McMaster University

Integrated Biomedical Engineering & Health Sciences Program

IBEHS - 4QZ3
Modelling of Biological Systems

Lecture 8

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Todays Aims…

Fourier Transforms

STFT

Wavelets

Fourier Transform

Process that breaks down functions depending on space or time into functions depending on spatial or temporal frequency

Time domain → Frequency Domain

Break signal down into sum of sines and cosines

Can be continuous or discrete

https://en.wikipedia.org/wiki/Fourier_transform

Jean-Baptiste Joseph Fourier

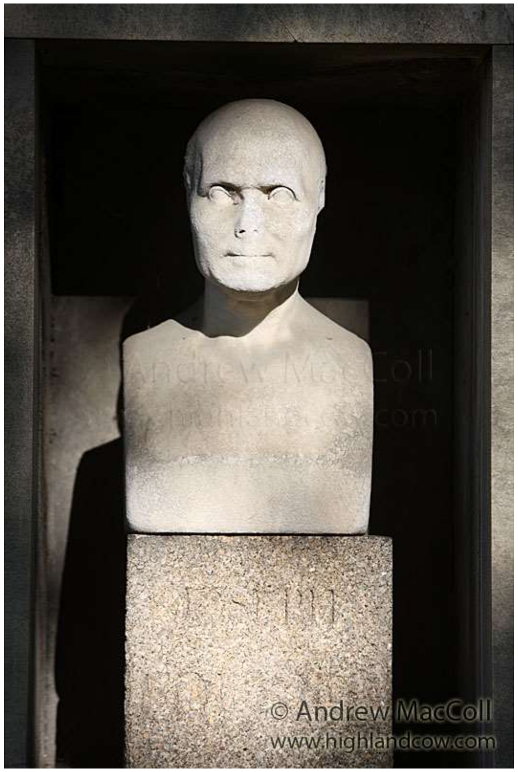
Jean-Baptiste Joseph Fourier

(21 March 1768 – 16 May 1830)

French mathematician and physicist

Fourier's law of conduction

Discovery of the greenhouse effect



Timeline

1807, J.B. Fourier:

- All periodic functions can be expressed as a weighted sum of trigonometric function
- Denied publication by Lagrange, Legendre and Laplace
- 1822: Fourier's work is finally published
- …
- …
- …
- …
- …
- …
- 1965, Cooley & Tukey: Fast Fourier Transform

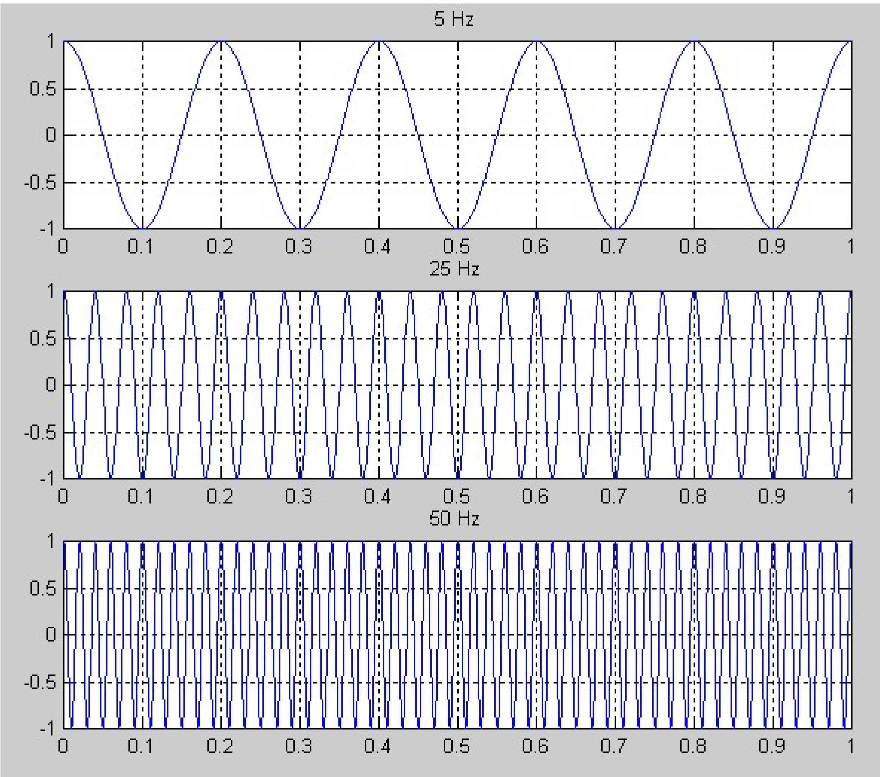
143 years

Fourier Transform (FT)

$X_1(t)=\cos(2p\cdot 5\cdot t)$

$X_2(t)=\cos(2p\cdot 25\cdot t)$

$X_3(t)=\cos(2p\cdot 50\cdot t)$

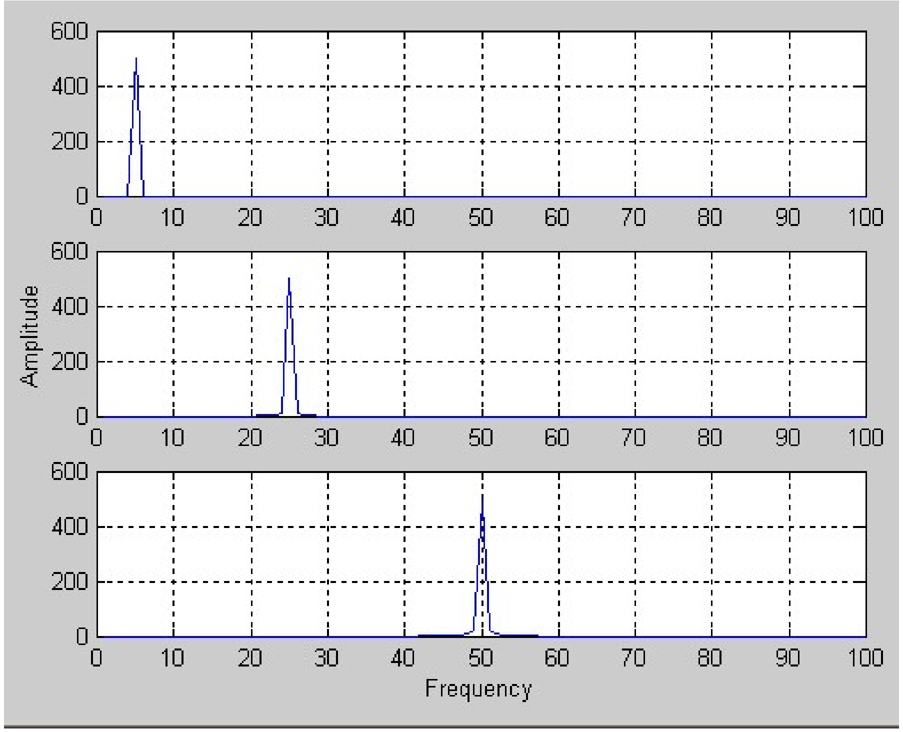


The Fourier Transform at work:

$$x_1(t) \xleftrightarrow{\mathcal{FT}} X_1(\omega)$$

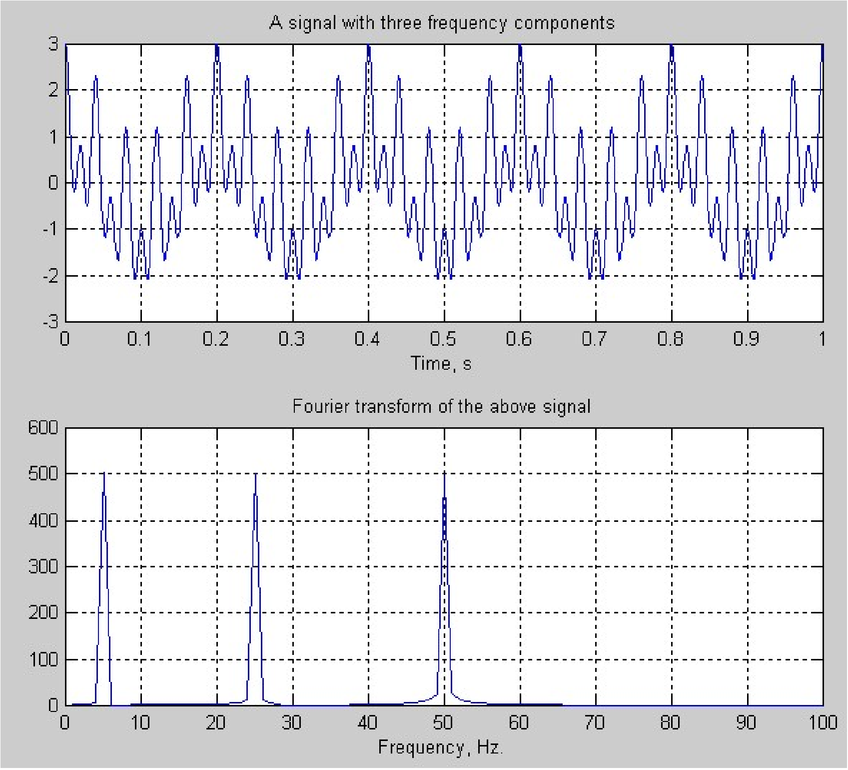
$$x_2(t) \xleftrightarrow{\mathcal{FT}} X_2(\omega)$$

$$x_3(t) \xleftrightarrow{\mathcal{FT}} X_3(\omega)$$



FT At Work

$$x_4(t) = \cos(2\pi \cdot 5 \cdot t) + \cos(2\pi \cdot 25 \cdot t) + \cos(2\pi \cdot 50 \cdot t)$$



$$x_4(t) \xleftrightarrow{\mathcal{FT}} X_4(\omega)$$

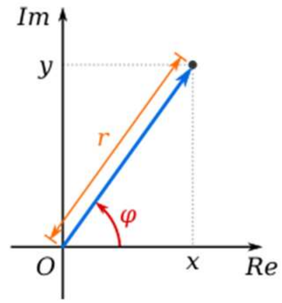
Fourier Transform

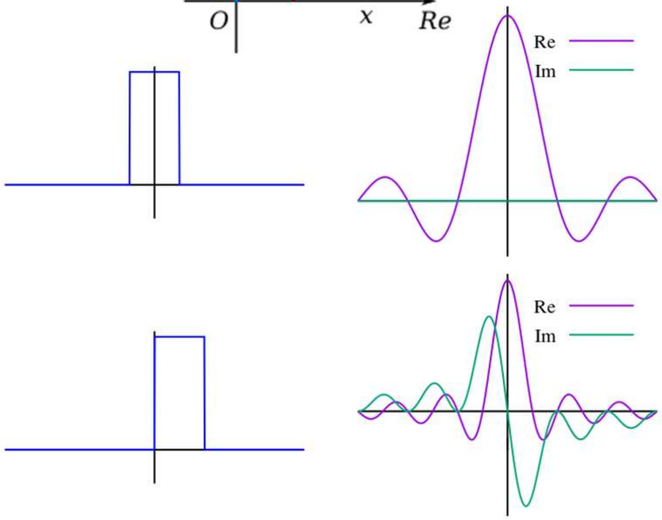
The amplitude is $|\hat{f}(\omega)|$

Phase is in $\arg \hat{f}(\omega)$ (Angle between positive real axis and point)

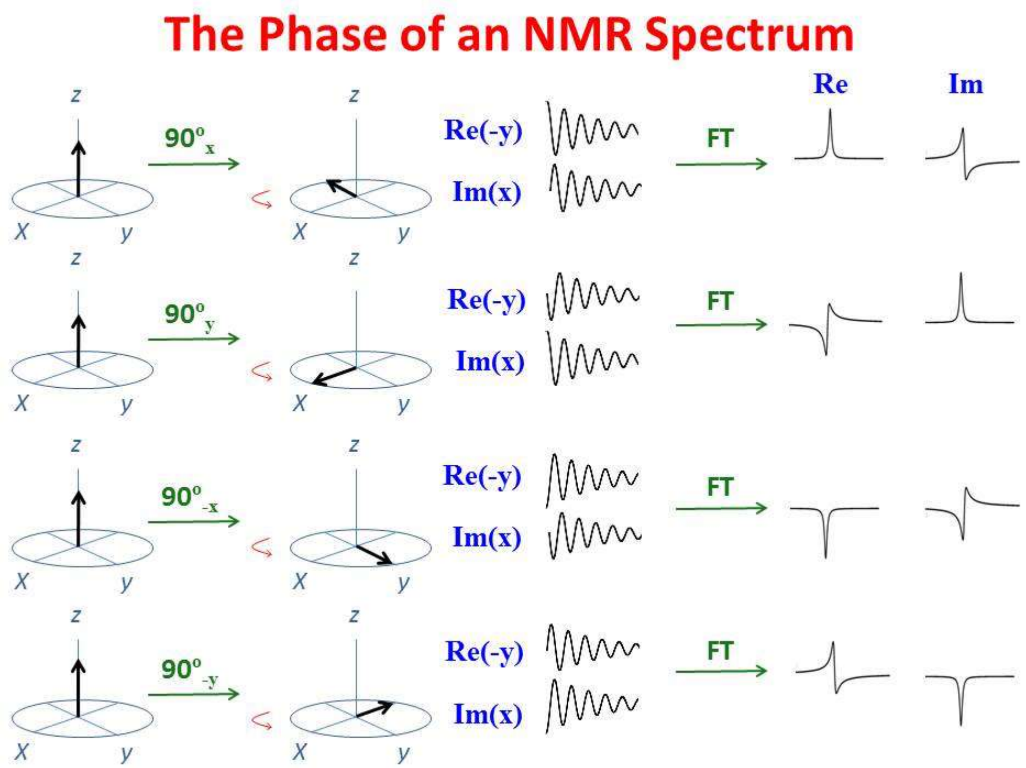
$f(\omega) = |\hat{f}(\omega)| e^{i \arg \hat{f}(\omega)}$

Shift in time is multiplication by $e^{-ia\omega}$ in the frequency domain

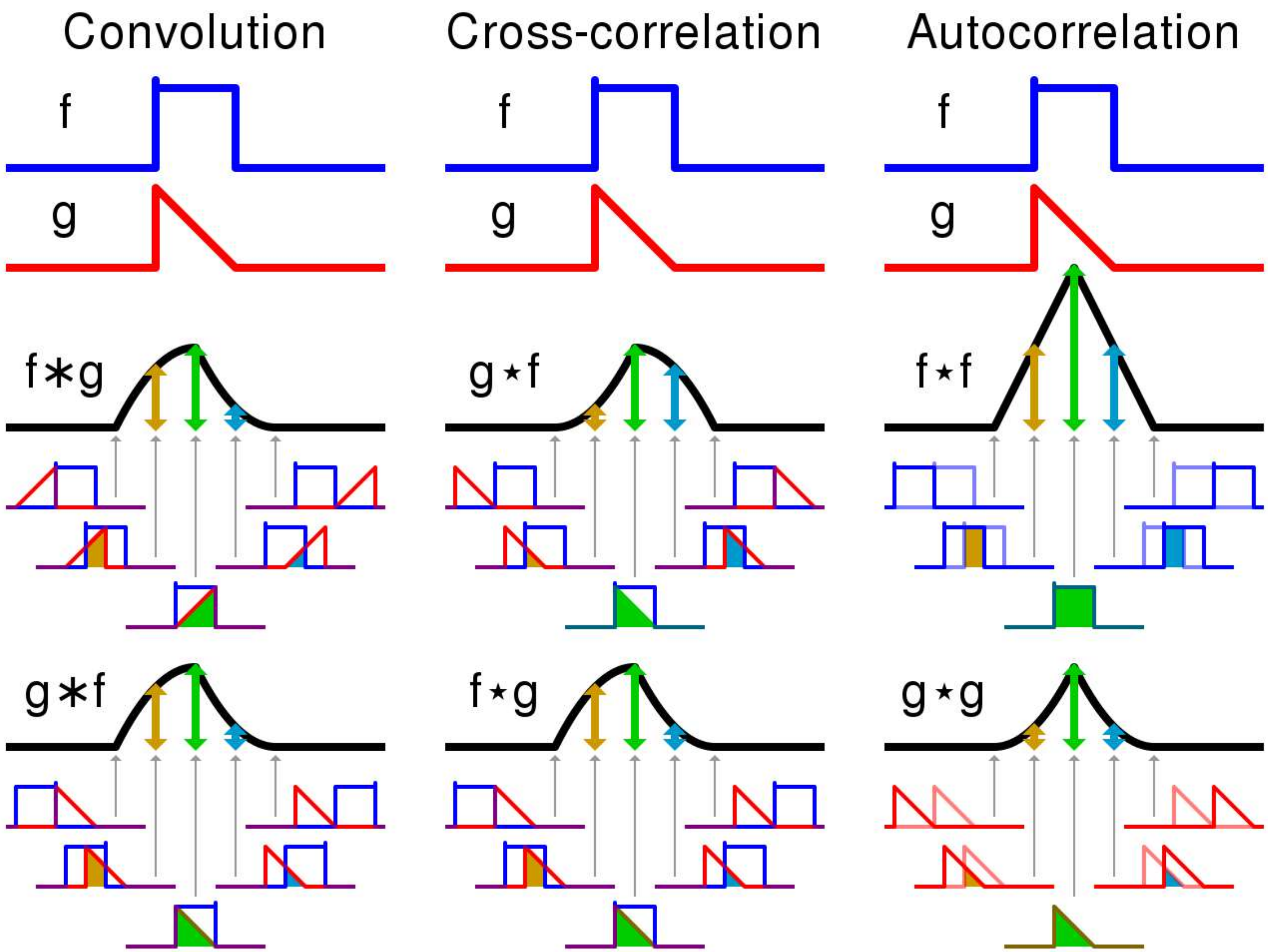




A great example showing effects of phase on the FT



http://u-of-o-nmr-facility.blogspot.com/2010/11/phase-of-nmr-spectrum.html



Convolution
Cross-correlation
Autocorrelation
f
g
f∗g
g∗f
g⋆f
f⋆g
f⋆f
g⋆g

Data Truncation (Windowing)

A digitized waveform must necessarily be truncated to the length of the memory storage array, a process described as "windowing."

The windowing process can be thought of as multiplying the data by some shaped function.

