A System Level Approach to Controller Synthesis

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***Abstract*—Biological and advanced cyberphysical control systems often have limited, sparse, uncertain, and distributed communication and computing in addition to sensing and actuation. Fortunately, the corresponding plants and performance requirements are also sparse and structured, and this must be exploited to make constrained controller design feasible and tractable. We introduce a new "system level" (SL) approach involving three complementary SL elements. System Level Parameterizations (SLPs) generalize state space and Youla parameterizations of all stabilizing controllers and the responses they achieve, and combine with System Level Constraints (SLCs) to parameterize the largest known class of constrained stabilizing controllers that admit a convex characterization, generalizing quadratic invariance (QI). SLPs also lead to a generalization of detectability and stabilizability, suggesting the existence of a rich separation structure, that when combined with SLCs, is naturally applicable to structurally constrained controllers and systems. We further provide a catalog of useful SLCs, most importantly including sparsity, delay, and locality constraints on both communication and computing internal to the controller, and external system performance. The resulting System Level Synthesis (SLS) problems that arise define the broadest known class of constrained optimal control problems that can be solved using convex programming. An example illustrates how this system level approach can systematically explore tradeoffs in controller performance, robustness, and synthesis/implementation complexity.**

***Index Terms*—constrained & structured optimal control, decentralized control, large-scale systems, system level parameterization, system level constraint, system level synthesis**

Preliminaries & Notation: We use lower and upper case Latin letters such as x and A to denote vectors and matrices, respectively, and lower and upper case boldface Latin letters such as $\mathbf{x}$ and $\mathbf{G}$ to denote signals and transfer matrices, respectively. We use calligraphic letters such as $\mathcal{S}$ to denote sets.

In the interest of clarity, we work with discrete time linear time invariant systems, but unless stated otherwise, all results extend naturally to the continuous time setting. We use standard definitions of the Hardy spaces $\mathcal{H}_2$ and $\mathcal{H}_\infty$, and denote their restriction to the set of real-rational proper transfer matrices by $\mathcal{RH}_2$ and $\mathcal{RH}_\infty$. We use $G[i]$ to denote the ith spectral component of a transfer function $\mathbf{G}$, i.e., $\mathbf{G}(z) = \sum_{i=0}^{\infty} \frac{1}{z^i} G[i]$ for $|z| > 1$. Finally, we use $\mathcal{F}_T$ to denote the space of finite impulse response (FIR) transfer matrices with horizon T, i.e., $\mathcal{F}_T := \{\mathbf{G} \in \mathcal{RH}_\infty \,|\, \mathbf{G} = \sum_{i=0}^{T} \frac{1}{z^i} G[i]\}$.

I. Introduction

The Youla parameterization [7] represented an important shift towards a system level approach to optimal controller synthesis. Youla showed that there exists an isomorphism between a stabilizing controller and the resulting closed loop system response from sensors to actuators – therefore rather than synthesizing the controller itself, this system response (or Youla parameter) could be designed directly. The advantage of this approach is that an affine expression of the Youla parameter describes all achievable responses of the closed loop system, allowing for system behavior to be directly optimized. Together with state-space methods, this contribution played a major role in shifting controller synthesis from an ad hoc, loop-at-a-time tuning process to a principled one with well defined notions of optimality. Indeed, this approach proved very powerful, and paved the way for the foundational results of robust and optimal control that would follow [8].

This paper presents an approach that is inspired by the system level thinking pioneered by Youla: rather than directly designing only the feedback loop between sensors and actuators, we propose directly designing the *entire closed loop response of the system*, as captured by the maps from process and measurement disturbances to control actions and states – as such, we call the proposed method a system level approach to controller synthesis. A distinction between our approach and Youla's is that we explicitly model the internal delay structure of the feedback system, whereas Youla (and contemporary state-space methods) hid the internal structure of the controller, and focused instead on its input-output behavior. This focus on controller input-output behavior was natural for the problems of that era (often motivated by aerospace and process control applications), where systems had a single logically centralized controller with global access to sensor measurements and global control over actuators.

In contrast, modern cyber-physical systems (CPS) are large-scale, physically distributed, and interconnected. Rather than a logically centralized controller, these systems are composed of several sub-controllers, each equipped with their own sensors and actuators – these sub-controllers then exchange locally available information (such as sensor measurements or applied control actions) via a communication network. It follows that the information exchanged between sub-controllers is constrained by the delay, bandwidth and reliability properties of this communication network, ultimately manifesting as information asymmetry among sub-controllers of the system. It is this information asymmetry, as imposed by the underlying communication network, that lies at the heart of what makes distributed optimal controller synthesis challenging [9]–[14].

A defining feature of CPS is that controllers have internal delays, as specified by the exchange of information between constituent sub-controllers. These delays thus needed to be reintroduced into Youla and state-space based synthesis methods; methods that aimed to hide the internals of the controller from the system engineer. Further, there was reason to suspect

Preliminary versions of this work can be found in [1]–[6].

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Thanks to funding from AFOSR and NSF and gifts from Huawei and Google.

that introducing such information asymmetry into the optimal control problem lead to intractable synthesis tasks [15], [16].

Despite these apparent technical and conceptual challenges, a body of work [10]–[14], [17], [18] that began in the early 2000s, and culminated with the introduction of quadratic invariance (QI) in the seminal paper [11], showed that for a large class of practically relevant systems, such internal structure could be incorporated into the Youla parameterization and still preserve the convexity of the optimal controller synthesis task. Informally, a system is quadratically invariant if sub-controllers are able to exchange information with each other faster than their control actions propagate through the CPS [19]. Even more remarkable is that this condition is tight, in the sense that QI is a necessary [20] and sufficient [11] condition for subspace constraints (defined by, for example, communication delays) on the controller to be enforceable via convex constraints on the Youla parameter.

The identification of QI as a useful condition for determining the tractability of a distributed optimal control problem led to an explosion of synthesis results in this area [21]–[29]. These results showed that the robust and optimal control methods that proved so powerful for centralized systems could be ported to distributed settings. However, they also made clear that the synthesis and implementation of QI distributed optimal controllers did not scale gracefully with the size of the underlying CPS. In particular, a QI distributed optimal controller is at least as expensive to compute as its centralized counterpart (c.f., the solutions presented in [21]–[29]), and can be more difficult to implement (c.f., the message passing implementation suggested in [29]).

We show in Section II that the QI framework, which adapts the Youla parameterization to a distributed setting, fails to capture certain constraints that are needed for optimal controller synthesis to scale to arbitrarily large systems. In particular, when the underlying physical system has a strongly connected network topology,[1] the QI framework does not allow for localized controllers, in which local sub-controllers only access a subset of system-wide measurements (c.f., Section IV-H), to be synthesized using convex programming – perhaps counter-intuitively, this statement holds true even when sub-controllers can exchange information with no delay (c.f., Example 1). Although this may seem surprising, note that implicit to the Youla parameterization is that sub-controllers can only exchange locally collected measurements with each other, and not, for instance, locally applied control actions. This restriction has no consequences in centralized applications, but can complicate controller implementation and synthesis in a distributed setting.

To overcome this limitation, we propose the system level approach to controller synthesis, which is composed of three elements: System Level Parameterizations (SLPs), System Level Constraints (SLCs) and System Level Synthesis (SLS) problems. Our specific contributions are outlined below.

[1]We say that a plant is strongly connected if the state of any subsystem can eventually alter the state of all other subsystems.

A. Contributions

This paper presents novel theoretical and computational contributions to the area of constrained optimal controller synthesis. In particular, we

- define and analyze the system level approach to controller synthesis, which is built around novel SLPs of all stabilizing controllers and the closed loop responses that they achieve;
- show that SLPs allow us to constrain the closed loop response of the system to lie in arbitrary sets: we call such constraints on the system SLCs. If these SLCs admit a convex representation, then the resulting set of constrained system responses admits a convex representation as well;
- show that such constrained system responses can be used to directly implement a controller achieving them – in particular, any SLC imposed on the system response imposes a corresponding SLC on the internal structure of the resulting controller;
- show that the set of constrained stabilizing controllers that can be efficiently parameterized using SLPs and SLCs is a strict superset of those that can be parameterized using quadratic invariance – hence we provide a generalization of the QI framework, characterizing the broadest known class of constrained controllers that admit a convex parameterization;
- provide a catalog of SLCs that admit a convex representation: highlights include general convex constraints on the Youla parameter (QI subspace constraints being a special case thereof), robustness and architectural constraints on the controller, as well as multi-objective performance constraints and spatiotemporal constraints on the system response;
- formulate and analyze the SLS problem, which exploits SLPs and SLCs to define the broadest known class of constrained optimal control problems that can be solved using convex programming. We show that the optimal control problems considered in the QI literature [10], as well as the recently defined localized optimal control framework [4] are all special cases of SLS problems.

B. Organization of Section II

In Section II, we define the system model considered in this paper, and review relevant results from the distributed optimal control and QI literature. We then provide a motivating example as to why moving beyond QI systems may be desirable, before presenting a survey of our main results. We end the section with the organization of the remainder of the paper.

II. Problem Statement and Main Results

A. System Model

We consider discrete time linear time invariant (LTI) systems of the form

$$x[t+1] = Ax[t] + B_1 w[t] + B_2 u[t] \tag{1a}$$
$$\bar{z}[t] = C_1 x[t] + D_{11} w[t] + D_{12} u[t] \tag{1b}$$
$$y[t] = C_2 x[t] + D_{21} w[t] + D_{22} u[t] \tag{1c}$$

where x, u, w, y, $\bar{z}$ are the state vector, control action, external disturbance, measurement, and regulated output, respectively. Equation (1) can be written in state space form as

$$\mathbf{P} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix}$$

where $\mathbf{P}_{ij} = C_i(zI - A)^{-1}B_j + D_{ij}$. We refer to $\mathbf{P}$ as the open loop plant model.

Consider a dynamic output feedback control law $\mathbf{u} = \mathbf{K}\mathbf{y}$. The controller $\mathbf{K}$ is assumed to have the state space realization

$$\xi[t+1] = A_k\xi[t] + B_k y[t] \tag{2a}$$
$$u[t] = C_k\xi[t] + D_k y[t], \tag{2b}$$

where ξ is the internal state of the controller. We have $\mathbf{K} = C_k(zI - A_k)^{-1}B_k + D_k$. A schematic diagram of the interconnection of the plant $\mathbf{P}$ and the controller $\mathbf{K}$ is shown in Figure 1.

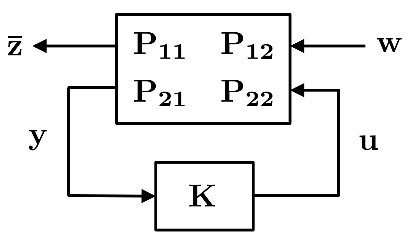


Fig. 1. Interconnection of the plant $\mathbf{P}$ and controller $\mathbf{K}$.

The following assumptions are made throughout the paper.

Assumption 1: The interconnection in Figure 1 is well-posed – the matrix $(I - D_{22}D_k)$ is invertible.

Assumption 2: Both the plant and the controller realizations are stabilizable and detectable; i.e., (A, B_2) and (A_k, B_k) are stabilizable, and (A, C_2) and (A_k, C_k) are detectable.

B. Structured Controller Synthesis, Youla, and QI

We follow the paradigm adopted in [11], [21]–[28], and focus on information asymmetry introduced by delays in the communication network – this is a reasonable modeling assumption when one has dedicated physical communication channels (e.g., fiber optic channels), but may not be valid under wireless settings. In the references cited above, locally acquired measurements are exchanged between sub-controllers subject to delays imposed by the communication network,[2] which manifest as subspace constraints on the controller itself.

We consider the following optimal structured controller synthesis task, as defined in [11], [20], [30], [31]:

$$\begin{array}{ll} \underset{\mathbf{K}}{\text{minimize}} & \|\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}\| \\ \text{subject to} & \mathbf{K} \text{ internally stabilizes } \mathbf{P} \\ & \mathbf{K} \in \mathcal{C}, \end{array} \tag{3}$$

[2]Note that this delay may range from 0, modeling instantaneous communication between sub-controllers, to infinite, modeling no communication between sub-controllers.

for $\mathcal{C}$ a subspace. This subspace can enforce, for instance, the information sharing constraints imposed on the controller $\mathbf{K}$ by the underlying communication network, as described above.

A synthesis of the main results of these papers can be expressed as follows: if the subspace $\mathcal{C}$ is quadratically invariant with respect to $\mathbf{P}_{22}$ [11], then the set of all stabilizing controllers lying in subspace $\mathcal{C}$ can be parameterized by those stable transfer matrices $\mathbf{Q} \in \mathcal{RH}_\infty$ satisfying $\mathfrak{M}(\mathbf{Q}) \in \mathcal{C}$, for $\mathfrak{M}$ an invertible affine map defined in terms of an arbitrary doubly co-prime factorization of the plant $\mathbf{P}$ (we defer a more detailed review of this material to Appendix A). Further, these conditions can be viewed as tight, in the sense that quadratic invariance is also a necessary condition [20], [30] for a subspace constraint $\mathcal{C}$ on the controller $\mathbf{K}$ to be enforced on the Youla parameter $\mathbf{Q}$ in a convex manner.

