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FUZZY SYSTEMS AND THEIR MATHEMATICS

Exponential negation of a probability distribution

Qinyuan Wu[1,2] · Yong Deng[1,3,4] · Neal Xiong[5]

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Abstract

Negation operation is important in intelligent information processing. Different existing arithmetic negation, an exponential negation is presented in this paper. The new negation can be seen as a kind of geometry negation. Some basic properties of the proposed negation are investigated, and we find that the fix point is the uniform probability distribution, which reaches the maximum entropy. The proposed exponential negation is an entropy increase operation, and all the probability distributions will converge to the uniform distribution after multiple negation iterations. The convergence speed of the proposed negation is also faster than the existed negation. The number of iterations of convergence is inversely proportional to the number of elements in the distribution. Some numerical examples are used to illustrate the efficiency of the proposed negation.

Keywords Negation · Exponential negation · Probability distribution · Entropy

1 Introduction

Knowledge representation and uncertainty measures are important issues in artificial intelligence (Kanal and Lemmer 2014; Fu et al. 2020; Xu et al. 2018; Fei et al. 2019). The probability distribution is an efficient way to describe uncertainty, which has achieved remarkable results in artificial intelligence (Solomonoff 1986), quantum science (Pitowsky 1989), and many other scientific fields (Huelsenbeck et al. 2001; Feller 2008). However, sometimes it is very difficult to analyze the knowledge directly according to the probability distribution itself. A possible way is to analyze the probability distribution from the negation. Negation is an efficient way to present information from the different opposite side (Anjaria 2020). For example, in the process of proving mathematical theorems, contradiction is usually an effective method. Through this example, we can find that negation can show us the other side of the higher order of knowledge and help us indirectly acquire the knowledge we need.

The negation operation has been heavily studied in previous work. In previous researches, Zadeh first proposed the negation of probability distribution in his BISC (Berkeley Initiative in Soft Computing) blog. Negation can be seen as a special transformation to get the higher-order information behind the probability distribution, and it is a different information representation from the negative perspective. After that, Yager proposed an important method of negation which has the maximum entropy allocation (Yager 2014). The basic framework of Yager's negation is subtracting a probability by 1. To some extent, Yager got the expression of negation based on subtraction, which can be seen as a kind of arithmetic negation. Yager also indicated some basic properties of this negation which gets some superior properties that accord with our intuitive understanding of the laws of nature. Inspired by Yager's negation, the negation of joint and marginal probability distributions in the binary case is proposed (Srivastava and Maheshwari 2018). To study more about the nature of negation, uncertainty metrics are developed to measure the uncertainty associated with negative probability distributions (Srivastava and Kaur 2019). A new method is also proposed based on Tsallis entropy (Zhang et al. 2020) which will degenerate to Yager's method in some

✉ Yong Deng
dengentropy@uestc.edu.cn

1 Institute of Fundamental and Frontier Science, University of Electronic Science and Technology of China, Chengdu 610054, China

2 Yingcai Honors College, University of Electronic Science and Technology of China, Chengdu, China

3 School of Education, Shaanxi Normal University, Xi'an 710062, China

4 School of Knowledge Science, Japan Advanced Institute of Science and Technology, Nomi, Ishikawa 923-1211, Japan

5 Department of Mathematics and Computer Science, Northeastern State University, 611 N. Grand Ave, #323, Webb center, Tahlequah, OK 74464, USA

special cases. The negation can be also explored to have a significant role in evidence theory. The basic probability assignment(BPA) in evidence theory is similar to the probability distribution in probability theory, some researchers used the negation of BPA to get the higher-order information or proposed some method to measure the uncertainty of BPA based on the negation and entropy. To verify the new entropy with good performance in uncertainty measurement, a new Pythagoras fuzzy number negation method is proposed and the new method can be applied in a service supplier selection system (Mao and Cai 2020). Deng and Jiang proposed a negation transformation for belief structure which is based on maximum uncertainty allocation (Deng and Jiang 2020). By introducing Dempster–Shafer's theory, the negation of probability is extended to change the allocation. Some methods for the negation of BPA are proposed to be seen as a new perspective of uncertainty measure (Yin et al. 2018). Some matrix methods for the negation of BPA have also been proposed (Luo and Deng 2019). Some researchers also proposed the negation of BPA to get the belief interval approach and apply it on the medical pattern recognition (Mao and Deng 2021). Based on the negation of BPA, a method for conflict management of evidence theory is proposed. From another expansion point, some researchers used Z number to describe the real world, and the negation of Z number is also an effective tool to obtain some new information (Liu et al. 2020). Also, having an excellent negative method of the probability or BPA can help improve the decision-making performance of the decision-making system (Xiao 2021a).