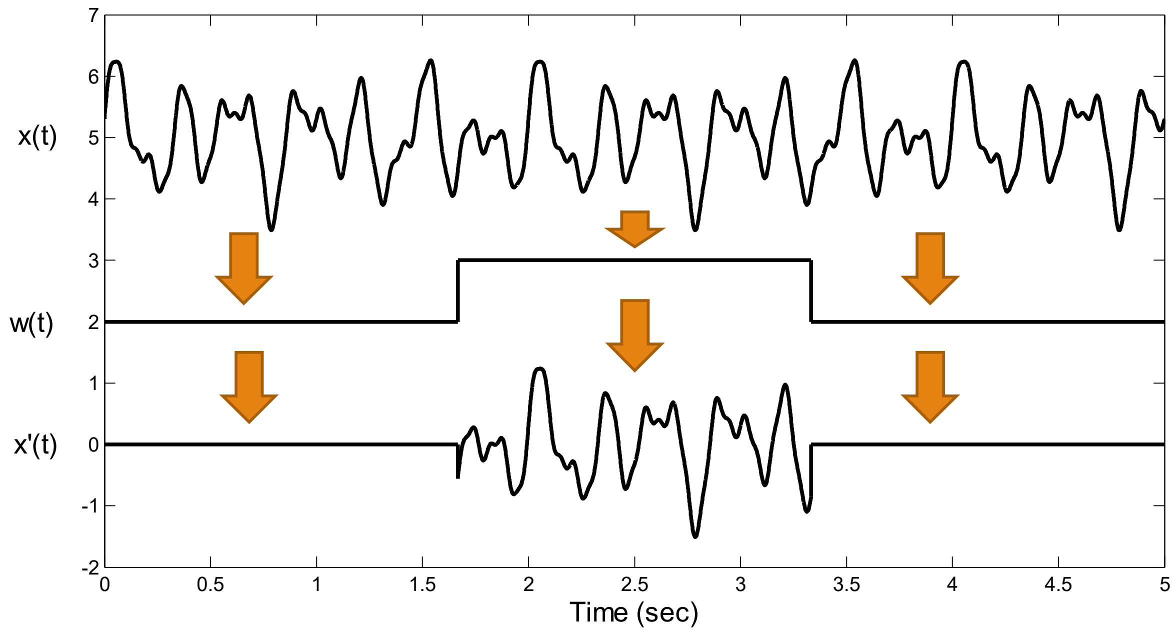
If the waveform is simply truncated (as often the case), then the window shape is rectangular.

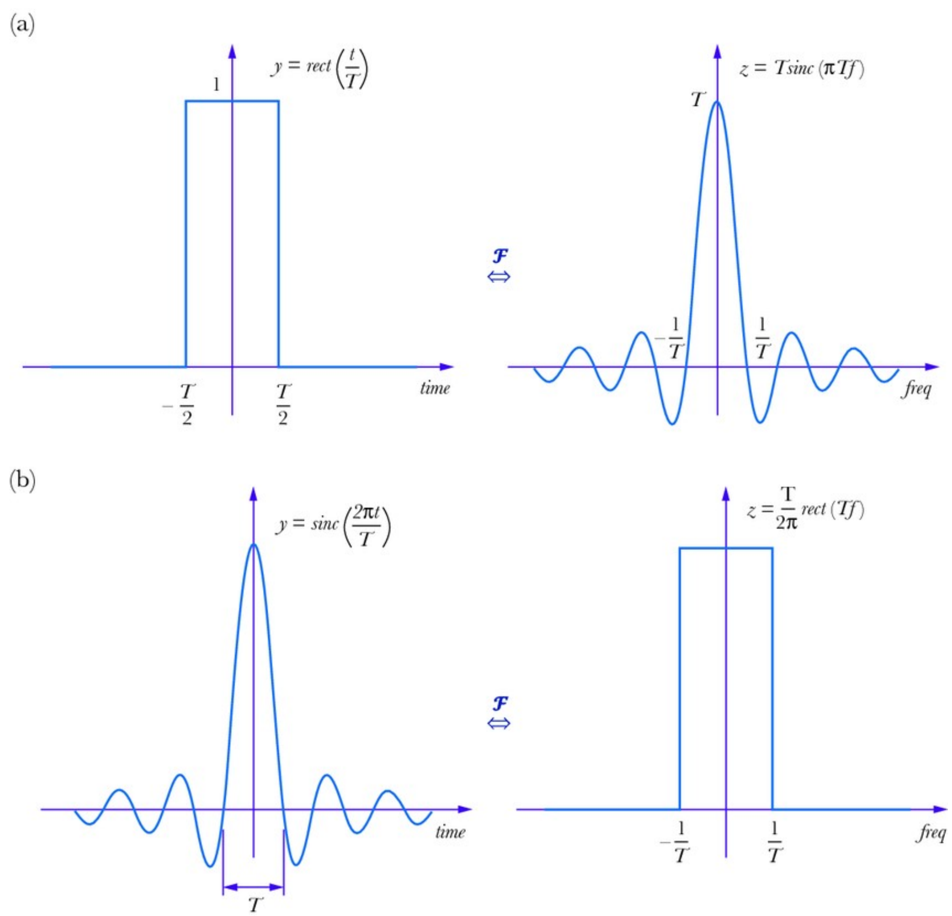
Multiplying in the time domain is equivalent to convolution in the frequency domain (and vice versa)

Windowing

Truncation is the same as multiplying the data by a function that is 1. for the length of the data and 0.0 everywhere else.

This function is called a 'rectangular window.'





Windowing in Time Domain

Windowing is multiplication in the time domain: the signal is multiplied by the window.

Multiplication in the time domain is like convolution in the frequency domain (and vise versa).

Every point in the frequency domain is modified by the <u>frequency characteristics</u> of the window function.

Windowing (cont)

The ideal window would have a spectrum that was 1.0 at $f = 0$ Hz and 0 everywhere else.

The width of a window's spectrum at $f = 0$ Hz is termed the main lobe.

The nonzero height of a window's spectrum away from $f = 0$ Hz is termed the sidelobe.

The wider the mainlobe the less accurate the frequency resolution: nearby spectral points become averaged together.

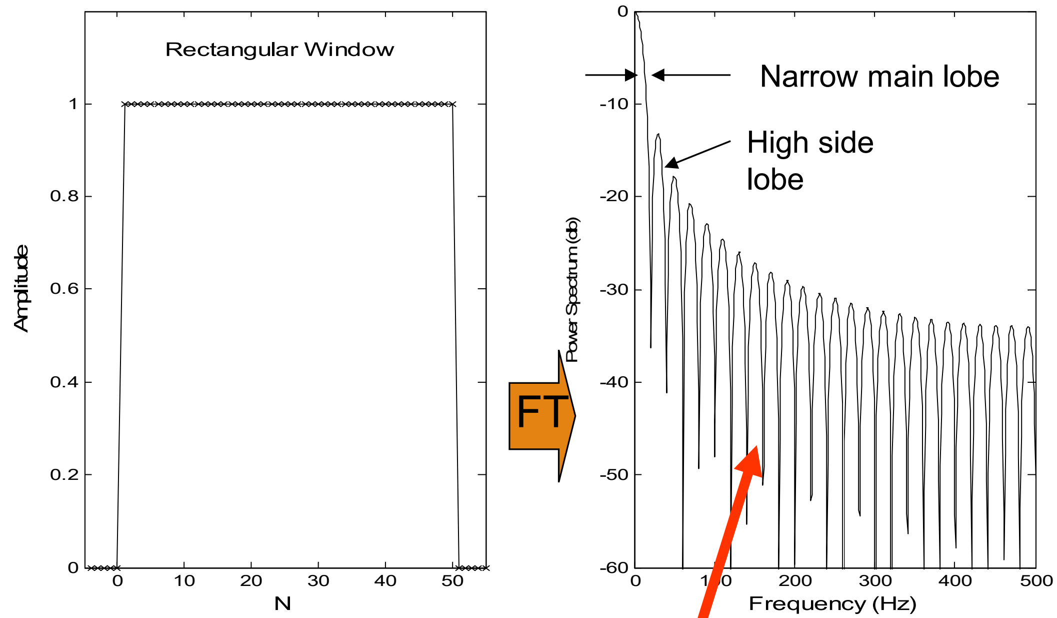
The higher the sidelobes the more adjacent frequencies influence the spectrum and are merged into the spectrum.

Windowing

Each point in a spectrum obtained from windowed signal is influenced by adjacent frequencies.

The spectrum of a rectangular window has high '<u>sidelobes</u>' (15 dB) which mixes nearby frequencies into each calculated spectral point.

- also has a narrow '<u>mainlobe</u>' so that its frequency resolution will be 'good'

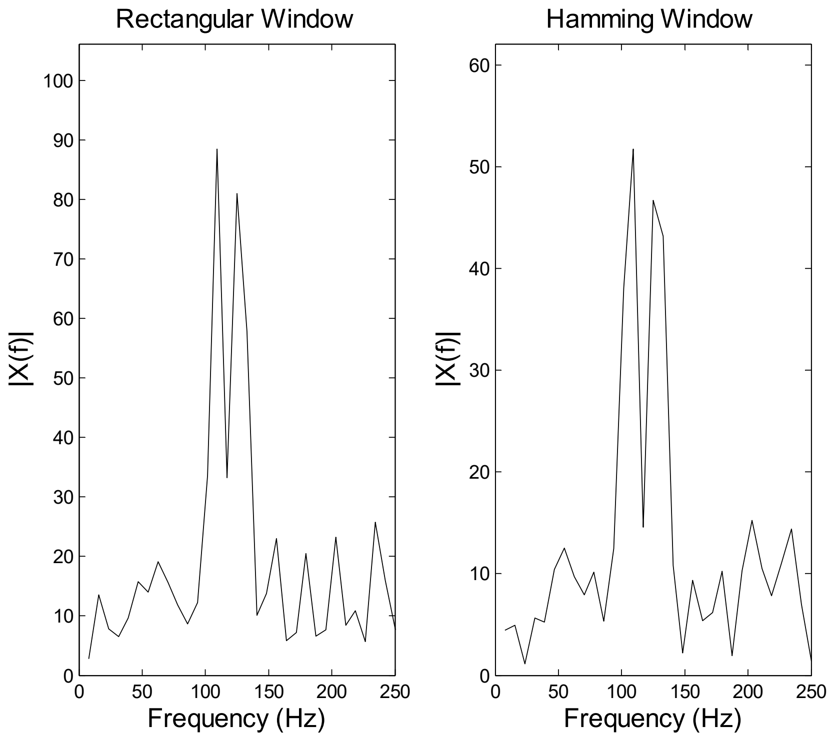


Every spectral point in the FFT is convolved with this curve

Windowing

If the data set is fairly long (perhaps 256 points or more), the benefits of a non-rectangular window are slight.

Figure shows spectra obtained with a rectangular and Hamming window.
They are nearly the same except for a scale difference produced by the Hamming window.



Power Spectrum

In the direct approach, the power spectrum is calculated as the magnitude squared of the Fourier transform of the waveform of interest:

$$PS(f) = |X(f)|^2$$

The power spectrum does not contain phase information so the power spectrum is not a bilateral transformation -- it is not possible to reconstruct the signal from the power spectrum.

Spectral Averaging: Periodogram

Just as time signals can be averaged, Power Spectra can be averaged.

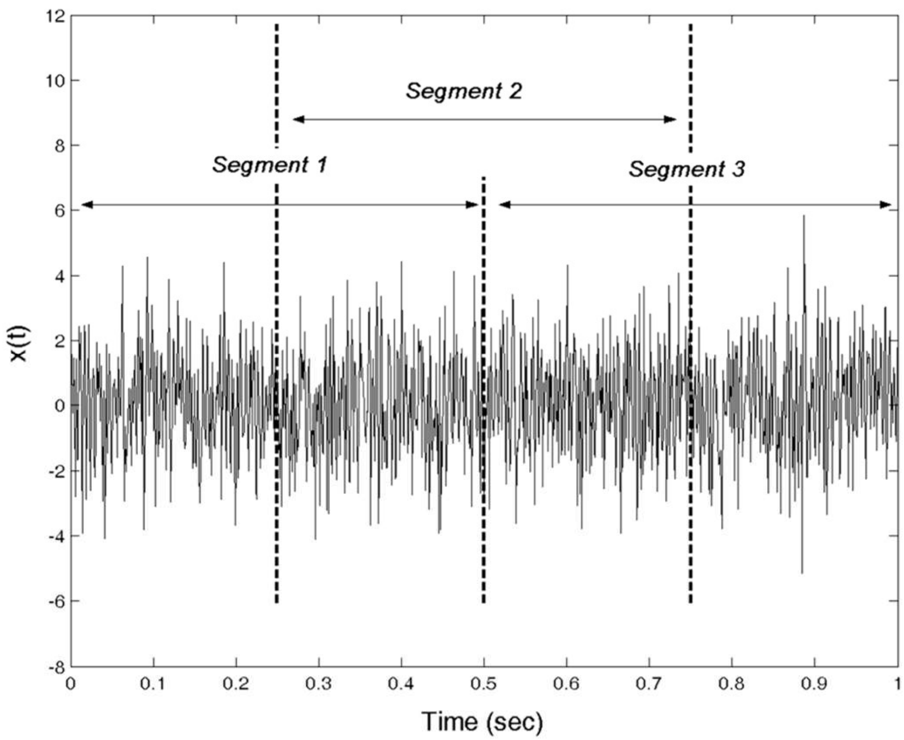
Even if only one signal is available, isolated segments of the data can be used.

The Power Spectra determined from each segment is averaged to produce a spectrum that better represents the broad, or global, features of the spectrum.

When the Power Spectrum is based on a direct application of the Fourier Transform followed by averaging, it is commonly referred to as an averaged periodogram.

Spectral Averaging: Welch Method

One of the most common methods to evaluate the average periodogram is attributed to Welch which uses overlapping segments.
(here with 50% overlap)



A shaping window is often applied to each segment because the segments tend to be short.

Spectral Averaging (cont)

Averaging spectra can only be applied to the magnitude spectrum or power spectrum because magnitude spectra are insensitive to time translation.

Averaged periodograms traditionally average spectra from half-overlapping segments, i.e., segments that overlap by 50%

Frequency resolution $\propto$ fs/N

Fs = sampling frequency

N is # of samples in a segment,

Stationary and Non-stationary Signals

FT identifies all spectral components present in the signal, however it does not provide any information regarding the temporal (time) localization of these components.

Why?

Stationary signals consist of spectral components that do not change in time

- all spectral components exist at all times
- no need to know any time information
- FT works well for stationary signals

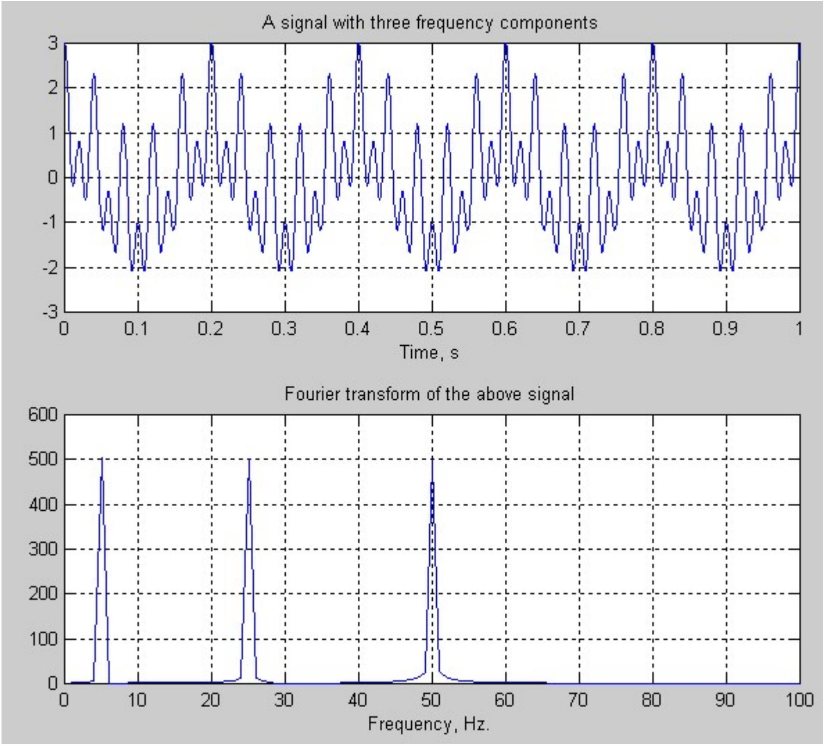
However, non-stationary signals consists of time varying spectral components

- How do we find out which spectral component appears when?
- FT only provides *what spectral components exist* , not where in time they are located.
- Need some other ways to determine *time localization of spectral components*

Recall: FT At Work

$$x_4(t) = \cos(2\pi \cdot 5 \cdot t) + \cos(2\pi \cdot 25 \cdot t) + \cos(2\pi \cdot 50 \cdot t)$$

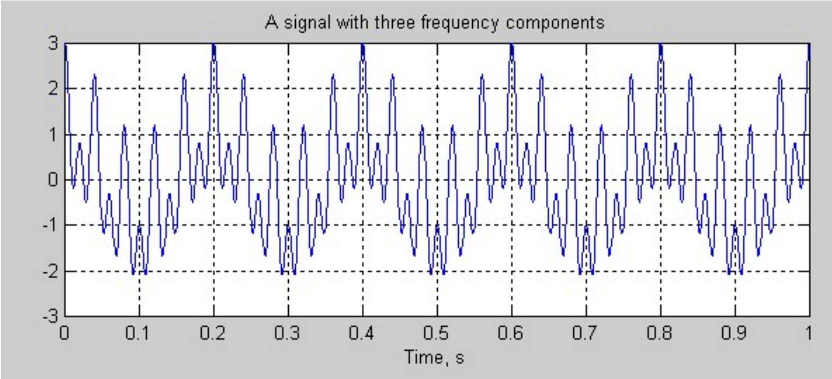
$$x_4(t) \xleftrightarrow{\mathcal{FT}} X_4(\omega)$$



Stationary and Non-stationary Signals

Stationary signals' spectral characteristics do not change with time

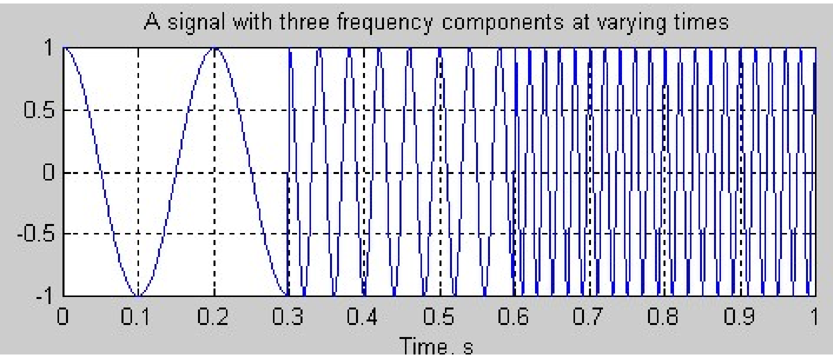
$$x_4(t) = \cos(2\pi \cdot 5 \cdot t) + \cos(2\pi \cdot 25 \cdot t) + \cos(2\pi \cdot 50 \cdot t)$$