This allows the optimal control problem (3) to be recast as the following convex model matching problem:

$$\begin{array}{ll} \underset{\mathbf{Q}}{\text{minimize}} & \|\mathbf{T}_{11} + \mathbf{T}_{12}\mathbf{Q}\mathbf{T}_{21}\| \\ \text{subject to} & \mathbf{Q} \in \mathcal{RH}_\infty \\ & \mathfrak{M}(\mathbf{Q}) \in \mathcal{C}, \end{array} \tag{4}$$

where the transfer matrices $\mathbf{T}_{ij}$ can be expressed in terms of the original plant $\mathbf{P}_{ij}$ and the doubly co-prime factorization used to construct the map $\mathfrak{M}$ (c.f., Appendix A).

C. Beyond QI

We now present a simple example showing how the above framework, built around the Youla parameterization, fails to capture an "obvious" structured controller. We return to this example at the end of this section to show that our system level approach naturally recovers said obvious controller.

Example 1: Consider the optimal control problem:

$$\begin{array}{ll} \underset{u}{\text{minimize}} & \lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T} \mathbb{E}\|x[t]\|_2^2 \\ \text{subject to} & x[t+1] = Ax[t] + u[t] + w[t], \end{array} \tag{5}$$

with disturbance $w[t] \overset{\text{i.i.d}}{\sim} \mathcal{N}(0, I)$. We assume full state-feedback, i.e., the control action at time t can be expressed as $u[t] = f(x[0:t])$ for some function f. An optimal control policy $u^\star$ for this LQR problem is easily seen to be given by $u^\star[t] = -Ax[t]$.

Further suppose that the state matrix A is sparse and let its support define the adjacency matrix of a graph $\mathcal{G}$ for which we identify the ith node with the corresponding state/control pair (x_i, u_i). In this case, we have that the optimal control policy $u^\star$ can be implemented in a *localized* manner. In particular, in order to implement the state feedback policy for the ith actuator u_i, only those states x_j for which $A_{ij} \neq 0$ need to be collected – thus only those states corresponding to immediate neighbors of node i in the graph $\mathcal{G}$, i.e., only *local* states, need to be collected to compute the corresponding control action, leading to a localized implementation. As we discuss in more detail in Section IV-H and our companion paper [32], the idea of locality is essential to allowing controller synthesis and implementation to scale to arbitrarily large systems, and hence such a structured controller is desirable.

Now suppose that we naively attempt to solve optimal control problem (5) by converting it to its equivalent $\mathcal{H}_2$

model matching problem and constraining the controller $\mathbf{K}$ to have the same support as A, i.e., $\mathbf{K} = \sum_{t=0}^{\infty} \frac{1}{z^t} K[t]$, $\operatorname{supp}(K[t]) \subset \operatorname{supp}(A)$. If the graph $\mathcal{G}$ is strongly connected, then the conditions in [19] imply that the corresponding distributed optimal control problem is not quadratically invariant. The results of [20] further allow us to conclude that computing such a structured controller cannot be done using convex programming when using the Youla parameterization described in the previous subsection.

D. A System Level Approach

The rest of the paper is devoted to defining and analyzing the system level approach to controller synthesis, centered around the notion of a *system response*. We collect here a summary of our main results, and show how they can be used to pose a novel SLS problem that significantly generalizes that defined in equation (3).

For a LTI system with dynamics given by (1), we define a *system response* $\{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}\}$ to be the maps satisfying

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta_x} \\ \boldsymbol{\delta_y} \end{bmatrix}, \tag{6}$$

where $\boldsymbol{\delta_x} = B_1 \mathbf{w}$ is the disturbance on the state vector, and $\boldsymbol{\delta_y} = D_{21} \mathbf{w}$ is the disturbance on the measurement.

We say that a system response $\{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}\}$ is *stable and achievable* with respect to a plant $\mathbf{P}$ if there exists an internally stabilizing controller $\mathbf{K}$ such that the interconnection illustrated in Figure 1 leads to closed loop behavior consistent with equation (6).

In Section III-B, Theorem 2, we show that a system response $\{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}\}$ is stable and achievable with respect to a strictly proper plant $\mathbf{P}$ with realization (1) if and only if it lies in the affine subspace described by:

$$\begin{bmatrix} zI - A & -B_2 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix} \tag{7}$$

$$\begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} zI - A \\ -C_2 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{8}$$

$$\mathbf{R}, \mathbf{M}, \mathbf{N} \in \frac{1}{z}\mathcal{RH}_\infty, \quad \mathbf{L} \in \mathcal{RH}_\infty. \tag{9}$$

As the above characterizes all stable and achievable system responses, we call it a SLP.

In addition, for such a stable achievable system response $\{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}\}$, a controller that leads to these closed loop maps is given by $\mathbf{K} = \mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N}$, and can be implemented as:

$$\begin{aligned} \boldsymbol{\beta} &= (I - z\mathbf{R})\boldsymbol{\beta} - \mathbf{N}\mathbf{y} \\ \mathbf{u} &= z\mathbf{M}\boldsymbol{\beta} + \mathbf{L}\mathbf{y}, \end{aligned} \tag{10}$$

for $\boldsymbol{\beta}$ the internal state of the stabilizing controller.[3] A block diagram of the controller structure is shown in Figure 2.

Notice that any sparsity structure imposed on the system response $\{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}\}$ translates directly to the sparsity structure of the controller implementation – hence information sharing constraints on the measured output $\mathbf{y}$ and controller state $\boldsymbol{\beta}$ can be imposed via subspace constraints on the system response $\{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}\}$. As the controller is implemented directly using these transfer matrices, we are further no longer limited to subspace constraints, and can in fact impose arbitrary SLCs on the closed loop response of the system, and by extension the controller implementation – we provide a catalog of useful SLCs in Section IV. We also show in Section IV-A and IV-B that by combining appropriate SLCs with the SLP (7) - (9), we recover all structured controllers that can be parameterized using the Youla parameter and quadratic invariance.

[3] Although not apparent, $(I - z\mathbf{R}) \in \frac{1}{z}\mathcal{RH}_\infty$, and hence the suggested controller implementation is causal. See Section III-B for further details.

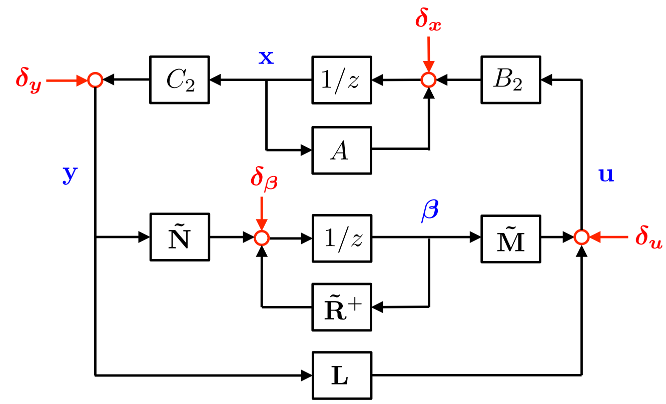


Fig. 2. The proposed output feedback controller structure, with $\tilde{\mathbf{R}}^+ = z\tilde{\mathbf{R}} = z(I - z\mathbf{R})$, $\tilde{\mathbf{M}} = z\mathbf{M}$, and $\tilde{\mathbf{N}} = -z\mathbf{N}$.

Let $\mathcal{S}$ denote such a SLC, and assume that it admits a convex representation. Further, let $g(\cdot)$ be a convex functional. Then the solution to the SLS problem:

$$\begin{aligned} \underset{\{\mathbf{R},\mathbf{M},\mathbf{N},\mathbf{L}\}}{\text{minimize}} \quad & g(\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}) && \text{(11a)} \\ \text{subject to} \quad & \text{equations (7) – (9)} && \text{(11b)} \\ & \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \in \mathcal{S} && \text{(11c)} \end{aligned}$$

can be found via convex programming. Further, as we briefly discuss in Section IV-H, and develop in great detail in our companion paper [32], if the SLC and the objective $g(\cdot)$ are localized and separable, then a structured optimal controller can be synthesized and implemented in a scalable manner, i.e., with $O(1)$ parallel computational and implementation complexity relative to the size of the overall system.

Example 2 (Example 1 cont'd): We now return to the motivating example introduced above to provide a preview of the usefulness of the system level approach to controller synthesis. In the case of a full control ($B_2 = I$) state-feedback ($C_2 = I$, $D_{21} = 0$) problem, the conditions (7)-(9) simplify to $(zI - A)\mathbf{R} - \mathbf{M} = I$, $\mathbf{R}, \mathbf{M} \in \frac{1}{z}\mathcal{RH}_\infty$ (c.f. Section III-A), and a controller achieving the desired response is given by $\mathbf{K} = \mathbf{M}\mathbf{R}^{-1}$. Further this controller can be implemented as

$$\hat{\mathbf{w}} = \mathbf{x} - \hat{\mathbf{x}}, \ \mathbf{u} = z\mathbf{M}\hat{\mathbf{w}}, \ \hat{\mathbf{x}} = (z\mathbf{R} - I)\hat{\mathbf{w}}.$$

Again, suppose that we wish to synthesize an optimal controller that has a communication topology given by the support of A – from the above implementation, it suffices to constrain the support of transfer matrices $\mathbf{R}$ and $\mathbf{M}$ to be a subset of that of A. It can be checked that $\mathbf{R} = \frac{1}{z}I$, and $\mathbf{M} = -\frac{1}{z}A$

satisfy the above constraints, and recover the globally optimal controller $\mathbf{K} = -A$. Recall that this controller cannot be computed using quadratic invariance and the Youla parameterization if the graph with adjacency matrix defined by the support of A is strongly connected and sparse.

E. Paper Structure

The rest of this paper is structured as follows. In Section III we define and analyze SLPs for state and output feedback problems, and provide a characterization of stable and achievable system responses (7)-(9). We show that SLPs also give a characterization of all internally stabilizing controllers, and that the controller implementation (10) achieves the desired system response. In Section IV we provide a catalog of SLCs that can be imposed on the system responses parameterized by the SLPs described in the previous section – in particular, we show that by appropriately selecting these SLCs, we can provide convex characterizations of all stabilizing controllers satisfying QI subspace constraints, convex constraints on the Youla parameter, finite impulse response (FIR) constraints, sparsity constraints, spatiotemporal constraints [1]–[4], controller internal robustness constraints, multi-objective performance constraints, controller architecture constraints [5], [33], [34], and any combination thereof. In Section V, we define and analyze the SLS problem (11), which incorporates SLPs and SLCs into an optimal control problem, and show that the structured optimal control problem (3) is a special case of SLS. We end with an application of the system level approach to exploring tradeoffs between controller performance, robustness and implementation complexity in Section VI, followed by conclusions in Section VII.

III. System Level Parameterization

In this section, we show that the affine subspace defined by the constraints (7) - (9) parameterizes all stable achievable system responses $\{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}\}$, and that the controller $\mathbf{K} = \mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N}$, which admits a realization as described in (10), parameterizes all internally stabilizing controllers for a strictly proper plant $\mathbf{P}_{22}$.[4] We begin by analyzing the state feedback case, as it admits a simpler characterization and allows us to provide intuition about the construction of a controller that achieves a desired system response. With this intuition in hand, we present our results for the output feedback setting, which is the main focus of this paper.

A. State Feedback

We consider a state feedback problem with plant model given by

$$\mathbf{P} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ I & 0 & 0 \end{array}\right]. \tag{12}$$

The z-transform of the state dynamics (1a) is given by

$$(zI - A)\mathbf{x} = B_2\mathbf{u} + \boldsymbol{\delta_x}, \tag{13}$$

[4]The non-strictly proper case is discussed in Section III-C.

where we let $\boldsymbol{\delta_x} := B_1\mathbf{w}$ denote the disturbance affecting the state.

We define $\mathbf{R}$ to be the system response mapping the external disturbance $\boldsymbol{\delta_x}$ to the state $\mathbf{x}$, and $\mathbf{M}$ to be the system response mapping the disturbance $\boldsymbol{\delta_x}$ to the control action $\mathbf{u}$. By substituting a dynamic state feedback control rule $\mathbf{u} = \mathbf{K}\mathbf{x}$ into (13), we can write the system response $\{\mathbf{R}, \mathbf{M}\}$ as a function of the controller $\mathbf{K}$ as

$$\begin{aligned} \mathbf{R} &= (zI - A - B_2\mathbf{K})^{-1} \\ \mathbf{M} &= \mathbf{K}(zI - A - B_2\mathbf{K})^{-1}. \end{aligned} \tag{14}$$

The main result of this subsection is an algebraic characterization of the set $\{\mathbf{R}, \mathbf{M}\}$ of state-feedback system responses that are achievable by an internally stabilizing controller $\mathbf{K}$, as stated in the following theorem.

Theorem 1: For the state feedback system (12), the following are true:

(a) The affine subspace defined by

$$\begin{bmatrix} zI - A & -B_2 \end{bmatrix} \begin{bmatrix} \mathbf{R} \\ \mathbf{M} \end{bmatrix} = I \tag{15a}$$

$$\mathbf{R}, \mathbf{M} \in \frac{1}{z}\mathcal{RH}_\infty \tag{15b}$$

parameterizes all system responses from $\boldsymbol{\delta_x}$ to $(\mathbf{x}, \mathbf{u})$, as defined in (14), achievable by an internally stabilizing state feedback controller $\mathbf{K}$.

