Hard limits on robust control over delayed and quantized communications

Yorie Nakahira, John C. Doyle, Nikolai Matni

***Abstract*— This paper addresses a fundamental tradeoff when robust control is done using communication with lower bounds on delay ($\propto$ 1/speed) and quantization error. Animal sensorimotor control systems have a particularly interesting tradeoff between low delay versus low quantization error in nervous system communications, motivating a minimal model to explore the resulting impact on robust control. This yields surprisingly simple, tight, analytic bounds with clear interpretations and insights regarding hard tradeoffs as well as optimal coding and control strategies. These results appear more biologically relevant and extremely different from insights from information theory, which has dominated theoretical neuroscience. The simple analytic results and their proofs extend to more general models at the expense of less insight and nontrivial (but still scalable) computation. They are also relevant, though less dramatically, to certain cyberphysical systems.**

I. Introduction

The subject of control with communication constraints is almost overwhelmingly rich and provides a holistic perspective when looking at the design space involving fundamental trade offs between performance and physical limitations relevant to biological or cyberphysical systems. Realistic limits on communication, such as delay and channel capacity, are typically introduced in a feedback system and then stability, fundamental limitation in performance, and/or system design is studied. The first category deals with the minimum communication capacity that is required to stabilize a feedback system under various setting. [1][2][3]. The literature [4] [5] also considers the stability issue with both delay and quantization. The second category gives fundamental limitation in control under communication constraints, in which the information theory are introduced into feedback control. Related work includes [6] [7]. The third category proposes controller under delay or quantization [8][9] [10][11][12]. Meanwhile, various quantization schemes are proposed including, uniform , adaptive [13][14], memoryless [15], and logarithmic [14]. See [16][17] for general review of this fields. A recent thesis gives an insightful overview as well ([18]). There is an equally vast literature on distributed control over networks with sparse and delayed communication (see references in [19]).

This is a very thin and woefully incomplete literature review, but our aim is less to add to this literature than to essentially reboot with minimal models, motivated in part by neuroscience, that give clear and broadly accessible insights on tradeoffs most relevant to biology and engineering. The motivating literature, particularly in neuroscience, is even more overwhelmingly diverse, and we also are seeking simple models and clear, acessible results. The hope is that we can help build a maximally accessible bridge between the above rich control theory and new applications. A secondary goal is a framework that is scalable to more realistic and complex models and objectives.

Delay tradeoffs have a long history in neuroscience. Over 100 years ago Ramon y Cajal suggested that there exists a tradeoff between conserving materials (wire lengths of nerve fiber) and signaling delay [20]. Recently [21] considers tradeoffs in conduction delay, energy, and spike timing. In [22], it has been suggested that a star like network would minimize the signaling delay (more efficient in communication), while having large total lengths (high energy consumption). [23] studies how this trade-off between wiring delay and total wire length influences the formation of modules and hubs in neural systems. However, in theory, axons are usually assumed to be uniform wires, and the signaling delay is considered to be proportional only to wiring lengths. In reality, human axon propagation speeds vary by many orders of magnitude. While it is taken for granted that large, fast, costly axons are needed for low delay decision and control, there is little formal analysis of what tradeoffs would lead to the observed extreme size and delay heterogeneity in humans and other animals. This is likely due in part to the dominant role that information theory has played in neuroscience (e.g. see [24],[25]), and resulting necessary focus on mutual information rather than low latency.

Again, our aim here is not to so much add to this neuroscience literature but to reboot with a minimal but biologically relevant model of tradeoffs between delay and quantization error in neural communications, and then derive tight bounds on the resulting limits on robustness of control systems relying on such communications. The main result can actually be applied to any delay/quantization tradeoff and is also stated in this generality. Section II sets up the notation, and Section III has the main results and implications. Section IV gives the proofs of the main results and Section V sketches extensions to more general problems.

II. Preliminary

Notations

- $\mathbb{N}$ is a set of non-negative integers.
- For a discrete set Σ, $|\Sigma|$ is the cardinality of the set.
- The infinity norm of an infinite sequence $\{x(t)\}$ is defined as

$$\|x\|_\infty \triangleq \sup_{t\in\mathbb{N}} \max_i |x_i(t)|.$$

We use $x(0:T)$ as the restriction of this sequence up to time T. We denote l_∞ as the space of infinite sequences with bounded infinity norm.

- We use $\log$ as the logarithm to base 2.