How to reasonably negate the probability distribution is still an open issue. All the existing negation methods have a similar structure with Yager's negation, whose aim is to achieve the maximum entropy in the final status, which is the uniform distribution. Shannon entropy is a corresponding information volume for a given probability distribution, a distribution with the maximum entropy means the distribution has the least information. For a probability which has a certain number of elements, the uniform distribution has the maximum entropy. There are some researches to use different entropy to measure the information volume, as the information volume of mass function (Deng 2020). Also, the entropy can also be seen as a measure of uncertainty, especially for the basic probability assignment, some other kind of entropy can be the measure of uncertainty, like the Deng entropy (Deng 2016; Song and Deng 2021), belief entropy (Chen et al. 2021; Qiang and Deng 2021), belief eXtropy (Zhou and Deng 2021), Tsallis eXtropy (Xue and Deng 2021) and so on. Through entropy, we can quantify the amount of information and uncertainty contained in probability distributions or BPA and then obtain the interval uncertainty value of a given uncertain object successfully and make a decision effectively (Xue and Deng 2021). Negation of a distribution is a way to find the higher-order information of distribution; after the iteration of negation, more and more information is obtained and the distribution will contain less information. After each iteration, the information volume contained in the probability distribution decreases as we acquire new information. Some researchers used to use the negation of probability to handle the multi-criterion decision making (MCDM) and classification problems (Gao and Deng 2021). Some researchers also find a complex mass function to predict the inference effect under the measure of uncertainty (Xiao 2021b). Entropy has also played a great role in solving the problem of conflicting evidence fusion and some dynamic models as a measurement tool of uncertainty and information (Xiong et al. 2021; Gao et al. 2021). When the information volume is reduced to the minimum state, that is, when the entropy is evenly distributed with the maximum, no new information can be obtained. In that case, the probability distribution has to converge to the uniform distribution after the iterated negation, and the convergence of the maximum entropy would be the most important property of the negation. However, one shortcoming is that the maximum entropy cannot be obtained when only two elements. For example, a probability is $P(A) = 0.7$, $P(B) = 0.3$. All the existing methods will obtain the result either $P(A) = 0.7$, $P(B) = 0.3$ or $P(A) = 0.3$, $P(B) = 0.7$. None of existing methods can achieve the maximum entropy. To address this issue, an exponential negation is proposed in this article. We solve this problem based on the exponent function. In short, Yager's negation can be seen as arithmetic negation and the proposed method can be seen as geometry negation to some degree. Although the form of the two transformations is different, the transformation we proposed has the same properties as the transformation of Yager in many fundamental ways. Especially, the two negations are completely different when we have a negation transformation on binary probability distribution. And the result of exponential negation can better meet our intuitive understanding.

The paper is structured as follows: Sect. 2 introduces Yager's negation. The exponential negation is presented in Sect. 3, and we also investigate some basic properties of the negation. In Sect. 4, we give some numerical examples to support the illustrated negation. We get the conclusion in Sect. 5.

2 Preliminaries

Knowledge representation and uncertainty measure are important issues in artificial intelligence (Fujita and Ko 2020; Liu et al. 2017; Zhou et al. 2020; Meng et al. 2020). A lot of methods have been presented, including intuitionistic fuzzy sets (Garg and Kumar 2019), Z-number (Jiang et al. 2020), evidence theory (Song et al. 2020), D numbers (Liu et al. 2020), evidential reasoning (Zhou et al. 2017; Liao et al.

2020), complex evidence theory, which are applied in classification (Liu et al. 2018; Data classification using evidence reasoning rule 2017; Li et al. 2020), information fusion (Lai et al. 2020), medical diagnosis (Cao et al. 2019; Wang et al. 2017), intrusion detection (Mi et al. 2018), reliability analysis (Mi et al. 2018), risk analysis and assessment (Pan et al. 2020; Zhang et al. 2015; Pan et al. 2019; Wang et al. 2021; Zhou and Su 2020), and decision making (Tang et al. 2020; Feng et al. 2020; Fu et al. 2020).