(b) For any transfer matrices $\{\mathbf{R}, \mathbf{M}\}$ satisfying (15), the controller $\mathbf{K} = \mathbf{M}\mathbf{R}^{-1}$ is internally stabilizing and achieves the desired system response (14).

The rest of this subsection is devoted to proving the claims made in Theorem 1.

Necessity: The necessity of a stable and achievable system response $\{\mathbf{R}, \mathbf{M}\}$ lying in the affine subspace (15) follows from rote calculation (see Lemma 7 in Appendix B). Here we instead attempt to provide some intuition about the implications of the conditions (15): in particular, we show that there is a natural connection between these conditions and the stabilizability of system (12).

Lemma 1: The pair (A, B_2) is stabilizable if and only if the affine subspace defined by (15) is non-empty.

Proof: We first show that the stabilizability of (A, B_2) implies that there exist transfer matrices $\mathbf{R}, \mathbf{M} \in \frac{1}{z}\mathcal{RH}_\infty$ satisfying equation (15a). From the definition of stabilizability, there exists a matrix F such that $A + B_2F$ is a stable matrix. Substituting the state feedback control law $u = Fx$ into (13), we have $\mathbf{x} = (zI - A - B_2F)^{-1}\boldsymbol{\delta_x}$ and $\mathbf{u} = F(zI - A - B_2F)^{-1}\boldsymbol{\delta_x}$. The system response is given by $\mathbf{R} = (zI - A - B_2F)^{-1}$ and $\mathbf{M} = F(zI - A - B_2F)^{-1}$, which lie in $\frac{1}{z}\mathcal{RH}_\infty$ and are a solution to (15a).

For the opposite direction, we note that $\mathbf{R}, \mathbf{M} \in \mathcal{RH}_\infty$ implies that these transfer matrices do not have poles outside the unit circle $|z| \geq 1$. From (15a), we further observe that $\begin{bmatrix} zI - A & -B_2 \end{bmatrix}$ is right invertible in the region where $\mathbf{R}$ and $\mathbf{M}$ do not have poles, with $\begin{bmatrix} \mathbf{R}^\top & \mathbf{M}^\top \end{bmatrix}^\top$ being its right inverse. This then implies that $\begin{bmatrix} zI - A & -B_2 \end{bmatrix}$ has full row rank for all $|z| \geq 1$. This is equivalent to the PBH test [35] for stabilizability, proving the claim. ∎

Thus Lemma 1 provides an alternative definition of (state feedback) stabilizability via the conditions described in (15) – in particular, stable achievable responses exist only if the state feedback system is stabilizable.

Sufficiency: Here we show that for any system response $\{\mathbf{R}, \mathbf{M}\}$ lying in the affine subspace (15), we can construct an internally stabilizing controller $\mathbf{K}$ that leads to the desired system response (14).

A partial solution is provided in our prior work [2], where we give a construction for finite impulse response (FIR) system responses $\{\mathbf{R}, \mathbf{M}\}$. Here we extend these results to infinite impulse response (IIR) system responses, and provide a proof of internal stability for the proposed controller structure.

In [2], we considered FIR system responses of horizon T, i.e., $\mathbf{R}, \mathbf{M} \in \mathcal{F}_T$, and proposed the following disturbance-based controller implementation:

$$\hat{\delta_x}[t] = x[t] - \hat{x}[t] \tag{16a}$$

$$u[t] = \sum_{\tau=0}^{T-1} M[\tau+1]\hat{\delta_x}[t-\tau] \tag{16b}$$

$$\hat{x}[t+1] = \sum_{\tau=0}^{T-2} R[\tau+2]\hat{\delta_x}[t-\tau]. \tag{16c}$$

The internal states of the controller (16) should be interpreted as follows: $\hat{\delta_x}$ is the controller estimate of the state disturbance, and $\hat{x}$ is a desired or reference state trajectory. The estimated disturbance $\hat{\delta_x}[t]$ is computed by taking the difference between the current state measurement $x[t]$ and the current reference state value $\hat{x}[t]$. The control action $u[t]$ and the next reference state value $\hat{x}[t+1]$ are then computed using past estimated disturbances $\hat{\delta_x}[t-T+1], \dots, \hat{\delta_x}[t]$.

Taking the z-transform of equations (16), we obtain their representation in the frequency domain

$$\hat{\boldsymbol{\delta_x}} = \mathbf{x} - \hat{\mathbf{x}} \tag{17a}$$

$$\mathbf{u} = z\mathbf{M}\hat{\boldsymbol{\delta_x}} \tag{17b}$$

$$\hat{\mathbf{x}} = (z\mathbf{R} - I)\hat{\boldsymbol{\delta_x}}. \tag{17c}$$

Combining equations (17) with (13) and (15), one can verify that the estimated disturbance $\hat{\delta_x}[t]$ indeed reconstructs the true disturbance $\delta_x[t-1]$ that perturbed the plant at time $t-1$; hence $\hat{\boldsymbol{\delta_x}} = z^{-1}\boldsymbol{\delta_x}$. It is then straightforward to show that the desired system response $\{\mathbf{R}, \mathbf{M}\}$ satisfying $\mathbf{x} = \mathbf{R}\boldsymbol{\delta_x}$ and $\mathbf{u} = \mathbf{M}\boldsymbol{\delta_x}$ is achieved. Note that the previous argument holds for any FIR horizon T as well as for $T = \infty$.

Remark 1: From (17), the control action $\mathbf{u}$ can be expressed as $\mathbf{u} = \mathbf{M}\mathbf{R}^{-1}\mathbf{x}$. We can therefore also implement the controller defined in (17) via the dynamic state feedback gain $\mathbf{K} = \mathbf{M}\mathbf{R}^{-1}$.[5] However, we show in Section IV that the disturbance-based implementation in (17) has significant advantages over a traditional state feedback implementation – specifically, this implementation allows us to connect constraints imposed on the system response to constraints on the controller implementation.

[5]As $\mathbf{R}$ is strictly proper, $\mathbf{R}^{-1}$ is not proper. However, $\mathbf{K} = \mathbf{M}\mathbf{R}^{-1}$ can be verified to always be proper.

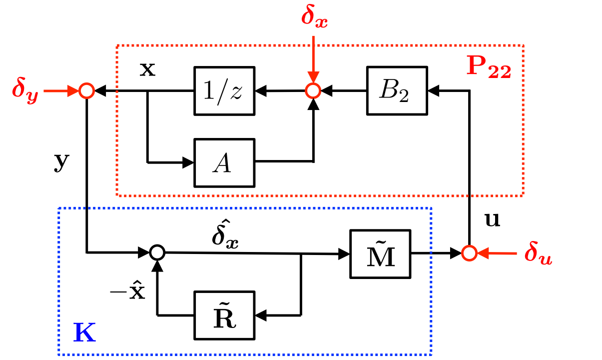


Fig. 3. The proposed state feedback controller structure, with $\tilde{\mathbf{R}} = I - z\mathbf{R}$ and $\tilde{\mathbf{M}} = z\mathbf{M}$.

It remains to be shown that the controller implementation (17) internally stabilizes the plant (12). We consider the block diagram shown in Figure 3, where here $\tilde{\mathbf{R}} = I - z\mathbf{R}$ and $\tilde{\mathbf{M}} = z\mathbf{M}$. It can be checked that $z\tilde{\mathbf{R}}, \tilde{\mathbf{M}} \in \mathcal{RH}_\infty$, and hence the internal feedback loop between $\hat{\boldsymbol{\delta_x}}$ and the reference state trajectory $\hat{\mathbf{x}}$ is well defined.

As is standard, we introduce external perturbations δ_x, δ_y, and δ_u into the system and note that the perturbations entering other links of the block diagram can be expressed as a combination of $(\boldsymbol{\delta_x}, \boldsymbol{\delta_y}, \boldsymbol{\delta_u})$ being acted upon by some stable transfer matrices.[6] Hence the standard definition of internal stability applies, and we can use a bounded-input bounded-output argument (e.g., Lemma 5.3 in [36]) to conclude that it suffices to check the stability of the nine closed loop transfer matrices from perturbations $(\boldsymbol{\delta_x}, \boldsymbol{\delta_y}, \boldsymbol{\delta_u})$ to the internal variables $(\mathbf{x}, \mathbf{u}, \hat{\boldsymbol{\delta_x}})$ to determine the internal stability of the structure as a whole.

As all blocks in Figure 3 are stable filters, it follows that if the origin $(x, \hat{\delta_x}) = (0, 0)$ is asymptotically stable then any other signals in the block diagram will decay asymptotically. This is equivalent to the conventional notion of internal stability [36], which we recall here for the reader before stating and proving that the proposed controller implementation is internally stabilizing.

Definition 1: The interconnection in Figure 3 is internally stable if the origin $(x, \hat{\delta_x}) = (0, 0)$ is asymptotically stable, i.e., $x[t], \hat{\delta_x}[t] \to 0$ for $t \to \infty$ for all initial conditions when the external perturbations $\delta_x, \delta_y, \delta_u$ in Figure 3 are set to 0.

Lemma 2: Consider the state feedback system (12). Given any system response $\{\mathbf{R}, \mathbf{M}\}$ lying in the affine subspace described by (15), the state feedback controller $\mathbf{K} = \mathbf{M}\mathbf{R}^{-1}$, with structure shown in Figure 3, internally stabilizes the plant. In addition, the desired system response, as specified by $\mathbf{x} = \mathbf{R}\boldsymbol{\delta_x}$ and $\mathbf{u} = \mathbf{M}\boldsymbol{\delta_x}$, is achieved.

Proof: We first note that from Figure 3, we can express the state feedback controller $\mathbf{K}$ as $\mathbf{K} = \tilde{\mathbf{M}}(I - \tilde{\mathbf{R}})^{-1} = (z\mathbf{M})(z\mathbf{R})^{-1} = \mathbf{M}\mathbf{R}^{-1}$. Now, for any system response $\{\mathbf{R}, \mathbf{M}\}$ lying in the affine subspace described by (15), we construct a controller using the structure given in Figure 3. To show that the constructed controller internally stabilizes the

[6]The matrix A may define an unstable system, but viewed as an element of $\mathcal{F}_0$, defines a stable (FIR) transfer matrix.

plant, we list the following equations from Figure 3:

$$\begin{aligned} z\mathbf{x} &= A\mathbf{x} + B_2\mathbf{u} + \boldsymbol{\delta_x} \\ \mathbf{u} &= \tilde{\mathbf{M}}\hat{\boldsymbol{\delta_x}} + \boldsymbol{\delta_u} \\ \hat{\boldsymbol{\delta_x}} &= \mathbf{x} + \boldsymbol{\delta_y} + \tilde{\mathbf{R}}\hat{\boldsymbol{\delta_x}}. \end{aligned}$$

Routine calculations show that the closed loop transfer matrices from $(\boldsymbol{\delta_x}, \boldsymbol{\delta_y}, \boldsymbol{\delta_u})$ to $(\mathbf{x}, \mathbf{u}, \hat{\boldsymbol{\delta_x}})$ are given by

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \\ \hat{\boldsymbol{\delta_x}} \end{bmatrix} = \begin{bmatrix} \mathbf{R} & -\tilde{\mathbf{R}} - \mathbf{R}A & \mathbf{R}B_2 \\ \mathbf{M} & \tilde{\mathbf{M}} - \mathbf{M}A & I + \mathbf{M}B_2 \\ \frac{1}{z}I & I - \frac{1}{z}A & \frac{1}{z}B_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta_x} \\ \boldsymbol{\delta_y} \\ \boldsymbol{\delta_u} \end{bmatrix}. \quad (18)$$

As all nine transfer matrices in (18) are stable, the implementation in Figure 3 is internally stable. Furthermore, the desired system response $\{\mathbf{R}, \mathbf{M}\}$, from $\boldsymbol{\delta_x}$ to $(\mathbf{x}, \mathbf{u})$, is achieved. ■

Summary and corollary: Theorem 1 provides a necessary and sufficient condition for the system response $\{\mathbf{R}, \mathbf{M}\}$ to be stable and achievable, in that elements of the affine subspace defined by (15) parameterize all stable system responses achievable via state-feedback, as well as the internally stabilizing controllers that achieve them.

We note that the analysis for the state feedback problem can be applied to the state estimation problem by considering the dual to a full control system (c.f., §16.5 in [36]). For instance, the following corollary to Lemma 1 gives an alternative definition of the detectability of pair (A, C_2) [6].

Corollary 1: The pair (A, C_2) is detectable if and only if the following conditions are feasible:

$$\begin{bmatrix} \mathbf{R} & \mathbf{N} \end{bmatrix} \begin{bmatrix} zI - A \\ -C_2 \end{bmatrix} = I \quad (19a)$$

$$\mathbf{R}, \mathbf{N} \in \frac{1}{z}\mathcal{RH}_\infty. \quad (19b)$$

A parameterization of all detectable observers can be constructed using the affine subspace (19) in a manner analogous to that described above.