A. Systems

Consider an strictly proper SISO LTI system whose transfer function is of the form:

$$P(z) = \frac{z^{n-1}}{z^n + a_0 z^{n-1} + \cdots a_n}.$$

The controllable canonical state space form for the transfer function $P(z)$ is defined by

$$\begin{aligned} \mathbf{x}(t+1) &= A\mathbf{x}(t) + Bu(t) \\ y(t) &= C\mathbf{x}(t), \end{aligned} \tag{1}$$

where $u(t)$ is the time domain system input, $y(t)$ is the output, $\mathbf{x}(t)$ is the state with initial condition $\mathbf{x}(0) = 0$. The matrices A, B, C are defined as:

$$A = \begin{bmatrix} 0 & 1 & & \\ & \ddots & & \\ & & & 1 \\ -a_n & a_{n-1} & \cdots & -a_0 \end{bmatrix} B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} C = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}' \tag{2}$$

The structure of the matrix A permits the state $\mathbf{x}(t)$ to be written as $\mathbf{x}(t) = \begin{bmatrix} x(t-n) & \cdots & x(t) \end{bmatrix}'$. We define the impulse response h_t as:

$$h_t \triangleq CA^{t-1}B, \tag{3}$$

by which the output can be calculated by $y(t) = \sum_{i=0}^{t-1} h_{t-i} u(i)$.

B. Communication channel and quantizer

Definition 1: A errorless communication channel with capacity R and delay d is defined by a pair of encoder and decoder $(\mathcal{E}, \mathcal{D})$ that can transmit $\lfloor 2^R \rfloor$ symbols denoted as $q \in \Sigma, |\Sigma| = \lfloor 2^R \rfloor$ with delay d. The encoder is an infinite sequence of mappings $\mathcal{E} \triangleq \{\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, \cdots\}$, where each $\mathcal{E}_t : (\mathbb{R}^t, \mathbb{R}^{t-1}) \to \Sigma$ maps past and current channel inputs $y(0:t)$ and past codeword $q(0:t-1)$ to a codeword $q(t)$:

$$q(t) = \mathcal{E}_t(y(0:t), q(0:t-1)).$$

The decoder is an infinite sequence of mappings $\mathcal{E} \triangleq \{\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2, \cdots\}$, where each $\mathcal{D}_t : \{1, \cdots, 2^R\} \to \mathbb{R}$ maps the delayed codeword $q(0:t-d)$ to the channel output:

$$u(t) = \mathcal{D}_t(q(0:t))$$

Definition 2: A uniform quantizer with rate $\log(M)$ in the interval $[-L, L] \subset \mathbb{R}$ is defined by a pair of maps $Q_{\{M,L\}} : \mathbb{R} \to \{1, \cdots, M\}$ and $Q^{-1}_{\{M,L\}} : \{1, \cdots, M\} \to \{y_1, \cdots, y_M\}$ such that

$$Q^{-1}_{\{M,L\}} Q_{\{M,L\}} x = \mathrm{sgn(x)} \Delta \left(\left\lfloor \frac{x}{\Delta} \right\rfloor + 0.5 \right) \tag{4}$$

where $x \in [-L, L]$ is the input and $\Delta = \frac{2L}{M}$ is the step size of the quantizer.

III. Motivation and Model Framework

Axon delays Consider an axon which transmits electrical impulses between neuron's cell bodies. The signaling speed of an axon is roughly proportional to its radius r, so the signal transmission time d is roughly

$$d \propto \frac{1}{r}. \tag{5}$$

If there is a total cross-sectional area $\mathcal{A}$ in which fibers (bundles of axons) can be built, and the overall length is fixed, then the number of identical (and assumed myelinated) axons that can be contained in this region is proportional to the total area divided by the area ($\propto \pi r^2$) of each fiber:

$$R \propto \frac{\mathcal{A}}{\pi r^2}. \tag{6}$$

Combining (5) and (6) yields the relation between signal transmission time d and the number of axons in a fiber R:

$$R \propto d^2 \mathcal{A} \tag{7}$$

Sensorimotor control This admittedly rough tradeoff between axon delay and number within a fiber of fixed area is relevant to a greatly simplified model of communication in sensorimotor control. Suppose that the delay in feedback from sensors to controllers to actuators is dominated by axon delays, and that the fibers are of fixed lengths and cross sectional areas. This is quite unrealistic for sensing like vision that involves substantial cortical computation, but is more realistic for the fast reflexes involved in the fastest control loops (like the vestibular ocular reflex, VOR, or stretch reflexes). Even here, the control is highly distributed, with varied delays also between compute/control components. Nevertheless, in all cases there is a tradeoff between delay in action, and the cost to build and maintain the intermediate sensory, neural, and actuator components. The simplest possible model that captures such a tradeoff, and is compatible with other results in distributed control, is to assume there is a single homogeneous fiber of fixed area between sensor and actuator over which control is done. In fact, our results apply to any R and d so are not restricted to the tradeoff in 7, but this leads to very interesting and concrete results.

