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Localized Distributed Optimal Control with Output Feedback and Communication Delays

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Abstract— **This paper presents an output feedback control scheme for localizable distributed systems subject to delay, that is to say systems for which the effect of both process noise and sensor noise can be localized in closed loop despite communications delays between controllers. By reformulating the distributed optimal control problem in terms of the closed loop transfer matrices from sensor and process noise to controlled output, we cast the optimal localized distributed control problem as a finite dimensional affinely constrained convex program. We additionally show how to synthesize the controller achieving the desired closed loop response, and that the controller can be implemented in a localized and thus scalable manner, which is essential when applying the scheme to large scale systems. Simulation shows that for certain systems, our optimal controller, with its constraints on locality, settling time, and communication delay, can achieve similar performance to a centralized optimal one.**

I. Introduction

A fundamental challenge in controller design is balancing the trade-off between closed loop system performance and controller complexity. This issue becomes one that cannot be ignored for large scale distributed systems, as simple local controllers generated by heuristics cannot guarantee global performance (or at times even stability), whereas a traditional centralized controller simply cannot be implemented.

Specifically, the unrealistic assumption behind traditional centralized optimal control is that information is *instantaneously* shared among *all* controllers. In order to incorporate realistic communication constraints into the control design process, the field of distributed (decentralized) optimal control has emerged. These communication constraints often manifest themselves in the optimal control problem as subspace constraints on the controller, such as delay and sparsity constraints.

It was soon realized, however, that the tractability of the distributed optimal control problem depends on the imposed information structure relative to the dynamics of the plant. For a completely decentralized information structure (infinite communication delay between controllers), the problem is known to be NP-hard in the worst case [1], [2]. If the information sharing constraint is quadratically invariant (QI) with respect to the plant, then the distributed optimal control problem can be formulated in a convex manner [3], [4] by Youla parametrization [5]. With the identification of quadratic invariance as an appropriate means of convexifying the distributed optimal control problem, much progress has been made in finding finite dimensional reformulations of the optimal control problem.

In the case of QI constraints induced by strongly connected communication graphs, computing the optimal controller has been reduced to solving a finite dimensional convex program in [6], [7] ($\mathcal{H}_2$ criteria) and [8] ($\mathcal{H}_\infty$ criteria). Roughly speaking, constraints imposed by communication networks that allow exchange of information between controllers faster than dynamics propagate through the plant can be handled in a convex way. In the case of sparsity constrained controllers, specific structures have been explicitly solved: including two-player [9], [10], triangular [11], [12], partially nested [13], and poset-causal [14], [15] systems. It should also be noted that similar approaches to convex distributed optimal controller synthesis have been applied to spatially invariant systems satisfying a funnel causality property [16], [17].

As promising and impressive as all of the results have been, they ignore one key aspect of applying distributed control to large scale systems: *scalability*. Although the computed controllers respect the information sharing constraints of the problem, in all of the above, the resulting optimal controller is as difficult to compute as its centralized counterpart, and even more demanding to implement! In fact, for a plant with a strongly connected topology, even as simple as a chain, the QI condition implies that local measurements need to be shared among *all local controllers* in the network (which can be inferred from [4] and [18]). This then implies that finding an optimal sparse controller K (which is defined as the transfer function from measurement to control action) for a strongly connected plant cannot be formulated in a convex manner in the Youla domain [19].

Based on this discussion, we argue that the closed loop behavior achieved by these distributed optimal controllers should be viewed as fundamental limits on performance, that we should strive to attain via more *scalable* and *realistically implementable* control schemes.

The idea of designing localized (sparse) controllers to improve scalability and ease of implementation is not a new one. For example, regularization or approximation have been used in hopes of finding a sparse controller, such as sparsity-promoting control [20], [21], spatial truncation [22], or on a spatial decaying system [23].

The key observation in our work that distinguishes us

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from existing results in the literature is that a localized (and thus sparse and scalable) controller implementation does not necessarily imply that K (which we recall is the transfer function from measurements to control actions) needs to be sparse. In fact, by using implicit implementations of the controller (which allow the communication of both *measurements and control actions*), we show how localized closed loop responses and implementations can indeed be compatible with each other, and computed in a convex way.

Specifically, in [24], we introduced the notion of a state feedback localizable system, that is to say systems for which a localized closed loop response exists. We derived a direct relation between the closed loop response of the system and the controller implementation achieving the desired closed loop. Building on these results, in [25] we showed how the computation and implementation of a LQR optimal localizing state feedback law can be done in a completely localized (and hence scalable) manner. Surprisingly, simulations showed that the proposed localized controller achieved performance similar to that of a centralized optimal one in terms of the LQR cost, while being far superior in terms of the sparsity of implementation and the closed loop response. We emphasize once again that this latter property is key for the scalability of this scheme.

