Midterm Review

Convex sets

a set C is convex if the following holds

$$x, y \in C, \quad 0 \leq \theta \leq 1 \implies \theta x + (1-\theta) y \in C$$

demonstrating a set is convex

- apply definition above
- show C is obtained by convexity-preserving operations of known convex sets
- show C is a sublevel set of a convex function

Convex functions

$f : \mathbf{R}^n \to \mathbf{R}$ is convex if $\mathbf{dom}\, f$ is a convex set and

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$

for all $x, y \in \mathbf{dom}\, f$, $0 \leq \theta \leq 1$

demonstrating a function is convex

- apply definition
- for twice differentiable functions, show $\nabla^2 f(x) \succeq 0$
- show that f is obtained by operations that preserve convexity

Convexity via composition

scalar composition
composition of $g : \mathbf{R}^n \to \mathbf{R}$ and $h : \mathbf{R} \to \mathbf{R}$:

$$f(x) = h(g(x))$$

f is convex if $\begin{array}{l} g \text{ convex, } h \text{ convex, } \tilde{h} \text{ nondecreasing} \\ g \text{ concave, } h \text{ convex, } \tilde{h} \text{ nonincreasing} \end{array}$

vector composition
similarly, composition of $g : \mathbf{R}^n \to \mathbf{R}^k$ and $h : \mathbf{R}^k \to \mathbf{R}$:

$$f(x) = h(g(x)) = h(g_1(x), g_2(x), \ldots, g_k(x))$$

f is convex if for each i $\begin{array}{l} g_i \text{ convex, } h \text{ convex, } \tilde{h} \text{ nondecreasing in arg. } i \\ g_i \text{ concave, } h \text{ convex, } \tilde{h} \text{ nonincreasing in arg. } i \end{array}$

Convex or not?

are the following functions convex, concave, or neither?

- $f(x,y) = 1/\sqrt{xy}$, with $\mathbf{dom}\, f = \mathbf{R}^2_{++}$
- $f(x) = \log(a^T x - b)$, with $\mathbf{dom}\, f = \{x \mid a^T x > b\}$
- $f(x) = \log(1 + 1/x)$, with $\mathbf{dom}\, f = \mathbf{R}_{++}$
- $f(x,y) = -x^{3/2}/\sqrt{y}$, with $\mathbf{dom}\, f = \mathbf{R}^2_{++}$
- $f(\theta) = \log\det\theta - \mathbf{tr}(S\theta)$, with $\mathbf{dom}\, f = \mathbf{S}^n_{++}$ and $S \in \mathbf{S}^n$
- $f(x,u,v) = \log(v - x^T x/u)$, $\mathbf{dom}\, f = \{(x,u,v) \mid uv > x^T x, u > 0\}$

Another example

show f is convex, with $\mathbf{dom}\, f = \{(x,y,z) \in \mathbf{R}^3 \mid x > 0, y+1 > 0\}$

$$f(x,y,z) = \frac{(x-z)^2}{y+1} + \max\left\{1 - y + \frac{1}{\sqrt{x}}, e^z\right\}$$

solution.

- $\frac{(x-z)^2}{y+1}$ is composition of quadratic-over-linear function $\frac{s^2}{t}$ with affine function that maps (x,y,z) to $(x-z, y+1)$, so is convex
- $\frac{1}{\sqrt{x}}$ is convex in x
- $1-y$ is affine, so $1 - y + \frac{1}{\sqrt{x}}$ is convex in x and y
- e^z is exponential, so convex in z
- $\max$ term is convex, since its arguments are
- sum of left and right terms is convex

More examples

(from summer 2018 midterm) let $f, g : \mathbf{R} \to \mathbf{R}$ be given

- if f, g convex, is $(f(x) + g(y))^2$ convex?
- if f, g positive and concave, is $\sqrt{f(x)g(x)}$ concave?

- if $A \prec 0$ is $\{x \in \mathbf{R}^n \mid x^T A^{-1} x \geq 0\}$ convex?
- is $\{(u, v) \in \mathbf{R}^2 \mid \cos(u + v) \geq \sqrt{2}/2, u^2 + v^2 \leq \pi^2/4\}$ convex?

(full solutions will be posted online)