The discrete time model below implicitly assumes that the axons work as a *population code* so that a discrete time communication signal is R bits at each time (arriving with delay d), and given these constraints, can use any code on these R bits. Maximizing information transfer typically leads to coding across time, but optimal control leads to codes that are only across axon space and not time. (Note to reviewers: There is overwhelming evidence from experimental neuroscience that this model is plausible if not how reflexes actually work. What appears implausible but apparently is not is that bundles of axons can maintain temporal coherence of action potentials, and that noise is less important than total delay. Unfortunately, the literature on this is vast and cryptic. If this paper is accepted we will write and post a tutorial on the relevant neuroscience, which is unfortunately much too long for the paper. We have vetted these assumptions with a variety of neuroscientists and they have agreed they are plausible starting points, and they will help with a tutorial.)

Modelling as Feedback System We model the nervous system described above as a feedback system of the form:

$$Y(z) = P(z)(U(z) + W(z)), \tag{8}$$

where $P(z)$ is the plant transfer function, $W(z)$ is the disturbance, $U(z)$ is the control action, and $Y(z)$ is the plant output. There exits an error-less but delayed communication channel (definition 1) between the plant output $Y(z)$ and the control action $U(z)$. This system is shown in figure 1.

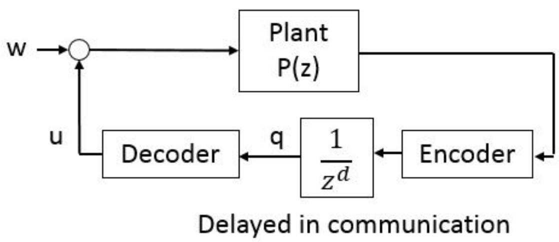

Fig. 1: Feedback system I

The signal transmission time can be interpreted as the communication delay in system (8), whereas the number of axons can be interpreted as the channel capacity. Accordingly, we make following assumptions for the communication channel:

A. The error-less communication channel has channel capacity R and delay d.
B. The decoder of the error-less communication channel is memory-less, i.e., for any $t \in \mathbb{N}$, $u(t) = \mathcal{D}(q(t))$
C. We will particularly focus on the special case where delay in communication and channel capacity satisfies the following relation:

$$d = \lceil p\sqrt{R} \rceil \tag{9}$$

for some scaling $p > 0$. Note that *small* p gives small delay and thus requires large total area.

Assumption C comes from the relation (7), where the scaling variable can be related to the total area by $p \propto \mathcal{A}^{-1}$.

In order to study the impact of communication constraints on the feedback system (8), we introduce the performance criteria:

$$\eta = \inf_{\mathcal{D},\mathcal{E}} \sup_{w} ||y||_\infty. \tag{10}$$

This performance criteria resembles the standard L_1 optimal control problem, except that there exits a communication channel between the plant output and the control. We make another assumption to put the system into L_1 optimal control framework:

D. The noise is l_∞ bounded with a known bound. Without loss of generality, we assume $||w||_\infty \leq 1$.

IV. Main Results

A. Fundamental Limitations

In this section, we provide an analytic bound for the performance criteria (10) in terms of the parameters of the plant, channel capacity, and communication delay.

Theorem 1: Consider the feedback system (8) with the plant

$$P(z) = \frac{1}{z - a}, \tag{11}$$

and assumption A,B, and D. When $|a| < \lfloor 2^R \rfloor$, the optimal performance is given by

$$\inf_{\mathcal{E},\mathcal{D}} \sup_{w} ||y||_\infty = \sum_{i=0}^{d} |a^i| + (\lfloor 2^R \rfloor - |a|)^{-1} |a|^{d+1} \tag{12}$$

When $|a| \geq \lfloor 2^R \rfloor$, there does not exits any control law that stabilizes the system.

Proof: (outline) The feedback system (8) with plant $P(z)$ and delay in communication can be equivalently model by a feedback system with delayed plant $\frac{1}{z^d}P(z)$ and a communication channel with no delay (figure 5). Without loss of generality, we use this equivalent system during the derivation.