This paper extends our previous results on localized distributed control to output feedback in a generalized framework. We characterize output feedback localizable system by the feasibility of a set of linear equation, and formulate the localized distributed optimal control problem as a finite dimensional affinely constrained convex program. Then, we propose two different localized controller implementations achieving the desired closed loop response. Simulation shows that our localized (scalable) controller implementation can achieve similar transient response to a centralized optimal one.

The rest of this paper is organized as follows. Section II introduces the system model, spatio-temporal constraints, and distributed optimal control. In Section III, we reformulate the distributed optimal control problem in terms of the closed loop transfer matrix and formally define localizability for output feedback systems. We then formulate the optimal localized distributed control problem as a finite dimensional affinely constrained convex program. In Section IV, we propose two different ways of implementing the localized controller based on the solution obtained from the optimization problem. Simulation results are shown in Section V to illustrate the effectiveness of our method. Lastly, Section VI ends with conclusions and offers some future research directions.

II. Preliminaries

In this section, we describe the system model, spatio-temporal constraints, and the traditional formulation for distributed optimal control.

A. System Model

Consider a discrete time distributed system described by

$$\begin{aligned} x[k+1] &= Ax[k] + B_2 u[k] + \delta_x[k] \\ y[k] &= C_2 x[k] + \delta_y[k] \end{aligned} \tag{1}$$

where $x = (x_i)$, $u = (u_i)$, and $y = (y_i)$ are stacked vectors of local state, control, and measurement, respectively, and $\delta_x = (\delta_x)_i$ and $\delta_y = (\delta_y)_i$ are the stacked perturbations on each state and measurement, respectively. These perturbation may represent the process noise and sensor noise of the system, but may include other sources of disturbance. For example, actuator saturation and/or quantization on the control action can be modeled as $\delta_x = B_2\delta_u$ for appropriately defined δ_u.

We also assume that the plant is open-loop stable – this assumption is required for technical reasons, but we believe that this is symptomatic of our derivation, and can be lifted. This is the subject of current work.

The objective is to design a distributed controller that generates a control signal u, based on the measurements y (subject to the information sharing constraints of the system) such that the closed loop transfer functions from the perturbations (δ_x, δ_y) to (x, u) satisfy some spatio-temporal constraints (which will be defined in the next subsection). We will show that if the closed loop transfer functions can be localized, then the controller can be implemented in a localized manner. Throughout this paper, we use $L[i]$ to denote the i-th spectral component of any transfer function L, $L = \sum_{i=0}^{\infty} \frac{1}{z^i} L[i]$. In time domain, the order set $\{L[i]\}_{i=0}^{\infty}$ describes the impulse response of the system.

B. Spatio-Temporal Constraints

Let $\text{sp}(\cdot) : R^{m\times n} \to \{0,1\}^{m\times n}$ be the support operator, where $\{\text{sp}(A)\}_{ij} = 1$ if and only if $A_{ij} \neq 0$, and 0 otherwise. We define $\mathcal{S}_1 \bigcup \mathcal{S}_2$ the entry-wise OR operation for two binary matrices $\mathcal{S}_1, \mathcal{S}_2 \in \{0,1\}^{m\times n}$. In addition, we write $\mathcal{S}_1 \subseteq \mathcal{S}_2$ when $\mathcal{S}_1 \bigcup \mathcal{S}_2 = \mathcal{S}_2$. The product $\mathcal{S}_1 = \mathcal{S}_2\mathcal{S}_3$ with binary matrices of compatible dimension is defined by the rule

$$\begin{aligned} &(\mathcal{S}_1)_{ij} = 1 \textit{ iff there exists a } k \textit{ such that} \\ &(\mathcal{S}_2)_{ik} = 1 \textit{ and } (\mathcal{S}_3)_{kj} = 1. \end{aligned}$$

For a square binary matrix $\mathcal{S}_0$, we define the power $\mathcal{S}_0^{i+1} := \mathcal{S}_0^i \mathcal{S}_0$ for all positive integer i, and $\mathcal{S}_0^0 = I$. If $\mathcal{S}_0$ is the support of the adjacency matrix of a graph, the distance from state k to j can be defined as

$$\text{dist}_{\mathcal{S}_0}(k \to j) := \min\{i \in \mathbb{N} \cup 0 \mid \left(\mathcal{S}_0^i\right)_{jk} \neq 0\}.$$

Let $\mathcal{RH}_\infty$ be the set of all real-rational stable proper transfer functions. The spatio-temporal constraints for a transfer function R is described by a constraint space $\mathcal{S}_R := \sum_{i=0}^{\infty} \frac{1}{z^i} \mathcal{S}_R[i]$, where $\{\mathcal{S}_R[i]\}$ is an ordered set of binary matrices. We say that a transfer function R is in $\mathcal{S}_R$ if and only if $\text{sp}(R[i]) \subseteq \mathcal{S}_R[i]$ for all i, which we denote by $R \in \mathcal{S}_R$.

