metric distance between different rankings of the same set of DAs, which utilises the AHP. Ray and Triantaphyllou (1998) acknowledge the possibility of disagreements in GDM, and the need to better understand the magnitude of disagreements in a hierarchical group of DMs. Xu et al. (2001) construct an MCDM ranking procedure based on the distance between partial pre-orders of DAs, they include the descriptions of other measures including; intensity of divergence (Roy and Slowinski 1993) and disagreement measure (Ben Khélifa and Martel 2001). See also Cook and Kress (1985), Ali et al. (1986) and Bryson (1996) for further information regarding such distance measures.

Using DS/AHP, the iterative aggregation of individual BOEs to a group BOE confers clusters (coalitions) of individuals within which their opinions have been combined. Moreover, using a single linkage dendrite approach to clustering, it allows a visual interpretation to the order of the combination of individual BOEs to confer a group decision. The resultant dendogram elucidates the possible existence of inter-group alliances of DMs. Considered another way, the addition of an individual to an existing cluster of individuals is based on the least compromise that would be necessary between the individual DM's judgements and that of the cluster in question (of DMs).

Hinsz (1999) exposits GDM with preference judgements through the theory of social decision schemes (with quantitative responses), it is argued that the group decision process would be characterised by compromise, with group members changing their position to that adopted by the group (for which their evidence has contributed to). The notion of a 'majority decision' is considered a powerful tool in GDM (*ibid.*), and a minimum majority decision rule utilised to find the required majority of group members (see also Murnighan 1978; Davis et al. 1997). Within DS/AHP, the distance measure defined is used to identify group members which would require the least compromise between their judgements, consequently building a majority decision (or plurality decision, see Hinsz 1999). Moreover, an existing cluster of individual DMs is identified which has an associated intermediate group BOE. Based on this intermediate group BOE final results may be identified that are different to those from a simple combination of the individual BOEs associated with all the group members.

The structure of the rest of the paper is as follows. Section 2 describes the DS/AHP method of MCDM through a large example problem. Section 3 investigates the distance measure between BOEs, including the utilisation of a dendogram. Section 4 considers the distance measure between individual BOEs in identifying inter-group alliances and a final group decision through a 'majority rule' approach. In Section 5, conclusions are presented as well as directions for future research. In Appendix A, details of DST necessary in the analysis in this paper are presented and illustrated through an example.

2. The DS/AHP Method of MCDM

This section undertakes an exposition of the DS/AHP method of MCDM, including its analytic processes based on DST (for those readers not familiar with DST, Appendix A presents relevant details). With a normative based exposition of DS/AHP throughout this study an illustrative problem is utilised (Bell et al. 1988). There are six DMs considered ($\mathrm{DM}_1, \mathrm{DM}_2, \ldots, \mathrm{DM}_6$), each have made a series of preference judgements on three criteria, labelled X, Y and Z, associated with a number of DAs to a specific focus – Best DA(s). The preference judgements are on 12 DAs, labelled A, B, C, D, E, F, G, H, I, J, K and L, which within DS/AHP make up the frame of discernment Θ.

Due to the size of this problem, a seven unit scale was utilised ($\nu_1 = 2, \nu_2 = 3, \ldots, \nu_7 = 8$), defined over the verbal statement range of moderately to extremely preferred (see Beynon 2002). These statements of preference are employed on identified subsets of DAs with respect to their comparison to all the DAs considered (Θ). A series of judgements hypothetically made by the six DMs are reported in Figure 1. These judgements illustrate, using DS/AHP, a DM can make weak (partial) pre-orderings of DAs (in terms of preference) over the different criteria (Ben Khélifa and Martel 2001). This includes that not all DAs are necessary to be included in the identified subsets of DAs given distinct levels of preference. These facets of the DS/AHP contribute to the antecedents of ignorance inherent in this approach (as noted in Ozdemir and Saaty 2005). Beynon (2005b) identify these facets as advantages associated with the use of DS/AHP, especially when the DM has doubt/hesitancy on their judgement making due to the consequential nature of a problem considered (see Lipshitz and Strauss 1997).

To describe the judgements reported in Figure 1, the DM_1 has identified three subsets of DAs over the X criterion, namely $\{B, F\}$, $\{C, H, K\}$ and $\{G, L\}$, which are assigned the scale values 7, 4 and 2, respectively. For the case of $\{G, L\}$ assigned the scale value 2, it would have been judged moderately preferred when compared to all the DAs considered (the frame of discernment Θ). Importantly, each identified preference is to a whole subset of DAs and not to the specific individual DAs in the subset. The necessary criteria priority values (CPVs), must be evaluated before a series of bodies of evidence are constructed (BOEs, see Appendix A), which represent the judgements on the criteria by a DM. A CPV represents the weight of knowledge/importance each criterion is perceived to have by a DM with respect to the decision problem considered (as in the AHP and SMART, these CPVs sum to one for each DM). In this hypothetical problem the CPV associated with each criterion and DM are presented in Figure 1 (for DM_1 the criteria X, Y and Z have the CPVs, 0.3, 0.5 and 0.2).

Beynon (2002) showed that if d focal elements (subsets of DAs) $s_1, s_2, \ldots, s_d$ are identified on a criterion with the respective scale values $a_1, a_2, \ldots, a_d$, then the associated

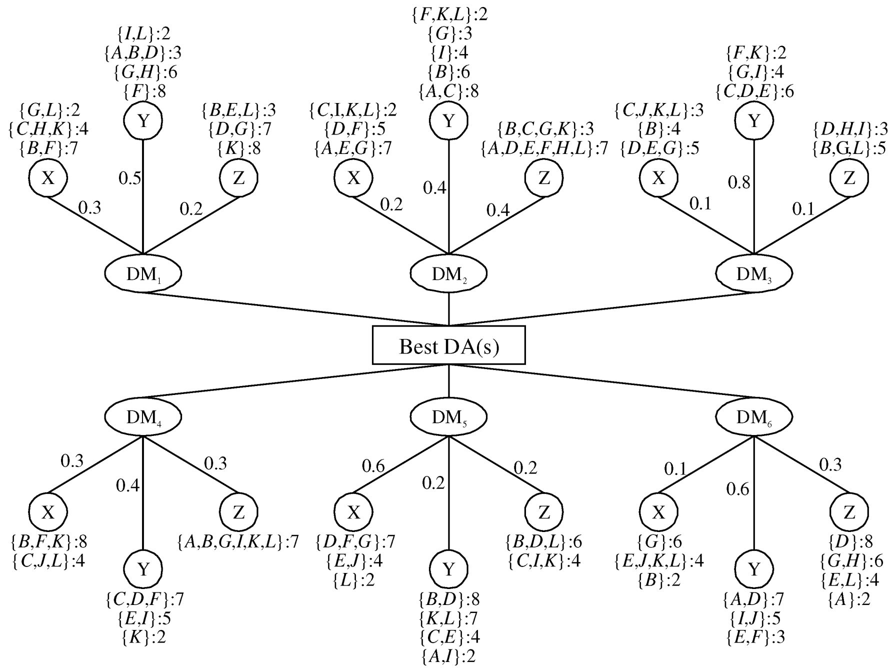

Figure 1. Preference judgements made by $DM_1, DM_2, \ldots, DM_6$.