We first introduce a state x to be defined as:

$$x(t) \triangleq y(t + d). \tag{13}$$

The feedback system (8) can be written as

$$x(t) = ax(t - 1) + w(t - 1) + u(t - 1)$$

The state x can be decomposed into the following two terms: $x(t) = x_d(t) + e(t)$, where $x_d(t)$ is the uncertainty due to delay and $e(t)$ is the uncertainties due to limited channel capacity.

For the delay term $x_d(t)$, since the output measurement $y(t-1)$ is a function of $w(0 : t - d - 2)$, the controller at time $t-1$ cannot infer the disturbances $w(t-d-1 : t-1)$. Hence, the disturbance $w(t-d-1 : t-1)$ cannot be canceled at time $t-1$, it will be present in $x(t)$. We denote this term as $x_d(t)$, which permits an explicit formulation:

$$x_d(t) \triangleq \sum_{i=1}^{d+1} a^{i-1} w(t - i) \tag{14}$$

For the uncertainty due to limited channel capacity, we can write it as

$$e(t) \triangleq x(t) - x_d(t), \tag{15}$$

The two term $x_d(t)$ and the term $e(t)$ are not related in the sense that: $x_d(t)$ is a linear combination of $w(t - d - 1 : t - 1)$, $e(t)$ is merely due to control error caused by limited information of $w(0 : t - d - 2)$. Hence, the supremum of $\sup_w |x_d(t)|$ and $\sup_w |e(t)|$ are simultaneously attainable by a single sequence of $w(t)$. Therefore, the system performance can be decomposed into:

$$\sup_{w} \|y\|_\infty \quad = \quad \sup_{w} \|x_d\|_\infty + \sup_{w} \|e\|_\infty$$

Since the term x_d cannot be canceled by any control action, we have the condition:

$$\inf_{\mathcal{E},\mathcal{D}} \sup_w \|y\|_\infty = \sup_w \|x_d\|_\infty + \inf_{\mathcal{E},\mathcal{D}} \sup_w \|e\|_\infty. \quad (16)$$

It can be shown that (see section V—)

$$\begin{aligned} \sup_w \|x_d\|_\infty &= \sum_{i=0}^{d} |a^i| \\ \sup_w \|e\|_\infty &= (\lfloor 2^R \rfloor - |a|)^{-1} |a|^{d+1} \end{aligned}$$

■

Corollary 1: Consider the feedback system (8) with the plant transfer function (11) and assumption A-D. Additionally assume $|a| < \lfloor 2^R \rfloor$. When $a \neq 1$, the optimal performance is given by

$$\inf_{\mathcal{E},\mathcal{D}} \sup_w ||y||_\infty = \quad (17)$$

$$\frac{|a|^{\lceil p\sqrt{R}\rceil+1} - 1}{|a| - 1} + (\lfloor 2^R \rfloor - |a|)^{-1} |a|^{\lceil p\sqrt{R}\rceil+1}.$$

When $a = 1$, the optimal performance is given by

$$\inf_{\mathcal{E},\mathcal{D}} \sup_w ||y||_\infty = \quad (18)$$

$$\lceil p\sqrt{R}\rceil + 1 + (\lfloor 2^R \rfloor - 1)^{-1}. \quad (19)$$

B. Implications

Impact of delay We consider the impact of delay on performance. Figure 2 shows the optimal performance $\eta = \inf_{\mathcal{E},\mathcal{D}} \sup_w ||y||_\infty$ for stable, marginally stable and unstable plants. Increment in delay greatly degrade the performance for unstable system. In contrast, large delay has less dramatic impact on the performance for stable plants, since the limit goes to the open loop norm. Note that relative to delay, even small values of R appear to make little difference.

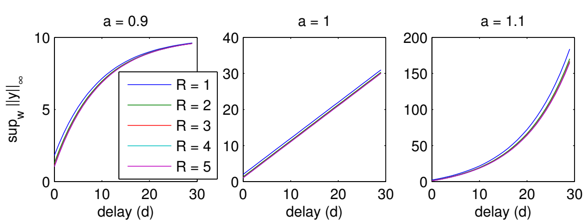

Fig. 2: Optimal Performance with varying delay (d) and channel capacity (R).

Impact of channel capacity To clarify the impact of channel capacity on performance, we focus on the low R regime. Figure 3 implies the increment in communication capacity has negligible effect on the performance when $2^R \gg |a|$.