Non-stationary signals have time varying spectra

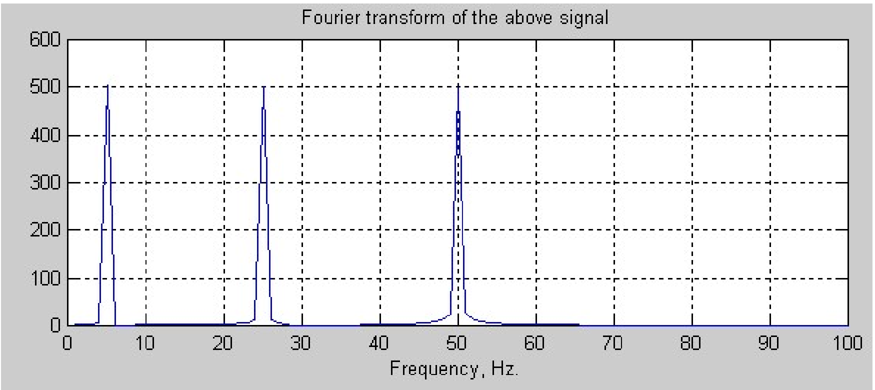
$$x_5(t) = [x_1 \oplus x_2 \oplus x_3]$$

$\oplus$ Concatenation

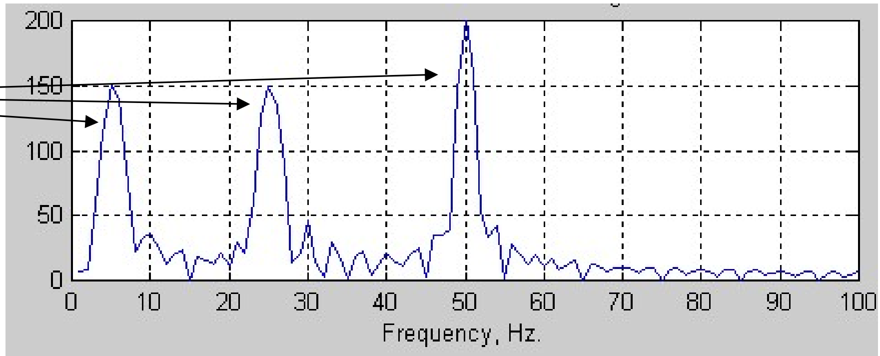


Stationary vs. Non-Stationary

$X_4(\omega)$



$X_5(\omega)$

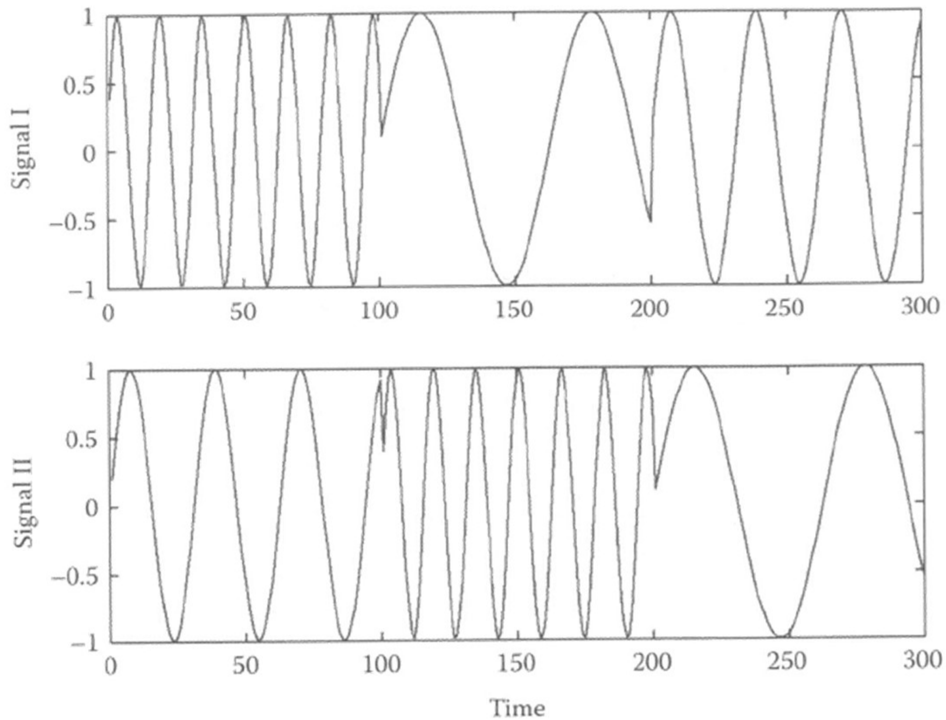


Perfect knowledge of what frequencies exist, but no information about where these frequencies are located in time

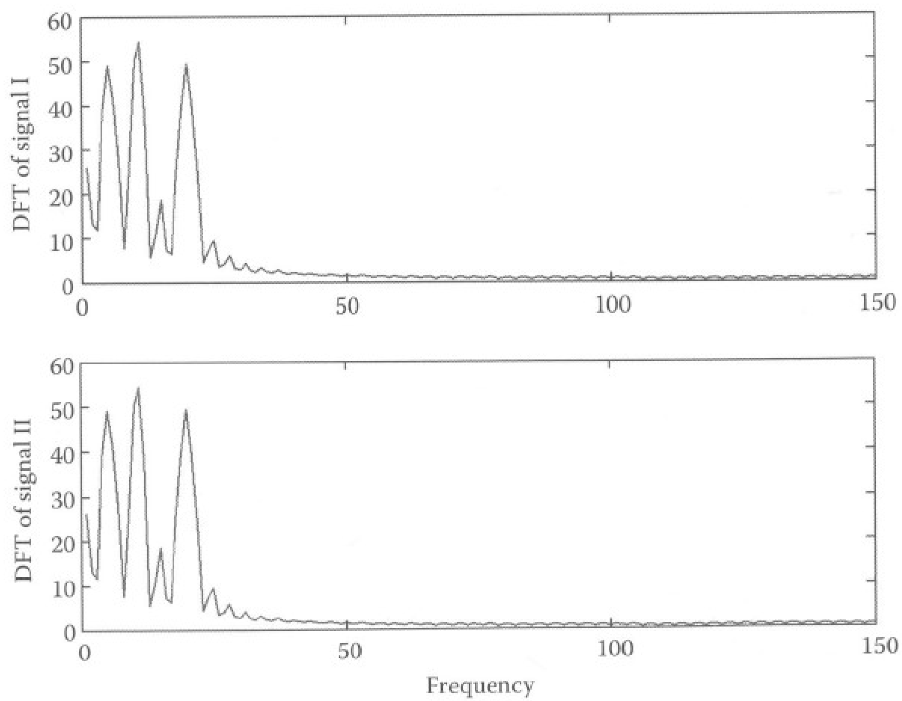
Shortcomings of the FT

- Sinusoids and exponentials
  - Stretch into infinity in time, → no time localization
  - Instantaneous in frequency, → perfect spectral localization
  - *Global* analysis does not allow analysis of non-stationary signals
- Need a *local* analysis scheme for a time-frequency representation (TFR) of nonstationary signals
  - Windowed F.T. (AKA Short Time FT, STFT) : Segmenting the signal into narrow time intervals, narrow enough to be considered stationary, and then take the FT of each segment, (Gabor 1946).
  - Followed by other TFRs, which differed from each other by the selection of the windowing function

Example: What will FTs look like?



Example: FTs are identical



Short Time Fourier Transform

define a new version of the FT in which the time localization

is preserved. This attempt leads us to the definition of a particular form of the FT called the short-time Fourier transform or STFT.

For a signal x(t), the STFT is defined as

$$X_{\mathrm{STFT}}(a, f) = \int_{-\infty}^{+\infty} x(t)g^*(t-a)e^{-j2\pi ft}dt$$

STFT

Time parameter

Frequency parameter

Signal to be analyzed

FT Kernel (basis function)

$$X_{STFT}(a, f) = \int_{-\infty}^{+\infty} x(t) g^*(t - a) e^{-j2\pi ft} dt$$

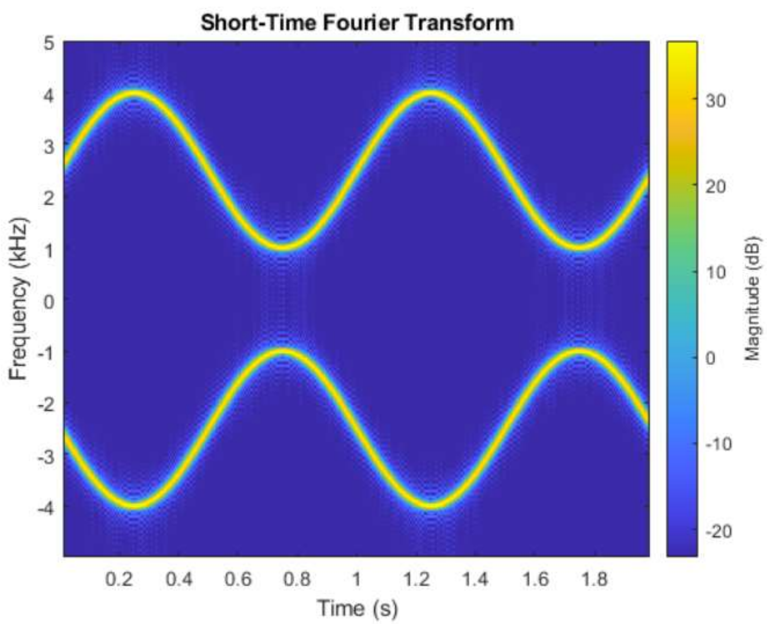
STFT of signal x(t): Computed for each window centered at t=a

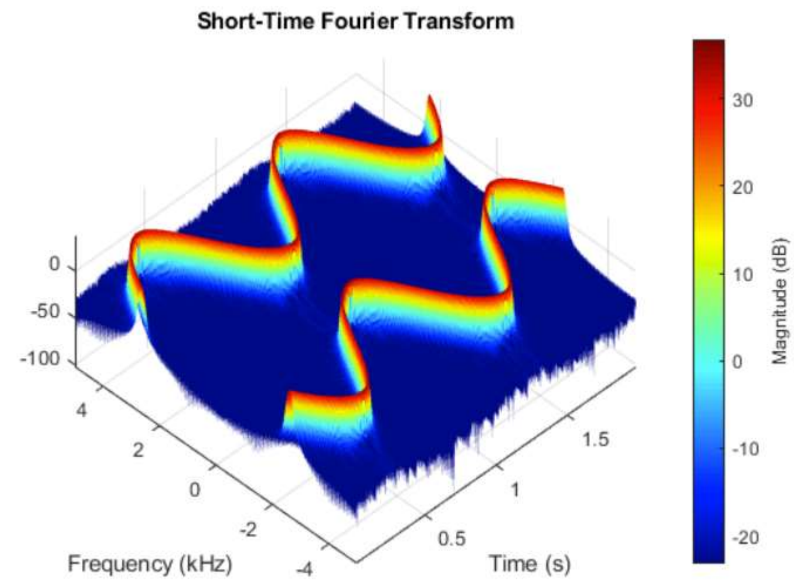
Windowing function

Windowing function centered at t=a

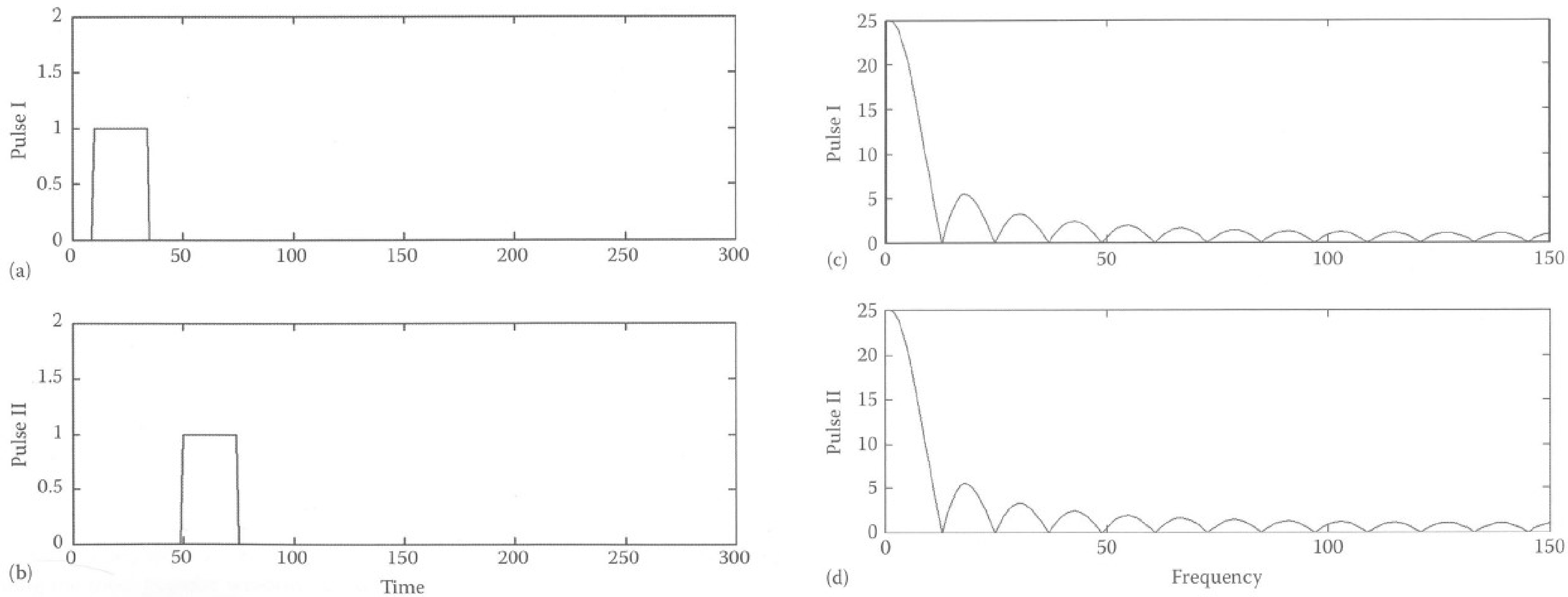
MATLAB

```
stft(x,fs,'Window',kaiser(256,5),'OverlapLength',220,'FFTLength',512);
```





Windowing



Pulse I
Pulse II
Time
Frequency
(a)
(b)
(c)
(d)
0
0.5
1
1.5
2
50
100
150
200
250
300
5
10
15
20
25

Windowing

$$X_{\mathrm{STFT}}(a, f) = \int_{-\infty}^{+\infty} x(t) g^*(t - a) e^{-j2\pi ft} dt$$

By definition, g(t - a) is a shifted version of a time window (gate) g(t) that extracts a portion of the signal x(t)

i.e. the “gate” g(t - a), having a limited time span, selects and extracts only a portion of the signal x(t) to be analyzed by the FT.

This time window is often a real-time function, and, therefore,

$$g^*(t - a) = g(t - a)$$

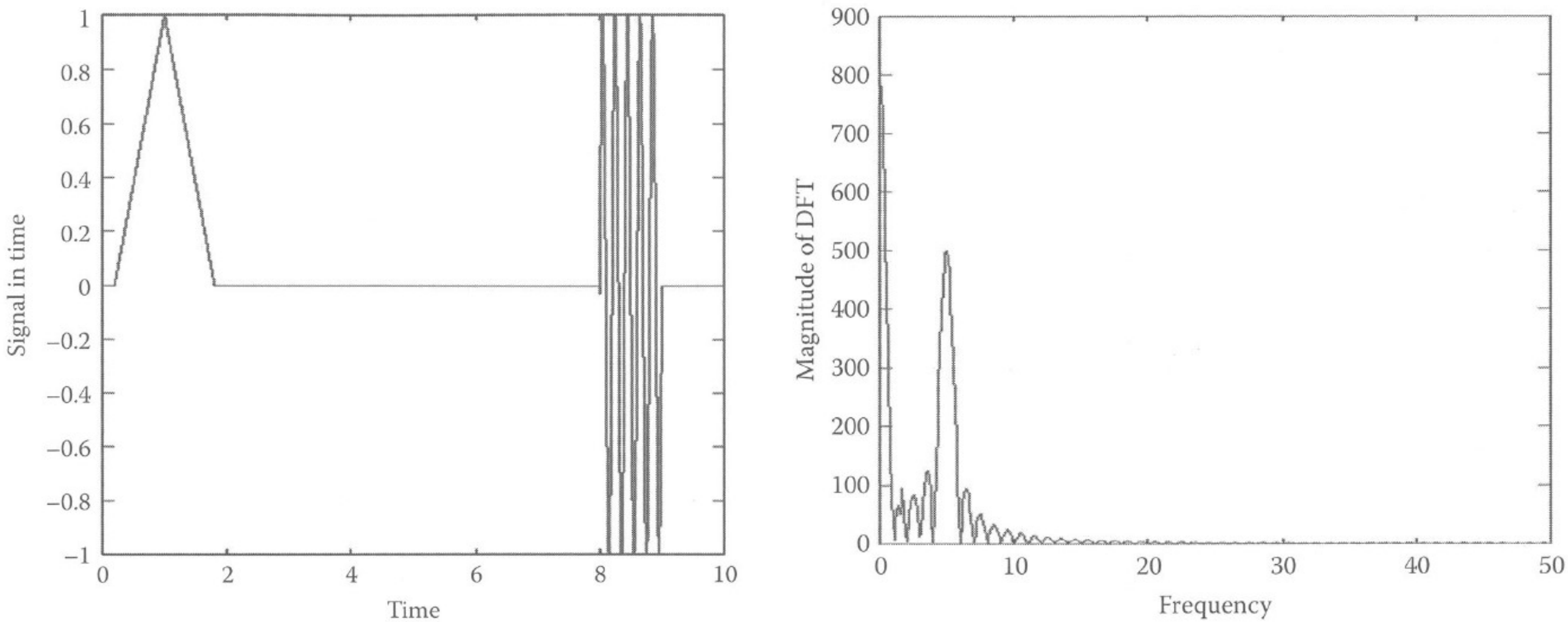
Windowing

So for the STFT, a time window selects a portion of $x(t)$ and then the regular old FT is calculated for this selected part of the signal.