In this paper, we consider finite impulse response (FIR) constraints, locality constraints, and communication delay constraints – the combination of these three constraints allow for the control problem to be formulated as a finite dimensional convex program that leads to a scalable implementation, all the while respecting the information sharing constraints of the system.

1) FIR constraints: If $\mathcal{S}_R[i] = \mathbf{0}$ for all $i > T$, we say that $\mathcal{S}_R$ is finite in time T, and any transfer function in $\mathcal{S}_R$ has a FIR.

2) Locality constraints: We describe locality constraints in terms of (A, d) sparseness, which is defined as follows.

Definition 1: A real matrix X is (A, d) *sparse* if

$$\text{sp}\,(X) \subseteq \bigcup_{i=0}^{d} \text{sp}\,(A)^i.$$

A constraint space $\mathcal{S}_x$ is (A, d) sparse if and only if $\mathcal{S}_x[i]$ is (A, d) sparse for all i.[1] The (A, d) sparseness for a transfer function can be defined in a similar manner.

The definition of (A, d) sparseness has a fairly intuitive interpretation in terms of the distance function on the topology of A. Define $\mathcal{E}_{(j,d)}$ the set of all (possibly) nonzero elements in j-th row of $\bigcup_{i=0}^{d} \text{sp}\,(A)^i$, and $\mathcal{F}_{(j,d)}$ the set of all (possibly) nonzero elements in j-th column of $\bigcup_{i=0}^{d} \text{sp}\,(A)^i$. It is straightforward to show that

$$\begin{aligned}\mathcal{E}_{(j,d)} &= \{s \,|\, \text{dist}_{\text{sp}(A)}\,(s \rightarrow j) \leq d\} \\ \mathcal{F}_{(j,d)} &= \{s \,|\, \text{dist}_{\text{sp}(A)}\,(j \rightarrow s) \leq d\}.\end{aligned}$$

In other words, (A, d) sparseness provides a natural measure of sparsity with respect to the underlying topology of a given matrix A, which in our case will be intimately related to how perturbations spread across the physical topology of the plant.

3) Communication delay constraints: In our control scheme, the required communication delay constraints depend on the implementation of the controller, and therefore, we hold off on defining these until after we have introduced the controller implementation in Section IV.

C. Distributed Optimal Control Problem

Consider a stable discrete time plant model given by

$$P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}. \qquad (2)$$

Let K be the transfer function from y to u, i.e. $u = Ky$. The optimal control problem of minimizing the norm[2] of the closed loop transfer function from the disturbances to the controlled output is formulated as

$$\begin{array}{ll} \underset{K}{\text{minimize}} & ||P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}|| \\ \text{subject to} & K \in \mathcal{S}_c \\ & K(I - P_{22}K)^{-1} \in \mathcal{RH}_\infty \end{array} \qquad (3)$$

where $\mathcal{S}_c$ is the information sharing constraint that K must satisfy in order to be implementable on the communication network of the system. As can be seen, this problem is not convex in K, and as alluded to in the introduction, only certain classes of $\mathcal{S}_c$ can be reformulated in a convex manner.

The information sharing constraint $\mathcal{S}_c$ is said to be QI [3] under P_{22} if $KP_{22}K \in \mathcal{S}_c$ for all $K \in \mathcal{S}_c$. If $\mathcal{S}_c$ is QI under P_{22}, then $K \in \mathcal{S}_c$ is equivalent to $Q \in \mathcal{S}_c \cap \mathcal{RH}_\infty$, where $Q = K(I - P_{22}K)^{-1}$ is the Youla parametrization [3]. Thus, we can convexify (3) by a change of variable as

$$\begin{array}{ll} \underset{Q}{\text{minimize}} & ||P_{11} + P_{12}QP_{21}|| \\ \text{subject to} & Q \in \mathcal{S}_c \cap \mathcal{RH}_\infty. \end{array} \qquad (4)$$

The controller is then implemented by $K = (I + QP_{22})^{-1}Q$, and $K \in \mathcal{S}_c$.

As discussed previously, the primary limitation of this formulation is that locality constraints are not QI for strongly connected plants – thus the aforementioned reparameterization does not yield a K that satisfies the locality constraints, even if such a Q could be computed. However, it is important to note that this does not mean that the controller cannot be implemented in a localized way. In the next section, we will reformulate (4) in terms of the closed loop transfer matrix from the perturbations (δ_x, δ_y) to (x, u), upon which we can impose the locality constraint. We will show that the localized distributed controller can be synthesized directly from this closed loop transfer matrix.

III. Localized Distributed Optimal Control

In this section, we reformulate the distributed optimal control problem in terms of the closed loop transfer matrix. Then, we define localizability by imposing the spatio-temporal constraints on the closed loop transfer matrix. This section ends with the general formulation of localized distributed optimal control problem.

