ELSEVIER

Fuzzy Sets and Systems 118 (2001) 467–477

FUZZY sets and systems

www.elsevier.com/locate/fss

Entropy for intuitionistic fuzzy sets

Eulalia Szmidt, Janusz Kacprzyk *

Systems Research Institute, Polish Academy of Sciences ul. Newelska 6, 01-447 Warsaw, Poland

Received September 1997; received in revised form August 1998

Abstract

A non-probabilistic-type entropy measure for intuitionistic fuzzy sets is proposed. It is a result of a geometric interpretation of intuitionistic fuzzy sets and uses a ratio of distances between them proposed in Szmidt and Kacprzyk (to appear). It is also shown that the proposed measure can be defined in terms of the ratio of intuitionistic fuzzy cardinalities: of $F \cap F^c$ and $F \cup F^c$.

Keywords: Distance between intuitionistic fuzzy sets; Cardinality of intuitionistic fuzzy set; Entropy of intuitionistic fuzzy set

1. Introduction

Fuzziness, a feature of imperfect information, results from the lack of crisp distinction between the elements belonging and not belonging to a set (i.e. the boundaries of the set under consideration are not sharply defined). A measure of fuzziness often used and cited in the literature is an entropy first mentioned in 1965 by Zadeh [22]. The name entropy was chosen due to an intrinsic similarity of equations to the ones in the Shannon entropy [7]. However, the two functions measure fundamentally different types of uncertainty. Basically, the Shannon entropy measures the average uncertainty in bits associated with the prediction of outcomes in a random experiment.

In 1972, De Luca and Termini [14] introduced some requirements which capture our intuitive comprehension of the degree of fuzziness. Kaufmann [9] proposed to measure the degree of fuzziness of any fuzzy set A by a metric distance between its membership function and the membership function (characteristic function) of its nearest crisp set. Another way given by Yager [20] was to view the degree of fuzziness in terms of a lack of distinction between the fuzzy set and its complement. Kosko [11–13] investigated the fuzzy entropy in relation to a measure of subsethood.

In this paper, we propose a measure of fuzziness for intuitionistic fuzzy sets introduced by Atanassov [1–5]. The measure of entropy is a result of a geometric interpretation of intuitionistic fuzzy sets and basically uses

* Corresponding author.
E-mail address: kacprzyk@ibspan.waw.pl (J. Kacprzyk).

PII: S0165-0114(98)00402-3

a ratio of distances between them [19]. It is also shown that the proposed measure can be stated as the ratio of intuitionistic fuzzy cardinalities: that of $F \cap F^c$ and that of $F \cup F^c$, where F^c is the complement of F. For a different approach we refer the reader to Burillo and Bustince [6].

2. Intuitionistic fuzzy sets – a geometric interpretation

In this section, we will present those aspects of intuitionistic fuzzy sets which will be needed in our next discussion.

Definition 1. A fuzzy set A' in $X = \{x\}$ may be given as [21]

$$A' = \{\langle x, \mu_{A'}(x)\rangle \mid x \in X\}, \tag{1}$$

where $\mu_{A'} : X \to [0,1]$ is the membership function of A'; $\mu_{A'}(x) \in [0,1]$ is the degree of membership of $x \in X$ in A'.

Definition 2. An intuitionistic fuzzy set A in X is given by [1–5]

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}, \tag{2}$$

where

$$\mu_A : X \to [0,1] \qquad \nu_A : X \to [0,1]$$

with the condition

$$0 \leqslant \mu_A(x) + \nu_A(x) \leqslant 1 \quad \forall x \in X.$$

The numbers $\mu_A(x), \nu_A(x) \in [0,1]$ denote the degree of membership and non-membership of x to A, respectively.

Obviously, each fuzzy set A' in X may be represented as the following intuitionistic fuzzy set:

$$A = \{\langle x, \mu_{A'}(x),\ 1 - \mu_{A'}(x)\rangle \mid x \in X\}. \tag{3}$$

For each intuitionistic fuzzy set in X, we will call

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x), \tag{4}$$

the *intuitionistic index* of x in A. It is a hesitancy degree of x to A [1–5].