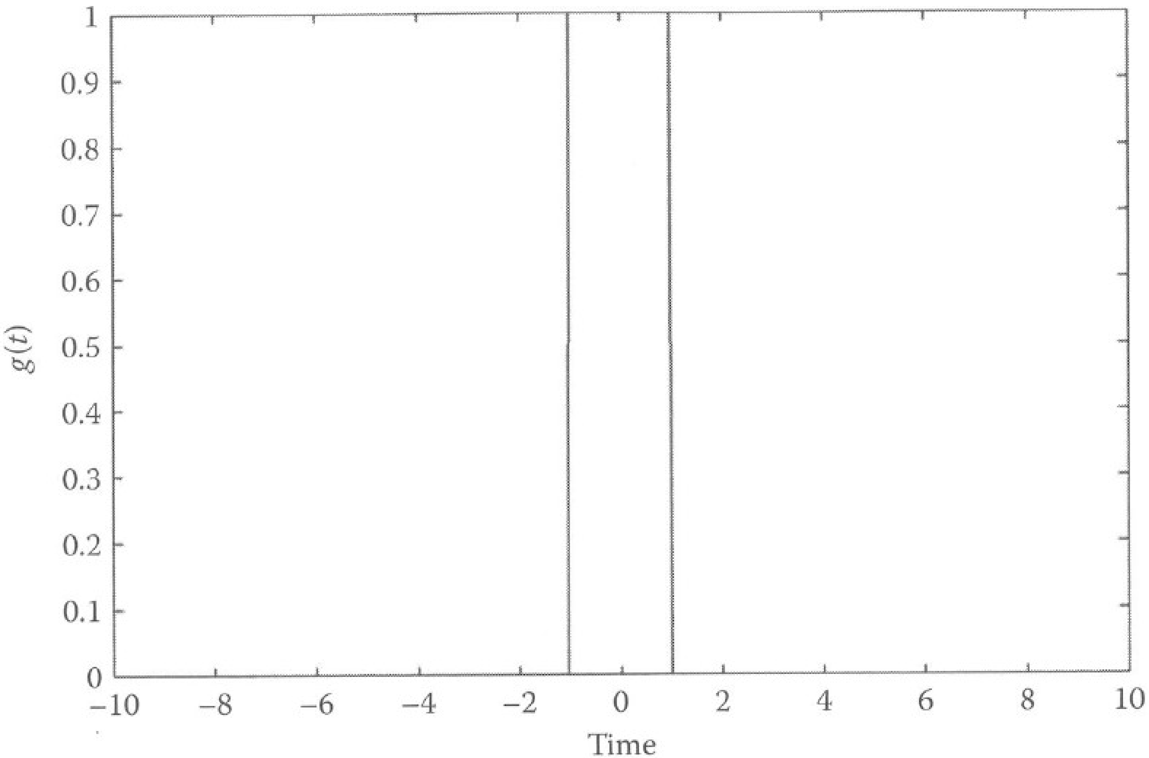
- By changing the amount of shift parameter a, one obtains not only the FT of every part of the signal, but also the time localization of each part as these portions are extracted at known time intervals identified by the shift factor a

STFT has two parameters, f and a. So, there is more computation (compared to FT)

STFT: Example



Window function to be used by the STFT



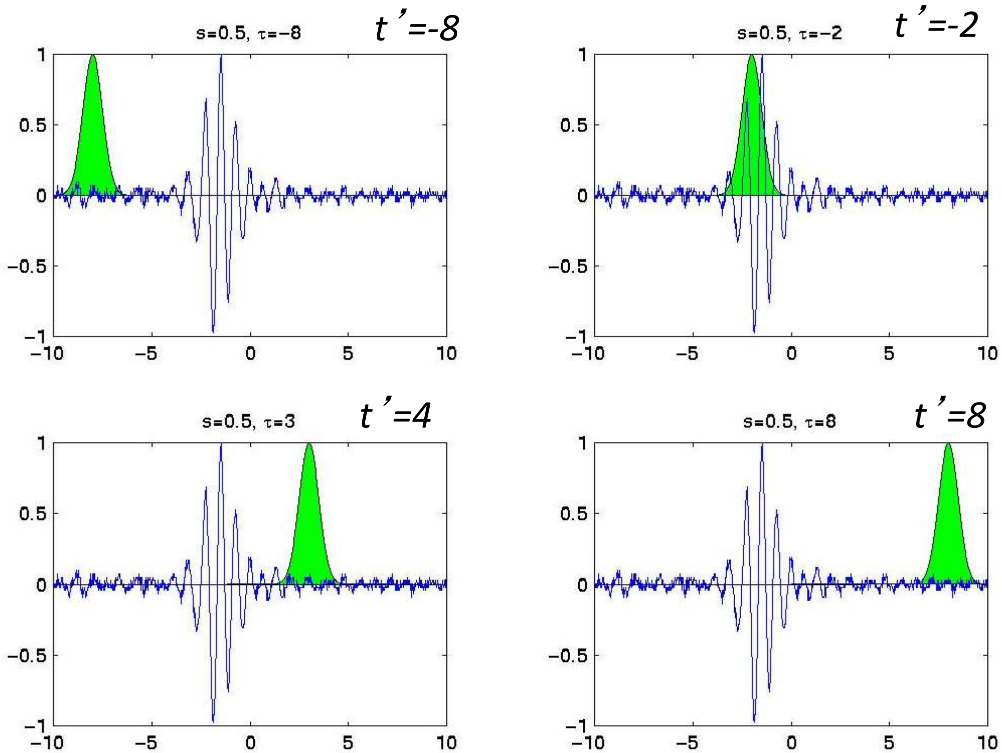
Short Time Fourier Transform (STFT)

1. Choose a window function of finite length
2. Place the window on top of the signal at t=0
3. Truncate the signal using this window
4. Compute the FT of the truncated signal, save.
5. Incrementally slide the window to the right
6. Go to step 3, until window reaches the end of the signal

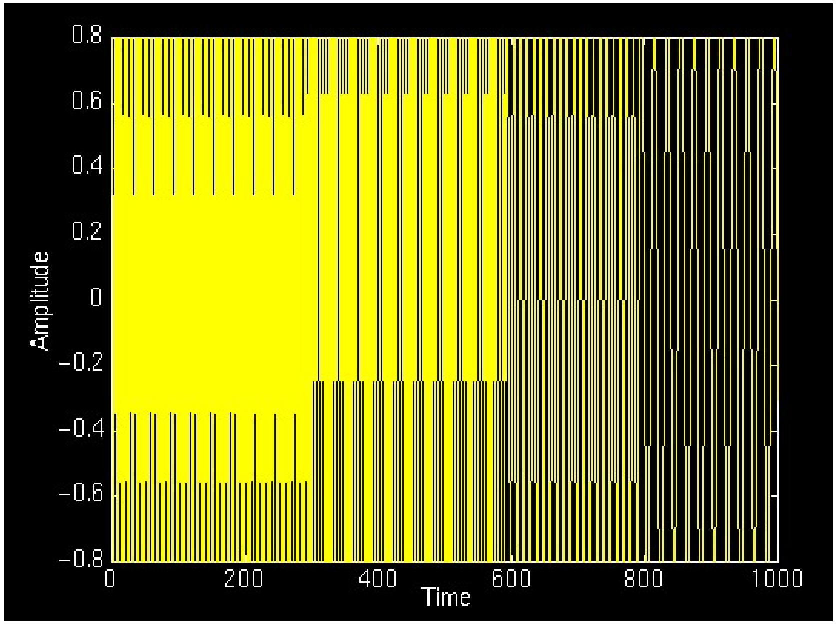
For each time location where the window is centered, we obtain a different FT

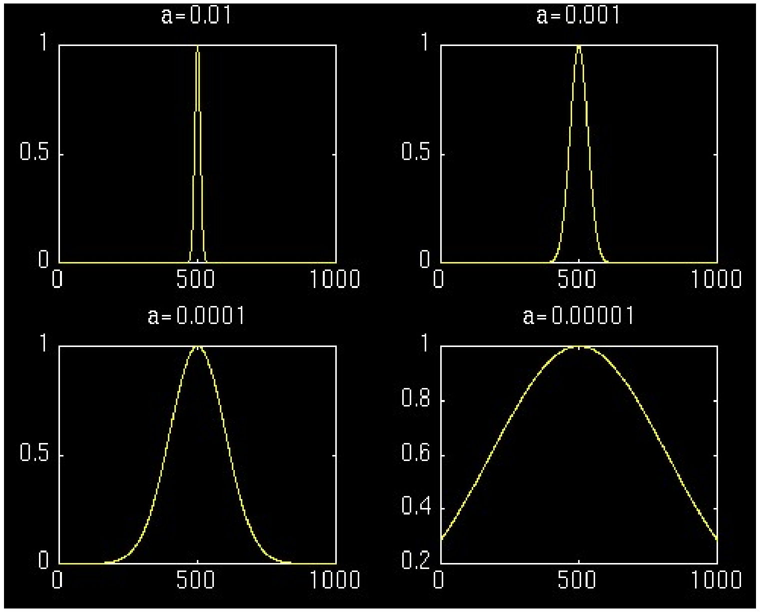
- Hence, each FT provides the spectral information of a separate time-slice of the signal, providing simultaneous time and frequency information

STFT

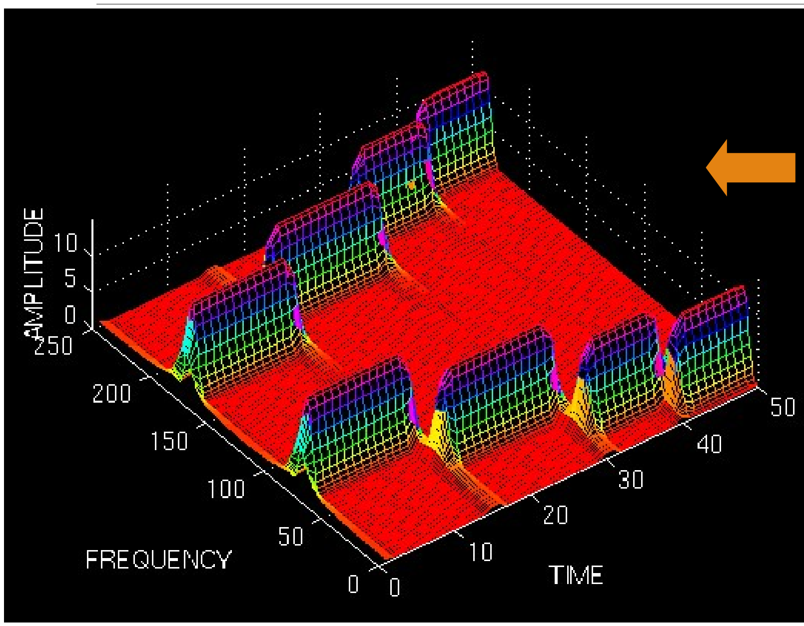


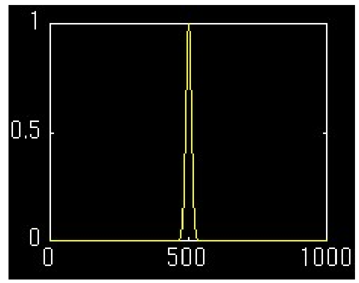
Different Sized Windows

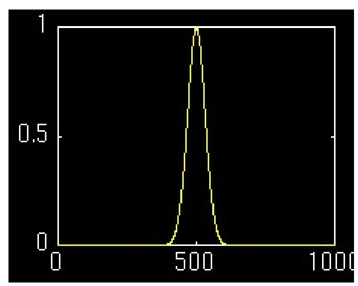


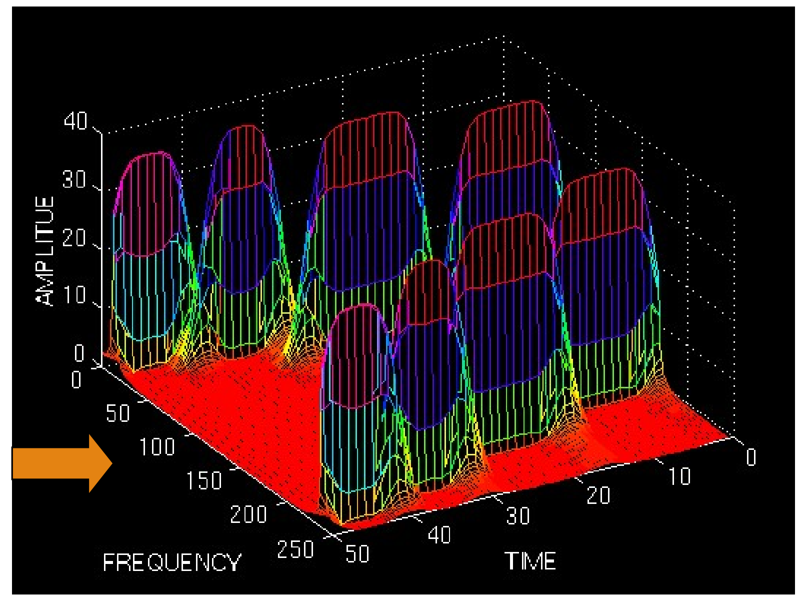


STFT At Work

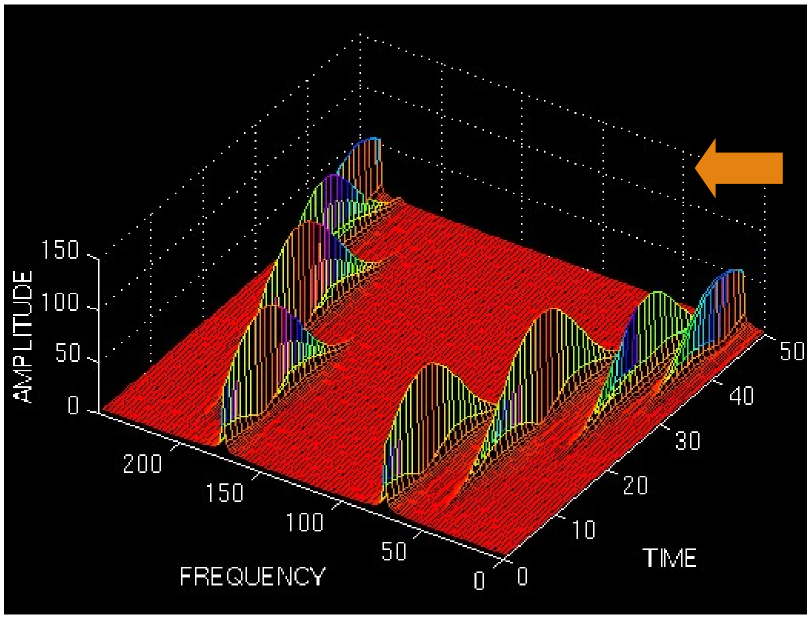

AMPLITUDE
10
5
0
250
200
150
100
50
0
FREQUENCY
0
10
20
30
40
50
TIME


1
0.5
0
0
500
1000


1
0.5
0
0
500
1000


40
30
20
10
0
AMPLITUE
0
50
100
150
200
250
FREQUENCY
50
40
30
20
10
0
TIME

STFT At Work


AMPLITUDE
150
100
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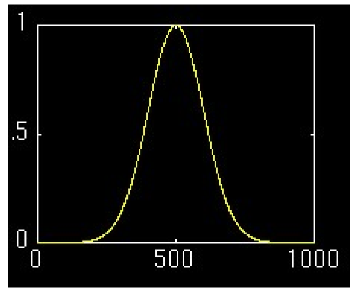
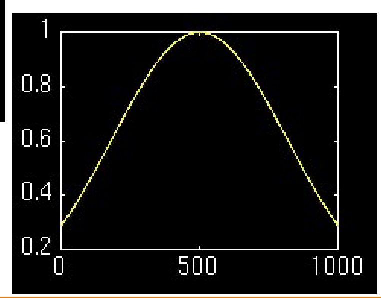
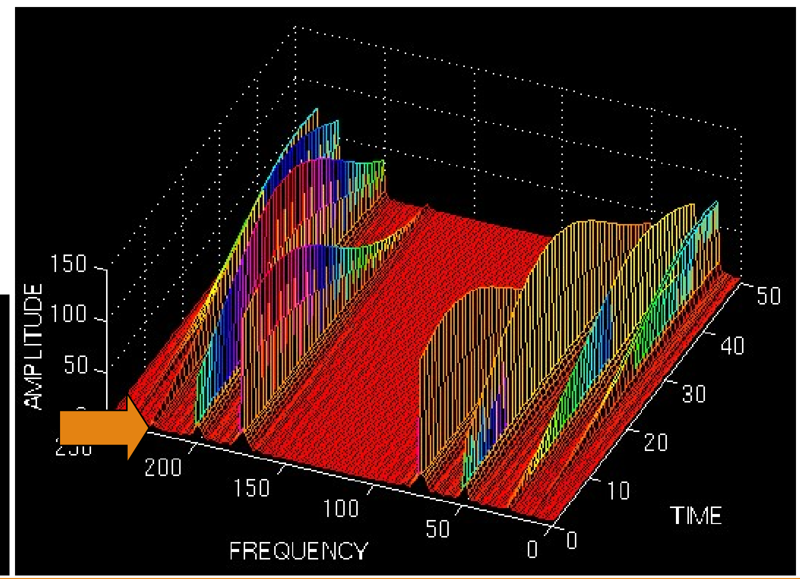
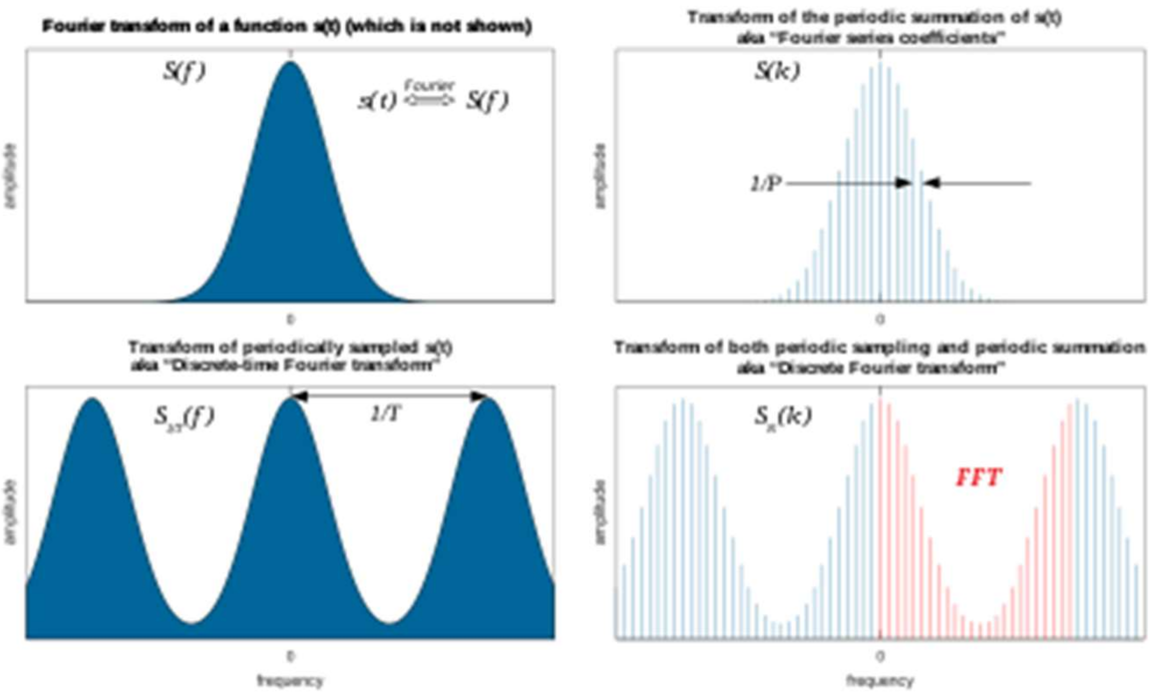
200
150
100
50
0
FREQUENCY
0
10
20
30
40
50
TIME