criterion BOE, defined $m(\cdot)$, is made up of a series of mass values given by:

$$m(s_i) = \frac{a_i p}{\sum_{j=1}^{d} a_j p + \sqrt{d}}, \quad i = 1, 2, \ldots d \quad \text{and} \quad m(\Theta) = \frac{\sqrt{d}}{\sum_{j=1}^{d} a_j p + \sqrt{d}},$$

where Θ is the frame of discernment and p is the associated CPV. These expressions represent the preference values between each subset of DAs and Θ, and not between subsets of DAs. That is, no effort is made to explicitly utilise comparative preferences between the identified subsets of DAs, and is a factor of the incompleteness facet of ignorance inherent in DS/AHP (as highlighted in Section 1). It follows, criterion BOEs can be constructed for all the criteria the six DMs have separately made preference judgements on. In the case of DM_1 (from Figure 1), for the X criterion with a general CPV $p_{1,X}$, $s_1 = \{B, F\}$, $s_2 = \{C, H, K\}$ and $s_3 = \{G, L\}$ also $a_1 = 7$, $a_2 = 4$ and $a_3 = 2$, then the associated criterion BOE, defined $m_{1,X}(\cdot)$, is made up of the mass values:

$$m_{1,X}(s_1) = m_{1,X}(\{B, F\}) = \frac{7p_{1,X}}{13p_{1,X} + \sqrt{3}}, \quad m_{1,X}(\{C, H, K\}) = \frac{4p_{1,X}}{13p_{1,X} + \sqrt{3}},$$

$$m_{1,X}(\{G, L\}) = \frac{2p_{1,X}}{13p_{1,X} + \sqrt{3}} \quad \text{and} \quad m_{1,X}(\Theta) = \frac{\sqrt{3}}{13p_{1,X} + \sqrt{3}}.$$

Similar criterion BOEs can be constructed from the judgements made on the other criteria Y and Z. With the general CPVs $p_{1,\mathrm{Y}}$ and $p_{1,\mathrm{Z}}$, their criterion BOEs $m_{1,\mathrm{Y}}(\cdot)$ and $m_{1,\mathrm{Z}}(\cdot)$ are:

$$m_{1,\mathrm{Y}}(\{F\}) = \frac{8p_{1,\mathrm{Y}}}{19p_{1,\mathrm{Y}} + \sqrt{4}}, \quad m_{1,\mathrm{Y}}(\{G, H\}) = \frac{6p_{1,\mathrm{Y}}}{19p_{1,Y} + \sqrt{4}},$$
$$m_{1,\mathrm{Y}}(\{A, B, D\}) = \frac{3p_{1,\mathrm{Y}}}{19p_{1,\mathrm{Y}} + \sqrt{4}}, \quad m_{1,\mathrm{Y}}(\{I, L\}) = \frac{2p_{1,\mathrm{Y}}}{19p_{1,\mathrm{Y}} + \sqrt{4}} \quad \text{and}$$
$$m_{1,\mathrm{Y}}(\Theta) = \frac{\sqrt{4}}{19p_{1,\mathrm{Y}} + \sqrt{4}},$$

and

$$m_{1,\mathrm{Z}}(\{K\}) = \frac{8p_{1,\mathrm{Z}}}{18p_{1,\mathrm{Z}} + \sqrt{3}}, \quad m_{1,\mathrm{Z}}(\{D, G\}) = \frac{7p_{1,\mathrm{Z}}}{18p_{1,\mathrm{Z}} + \sqrt{3}},$$
$$m_{1,\mathrm{Z}}(\{B, E, L\}) = \frac{3p_{1,\mathrm{Z}}}{18p_{1,\mathrm{Z}} + \sqrt{3}} \quad \text{and} \quad m_{1,\mathrm{Z}}(\Theta) = \frac{\sqrt{3}}{18p_{1,\mathrm{Z}} + \sqrt{3}}.$$

Using the specific CPVs for each of these criteria given in Figure 1, $p_{1,\mathrm{X}} = 0.3$, $p_{1,\mathrm{Y}} = 0.5$ and $p_{1,\mathrm{Z}} = 0.2$, then:

$$m_{1,X}(\{B, F\}) = 0.3729, \quad m_{1,\mathrm{X}}(\{C, H, K\}) = 0.2131,$$
$$m_{1,\mathrm{X}}(\{G, L\}) = 0.1065 \quad \text{and} \quad m_{1,\mathrm{X}}(\Theta) = 0.3075,$$
$$m_{1,\mathrm{Y}}(\{F\}) = 0.3478, \quad m_{1,\mathrm{Y}}(\{G, H\}) = 0.2609, \quad m_{1,\mathrm{Y}}(\{A, B, D\}) = 0.1304,$$
$$m_{1,\mathrm{Y}}(\{I, L\}) = 0.0870 \quad \text{and} \quad m_{1,\mathrm{Y}}(\Theta) = 0.1739,$$
$$m_{1,\mathrm{Z}}(\{K\}) = 0.3001, \quad m_{1,\mathrm{Z}}(\{D, G\}) = 0.2626,$$
$$m_{1,\mathrm{Z}}(\{B, E, L\}) = 0.1125 \quad \text{and} \quad m_{1,\mathrm{Z}}(\Theta) = 0.3248.$$

To elucidate the role played by a CPV, the respective mass values in the criterion BOEs $m_{1,\mathrm{X}}(\cdot)$, $m_{1,\mathrm{Y}}(\cdot)$ and $m_{1,\mathrm{Z}}(\cdot)$ are presented in Figure 2, for when $p_{1,\mathrm{X}}$, $p_{1,\mathrm{Y}}$ and $p_{1,\mathrm{Z}}$ go from 0 to 1 (from none to a good level of knowledge/importance).

In Figure 2, as a CPV increases in value the local ignorance values $m_{1,\mathrm{X}}(\Theta)$, $m_{1,\mathrm{Y}}(\Theta)$ and $m_{1,\mathrm{Z}}(\Theta)$ all decrease, and the mass values (preference) of the identified subsets of DAs all increase. This asymmetric relationship between the levels of ignorance and preference of subsets of DAs is expected as a CPV increases. For the three criteria X, Y and Z, with $p_{1,\mathrm{X}} = 0.3$, $p_{1,\mathrm{Y}} = 0.5$ and $p_{1,\mathrm{Z}} = 0.2$, the respective mass values evaluated previously are confirmed.