It is obvious that

$$0 \leqslant \pi_A(x) \leqslant 1 \quad \text{for each } x \in X.$$

For each fuzzy set A' in X, evidently, we have

$$\pi_A(x) = 1 - \mu_A(x) - [1 - \mu_A(x)] = 0 \quad \text{for each } x \in X.$$

A geometric interpretation of intuitionistic fuzzy sets and fuzzy sets is presented in Fig. 1 which summarizes considerations presented in [19]. Basically, it should be meant as follows. An intuitionistic fuzzy set X is mapped into the triangle ABD in that each element of X corresponds to an element of ABD – in Fig. 1, as an example, a point $x' \in ABD$ corresponding to $x \in X$ is marked (the values of $\mu_i(x), \nu_i(x), \pi_i(x)$ fulfill Eq. (4)). When π_i is equal 0, then $\mu_i + \nu_i = 1$. In Fig. 1, this condition is fulfilled only on the segment AB. Segment AB may be therefore viewed to represent a fuzzy set.

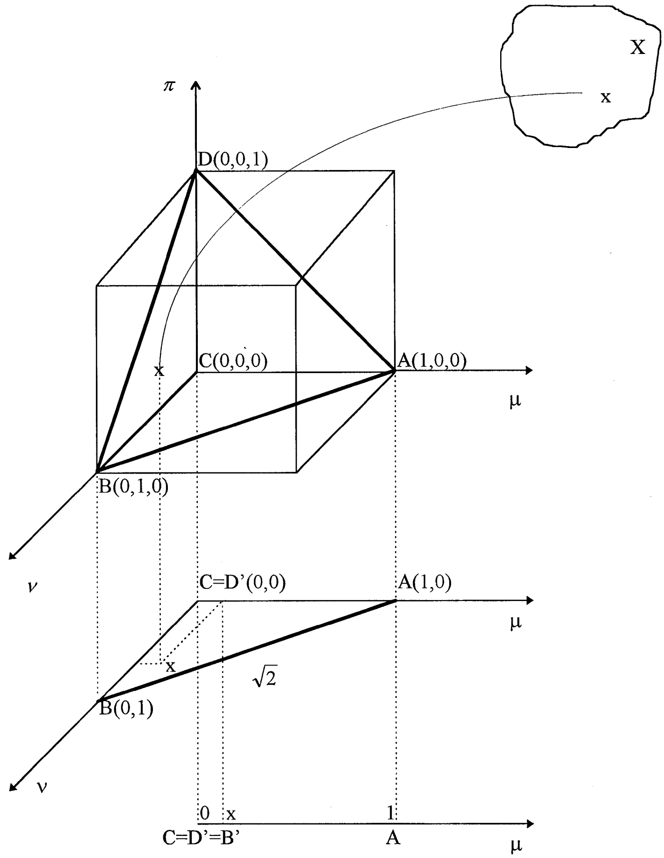

Fig. 1. A geometrical interpretation of an intuitionistic fuzzy set.

The orthogonal projection of the triangle ABD gives the representation of an intuitionistic fuzzy set on the plane. (The orthogonal projection transfers $x' \in ABD$ into $x'' \in ABC$.) The interior of the triangle $ABC = ABD'$ is the area where $\pi > 0$. Segment AB represents a fuzzy set described by two parameters: μ and ν.

The orthogonal projection of the segment AB on the axis μ (the segment $[0,1]$ is only considered) gives the fuzzy set represented by one parameter μ only. (The orthogonal projection transfers $x'' \in ABC$ into $x''' \in CA$.)

As it was shown in [19], distances between intuitionistic fuzzy sets should be calculated taking into account three parameters describing an intuitionistic fuzzy set.

The most popular distances between intuitionistic fuzzy sets A, B in $X = \{x_1, x_2, \ldots, x_n\}$ are [19]:

- *The Hamming distance*:

$$d_{\mathrm{IFS}}(A,B) = \sum_{i=1}^{n} (|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|). \tag{5}$$

- *The normalized Hamming distance*:

$$l_{\mathrm{IFS}}(A,B) = \frac{1}{n} \sum_{i=1}^{n} (|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| + |\pi_A(x_i) - \pi_B(x_i)|). \tag{6}$$

- *The Euclidean distance*:

$$e_{\mathrm{IFS}}(A,B)=\sqrt{\sum_{i=1}^{n}(\mu_A(x_i)-\mu_B(x_i))^2+(v_A(x_i)-v_B(x_i))^2+(\pi_A(x_i)-\pi_B(x_i))^2}. \tag{7}$$