A. Reformulation of Distributed Optimal Control Problem

Taking the z-transform of (1), we have

$$\begin{aligned}(zI - A)x &= B_2u + \delta_x \\ y &= C_2x + \delta_y.\end{aligned}$$

Assume that there exists a stabilizing real rational (strictly) proper K such that the control rule $u = Ky$ stabilizes the plant. We then have

$$(zI - A - B_2KC_2)x = \delta_x + B_2K\delta_y$$

As the controller is stabilizing, the inverse of $(zI - A - B_2KC_2)$ exists. We define $R = (zI - A - B_2KC_2)^{-1}$, which leads to

$$\begin{aligned}x &= R\delta_x + RB_2K\delta_y \\ u &= KC_2R\delta_x + (K + KC_2RB_2K)\delta_y.\end{aligned}$$

The closed loop transfer function from (δ_x, δ_y) to (x, u) can be summarized as

$$\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} R & RB_2K \\ KC_2R & K + KC_2RB_2K \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix}. \qquad (5)$$

[1]For a large d, it is possible that $\bigcup_{i=0}^{d} \text{sp}\,(A)^i = \mathbf{1}$ - this simply means that the "localized" region is the entire system.

[2]Typical choices for the norm include $\mathcal{H}_2$ and $\mathcal{H}_\infty$, although our formulation is amenable to any convex measure of closed loop performance.

Before formally defining the closed loop transfer matrix, we will derive a relation between (5) and the Youla parametrization $Q = K(I - P_{22}K)^{-1}$.

Lemma 1: $Q = K + KC_2RB_2K$.

Proof: From the definition of R, we have

$$(zI - A - B_2KC_2)R = I.$$

As we assume the plant is open-loop stable, $(zI - A)^{-1}$ is well-defined. Moving B_2KC_2R to the right-hand-side (RHS) and multiplying $(zI-A)^{-1}$ on both side, it is straightforward to derive that

$$R = (zI - A)^{-1}(I + B_2KC_2R). \tag{6}$$

Then,

$$\begin{aligned} C_2R &= C_2(zI - A)^{-1} + C_2(zI - A)^{-1}B_2KC_2R \\ &= C_2(zI - A)^{-1} + P_{22}KC_2R. \end{aligned}$$

Moving $P_{22}KC_2R$ to the left-hand-side (LHS) and multiplying $(I - P_{22}K)^{-1}$ on both sides, we have

$$C_2R = (I - P_{22}K)^{-1}C_2(zI - A)^{-1}. \tag{7}$$

Finally,

$$\begin{aligned} & K + KC_2RB_2K \\ = \;& K + K(I - P_{22}K)^{-1}C_2(zI - A)^{-1}B_2K \\ = \;& K + K(I - P_{22}K)^{-1}P_{22}K \\ = \;& K(I - P_{22}K)^{-1} \\ = \;& Q. \end{aligned}$$

■

We can then formally define the closed loop transfer matrix as follows.

Definition 2: For the system with dynamics described by (1), a stable block transfer matrix with structure $\begin{bmatrix} R & N \\ M & Q \end{bmatrix}$ is a closed loop transfer matrix if and only if there exists a real rational (strictly) proper K such that

$$\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} R & N \\ M & Q \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix}, \tag{8}$$

where

$$\begin{aligned} R &= (zI - A - B_2KC_2)^{-1} \\ N &= RB_2K \\ M &= KC_2R \\ Q &= K + KC_2RB_2K. \end{aligned}$$

The following lemma states the algebraic relations that a closed loop transfer matrix must satisfy.

Lemma 2: $\begin{bmatrix} R & N \\ M & Q \end{bmatrix}$ is a closed loop transfer matrix for (1) if and only if R, N, M are strictly proper, and $R, N, M, Q \in \mathcal{RH}_\infty$ satisfy the following equations.

$$\begin{bmatrix} zI - A & -B_2 \end{bmatrix} \begin{bmatrix} R & N \\ M & Q \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix} \tag{9}$$

$$\begin{bmatrix} R & N \\ M & Q \end{bmatrix} \begin{bmatrix} zI - A \\ -C_2 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{10}$$

Proof: Assume that $\begin{bmatrix} R & N \\ M & Q \end{bmatrix}$ is a closed loop transfer matrix. From the definition, it is straightforward to show that R, N, M are strictly proper. Multiplying $(zI - A)$ on both sides of (6), we have

$$(zI - A)R = I + B_2KC_2R = I + B_2M.$$

From the definition,

$$\begin{aligned} (zI - A)N &= (zI - A)RB_2K \\ &= B_2K + B_2KC_2RB_2K \\ &= B_2Q, \end{aligned}$$

so (9) is satisfied. Similarly, as $R(zI - A - B_2KC_2) = I$, it is straightforward to derive that $R(zI - A) = I + NC_2$. From the definition and (7), we have

$$M(zI - A) = KC_2R(zI - A) = QC_2,$$

so (10) is also satisfied.