While the criterion BOEs can be further investigated, the intermediate goal to be achieved is the construction of an individual BOE for each DM, found from the combination of the respective three criterion BOEs. For DM_1, using Dempster's combination rule (see Appendix A), the three criterion BOEs $m_{1,\mathrm{X}}(\cdot)$, $m_{1,\mathrm{Y}}(\cdot)$ and $m_{1,\mathrm{Z}}(\cdot)$ are combined into an

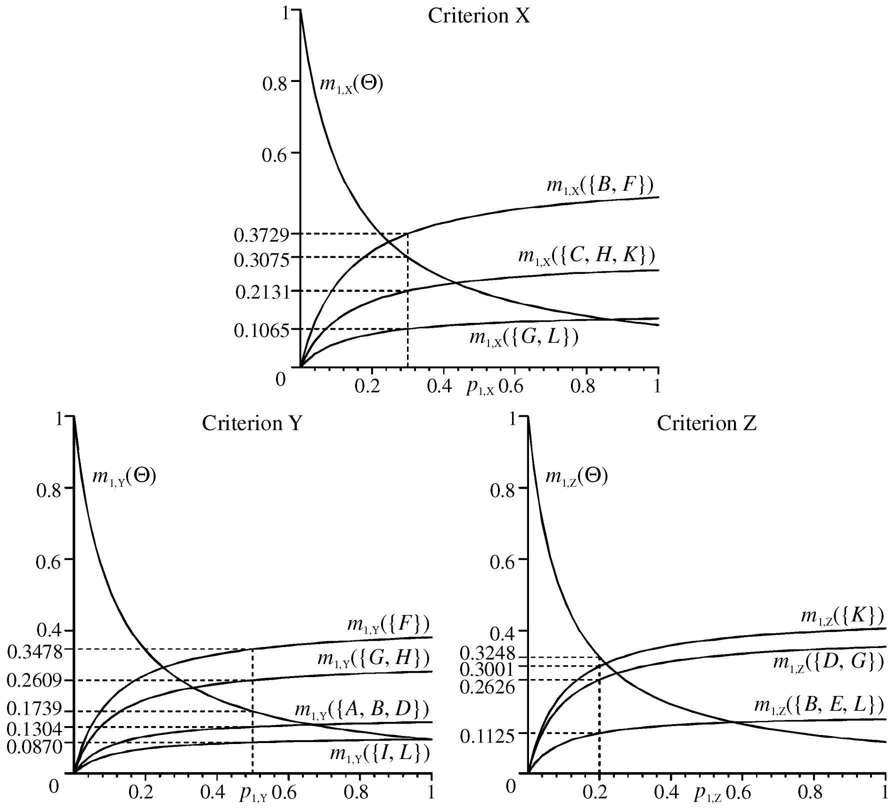

Figure 2. Graphs of mass values and focal elements on the X, Y and Z criteria for DM_1.

individual BOE, defined $m_1(\cdot)$. In Table 1, 16 different focal elements (including Θ) make up the individual BOE $m_1(\cdot)$, for example $m_1(\{B, F\}) = 0.0617$, implying the exact belief in the subset of DAs $\{B, F\}$ containing the best DA(s) is 0.0617 (the level of ignorance $m_1(\Theta)$ is 0.0509).

The information in Table 1 forms the basis for identifying the best DA(s) for a DM, the focus of this example. To consider the confidence in subsets of DAs containing the best DA(s), the belief and plausibility measures are utilised on the individual BOE $m_1(\cdot)$

Table 1. Individual subsets of DA and mass values which make up the individual BOE $m_1(\cdot)$.

$\{B\}$, 0.0969	$\{H\}$, 0.0529	$\{D, G\}$, 0.0411	$\{A, B, D\}$, 0.0382
$\{D\}$, 0.0308	$\{K\}$, 0.0796	$\{G, H\}$, 0.0763	$\{B, E, L\}$, 0.0176
$\{F\}$, 0.2251	$\{L\}$, 0.0268	$\{G, L\}$, 0.0176	$\{C, H, K\}$, 0.0353
$\{G\}$, 0.1237	$\{B, F\}$, 0.0617	$\{I, L\}$, 0.0254	Θ, 0.0509

Table 2. Subsets of DAs with largest *Bel* and *Pls* values from the individual BOE $m_1(\cdot)$.

Size of DA subset	Belief (*Bel*)	Plausibility (*Pls*)
1	$\{F\}$, 0.2251	$\{F\}$, 0.3377
2	$\{B, F\}$, 0.3837	$\{F, G\}$, 0.5966
3	$\{B, F, G\}$, 0.5075	$\{B, F, G\}$, 0.7492
4	$\{B, F, G, H\}$, 0.6367	$\{B, F, G, K\}$, 0.8640
5	$\{B, F, G, H, K\}$, 0.7162	$\{B, F, G, H, K\}$, 0.9169
6	$\{B, D, F, G, H, K\}$, 0.7882	$\{B, F, G, H, K, L\}$, 0.9692

constructed for DM_1 (see Appendix A). Since these measures can be found for each possible subset of Θ, Table 2 reports only those subsets of DAs that have the largest belief and plausibility values from all those subsets of DAs of the same size. With 12 DAs in the frame of discernment, identified subsets of size one to six are considered.

To illustrate the results in Table 2, if all subsets of DAs made up of three DAs are considered, the subset $\{B, F, G\}$ is identified as having the largest belief and plausibility values. These results highlight the use of the DS/AHP to identify a reduced number of DAs to possibly further consider. If there is the requirement to only identify the single best DA, the measures of belief and plausibility both indicate the DA F is best, based on all the judgements made by the DM_1. Similar results can be found for the other DMs; DM_2, $\text{DM}_3, \ldots, \text{DM}_6$, based on their series of judgements over the three different criteria (see later).

These results along with the original judgements made by the DM_1 allow an opportunity to relate the DS/AHP method to two of the prominent social choice axioms, centrepieces for judging the realism with decision-making tools (Arrow 1950, 1963). The first axiom considered is termed 'Pareto optimality', whereby if two pieces of evidence separately prefer the same DA to another, then the combined evidence should similarly prefer the identified preferred DA. In this example for DM_1, over the criteria X and Y the DA F is consistently more preferred than H, on the Z criterion neither F nor H are identified for preference (see Figure 1). The results in Table 2, indicate the combined evidence from all three criterion has identified the DA F is more preferred than the DA H. While only an example of the satisfaction of the Pareto optimality axiom, the small example in Appendix A (Table A1) illustrates Dempster's rule of combination is multiplicative in nature and will it is suspected satisfy generally this axiom.

The second axiom concerned is termed 'Independence of irrelevant DAs', whereby if a DA is eliminated from consideration, then the new group ordering for the remaining DAs should be equivalent to the original group ordering for the same DAs. Within DS/AHP, the notion of an irrelevant DA has two meanings; (i) it would not have been included in any identified subsets of DAs, hence its only presence is in Θ and the level of ignorance, or (ii) it was never included in the frame of discernment from the start. The interpretations to each of these notions of an irrelevant DA do not affect the identification of best DA(s) using DS/AHP. Further, the notion of an ordering of DAs is compounded here, with an ordering of same size subsets of DAs based on belief and plausibility measures the issue with DS/AHP.

The effect of an irrelevant DA is considered in the above example, concentrating on DM_1. If the DA B was removed from the judgements presented in Figure 1, the focal element $\{B\}$ which originally exists in the individual BOE $m_1(\cdot)$ would not be present (see Table 1), its associated mass value would be dissipated amongst the other mass values (including ignorance). Also, focal elements that include the DA B would have it removed. The effect of these changes on the identified subsets of DAs in Table 2 is that they are similar but with the DA B not present. The belief (*Bel*) based results remain consistent, namely $\{F, G\}$, $\{F, G, H\}$ etc., become the best subsets of DAs of size two, three etc., found from the rows below these sizes in Table 2 (with the DA B removed). Similar consistency was found when the other DAs were individually removed, indicating for this example using the belief measure, the 'Independence of irrelevant DAs' axiom holds.