1
.5
0
0
500
1000


1
0.8
0.6
0.4
0.2
0
500
1000

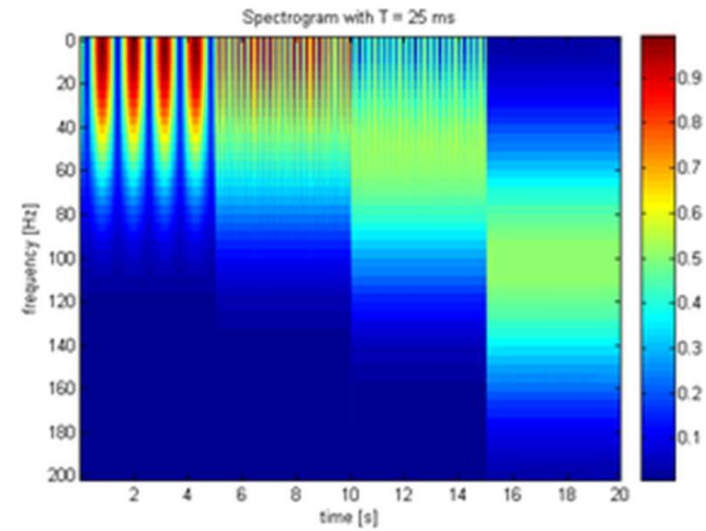

AMPLITUDE
150
100
50
200
150
100
50
0
FREQUENCY
0
10
20
30
40
50
TIME

Continuous vs Discrete



$$\mathbf{STFT}\{x[n]\}(m,\omega) \equiv X(m,\omega) = \sum_{n=-\infty}^{\infty} x[n]w[n-m]e^{-j\omega m}$$

$$\mathbf{STFT}\{x(t)\}(\tau,\omega) \equiv X(\tau,\omega) = \int_{-\infty}^{\infty} x(t)w(t-\tau)e^{-i\omega t}\,dt$$



STFT

STFT provides the time information by computing a different FTs for consecutive time intervals, and then putting them together

- Time-Frequency Representation (TFR)
- Maps 1-D time domain signals to 2-D time-frequency signals

Consecutive time intervals of the signal are obtained by truncating the signal using a sliding windowing function

How to choose the windowing function?

- What shape? Rectangular, Gaussian, Elliptic…?
- How wide?
  - Wider window require less time steps → low time resolution
  - Also, window should be narrow enough to make sure that the portion of the signal falling within the window is stationary
  - Can we choose an arbitrarily narrow window…?

Selection of STFT Window

Two extreme cases:

<u>*W(t)* infinitely long</u>: $W(t) = 1$

➔ STFT turns into FT, providing excellent frequency information (good frequency resolution), but no time info.

Wide analysis window➔ poor time resolution, good frequency resolution

$$X_{\mathrm{STFT}}(a, f) = \int_{-\infty}^{+\infty} x(t) g^*(t-a) e^{-j2\pi ft} dt \qquad XSTFT(a, f) = \int_t \left[x(t) \cdot \delta(t-a) \right] \cdot e^{-j\omega t} \, dt = x(a) \cdot e^{-j\omega a}$$

Selection of STFT Window

<u>*W(t)* infinitely short</u>: $W(t) = \delta(t)$

➔ STFT then gives the time signal back, with a phase factor. Excellent time information (good time resolution), but no frequency information

Narrow analysis window → good time resolution, poor frequency resolution

Once the window is chosen, the resolution is set for both time and frequency.

Heisenberg Principle

$$\Delta t \cdot \Delta f \geq \frac{1}{4\pi}$$

TIME RESOLUTION:

How well two spikes in time can be separated from each other in the transform domain

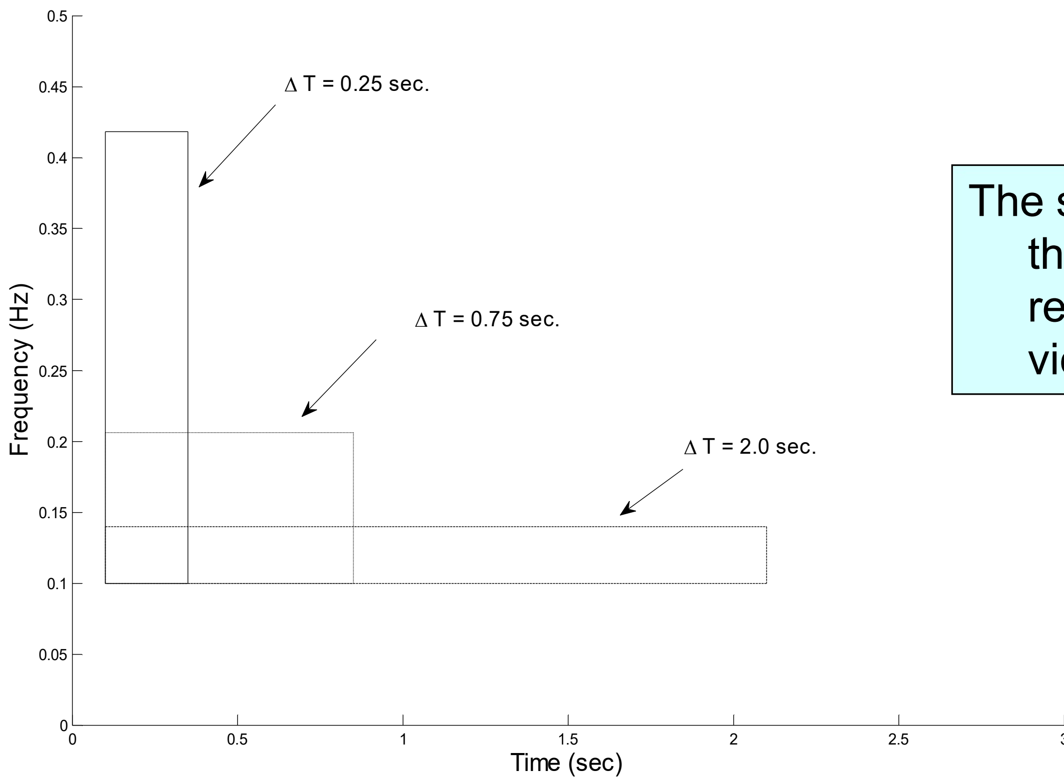
FREQUENCY RESOLUTION:

How well two spectral components can be separated from each other in the transform domain

Both time and frequency resolutions cannot be arbitrarily high!!!

We cannot precisely know at what time instance a frequency component is located. We can only know what *interval of frequencies* are present in which *time intervals*

Time Frequency Tradeoff



The smallest time resolution gives the poorest frequency resolution (black rectangle) and vice versa (dashed rectangle).

STFT Conclusion

Despite the favorable properties of the STFT, this transform is not the best solution to address the time and frequency localization of signal events.

shortcomings of the STFT,

- choice of the window length:
  - A too short window may not capture the entire duration of an event
  - too long window may capture two or more events in the same shift.
- nature of the basis functions used in the FT, i.e., complex exponentials.
  - The term e-i2πft describes sinusoidal variations in real and complex spaces. Such functions are not limited in time span or duration

WHAT TO DO NOW?

The Wavelet Transform

Overcomes the preset resolution problem of the STFT by using a variable length window

Analysis windows of different lengths are used for different frequencies:

- Analysis of high frequencies➔ Use narrower windows for better time resolution
- Analysis of low frequencies ➔ Use wider windows for better frequency resolution

This works well, if the signal to be analyzed mainly consists of slowly varying characteristics with occasional short high frequency bursts.

Heisenberg principle still holds!!!

The function used to window the signal is called *the wavelet*

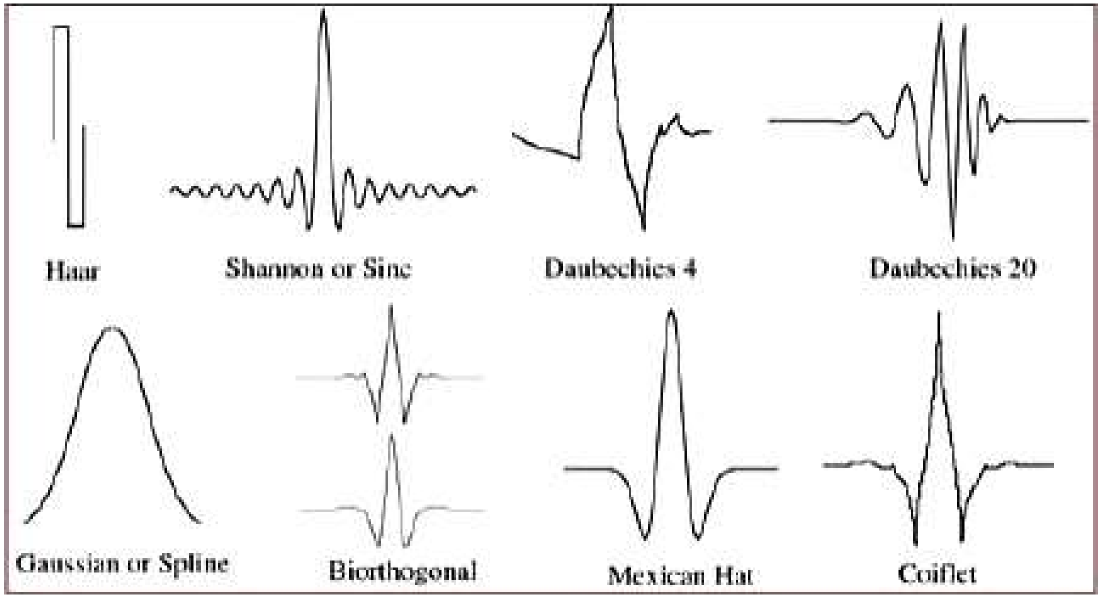
Wavelets

Wave-like function

Amplitude starts and ends at 0

Crafted with specific functions in mind

Can be used for 1d or 2d signals
- i.e. audio signals, eeg signals etc
- Images



Mother and Father's of Wavelets

- Haar – started the train of thought
- Gabor – Gabor atoms
- Zweig – CWT
- Gopillard, Grossman, Morlet – Refined Zweigs work
- Jan-Olov Strömberg – DWT
- Ingrid Daubechies – orthogonal wavelet
- JPEG 2000 – image compression

Wavelet “Frequency”

Before introducing the Wavelet Transform (WT), let’s take a closer look at the basic definition of "frequency”:

- the fundamental concept of the Fourier Transform (FT)
- uses periodic time-unlimited functions
- based on what we have seen we need to use time-limited basis functions for our new (i.e. WT) transform

- not using periodic sinusoidal basis functions for the new transform, we need to think of a concept that replaces frequency

- Time-limited basis functions are obviously not periodic, and therefore, we need to invent a new concept that can represent a concept similar to frequency

Wavelet “Frequency”

- Consider a sinusoidal basis function with

frequency = 0.1 Hz.

- another basis function in the Fourier decomposition of this signal would be the 2nd harmonic (i.e., a sinusoidal basis function with frequency = 0.2 Hz).

- harmonic relations among the basis signals is the fundamental concept of signal transformation and decomposition

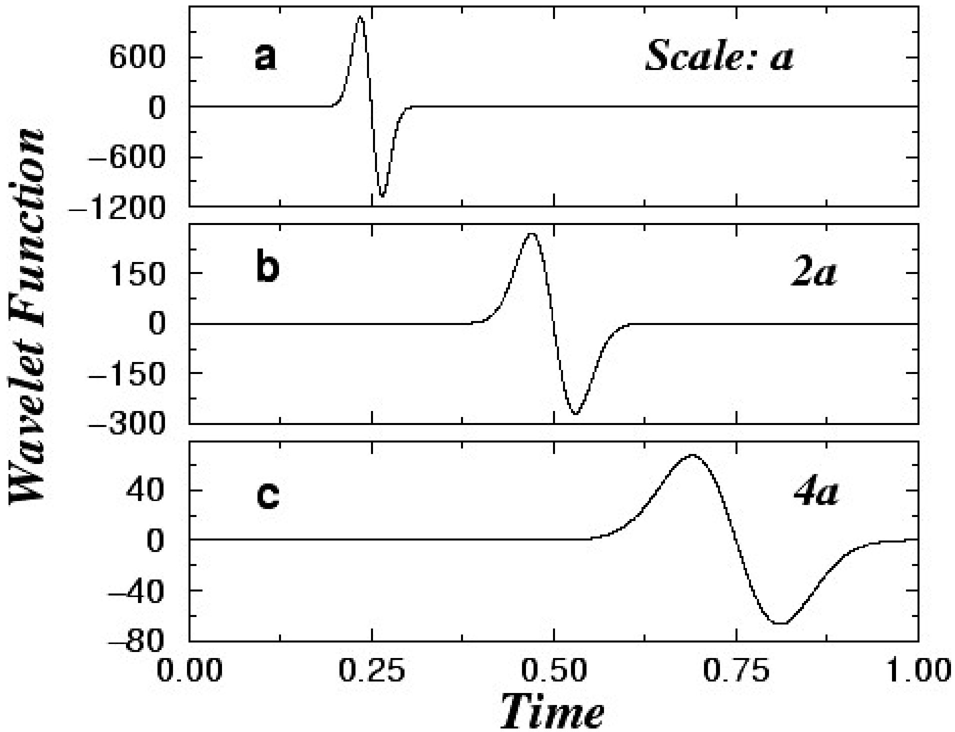
Therefore, the relation among harmonics is something that can somehow represent a new concept that will replace frequency.

Wavelet "Frequency"

- make the following important observation about harmonics:
  - by warping the time axis "t" one can obtain the harmonics from the original signal
  - E.g. By replacing the time axis "t" in the original signal with "2t" time axis results in the second harmonic.

- This is essentially "scaling" the signal in time to generate other basis functions.

- main characteristic of harmonic frequencies can be drawn from a more general concept: "scale"



Wavelet Scale

- unlike frequency (defined only for periodic signals) scale is equally applicable to nonperiodic signals.

- Using scale as a variable, the new transform, which will be based on time-limited basis function, can be meaningfully applied to both time-unlimited and time-limited signals.

Continuous Wavelet Transform

$$W_{\Psi,X}(a,b) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{+\infty} x(t)\Psi*\left(\frac{t-b}{a}\right)dt, \qquad a \neq 0$$

Ψ(t) is a function with limited duration in time

b is a shifting parameter. Translates the function across x(t) (like the t in the STFT) a is a time scaling parameter (replaces frequency parameter f)

So, the basis functions of the CWT are the shifted and scaled version of the probing function Ψ(t) (mother wavelet)

The * indicates complex conjugation and dividing by $\sqrt{|a|}$ normalizes energy.

Continuous Wavelet Transform

Wavelets are a class of basis functions that incorporate two parameters:

1. *translation* in time
2. *scaling* in time

main point is to accomodate temporal information

(e.g. crucial in evoked responses, aka event related potential (ERP) analysis)

Another definition:

A wavelet is an oscillating function whose energy is concentrated in time to better represent transient and nonstationary signals.

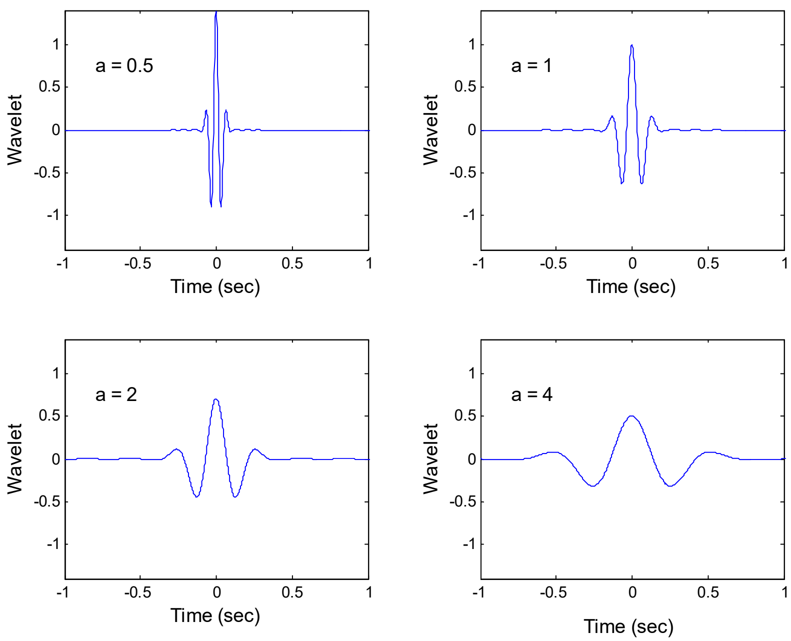
Wavelet Families - Morlet

The wavelet shown is the ‘Morlet Wavelet’ described by the equation:

$$\psi(t) = e^{-t^2} \cos\left(\pi \sqrt{\frac{2}{\ln 2}} t\right)$$

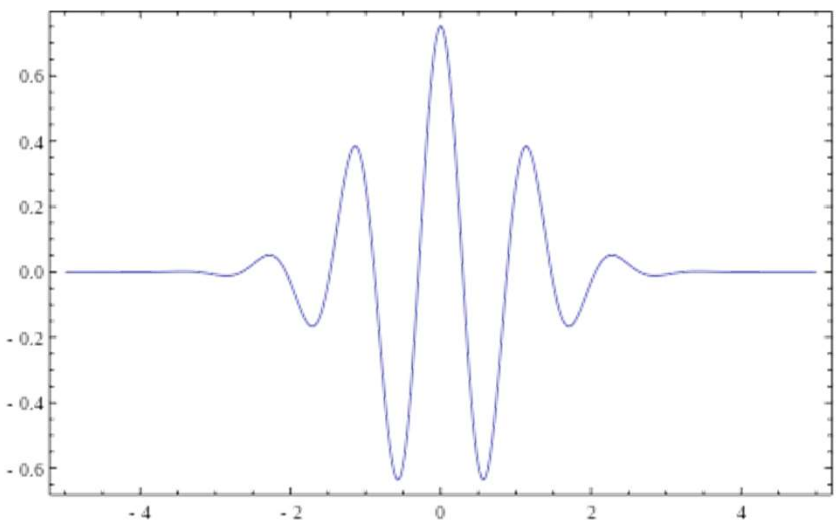
If b = 0, and a = 1, then the wavelet is in its basic form, the "mother wavelet."