Assume that there exists $R, N, M, Q \in \mathcal{RH}_\infty$ satisfying (9) and (10): we show that $\begin{bmatrix} R & N \\ M & Q \end{bmatrix}$ is a closed loop transfer matrix by reconstructing K. Define $K_s = Q - MR^{-1}N$, which implies

$$B_2K_sC_2 = B_2QC_2 - B_2MR^{-1}NC_2.$$

Substituting B_2M and NC_2 from (9) and (10), we derive

$$B_2K_sC_2 = B_2QC_2 - R^{-1} + 2(zI - A) - (zI - A)R(zI - A).$$

In addition, we can derive $B_2QC_2 = (zI - A)R(zI - A) - (zI - A)$ directly from (9) and (10). Combining these two identities, it is straightforward to show that

$$R = (zI - A - B_2K_sC_2)^{-1}.$$

Similarly, $B_2K_s = B_2Q - B_2MR^{-1}N$. Substituting B_2M from (9), we derive

$$\begin{aligned} B_2K_s &= B_2Q - (zI - A)N + R^{-1}N \\ &= R^{-1}N. \end{aligned}$$

So $N = RB_2K_s$. Following a similar procedure, we can show that M and Q also satisfy the definition of a closed loop transfer matrix. Thus $K = K_s$ and the proof is completed.

■

From Lemma 2, the distributed optimal control problem in (4) can be reformulated in terms of the closed loop transfer matrix as

$$\begin{aligned} \underset{\{R,M,N,Q \in \mathcal{RH}_\infty\}}{\text{minimize}} \quad & \left|\left| \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} R & N \\ M & Q \end{bmatrix} \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} \right|\right| \\ \text{subject to} \quad & (9), (10), Q \in \mathcal{S}_c \end{aligned} \tag{11}$$

for some strictly proper transfer functions R, N, and M. By applying Lemma 2, we use the constraints (9) and (10) to ensure that the relation (8) is satisfied. The objective in (11) follows directly from the relation (8), so the two formulations (4) and (11) are equivalent.

The advantage of the formulation (11) is that the objective function is convex with respect to the closed loop transfer matrix. We can further impose the sparsity constraints on the closed loop transfer matrix to localize the closed loop response, and the optimization problem is still convex. Once the closed loop response is localized, the controller can be implemented by a combination of those localized transfer functions. Note that we do not need to solve for K in the synthesis procedure, nor in order to actually implement the controller (which will be shown in Section IV) – this in effect allows us to bypass the non-convexity issues introduced by the potentially non-QI constraints that we are imposing in order to maintain locality.

B. Output Feedback Localizability

We now incorporate spatio-temporal constraints into (11). We define a (d,T) localized FIR constraint for a system as follows.

Definition 3: The constraint space quadruple $(\mathcal{S}_R, \mathcal{S}_N, \mathcal{S}_M, \mathcal{S}_Q)$ is a (d,T) localized FIR constraint for system (1) if and only if

1) $\mathcal{S}_R, \mathcal{S}_N, \mathcal{S}_M, \mathcal{S}_Q$ are finite in time T.
2) $\mathcal{S}_R$ is (A,d) sparse.
3) $\mathcal{S}_N \text{sp}(C_2)$ is $(A,d+1)$ sparse.
4) $\text{sp}(B_2)\mathcal{S}_M$ is $(A,d+1)$ sparse.
5) $\text{sp}(B_2)\mathcal{S}_Q \text{sp}(C_2)$ is $(A,d+2)$ sparse.

The purpose of a (d,T) localized FIR constraint is to constrain the closed loop transfer matrix to lie inside a prescribed space-time region. For instance, if $R \in \mathcal{S}_R$, then each local state perturbation $(\delta_x)_j$ can only affect the states in the set $\mathcal{F}_{(j,d)}$ with time T in closed loop. On the other hand, each state $(x)_j$ can only be affected by the state perturbation in the set $\mathcal{E}_{(j,d)}$.

It is also noteworthy that the condition in Definition 3 can be simplified for some specialized plant models. For example, for the scalar sub-system model[3] considered in [24], we only need to impose the constraint that $\mathcal{S}_R$ is (A,d) sparse and finite in time T as all other conditions will then be automatically satisfied.

Finally, it is straightforward to define output feedback localizability for a system as follows.

Definition 4: Assume that $(\mathcal{S}_R, \mathcal{S}_N, \mathcal{S}_M, \mathcal{S}_Q)$ is a (d,T) localized FIR constraint for (1). The system (1) is (d,T) localizable by $(\mathcal{S}_R, \mathcal{S}_N, \mathcal{S}_M, \mathcal{S}_Q)$ if and only if there exists a closed loop transfer matrix $\begin{bmatrix} R & N \\ M & Q \end{bmatrix}$ for (1) such that

$$\begin{bmatrix} R & N \\ M & Q \end{bmatrix} \in \begin{bmatrix} \mathcal{S}_R & \mathcal{S}_N \\ \mathcal{S}_M & \mathcal{S}_Q \end{bmatrix}. \tag{12}$$

Note that (12) implicitly imply that $R,N,M,Q \in \mathcal{RH}_\infty$ as FIR transfer functions are always stable.