Next, we fix delay and compare how $||x_d||_\infty$ and $||e||_\infty$ vary for two feedback systems with their plant transfer functions given by:

$$\begin{aligned} P_1(z) &= \frac{z}{z^2 - 0.1z - 10} \\ P_2(z) &= \frac{z}{z^2 - 0.1z}, \end{aligned}$$

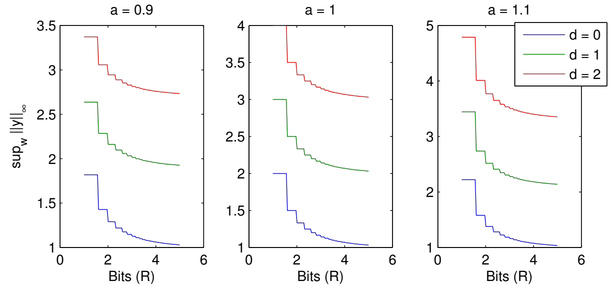

Fig. 3: Optimal Performance of different channel capacity

where $P_1(z)$ has unstable poles at $z = 3.2, -3.1$, and $P_2(z)$ has stable poles at $z = 0.1, 0$. Following table shows that the term $\sup_w ||x_d||_\infty$ is a function of $a(0 : d)$, and are same for both plants. However, the value of $\sup_w ||e||_\infty$ is a function of $a(0 : d)$, which gives quite different performance limitations the the two plants.

	system 1	system 2
$\inf_{\mathcal{E},\mathcal{D}} \sup_w \|\|y\|\|_\infty$	4.02	1.67
$\sup_w \|\|x_d\|\|_\infty$	1.10	1.10
$\sup_w \|\|e\|\|_\infty$	2.92	0.57

Trade-off of delay and channel capacity We consider the trade-off between delay and channel capacity under assumption C, and its impact on the system performance. Figure 4 (with $a = 1$) and compares the performance degradation due to delay $\sup_w ||x_d||_\infty$, the performance degradation due to channel capacity $\sup_w ||e||_\infty$, with the overall performance. When the scaling variable $p \propto \mathcal{A}^{-\frac{1}{2}}$ in (9) is small, within the plotted range, the performance monotonically increases (i.e., $\sup_w ||y||_\infty$ monotonically decreases) as channel capacity increases. This implies that, in axon's example, when the feasible area containing fibers is large, the channel capacity can be increased without performance degradation in $\sum_w ||x_d||_\infty$ up to the level that the delay is smaller than one time step. When p is moderately large, as the channel capacity increase, the term $\sup_w ||x_d||_\infty$ increases and the term $\sup_w ||e||_\infty$ decreases. The optimum performance lies in something around $R = 3$, at which both terms have competing sizes. When p is large, small increment in channel capacity would results in large increment in $\sup_w ||x_d||_\infty$. Therefore, the optimal performance is when the channel capacity is minimally stabilizing, in order to minimize delay. Note that this is, effectively, nearly *minimizing* the total information transfer.

V. Proof for Main Theorem

Proof: (Theorem 1, continued from proof sketch)

Continued from the proof sketch presented in section IV, now we look into how the volume of the uncertainty in $e(t)$ evolve over time in order to calculate $\inf_{\mathcal{E},\mathcal{D}} \sup_w \|e\|_\infty$. Using $x(t) = x_d(t) + e(t)$, we have

$$\begin{aligned} x(t+1) &= ax(t) + w(t) + u(t) \\ x_d(t+1) + e(t+1) &= ax_d(t) + ae(t) + w(t) + u(t). \end{aligned}$$

Since $ax_d(t) + w(t) - x_d(t+1) = a^{d+1}w(t-d)$, we have

$$e(t+1) = ae(t) + a^{d+1}w(t-d) + u(t). \tag{20}$$

Therefore, the optimal control action with perfect communication is:

$$u^*(t) = -a^{d+1}w(t-d) - ae(t). \tag{21}$$

Assume L_t to be the upper bound of the absolute value of $e(t)$, i.e., $|e(t)| \leq L_t$, the absolute value of $u^*(t)$ can be bounded by

$$|u^*(t)| \leq |a|L_t + \Upsilon, \tag{22}$$

where Υ is the least upper bound of the absolute value of $-a^{d+1}w(t-d)$:

$$\Upsilon \triangleq \sup_w |-a^{d+1}w(t-d)| = |a^{d+1}|,$$

since $\|w\|_\infty \leq 1$. If the channel uniformly quantizes the optimal control action $u^*(t)$ in the interval $[-\Psi_t, \Psi_t]$ with $\Psi_t = |a|L_t + \Upsilon$, i.e.,

$$u(t) = Q^{-1}_{\{R,\Psi_t\}} Q_{\{R,\Psi_t\}} u^*(t), \tag{23}$$

with $Q_{\{R,\Psi_t\}}$ in (4), then the control error due to quantizations at $t+1$ can be bounded by

$$|e(t+1)| = |u(t) - u^*(t)| \leq \frac{\Psi_t}{\lfloor 2^R \rfloor}$$

We can bound $|e(t+1)|$ by $L_{t+1} = \frac{\Psi_t}{2^R}$, which gives a recursive relation for the upper bounds of $|e(\cdot)|$:

$$L_{t+1} = \frac{|a|}{\lfloor 2^R \rfloor} L_{t-1} + \frac{\Upsilon}{\lfloor 2^R \rfloor} \tag{24}$$