In the case of the plausibility (*Pls*) measure, similar consistency was evident for low sized identified best subsets of DAs. However, when not considering the DA B, the best subset of four DAs would be $\{F, G, K, L\}$, but from Table 2 the best subset of five DAs is $\{B, F, G, H, K\}$, these last two subsets of DAs are not the same when B is ignored. In this example, the inclusion of the DA L rather than H is due to the $\{B, E, L\}$ focal element included in $m_1(\cdot)$ (see Table 1). It contributes to the $Pls(\{B, F, G, H, K\})$ value, with B removed it does not contribute to $Pls(\{F, G, H, K\})$ but still contributes to $Pls(\{F, G, K, L\})$, hence this latter focal elements' identification as the best subset of four DAs based on plausibility.

These initial findings illustrate how two of the well-known social choice axioms can be discussed with respect to DS/AHP. Here, a concern has been expressed on the validity of the 'Independence of irrelevant DAs' axiom when the plausibility measure is considered. However, the ranking issue considered is not the simple ranking of the single DAs, but the identification of subsets of DAs of different sizes. Hence, the general relevance of these arguments presented should not be taken too formally, but considered a start towards the explicit validity of these axioms with respect to DS/AHP. Indeed, it is a natural direction for future research in this area.

3. Utilisation of the Distance Measure within DS/AHP

The distance measure $d_{\mathrm{BOE}}(\cdot, \cdot)$ described in Appendix A quantifies the level of similarity between pairs of BOEs. Here, this distance measure is applied to the criterion BOEs constructed from the judgements made by DM_1. With three criteria considered by DM_1, it is possible to find the respective distances between pairs of the three criterion BOEs $m_{1,\mathrm{X}}(\cdot)$, $m_{1,\mathrm{Y}}(\cdot)$ and $m_{1,\mathrm{Z}}(\cdot)$. That is, how much in agreement are the judgements (identified subsets of DAs and mass values), made by DM_1 over any two criteria. This starts with consideration of the effect on the distance between two criterion BOEs, as their respective CPVs change.

Since the CPVs sum to one (for a single DM), for any two criterion BOEs their respective CPVs should sum to less than or equal to one. In the case of the criteria X and Y, their associated distance measure $d_{\mathrm{BOE}}(m_{1,\mathrm{X}}, m_{1,\mathrm{Y}})$ would be restricted with the constraint $p_{1,\mathrm{X}} + p_{1,\mathrm{Y}} \leq 1$ on the CPVs used in the respective criterion BOEs ($p_{1,\mathrm{X}} + p_{1,\mathrm{Y}} = 1$ implies

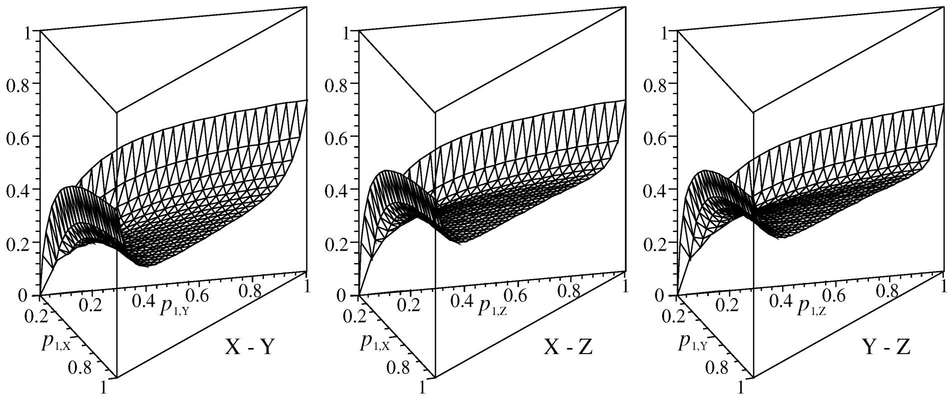

Figure 3. Graphs of distance measure between pairs of criterion BOEs for DM_1.

$p_{1,Z} = 0$). Figure 3 shows the graphs of the distance measure $d_{BOE}(m_{1,X}, m_{1,Y})$, and those for the other two permutations of pairs of criteria, namely $d_{BOE}(m_{1,X}, m_{1,Z})$ and $d_{BOE}(m_{1,Y}, m_{1,Z})$.

The graphs in Figure 3 show similarity in their structure, as the pairs of CPVs tend to zero so the levels of local ignorance in each criterion BOE increases. Since ignorance is associated with the same set of DAs (Θ), for each criterion BOE then it is expected the associated distances should tend to zero (see also Figure 2). As the respective CPVs increase, so the distance between the criterion BOEs increases (since more mass value assigned to the identified subsets of DAs). The largest distance value shown is away from the value one (its maximum). This is because, within each criterion BOE a level of local ignorance exists, hence a level of similarity will be inherent.

Using the specific CPVs for the three criteria ($p_{1,X} = 0.3$, $p_{1,Y} = 0.5$ and $p_{1,Z} = 0.2$), then the distance measures between pairs of criterion BOEs are $d_{BOE}(m_{1,X}, m_{1,Y}) = 0.3349$, $d_{BOE}(m_{1,Y}, m_{1,Z}) = 0.4047$ and $d_{BOE}(m_{1,X}, m_{1,Z}) = 0.3730$. It follows, the judgements made by the DM_1 on the criteria X and Y are most similar. If a policy of combining the evidence from the more similar sources of evidence first is adhered to, then the criterion BOEs $m_{1,X}(\cdot)$ and $m_{1,Y}(\cdot)$ should be first combined (defined $m_{1,(X,Y)}$). The subsequent distance measure between this new BOE $m_{1,(X,Y)}$ and the remaining criterion BOE $m_{1,Z}(\cdot)$ can be calculated, in this case $d_{BOE}(m_{1,(X,Y)}, m_{1,Z}) = 0.4341$.

This process of combining sources of evidence (BOEs) in a step-by-step way follows a single linkage dendrite approach, which merges nearest neighbours, where the term nearest neighbour connotes the lowest distance between BOEs. Within a resultant dendogram, the branches in the tree represent clusters, which come together at nodes whose positions along the distance axis indicate the level at which the combination occurs (of clusters). This is an agglomerate method of clustering, which here progressively combines the BOEs according to the distance measure $d_{BOE}(\cdot, \cdot)$ in such a way that whenever two BOEs belong to the same cluster at some level they remain together at all higher levels (see Gowda and Krishna, 1978).

For DM_1, the dendogram showing the iterative process of combining the evidence from the three criteria X, Y and Z is reported in Figure 4. The evidence from the criteria X and

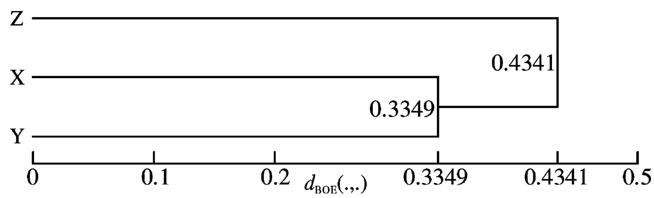

Figure 4. Dendogram representation of the ordered combination of the criterion BOEs for DM_1.

Y were first combined ($d_{\mathrm{BOE}}(m_{1,\mathrm{X}}, m_{1,\mathrm{Y}}) = 0.3349$), followed by its combination with the evidence from the criterion Z ($d_{\mathrm{BOE}}(m_{1,(\mathrm{X},\mathrm{Y})}, m_{1,\mathrm{Z}}) = 0.4341$), as described previously. While the advantage of using a dendogram is not particularly elucidated here (see later), the process of its construction has been exposited, since it illustrates well the iterative process for the combination of the different BOEs based on the distance measure $d_{\mathrm{BOE}}(\cdot, \cdot)$.