B. Output Feedback with $D_{22} = 0$

We now extend the arguments of the previous subsection to the output feedback setting, and begin by considering the case of a strictly proper plant

$$\mathbf{P} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]. \quad (20)$$

Letting $\delta_x[t] = B_1 w[t]$ denote the disturbance on the state, and $\delta_y[t] = D_{21}w[t]$ denote the disturbance on the measurement, the dynamics defined by plant (20) can be written as

$$\begin{aligned} x[t+1] &= Ax[t] + B_2u[t] + \delta_x[t] \\ y[t] &= C_2x[t] + \delta_y[t]. \end{aligned} \quad (21)$$

Analogous to the state-feedback case, we define a system response $\{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}\}$ from perturbations $(\boldsymbol{\delta_x}, \boldsymbol{\delta_y})$ to state and control inputs $(\mathbf{x}, \mathbf{u})$ via the following relation:

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta_x} \\ \boldsymbol{\delta_y} \end{bmatrix}. \quad (22)$$

Substituting the output feedback control law $\mathbf{u} = \mathbf{K}\mathbf{y}$ into the z-transform of system equation (21), we obtain

$$(zI - A - B_2\mathbf{K}C_2)\mathbf{x} = \boldsymbol{\delta_x} + B_2\mathbf{K}\boldsymbol{\delta_y}.$$

For a proper controller $\mathbf{K}$, the transfer matrix $(zI - A - B_2\mathbf{K}C_2)$ is always invertible, hence we obtain the following expressions for the system response (22) in terms of an output feedback controller $\mathbf{K}$:

$$\begin{aligned} \mathbf{R} &= (zI - A - B_2\mathbf{K}C_2)^{-1} \\ \mathbf{M} &= \mathbf{K}C_2\mathbf{R} \\ \mathbf{N} &= \mathbf{R}B_2\mathbf{K} \\ \mathbf{L} &= \mathbf{K} + \mathbf{K}C_2\mathbf{R}B_2\mathbf{K}. \end{aligned} \quad (23)$$

We now present one of the main results of the paper: an algebraic characterization of the set $\{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}\}$ of output-feedback system responses that are achievable by an internally stabilizing controller $\mathbf{K}$, as stated in the following theorem.

Theorem 2: For the output feedback system (20), the following are true:

(a) The affine subspace described by:

$$\begin{bmatrix} zI - A & -B_2 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix} \quad (24a)$$

$$\begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} zI - A \\ -C_2 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} \quad (24b)$$

$$\mathbf{R}, \mathbf{M}, \mathbf{N} \in \frac{1}{z}\mathcal{RH}_\infty, \quad \mathbf{L} \in \mathcal{RH}_\infty \quad (24c)$$

parameterizes all system responses (23) achievable by an internally stabilizing controller $\mathbf{K}$.

(b) For any transfer matrices $\{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}\}$ satisfying (24), the controller $\mathbf{K} = \mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N}$ is internally stabilizing and achieves the desired response (23).

The argument is similar in spirit to that of the state-feedback case, but proofs of intermediary results are more cumbersome, and hence are relegated to Appendix B.

Necessity: As was the case for the state-feedback setting, the necessity of a stable and achievable system response $\{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}\}$ lying in the affine subspace (24) follows from rote calculation (see Lemma 8 in Appendix B). Further, there is also a natural connection between the feasibility of conditions (24) and the stabilizability and detectability of system (20), as stated in the following lemma.

Lemma 3: The triple (A, B_2, C_2) is stabilizable and detectable if and only if the affine subspace described by (24) is non-empty.

Thus Lemma 3 provides an alternative characterization of stabilizability and detectability via the conditions described in (24) – in particular, stable achievable system responses (23) exist only if the output feedback system is stabilizable and detectable.

Sufficiency: Here we show that for any system response $\{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}\}$ lying in the affine subspace (24), there exists an internally stabilizing controller $\mathbf{K}$ that leads to the desired system response (23). From the relations in (23), we notice the identity $\mathbf{K} = \mathbf{L} - \mathbf{K}C_2\mathbf{R}B_2\mathbf{K} = \mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N}$. This relation leads to the controller structure given in Figure 2, with $\tilde{\mathbf{R}}^+ = z\tilde{\mathbf{R}} = z(I - z\mathbf{R})$, $\tilde{\mathbf{M}} = z\mathbf{M}$, and $\tilde{\mathbf{N}} = -z\mathbf{N}$. As was the case for the state feedback setting, it can be verified that $\tilde{\mathbf{R}}^{+}, \tilde{\mathbf{M}}$, and $\tilde{\mathbf{N}}$ are all in $\mathcal{RH}_\infty$. Therefore, the structure given in Figure 2 is well defined. The controller implementation of Figure 2 is governed by the following equations:

$$\begin{aligned} z\boldsymbol{\beta} &= \tilde{\mathbf{R}}^{+}\boldsymbol{\beta} + \tilde{\mathbf{N}}\mathbf{y} \\ \mathbf{u} &= \tilde{\mathbf{M}}\boldsymbol{\beta} + \mathbf{L}\mathbf{y}. \end{aligned} \tag{25}$$

The control implementation equations (25) can be interpreted as an extension of the state-space realization (2) of a controller $\mathbf{K}$. In particular, in the realization equations (25) we allow the constant matrices A_K, B_K, C_K, D_K of the state-space realization (2) to be stable proper transfer matrices $\tilde{\mathbf{R}}^{+}, \tilde{\mathbf{M}}, \tilde{\mathbf{N}}, \mathbf{L}$. The benefit of this implementation is that arbitrary convex constraints imposed on the transfer matrices $\tilde{\mathbf{R}}^{+}, \tilde{\mathbf{M}}, \tilde{\mathbf{N}}, \mathbf{L}$ carry over directly to the controller implementation. We show in Section IV that this allows for a class of structural (locality) constraints to be imposed on the system response (and hence the controller) that are crucial for extending controller synthesis methods to large-scale systems. In contrast, we recall that imposing general convex constraints on the controller $\mathbf{K}$ or its state-space realization A_K, B_K, C_K, D_K cannot be done in a computationally efficient manner.

What remains to be shown is that the proposed controller implementation (25) is internally stabilizing and achieves the desired system response (23). As was the case for the state feedback setting, all of the blocks in Figure 2 are stable filters – thus, as long as the origin $(x, \beta) = (0, 0)$ is asymptotically stable, all signals internal to the block diagram will decay to zero. To check the internal stability of the structure, we introduce external perturbations $\boldsymbol{\delta_x}, \boldsymbol{\delta_y}$, $\boldsymbol{\delta_u}$, and $\boldsymbol{\delta_\beta}$ to the system. The perturbations appearing on other links of the block diagram can all be expressed as a combination of the perturbations $(\boldsymbol{\delta_x}, \boldsymbol{\delta_y}, \boldsymbol{\delta_u}, \boldsymbol{\delta_\beta})$ being acted upon by some stable transfer matrices, and so it suffices to check the input-output stability of the closed loop transfer matrices from perturbations $(\boldsymbol{\delta_x}, \boldsymbol{\delta_y}, \boldsymbol{\delta_u}, \boldsymbol{\delta_\beta})$ to controller signals $(\mathbf{x}, \mathbf{u}, \mathbf{y}, \boldsymbol{\beta})$ to determine the internal stability of the structure [36].

With this discussion in mind, we formally define internal stability for the controller structure of Figure 2, and state and prove the sufficiency of the conditions stated in Theorem 2.

Definition 2: The interconnection in Figure 2 is said to be internally stable if the origin $(x, \beta) = (0, 0)$ is asymptotically stable, i.e., $x[t], \beta[t] \to 0$ for $t \to \infty$ from any initial condition when the perturbations $\delta_x, \delta_y, \delta_u, \delta_\beta$ in Figure 2 are 0.

Lemma 4: Consider the output feedback system (20). For any system response $\{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}\}$ lying in the affine subspace defined by (24), the controller $\mathbf{K} = \mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N}$ (with structure shown in Figure 2) internally stabilizes the plant. In addition, the desired system response, as specified by $\mathbf{x} = \mathbf{R}\boldsymbol{\delta_x} + \mathbf{N}\boldsymbol{\delta_y}$ and $\mathbf{u} = \mathbf{M}\boldsymbol{\delta_x} + \mathbf{L}\boldsymbol{\delta_y}$, is achieved.

Proof: For any system response $\{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}\}$ lying in the affine subspace defined by (24), we construct a controller using the structure given in Figure 2. We now check the stability of the closed loop transfer matrices from the perturbations $(\boldsymbol{\delta_x}, \boldsymbol{\delta_y}, \boldsymbol{\delta_u}, \boldsymbol{\delta_\beta})$ to the internal variables $(\mathbf{x}, \mathbf{u}, \mathbf{y}, \boldsymbol{\beta})$. We have the following equations from Figure 2:

$$\begin{aligned} z\mathbf{x} &= A\mathbf{x} + B_2\mathbf{u} + \boldsymbol{\delta_x} \\ \mathbf{y} &= C_2\mathbf{x} + \boldsymbol{\delta_y} \\ z\boldsymbol{\beta} &= \tilde{\mathbf{R}}^{+}\boldsymbol{\beta} + \tilde{\mathbf{N}}\mathbf{y} + \boldsymbol{\delta_\beta} \\ \mathbf{u} &= \tilde{\mathbf{M}}\boldsymbol{\beta} + \mathbf{L}\mathbf{y} + \boldsymbol{\delta_u}. \end{aligned}$$

Combining these equations with the relations in (24a) - (24b), we summarize the closed loop transfer matrices from $(\boldsymbol{\delta_x}, \boldsymbol{\delta_y}, \boldsymbol{\delta_u}, \boldsymbol{\delta_\beta})$ to $(\mathbf{x}, \mathbf{u}, \mathbf{y}, \boldsymbol{\beta})$ in Table I.

TABLE I
CLOSED LOOP MAPS FROM PERTURBATIONS TO INTERNAL VARIABLES

	$\boldsymbol{\delta_x}$	$\boldsymbol{\delta_y}$	$\boldsymbol{\delta_u}$	$\boldsymbol{\delta_\beta}$
$\mathbf{x}$	$\mathbf{R}$	$\mathbf{N}$	$\mathbf{R}B_2$	$\frac{1}{z}\mathbf{N}C_2$
$\mathbf{u}$	$\mathbf{M}$	$\mathbf{L}$	$I + \mathbf{M}B_2$	$\frac{1}{z}\mathbf{L}C_2$
$\mathbf{y}$	$C_2\mathbf{R}$	$I + C_2\mathbf{N}$	$C_2\mathbf{R}B_2$	$\frac{1}{z}C_2\mathbf{N}C_2$
$\boldsymbol{\beta}$	$-\frac{1}{z}B_2\mathbf{M}$	$-\frac{1}{z}B_2\mathbf{L}$	$-\frac{1}{z}B_2\mathbf{M}B_2$	$\frac{1}{z}I - \frac{1}{z^2}(A + B_2\mathbf{L}C_2)$

Equation (24c) implies that all sixteen transfer matrices in Table I are stable, so the implementation in Figure 2 is internally stable. Furthermore, the desired system response from $(\boldsymbol{\delta_x}, \boldsymbol{\delta_y})$ to $(\mathbf{x}, \mathbf{u})$ is achieved. ■

Alternative controller implementations: We conclude this subsection by showing that care must be taken when dealing with open loop unstable systems, and in particular, we demonstrate the necessity of considering perturbations on the internal controller state $\boldsymbol{\beta}$ and control input $\mathbf{u}$ for unstable plants. Such perturbations can arise, for instance, from using floating point arithmetic within the controller, or from quantization at the actuators.

In particular, if we set $\boldsymbol{\delta_u}$ and $\boldsymbol{\delta_\beta}$ to 0, it follows that $\boldsymbol{\beta} = -\frac{1}{z}B_2\mathbf{u}$ (c.f., Table I). This leads to a simpler controller implementation given by $\mathbf{u} = \mathbf{L}\mathbf{y} - \mathbf{M}B_2\mathbf{u}$, with the corresponding controller structure shown in Figure 4(a). This implementation can also be obtained from the identity $\mathbf{K} = (I + \mathbf{M}B_2)^{-1}\mathbf{L}$, which follows from the relations in (23). Unfortunately, as shown below, this implementation is internally stable only when the open loop plant is stable.

For the controller implementation and structure shown in Figure 4(a), the closed loop transfer matrices from perturbations to the internal variables are given by

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{N} & \mathbf{R}B_2 & (zI - A)^{-1}B_2 \\ \mathbf{M} & \mathbf{L} & I + \mathbf{M}B_2 & I \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta_x} \\ \boldsymbol{\delta_y} \\ \boldsymbol{\delta_u} \\ \boldsymbol{\delta_\beta} \end{bmatrix}. \tag{26}$$

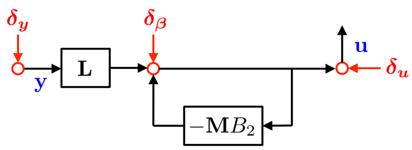
When A defines a stable system, the implementation in Figure 4(a) is internally stable. However, when the open loop plant is unstable (and the realization (A, B_2) is stabilizable), the transfer matrix $(zI - A)^{-1}B_2$ is unstable. From (26), the effect of the perturbation $\boldsymbol{\delta_\beta}$ can lead to instability of the closed loop system. This structure thus shows the necessity of introducing and analyzing the effects of perturbations $\boldsymbol{\delta_\beta}$ on the controller internal state.