2.1 Yager's negation

A maximum entropy negation of a probability distribution is proposed by Yager (2014). Suppose the frame of reference is the set $X = \{x_1, x_2, \ldots, x_n\}$, $P = \{p_1, p_2, \ldots, p_n\}$ is a probability distribution on X. Yager's negation can be expressed as follows (Yager 2014):

$$\overline{p_i} = \frac{1 - p_i}{n - 1} \tag{1}$$

This negation satisfies some basic properties of probability:

$$\sum_i \overline{p_i} = 1 \tag{2}$$

$$\overline{p_i} \in [0, 1] \tag{3}$$

For many other negation operations like real numbers' negation of the negation of a matrix, the final result which has been negative twice is equal to the initial one. However, Yager's negation is not involutionary.

$$\overline{\overline{p_i}} \neq p_i \tag{4}$$

In further research, Yager indicated the reason for this unusual property. Yager's negation operation will increase the entropy of a system, and then he raised a new method to measure the entropy of a distribution. The entropy of the distribution and the negative distribution is measured following these two formulas:

$$H(P) = \sum_i (1 - p_i)(p_i) = 1 - \sum_i p_i^2 \tag{5}$$

$$H(\overline{P}) = \sum_i (1 - \overline{p_i})(\overline{p_i}) = 1 - \sum_i \overline{p_i}^2 \tag{6}$$

Then, the entropy increasing in this negation process is shown as follows:

$$\begin{aligned} H(\overline{P}) - H(P) &= \sum_i p_i^2 - \sum_i \overline{p_i}^2 \\ &= \sum_i p_i^2 - \frac{1}{(n-1)^2} \sum_i (1 - 2p_i + p_i^2) \\ &= \frac{(n-2)}{(n-1)^2} (n \sum_i p_i^2 - 1) \end{aligned} \tag{7}$$

As can be seen in Eq. (7), it is positive and the negation is an entropy increasing operation.

3 Exponential negation

In this section, a new method of negation of a probability distribution is proposed and some properties of it will be discussed.

Suppose a set of random variables, $X = \{x_1, x_2, \ldots, x_n\}$. It is a mathematical abstract representation of event $A = \{A_1, A_2, \ldots, A_n\}$. And let $P = \{p_1, p_2, \ldots, p_n\}$ as a probability distribution on X. p_i represents the probability of occurrence of event A_i, and each p_i corresponds to an event. For a probability distribution, two conditions are satisfied as follows:

$$\sum p_i = 1 \tag{8}$$

$$p_i \in [0, 1] \tag{9}$$

Then, the distribution $\overline{P} = \{\overline{p_1}, \overline{p_2}, \ldots, \overline{p_n}\}$ is used to describe the negation of distribution P. $\overline{p_i}$ indicates the information or knowledge of 'not A_i.' And each negation probability $\overline{p_i}$ has a corresponding 'not A_i.' In Yager's negation, he used the idea of additive inverse to show the information behind the probability distribution, so Yager's negation has a structure with $1 - p_i$. In the exponential negation, we used the idea of the multiplicative inverse to find the information of negation; in that case, the exponential negation has a structure of $\frac{1}{p_i}$; considering that in the real world, $p_i = 0$ is still meaningful, we use the exponential structure to avoid the meaningless situation, where the denominator cannot be 0. Taking into account the normalization of the probabilities, we end up with Definition 1, which is the simplest of the structures that meet the requirements. Therefore, Definition 1 is the probability distribution of negative information contained in each element of the probability distribution. It is a negative transformation of a probability distribution.

Definition 1 Given a probability distribution $P = \{p_1, p_2, \ldots, p_n\}$ on $X = \{x_1, x_2, \ldots, x_n\}$, the corresponding exponential negation of the probability distribution is defined as follows:

$$\begin{aligned}\overline{p_i} &= \frac{e^{-p_i}}{\sum_i e^{-p_i}} \\ &= (\sum_{i=1}^{n} e^{-p_i})^{-1} e^{-p_i} \\ &= Ae^{-p_i}\end{aligned} \tag{10}$$

where e is the natural base.

A is a normal number for every certain probability distribution, shown as follows:

$$A = (\sum_{i=1}^{n} e^{-p_i})^{-1}$$

$\overline{P}$ is still a probability distribution and satisfied as following conditions.