4. Utilisation of Distance Measure and Dendrite Approach in GDM

In Figure 1, the example problem presented includes six DMs with their associated judgements made on the same three criteria. The results of combining the evidence from the criteria for each DM are in the form of the individual BOEs; $m_1(\cdot), m_2(\cdot), \ldots, m_6(\cdot)$ in this case (see Table 1). It is then possible to find the distance between any two individual BOEs associated with the evidence from pairs of DMs, see Table 3.

In Table 3, the distance between each pair of DMs' individual BOEs is reported, one of these is the largest value $d_{\mathrm{BOE}}(m_4, m_6) = 0.3964$ (underlined), which indicates the DMs, DM_4 and DM_6, are the most different in terms of the judgements they made. The lowest distance between two individual BOEs is shown in bold, in this case it is between DM_5 and DM_6, with $d_{\mathrm{BOE}}(m_5, m_6) = 0.2185$. That is, DM_5 and DM_6 show most agreement (consensus) in the judgements they have made. Returning to the process of the combination of evidence from the individuals in the decision-making group, Lai et al. (2002) suggested in the case of no consensus in a decision-making group that they may choose to compromise on judgements.

Table 3. Distance measure values between pairs of individual BOEs.

Distance	DM_1: $m_1(\cdot)$	DM_2: $m_2(\cdot)$	DM_3: $m_3(\cdot)$	DM_4: $m_4(\cdot)$	DM_5: $m_5(\cdot)$	DM_6: $m_6(\cdot)$
DM_1: $m_1(\cdot)$	1					
DM_2: $m_2(\cdot)$	0.2999	1				
DM_3: $m_3(\cdot)$	0.3106	0.3317	1			
DM_4: $m_4(\cdot)$	0.2702	0.3327	0.3431	1		
DM_5: $m_5(\cdot)$	0.3499	0.3769	0.3039	0.3734	1	
DM_6: $m_6(\cdot)$	0.3472	0.3069	0.2915	<u>0.3964</u>	**0.2185**	1

Table 4. Distance measure values between $m_{(5,6)}(\cdot)$ and other individual BOEs.

Distance	$m_1(\cdot)$	$m_2(\cdot)$	$m_3(\cdot)$	$m_4(\cdot)$	$m_{(5,6)}(\cdot)$
$m_1(\cdot)$	1				
$m_2(\cdot)$	0.2999	1			
$m_3(\cdot)$	0.3106	0.3317	1		
$m_4(\cdot)$	**0.2702**	0.3327	0.3431	1	
$m_{(5,6)}(\cdot)$	0.4841	0.5019	0.4170	<u>0.5285</u>	1

In a process of amalgamating most similar views then the two individual BOEs $m_5(\cdot)$ and $m_6(\cdot)$ should be first combined using Dempster's rule of combination. Defining the combined evidence from DM_5 and DM_6 as $DM_{(5,6)}$, then the distance between the resultant new BOE $m_{(5,6)}(\cdot)$ and the individual BOEs of the other four remaining DMs can be calculated, see Table 4. Also included are the previously calculated distances between the other four DMs, as reported in Table 3. Within Table 4, given in bold (and underlined) is the lowest (and largest) distance value, in this case between DM_1 and DM_4 (and between DM_4 and $DM_{(5,6)}$).

This process of combining nearest neighbour BOEs (based on lowest between distance) and then the calculation of new distance measures is continued until the combination of the evidence from all the DMs is achieved. That is, from Table 4 the evidence from the individuals or clusters of individuals next to combine are DM_1 and DM_4, resulting in a new BOE $m_{(1,4)}(\cdot)$, this process then continues, see Table 5.

The results in Table 5 report the iterative process of the combination of the evidence from the individuals in a decision-making group. This combination is based on the level of distance between certain BOEs. The iteration of the combination of the evidence from DMs can be presented in the form of a dendogram, see Figure 5 (using the values in Tables 3–5).

Table 5. Successive distance measure tables.

Distance	$DM_{(1,4)}$	DM_2	DM_3	$DM_{(5,6)}$
$DM_{(1,4)}$	1			
DM_2	**0.3262**	1		
DM_3	0.3605	0.3317	1	
$DM_{(5,6)}$	<u>0.5164</u>	0.5019	0.4170	1

Distance	$DM_{((1,4),2)}$	DM_3	$DM_{(5,6)}$
$DM_{((1,4),2)}$	1		
DM_3	**0.3649**	1	
$DM_{(5,6)}$	<u>0.5116</u>	0.4170	1

Distance	$DM_{(((1,4),2),3)}$	$DM_{(5,6)}$
$DM_{(((1,4),2),3)}$	1	
$DM_{(5,6)}$	<u>**0.4568**</u>	1

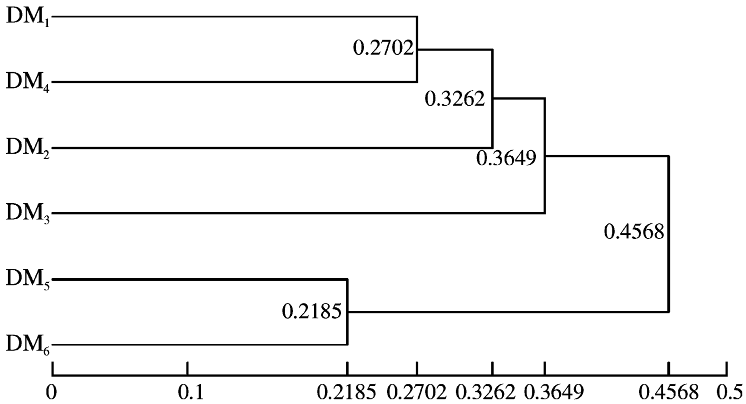

Figure 5. Dendogram of iterative aggregation of evidence from all DMs.

In general, the dendogram allows a visualisation of the iterative combination process, based on the consecutive least compromise inherent between the judgements of the members of the clusters of DMs. That is, by always combining the evidence in the BOEs from the two most similar clusters of DMs, at each stage it would require the lowest level of compromise between DMs for the resultant cluster to support the subsequent findings (resultant BOE). It is noted that the order of the combination of BOEs from a number of DMs does not affect the final BOE from their combined evidence. In this case it was found the final *group* BOE, defined $m_{1-6}(\cdot)$, for the combined evidence was made up of 74 different subsets of Θ (including Θ), with positive mass values associated with them (for brevity they are not reported in this paper).

Following the example description on the individual DM$_1$ in Section 2, the identification of best subsets of DAs from the judgements of all group members is next considered. The belief and plausibility measures are again utilised on the group BOE $m_{1-6}(\cdot)$ (from Table 5 it is $m_{((((1,4),2),3),(5,6))}(\cdot)$). Table 6 reports those subsets of DAs that have the largest belief and plausibility values from all those subsets of DAs of the same size (similar to that for DM$_1$ given in Table 2).

From Table 6, the DA D would be identified as the single best DA based on the judgements of the six DMs, irrespective of whether belief or plausibility values were considered. These

Table 6. Subsets of DAs with largest *Bel* and *Pls* values from the individual BOE $m_1(\cdot)$.