Alternatively, if we start with the identity $\mathbf{K} = \mathbf{L}(I + C_2\mathbf{N})^{-1}$, which also follows from (23), we obtain the controller structure shown in Figure 4(b). The closed loop map from perturbations to internal signals is then given by

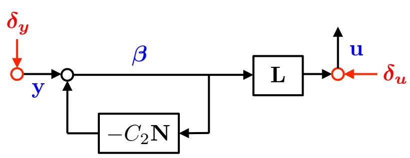
$$\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \\ \boldsymbol{\beta} \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{N} & \mathbf{R}B_2 \\ \mathbf{M} & \mathbf{L} & I + \mathbf{M}B_2 \\ C_2(zI - A)^{-1} & I & C_2(zI-A)^{-1}B_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta_x} \\ \boldsymbol{\delta_y} \\ \boldsymbol{\delta_u} \end{bmatrix}.$$

As can be seen, the controller implementation is once again internally stable only when the open loop plant is stable (if the realization (A, C_2) is detectable). This structure thus shows the necessity of introducing and analyzing the effects of perturbations on the controller internal state $\boldsymbol{\beta}$.

Of course, when the open loop system is stable, the controller structures illustrated below may be appealing as they are simpler and easier to implement.



(a) Structure 1



(b) Structure 2

Fig. 4. Alternative controller structures for stable systems.

Summary: Theorem 2 provides a necessary and sufficient condition for the system response $\{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}\}$ to be stable and achievable, in that elements of the affine subspace defined by (24) parameterize all stable achievable system responses, as well as all internally stabilizing controllers that achieve them.

C. Output Feedback with $D_{22} \neq 0$

Finally, for a general proper plant model (1) with $D_{22} \neq 0$, we define a new measurement $\bar{y}[t] = y[t] - D_{22}u[t]$. This leads to the controller structure shown in Figure 5. In this case, the closed loop transfer matrices from $\boldsymbol{\delta_u}$ to the internal variables become

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \\ \mathbf{y} \\ \boldsymbol{\beta} \end{bmatrix} = \begin{bmatrix} \mathbf{R}B_2 + \mathbf{N}D_{22} \\ I + \mathbf{M}B_2 + \mathbf{L}D_{22} \\ C_2\mathbf{R}B_2 + D_{22} + C_2\mathbf{N}D_{22} \\ -\frac{1}{z}B_2(\mathbf{M}B_2 + \mathbf{L}D_{22}) \end{bmatrix} \boldsymbol{\delta_u}.$$

The remaining entries of Table I remain the same. Therefore, the controller structure shown in Figure 5 internally stabilizes the plant.

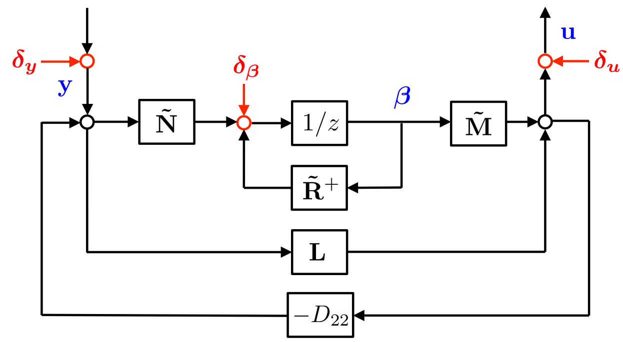


Fig. 5. The proposed output feedback controller structure for $D_{22} \neq 0$.

IV. System Level Constraints

An advantage of the parameterizations described in the previous section is that they allow us to impose additional constraints on the system response. These constraints may be in the form of structural constraints on the response (and the corresponding controller implementation), or may capture a suitable measure of system performance: in this section, we provide a catalog of useful SLCs that can be naturally incorporated into the SLPs described in the previous section. In particular, we show that QI subspace constraints are a special case of SLCs, and as such, we provide here a description of the largest known class of constrained stabilizing controllers that admit a convex parameterization.

A. Constraints on the Youla Parameter

We show that any constraint imposed on the Youla parameter can be translated into a SLC, and vice versa. In particular, if this constraint is convex, then so is the corresponding SLC.

Consider the following modification of the standard Youla parameterization, which characterizes a set of constrained internally stabilizing controllers $\mathbf{K}$ for a plant (20):

$$\mathbf{K} = \mathfrak{Y}(\mathbf{Q})[\mathfrak{X}(\mathbf{Q})]^{-1}, \ \ \mathbf{Q} \in \mathcal{Q} \cap \mathcal{RH}_\infty. \tag{27}$$

Here $\mathfrak{Y}$ and $\mathfrak{X}$ are affine maps defined in terms of a doubly coprime factorization of the plant (20) (c.f. Appendix A and §5.4 of [36]), and $\mathcal{Q}$ is an arbitrary set – if we take $\mathcal{Q} = \mathcal{RH}_\infty$, we recover the standard Youla parameterization. By appropriately varying the set $\mathcal{Q}$, one can then characterize all possible constrained internally stabilizing controllers,[7] and hence this formulation is as general as possible. We now show that an equivalent parameterization can be given in terms of a SLC.

Theorem 3: The set of constrained internally stabilizing controllers described by (27) can be equivalently expressed as $\mathbf{K} = \mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N}$, where the system response $\{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}\}$ lies in the set

$$\{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L} \,|\ \text{(24a) - (24c) hold}, \mathbf{L} \in \mathfrak{M}(\mathcal{Q})\}, \tag{28}$$

for $\mathfrak{M}$ an invertible affine map as defined in Appendix A. Further, this parameterization is convex if and only if $\mathcal{Q}$ is convex.

[7] In particular, to ensure that $\mathbf{K} \in \mathcal{C}$, it suffices to enforce that $\mathfrak{Y}(\mathbf{Q})[\mathfrak{X}(\mathbf{Q})]^{-1} \in \mathcal{C}$.

We note that when the system response $\{\mathbf{R},\mathbf{M},\mathbf{N},\mathbf{L}\}$ is constrained to lie within a convex set, the transformation $\mathbf{K} = \mathbf{L} - \mathbf{M}\mathbf{R}^{-1}\mathbf{N}$ implies that the corresponding set of parameterized controllers $\mathbf{K}$ can be non-convex. Therefore it is possible to parameterize the set of internally stabilizing controllers $\mathbf{K}$ lying in certain non-convex sets $\mathcal{C}$ using the parameterization described in Theorem 3 and appropriately selected additional convex constraints on the system response $\{\mathbf{R},\mathbf{M},\mathbf{N},\mathbf{L}\}$.

In order to prove this result, we first need to understand the relationship between the controller $\mathbf{K}$, the Youla parameter $\mathbf{Q}$, and the system response $\{\mathbf{R},\mathbf{M},\mathbf{N},\mathbf{L}\}$.

Lemma 5: Let $\mathbf{L}$ be defined as in (23), and the invertible affine map $\mathfrak{M}$ be defined as in Appendix A. We then have that

$$\mathbf{L} = \mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1} = \mathfrak{M}(\mathbf{Q}). \tag{29}$$

Proof: From the equations $\mathbf{u} = \mathbf{K}\mathbf{y}$ and $\mathbf{y} = \mathbf{P}_{21}\mathbf{w} + \mathbf{P}_{22}\mathbf{u}$, we can eliminate $\mathbf{u}$ and express $\mathbf{y}$ as $\mathbf{y} = (I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}\mathbf{w}$. We then have that

$$\mathbf{u} = \mathbf{K}\mathbf{y} = \mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}\mathbf{w}. \tag{30}$$

Recall that we define $\boldsymbol{\delta_x} = B_1\mathbf{w}$ and $\boldsymbol{\delta_y} = D_{21}\mathbf{w}$. As a result, we have $\mathbf{P}_{21}\mathbf{w} = C_2(zI - A)^{-1}\boldsymbol{\delta_x} + \boldsymbol{\delta_y}$. Substituting this identity into (30) yields

$$\mathbf{u} = \mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}[C_2(zI - A)^{-1}\boldsymbol{\delta_x} + \boldsymbol{\delta_y}]. \tag{31}$$

By definition, $\mathbf{L}$ is the closed loop mapping from $\boldsymbol{\delta_y}$ to $\mathbf{u}$. Equation (31) then implies that $\mathbf{L} = \mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}$. From [31], [37] (c.f. Appendix A), we have $\mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1} = \mathfrak{M}(\mathbf{Q})$, which completes the proof. ∎

Proof of Theorem 3: The equivalence between the parameterizations (27) and (28) is readily obtained from Lemma 5. As $\mathfrak{M}$ is an invertible affine mapping between $\mathbf{L}$ and $\mathbf{Q}$, any convex constraint imposed on the Youla parameter $\mathbf{Q}$ can be equivalently translated into a convex SLC imposed on $\mathbf{L}$, and vice versa. ∎

B. Quadratically Invariant Subspace Constraints

Recall that for a subspace $\mathcal{C}$ that is quadratically invariant with respect to a plant $\mathbf{P}_{22}$, the set of internally stabilizing controllers $\mathbf{K}$ that lie within the subspace $\mathcal{C}$ can be expressed as the set of stable transfer matrices $\mathbf{Q} \in \mathcal{RH}_\infty$ satisfying $\mathfrak{M}(\mathbf{Q}) \in \mathcal{C}$, for $\mathfrak{M}$ and invertible affine map. (c.f. Section II-B, problem (4) and Appendix A). We therefore have the following corollary to Theorem 3.

Corollary 2: Let $\mathcal{C}$ be a subspace constraint that is quadratically invariant with respect to $\mathbf{P}_{22}$. Then the set of internally stabilizing controllers satisfying $\mathbf{K} \in \mathcal{C}$ can be parameterized as in Theorem 3 with $\mathbf{L} \in \mathfrak{M}(\mathcal{Q}) = \mathcal{C}$.

Proof: From Lemma 5, we have $\mathbf{L} = \mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}$. Invoking Theorem 14 of [11], we have that $\mathbf{K} \in \mathcal{C}$ if and only if $\mathbf{L} = \mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1} \in \mathcal{C}$. The claim then follows immediately from Theorem 3. ∎

Note that Corollary 2 holds true for stable and unstable plants $\mathbf{P}$. Therefore, in order to parameterize the set of internally stabilizing controllers lying in $\mathcal{C}$, we do not need to assume the existence of an initial strongly stabilizing controller as in [11] nor do we need to perform a doubly coprime factorization as in [31]. Thus we see that QI subspace constraints are a special case of SLCs.

Finally, we note that in [20] and [30], the authors show that quadratic invariance is necessary for a subspace constraint $\mathcal{C}$ on the controller $\mathbf{K}$ to be enforceable via a convex constraint on the Youla parameter $\mathbf{Q}$. However, when $\mathcal{C}$ is not a subspace constraint, no general methods exist to determine whether the set of internally stabilizing controllers lying in $\mathcal{C}$ admits a convex representation. In contrast, determining the convexity of a SLC is trivial.

C. System Performance Constraints

Let $g(\cdot)$ be a functional of the system response – it then follows that all internally stabilizing controllers satisfying a performance level, as specified by a scalar γ, are given by transfer matrices $\{\mathbf{R},\mathbf{M},\mathbf{N},\mathbf{L}\}$ satisfying the conditions of Theorem 2 and the SLC

$$g(\mathbf{R},\mathbf{M},\mathbf{N},\mathbf{L}) \leq \gamma. \tag{32}$$

Further, recall that the sublevel set of a convex functional is a convex set, and hence if g is convex, then so is the SLC (32). A particularly useful choice of convex functional is

$$g(\mathbf{R},\mathbf{M},\mathbf{N},\mathbf{L}) = \left\| \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} + D_{11} \right\|, \tag{33}$$

for a system norm $\|\cdot\|$, which is equivalent to the objective function of the decentralized optimal control problem (3). Thus by imposing several performance SLCs (33) with different choices of norm, one can naturally formulate multi-objective optimal control problems.

Remark 2: For a continuous time system with the system norm $\|\cdot\|$ in (33) chosen to be the $\mathcal{H}_2$ norm, the closed loop transfer matrix in (33) needs to be strictly proper, i.e., $D_{12}L[0]D_{21} + D_{11} = 0$.

D. Controller Robustness Constraints

Suppose that the controller is to be implemented using limited hardware, thus introducing non-negligible quantization (or other errors) to the internally computed signals: this can be modeled via an internal additive noise $\boldsymbol{\delta_\beta}$ in the controller structure (c.f., Figure 2). In this case, we may wish to design a controller that further limits the effects of these perturbations on the system: to do so, we can impose a performance SLC on the closed loop transfer matrices specified in the rightmost column of Table I. Note that it is not clear how to impose such controller robustness constraints using the Youla parameterization.

E. Controller Architecture Constraints

The controller implementation (25) also allows us to naturally control the number of actuators and sensors used by a controller – this can be useful when designing controllers for large-scale systems that use a limited number of hardware resources (c.f., Section V-D). In particular, assume that implementation (25) parameterizing stabilizing controllers that use all possible actuators and sensors. It then suffices to constrain the number of non-zero rows of the transfer matrix $[\tilde{\mathbf{M}}, \mathbf{L}]$ to limit the number of actuators used by the controller, and similarly, the number of non-zero columns of the transfer matrix $[\tilde{\mathbf{N}}^\top, \mathbf{L}^\top]^\top$ to limit the number of sensors used by the controller. As stated, these constraints are non-convex, but recently proposed convex relaxations [33], [34] can be used in their stead to impose convex SLCs on the controller architecture.