$$\overline{p_i} \in [0, 1] \tag{11}$$

$$\sum_i \overline{p_i} = 1 \tag{12}$$

After one time of the negation, we can get the negative probability distribution of P, which is denoted by $\overline{P}$. $\overline{P}$ is also a probability distribution. An example can be shown to explain more clearly. There is a probability distribution $P = p_1, p_2$ about whether it will rain. There is a probability of p_1 that it will rain, and a probability of p_2 that it will not rain. From the traditional probability theory, if we get the distribution, the probability of rain is determined by this probability distribution, and negation of the probability of 'rain' is $1 - p_1$, which is just the probability of 'not rain.' However, in the real world, the prediction is not so accurate and certain; there is some uncertain information contained in the distribution. From the perspective of the evidence theory, the researchers divide the whole probability 1 into three parts, 'RAIN,' 'NOT RAIN' and 'RAIN or NOT RAIN,' to show the uncertain information. In this case, the proposed negation of probability distribution, $\overline{P} = \overline{p_1}, \overline{p_2}$, can be seen as the negative result of reasonable cutting of 'RAIN or NOT RAIN' part. $\overline{p_1}$ is for 'NOT RAIN' and some parts of 'RAIN or NOT RAIN,' $\overline{p_2}$ is for 'RAIN' and some parts of 'RAIN or NOT RAIN.' When $\overline{P}$ gets the negation version, part of the uncertain information is divided again. Therefore, after some iterative negative operations are performed, there is less and less uncertain information in the probability distribution, and finally, the maximum entropy state is reached. Since the more elements contained in the probability distribution, the more uncertain information, and the more difficult it is to segment the uncertain information. Therefore, more iterations are required to achieve the uniform distribution of maximum entropy.

Some numerical examples of the exponential negation are shown as follows.

Example 1 Give a probability distribution as follows:

$$\begin{aligned} P: p_1 &= 1 \\ p_i &= 0, (i \neq 1, i = 2, 3, ..., n)\end{aligned}$$

Its corresponding negation, according to Iq 1, is shown as follows.

$$\begin{aligned} \overline{P}: \overline{p_1} &= (e^{-1} + n - 1)^{-1} e^{-1} \\ \overline{p_i} &= (e^{-1} + n - 1)^{-1}, (i \neq 1, i = 2, 3, ..., n)\end{aligned}$$

Considering $n = 2$, we can get the result as:

$$\begin{aligned} \overline{P}: \overline{p_1} &= (e^{-1} + 1)^{-1} e^{-1} = 0.2689 \\ \overline{p_i} &= (e^{-1} + 1)^{-1} = 0.7311, (i \neq 1, i = 2, 3, ..., n)\end{aligned}$$

Specially, when $n = 2$, $p_1 = 1$, $p_2 = 0$, this distribution is a very special case. This binary distribution means a certain system which gets the minimum entropy and maximum information. Performing an iterative negation operation on this system will increase its entropy and converge to a uniform distribution. This is a reasonable result following the laws of nature; it means the maximum entropy.

Example 2 Give a probability distribution as follows:

$$\begin{aligned} P: p_1 &= p_2 = \frac{1}{2} \\ p_i &= 0, (i \neq 1, 2, i = 3, ..., n)\end{aligned}$$

Its corresponding negation, according to Iq 1, is shown as follows.

$$\begin{aligned} \overline{P}: \overline{p_1} &= p_2 = (2e^{-1} + n - 2)^{-1} e^{-\frac{1}{2}} \\ \overline{p_i} &= (2e^{-1} + n - 2)^{-1}, (i \neq 1, i = 2, 3, ..., n)\end{aligned}$$

Considering $n = 2$, we can get the result as:

$$\begin{aligned} P &: p_1 = p_2 = \frac{1}{2} \\ \overline{P} &: \overline{p_1} = p_2 = (2e^{-1})^{-1} e^{-1/2} = \frac{1}{2}\end{aligned}$$

The negation of P is also P. From this result, we can guess that the uniform distribution is a fix point of the negation.

Example 3 Give a probability distribution as follows:

$$P = 1$$
$$\overline{P} = 1$$

Consider the situation that the probability distribution only has one element, which is $P = 1$, then the negation of P is $\overline{P} = 1$. The proposed exponential negation is meaningless. For the reason that the special probability distribution P in Example 3 does not have the uncertain information, therefore, the exponential negation we proposed will have no effect on it.

It is similar to the case of absolute zero in thermodynamics; the object molecules have no kinetic energy, but potential energy still exists, and the internal energy is at the minimum at this time. When the probability is 1, the course of events is already determined and there is no uncertainty; the entropy of this probability distribution also reaches maximum 0. Therefore, it is reasonable for the negative probability to be unchanged.

Furthermore, the proposed exponential negation also has the property of order reversal. If $p_i > p_j$, we can find that $\overline{p_i} < \overline{p_j}$ from the definition. It is obvious to get such a result. If the probability of event A_1 is greater than event A_2, it is reasonable that the probability of 'not A_1' is smaller than 'not A_2' from experience.

Also, we note that this negation is not involutionary and coincides with Yager's negation (Yager 2014) and Heyting intuitionistic logic (Heyting 1966).

$$\begin{aligned}\overline{\overline{p_i}} &= A_1 e^{-\overline{p_i}} = (\sum_i e^{-\overline{p_i}})^{-1} e^{-\overline{p_i}} \\ &= (\sum_i e^{-\frac{e^{-p_i}}{\sum_i e^{-p_i}}})^{-1} e^{-\frac{e^{-p_i}}{\sum_i e^{-p_i}}} \neq p_i\end{aligned} \tag{13}$$

This property can be understood by Example 2; the distribution will converge to a uniform distribution. If the operation is involutionary, we will never get the uniform distribution in this example.

