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Information volume of mass function based on extropy

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Abstract

The information volume of mass function based on extropy is proposed in this paper. Although the information volume of the probability distribution can be calculated by Shannon entropy, how to calculate the information of the mass function is still being explored. Recently, the concept of extropy was proposed by Lad et al. Based on extropy, the information volume of mass function is proposed in this paper. For a basic probability assignment function (BPA), if the focal elements of the frame of discernment (FOD) are all single elements, the information volume proposed in this paper is equal to the corresponding extropy. Otherwise, the information volume is greater than the corresponding extropy. Besides, when the cardinality of the FOD is identical, both the total uncertainty case and the mass function distribution of the maximum extropy have the same information volume. More precisely, the distribution of the latter can be regarded as the former obtained by decomposing the BPA once. Finally, the experiment proves that the maximum information volume increases with the increase in the cardinality of the FOD, and has the same limit value $\log_2 e$ as the maximum extropy. Some numerical examples are given to prove the nature of the information volume.

Keywords Information volume · Extropy · Mass function · Maximum extropy

1 Introduction

The presentation of information is usually imprecise. Many methods have been proposed for the problem of knowledge expression of sources information (Gao et al. 2021; Xiao 2021c), such as Dempster–Shafer theory (DST) (Chen et al. 2021; Dempster 1967; Song and Deng 2021; Xue and Deng 2021b), fuzzy set (Deng and Deng 2021; Zadeh 1965), D numbers (Deng and Jiang 2019). Moreover, DST is one of the most important methods. DST satisfies weaker conditions than Bayesian probability theory, and it can represent the uncertainty of information.

Entropy plays an important role in uncertainty measurement (Xiao 2020, 2021b). After Shannon entropy (Shannon 1948) was proposed, many entropies have been proposed, such as Renyi entropy (Rényi 1961), Tsallis entropy (Tsallis 1988), Deng entropy (Deng 2020b), and fuzzy entropy (Cao et al. 2020). Recently, the concept of extropy was proposed by (Lad et al. 2015). As the complementary dual of entropy, extropy can also measure uncertainty (Singh et al. 2019; Srivastava and Kaur 2019).

After extropy was proposed, many dual forms of entropy were subsequently proposed, such as Deng extropy (Buono and Longobardi 2020), NCEX (Negative Cumulative Extropy) (Tahmasebi and Toomaj 2020), residual extropy (Jose and Sathar 2019; Qiu and Jia 2018), cumulative residual extropy (Jahanshahi et al. 2020), Tsallis extropy (Xue and Deng 2021a), Survival extropy (Sathar and Nair 2021) and quantum extropy (Almarashi et al. 2020). The application of extropy involves many fields, such as forecast distribution (Lad et al. 2018), pattern recognition (Becerra et al. 2018, 2020), lifetime distribution (Kamari and Buono 2020; Krishnan et al. 2020), logical operations (Gilio and Sanfilippo 2019; Maya and Irshad 2019; Sanfilippo et al. 2020), multivariate dependence (Bortot et al. 2020; James and Crutchfield 2017; Weiss 2019), estimators of random variables (Al-Labadi and Berry 2020; Noughabi and Jarrahiferiz 2019; Yang et al. 2019) and order statistics (Noughabi and Jarrahiferiz 2020; Qiu 2017; Raqab and Qiu 2019).

Shannon entropy can calculate the information volume in the probability distribution, but what method should be used

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for the information volume of a BPA is still being explored. Deng (Deng 2020a) proposed the information volume of mass function based on Deng entropy (Deng 2020b). Based on extropy, the information volume of mass function is proposed in this paper. For a BPA, when the focal elements of FOD are all single elements, the information volume proposed in this paper is equal to the corresponding extropy. Otherwise, the information volume is greater than the corresponding extropy. In addition, when the cardinality of FOD is equal, both the $m(\text{FOD}) = 1$ case and the BPA distribution of the maximum extropy have the identical information volume.

The paper structure is as follows. Section 2 briefly introduces some preliminaries, including evidence theory (Dempster 1967; Xiao 2021a; Xiong et al. 2021), Shannon entropy (Shannon 1948), extropy (Lad et al. 2015) and Deng entropy (Deng 2020b). In Sect. 3, the method for calculating information volume based on extropy is proposed, including calculation steps and recursive algorithms. In Sect. 4, some numerical examples are presented to show the calculation of information volume. In addition, the maximum information volume and its limit are also studied. Section 5 concludes this paper.