Under the assumption $|a| < \lfloor 2^R \rfloor$, the recursion (24) have a unique fixed point L^* which satisfies the following equality:

$$L^* = \frac{|a|}{\lfloor 2^R \rfloor} L^* + \frac{\Upsilon}{\lfloor 2^R \rfloor}. \tag{25}$$

We have $L_0 = 0$ by initialization. The value L_t will monotonically increase toward L^*, i.e., $\lim_{t\to\infty} L_t = L^*$.

Conversely, the equality in (22) can be attained for some w. Therefore, the volume of the uncertainty in $e(t)$ will

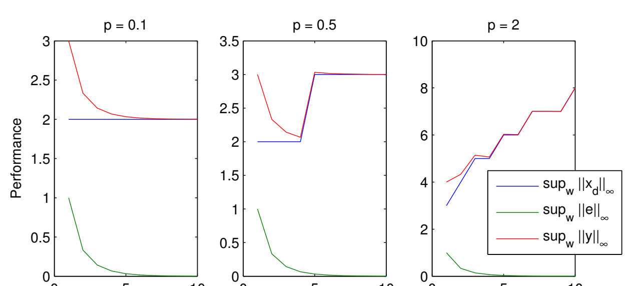

Fig. 4: The impact of $\sup_w ||x_d||_\infty$ and $\sup_w ||e||_\infty$ under different channel capacity R.

converge to a unique fixed value, which is also the *least* upper bound for $\|e\|_\infty$, i.e.,

$$\inf_{\mathcal{E},\mathcal{D}} \sup_w ||e||_\infty = L^*. \tag{26}$$

If $|a| \geq \lfloor 2^R \rfloor$, then the uncertainty of $e(t)$ grows with time. In this case, there does not exits an memoryless decoding scheme to achieve stability. ■

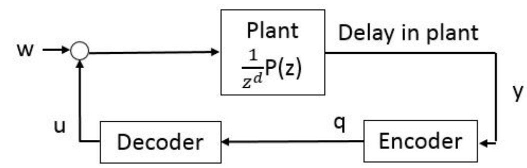

Fig. 5: Equivalent system for the feedback system with assumption A-D.

Remark 1: The infimum in (26) is achieved by the control law in (23), which uses *time invariant* uniform quantizers on the interval $[-\Psi^*, \Psi^*]$ with $\Psi^* = |a|L^* + \Upsilon$.

Proof: (Corollary 1) From Theorem 1, the optimal performance can be written as:

$$\inf_{\mathcal{E},\mathcal{D}} \sup_{w\in[\![w]\!]} ||y||_\infty = \sup_w ||x_d||_\infty + \inf_{\mathcal{E},\mathcal{D}} \sup_w ||e||_\infty. \tag{27}$$

Notice that

$$\begin{aligned} \sup_{w\in[\![w]\!]} ||x_d||_\infty &= \sum_{i=0}^{d} |a^i| = \frac{|a|^{d+1}-1}{|a|-1} \\ &= \frac{|a|^{\lceil p\sqrt{R}\rceil+1}-1}{|a|-1}, \end{aligned}$$

where the last inequality uses (9) from assumption C. By substituting (9) into $\inf_{\mathcal{E},\mathcal{D}} \sup_w ||e||_\infty$, we arrive (17). ■

VI. Extensions

Using similar ideas, following theorem can be proved as a general case.

Theorem 2: Consider the feedback system (8) with the plant

$$P(z) = \frac{z^{n-1}}{z^n + a_0 z^{n-1} + \cdots + a_n}.$$

assumption A-D. If the following relations holds:

$$|a_0| + |a_1| + \cdots + |a_n| < \lfloor 2^R \rfloor, \tag{28}$$

then the system performance satisfies

$$\begin{aligned} &\inf_{(\mathcal{D},\mathcal{E})} \sup_w ||y||_\infty \\ &\leq \sum_{i=1}^{d+1} |h_i| + \Big(\lfloor 2^R \rfloor - |a_0| - |a_1| - \cdots - |a_n|\Big)^{-1} \Upsilon. \end{aligned} \tag{29}$$

The term h_t is the impulse response of the plant defined by (3). The term Υ is defined by

$$\Upsilon \triangleq (n-d)|h_{d+2}| + \sum_{i=1}^{d} \Big| \sum_{j=1}^{i} a_{n+i-j} h_{d+1-i+j} \Big|. \tag{30}$$

The equality in (29) holds when degree of the plant is equal or less than 2, i.e., $n \leq 2$.