F. Positivity Constraints

It has recently been observed that (internally) positive systems are amenable to efficient analysis and synthesis techniques (c.f., [38] and the references therein). Therefore it may be desirable to synthesize a controller that either preserves or enforces positivity of the resulting closed loop system. We can enforce this condition via the SLC that the elements

$$\left\{ \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} R[t] & N[t] \\ M[t] & L[t] \end{bmatrix} \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} \right\}_{t=1}^{\infty}$$

and the matrix $(D_{12}L[0]D_{21} + D_{11})$ are all element-wise nonnegative matrices. This SLC is easily seen to be convex.

G. FIR Constraints

Given the parameterization of stabilizing controllers of Theorem 2, it is trivial to enforce that a system response be FIR with horizon T via the following SLC

$$\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L} \in \mathcal{F}_T. \tag{34}$$

We argue that imposing a FIR SLC is beneficial in the following ways:

(a) The closed loop response to an impulse disturbance is FIR of horizon T, where T can be set by the control designer. As such, the settling time of the system can be accurately tuned.
(b) The controller achieving the desired system response can be implemented using the FIR filter banks $\tilde{\mathbf{R}}^+, \tilde{\mathbf{M}}, \tilde{\mathbf{N}}, \mathbf{L} \in \mathcal{F}_T$, as illustrated in Figure 2. This simplicity of implementation is extremely helpful when applying these methods in practice.
(c) When a FIR SLC is imposed, the resulting set of stable achievable system responses and corresponding controllers admit a finite dimensional representation – specifically, the constraints specified in Theorem 2 only need to be applied to the impulse response elements $\{R[t], M[t], N[t], L[t]\}_{t=0}^T$.

Remark 3: It should be noted that the computational benefits claimed above hold only for discrete time systems. For continuous time systems, a FIR transfer matrix is still an infinite dimensional object, and hence the resulting parameterizations and constraints are in general infinite dimensional as well.

We defer discussing how to select the FIR horizon T in the context of an optimal control problem to our companion paper [32], and present here instead connections between the feasibility of the SLC (34) and the controllability, reachability and observability of a system (20).

Recall that the feasibility of condition (15) provides an alternative characterization of the stabilizability of the system (A, B_2). We now give an alternative characterization of the controllability of (A, B_2), in terms of condition (15) and a FIR SLC.

Before proceeding, we recall the notions of *controllability* and *reachability* in the sense of Kalman [39]. Given a positive integer T, we say that a system (A, B_2) with dynamics $x[t+1] = Ax[t] + B_2 u[t]$ is T-step controllable if we can select a sequence of control actions $\{u[t]\}_{t=0}^{T-1}$ to drive the state $x[T]$ to 0 from any initial condition $x[0]$. If we can drive the state $x[T]$ to an arbitrary value from any initial condition $x[0]$, then the system is said to be T-step reachable. A system (A, B_2) is said to be controllable (reachable) if it is T-step controllable (reachable) for some finite T. Therefore, to check whether a system (A, B_2) is T-step controllable, it suffices to verify that the impulse responses from $\boldsymbol{\delta_x}$ to $(\mathbf{x}, \mathbf{u})$ in (13) are FIR of horizon T. It is therefore clear that a pair (A, B_2) is T-step controllable if and only if the following equations are feasible for some finite T:

$$\begin{bmatrix} zI - A & -B_2 \end{bmatrix} \begin{bmatrix} \mathbf{R} \\ \mathbf{M} \end{bmatrix} = I \tag{35a}$$

$$\mathbf{R}, \mathbf{M} \in \mathcal{F}_T \cap \frac{1}{z}\mathcal{RH}_\infty. \tag{35b}$$

The conditions for reachability are slightly more restrictive than those for controllability.

Lemma 6: The pair (A, B_2) is reachable if and only if $\begin{bmatrix} A & B_2 \end{bmatrix}$ is full rank and (35) is feasible for a finite T.

Proof: See Appendix B. ∎

A similar argument leads to an alternative characterization of the observability of a pair (A, C_2): it suffices to check the feasibility of conditions (19) with the added FIR SLC that $\mathbf{R}, \mathbf{N} \in \mathcal{F}_T$ for some T.

Finally we note that when the triple (A, B_2, C_2) is controllable and observable, we can use equation (45) in Appendix B to construct a FIR system response $\{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}\}$ that satisfies the conditions (24) of Theorem 2 – hence, the controllability and observability of a triple (A, B_2, C_2) is a necessary and sufficient condition for the existence of a FIR system response $\{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}\}$ for some horizon T.

H. Subspace and Sparsity Constraints

Let $\mathcal{L}$ be a subspace of $\mathcal{RH}_\infty$. We can then parameterize all stable achievable system responses that lie in this subspace by adding the following SLC to the parameterization of Theorem 2:

$$\begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \in \mathcal{L}. \tag{36}$$

Of particular interest are subspaces $\mathcal{L}$ that define transfer matrices of sparse support. An immediate benefit of enforcing such sparsity constraints on the system response is that implementing the resulting controller (25) can be done in a *localized way*, i.e., each controller state β_i and control action u_i can be computed using a local subset (as defined by the support of the system response) of the global controller state β and sensor measurements y. For this reason, we refer to the constraint (36) as a *localized* SLC when it defines a subspace with sparse support. Further, as we briefly highlight in the next section, and discuss in detail in our companion paper [32], such localized constraints also allow for the resulting system response to be computed in a localized way, i.e., the global computation decomposes naturally into decoupled subproblems that depend only on local sub-matrices of the state-space representation (1). Clearly, both of these features are extremely desirable when computing controllers for large-scale systems.

Selecting an appropriate (feasible) localized SLC, as defined by the subspace $\mathcal{L}$, is a subtle task: it depends on an interplay between actuator and sensor density, information exchange delay and disturbance propagation delay. Formally defining and analyzing a procedure for designing a localized SLC is beyond the scope of this paper: as such, we refer the reader to our recent paper [5], in which we present a method that allows for the joint design of an actuator architecture and corresponding feasible localized SLC. We generalize these methods to the output feedback setting in our companion paper [32], where we show that actuation and sensing architectures, as well as feasible localized SLCs, can be co-designed using convex programming.

We informally illustrate some of these concepts in the example below, which builds on the motivating example Example 2 of Section II.

Example 3: Consider the optimal control problem (5) in Example 1, and assume that there is unit measurement delay, i.e., the control action at time t can be expressed as $u[t] = f(x[0:t-1])$ for some function f. The optimal system response for this delayed centralized optimal control problem is given by

$$\mathbf{R} = \frac{1}{z} I + \frac{1}{z^2} A, \quad \mathbf{M} = -\frac{1}{z^2} A^2, \tag{37}$$

with the optimal state-feedback controller given by $\mathbf{K} = -A^2(zI + A)^{-1}$. Note that the support of the system response elements $\mathbf{R}$ and $\mathbf{M}$ is defined by the support of A and A^2, respectively. As a concrete example, let the state matrix A be tridiagonal (hence its support defines the adjacency matrix of a chain): it then follows that the transfer matrix $\mathbf{R}$ is tridiagonal and $\mathbf{M}$ is pentadiagonal.

Using Lemma 2, the controller achieving the desired system response can be implemented using the FIR transfer matrices $\tilde{\mathbf{R}}$ and $\tilde{\mathbf{M}}$ (which have the same support as $\mathbf{R}$ and $\mathbf{M}$), via the controller structure in Figure 3. This implementation is localized as each node i needs only collect its first and second neighbors' estimated disturbances $\hat{w}_j$ to compute its control action u_i and reference state value $\hat{x}_i$. In contrast, notice that implementing the control policy $\mathbf{u} = \mathbf{K}\mathbf{y}$, with controller $\mathbf{K} = \mathbf{M}\mathbf{R}^{-1} = -A^2(zI + A)^{-1}$, requires each node to collect measurements y from every other node in the system, as $\mathbf{K}$ is dense.

As we highlighted in Example 1 and the discussion of Section II, imposing sparsity constraints on the controller $\mathbf{K}$ violates the conditions of quadratic invariance when the underlying system is strongly connected, and hence cannot be done using convex constraints on the Youla parameter. In contrast, imposing a localized SLC on the system response elements $\mathbf{R}$ and $\mathbf{M}$ is always convex, and in this case allows us to recover the centralized optimal solution.

I. *Intersections of SLCs and Spatiotemporal Constraints*

Another major benefit of SLCs is that several such constraints can be imposed on the system response at once. Further, as convex sets are closed under intersection, convex SLCs are also closed under intersection. To illustrate the usefulness of this property, consider the intersection of a QI subspace SLC, a FIR SLC and a localized SLC. The resulting SLC can be interpreted as enforcing a spatiotemporal constraint on the system response and its corresponding controller, as we explain using the chain example previously described.

Figure 6 shows a diagram of the system response to a particular disturbance $(\boldsymbol{\delta_x})_i$. In this figure, the vertical axis denotes the spatial coordinate of a state in the chain, and the horizontal axis denotes time: hence we refer to this figure as a space-time diagram. Depicted are the three components of the spatiotemporal constraint, namely the communication delay imposed on the controller via the QI subspace SLC, the deadbeat response of the system to the disturbance imposed by the FIR SLC, and the localized region affected by the disturbance $(\boldsymbol{\delta_x})_i$ imposed by the localized SLC.

When the effect of each disturbance $(\boldsymbol{\delta_x})_i$ can be localized within such a spatiotemporal SLC, the system is said to be *localizable* (c.f., [2], [4]). Recall that the controllability and the observability of a system is determined by the existence of an FIR system response. Similarly, the localizability of a system is determined by the existence of a system response satisfying a spatiotemporal SLC – in this sense, localizability can be viewed as a natural generalization of controllability and observability to the spatiotemporal domain.

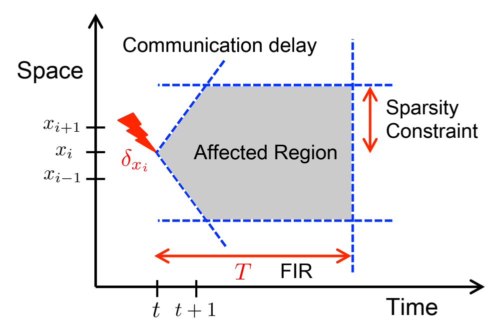


Fig. 6. Space time diagram for a single disturbance striking the chain described in Example 3.

V. System Level Synthesis

We build on the results of the previous sections to formulate the SLS problem. We show that by combining appropriate SLPs and SLCs, the largest known class of convex structured optimal control problems can be formulated. As a special case, we show that we recover all possible structured optimal control problems of the form (3) that admit a convex representation in the Youla domain.

A. General Formulation

Let $g(\cdot)$ be a functional capturing a desired measure of the performance of the system (as described in Section IV-C), and let $\mathcal{S}$ be a SLC. We then pose the SLS problem as

$$\begin{aligned} \underset{\{\mathbf{R},\mathbf{M},\mathbf{N},\mathbf{L}\}}{\text{minimize}} \quad & g(\mathbf{R},\mathbf{M},\mathbf{N},\mathbf{L}) \\ \text{subject to} \quad & (24a) - (24c) \\ & \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \in \mathcal{S}. \end{aligned} \tag{38}$$

For g a convex functional and $\mathcal{S}$ a convex set,[8] the resulting SLS problem is a convex optimization problem.

Remark 4: For a state feedback problem, the SLS problem can be simplified to

$$\begin{aligned} \underset{\{\mathbf{R},\mathbf{M}\}}{\text{minimize}} \quad & g(\mathbf{R},\mathbf{M}) \\ \text{subject to} \quad & (15a) - (15b) \\ & \begin{bmatrix} \mathbf{R} \\ \mathbf{M} \end{bmatrix} \in \mathcal{S}. \end{aligned} \tag{39}$$

B. Distributed Optimal Control

Here we show that by combining an appropriate SLC with the SLP described in Theorem 2, we recover the distributed optimal control formulation (3) as a special case of the SLS problem (38). Recall that the objective function in (3) is specified by a suitably chosen system norm measuring the size of the closed loop transfer matrix from the external disturbance $\mathbf{w}$ to the regulated output $\bar{\mathbf{z}}$. Therefore it suffices to select the objective functional g to be as described in equation (33), and to select the SLC constraint set $\mathcal{S}$ as described in Corollary 2. The resulting SLS problem

$$\begin{aligned} \underset{\{\mathbf{R},\mathbf{M},\mathbf{N},\mathbf{L}\}}{\text{minimize}} \quad & \left\| \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} + D_{11} \right\| \\ \text{subject to} \quad & (24a) - (24c) \\ & \mathbf{L} \in \mathcal{C} \end{aligned} \tag{40}$$

is then equivalent to the distributed optimal control problem (3) when the subspace $\mathcal{C}$ is QI with respect to the plant $\mathbf{P}_{22}$. Thus all decentralized optimal control problems that can be formulated as convex optimization problems in the Youla domain are special cases of the SLS problem (38).

C. Localized LQG Control

In [2], [4] we posed and solved a localized LQG optimal control problem. It can be recovered as a special case of the SLS problem (38) by selecting the functional g to be of the form (33) (with the system norm $\|\cdot\|$ chosen to be the $\mathcal{H}_2$ norm), and selecting the constraint set $\mathcal{S}$ to be a spatiotemporal SLC. In the case of a state-feedback problem [2], the resulting SLS problem is of the form

$$\begin{aligned} \underset{\{\mathbf{R},\mathbf{M}\}}{\text{minimize}} \quad & \|C_1\mathbf{R} + D_{12}\mathbf{M}\|_{\mathcal{H}_2}^2 \\ \text{subject to} \quad & (15a) - (15b) \\ & \begin{bmatrix} \mathbf{R} \\ \mathbf{M} \end{bmatrix} \in \mathcal{C} \cap \mathcal{L} \cap \mathcal{F}_T, \end{aligned} \tag{41}$$

for $\mathcal{C}$ a QI subspace SLC, $\mathcal{L}$ a sparsity SLC, and $\mathcal{F}_T$ a FIR SLC.