2 Preliminaries

2.1 Dempster–Shafer evidence theory

DST defined a space composed of mutually exclusive elements, called frame of discernment (FOD), namely (Dempster 1967)

$$\Theta = \{\theta_1, \theta_2, \ldots, \theta_N\}. \tag{1}$$

The power set of Θ is indicated by $P(\Theta)$, namely (Dempster 1967)

$$\begin{aligned} P(\Theta) = \ & \{\emptyset, \{\theta_1\}, \{\theta_2\}, \ldots, \{\theta_N\}, \{\theta_1, \theta_2\}, \\ & \{\theta_1, \theta_3\}, \ldots, \{\theta_1, \theta_N\}, \ldots, \Theta\}. \end{aligned} \tag{2}$$

For a frame of discernment Θ, the mass function mz is a mapping from 2^{Θ} to [0, 1], namely (Dempster 1967)

$$m : 2^{\Theta} \rightarrow [0, 1], \tag{3}$$

which satisfies the following condition (Dempster 1967):

$$m(\emptyset) = 0 \quad \text{and} \quad \sum_{A \in P(\Theta)} m(A) = 1. \tag{4}$$

If $m(A) > 0$ and A is a non-empty set, then A is called the focal element.

The indicators that measure uncertainty in DST include the belief function (Bel) and plausibility function (Pl), which are defined as (Yager 1983)

$$\text{Bel}(X) = \sum_{Y \subseteq X} m(Y), \tag{5}$$

$$\text{Pl}(X) = \sum_{Y \cap X \neq \emptyset} m(Y), \tag{6}$$

where Bel(X) represents the full support for X, and Pl(X) represents the maximum support for X.

2.2 Shannon entropy

Shannon proposed the concept of Shannon entropy (Shannon 1948) and quantified the information with bits. At the same time, the uncertainty of information is also quantified. Given a probability distribution $P = \{p_1, p_2, \ldots, p_n\}$ under a probability space $X = \{x_1, x_2, \ldots, x_n\}$, the definition of entropy under P is as follows (Shannon 1948):

$$H(P) = -\sum_{i=1}^{n} p_i \log_2 p_i. \tag{7}$$

Shannon entropy is used to evaluate the degree of disorder of the probability system. Correspondingly, the disorder measurement in DST uses dissonance measure (D_{Y}) (Yager 1983) and confusion measure (C_{H}) (Hohle 1987), namely

$$C_{\text{H}}(m) = -\sum_{A \subseteq \Theta} m(A) \log_2(\text{Bel}(A)), \tag{8}$$

$$D_{\text{Y}}(m) = -\sum_{A \subseteq \Theta} m(A) \log_2(\text{Pl}(A)). \tag{9}$$

2.3 Extropy

Recently, Lad et al. (2015) proposed a kind of entropy that is symmetric to Shannon entropy, which is called extropy. Given a probability distribution $P = \{p_1, p_2, \ldots, p_n\}$ under a probability space $X = \{x_1, x_2, \ldots, x_n\}$, the definition of extropy under P is as follows (Lad et al. 2015):

$$J(P) = -\sum_{i=1}^{n} (1 - p_i) \log_2(1 - p_i). \tag{10}$$

When the probability is uniformly distributed, the maximum extropy and its limit are as follows (Lad et al. 2015):

$$J\left(\frac{1}{n},\dots,\frac{1}{n}\right) = (n-1)\log_2\frac{n}{n-1}, \tag{11}$$

$$\lim_{n\to+\infty}(n-1)\log_2\frac{n}{n-1} = \lim_{n\to+\infty}\log_2\left(1+\frac{1}{n-1}\right)^{n-1} = \log_2 e. \tag{12}$$

It can be seen that the maximum extropy value is $\log_2 e$ (Lad et al. 2015).