Size of DA subset	Belief (*Bel*)	Plausibility (*Pls*)
1	$\{D\}$, 0.5905	$\{D\}$, 0.5930
2	$\{D, G\}$, 0.7168	$\{D, G\}$, 0.7198
3	$\{D, F, G\}$, 0.8373	$\{D, F, G\}$, 0.8395
4	$\{D, F, G, L\}$, 0.8752	$\{D, F, G, L\}$, 0.8781
5	$\{B, D, F, G, L\}$, 0.9057	$\{B, D, F, G, L\}$, 0.9085
6	$\{B, D, E, F, G, L\}$, 0.9360	$\{B, D, E, F, G, L\}$, 0.9385

Table 7. Best subsets of DAs of size 1 and 2 for individual DMs.

	DM_1		DM_2		DM_3	
Size	Belief	Plausibility	Belief	Plausibility	Belief	Plausibility
1	$\{F\}$, 0.2251	$\{F\}$, 0.3377	$\{A\}$, 0.2413	$\{A\}$, 0.5034	$\{G\}$, 0.1224	$\{D\}$, 0.4880
2	$\{B, F\}$, 0.3837	$\{G, F\}$, 0.5966	$\{A, C\}$, 0.3665	$\{A, F\}$, 0.6341	$\{I, G\}$, 0.3079	$\{D, G\}$, 0.7927
	DM_4		DM_5		DM_6	
Size	Belief	Plausibility	Belief	Plausibility	Belief	Plausibility
1	$\{K\}$, 0.1598	$\{K\}$, 0.5058	$\{D\}$, 0.3017	$\{D\}$, 0.6091	$\{D\}$, 0.2671	$\{D\}$, 0.4898
2	$\{B, K\}$, 0.3073	$\{F, K\}$, 0.6862	$\{D, L\}$, 0.4498	$\{D, L\}$, 0.7947	$\{D, A\}$, 0.4916	$\{D, E\}$, 0.6858

findings would imply that the opinions of DM_1 reported in Table 2 are not fully represented in the overall results from the six DMs. That is, DM_1 would individually select the DA F as the best DA (based on belief or plausibility values – see Table 2), but the group of six DMs would choose the best DA D. To see if this is the case for the other DMs in the group who have made individual judgements, Table 7 reports the best subsets of DAs (of size one and two DAs), along with the associated mass values, for each of the individual DMs.

The results in Table 7 show the subsets of DAs identified with highest belief and plausibility values vary amongst the six DMs. In the case of a single best DA identified (and only belief values considered), only two DMs have the same opinion, namely DM_5 and DM_6, whom both identify D as the best DA, with the DAs A, F, G and K separately identified by the other DMs. When also considering the results in Table 6, it shows only two DMs, DM_5 and DM_6, would have agreed with the final group decision. Table 7 also highlights the variety of different subsets of two DAs that would be considered most preferred (best) by each DM.

This identified discord in group and individual DM results asks the question if this approach to a final decision is fair. The approach taken here is to offer a route to identifying the best DA(s), which may appease a certain proportion of the group of DMs making the decision. This could be simply ‘majority rule’, where once a majority of group members has agreed on their decision then this decision is taken as representative of the whole group. As Crott and Zuber (1983) define, if more than 50% of group members give the first rank to the same DA, then this DA will become the group’s decision. To achieve this majority rule approach in DS/AHP, the role of the distance measure between two groups of DMs and the dendogram exposited previously are utilised. The dendogram describing the process of the combination of the evidence from the six DMs (see Figure 5) is again presented in Figure 6, now with the inclusion of identified best subsets of DAs of sizes one and two (based on the belief and plausibility values – in that order).

In Figure 6 (as reported before), DM_5 and DM_6 are in most agreement with each other, so their evidence was combined first (forming a subgroup BOE). From Table 7, DM_5 and DM_6 both individually favour the DA D, the evidence from their subgroup BOE understandably also identifies the best DA is D (with a larger belief value of 0.6254 than their two individual

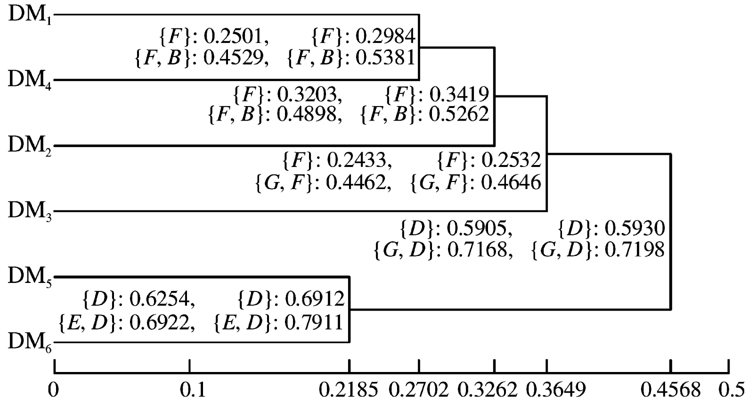

Figure 6. Dendogram of clustering of DMs showing best subsets of DAs of size one and two.

belief values). Figure 6 shows as the process of combining the BOEs from individuals or groups of individuals whose evidence is in most agreement is followed, the DA which would be considered best (based on belief and plausibility).

Figure 6 clearly shows how the results in Table 6 on the best DA(s) of sizes one and two were based mostly on the judgements from DM$_5$ and DM$_6$. That is, after the aggregation of the first two DMs, DM$_5$ and DM$_6$, with the subsequent best DA D identified, the next three processes of aggregation acting on the evidence from DM$_1$, DM$_4$, DM$_2$ and DM$_3$ (in that order), consistently identified the best DA F from their combination. Based on this series of aggregations, if a 'majority rule' approach was taken then this subgroup of DMs would identify the best DA F. This is since the four DMs, DM$_1$, DM$_4$, DM$_2$ and DM$_3$, together make up 66.67% of the group and as a majority, the results from their combined judgements could be defined the group's final decision.

These results from the 'majority rule' approach and those in Table 6 and those with full group aggregation, are different. The majority rule approach would ignore the judgements of a proportion of the group members. Considered more dramatically, these results elucidate the least negotiation that would be necessary for this group of four DMs to become an inter-group alliance from which their majority rule would deliver the group's final decision. Perhaps an interactive GDM environment would need to exist to allow those ignored to re-evaluate their judgements, to see if a larger consensus may become apparent.

Throughout this analysis a level of equal importance has existed between the members of the decision-making group. DeSanctis and Gallupe (1987) advocate that with the utilisation of group decision support systems, equality of participation in GDM will be encouraged and dominance of individuals or subgroups discouraged, with member influence more distributed and decision quality improved. It follows, the utilisation of this clustering and associated dendogram in the DS/AHP approach could be used in an iterative tool for GDM, enabling members to change their opinions in an attempt to reach better overall consensus.

With respect to the social choice axioms which are often considered to under pin the realism of a decision making model, their brief non-exhaustive description given in Section 2 is pertinent to the results here. Due to the homogeneity of the aggregation processes in the combination of evidence from criteria to individual and individual to a group decision, the validity of the 'Pareto optimality' and 'Independence of irrelevant DAs' axioms can be similarly argued as before.