If a > 1, the wavelet is stretched along the time axis, and if a < 1, the wavelet is contracted.

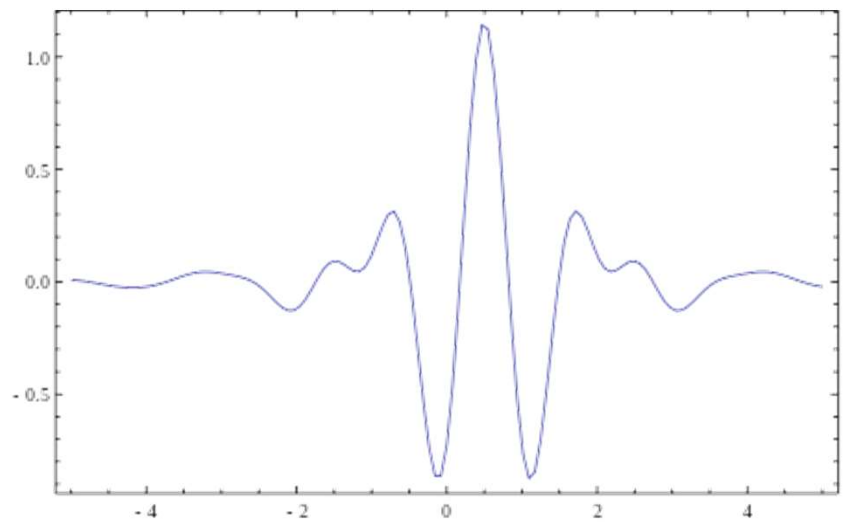


Wavelet Families - Meyer

Morlet Wavelet (real valued)



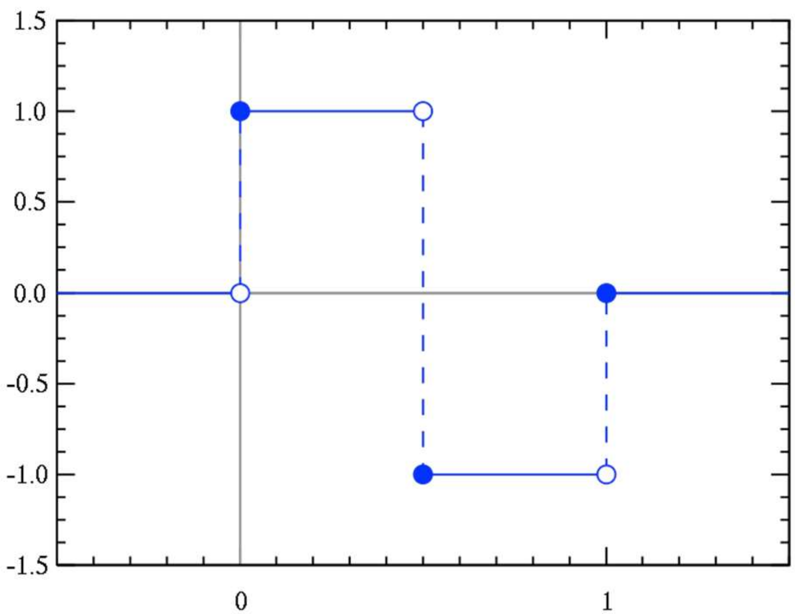
Meyer Wavelet



http://en.wikipedia.org/wiki/Wavelet

http://en.wikipedia.org/wiki/Haar_wavelet

Wavelet Families - Haar



Haar wavelet mother wavelet function:

$$\Psi(t) = \begin{cases} 1 & 0 \leq t < 0.5 \\ -1 & 0.5 \leq t < 1 \\ 0 & otherwise \end{cases}$$

Haar wavelet scaling function:

$$\phi(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & otherwise \end{cases}$$

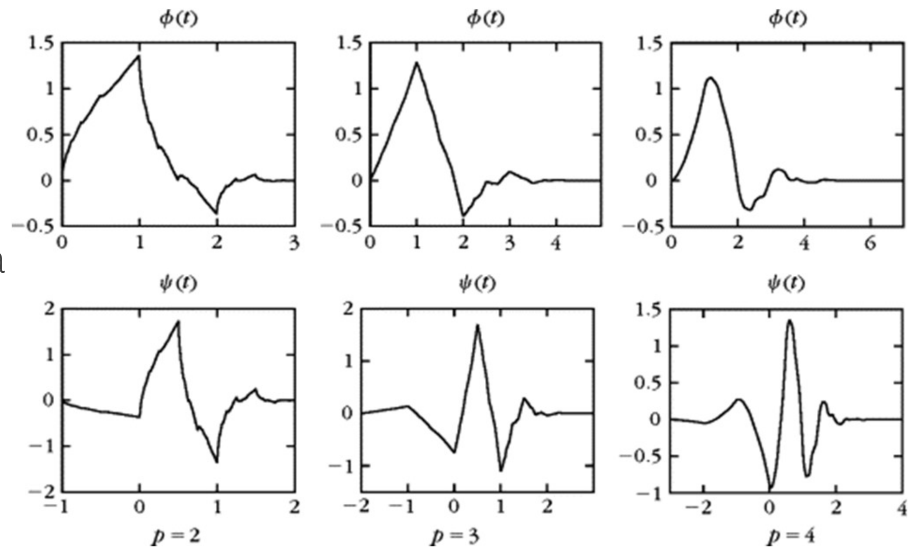
Wavelet Families - Daubechies

- Daubechies (dbX) wavelets are among the most popular mother wavelets that are commonly used in signal and image processing.

- The index "X" in dbX identifies the exact formulation of the function

- all dbX functions look more or less similar, db2 is a simpler mother wavelet than db3, db4, etc.

As a rule of thumb, more complex mother wavelets may be needed to analyze more complex signals.

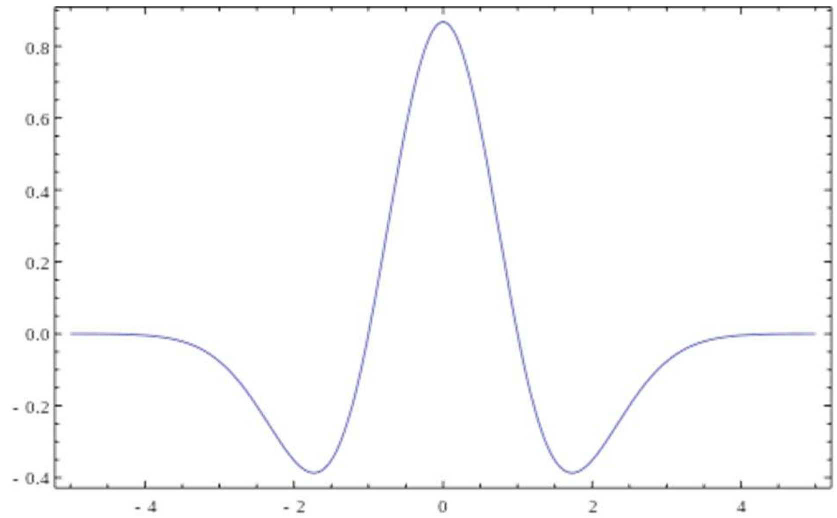


NOTES

- every choice of mother wavelet gives a particular CWT

- Any choice of mother wavelet gives certain unique properties that make the resulting transformation a suitable choice for a particular task. E.g. "Mexican Hat",

Infinite number of transformations

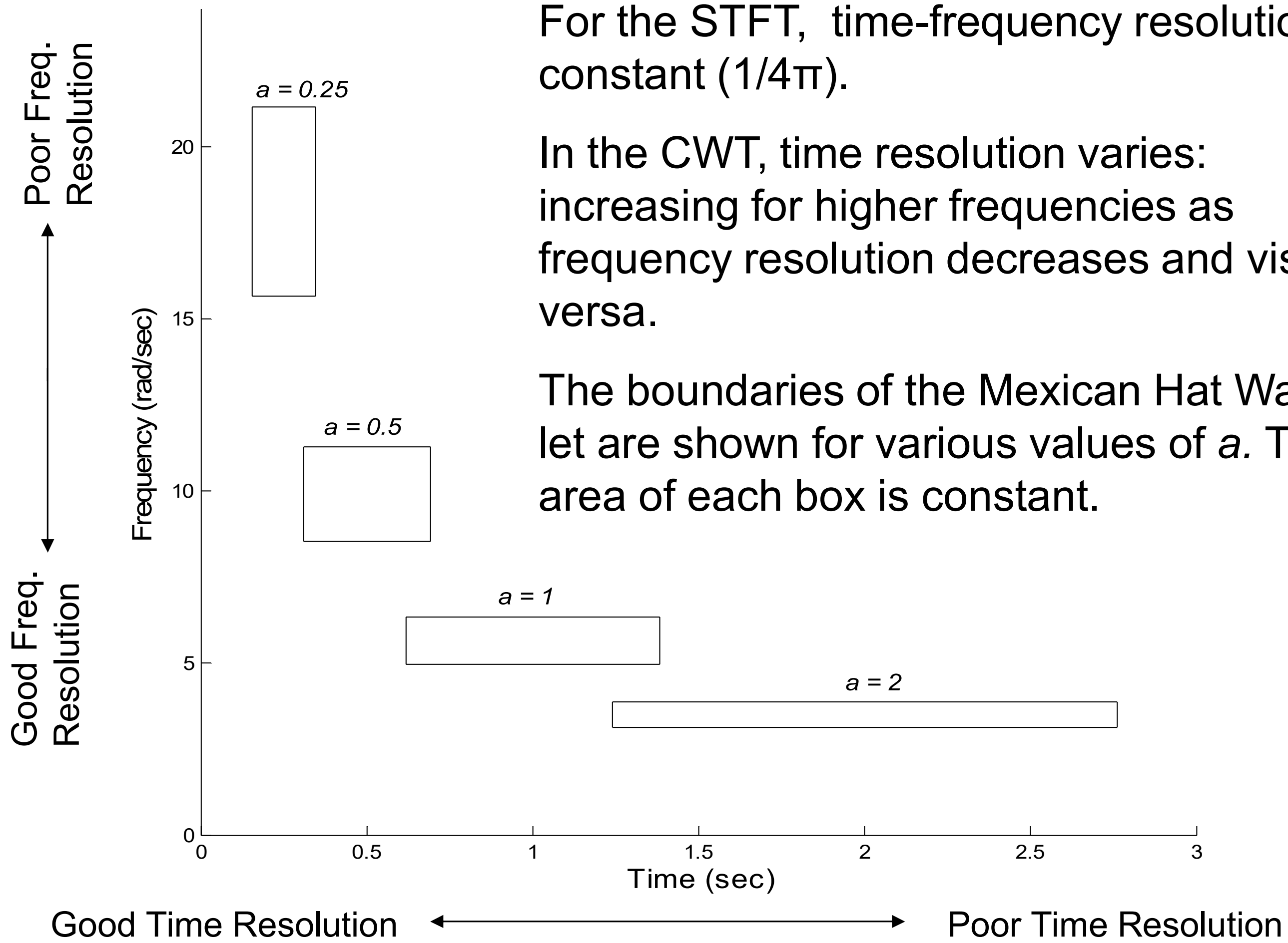


Wavelets: Time-Frequency Trade-off

Built in tradeoffbetween time and frequency:

$$\Delta\omega\psi(a)\ \Delta t\psi(a) = \Delta\omega\psi\ \Delta t\psi = \text{constant} \geq 0.5$$

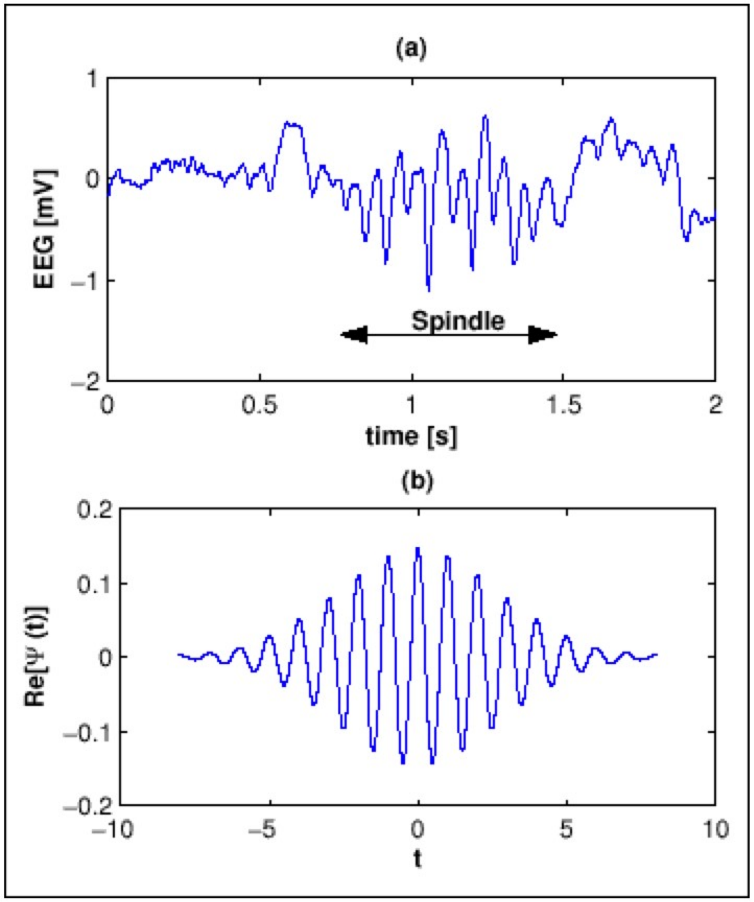
This trade-off is illustrated in the next slide which shows the time-frequency boundaries for the Mexican hat wavelet for various values of a.



For the STFT, time-frequency resolution is constant (1/4π).

In the CWT, time resolution varies: increasing for higher frequencies as frequency resolution decreases and vise versa.

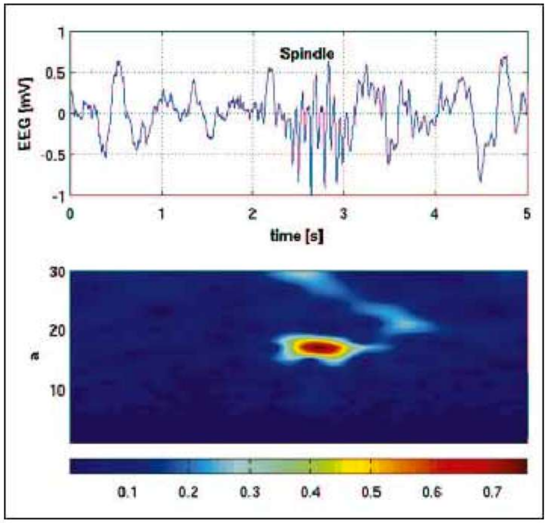
The boundaries of the Mexican Hat Wavelet are shown for various values of *a.* The area of each box is constant.