Then, we figure out that uniform probability distribution is a fix point of the exponential negation, and we will use Shannon entropy (Shannon 1948) to explain it. Also, we will give some numerical examples to show the details and help understand. Considering a uniform probability distribution: $P = \{p_i | p_i = \frac{1}{n}, (i = 1, 2, ..., n)\}$. By definition of the proposed exponential negation, we can calculate the negation distribution as:

$$\overline{p_i} = \frac{e^{-p_i}}{\sum_i e^{-p_i}} = \frac{e^{\frac{1}{n}}}{ne^{\frac{1}{n}}} = \frac{1}{n} \tag{14}$$

So, we can get the negative distribution as: $\overline{P} = \{\overline{p_i} | \overline{p_i} = \frac{1}{n}, (i = 1, 2, ..., n)\}$ From the first perspective, we can easily note that when $p_1 = p_2 = ... = p_i = p_j = ... = p_n$, there is no doubt that $\overline{p_1} = \overline{p_2} = ... = \overline{p_i} = \overline{p_j} = ... = \overline{p_n}$ by the definition. And following the basic character, $\sum_i \overline{p_i} = 1$, so the negation of uniform distribution is also the same uniform distribution.

From the second perspective, we explain it by Shannon entropy (Shannon 1948). Shannon's entropy is defined as follows:

$$H(P) = -\sum_i p_i ln(p_i) \tag{15}$$

$H(P)$ is a significant measurement for information of a probability distribution. For a probability distribution we obtained, the greater entropy means the less information. Shannon entropy can also be seen as information volume (Deng 2020; Deng and Deng 2021) It is easy to prove that a uniform probability distribution has the maximum entropy in all distributions.

Theorem 1 *Applying the proposed exponential negation to the probability distribution causes it to converge to the maximum entropy state.*

Proof

$$\begin{aligned}H(P) &= -\sum_i p_i ln(p_i) \\ H(\overline{P}) &= -\sum_i Ae^{-p_i} ln(Ae^{-p_i}) \\ H(\overline{P}) - H(P) &= -\sum_i Ae^{-p_i} ln(Ae^{-p_i}) + \sum_i p_i ln(p_i) \\ &= \sum_i [-Ae^{-p_i} ln(Ae^{-p_i}) + p_i ln(p_i)] \\ &= \sum_i [\left(\sum_i e^{-p_i}\right)^{-1} e^{-p_i} \\ &\quad \times \left(ln \sum_i e^{-p_i} + p_i\right) + p_i ln p_i] \geq 0\end{aligned} \tag{16}$$

□

The result which is always positive or equal to zero shows that every negative distribution will never have smaller entropy than the original one. When $p_i = \frac{1}{n}$, $H(\overline{P}) - H(P) = 0$; the uniform distribution has the maximum entropy already, and the entropy cannot increase anymore. So, after the negation operation, it will stay at the maximum and become the fixed point of our negation.

This is also a piece of strong evidence to show that our negation could reflect some essence of the real world. The second law of thermodynamics (Callen 1998) claims a theory that for an isolated system, it always changes in the direction of entropy increase. Our negation operation is consistent with The second law of thermodynamics. According to the above reasoning, we can easily infer that after several negation iterations of entropy increase, the distribution will approach the fixed point. In another perspective, all the probability distributions will converge to a uniform distribution.

Theorem 2 *The more elements in the probability distribution, the fewer iterations are needed to reach the maximum entropy state, i.e., the faster the convergence speed, and the fewer elements in the probability distribution, the slower the convergence speed.*

The more elements in the probability distribution, more uncertain information the probability distribution contains. From the perspective of the evidence theory, it can be seen as the more subsets if there are more elements, which can show different uncertainty. In that case, it is more suitable to use the proposed exponential negation to reduce the uncertainty. A negative probability transformation can find more uncertain information and the probability will converge at a fast speed. Theorem 2 is also illustrated by numerical examples in the next section.

4 Numeral examples

In this section, three numeral examples are shown and compared with Yager's method. In short, the proposed exponential negation has the following desirable properties.

In all tables, P_i means the distribution and n means the number of negation iteration. Suppose three representative distributions for numerical calculation.

1. Any negation iteration before the final status increase entropy.
2. The final status will achieve the fixed point, which means all the probability distribution will converge to the uniform distribution.
3. Probability distributions with more elements converge to maximum entropy more quickly.