- *The normalized Euclidean distance*:

$$q_{\mathrm{IFS}}(A,B)=\sqrt{\frac{1}{n}\sum_{i=1}^{n}(\mu_A(x_i)-\mu_B(x_i))^2+(v_A(x_i)-v_B(x_i))^2+(\pi_A(x_i)-\pi_B(x_i))^2}. \tag{8}$$

For Eqs. (5)–(8), there hold:

$$0\leqslant d_{\mathrm{IFS}}(A,B)\leqslant 2n, \tag{9}$$

$$0\leqslant l_{\mathrm{IFS}}(A,B)\leqslant 2, \tag{10}$$

$$0\leqslant e_{\mathrm{IFS}}(A,B)\leqslant \sqrt{2n}, \tag{11}$$

$$0\leqslant q_{\mathrm{IFS}}(A,B)\leqslant \sqrt{2}. \tag{12}$$

On the other hand, the most widely used distances for fuzzy sets A,B in $X=\{x_1,x_2,\dots,x_n\}$ are [8]:

- *The Hamming distance* $d(A,B)$:

$$d(A,B)=\sum_{i=1}^{n}|\mu_A(x_i)-\mu_B(x_i)|. \tag{13}$$

- *The normalized Hamming distance* $l(A,B)$:

$$l(A,B)=\frac{1}{n}\sum_{i=1}^{n}|\mu_A(x_i)-\mu_B(x_i)|. \tag{14}$$

- *The Euclidean distance* $e(A,B)$:

$$e(A,B):=\sqrt{\sum_{i=1}^{n}(\mu_A(x_i)-\mu_B(x_i))^2}. \tag{15}$$

- *The normalized Euclidean distance* $q(A,B)$:

$$q(A,B):=\sqrt{\frac{1}{n}\sum_{i=1}^{n}(\mu_A(x_i)-\mu_B(x_i))^2}. \tag{16}$$

For Eqs. (13)–(16), there hold:

$$0\leqslant d_{\mathrm{IFS}}(A,B)\leqslant n, \tag{17}$$

$$0\leqslant l_{\mathrm{IFS}}(A,B)\leqslant 1, \tag{18}$$

$$0\leqslant e_{\mathrm{IFS}}(A,B)\leqslant \sqrt{n}, \tag{19}$$

$$0\leqslant q_{\mathrm{IFS}}(A,B)\leqslant 1. \tag{20}$$

In our further considerations on entropy for intuitionistic fuzzy sets, besides the distances the concept of cardinality of an intuitionistic fuzzy set will also be useful.

Definition 3. Let A be an intuitionistic fuzzy set in X. First, we define the following two cardinalities of an intuitionistic fuzzy set:

- the least ("sure") cardinality of A is equal to the so-called sigma-count (cf. [23,24]), and is called here the $\min\sum Count$ (min-sigma-count):

$$\min\sum Count(A) = \sum_{i=1}^{n} \mu_A(x_i); \tag{21}$$

- the biggest cardinality of A, which is possible due to π_A, is called the $\max\sum Count$ (max-sigma-count), and is equal to

$$\max\sum Count(A) = \sum_{i=1}^{n} (\mu_A(x_i) + \pi_A(x_i)) \tag{22}$$

and, clearly, for A^c we have

$$\min\sum Count(A^c) = \sum_{i=1}^{n} \nu_A(x_i), \tag{23}$$

$$\max\sum Count(A^c) = \sum_{i=1}^{n} (\nu_A(x_i) + \pi_A(x_i)). \tag{24}$$

Then the cardinality of an intuitionistic fuzzy set is defined as the interval

$$\operatorname{card} A = \left[\min\sum Count(A), \max\sum Count(A)\right]. \tag{25}$$

Remark. In the above formulas (21)–(25), for $i=1$, we will use later, for simplicity, the following symbols: $\min Count(A)$ instead of $\min\sum Count(A)$, $\max Count(A)$ instead of $\max\sum Count(A)$, $\min Count(A^c)$ instead of $\min\sum Count(A^c)$, $\max Count(A^c)$ instead of $\max\sum Count(A^c)$.