2.4 Deng entropy

How to measure uncertainty in evidence theory is a hot topic. Deng entropy (Deng 2020b) considers both the non-specificity and discord (Jousselme et al. 2005) of mass function when dealing with uncertain problems. Assume the frame of discernment $\Theta = \{\theta_1, \theta_2, \dots, \theta_N\}$, the definition of Deng entropy is as follows (Deng 2020b):

$$E_d = -\sum_{A\subseteq\Theta} m(A)\log_2\frac{m(A)}{2^{|A|}-1}, \tag{13}$$

where $|A|$ represents the cardinality of set A.

When the mass function satisfies the distribution: $m(X) = \frac{2^{|X|}-1}{\sum_{Y\subseteq\Theta}2^{|Y|}-1}$, $X, Y \subseteq \Theta$, Deng entropy gets the maximum value $\log_2\sum_{X\subseteq\Theta}\left(2^{|X|}-1\right)$ (Deng 2020b).

On the basis of Deng entropy (Deng 2020b), Deng obtained the information volume of the mass function by splitting the mass function according to the maximum Deng entropy distribution (Deng 2020a).

3 The proposed model

In this section, based on extropy, the information volume of mass function ($H_{\mathrm{EX-IV-mass}}(m)$) is defined.

Definition 3.1 Information volume of mass function

Let the FOD $= \{a_1, a_2, \dots, a_n\}$. Use index i to denote the times of this loop, and use $m(A_i)$ to denote different mass functions of different loops. Based on extropy, the information volume of mass function can be calculated by the following steps:

step 1: Input the mass function $m(A_0)$.

step 2: Repeatedly split those focal elements whose cardinality is greater than 1, and evenly distribute their mass values to its subsets until the extropy stabilizes. The specific steps can be summarized as: repeat the operations from step 2-1 to step 2-3 until the corresponding extropy converges.

step 2-1: Focus on the element whose cardinality is larger than 1, namely $|A_i| > 1$. And then, separate its mass function equally. If $|A_i| = N$, $m(A_i)$ will be divided into $2^N - 1$ parts, and the separating proportion is $1/\left(2^N-1\right) : 1/\left(2^N-1\right) : \dots : 1/\left(2^N-1\right)$.

For example, given a focal element $A_{i-1} = \{a_x, a_y\}$ and its mass function $m(A_{i-1})$, the separating proportion is that $1/3 : 1/3 : 1/3$. The ith times of separation divides $m(A_{i-1})$ and yields the following new mass function: $m(P_i), m(Q_i), m(R_i)$, where $P_i = \{a_x\}$, $Q_i = \{a_y\}$ and $R_i = \{a_x, a_y\}$. In addition, they satisfy these equations:

$$m(P_i) + m(Q_i) + m(R_i) = m(A_{i-1}), \tag{14}$$

$$m(P_i) : m(Q_i) : m(R_i) = 1/3 : 1/3 : 1/3. \tag{15}$$

step 2-2: Calculate the sum of extropy corresponding to all mass values that have not been split. The result is denoted as $H_i(m)$.

step 2-3: Calculate $\triangle i = H_i(m) - H_{i-1}(m) < \varepsilon$. When $\triangle i$ meets following condition, the loop ends.

step 3: Output $H_{\mathrm{EX-IV-mass}}(m) = H_i(m)$, which is the information volume of the BPA.

Algorithm 3.1 For a mass function, the specific algorithm for solving its $H_{\mathrm{EX-IV-mass}}(m)$ is as follows.

Assume the frame of discernment $X = \{x_1, x_2, \dots, x_N\}$, $|X| = N$ represents the cardinality of X. The mass function of X is represented by m. When m is split i times, calculate the sum of extropy corresponding to all mass values that have not been split, denoted as $H_i(m)$. $H_i(m) = \sum_{|A|=1, A\subseteq X} f1(m(A)) + \sum_{|B|=2, B\subseteq X} f2(m(B), i) + \dots + \sum_{|S|=N-1, S\subseteq X} f(N-1)(m(S), i) + fN(m(X), i)$. The definitions of $f1$, $f2$, $f3$, fN are shown in Tables 1, 2, 3 and 4.

For function $fj(m(T), i)(j = |T|, T \subseteq X, j = 2, \dots, N$, $i = 0, 1, 2, \dots)$, i represents the number of splits of T. If i is large ($i = 31$ in the experiment), $H_i(m)$ will converge to $H_{\mathrm{EX-IV-mass}}(m)$. In addition, A in $f1(m(A))$ has only one element, so A does not split and the function omits the number of splits parameter i.