Conversely, if (28) holds, then the system performance satisfies

$$\inf_{(\mathcal{D},\mathcal{E})} \sup_{w \in [\![w]\!]} ||y(t)||_\infty \geq \sum_{i=1}^{d+1} |h_i| + \left(\lfloor 2^R \rfloor\right)^{-1} \Upsilon. \quad (31)$$

APPENDIX

Proof: (Theorem 2) We use the bold front $\mathbf{x}$ to denote the state of the system in state space notation (A, B, C) which is a vector stacked by individual state denoted by lower case letters x.

The state x is composed of two term: x_d and $e(t)$, defined similarly in the proof of theorem 1. By the same manner, we define

$$x_d(t) \triangleq \sum_{i=1}^{d+1} h(i) w(t-i) \quad (32)$$

$$e(t) \triangleq x(t) - x_d(t) \quad (33)$$

Similarly, the term $x_d(t)$ comes from $w(t-d:t)$ while the term $e(t)$ comes from $w(0:t-d-1)$. Since $x_d(t)$ and $e(t)$ are independent,

$$\inf_{\mathcal{E},\mathcal{D}} \sup_w ||y||_\infty \sup_w ||x_d||_\infty + \inf_{\mathcal{E},\mathcal{D}} \sup_w ||e||_\infty$$

We can decompose the evolution of the state $x(t+1)$ in terms of 4 components:

$$\begin{aligned} & x(t+1) \\ = \; & -\begin{bmatrix} a_n & \cdots & a_0 \end{bmatrix} \begin{bmatrix} x_d(t-n) + e(t-n) \\ \vdots \\ x_d(t) + e(t) \end{bmatrix} + w(t) \\ = \; & W_d(t) + W_m(t) + E(t) + u(t) \end{aligned} \quad (34)$$

The term

$$\begin{aligned} W_d(t) \; & \triangleq \; -\begin{bmatrix} a_n & \cdots & a_0 \end{bmatrix} \begin{bmatrix} x_d(t-n) \\ \vdots \\ x_d(t) \end{bmatrix} + w(t) \\ & = \; -\begin{bmatrix} a_n \\ \vdots \\ a_0 \end{bmatrix}' \begin{bmatrix} 0 & & \\ h_1 w_{t-d} & & \\ & \ddots & \\ h_1 w_{t-1} + \cdots h_{d+1} w_{t-d} & & \end{bmatrix} + w(t) \end{aligned}$$

consists of the sum of the distrubances $w(t-1:t-d-2)$, whose information cannot be access by the controller at time t. We wrote $w(t)$ as w_t due to space constraints. Substitute $h(k+2) = -a_k h_1 - a_{k-1} h_2 \cdots - a_0 h_{k+1}$ for $k \geq d$ into $W_d(t)$, we arrive

$$x_d(t+1) = W_d(t),$$

The term

$$W_m(t) = \\ -\begin{bmatrix} a_n \\ \vdots \\ a_0 \end{bmatrix}' \begin{bmatrix} h_1 w_{t-n-1} \cdots h_d w_{t-n-d-2} + h_{d+1} w_{t-d-n-1} \\ \vdots \\ h_1 w_{t-d-1} \cdots h_d w_{t-d-n-1} + h_{d+1} w_{t-d-n-2} \\ \ddots \\ h_d w_{t-d-1} + h_{d+1} w_{t-d-2} \\ h_{d+1} w_{t-d-1} \end{bmatrix}$$

is the sum of the term caused by $w(t-d-1:t-d-n-1)$, which can be measured at time t. Since the state $x(t+1)$ is affected by $x(t-n:t)$, this term is present in $A\mathbf{x}(t)$.

The term

$$E(t) = -a_n e(t-n) - \cdots - a_1 e(t-1) - a_0 e(t)$$

is caused by the error in control due to communication constraint from $t-n$ to t.

The performance degradation from delayed communication with infinite capacity is

$$e(t+1) = W_m(t) + E(t) + u(t) \quad (35)$$

We now construct a upper bound for $\sup_w ||e||_\infty$.