3. Entropy

De Luca and Termini [14] first axiomatized non-probabilistic entropy. The De Luca–Termini axioms formulated for intuitionistic fuzzy sets are intuitive and have been widely employed in the fuzzy literature. They were formulated in the following way. Let E be a set-to-point mapping $E : F(2^x) \to [0,1]$. Hence E is a fuzzy set defined on fuzzy sets. E is an entropy measure if it satisfies the four De Luca and Termini axioms:

1. $E(A) = 0$ iff $A \in 2^x$ (A non-fuzzy), (26)

2. $E(A) = 1$ iff $\mu_A(x_i) = 0.5$ for all i, (27)

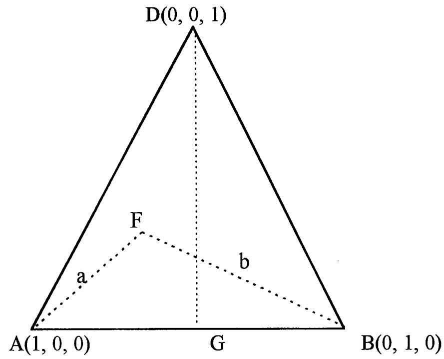

Fig. 2. The triangle ABD (Fig. 1) explaining a ratio-based measure of fuzziness.

3. $E(A) \leqslant E(B)$ if A is less fuzzy than B, i.e., if

$$\mu_A(x) \leqslant \mu_B(x) \text{ when } \mu_B(x) \leqslant 0.5 \quad \text{and} \quad \mu_A(x) \geqslant \mu_B(x) \text{ when } \mu_B(x) \geqslant 0.5, \tag{28}$$

4. $E(A) = E(A^c)$. (29)

Since the De Luca and Termini axioms (26)–(29) were formulated for fuzzy sets (given only by their membership functions, and describing the situation depicted by the segment AB in Fig. 2), they are expressed for the intuitionistic fuzzy sets as follows:

1. $E(A) = 0$ iff $A \in 2^x$ (A non-fuzzy), (30)

2. $E(A) = 1$ iff $m_A(x_i) = v_A(x_i)$ for all i, (31)

3. $E(A) \leqslant E(B)$ if A is less fuzzy than B, i.e.,

$$\mu_A(x) \leqslant \mu_B(x) \text{ and } v_A(x) \geqslant v_B(x) \quad \text{for } \mu_B(x) \leqslant v_B(x)$$

or

$$\mu_A(x) \geqslant \mu_B(x) \text{ and } v_A(x) \leqslant v_B(x) \quad \text{for } \mu_B(x) \geqslant v_B(x), \tag{32}$$

4. $E(A) = E(A^c)$. (33)

Differences with Eqs. (26)–(29) occur in axioms 2 and 3 as axiom 2 must be fulfilled now not only for point G (Fig. 2), but for the whole segment DG.

The fuzziness of a fuzzy set, or its entropy, answers the question: how fuzzy is a fuzzy set? The same question may be posed in case of an intuitionistic fuzzy set. Having in mind our geometric interpretation of intuitionistic fuzzy sets (Fig. 1), let us concentrate on the triangle ABD – Fig. 2.

A non-fuzzy set (a crisp set) corresponds to the point A (the element fully belongs to it as $(\mu_A, v_A, \pi_A) = (1, 0, 0)$) and at point B (the element fully does not belong to it as $(\mu_B, v_B, \pi_B) = (0, 1, 0)$). Points A and B representing a crisp set have the degree of fuzziness equal to 0.

A fuzzy set corresponds now to the segment AB. When we move from point A towards point B (along the segment AB, we go through points for which the membership function decreases (from 1 at point A to 0 at point B), the non-membership function increases (from 0 at point A to 1 at point B)). For the midpoint G (Fig. 2) the values of both the membership and non-membership functions are equal to 0.5. So, the midpoint

G has the degree of fuzziness equal to 100% (we do not know if point G belongs or if does not belong to our set). On the segment AG the degree of fuzziness grows (from 0% at A to 100% at G). The same situation occurs on the segment BG. The degree of fuzziness is equal 0% at B, grows towards G (where it is equal 100%).