4 Numerical examples and discussion

In this section, some examples are given to illustrate the calculation of the $H_{\mathrm{EX-IV-mass}}(m)$. In addition, some properties of the maximum $H_{\mathrm{EX-IV-mass}}(m)$ are also proposed.

Example 4.1 Assume the FOD $= \{a, b, c\}$, let the mass function $m(\{a\}) = m(\{b\}) = m(\{c\}) = 1/3$. It can be seen that

Table 1 Definition of function $f1$

$f1(m(A))$	Explian
return $-(1-m(A))\log_2(1-m(A))$	$\lvert A\rvert=1$, $A\subseteq X$, so $m(A)$ doesn't split

Table 2 Definition of function $f2$

$f2(m(B),i)$	Explian
if $i==0$:	$\lvert B\rvert=2$, $B\subseteq X$, i represents the number of splits. If $i=0$, B will not be split. Otherwise, $m(B)$ will be evenly divided into $2^{\lvert B\rvert}-1=3$ parts
return $f1(m(B))$	
else:	
return $2\times f1\left(\frac{m(B)}{3}\right)+f2\left(\frac{m(B)}{3},i-1\right)$	

Table 3 Definition of function $f3$

$f3(m(C),i)$	Explian
if $i==0$:	$\lvert C\rvert=3$, $C\subseteq X$, i represents the number of splits. If $i=0$, C will not be split. Otherwise, $m(C)$ will be evenly divided into $2^{\lvert C\rvert}-1=7$ parts
return $f1(m(C))$	
else:	
return $3\times f1\left(\frac{m(C)}{7}\right)+3\times f2\left(\frac{m(C)}{7},i-1\right)$ $+f3\left(\frac{m(C)}{7},i-1\right)$	

Table 4 Definition of function fN

$fN(m(X),i)$	Explian
if $i==0$:	$\lvert X\rvert=N$, n represents the number of splits. If $i=0$, X will not be split. Otherwise, $m(X)$ will be evenly divided into $2^{\lvert X\rvert}-1=2^N-1$ parts
return $f1(m(X))$	
else:	
return $C_N^1\times f1\left(\frac{m(X)}{2^N-1}\right)+C_N^2\times f2\left(\frac{m(X)}{2^N-1},i-1\right)$ $+\cdots+C_N^{N-1}\times f(N-1)\left(\frac{m(X)}{2^N-1},i-1\right)$ $+C_N^N\times fN\left(\frac{m(X)}{2^N-1},i-1\right)$	

the focus elements in this example are all single elements, so skip step 2-1 and directly calculate extropy.

$$
\begin{aligned}
H_i(m) = &-\left(1-\frac{1}{3}\right)\log_2\left(1-\frac{1}{3}\right)\\
&-\left(1-\frac{1}{3}\right)\log_2\left(1-\frac{1}{3}\right)\\
&-\left(1-\frac{1}{3}\right)\log_2\left(1-\frac{1}{3}\right)=1.1699.
\end{aligned}
$$

This kind of BPA with all focal elements containing only a single element degenerates into a probability distribution: $p_1=p_2=p_3=1/3$. Actually, when the mass function degenerates into the probability distribution, the value of $H_{\mathrm{EX-IV-mass}}(m)$ is equal to the extropy.

Example 4.2 Assume the FOD $=\{a,b,c\}$, let the mass function $m(\{a\})=1/14$, $m(\{b\})=1/7$, $m(\{c\})=3/14$, $m(\{a,b\})=3/14$, $m(\{a,c\})=1/7$, $m(\{b,c\})=1/14$, $m(\{a,b,c\})=1/7$.

Table 5 Convergence procedure of $H_i(m)$

i	$H_i(m)$
0	1.3172
1	1.3680
2	1.3728
3	1.3733
4	1.3734

Table 6 Convergence procedure of $H_i(m)$

i	$H_i(m)$
0	0
1	1.1699
2	1.2331
3	1.2393
4	1.2400
5	1.2401

As shown in Table 5, $H_i(m) - H_{i-1}(m) < 0.001$, which means that $H_i(m)$ finally converges to 1.3734. Hence, the information volume of this mass function is $H_{\mathrm{EX-IV-mass}}(m) = 1.3734$. The convergence procedure of $H_i(m)$ is listed in Table 5.