[3] A scalar sub-system model is one in which B_2 has full column rank with exactly one non-zero entry per column and C_2 has full row rank with exactly one non-zero entry per row.

C. General Formulation

From the definition of localizability, we can incorporate the (d,T) localized FIR constraint into (11) and formulate the general localized distributed optimal control problem as

$$\begin{aligned} \underset{\{R,N,M,Q\}}{\text{minimize}} \quad & f(R,N,M,Q) \\ \text{subject to} \quad & (9), (10), (12) \end{aligned} \tag{13}$$

for some convex function f. Note that all of the constraints in (13) are affine, and the constraint (12) further makes this optimization problem finite dimensional, which significantly simplifies the computational aspects of the problem.

Many distributed optimal control problems fall within the framework of (13) by choosing f appropriately. For example, the localized distributed $\mathcal{H}_2$ optimal control problem for (2) is given by

$$\begin{aligned} \underset{\{R,M,N,Q\}}{\text{minimize}} \quad & || \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} R & N \\ M & Q \end{bmatrix} \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} ||_{\mathcal{H}_2} \\ \text{subject to} \quad & \begin{bmatrix} zI - A & -B_2 \end{bmatrix} \begin{bmatrix} R & N \\ M & Q \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix} \\ & \begin{bmatrix} R & N \\ M & Q \end{bmatrix} \begin{bmatrix} zI - A \\ -C_2 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} \\ & \begin{bmatrix} R & N \\ M & Q \end{bmatrix} \in \begin{bmatrix} \mathcal{S}_R & \mathcal{S}_N \\ \mathcal{S}_M & \mathcal{S}_Q \end{bmatrix}. \end{aligned} \tag{14}$$

Localizability can be determined by the feasibility of (14). The localized distributed state feedback control in [24], [25] can be viewed as special cases of this formulation.

IV. Controller Implementation

In this section, we will show that there exists localized implementations achieving the desired localized closed-loop response. Specifically, we provide two different ways of implementing the localized controller based on the solution obtained from the optimization problem (13).

A. Implementation from Q-parametrization

From the Youla parametrization $K = (I + QP_{22})^{-1}Q$, $u = Ky$ leads to $(I + QP_{22})u = Qy$. Moving the $QP_{22}u$ term to the RHS, we have the implementation

$$\begin{aligned} u &= Qy - QP_{22}u \\ &= Qy - QC_2(zI - A)^{-1}B_2 u \\ &= Qy - MB_2 u \end{aligned} \tag{15}$$

As Q and M satisfy the locality constraint, (15) is a localized controller implementation achieving the desired closed loop response in (13). In other words, each local controller $(u)_i$ only needs to collect a subset of the measurements y and control actions u from a localized past space-time region in order to generate its current control action. To ensure that the implementation satisfies the communication delay constraints of the system, we impose delay constraints on Q (from y to u, as the usual case in distributed control) and on MB_2 (from u to u). Although we do not explicitly specify any requirements on the communication delay pattern, there is an implicit assumption that the communication speed between controllers is faster than propagation of disturbances through the plant, so that it is possible to localize the effect of the disturbance. Quantifying the explicit relationships needed between communication and computation delays and propagation delays to satisfy locality, as is done in [4] for QI constraints, is the subject of current work.

B. Alternative Implementation

An alternative implementation is based on the identity $K = Q - MR^{-1}N$ mentioned in the proof of Lemma 2, which seems to have better numerical property than (15). We use the technique introduced in [25] to implement the R^{-1} term in a localized way. Specifically, in frequency domain, the implementation is summarized as

$$\begin{aligned} u &= Qy - M\beta \\ \frac{1}{z}\beta &= \alpha - \alpha_r \\ \alpha &= Ny \\ \alpha_r &= (R - \frac{1}{z}I)\beta. \end{aligned} \tag{16}$$

Defining $\bar{\beta} = \frac{1}{z}\beta$, we may write this implementation in time domain, for a time index t, (16), as

$$\begin{aligned} &\bar{\beta}[t] = \alpha[t] - \alpha_r[t] \\ &u[t] = \sum_{\tau=0}^{T} Q[\tau]y[t-\tau] - \sum_{\tau=0}^{T-1} M[\tau+1]\bar{\beta}[t-\tau] \\ &\alpha[t+1] = \sum_{\tau=0}^{T-1} N[\tau+1]y[t-\tau] \\ &\alpha_r[t+1] = \sum_{\tau=0}^{T-2} R[\tau+2]\bar{\beta}[t-\tau]. \end{aligned} \tag{17}$$