An intuitionistic fuzzy set is represented by the triangle ABD and its interior. All points which are above the segment AB have a hesitancy margin greater than 0. The most undefined is point D. As the hesitancy margin for D is equal to 1, we cannot tell if this point belongs or does not belong to the set. The distance from D to A (full belonging) is equal to the distance to B (full non-belonging). So, the degree of fuzziness for D is equal to 100%. But the same situation occurs for all points x_i on the segment DG. For DG we have $\mu_{DG}(x_i)=\nu_{DG}(x_i)$, $\pi_{DG}(x_i)\geqslant 0$ (equality only for point G), and certainly $\mu_{DG}(x_i)+\nu_{DG}(x_i)+\pi_{DG}(x_i)=1$. For every $x_i \in DG$ we have: *distance* $(A,x_i)=$ *distance* (B,x_i).

This geometric representation of an intuitionistic fuzzy set motivates a ratio-based measure of fuzziness (a similar approach was proposed in [13] to calculate the entropy of fuzzy sets):

$$E(F)=\frac{a}{b}, \tag{34}$$

where a is a *distance* (F,F_{near}) from F to the nearer point F_{near} among A and B, and b is the *distance* (F,F_{far}) from F to the farther point F_{far} among A and B. The geometric interpretation confirms that Eq. (34) satisfies axioms (30)–(33).

An interpretation of entropy (34) can be as follows. This entropy measures the whole missing information which may be necessary to have no doubts when classifying the point F to the area of consideration, i.e. to say that F fully belongs (point A) or fully does not belong to our set (point B).

Formula (34) describes the degree of fuzziness for a single point belonging to an intuitionistic fuzzy set. For n points belonging to an intuitionistic fuzzy set we have

$$E=\frac{1}{n}\sum_{i=1}^{n}E(F_i). \tag{35}$$

Fortunately enough, while applying the Hamming distances in Eq. (34), the entropy of intuitionistic fuzzy sets is the ratio of the biggest cardinalities ($\max\sum Counts$) involving only F and F^{c}. The result is a generalized form of the entropy measure for fuzzy sets presented in [13].

Theorem 1. *A generalized entropy measure of an intuitionstic fuzzy set F of n elements is*

$$E(F)=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{\max Count(F_i\cap F_i^{\mathrm{c}})}{\max Count(F_i\cup F_i^{\mathrm{c}})}\right), \tag{36}$$

where [1–5]

$$F_i\cap F_i^{\mathrm{c}}=\left\langle \min(\mu_{F_i},\mu_{F_i}^{\mathrm{c}}),\ \max(\nu_{F_i},\nu_{F_i}^{\mathrm{c}})\right\rangle,$$

$$F_i\cup F_i^{\mathrm{c}}=\left\langle \max(\mu_{F_i},\mu_{F_i}^{\mathrm{c}}),\ \min(\nu_{F_i},\nu_{F_i}^{\mathrm{c}})\right\rangle.$$

Proof. Let

- F – a point having coordinates $\langle \mu_F,\nu_F,\pi_F\rangle$.
- $F^{\mathrm{c}}=\neg F$, a point having coordinates $\langle \mu_{F^{\mathrm{c}}},\nu_{F^{\mathrm{c}}},\pi_{F^{\mathrm{c}}}\rangle=\langle \nu_F,\mu_F,\pi_F\rangle$.
- $\bar{F}$ – the nearest non-fuzzy neighbor of F (i.e. point A for Fig. 3a, or point B for Fig. 3b),
- $\underline{F}$ – the farthest non-fuzzy neighbor of F (i.e. point B for Fig. 3a, or point A for Fig. 3b).

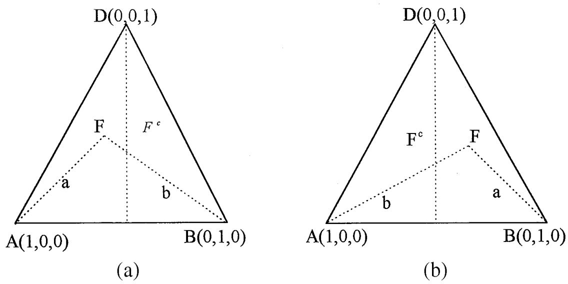

Fig. 3. (a) A case when point A is the nearest non-fuzzy neighbor, point B is the farthest non-fuzzy neighbor of F. (b) A case when point B is the nearest non-fuzzy neighbor, point A is the farthest non-fuzzy neighbor of F.