Example 4 Give a probability distribution as $P_1 = \{p_{11}, p_{12}\} = \{0, 1\}$. Tables 1 and 2 show the results after given some iterative exponential negations on the distribution. Figure 1 visualizes the process. Table 3 shows the results of Yager's negation, and Fig. 2 visualizes it.

For Yager's negation, a special result appears in this special case because $\overline{p_1} = 1 - p_2$ and $\overline{p_2} = 1 - p_1$, no matter how many iterations, the distribution will always alternate between these two distributions which do not reach the maximum entropy. In this distribution, Yager's negation will no longer cause an entropy increase. However, Fig. 1 shows that the distribution of the proposed exponential negation will finally converge to the uniform distribution. The proposed exponential negation is an entropy-increasing operation in this special case.

Example 5 Give a probability distribution as $P_2 = \{p_{21}, p_{22}, p_{23}\} = \{0.1, 0.4, 0.5\}$; Table 4 shows the results after given some iterative exponential negations on the distribution. Figure 3 visualizes the process. Table 5 shows the results of Yager's negation, and Fig. 4 visualizes it.

One advantage of the proposed negation is that it converges faster than Yager's negation is. In Example 5, the Yager's method and the proposed negation can both con-

Table 1 P_1 exponential negative iteration results 1

n / P_1	0	1	2	3	4	5
p_{11}	0	0.731	0.386	0.557	0.472	0.514
p_{12}	1	0.269	0.614	0.443	0.528	0.486

Table 2 P_1 exponential negative iteration results 2

n / P_1	6	7	8	9	10
p_{11}	0.493	0.504	0.498	0.501	0.500
p_{12}	0.507	0.496	0.502	0.499	0.500

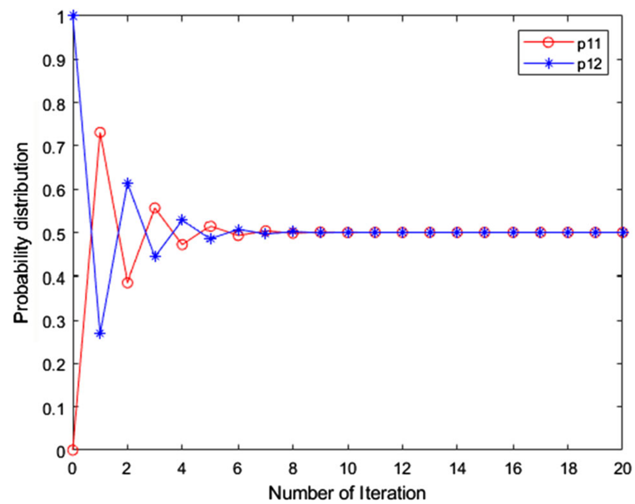

Fig. 1 Change of probability distribution P_1 with the number of exponential negative iterations

Table 3 P_1 Yager's negative iteration results Yager (2014)

n / P_1	0	1	2	3	4	5
p_{11}	0	1	0	1	0	1
p_{12}	1	0	1	0	1	0

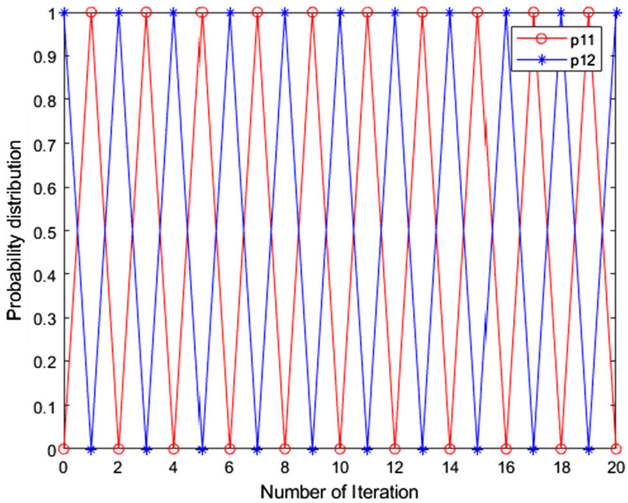

Fig. 2 Change of probability distribution P_1 with the number of Yager's negative iterations Yager (2014)

Table 4 P_2 exponential negative iterations results

n / P_1	0	1	2	3	4	5	6
p_{21}	0.100	0.414	0.306	0.342	0.330	0.334	0.333
p_{22}	0.400	0.307	0.342	0.331	0.334	0.333	0.333
p_{23}	0.500	0.278	0.352	0.327	0.335	0.332	0.333

