Gauss Elimination with Partial Pivoting: Examples

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Outline

Example 1

Example 2

Implementation notes

Example 1

Consider

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 4 & 6 \\ -1 & 2 & 5 \end{bmatrix}$$

The steps of GE with partial pivoting can be written as

$$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_1 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -2 \\ 2 & 4 & 6 \\ -1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & -2 \\ -1 & 2 & 5 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0.5 & 0 & 1 \end{bmatrix}$$

$$M_1 P_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0.5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & -2 \\ -1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 1 & -5 \\ 0 & 4 & 8 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P_2 M_1 P_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \\ 0 & 1 & -5 \\ 0 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 4 & 8 \\ 0 & 1 & -5 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -0.25 & 1 \end{bmatrix}$$

$$M_2 P_2 M_1 P_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -0.25 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \\ 0 & 4 & 8 \\ 0 & 1 & -5 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 4 & 8 \\ 0 & 0 & -7 \end{bmatrix} = U$$

How to obtain L and P? Denote $\widetilde{M}_1 = P_2 M_1 P_2^T$ and $P = P_2 P_1$. Then

$$\begin{aligned} M_2 P_2 M_1 P_1 A &= U \\ M_2 P_2 M_1 P_2^T P_2 P_1 A &= U, \quad \text{since } P_2^T P_2 = I \\ M_2 (P_2 M_1 P_2^T)(P_2 P_1) A &= U \\ M_2 \widetilde{M}_1 P A &= U \end{aligned}$$

Consider $\widetilde{M}_1$:

$$\widetilde{M}_1 = P_2 M_1 P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0.5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 0 & 1 \\ -0.5 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ -0.5 & 0 & 1 \end{bmatrix}$$

Then

$$
\begin{aligned}
PA &= \widetilde{M}_1^{-1} M_2^{-1} U \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ -0.5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0.25 & 1 \end{bmatrix} U \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ -0.5 & 0.25 & 1 \end{bmatrix} U \\
&= LU
\end{aligned}
$$

When inverting the M_i we flip the signs below the diagonal.

$$
\begin{aligned}
P = P_2 P_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

Example 2

Consider a 4×4 matrix A. Then GE elimination with partial pivoting can be written as

$$M_3 P_3 M_2 P_2 M_1 P_1 A = U$$

Using $P_3^T P_3 = I$ and $P_2^T P_3^T P_3 P_2 = I$, write the above as

$$\begin{aligned} U &= M_3 \underbrace{(P_3 M_2 P_3^T)}_{\widetilde{M}_2} (P_3 \underbrace{P_2 M_1 P_2^T P_3^T)}_{\widetilde{M}_1} \underbrace{(P_3 P_2 P_1)}_{P} A \\ &= M_3 \widetilde{M}_2 \widetilde{M}_1 P A \end{aligned}$$

Then

$$PA = \underbrace{\widetilde{M}_1^{-1} \widetilde{M}_2^{-1} M_3^{-1}}_{L} U = LU$$

Implementation notes

To implement the above, notice

- how the permutation matrices act on the M_i
- how their inverses are computed
- how L is formed
- how P is formed

The above can be implemented with < 20 lines in Matlab