Due to Eq. (34), we have

$$E(F) = \frac{a}{b} = \frac{d_{\mathrm{IFS}}(F, \bar{F})}{d_{\mathrm{IFS}}(F, \underline{F})} \tag{37}$$

and for the situation in Fig. 3a we have [using the Hamming distance (5)]

$$E(F) = \frac{|1-\mu_F| + |0-\nu_F| + |0-\pi_F|}{|0-\mu_F| + |1-\nu_F| + |0-\pi_F|}. \tag{38}$$

Having in mind that $\mu_F + \nu_F + \pi_F = 1$, from Eq. (38) we obtain

$$E(F) = \frac{\nu_F + \pi_F + \nu_F + \pi_F}{\mu_F + \mu_F + \pi_F + \pi_F} = \frac{2\nu_F + 2\pi_F}{2\mu_F + 2\pi_F} = \frac{\nu_F + \pi_F}{\mu_F + \pi_F} = \frac{\max Count(F^{\mathrm{c}})}{\max Count(F)}. \tag{39}$$

For multiple elements F_i $(i = 1, \ldots n)$ whose point A is their nearest fuzzy neighbor, Eq. (39) becomes owing to Eqs. (22), (24) and (35)

$$E = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\nu_{F_i} + \pi_{F_i}}{\mu_{F_i} + \pi_{F_i}}\right) = \frac{1}{n}\sum_{i=1}^{n}\frac{\max Count(F_i^{\mathrm{c}})}{\max Count(F_i)}. \tag{40}$$

For the situation in Fig. 3b we have

$$E(F) = \frac{|0-\mu_F| + |1-\nu_F| + |0-\pi_F|}{|1-\mu_F| + |0-\nu_F| + |0-\pi_F|}, \tag{41}$$

i.e. by following the previous line of reasoning, we obtain

$$E(F) = \frac{\mu_F + (\mu_F + \pi_F) + \pi_F}{(\nu_F + \pi_F) + \nu_F + \pi_F} = \frac{2\mu_F + 2\pi_F}{2\nu_F + 2\pi_F} = \frac{\mu_F + \pi_F}{\nu_F + \pi_F}. \tag{42}$$

Therefore, for multiple elements F_i $(i = 1, \ldots, n)$, we have

$$E = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\mu_{F_i} + \pi_{F_i}}{\nu_{F_i} + \pi_{F_i}}\right) = \frac{1}{n}\sum_{i=1}^{n}\frac{\max Count(F_i)}{\max Count(F_i^{\mathrm{c}})}. \tag{43}$$

In Eqs. (40) and (43) the numerator and the denominator are changed. If we take into account our assumption, i.e. the fact that for a point F we have:

- in Fig. 3a: $\mu_A(x) > \nu_A(x)$,
- in Fig. 3b: $\mu_A(x) < \nu_A(x)$,

(certainly, for $\mu_A(x) = \nu_A(x) \Rightarrow E = 1$), so for the situation in Fig. 3a we have

$$\max Count(F \cap F^c) = \max Count(\min(\mu_F(x_1),\ \mu_{F^c}(x_1)), \max(\nu_F(x_1), \nu_{F^c}(x_1))) = \max Count(F^c), \tag{44}$$

$$\max Count(F \cup F^c) = \max Count(\max(\mu_F(x_1),\ \mu_{F^c}(x_1)), \min(\nu_F(x_1), \nu_{F^c}(x_1))) = \max Count(F) \tag{45}$$

and a similar consideration for the situation in Fig. 3b gives

$$\max Count(F \cap F^c) = \max Count(F), \tag{46}$$

$$\max Count(F \cup F^c) = \max Count(F^c). \tag{47}$$

Formulas (44)–(47) lead to formulas (40) and (43) as

$$E = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{\max Count(F_i \cap F_i^c)}{\max Count(F_i \cup F_i^c)} \right). \qquad \square \tag{48}$$

Example 1. Let us calculate the entropy for an element F_1 with the coordinates

$$F_1 = (\tfrac{3}{4}, \tfrac{1}{6}, \tfrac{1}{12}). \tag{49}$$

Thus,

$$d(A, F_1) = |1 - \tfrac{3}{4}| + |0 - \tfrac{1}{6}| + |0 - \tfrac{1}{12}| = \tfrac{1}{2}, \qquad d(B, F_1) = |0 - \tfrac{3}{4}| + |1 - \tfrac{1}{6}| + |0 - \tfrac{1}{12}| = \tfrac{5}{3}$$

and

$$E(F_1) = \frac{d(A, F_1)}{d(B, F_1)} = \frac{1}{2} \cdot \frac{3}{5} = \frac{3}{10}. \tag{50}$$