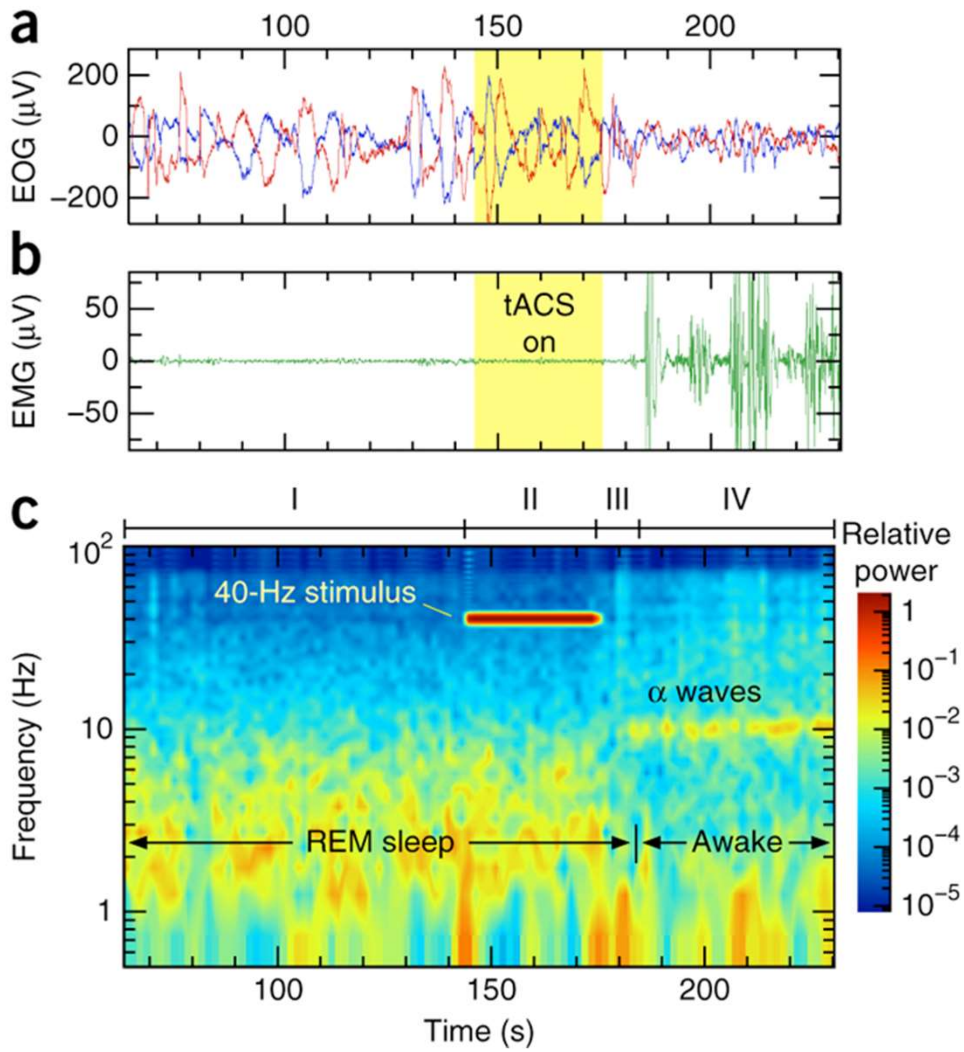
M. LATKA[1], A. KOZIK [2], J. JERNAJCZYK[1], B.J.WEST[3], W. JERNAJCZYK[4]

WAVELET MAPPING OF SLEEP SPINDLES IN YOUNG PATIENTS WITH EPILEPSY

[1]Institute of Physics, Wroclaw University of Technology, Wroclaw, Poland;
[2]Video EEG Laboratory, Department of Child Neurology, Marciniak Regional Medical Center, Wroclaw, Poland
[3]Mathematical & Information Science Directorate, Army Research Office, Research Triangle Park, NC, USA;
[4]Sleep Disorders Center, Department of Clinical Neurophysiology, Institute of Psychiatry and Neurology, Warsaw, Poland

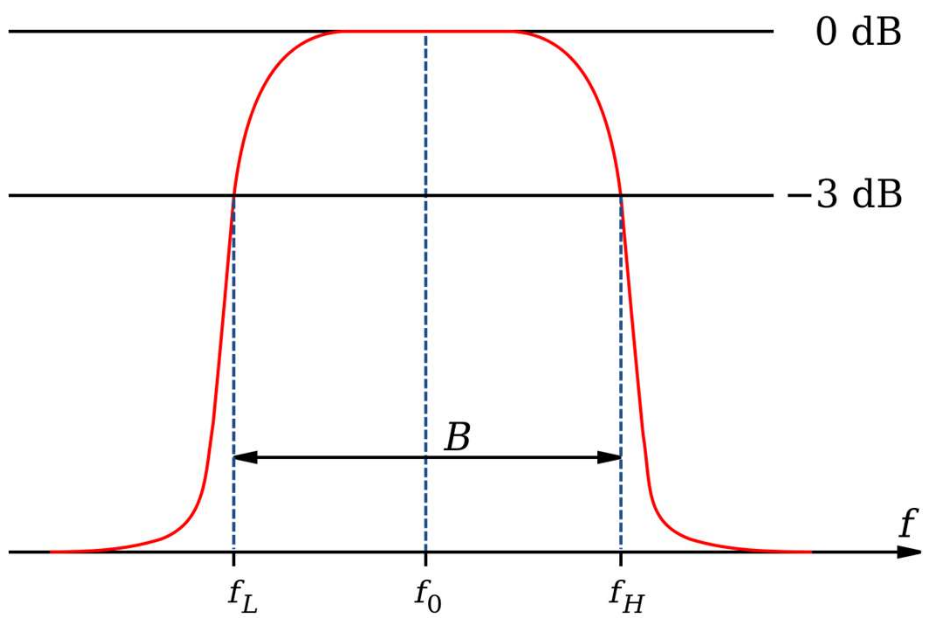


Morlet wavelet



(a) 2-channel EOG showing classical contralateral eye movements typical for REM sleep. Eye movements were synchronous before (phase I), during (phase II) and after stimulation (phase III), and only changed after awakening (phase IV).
(b) EMG activity unchanged until the subject awakened (phase IV), at which time it strongly increased signaling a loss of REM sleep atonia.
(c) Continuous wavelet transform of the recorded EEG signal at Fpz using the complex Morlet wavelet

Voss et al. (2014) Induction of self awareness in dreams through frontal low current stimulation of gamma activity. Nature Neuroscience. 17:810–812.



Other ways to view the CWT

The CWT can be interpreted as a *linear filtering* operation

- convolution between the signal x(t) and a filter with impulse response ψ *(-t/s)*

The CWT can be viewed as a type of bandpass analysis:

- where the scaling parameter *(s)* modifies *the center frequency* and the *bandwidth of a bandpass filter*

Wavelet rules

- By definition the mother wavelet must be limited in duration and looks like a decaying small wave.

- All other basis functions are the shifted and scaled version of the mother wavelet.

- In contrast to the STFT, instead of using sinusoidal functions in order to generate the basis functions, a mother wavelet is continuously shifted and scaled to create all basis functions in CWT.

ICWT

How to get back to our time domain signal?

→ The inverse continuous wavelet transform or ICWT:

$$x(t) = \frac{C_{\Psi}^{-1}}{a^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W_{\Psi,X}(a,b)\, \Psi\left(\frac{t-b}{a}\right) da\, db, \qquad a \neq 0$$

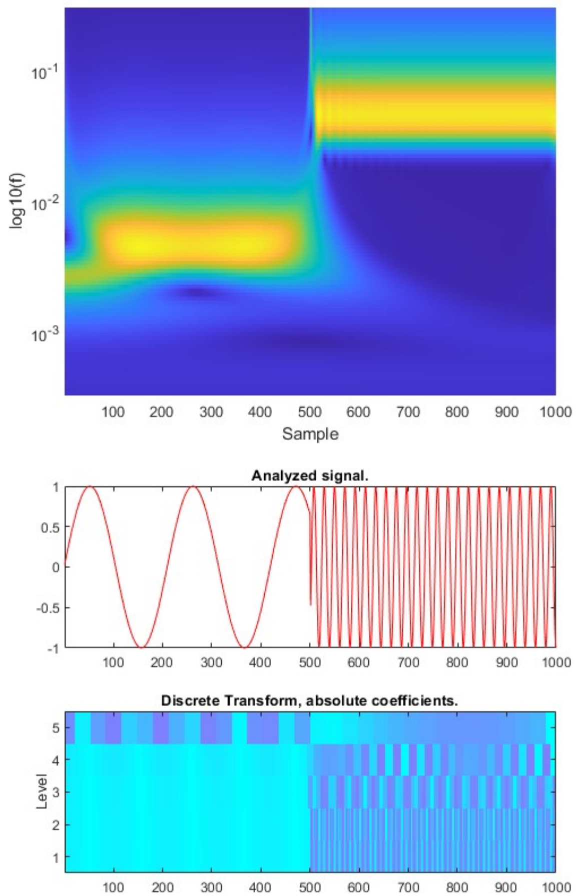
Here C-1Ψ is a constant depending on choice of mother wavelet Ψ(t).

How to choose a mother wavelet?

-an open problem, without a definite answer.

However, 2 "intuitive" rules of thumb are widely followed:

- (1) complex mother wavelets are needed for complex signals
- (2) the mother wavelet that resembles the general shape of the signal to be analyzed would be the more suitable choice



Discrete Wavelet Transform

- concerns are the same as applicable to the CFT

- CFT (and ICFT) are computationally more expensive

- The DWT applies only discrete shifts and scales on signals

Discrete Wavelet Transform

The CWT is highly redundant:

- many more coefficients are generated than needed uniquely to specify the signal.

If the application calls for recovery of the original signal, all of the coefficients will be required and the computational effort could be excessive.

The “Discrete Wavelet Transform” (DWT):

a) restricts the variation in translation and scale, usually to powers of 2;

b) downsamples lower scaled data.

Discrete Wavelet Transform

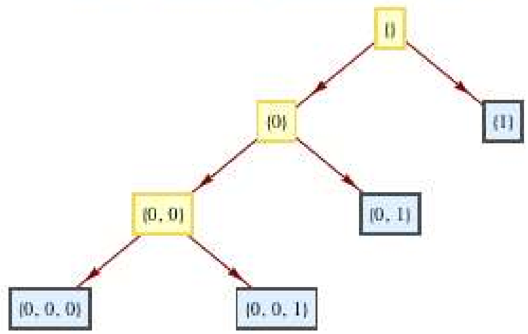
In the DWT, a new concept is introduced termed the "scaling function," a function that facilitates computation of the DWT.

To implement the DWT efficiently

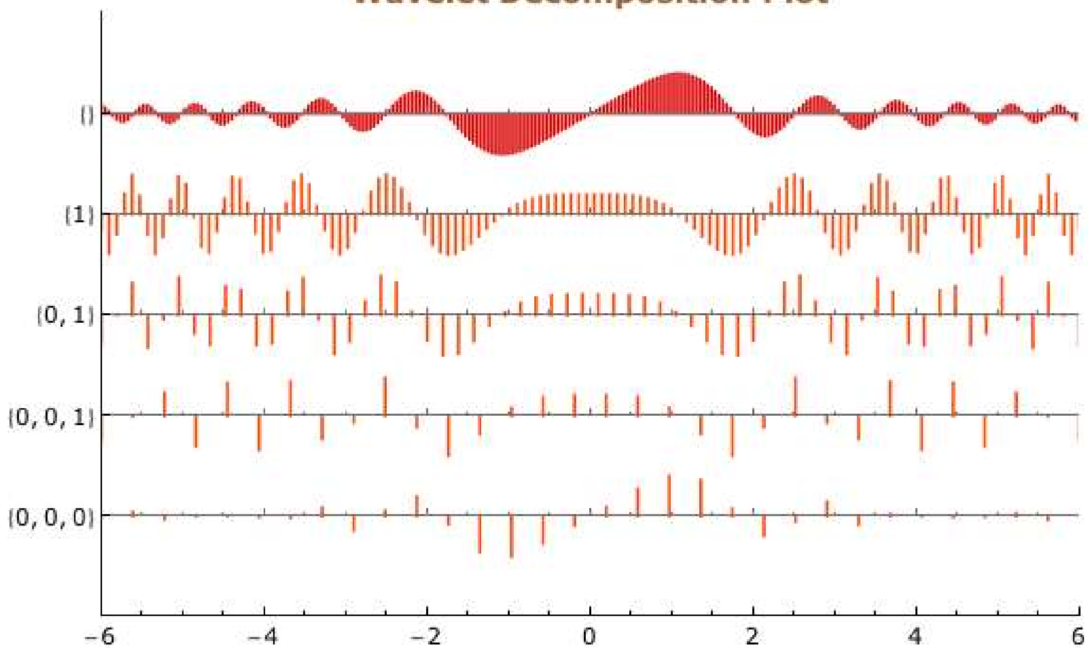
- the finest resolution is computed first.
- The computation then proceeds to courser resolutions,
- but rather than start over on the original waveform, the computation uses a smoothed version of the fine resolution waveform.

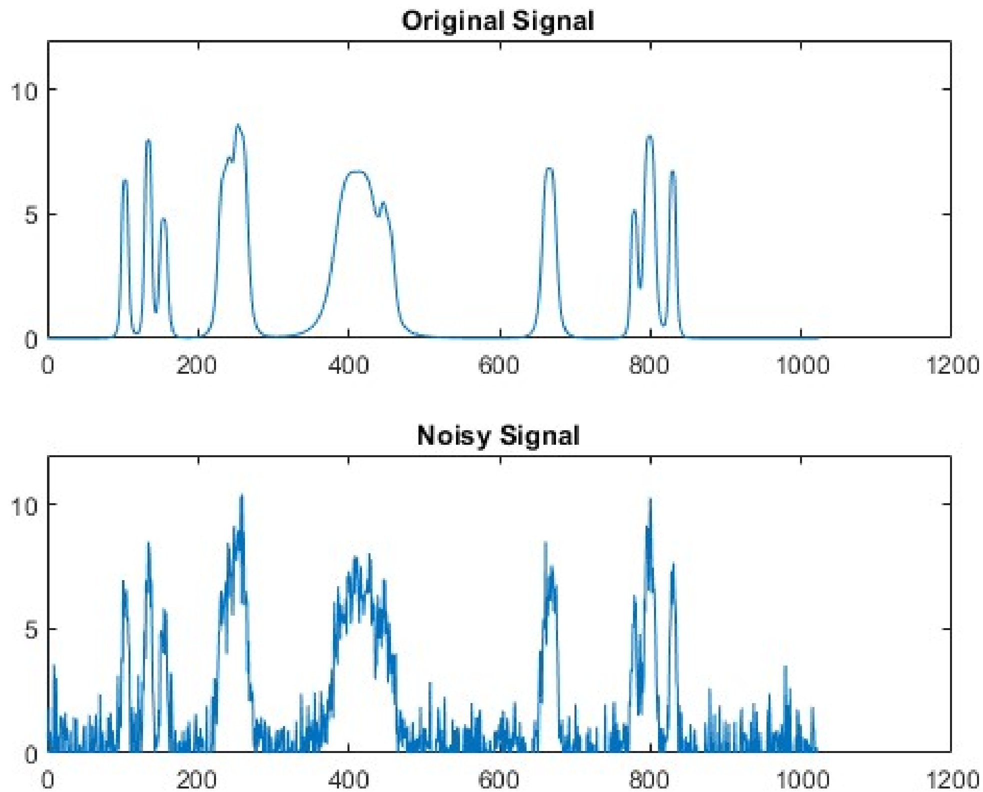
This smoothed version is obtained using a scaling function

LWT Decomposition Tree



Wavelet Decomposition Plot





Wavelet Denoising

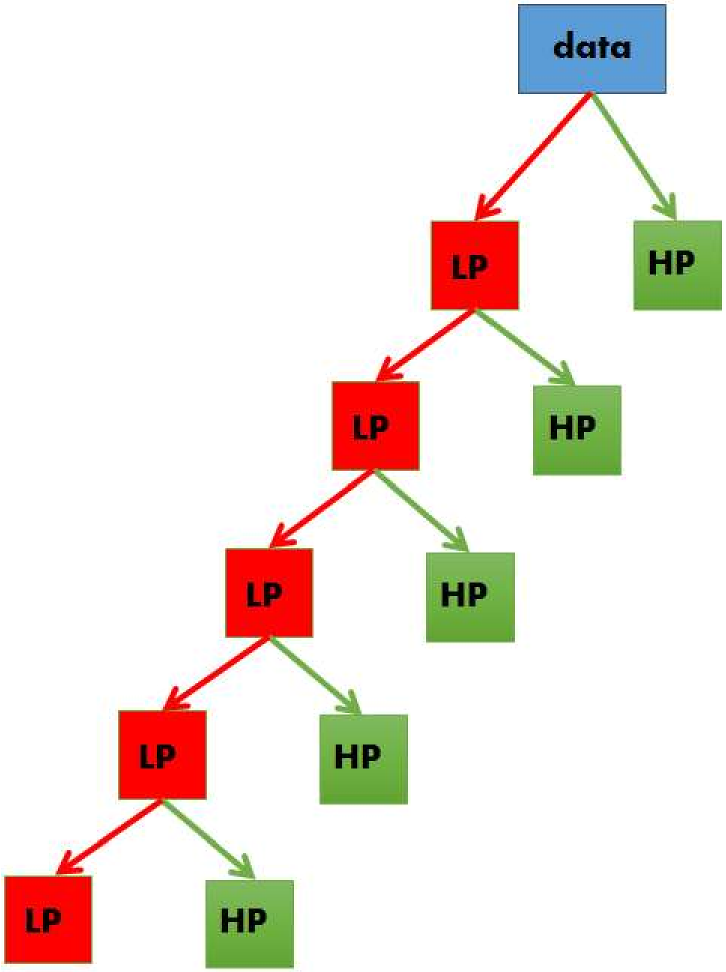
Discrete Wavelet Transform

The wavelet itself can be defined from the scaling function:

$$\Psi(t) = \sum_{n=-\infty}^{\infty} \sqrt{2}\, d(n)\phi(2t - n)$$

where d(n) is a series of scalars that are related to the waveform x(t) and that define the discrete wavelet in terms of the scaling function.

While the DWT can be implemented using the above equations, it is usually implemented using filter bank techniques.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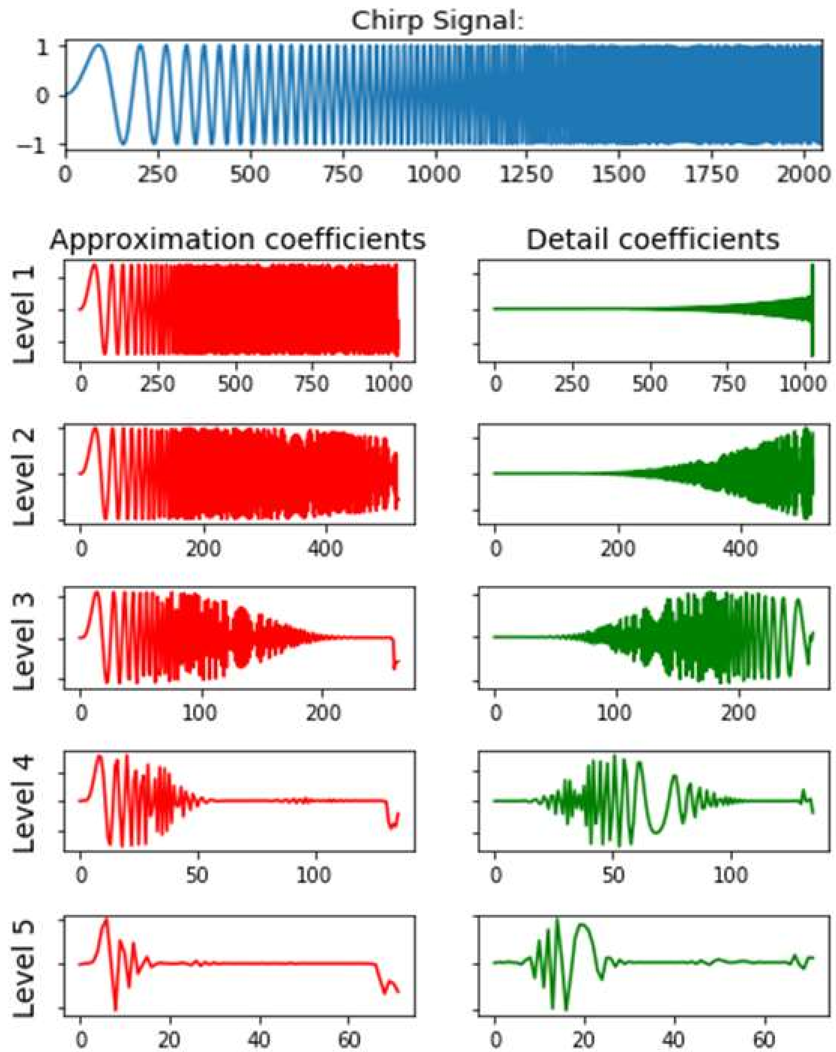

Chirp Signal:
Approximation coefficients
Detail coefficients
Level 1
Level 2
Level 3
Level 4
Level 5

Building the DWT:

$$\Psi_{jk}(t) = \frac{1}{\sqrt{a_{jk}}}\Psi\left(\frac{t-b_{jk}}{a_{jk}}\right) = a_0^{-j/2}\Psi(a_0^{-j}t-kT)$$

$$a_{jk} = a_0^j$$

$$b_{jk} = ka_0^jT$$

- T is the sampling time
- a_0 is a positive nonzero constant
- $\Psi(t)$ is continuous mother wavelet (a family of functions)
- $0 \leq j \leq (N-1)$, and $0 \leq k \leq (M-1)$

Building the DWT:

With the previous, the coefficients of the DWT can be calculated:

$$W_{jk} = \int_{-\infty}^{+\infty} x(t)\Psi_{jk}^{*}(t)dt$$

Which calculates a finite set of discrete coefficients directly from the signal.

$$x(t) = c\sum_{j=0}^{N-1}\sum_{k=0}^{M-1} W_{jk}\Psi_{jk}(t)$$

Building the DWT:

- c is a constant that depends on choice of mother wavelet.
- this equation can reconstruct the continuous signal directly from a set of discrete coefficients

$$x(t) = c \sum_{j=0}^{N-1} \sum_{k=0}^{M-1} W_{jk} \Psi_{jk}(t)$$

BUT, now….. how to choose the number of basis functions for a given signal (i.e. how many shifted and scaled versions of the mother wavelet are needed to decompose a signal)??