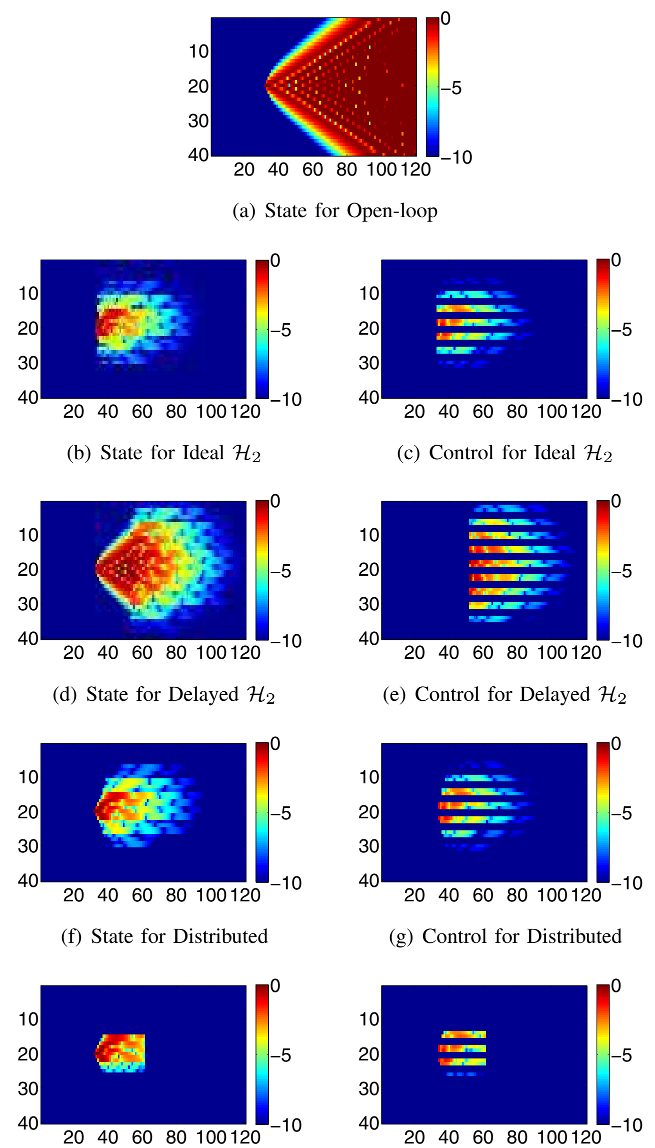
Similarly, as the controller is implemented by the transfer functions R, N, M, Q directly, (17) is a localized implementation. The communication delay constraint for this scheme depends on how the data is routed in the actual communication network. We need to impose the communication delay constraint on R, N, M, Q to ensure that the scheme (17) is *implementable* on that particular communication network.

Example 1: Assume that the communication network topology follows the plant topology, but with a speed $h > 1$ times faster than the plant propagation speed. Assume that each state has a computing unit that can receive, compute, and transmit the data. For the scalar sub-system model with sensing delay being 1 time step, a spatio-temporal constraint leading to (d, T) locality is then given by

$$\begin{aligned} \mathcal{S}_R &= \sum_{i=1}^{T} \frac{1}{z^i} \text{sp}\left(A\right)^{\min(d, \lfloor h(i-1) \rfloor)} \\ \mathcal{S}_N &= \sum_{i=1}^{T} \frac{1}{z^i} \text{sp}\left(A\right)^{\min(d+1, \lfloor h(i-1) \rfloor)} \text{sp}\left(C_2^\top\right) \\ \mathcal{S}_M &= \sum_{i=1}^{T} \frac{1}{z^i} \text{sp}\left(B_2^\top\right) \text{sp}\left(A\right)^{\min(d+1, \lfloor h(i-1) \rfloor)} \\ \mathcal{S}_Q &= \sum_{i=1}^{T} \frac{1}{z^i} \text{sp}\left(B_2^\top\right) \text{sp}\left(A\right)^{\min(d+2, \lfloor h(i-1) \rfloor)} \text{sp}\left(C_2^\top\right). \end{aligned} \tag{18}$$

The implementation (17) seems to work in a robust manner when the system is not *exactly* localized. This holds true even when the original plant is open-loop unstable, which will be illustrated in the simulation in the next section. Future work will investigate the robustness of this implementation subject to computation error and plant uncertainty. It is also necessary to establish a coherent theory for the implementation (17) on an open-loop unstable plant.

V. Performance Comparison



(a) State for Open-loop

(b) State for Ideal $\mathcal{H}_2$ (c) Control for Ideal $\mathcal{H}_2$

(d) State for Delayed $\mathcal{H}_2$ (e) Control for Delayed $\mathcal{H}_2$

(f) State for Distributed (g) Control for Distributed

(h) State for Localized (i) Control for Localized

Fig. 1. The log absolute value of state and control for a given disturbance. The horizontal axis represents time and the vertical axis represents state in space. The legend on the right shows the meaning of the colors.