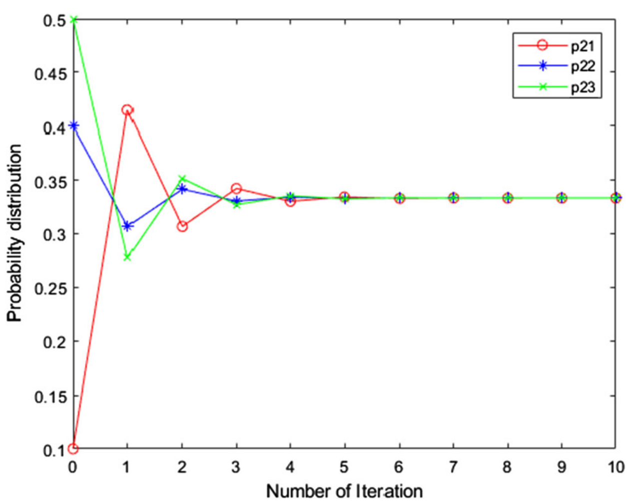

Fig. 3 Change of probability distribution P_2 with the number of exponential negative iterations

Table 5 P_2 Yager's negative iterations results Yager (2014)

n / P_1	0	1	2	3	4	5	6
p_{21}	0.100	0.450	0.275	0.363	0.319	0.341	0.330
p_{22}	0.400	0.300	0.350	0.325	0.338	0.331	0.334
p_{23}	0.500	0.250	0.375	0.313	0.344	0.328	0.336

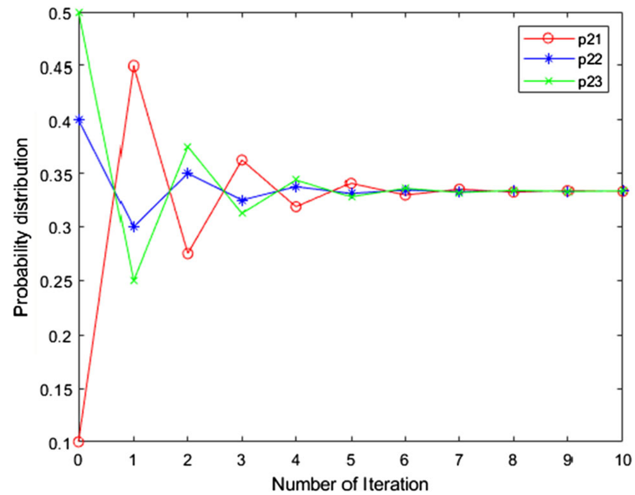

Fig. 4 Change of probability distribution P_2 with the number of Yager's negative iterations Yager (2014)

Table 6 P_3 exponential negative iteration results

n / P_1	0	1	2	3	4	5
p_{31}	0.100	0.224	0.195	0.201	0.200	0.200
p_{32}	0.130	0.218	0.197	0.201	0.200	0.200
p_{33}	0.170	0.209	0.198	0.200	0.200	0.200
p_{34}	0.300	0.184	0.203	0.199	0.200	0.200
p_{35}	0.400	0.166	0.207	0.199	0.200	0.200

verge to a uniform distribution. However, the convergence speed of Yager's negation is obviously slower than the convergence speed of our proposed negation. This is because the proposed exponential negation has a greater rate of change than Yager's negation, so it will be more efficient in obtaining information. When the precision reaches three decimal places, the tenth iteration of Yager's negation will converge to the uniform distribution, and our negation will converge to the uniform distribution of this precision after six iterations.

Example 6 Give a probability distribution as $P_3 = \{p_{31}, p_{32}, p_{33}, p_{34}, p_{35}\} = \{0.1, 0.13, 0.17, 0.3, 0.4\}$; Table 6 shows the results after being given some iterative exponential negations on the distribution. Figure 5 visualizes the process. Table 7 shows the results of Yager's negation, and Fig. 6 visualizes it.

It can be seen from Fig. 5 and Figure 6 that the convergence images of the two negations are very similar which means the convergence process of the two is similar. It can be further found from Tables 6 and 7 that in this distribution, in the case of three bits, they all converge to a uniform distribution in the fourth iteration. But after comparing more accurate values, we found that Yager's negation converged to a uniform distribution at the 10th time, and the exponen-