5. Conclusions

Realistically modelling the decision process of an individual with respect to multi-criteria decision making (MCDM) is an ongoing and often intractable problem. This is compounded when a group-decision making (GDM) environment is considered. That is, with a number of individuals making judgements from which a final group decision needs to be obtained. As a research problem, techniques such as the AHP have been introduced and developed to aid individuals in MCDM and GDM problems. More recently, the DS/AHP method has been introduced as an aid to MCDM in the presence of an acknowledged level of ignorance. This ignorance encapsulates the possible incompleteness and imprecision of the judgements made as well as the general uncertainty of the decision makers (DMs) to the problem in question.

In this paper, the notion of the distance between the judgements made by a DM over different criteria and between the DMs is investigated. In the context of MCDM this distance measure allows, at the criteria level, to identify criteria for which the judgements on subsets of DAs by a DM are similar. Perhaps more importantly within GDM, the distance measure can be used to identify members of a decision-making group with the most similar views. Hence allows some thought to the combination of individual DM's views, including its relation to negotiation and more particularly compromise. The results presented include the use of a dendogram to elucidate the progressive clustering of the evidence from individuals in a group. An understanding on the notion of inter-group alliances is exposited, as well as a 'majority rule' approach to the obtaining of a final group decision.

The identified benefits of the DS/AHP over specific techniques such as the AHP include; the making of judgements on subsets of DAs, the inherent allowance for ignorance and the homogeneity of the aggregation of evidence at different levels of the decision making hierarchy. These perceived benefits are viewed from the elucidation of an example hypothetical problem. It is this example problem that has allowed the first investigation of the validity of the DS/AHP method with respect to two of the well-known social choice axioms, namely 'Pareto optimality' and 'Independence of irrelevant DAs'. It is clearly evident that the further elucidation of the DS/AHP method with respect to these and the other social choice axioms is necessary. Indeed the analysis over subsets of DAs may require a re-formulisation of these axioms with respect to the DS/AHP.

Appendix A: Preliminaries of Dempster-Shafer Theory

In this Appendix, the fundamentals of DST necessary for the analytic processes in DS/AHP are presented and illustrated through an example (Beynon et al. 2000). Let

$\Theta = \{h_1, h_2, \ldots, h_n\}$ be a finite set of n mutually exhaustive and exclusive hypotheses (frame of discernment). A function $m: 2^\Theta \to [0, 1]$, produces a distribution of mass values such that:

$$m(\emptyset) = 0, (\emptyset - \text{empty set}) \quad \text{and} \quad \sum_{x \in 2^\emptyset} m(x) = 1 \ \ (2^\Theta \ \text{is the power set of}\ \Theta)\,.$$

Each proposition x is a member of 2^Θ for which $m(x) > 0$ and is called a focal element. The associated mass value $m(x)$ represents the level of exact belief that the best hypothesis is in x. The series of $m(\cdot)$ values that sum to one is called a body of evidence (BOE). Within DS/AHP, the initial focal elements considered are subsets of DAs, identified by a DM over the different criteria. The associated mass values are the levels of exact belief in the best DA(s) existing in the individual subsets of DAs.

DST also provides a method to combine the BOEs from different sources, using Dempster's rule of combination. This combination rule performs two operations, the intersection of the focal elements and the multiplication of the respective mass values in the BOEs. This rule assumes that these sources are independent, then the combination of two BOEs, $m_1(\cdot)$ and $m_2(\cdot)$, is through the function $m_1 \oplus m_2 : 2^\Theta \to [0, 1]$, defined by:

$$[m_1 \oplus m_2](x) = \begin{cases} 0 & x = \emptyset \\ \dfrac{\sum_{s_1 \cap s_2 = x} m_1(s_1) m_2(s_2)}{1 - \sum_{s_1 \cap s_2 = \emptyset} m_1(s_1) m_2(s_2)} & x \neq \emptyset \end{cases} \tag{1}$$

and produces a distribution of mass values on x, where s_1 and s_2 are focal elements in the BOEs $m_1(\cdot)$ and $m_2(\cdot)$, respectively. An important feature in the denominator part of $m_1 \oplus m_2$ is $\sum_{s_1 \cap s_2 = \emptyset} m_1(s_1) m_2(s_2)$, interpreted as a measure of conflict between the evidence (George and Pal 1996; Beynon 2002). The larger this measure, the more conflict in the sources of evidence, and subsequently the less sense there is in their combination (Hegarat-Mascle et al. 1997; Murphy 2000). Within DS/AHP, the initial use of the combination rule is with the aggregation of the judgements made by a DM over the different criteria into an *individual* BOE. In a GDM environment the same combination rule is used to combine the respective individual BOEs from the group members to construct a *group* BOE.

Based on a single BOE, certain measures of confidence may be defined. A *belief* measure is a function *Bel*: $2^\Theta \to [0, 1]$, and is drawn from the sum of the mass values associated with the focal elements that are subsets of the focal element in question, defined by $Bel(s_1) = \sum_{s_2 \subseteq s_1} m(s_2)$, for $s_1 \subseteq \Theta$. It represents the confidence that a proposition x lies in s_1 or any subset of s_1. A *plausibility* measure is a function *Pls*: $2^\Theta \to [0, 1]$, defined by $Pls(s_1) = \sum_{s_1 \cap s_2 = \emptyset} m(s_2)$, for $s_1 \subseteq \Theta$. Clearly $Pls(s_1)$ represents the extent to which we fail to disbelieve a proposition x lies in s_1.

With respect to a distance measure between two BOEs, in this paper the results from Jousselme et al. (2001) are used, for alternative measures see also Bauer (1997), Fixsen and Mahler (1997) and Zouhal and Denoeux (1998). In general, let $m_1(\cdot)$ and $m_2(\cdot)$ be two BOEs on the same frame of discernment Θ, containing the number $|\Theta|$ of mutually exclusive and exhaustive hypotheses in Θ. The distance between $m_1(\cdot)$ and $m_2(\cdot)$ is defined $d_{\mathrm{BOE}}(m_1, m_2)$ and given by:

$$d_{\mathrm{BOE}}(m_1, m_2) = \sqrt{\frac{1}{2}\left(\vec{m}_1 - \vec{m}_2\right)^T \underline{\underline{D}}\left(\vec{m}_1 - \vec{m}_2\right)},$$

where $\vec{m}_1$ and $\vec{m}_2$ are the respective BOEs in vector notation (each of size $2^{|\Theta|} \times 1$, see later) and $\underline{\underline{D}}$ is an $2^{|\Theta|} \times 2^{|\Theta|}$ matrix whose elements are $\underline{\underline{D}}(s_1, s_2) = \frac{|s_1 \cap s_2|}{|s_1 \cup s_2|}$ with $s_1, s_2 \in 2^{\Theta}$. Rather than considering the larger $2^{|\Theta|} \times 2^{|\Theta|}$ matrix, a smaller distance matrix $\underline{\underline{D}}$ is constructed in cooperation with smaller $\vec{m}_1$ and $\vec{m}_2$ vectors. That is, given 2^{Θ_1} and 2^{Θ_2} are the set of subsets of 2^{Θ} making up the focal elements of the BOEs $m_1(\cdot)$ and $m_2(\cdot)$, respectively. Then the *reduced* power set considered here, defined $2^{\Theta_{1\cup 2}}$, is the union of the sets 2^{Θ_1} and 2^{Θ_2}, i.e. $2^{\Theta_{1\cup 2}} = 2^{\Theta_1} \cup 2^{\Theta_2}$. It follows, the reduced size of $\underline{\underline{D}}$ is now $2^{|\Theta_{1\cup 2}|} \times 2^{|\Theta_{1\cup 2}|}$, with $\underline{\underline{D}}(s_1, s_2)$ for $s_1, s_2 \in 2^{\Theta_{1\cup 2}}$. The distance measure $d_{\mathrm{BOE}}(m_1, m_2)$ has domain between 0 and 1 (from Jousselme et al. 2001), when $d_{\mathrm{BOE}}(m_1, m_2)$ is near 0 it implies the BOEs are similar, and a value near 1 implying the BOEs are different (with no focal elements in common).