$$
\begin{aligned}
H_{EX}(m) = &-\left(1-\frac{1}{14}\right)\log_2\left(1-\frac{1}{14}\right)\times 2\\
&-\left(1-\frac{1}{7}\right)\log_2\left(1-\frac{1}{7}\right)\times 3\\
&-\left(1-\frac{3}{14}\right)\log_2\left(1-\frac{3}{14}\right)\times 2 = 1.3172.
\end{aligned}
$$

Compared $H_{\mathrm{EX-IV-mass}}(m)$ with $H_{\mathrm{EX}}(m)$ in this example, $H_{\mathrm{EX-IV-mass}}(m)$ is larger than $H_{\mathrm{EX}}(m)$.

Examples 4.1 and 4.2 illustrate that the extropy of BPA is not greater than its information volume.

Example 4.3 Assume the FOD $= \{a, b\}$, let the mass function $m(\{a, b\}) = 1$. The convergence procedure of $H_i(m)$ is listed in Table 6.

When there are only two elements in the FOD, the information volume under the maximum uncertainty is $H_{\mathrm{EX-IV-mass}}(m) = 1.2401$ (Fig. 1).

Example 4.4 Assume the FOD $= \{a, b\}$, let the mass function $m(\{a\}) = 1/3, m(\{a, b\}) = 1/3, m(\{b\}) = 1/3$. The convergence procedure of $H_i(m)$ is listed in Table 7. The calculation process of the information volume of the BPA is shown in Fig. 2.

$$
H_{EX} = -\left(1-\frac{1}{3}\right)\log_2\left(1-\frac{1}{3}\right)\times 3 = 1.1699.
$$

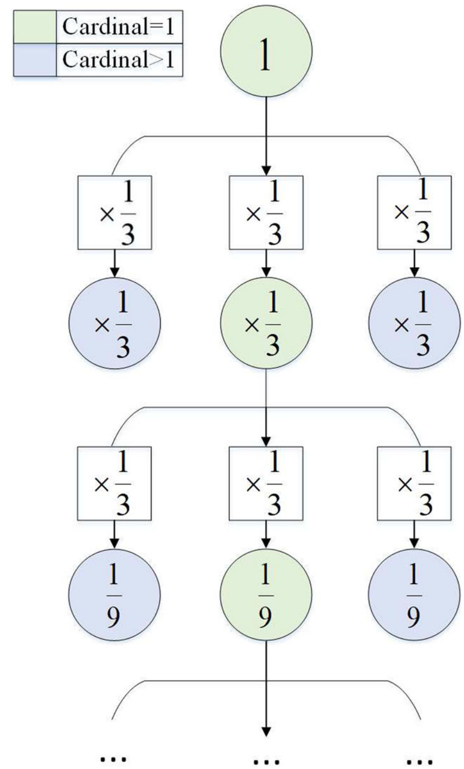

Fig. 1 Directed acyclic graphical model of Example 4.3

Table 7 Convergence procedure of $H_i(m)$

i	$H_i(m)$
0	1.1699
1	1.2331
2	1.2393
3	1.2400
4	1.2401

Compared Examples 4.3 with Example 4.4, it can be seen that although the BPA of the two examples are different, $m(\{a, b\}) = 1$ separates one step to get $m(\{a\}) = 1/3$, $m(\{b\}) = 1/3$, $m(\{a, b\}) = 1/3$. So the $H_{\mathrm{EX-IV-mass}}(m)$ of Examples 4.3 and 4.4 are the same.

Therefore, it can be concluded that when the cardinality of the FOD is identical, the BPA distribution of the maximum extropy and the total uncertainty case have identical information volume (Fig. 3).

More examples will be given to prove this conclusion.

Example 4.5 Assume the FOD $= \{a, b, c\}$, let the mass function $m(\{a, b, c\}) = 1$. As shown in Table 8, $H_i(m) - H_{i-1}(m) < 0.001$, which means that $H_i(m)$ finally converges to 1.3839. Hence, the information volume of this mass function is $H_{\mathrm{EX-IV-mass}}(m) = 1.3839$. The convergence procedure of $H_i(m)$ is listed in Table 8.