The observation that we make in [2] (and extend to the output feedback setting in [4]), is that the localized SLS problem (41) can be decomposed into a set of independent sub-problems solving for the columns $\mathbf{R}_i$ and $\mathbf{M}_i$ of the transfer matrices $\mathbf{R}$ and $\mathbf{M}$ – as these problems are independent, they can be solved in parallel. Further, the sparsity constraint $\mathcal{L}$ restricts each sub-problem to a local subset of the system model and states, as specified by the nonzero components of the corresponding column of the transfer matrices $\mathbf{R}$ and $\mathbf{M}$ (e.g., as was described in Example 1), allowing each of these sub-problems to be expressed in terms of optimization variables (and corresponding sub-matrices of the state-space realization (24)) that are of significantly smaller dimension than the global system response $\{\mathbf{R},\mathbf{M}\}$. Thus for a given feasible spatiotemporal SLC, the localized SLS problem (41) can be solved for arbitrarily large-scale systems, assuming that each sub-controller can solve its corresponding sub-problem in parallel.[9]

In our companion paper [32], we generalize all of these concepts to the system level approach to controller synthesis, and show that appropriate notions of separability for SLCs can be defined which allow for optimal controllers to be synthesized and implemented with order constant complexity (assuming parallel computation is available for each subproblem) relative to the global system size.

D. Regularization for Design and SLS

Throughout this paper our motivation has been to extend controller synthesis methods to large-scale systems. An equally important task when designing control systems in such settings is the design of the controller architecture itself, i.e., the placement of actuators, sensors and communication links between them. As we alluded to in the previous subsection, sufficiently dense actuation, sensing and communication architectures are necessary for a localized optimal control problem to be feasible. More generally, there is a tradeoff between closed loop performance and architectural cost, as denser controller architectures lead to better closed loop performance.

The regularization for design framework (RFD) was formulated to explore this tradeoff using convex programming by augmenting the objective function with a suitable convex regularizer that penalizes the use of actuators, sensors and communication links. The original RFD formulation allowed for controller architecture co-design in the Youla domain by exploiting QI properties of desirable architectures [33], [34], [40], [41], but was later ported to the localized optimal control framework [5]. Thus to integrate RFD with the system level approach, it suffices to add a suitable regularizer, as mentioned in Section IV-E and described in [5], [33], to the objective function of the SLS problem (38). We demonstrate the usefulness of combining RFD, locality and SLS in our companion paper [32].

[8] More generally, we only need the intersection of the set $\mathcal{S}$ and the restriction of the functional g to the affine subspace described in (24) to be convex.

[9] We also show how to co-design an actuation architecture and feasible corresponding spatiotemporal constraint in [5], and so the assumption of a feasible spatiotemporal constraint is a reasonable one.

E. Computational Complexity and Non-convex Optimization

A final advantage of the SLS problem (38) is that it is transparent to determine the computational complexity of the optimization problem. Specifically, the complexity of solving (38) is determined by the type of the objective function $g(\cdot)$ and the characterization of the intersection of the set $\mathcal{S}$ and the affine space (24a) - (24c). Further, when the SLS problem is non-convex, the direct nature of the formulation makes it straightforward to determine suitable convex relaxations or non-convex optimization techniques for the problem. In contrast, as discussed in [20], no general method exists to determine the computational complexity of the decentralized optimal control problem (3) for a general constraint set $\mathcal{C}$.

VI. System Level Tradeoffs

Here we show, using a smaller scale stylized example, the usefulness of the system level approach in exploring tradeoffs with respect to performance, implementation complexity and robustness. In our companion paper [32], we discuss how these methods can be extended to arbitrarily large systems, and perform more extensive computational studies on realistic system models with thousands of states.

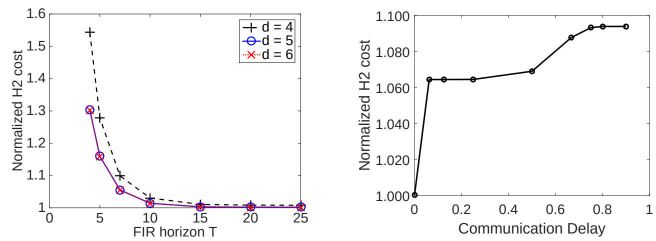
A. Performance vs. implementation complexity

We use a 100 node bi-directional chain with scalar subsystems for this numerical study. The dynamics of each subsystem i is given by

$$x_i[t+1] = \alpha(x_i[t] + \kappa x_{i-1}[t] + \kappa x_{i+1}[t]) + b_i u_i[t] + w_i[t]$$

for $i = 1, \cdots 100$, with $x_0 = x_{101} = 0$ as the boundary value. We can vary the parameter κ to adjust the coupling strength between subsystems, and vary the parameter α to adjust the instability of the open loop system. The value b_i is given by 1 if there is an actuator at subsystem i, and 0 otherwise. We place 40 actuators in the chain network, with actuator location specified by $i = 5j-4$ and $5j$ for $j = 1, \cdots, 20$. The objective function is given in the form of $\|\mathbf{x}\|_2^2 + \gamma\|\mathbf{u}\|_2^2$, where γ is the relative penalty between state deviation and control effort. If we choose $\gamma = 0$ and have $b_i = 1$ for all i, then this example reduces to the one in Example 1. We choose $\kappa = \gamma = 1$, and adjust α to make the spectral radius of A be 1.1 for the simulated example. Note that we use a plant with uniform, symmetric parameters and topology just for the convenience of illustration. Our method can be applied on plants with randomized parameters and arbitrary sparse interconnected topology.

Throughout, we impose a spatiotemporal SLC as described in Section IV-I and study the effects of choosing different sized localized regions, parameterized by d-hops of the physical plant topology (c.f., [4]) and the length of the FIR horizon – the system response $(\mathbf{R}, \mathbf{M})$ are band matrix with bandwidth d, and with FIR length T. As shown in Figure 7(a), appropriate choices of these parameters lead to no degradation in performance with respect to a centralized optimal controller, while leading to significant improvements in synthesis and implementation complexity – these complexity are in the order of $O(d^3)$.If we choose $(d, T) = (5, 15)$, then there is only 0.3% performance degradation compared to the centralized optimal one, which corresponds to $(d, T) = (99, \infty)$. The result shown in Figure 7(a) holds for a wide range of the parameters (κ, α, γ) of the plant. Specifically, we vary the coupling strength from 0.1 to 1, adjust the instability of the system from 1.1 to 3, change the relative penalty γ from 10^{-6} to 10^6, and change the actuator density from 20% to 100% – all the results are qualitatively similar to the one shown in Figure 7(a).



(a) Performance vs. FIR horizon T and localized region d

(b) Performance vs. communication speed ($(d, T) = (13, 20)$ for $t_c = 0.9$, $(d, T) = (8, 20)$ for the rest)

Fig. 7. Tradeoffs between performance and communication speed, localized region size, and FIR horizon (normalized with respect to the optimal centralized $\mathcal{H}_2$ cost).

B. Performance vs. communication delay

We next study the effects of communication delay on the performance. We assume that the communication network has the same topology as the physical network, and it takes time t_c for a sub-controller to transmit information to its direct neighbors. The delay t_c is normalized with respect to the sampling time of the discrete time system (1), and hence it may be non-integers in general. We adopt the following convention to handle fractional delays: if information is received by a sub-controller between two sampling times t and $t+1$, then it may be used by the sub-controller to compute its control action at time $t+1$. It is necessary to have $t_c < 1$ for the existence of a localized system response [5]. We choose $(d, T) = (8, 20)$ for the previous example, and study the tradeoff between communication delay t_c and the normalized $\mathcal{H}_2$ cost. As shown in Figure 7(b), communication delay only leads to slight degradation in performance. Note that the degradation is mostly contributed by the delay constraint. To verify this claim, we compare our localized controller with the QI optimal controller on a 40-state chain example – the QI method cannot scale to the 100-state example due to both memory issue and long computation time. Simulation shows that the localized FIR constraint $(d, T) = (8, 20)$ only leads to 0.03% degradation compared to a QI optimal controller with the same delay constraint.

VII. Conclusion

In this paper, we defined and analyzed the system level approach to controller synthesis, which consists of three elements: System Level Parameterizations (SLPs), System Level Constraints (SLCs), and System Level Synthesis (SLS) problems. We showed that all achievable and stable system responses can be characterized via the SLPs given in Theorems

1 and 2. We further showed that these system responses could be used to parameterize internally stabilizing controllers that achieved them, and proposed a novel controller implementation (25). We then argued that this novel controller implementation had the important benefit of allowing for SLCs to be naturally imposed on it, and showed in Section IV that using this controller structure and SLCs, we can characterize the broadest known class of constrained internally stabilizing controllers that admit a convex representation. Finally, we combined SLPs and SLCs to formulate the SLS problem, and showed that it recovered as a special case many well studied constrained optimal controller synthesis problems from the literature. In our companion paper [32], we show how to use the system level approach to controller synthesis to co-design controllers, system responses and actuation, sensing and communication architectures for large-scale networked systems.

Appendix A
Youla Parameterization and QI

Definition 3: A collection of stable transfer matrices, $\mathbf{U}_r, \mathbf{V}_r, \mathbf{X}_r, \mathbf{Y}_r, \mathbf{U}_l, \mathbf{V}_l, \mathbf{X}_l, \mathbf{Y}_l \in \mathcal{RH}_\infty$ defines a doubly co-prime factorization of $\mathbf{P}_{22}$ if $\mathbf{P}_{22} = \mathbf{V}_r \mathbf{U}_r^{-1} = \mathbf{U}_l^{-1} \mathbf{V}_l$ and

$$\begin{bmatrix} \mathbf{X}_l & -\mathbf{Y}_l \\ -\mathbf{V}_l & \mathbf{U}_l \end{bmatrix} \begin{bmatrix} \mathbf{U}_r & \mathbf{Y}_r \\ \mathbf{V}_r & \mathbf{X}_r \end{bmatrix} = I.$$

Such doubly co-prime factorizations can always be computed if $\mathbf{P}_{22}$ is stabilizable and detectable [36].

Let $\mathbf{Q}$ be the Youla parameter. When the subspace constraint $\mathcal{C}$ in (3) is quadratically invariant with respect to $\mathbf{P}_{22}$, the distributed optimal control problem (3) can be recast as the convex model matching problem [31], [37]:

$$\begin{aligned} \underset{\mathbf{Q}}{\text{minimize}} \quad & \|\mathbf{T}_{11} + \mathbf{T}_{12}\mathbf{Q}\mathbf{T}_{21}\| \\ \text{subject to} \quad & \mathbf{Q} \in \mathcal{RH}_\infty \\ & \mathfrak{M}(\mathbf{Q}) \in \mathcal{C}, \end{aligned} \tag{42}$$

where $\mathbf{T}_{11} = \mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{Y}_r\mathbf{U}_l\mathbf{P}_{21}$, $\mathbf{T}_{12} = -\mathbf{P}_{12}\mathbf{U}_r$, $\mathbf{T}_{21} = \mathbf{U}_l\mathbf{P}_{21}$, and $\mathfrak{M}(\mathbf{Q}) = \mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1} = (\mathbf{Y}_r - \mathbf{U}_r\mathbf{Q})\mathbf{U}_l$. By definition, we have $\mathbf{P}_{22} = \mathbf{V}_r\mathbf{U}_r^{-1} = \mathbf{U}_l^{-1}\mathbf{V}_l$. This implies that the transfer matrices $\mathbf{U}_r$ and $\mathbf{U}_l$ are both invertible. Therefore, $\mathfrak{M}$ is an invertible affine map of the Youla parameter $\mathbf{Q}$. Finally, the controller $\mathbf{K}$ can be expressed as $\mathbf{K} = \mathfrak{Y}(\mathbf{Q})[\mathfrak{X}(\mathbf{Q})]^{-1}$, with affine maps $\mathfrak{Y}(\mathbf{Q}) = (\mathbf{Y}_r - \mathbf{U}_r\mathbf{Q})$ and $\mathfrak{X}(\mathbf{Q}) = (\mathbf{X}_r - \mathbf{V}_r\mathbf{Q})$.

Appendix B
Proof of Lemmas

Lemma 7 (Necessity of conditions (15)*):* Consider the state feedback system (12). Let $(\mathbf{R}, \mathbf{M})$ be the system response achieved by an internally stabilizing controller $\mathbf{K}$. Then, $(\mathbf{R}, \mathbf{M})$ is a solution of (15).