We can obtain the same result using formula (48) and having in mind that

$$F_1^c = (\tfrac{1}{6}, \tfrac{3}{4}, \tfrac{1}{12}) \quad \text{and} \quad F_1 \cap F_1^c = \left\langle \tfrac{1}{6}, \tfrac{3}{4}, \tfrac{1}{12} \right\rangle = F_1^c,$$

$$\max Count(F_1 \cap F_1^c) = \tfrac{1}{6} + \tfrac{1}{12} = \tfrac{3}{12},$$

$$F_1 \cup F_1^c = \left\langle \tfrac{3}{4}, \tfrac{1}{6}, \tfrac{1}{12} \right\rangle = F_1,$$

$$\max Count(F_1 \cup F_1^c) = \tfrac{3}{4} + \tfrac{1}{12} = \tfrac{10}{12},$$

so that

$$E(F_1) = \frac{\max Count(F_1 \cap F_1^c)}{\max Count(F_1 \cup F_1^c)} = \frac{3}{10}, \tag{51}$$

i.e. the same value as from Eq. (47).

Let us consider another element F_2 with the coordinates

$$F_2 = (\tfrac{1}{2}, 0, \tfrac{1}{2}). \tag{52}$$

From Eq. (34) we have

$$E(F_2) = \frac{d(A,F_2)}{d(B,F_2)} = \frac{|1-\frac{1}{2}|+|0-0|+|0-\frac{1}{2}|}{|0-\frac{1}{2}|+|1-0|+|0-\frac{1}{2}|} = \frac{1}{2} \tag{53}$$

or having in mind that $F_2^c = (0, \frac{1}{2}, \frac{1}{2})$, we obtain

$$F_2 \cap F_2^c = \left\langle 0, \tfrac{1}{2}, \tfrac{1}{2} \right\rangle = F_2^c, \qquad F_2 \cup F_2^c = \left\langle \tfrac{1}{2}, 0, \tfrac{1}{2} \right\rangle = F_2$$

and

$$E(F_2) = \frac{\max Count(F_2 \cap F_2^c)}{\max Count(F_2 \cup F_2^c)} = \frac{\frac{1}{2}}{1} = \frac{1}{2}, \tag{54}$$

i.e. the same value as from Eq. (53).

For another point F_3 with the coordinates $F_3 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, we obtain due to Eq. (34)

$$E(F_3) = \frac{d(A,F_3)}{d(B,F_3)} = \frac{|1-\frac{1}{2}|+|0-\frac{1}{4}|+|0-\frac{1}{4}|}{|0-\frac{1}{2}|+|1-\frac{1}{4}|+|0-\frac{1}{4}|} = \frac{1}{\frac{3}{2}} = \frac{2}{3}, \tag{55}$$

or taking into account that $F_3^c = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$,

$$F_3 \cap F_3^c = \left\langle \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{4} \right\rangle = F_3^c, \qquad F_3 \cup F_3^c = \left\langle \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4} \right\rangle = F_3,$$

we obtain from Eq. (48)

$$E(F_3) = \frac{\max Count(F_3 \cap F_3^c)}{\max Count(F_3 \cup F_3^c)} = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}, \tag{56}$$

i.e. the same value as from Eq. (55).

It is worth noticing that despite the fact that the hesitation margin for F_2 is greater than that for F_3, the entropy of F_2 is less than the entropy of F_3. It can be explained simply via axiom (32). This case is also a good illustration of the nature of entropy. For the point F_2, the best case which can be achieved is a crisp point, i.e.

$$\langle \mu_{F_2} + \pi_{F_2},\ \nu_{F_2} \rangle = \langle 1, 0 \rangle \tag{57}$$

while for F_3, the best what can be attained is

$$\langle \mu_{F_3} + \pi_{F_3},\ \nu_{F_3} \rangle = \left\langle \tfrac{3}{4}, \tfrac{1}{4} \right\rangle. \tag{58}$$

Formula (57) means that in the best case F_2 can attain a crisp point A (Figs. 2 and 3), whereas F_3 (58) will never do this. So, a (degree of) fuzziness is bigger for F_3 than for F_2.