To illustrate the DST expressions introduced in this appendix, two example BOEs are considered, defined $m_1(\cdot)$ and $m_2(\cdot)$, over a frame of discernment $\Theta = \{x_1, x_2, x_3\}$ and given in vector form by $m_1(\cdot) = [\{x_1\}{:}\,0.3, \{x_1, x_2\}{:}\,0.4, \{x_1, x_2, x_3\}{:}\,0.3]$ and $m_2(\cdot) = [\{x_2\}{:}\,0.2, \{x_3\}{:}\,0.3, \{x_1, x_2, x_3\}{:}\,0.5]$. The two BOEs are made up of subsets of $\{x_1, x_2, x_3\}$ for which their associated mass values sum to one. Dempster's rule of combination is first illustrated with the combination of $m_1(\cdot)$ and $m_2(\cdot)$ to construct a new BOE defined $m_{12}(\cdot)$, see Table A1.

In Table A1, the intermediate stage of the combination of the BOEs $m_1(\cdot)$ and $m_2(\cdot)$ is presented. The resultant focal elements are found from the intersection of focal elements from the BOEs $m_1(\cdot)$ and $m_2(\cdot)$. The associated intermediate mass values are calculated from the multiplication of the respective mass values from $m_1(\cdot)$ and $m_2(\cdot)$. Within Table A1 the sum of the values associated with the empty intersection of focal elements is $\sum_{s_1 \cap s_2 = \emptyset} m_1(s_1)m_2(s_2) = 0.06 + 0.09 + 0.12 = 0.27$. The final mass values in the new BOE $m_{12}(\cdot)$ are found from the values associated with the focal elements derived in Table 1, each divided by the value $1 - 0.27 = 0.73$. In vector form the BOE $m_{12}(\cdot)$ is given by:

$$m_{12} = [\{x_1\}{:}\,0.2055, \{x_2\}{:}\,0.1918, \{x_3\}{:}\,0.1232, \{x_1, x_2\}{:}\,0.2740, \{x_1, x_2, x_3\}{:}\,0.2055].$$

To illustrate the construction of the mass values in the BOE $m_{12}(\cdot)$, $m_{12}(\{x_3\}) = 0.09/0.73 = 0.1232$ and $m_{12}(\{x_1, x_2, x_3\}) = 0.15/0.73 = 0.2055$. Using the BOE $m_{12}(\cdot)$,

Table A1. Intermediate stage of combination of BOEs $m_1(\cdot)$ and $m_2(\cdot)$

$m_2(\cdot)\backslash m_1(\cdot)$	$\{x_1\}$: 0.3	$\{x_1, x_2\}$: 0.4	$\{x_1, x_2, x_3\}$: 0.3
$\{x_2\}$: 0.2	$\emptyset$: 0.06	$\{x_2\}$: 0.08	$\{x_2\}$: 0.06
$\{x_3\}$: 0.3	$\emptyset$: 0.09	$\emptyset$: 0.12	$\{x_3\}$: 0.09
$\{x_1, x_2, x_3\}$: 0.5	$\{x_1\}$: 0.15	$\{x_1, x_2\}$: 0.2	$\{x_1, x_2, x_3\}$: 0.15

the calculation of the *Bel* and *Pls* measures can be elucidated. With consideration to the focal element $\{x_1, x_2\}$, it is found:

$$\begin{aligned}
Bel(\{x_1, x_2\}) &= \sum_{s_2 \subseteq \{x_1, x_2\}} m_{12}(s_2) = m_{12}(\{x_1\}) + m_{12}(\{x_2\}) + m_{12}(\{x_1, x_2\}), \\
&= 0.2055 + 0.1918 + 0.2740 = 0.6713,
\end{aligned}$$

and

$$\begin{aligned}
Pls(\{x_1, x_2\}) &= \sum_{\{x_1, x_2\} \cap s_2 \neq \emptyset} m_{12}(s_2) = m_{12}(\{x_1\}) + m_{12}(\{x_2\}) + m_{12}(\{x_1, x_2\}) \\
&\quad + m_{12}(\{x_1, x_2, x_3\}), \\
&= 0.2055 + 0.1918 + 0.2740 + 0.2055 = 0.8768.
\end{aligned}$$

The next expression considered is the distance between two BOEs, between $m_1(\cdot)$ and $m_2(\cdot)$. The union of the sets of focal elements of these BOEs results in the construction of the reduced power set $2^{\Theta_{1\cup 2}}$, which in vector form is $[\{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}]$. Hence the values inside the BOE vectors $\vec{m}_1$ and $\vec{m}_2$ and the distance matrix $\underline{\underline{D}}$ are given by:

$$\vec{m}_1 = \begin{pmatrix} 0.3 \\ 0 \\ 0 \\ 0.4 \\ 0.3 \end{pmatrix}, \quad \vec{m}_2 = \begin{pmatrix} 0 \\ 0.2 \\ 0.3 \\ 0 \\ 0.5 \end{pmatrix} \text{ and } \underline{\underline{D}} = \begin{pmatrix} 1 & 0 & 0 & 1/2 & 1/3 \\ 0 & 1 & 0 & 1/2 & 1/3 \\ 0 & 0 & 1 & 0 & 1/3 \\ 1/2 & 1/2 & 0 & 1 & 2/3 \\ 1/3 & 1/3 & 1/3 & 2/3 & 1 \end{pmatrix}.$$

It follows, $(\vec{m}_1 - \vec{m}_2)^T = (0.3 \;\; -0.2 \;\; -0.3 \;\; 0.4 \;\; -0.2)$ and

$$(\vec{m}_1 - \vec{m}_2) = \begin{pmatrix} 0.3 \\ -0.2 \\ -0.3 \\ 0.4 \\ -0.2 \end{pmatrix}$$

then the formula for the distance $d_{\text{BOE}}(m_1, m_2)$ can be written out as:

$$\begin{aligned}
&d_{\text{BOE}}(m_1, m_2) \\
&= \sqrt{\frac{1}{2}(0.3 \;\; -0.2 \;\; -0.3 \;\; 0.4 \;\; -0.2) \begin{pmatrix} 1 & 0 & 0 & 1/2 & 1/3 \\ 0 & & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 0 & 1/3 \\ 1/2 & 1/2 & 0 & 1 & 2/3 \\ 1/3 & 1/3 & 1/3 & 2/3 & 1 \end{pmatrix} \begin{pmatrix} 0.3 \\ -0.2 \\ -0.3 \\ 0.4 \\ -0.2 \end{pmatrix}}, \\
&= 0.4359.
\end{aligned}$$

When compared to the possible domain of $d_{\mathrm{BOE}}(m_1, m_2)$, it shows that for this example the two BOEs have a medium level of difference between them.

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