Proof: Consider an internally stabilizing controller $\mathbf{u} = \mathbf{K}\mathbf{x}$ with its state space realization given by (2) (with $\mathbf{x} = \mathbf{y}$ for state feedback). Combining (2) and (13), we have the closed loop system dynamics given by

$$\begin{bmatrix} z\mathbf{x} \\ z\boldsymbol{\xi} \end{bmatrix} = \begin{bmatrix} A + B_2D_k & B_2C_k \\ B_k & A_k \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\xi} \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} \boldsymbol{\delta_x}. \tag{43}$$

As the controller is internally stabilizing, we know that the state matrix in (43) is a stable matrix (Lemma 5.2 in [36]). The system response achieved by the control law $\mathbf{u} = \mathbf{K}\mathbf{x}$ is given by

$$\begin{bmatrix} \mathbf{R} \\ \mathbf{M} \end{bmatrix} = \left[\begin{array}{cc|c} A + B_2D_k & B_2C_k & I \\ B_k & A_k & 0 \\ \hline I & 0 & 0 \\ D_k & C_k & 0 \end{array} \right]. \tag{44}$$

It is clear that the system response (44) is strictly proper and stable, thus (15b) is satisfied. In addition, routine calculations show that system (44) satisfies the equality constraint (15a) for arbitrary (A_k, B_k, C_k, D_k). This completes the proof. ∎

Proof of Lemma 3: For one direction, note that the feasibility of (24) implies the feasibility of both (15) and (19). Using Lemma 1 and Corollary 1, we know that the triple (A, B_2, C_2) is stabilizable and detectable.

For the opposite direction, given a stabilizable (A, B_2) and a detectable (A, C_2), let $(\mathbf{R}_1, \mathbf{M}_1)$ be a feasible solution of (15) and $(\mathbf{R}_2, \mathbf{N}_2)$ be a feasible solution of (19). We use these to construct the feasible solution to (24)

$$\mathbf{R} = \mathbf{R}_1 + \mathbf{R}_2 - \mathbf{R}_1(zI - A)\mathbf{R}_2 \tag{45a}$$
$$\mathbf{M} = \mathbf{M}_1 - \mathbf{M}_1(zI - A)\mathbf{R}_2 \tag{45b}$$
$$\mathbf{N} = \mathbf{N}_2 - \mathbf{R}_1(zI - A)\mathbf{N}_2 \tag{45c}$$
$$\mathbf{L} = -\mathbf{M}_1(zI - A)\mathbf{N}_2, \tag{45d}$$

which completes the proof. ∎

Lemma 8 (Necessity of conditions (24)*):* Consider the output feedback system (20). Let $\{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}\}$, with $\mathbf{x} = \mathbf{R}\boldsymbol{\delta_x} + \mathbf{N}\boldsymbol{\delta_y}$ and $\mathbf{u} = \mathbf{M}\boldsymbol{\delta_x} + \mathbf{L}\boldsymbol{\delta_y}$, be the system response achieved by an internally stabilizing control law $\mathbf{u} = \mathbf{K}\mathbf{y}$. Then, $\{\mathbf{R}, \mathbf{M}, \mathbf{N}, \mathbf{L}\}$ lies in the affine subspace described by (24).

Proof: Consider an internally stabilizing controller $\mathbf{K}$ with state space realization (2). Combining (2) with the system equation (21), we obtain the closed loop dynamics

$$\begin{bmatrix} z\mathbf{x} \\ z\boldsymbol{\xi} \end{bmatrix} = \begin{bmatrix} A + B_2D_kC_2 & B_2C_k \\ B_kC_2 & A_k \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\xi} \end{bmatrix} + \begin{bmatrix} I & B_2D_k \\ 0 & B_k \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta_x} \\ \boldsymbol{\delta_y} \end{bmatrix}.$$

From the assumption that $\mathbf{K}$ is internally stabilizing, we know that the state matrix of the above equation is a stable matrix (Lemma 5.2 in [36]). The system response achieved by $\mathbf{u} = \mathbf{K}\mathbf{y}$ is given by

$$\begin{bmatrix} \mathbf{R} & \mathbf{N} \\ \mathbf{M} & \mathbf{L} \end{bmatrix} = \left[\begin{array}{cc|cc} A + B_2D_kC_2 & B_2C_k & I & B_2D_k \\ B_kC_2 & A_k & 0 & B_k \\ \hline I & 0 & 0 & 0 \\ D_kC_2 & C_k & 0 & D_k \end{array} \right], \tag{46}$$

which satisfies (24c). In addition, it can be shown by routine calculation that (46) satisfies both (24a) and (24b) for arbitrary (A_k, B_k, C_k, D_k). This completes the proof. ∎

Proof of Lemma 6: Assume that (35) is feasible. As $\mathbf{R}$ and $\mathbf{M}$ are FIRs, the poles of $\mathbf{R}$ and $\mathbf{M}$ must be located at the origin $z = 0$. From (35a), we observe that $\begin{bmatrix} zI - A & -B_2 \end{bmatrix}$ is right invertible in the region where $\mathbf{R}(z)$ and $\mathbf{M}(z)$ do not have poles, with $\begin{bmatrix} \mathbf{R}^\top & \mathbf{M}^\top \end{bmatrix}^\top$ being its right inverse. This means that $\begin{bmatrix} zI - A & -B_2 \end{bmatrix}$ has full row rank for all $|z| > 0$.

Combining with the assumption that $[A \quad B_2]$ has full row rank, we can use the PBH test [35] to show that (A, B_2) is reachable.

For the opposite, we apply the PBH test to show that $[A \quad B_2]$ must have full row rank. From the definition of reachability, it is straightforward to construct a feasible solution to (35). ∎

References

[1] Y.-S. Wang, N. Matni, S. You, and J. C. Doyle, "Localized distributed state feedback control with communication delays," in *2014 IEEE American Control Conference (ACC)*, June 2014.

[2] Y.-S. Wang, N. Matni, and J. C. Doyle, "Localized LQR optimal control," in *2014 53rd IEEE Conference on Decision and Control (CDC)*, 2014.

[3] Y.-S. Wang and N. Matni, "Localized distributed optimal control with output feedback and communication delays," in *IEEE 52nd Annual Allerton Conference on Communication, Control, and Computing*, 2014.

[4] ——, "Localized LQG optimal control for large-scale systems," in *2016 IEEE American Control Conference (ACC)*, 2016.

[5] Y.-S. Wang, N. Matni, and J. C. Doyle, "Localized LQR control with actuator regularization," in *2016 IEEE American Control Conference (ACC)*, 2016.

[6] Y.-S. Wang, S. You, and N. Matni, "Localized distributed Kalman filters for large-scale systems," in *5th IFAC Workshop on Distributed Estimation and Control in Networked Systems*, 2015.

[7] D. C. Youla, H. A. Jabr, and J. J. B. Jr., "Modern wiener-hopf design of optimal controllers-part ii: The multivariable case," *Automatic Control, IEEE Transactions on*, vol. 21, no. 3, pp. 319–338, 1976.

[8] J. C. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis, "State-space solutions to standard H_2 and H_∞ control problems," *IEEE Transactions on Automatic Control*, vol. 34, no. 8, pp. 831–847, Aug 1989.

[9] Y.-C. Ho and K.-C. Chu, "Team decision theory and information structures in optimal control problems–part i," *Automatic Control, IEEE Transactions on*, vol. 17, no. 1, pp. 15–22, 1972.

[10] A. Mahajan, N. Martins, M. Rotkowitz, and S. Yuksel, "Information structures in optimal decentralized control," in *Decision and Control (CDC), 2012 IEEE 51st Annual Conference on*, 2012, pp. 1291–1306.

[11] M. Rotkowitz and S. Lall, "A characterization of convex problems in decentralized control," *Automatic Control, IEEE Transactions on*, vol. 51, no. 2, pp. 274–286, 2006.

[12] B. Bamieh, F. Paganini, and M. A. Dahleh, "Distributed control of spatially invariant systems," *Automatic Control, IEEE Transactions on*, vol. 47, no. 7, pp. 1091–1107, 2002.

[13] B. Bamieh and P. G. Voulgaris, "A convex characterization of distributed control problems in spatially invariant systems with communication constraints," *Systems & Control Letters*, vol. 54, no. 6, pp. 575–583, 2005.

[14] A. Nayyar, A. Mahajan, and D. Teneketzis, "Decentralized stochastic control with partial history sharing: A common information approach," *IEEE Transactions on Automatic Control*, vol. 58, no. 7, pp. 1644–1658, July 2013.

[15] H. S. Witsenhausen, "A counterexample in stochastic optimum control," *SIAM Journal of Control*, vol. 6, no. 1, 1968.

[16] J. N. Tsitsiklis and M. Athans, "On the complexity of decentralized decision making and detection problems," in *IEEE Conference on Decision and Control (CDC)*, 1984.

[17] X. Qi, M. V. Salapaka, P. G. Voulgaris, and M. Khammash, "Structured optimal and robust control with multiple criteria: A convex solution," *Automatic Control, IEEE Transactions on*, vol. 49, no. 10, pp. 1623–1640, 2004.

[18] G. E. Dullerud and R. D'Andrea, "Distributed control of heterogeneous systems," *Automatic Control, IEEE Transactions on*, vol. 49, no. 12, pp. 2113–2128, 2004.

[19] M. Rotkowitz, R. Cogill, and S. Lall, "Convexity of optimal control over networks with delays and arbitrary topology," *Int. J. Syst., Control Commun.*, vol. 2, no. 1/2/3, pp. 30–54, Jan. 2010. [Online]. Available: http://dx.doi.org/10.1504/IJSCC.2010.031157

[20] L. Lessard and S. Lall, "Convexity of decentralized controller synthesis," *IEEE Transactions on Automatic Control, To appear*, 2016. [Online]. Available: http://arxiv.org/pdf/1305.5859v2.pdf

[21] ——, "Optimal controller synthesis for the decentralized two-player problem with output feedback," in *2012 IEEE American Control Conference (ACC)*, June 2012.

[22] P. Shah and P. A. Parrilo, "$\mathcal{H}_2$-optimal decentralized control over posets: A state space solution for state-feedback," in *Decision and Control (CDC), 2010 49th IEEE Conference on*, 2010.

[23] A. Lamperski and J. C. Doyle, "Output feedback $\mathcal{H}_2$ model matching for decentralized systems with delays," in *2013 IEEE American Control Conference (ACC)*, June 2013.

[24] L. Lessard, M. Kristalny, and A. Rantzer, "On structured realizability and stabilizability of linear systems," in *American Control Conference (ACC), 2013*, June 2013, pp. 5784–5790.

[25] C. W. Scherer, "Structured $\mathcal{H}_\infty$-optimal control for nested interconnections: A state-space solution," *Systems and Control Letters*, vol. 62, pp. 1105–1113, 2013.

[26] L. Lessard, "State-space solution to a minimum-entropy $\mathcal{H}_\infty$-optimal control problem with a nested information constraint," in *2014 53rd IEEE Conference on Decision and Control (CDC)*, 2014. [Online]. Available: http://arxiv.org/pdf/1403.5020v2.pdf

[27] N. Matni, "Distributed control subject to delays satisfying an $\mathcal{H}_\infty$ norm bound," in *2014 53rd IEEE Conference on Decision and Control (CDC)*, 2014. [Online]. Available: http://arxiv.org/pdf/1402.1559.pdf

[28] T. Tanaka and P. A. Parrilo, "Optimal output feedback architecture for triangular LQG problems," in *2014 IEEE American Control Conference (ACC)*, June 2014.

[29] A. Lamperski and L. Lessard, "Optimal decentralized state-feedback control with sparsity and delays," *Automatica*, vol. 58, pp. 143–151, 2015.

[30] L. Lessard and S. Lall, "Quadratic invariance is necessary and sufficient for convexity," in *Proceedings of the 2011 American Control Conference*, June 2011, pp. 5360–5362.

[31] Ş. Sabău and N. C. Martins, "Youla-like parametrizations subject to QI subspace constraints," *Automatic Control, IEEE Transactions on*, vol. 59, no. 6, pp. 1411–1422, 2014.

[32] Y.-S. Wang, N. Matni, and J. C. Doyle, "Localized system level synthesis for large-scale systems," *submitted to IEEE Transactions on Automatic Control*, 2016.

[33] N. Matni and V. Chandrasekaran, "Regularization for design," *IEEE Transactions on Automatic Control*, vol. PP, no. 99, pp. 1–1, 2016.

[34] ——, "Regularization for design," in *53rd IEEE Conference on Decision and Control*, Dec 2014, pp. 1111–1118.

[35] G. E. Dullerud and F. Paganini, *A Course In Robust Control Theory: A Convex Approach*. Springer-Verlag, 2000.

[36] K. Zhou, J. C. Doyle, and K. Glover, *Robust and optimal control*. Prentice Hall New Jersey, 1996.

[37] A. Lamperski and J. C. Doyle, "The $\mathcal{H}_2$ control problem for quadratically invariant systems with delays," *Automatic Control, IEEE Transactions on*, vol. 60, no. 7, pp. 1945–1950, 2015.

[38] A. Rantzer, "Scalable control of positive systems," *European Journal of Control*, vol. 24, pp. 72–80, 2015.

[39] R. E. Kalman, P. L. Falb, and M. A. Arbib, *Topics in mathematical system theory*. McGraw-Hill New York, 1969, vol. 33.

[40] N. Matni, "Communication delay co-design in h2 distributed control using atomic norm minimization," *IEEE Transactions on Control of Network Systems*, vol. PP, no. 99, pp. 1–1, 2015.

[41] ——, "Communication delay co-design in h2 decentralized control using atomic norm minimization," in *52nd IEEE Conference on Decision and Control*, Dec 2013, pp. 6522–6529.