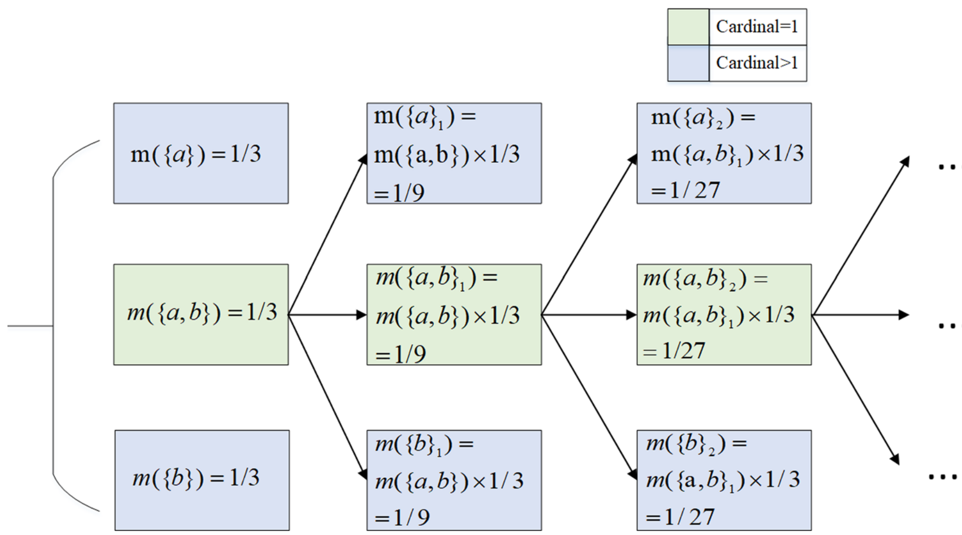

Fig. 2 Calculating procedure of Example 4.4

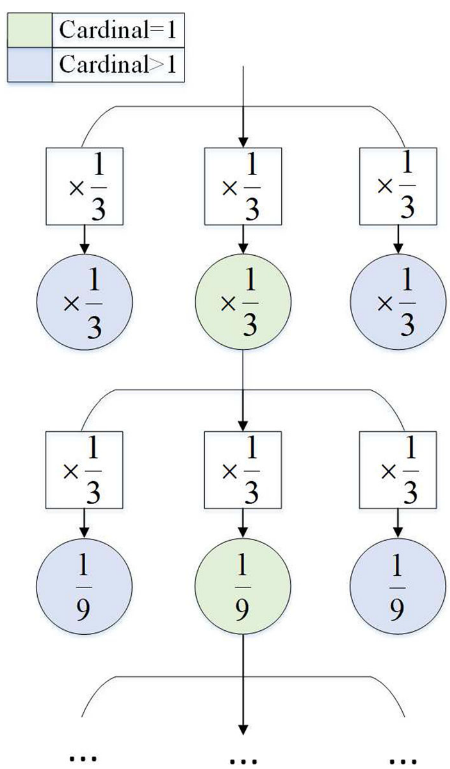

Fig. 3 Directed acyclic graphical model of Example 4.4

Table 8 Convergence procedure of $H_i(m)$

i	$H_i(m)$
0	0
1	1.3344
2	1.3792
3	1.3834
4	1.3838
5	1.3839

Table 9 Convergence procedure of $H_i(m)$

i	$H_i(m)$
0	1.3344
1	1.3792
2	1.3834
3	1.3838
4	1.3839

Example 4.6 Assume the FOD $=\{a, b, c\}$, let the mass function $m(\{a\})= m(\{b\})= m(\{c\})= m(\{a, b\})= m(\{a, c\})= m(\{b, c\})= m(\{a, b, c\}) = 1/7$. At this time, the mass function is uniformly distributed, and extropy takes the maximum value. As shown in Table 9, $H_i(m) - H_{i-1}(m) < 0.001$, which means that $H_i(m)$ finally converges to 1.3839. Hence, the information volume of this mass function is $H_{\text{EX-IV-mass}}(m) = 1.3839$. The convergence procedure of $H_i(m)$ is listed in Table 9.

Examples 4.5 and 4.6 further illustrate that if the cardinality of the FOD is identical, the information volume of the BPA distribution for the maximum extropy is equal to that of the total uncertainty case. From the perspective of uncertainty, this conclusion is consistent with common sense.