From Eq. (35) we can calculate the entropy of an intuitionistic fuzzy set $Z \subseteq X = \{F_1, F_2, F_3\}$. Taking into account Eqs. (50), (54), and (55) we have

$$E(Z) = \tfrac{1}{3}\{E(F_1) + E(F_2) + E(F_3)\} = \tfrac{1}{3}(\tfrac{3}{10} + \tfrac{1}{2} + \tfrac{2}{3}) = 0.49. \tag{59}$$

4. Concluding remarks

We have introduced a measure of entropy for an intuitionistic fuzzy set. This measure is consistent with similar considerations for ordinary fuzzy sets.

For further reading

The following references are also of interest to the reader: [10,15–18].

References

[1] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1) (1986) 87–96.
[2] K. Atanassov, More on intuitionistic fuzzy sets, Fuzzy Sets and Systems 33 (1) (1989) 37–46.
[3] K. Atanassov, New operations defined over the intuitionistic fuzzy sets, Fuzzy Sets and Systems 61 (2) (1994) 137–142.
[4] K. Atanassov, Operators over interval valued intuitionistic fuzzy sets, Fuzzy Sets and Systems 64 (1994) 159–174.
[5] K. Atanassov, Intuitionistic Fuzzy Sets. Theory and Applications, Physica-Verlag, Heidelberg/New York, 1999.
[6] P. Burillo, H. Bustince, Entropy on intuitionistic fuzzy sets and on interval-valued fuzzy sets, Fuzzy Sets and Systems 78 (1996) 305–316.
[7] E.T. Jaynes, Where do we stand on maximum entropy? in: Levine, Tribus (Eds.), The Maximum Entropy Formalism, MIT Press, Cambridge, MA.
[8] J. Kacprzyk, Multistage Fuzzy Control, Wiley, New York, 1997.
[9] A. Kaufmann, Introduction to the Theory of Fuzzy Subsets – vol. 1: Fundamental Theoretical Elements, Academic Press, New York, 1975.
[10] G.J. Klir, M.J. Wierman, Uncertainty-based information elements of generalized information theory, Lecture Notes in Fuzzy Mathematics and Computer Science, vol. 3, Creighton University, Omaha, Nebraska 68178, USA, 1997.
[11] B. Kosko, Fuzzy entropy and conditioning, Inform. Sci. 40 (2) (1986) 165–174.
[12] B. Kosko, Fuzziness vs. probability, Internat. J. General Systems 17 (2–3) (1990) 211–240.
[13] B. Kosko, Fuzzy Engineering, Prentice-Hall, Englewood Cliffs, NJ, 1997.
[14] A. De Luca, S. Termini, A definition of a non-probabilistic entropy in the setting of fuzzy sets theory, Inform. and Control 20 (1972) 301–312.
[15] E. Szmidt, J. Kacprzyk, Intuitionistic fuzzy sets in group decision making, Notes IFS 2 (1) (1996) 15–32.
[16] E. Szmidt, J. Kacprzyk, Group decision making via intuitionistic fuzzy sets, FUBEST'96, 9–11 October, Sofia, Bulgaria, 1996, pp. 107–112.
[17] E. Szmidt, J. Kacprzyk, Remarks on some applications of intuitionistic fuzzy sets in decision making, Notes on IFS 2 (3) (1996) 22–31.
[18] E. Szmidt, J. Kacprzyk, Intuitionistic fuzzy sets for more realistic group decision making, Proc. of TRANSITION'97, 18–21 June, Warsaw, Poland, 1997, pp. 430–433.
[19] E. Szmidt, J. Kacprzyk, On distances between intuitionistic fuzzy sets, Fuzzy Sets and Systems 114 (2000) 505–518.
[20] R.R. Yager, On the measure of fuzziness and negation. Part I: Membership in the unit interval, Internat. J. General Systems 5 (1979) 189–200.
[21] L.A. Zadeh, Fuzzy sets, Inform. and Control 8 (1965) 338–353.
[22] L.A. Zadeh, Fuzzy Sets and Systems, in: Proc. Symp. on Systems Theory, Polytechnic Institute of Brooklyn, New York, 1965, pp. 29–37.
[23] L.A. Zadeh, A computational approach to fuzzy quantifiers in natural languages, Comput. Math. Appl. 9 (1) (1983) 149–184.
[24] L.A. Zadeh, The role of fuzzy logic in the management of uncertainty in expert systems, Fuzzy Sets and Systems 11 (1983) 199–227.