In this section, we synthesize our localized distributed $\mathcal{H}_2$ optimal controller for a specific plant, and compare the performance with different classes of $\mathcal{H}_2$ optimal controllers. In particular, we consider the centralized, delayed centralized, and distributed $\mathcal{H}_2$ optimal controller with QI condition. These controllers are computed by the method in [7].

The test model has 40 states. There are 20 sensors located at the $(4n-3)$-th and $4n$-th states, 20 actuators located at the $(4n-2)$-th and $(4n-1)$-th states, $n = 1, \ldots, 10$. Specifically, A is a 40×40 tridiagonal matrix given by

$$A = \begin{bmatrix} 1 & 0.2 & \cdots & 0 \\ -0.2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0.2 \\ 0 & \cdots & -0.2 & 1 \end{bmatrix}.$$

The instability of the plant is quantified by the spectral radius of A, which is $\rho(A) = 1.0766 > 1$. B_2 is a 40×20 matrix with the $(4n-2, 2n-1)$-th and $(4n-1, 2n)$-th entries being 1 and zero elsewhere, C_2 a 20×40 matrix with the $(2n-1, 4n-3)$-th and $(2n, 4n)$-th entries being 1 and zero elsewhere, $n = 1, \ldots, 10$. The other matrices are given by

$$B_1 = \begin{bmatrix} I_{40\times 40} & \mathbf{0}_{40\times 20} \end{bmatrix}, D_{21} = \begin{bmatrix} \mathbf{0}_{20\times 40} & I_{20\times 20} \end{bmatrix}$$
$$C_1 = \begin{bmatrix} I_{40\times 40} \\ \mathbf{0}_{20\times 40} \end{bmatrix}, D_{12} = \begin{bmatrix} \mathbf{0}_{40\times 20} \\ I_{20\times 20} \end{bmatrix}.$$

Even though our method can be applied to an arbitrary plant topology, we use this simple plant model as it leads to easily visualized localized region.

For the localized controller, we choose $(d, T) = (7, 30)$. For all controllers with communication delay constraints, we assume that the communication network has the same topology as the physical network, but the speed is $h = 2$ times faster than the speed of plant propagation. For our localized distributed controllers, the spatio-temporal constraint is given by (18).

We illustrate the difference in closed loop response between different control schemes by plotting the space-time evolution of a single disturbance hitting the 20-th state. Figures 1(h) and 1(i) show that the effect of the disturbance is limited spatially and temporally in both state and control action.

Next, we calculate the $\mathcal{H}_2$ norm for each controller and summarize the results in Table I. Clearly, our localized control scheme can achieve similar performance to that of the distributed or even the global optimal one. Numerical evidence seem to suggest that this property holds for most plants, so long as the system is localizable. As the closed loop response of an ideal $\mathcal{H}_2$ control usually decays exponentially in both time and space, as indicated by Figures 1(b) and 1(c), it is generally possible to find a favorable (d, T) to synthesize the localized controller such that the closed loop transient response does not degrade much in terms of the $\mathcal{H}_2$ norm. In fact, our localized controller leads to a sparse closed loop response, which may be more desirable than the $\mathcal{H}_2$ optimal control in many real-world applications.

In summary, the simulation demonstrates that our localized controller, which is implemented in a localized and scalable manner, can achieve a localized closed loop response with transient performance close to that of one achieved by a centralized controller.

TABLE I
COMPARISON BETWEEN DIFFERENT CONTROL SCHEMES

	Ideal $\mathcal{H}_2$	Delayed	Distributed	Localized
Comm Speed	Inf	2	2	2
Control Time	Inf	Inf	Inf	30
Locality	Max(39)	Max(39)	Max(39)	7
$\mathcal{H}_2$ norm	25.5930	73.7056	27.3530	27.3960

VI. CONCLUSION

In this paper, we generalized our previous results on localized distributed control to the output feedback setting. We reformulated the distributed optimal control problem in terms of the closed loop transfer matrix, upon which we imposed appropriately defined spatio-temporal constraints. It was shown that this formulation of the localized distributed optimal control problem led to a finite dimensional affinely constrained convex program as in (14). Then, we synthesized the controller based on the solution obtained in the optimization problem as in (17), and showed that the controller is implemented in a localized and hence scalable way. Through numerical simulation, the localized distributed optimal controller is seen to achieve similar transient performance to a centralized optimal one.

There are many exciting directions for future work. Most pressing is to establish a theory for open-loop unstable and uncertain plants that justifies the apparent robustness of the implementation presented in Section IV-B. Another key distinction between the output feedback case and our previous results [24], [25] is that the controller synthesis problem is no longer localized – due to the inherent structure of the system. However, we believe that these synthesis optimizations will be amenable to distributed computation techniques such as the alternating direction method of multipliers [26]. Finally, as the entire synthesis problem is reduced to a finite dimensional convex program, it opens up the possibility of using regularization for design [27] for the co-design of actuator and sensor placement, as well as communication delays between controllers [28].

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