Example 4.7 Assume the frame of discernment $X = \{x_1, x_2, \ldots, x_N\}$, $|X| = N$ represents the cardinality of X. The mass function of X is represented by m. From the above examples, it can be seen that when $m(X) = 1$, $H_{\text{EX-IV-mass}}(m)$ takes the maximum value $H_{\text{EX-IV-mass}}(m) = fN(1, i)(i \to +\infty)$, $i = 31$ in this paper.

Let the probability space $X = \{x_1, x_2, \ldots, x_N\}$, and the corresponding probability distribution $P = \{\frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N}\}$, the extropy takes the maximum value $J_M(P)$ (Lad et al.

Table 10 Maximum information volume varies with $|X|$

$\lvert X\rvert$	$H_{\text{EX-IV-mass}}(m)$	$J_M(P)$ (Lad et al. 2015)	$H_{\text{DE-IV-mass}}(m)$ (Deng 2020a)	$E_{\text{MDE}}(m)$ (Deng 2020b)
1	0	0	0	0
2	1.240086	1	3.4273	2.3219
3	1.383100	1.1699	6.4698	4.2479
4	1.423646	1.2451	9.3714	6.0224
5	1.436327	1.2877	12.2060	7.7211
6	1.440551	1.3152	15.0055	9.3772
7	1.441974	1.3344	17.7857	11.0077
8	1.442453	1.3485	20.5550	12.6223
9	1.442614	1.3594	23.3181	14.2266
10	1.442668	1.3680	26.0776	15.8244
11	1.442685	1.3750	28.8350	17.4178

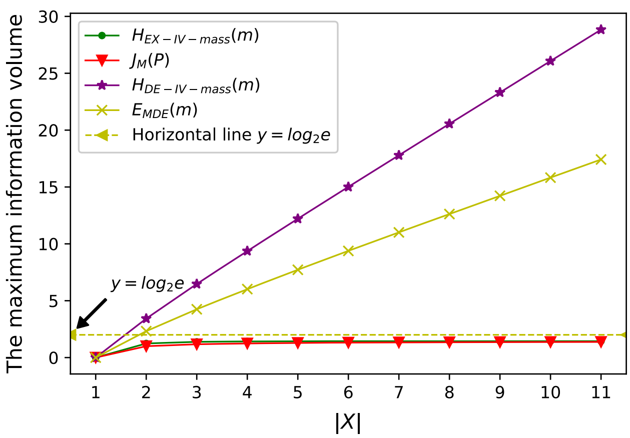

Fig. 4 Maximum information volume changes with $|X|$

2015). The information volume based on Deng entropy is represented by $H_{\text{DE-IV-mass}}(m)$ (Deng 2020a). $E_{\text{MDE}}(m)$ represents the maximum Deng entropy (Deng 2020b).

The maximum information volume calculated in four ways under different $|X|$ is shown in Table 10, and the corresponding figure is shown in Fig. 4.

Figure 4 shows that both $J_M(P)$ and $H_{\text{EX-IV-mass}}(m)$ are monotonically increasing, and the latter converges faster than the former. Both $H_{\text{DE-IV-mass}}(m)$ and $E_{\text{MDE}}(m)$ increase monotonously and increase greatly.

In fact, as $|X|$ increases, the limit of the maximum $H_{\text{EX-IV-mass}}(m)$ is $\log_2 e$. The proof will be given later.

Proof Let the frame of discernment $X = \{x_1, x_2, \ldots, x_N\}$($|X| = N$), and the corresponding mass function is m. From Algorithm 3.1 and Example 4.7, it can be seen that the maximum $H_{\text{EX-IV-mass}}(m) = fN(1, i)$,$(i \to +\infty)$.