Define $u^*(t) = -W_m(t) - E(t)$ to be the optimal control action before communication error comes in, which is only accessble from the encoder. Let $u(t)$ to be the actual control value, we will first bound $u(t) - u^* t(t)$.

Assume we know the upper bound of $e(t)$ is L_t for $t \leq T$, i.e., $|e(t)| \leq L_t$, then

$$\begin{aligned} & |u^*(t)| \\ \leq \; & |W_m(t)| + |E(t)| \\ \leq \; & \Big| \sum_{k=t-n-1}^{t-d-1} h_{d+2} w_k \\ & + \sum_{i=1}^{d} \sum_{j=1}^{i} a_{n+i-j} h_{d+1-i+j} w_{t-n-d+i-2} \Big| \quad (36) \\ & + |a_0 e(t) + a_1 e(t-1) \cdots a_n e(t-n)| \\ \leq \; & \sum_{k=t-n-1}^{t-d-1} |h_{d+2}||w_k| \\ & + \sum_{i=1}^{d} \Big| \sum_{j=1}^{i} a_{n+i-j} h_{d+1-i+j} \Big| |w_{t-n-d+i-2}| \quad (37) \\ & + |a_0||e(t)| + |a_1||e(t-1)| \cdots |a_n||e(t-n)| \\ \leq \; & \Big(|a_0| L_t + |a_1| L_{t-1} \cdots |a_n| L_{t-n} + \Upsilon \Big) \delta \quad (38) \end{aligned}$$

where the term Υ is defined as

$$\Upsilon \triangleq \sum_{k=t-n-1}^{t-d-1} |h_{d+2}| + \sum_{i=1}^{d} \Big| \sum_{j=1}^{i} a_{n+i-j} h_{d+1-i+j} \Big|. \quad (39)$$

We take an uniform quantizer in the interval $[-\Psi_t, \Psi_t]$ with $\Psi_t = |a_0| L_t + |a_1| L_{t-1} + \cdots + |a_n| L_{t-n} + \Upsilon$ to encode the value $u^*(t)$. As a result, the actual control value is

$$u(t) = Q^{-1}_{\{R,\Psi_t\}} Q_{\{R,\Psi_t\}} u^*(t) \quad (40)$$

with the error due to quantization at $t+1$ being

$$|e(t+1)| \quad \leq \quad \frac{\Psi_t}{2^R} = L_{t+1}$$

We arrive

$$L_{t+1} = \frac{|a_0|}{2^R} L_t + \frac{|a_1|}{2^R} L_{t-1} + \cdots + \frac{|a_n|}{2^R} L_{t-n} + \frac{\Upsilon}{2^R} \quad (41)$$

Define the mapping $g : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$:

$$g(\varepsilon) = \begin{bmatrix} 0 & 1 & & \\ & \ddots & & \\ & & & 1 \\ \frac{|a_n|}{2^R} & \frac{|a_{n-1}|}{2^R} & \cdots & \frac{|a_0|}{2^R} \end{bmatrix} \varepsilon + \frac{\Upsilon}{2^R}$$

It can be shown that the mapping g is a contraction under the condition (28) Equivalently, the recursion $\varepsilon_{t+1} = g(\varepsilon_t)$ converges to a unique point ε^* that satisfy $\varepsilon^* = g(\varepsilon^*)$. By setting $\varepsilon_t = \begin{bmatrix} L_{t-n} & \cdots & L_t \end{bmatrix}'$, we know the recursion (41) also converges to a unique point L^* that satisfies

$$L^* = \frac{|a_0|}{2^R} L^* + \frac{|a_1|}{2^R} L^* \cdots \frac{|a_n|}{2^R} L^* + \frac{\Upsilon}{2^R} \quad (42)$$

The size of the uncertainty in $e(t)$ will converge to a fixed volume which is gives an upper bound for $||e||_\infty$, i.e.,

$$\sup_w ||e||_\infty \leq L^*$$

Conversely, if the equality in (38) is achievable, then the optimal control laws is

$$u(t) = Q^{-1}_{\{R,\Psi^*\}} Q_{\{R,\Psi^*\}} u^*(t), \quad (43)$$

This can be achieved by having a time invariant uniform quantizer on the interval $[-\Psi^*, \Psi^*]$ where Ψ^* is defined as

$$\Psi^* \quad = \quad \big(|a_0| L^* \cdots |a_n| L^* + \Upsilon\big)\delta \quad (44)$$

with L^* being the unique fix point of the contraction mapping (41) satisfying (42).

■

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