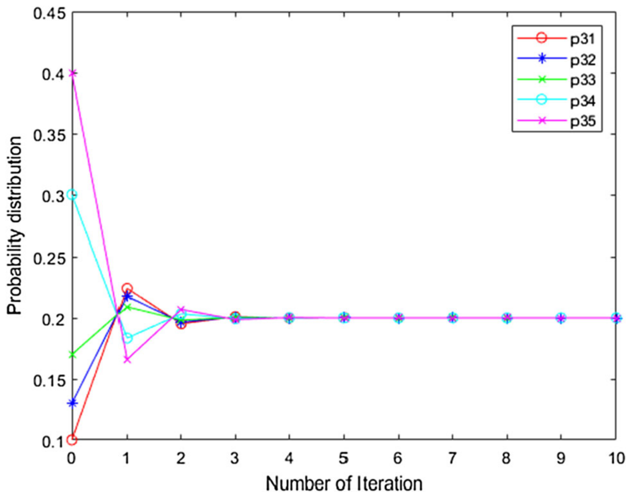

Fig. 5 Change of probability distribution P_3 with the number of exponential negative iterations

Table 7 P_3 Yager's negative iteration results Yager (2014)

n / P_1	0	1	2	3	4	5
p_{31}	0.100	0.225	0.194	0.202	0.200	0.200
p_{32}	0.130	0.218	0.196	0.201	0.200	0.200
p_{33}	0.170	0.208	0.198	0.200	0.200	0.200
p_{34}	0.300	0.175	0.206	0.198	0.200	0.200
p_{35}	0.400	0.150	0.213	0.197	0.201	0.200

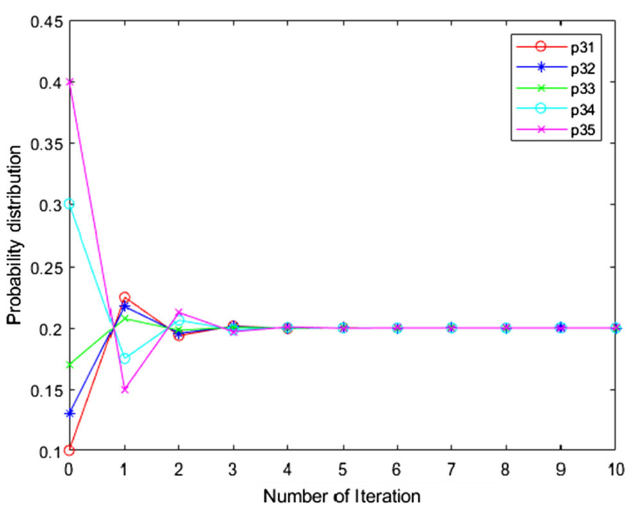

Fig. 6 Change of probability distribution P_3 with the number of Yager's negative iterations (Yager 2014)

tial negation converged to a uniform distribution at the 8th iteration.

In general, the more elements in the distribution, the more similar the convergence process of the two negations, but for the special distribution P_1, the situation will become completely different. The entropy in the proposed exponential negation will converge to a uniform distribution in this special binary case which is common in the real world. This property means that our proposed mathematical model can better reflect the actual situation.

We can see that all the examples converge to the uniform distribution with our exponential negation. In addition, an interesting point is that the numbers of iterations are different, the more elements the distribution has, the smaller number is required to converge to a uniform distribution. From Tables 1, 2, 4 and 6, it is easy to find that probability distributions with more elements converge to maximum entropy more quickly. P_1 with 2 elements converges in 10 times, P_2 with 3 elements converges in 6 times, and P_3 with 5 elements in 4 times. These numerical studies also validate Theorem 2. P_1 is required to be the biggest number in the three examples; we can infer that it also needs the biggest one in all distributions. As mentioned above, the entropy of P_1 is $H(P) = -\sum_i p_{1i} ln(p_{1i}) = 0$, $H(P) > 0$, P_1 has the minimum entropy in all distribution. So it needs to do more negation calculations to reach the maximum entropy.

5 Conclusion

In this article, a new negation method, called exponential negation, is presented. The proposed exponential negation has many desirable properties. For example, it is illustrated that all the probability distributions will converge to a uniform distribution after multiple negation iterations. The more elements the probability distribution has, the faster it converges. In addition, it can still converge very well even in special binary situations, which cannot be achieved by the previously proposed method. Its convergence speed is also faster than the traditional negative probability method, which means that it can get the negative information from the probability distribution more efficiently. Finally, it coincides with the second law of thermodynamics due to its entropy increase process. From the point of view of physical this operator is a correct multiplicative inverse operator, which we called negation and according to the numerical examples, also very effective for dealing with uncertain information. Above all, the proposed exponential negation has a strong potential to play a significant role in decision making, artificial intelligence, and other fields. Our future work will also put forward some more effective decision models based on this exponential negation.

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QW drafted the paper.
QW, YD and NX made important revisions to the paper.
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Declarations

Conflict of interest The authors declare that they have no conflict of interest. This article does not contain any studies with human participants or animals performed by any of the authors.

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