$$
\begin{aligned}
&\lim_{\substack{i\to+\infty\\ N\to+\infty}} fN(1,i)\\
&= C_N^1 \times f1\left(\frac{1}{2^N-1}\right) + C_N^2\\
&\quad \times f2\left(\frac{1}{2^N-1}, i-1\right) + \cdots\\
&\quad + C_N^{N-1} \times f(N-1)\left(\frac{1}{2^N-1}, i-1\right)\\
&\quad + C_N^N \times fN\left(\frac{1}{2^N-1}, i-1\right)\\
&< \lim_{N\to+\infty} C_N^1 \times f1\left(\frac{1}{2^N-1}\right)\\
&\quad + C_N^2 \times f1\left(\frac{1}{2^N-1}\right) + \cdots\\
&\quad + C_N^{N-1} \times f1\left(\frac{1}{2^N-1}\right) + C_N^N \times f1\left(\frac{1}{2^N-1}\right)\\
&= \lim_{N\to+\infty} \sum_{m=1}^{N} C_N^m \times f1\left(\frac{1}{2^N-1}\right)\\
&= \lim_{N\to+\infty} (2^N-1) \times f1\left(\frac{1}{2^N-1}\right)\\
&= \lim_{N\to+\infty} (2^N-1)\\
&\quad \times \left(-\left(1-\frac{1}{2^N-1}\right)\log_2\left(1-\frac{1}{2^N-1}\right)\right)\\
&= \lim_{N\to+\infty} -(2^N-1)\\
&\quad \times \log_2\left(1-\frac{1}{2^N-1}\right) + \log_2\left(1-\frac{1}{2^N-1}\right)\\
&\overset{t=2^N-1}{=\!=} \lim_{t\to+\infty} -t \times \log_2\left(1-\frac{1}{t}\right) + \log_2\left(1-\frac{1}{t}\right)\\
&= \lim_{t\to+\infty} \log_2\left(1+\frac{1}{-t}\right)^{-t} + 0 = \log_2 e = 1.442695.
\end{aligned}
$$

Table 11 Different mass function distributions

Case	Mass function
1	$m(x_1) = m(x_2) = m(x_3) = m(x_4) = \frac{1}{4}$
2	$m(x_1, x_2) = m(x_2, x_3) = m(x_3, x_4) = m(x_1, x_4) = \frac{1}{4}$
3	$m(x_1, x_2, x_3) = m(x_2, x_3, x_4) = m(x_1, x_2, x_4) = m(x_1, x_3, x_4) = \frac{1}{4}$
4	$m(x_1, x_2, x_3, x_4) = 1$

Table 12 Comparison of several kinds of entropy

Case	$H_{\mathrm{EX-IV-mass}}(m)$	C_{H} (Hohle 1987)	D_{Y} (Yager 1983)	$H_{\mathrm{DE-IV-mass}}(m)$ (Deng 2020a)	Deng entropy (Deng 2020b)
1	1.2451	1	1	2	2
2	1.3964	2	0.4150	5.4274	3.5850
3	1.4285	2	0	8.4701	4.8074
4	1.4236	0	0	9.3714	3.9069

□

Example 4.8 Let the frame of discernment $X = \{x_1, x_2, x_3, x_4\}$, the distributions of four different mass functions are shown in Table 11. Table 12 shows the values of five kinds of entropy in different mass functions.

The mass functions in the four cases are all uniformly distributed, so their disorder degrees are equal. However, the uncertainty in evidence theory also includes non-specificity, so the total uncertainty should be case $4 >$ case $3 >$ case $2 >$ case 1. Therefore, the three entropies of $H_{\mathrm{EX-IV-mass}}(m)$, $H_{DE-IV-mass}(m)$ and Deng entropy (Deng 2020b) perform better.

5 Conclusion

This paper proposes a method for calculating the information volume of BPA based on extropy, and studies the information volume and the maximum information volume through experiments and proofs. Through numerical examples, it can be seen that the information volume of BPA is related to the uncertainty of the extropy measurement and the initial distribution of BPA, and the maximum information volume is related to the cardinality of FOD. Some are summarized as follows:

(1) When the focal elements of FOD are all single elements, the information volume is equal to the corresponding extropy.
(2) For a BPA, in which not all focal elements are single elements, its information volume is greater than the corresponding extropy.
(3) When the cardinality of FOD is equal, both the maximum uncertainty situation and the BPA distribution of the maximum extropy have the identical information volume. The reason is that the BPA distribution of the former gets the latter after once split.
(4) The maximum information volume based on extropy increases monotonically with the cardinality of FOD, and the maximum value is $\log_2 e$.
(5) The method proposed in this paper does not perform well in non-specific measurement. The future work is to improve the formula so that it can better measure the uncertainty of the mass function.

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Declarations

Conflict of interest The authors state that there are no conflict of interest.

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