Discrete wavelet transform

CWT is highly redundant since a 1-dimensional function x(t) is transformed into a 2-dimensional function. Therefore, it is Ok to discretize them to some suitably chosen sample grid.

The most popular is dyadic sampling:

$s=2^{-j}$, $\tau = k2^{-j}$

With this sampling it is still possible to reconstruct exactly the signal x(t).

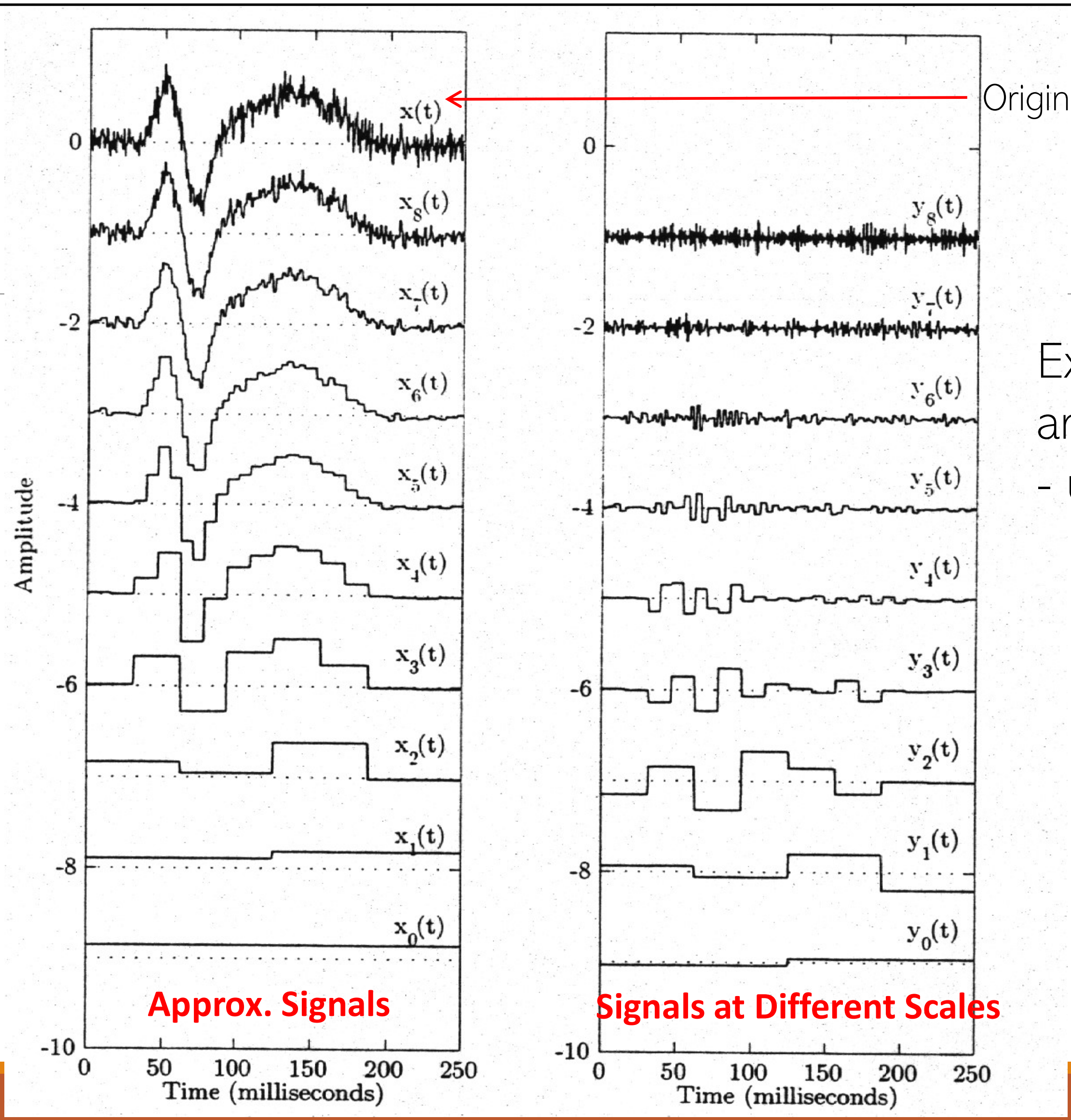
Multiresolution analysis

The signal can be viewed as the sum of:

1. a smooth ("coarse") part – reflects main features of the signal (approximation signal);
2. a detailed ("fine") part – faster fluctuations represent the details of the signal.

The separation of the signal into 2 parts is determined by the resolution.

Example of multi-resolution analysis
- using Haar



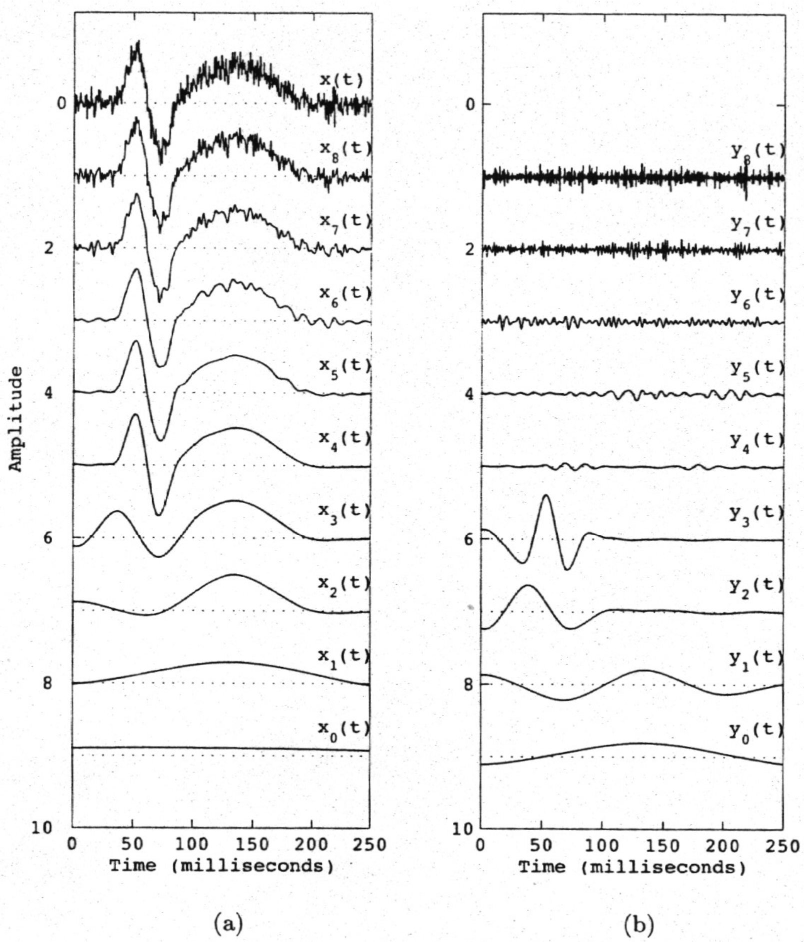


Figure 4.45: Multiresolution analysis of an evoked potential waveform using the Coiflet-4. (a) The approximation signals, and (b) the detail signals at different scales; the original signal is shown at the top left of the figure.

One more example but now with a smooth function Coiflet-4, you see, this one models the response somewhat better than Haar

What should you want from the scaling and wavelet function?

1. Orthonormality and compact support (concentrated in time, to give time resolution)
2. Smooth, if modeling or analyzing physiological responses (e.g., by requiring vanishing moments at certain scale): *Daubechies, Coiflets.*
3. Symmetric (hard to get, only Haar or sinc)

- Haar wavelet (square wave limited in time, superior time localization)
- Mexican hat (smooth)

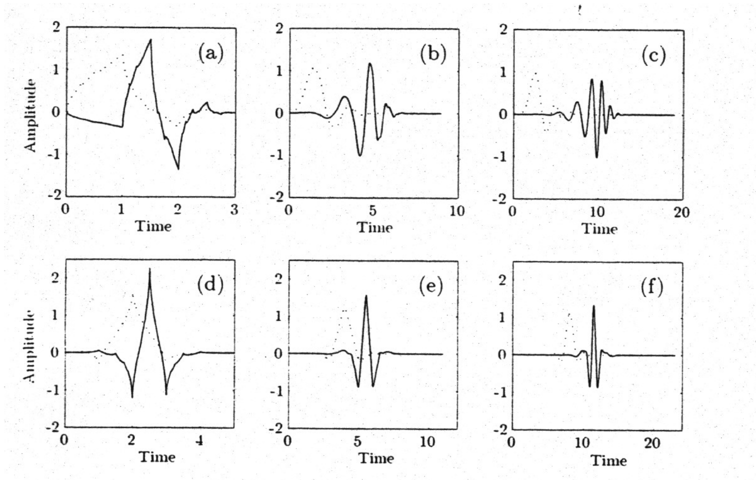


Figure 4.44: The scaling (dotted line) and wavelet function (solid line) for (a) Daubechies-2 (b) Daubechies-5, (c) Daubechies-10, (d) Coiflet-1, (e) Coiflet-2, and (f) Coiflet-4. Note that the time scales differ.

Applications of Wavelets

Compression

De-noising

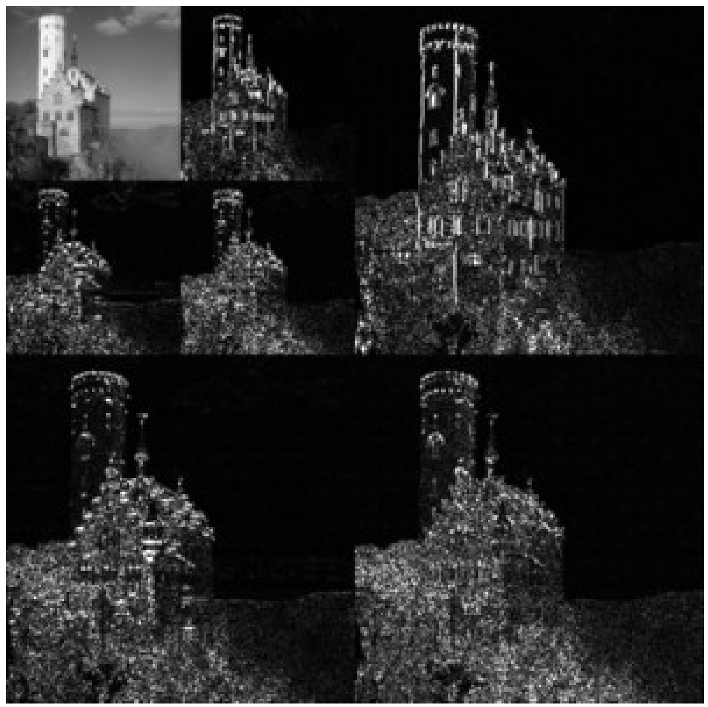
Feature Extraction

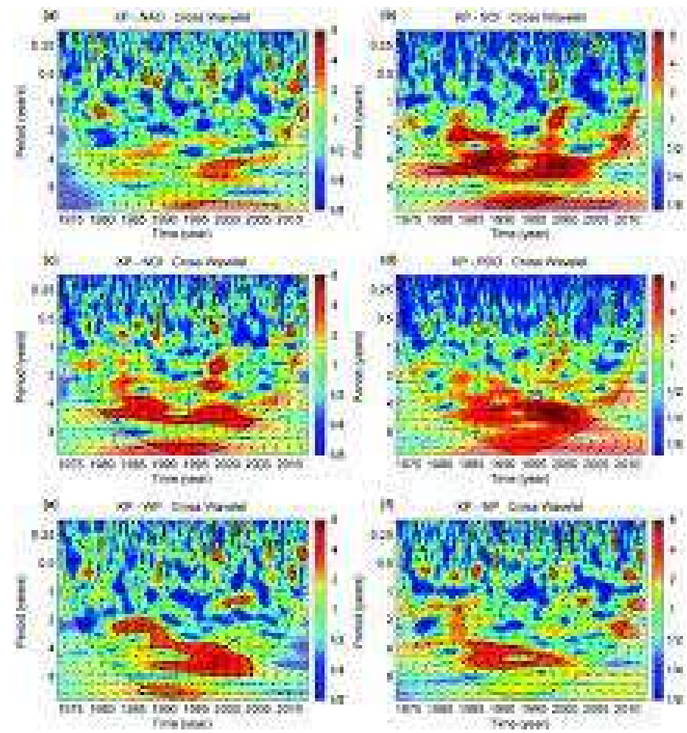
Discontinuity Detection

Distribution Estimation

Data analysis

- Biological data
- NDE data
- Financial data





Applications

Data Compression

Wavelet Shrinkage Denoising

Source and Channel Coding

Biomedical Engineering
- EEG, ECG, EMG, etc analysis
- MRI

Nondestructive Evaluation
- Ultrasonic data analysis for nuclear power plant pipe inspections
- Eddy current analysis for gas pipeline inspections

Numerical Solution of PDEs

Study of Distant Universes
- Galaxies form hierarchical structures at different scales

Applications

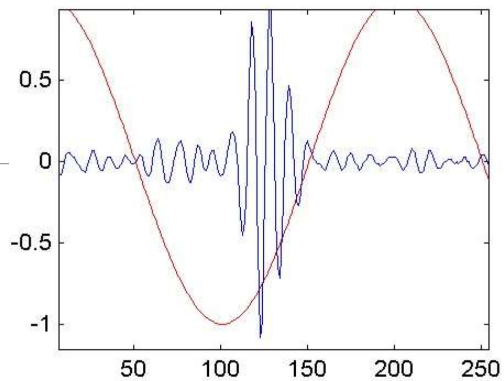
Wavelet Networks
- Real time learning of unknown functions
- Learning from sparse data

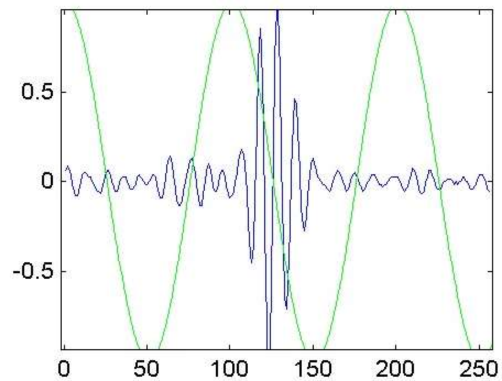
Turbulence Analysis
- Analysis of turbulent flow of low viscosity fluids flowing at high speeds

Topographic Data Analysis
- Analysis of geo-topographic data for reconnaissance / object identification

Fractals
- Daubechies wavelets: Perfect fit for analyzing fractals

Financial Analysis
- Time series analysis for stock market predictions

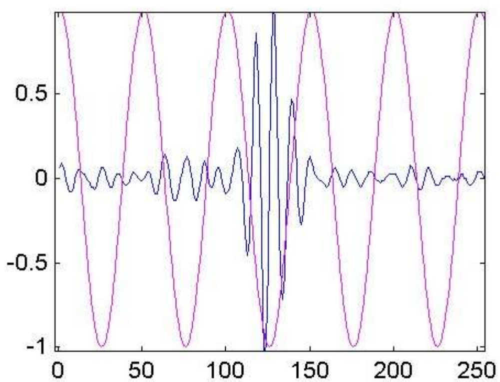


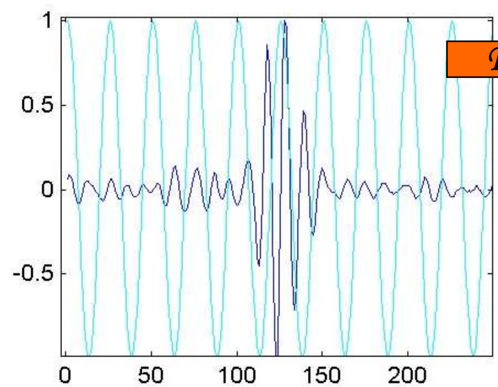


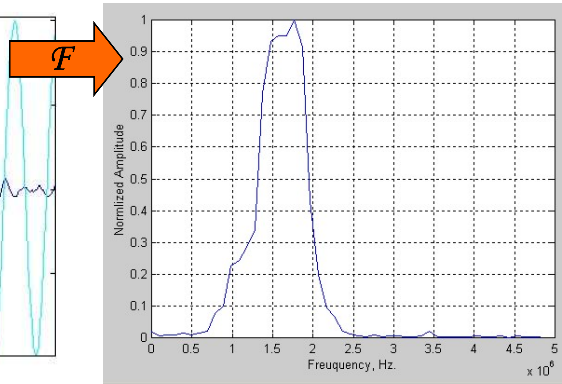
Complex exponentials (sinusoids) as basis functions:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t} dt$$

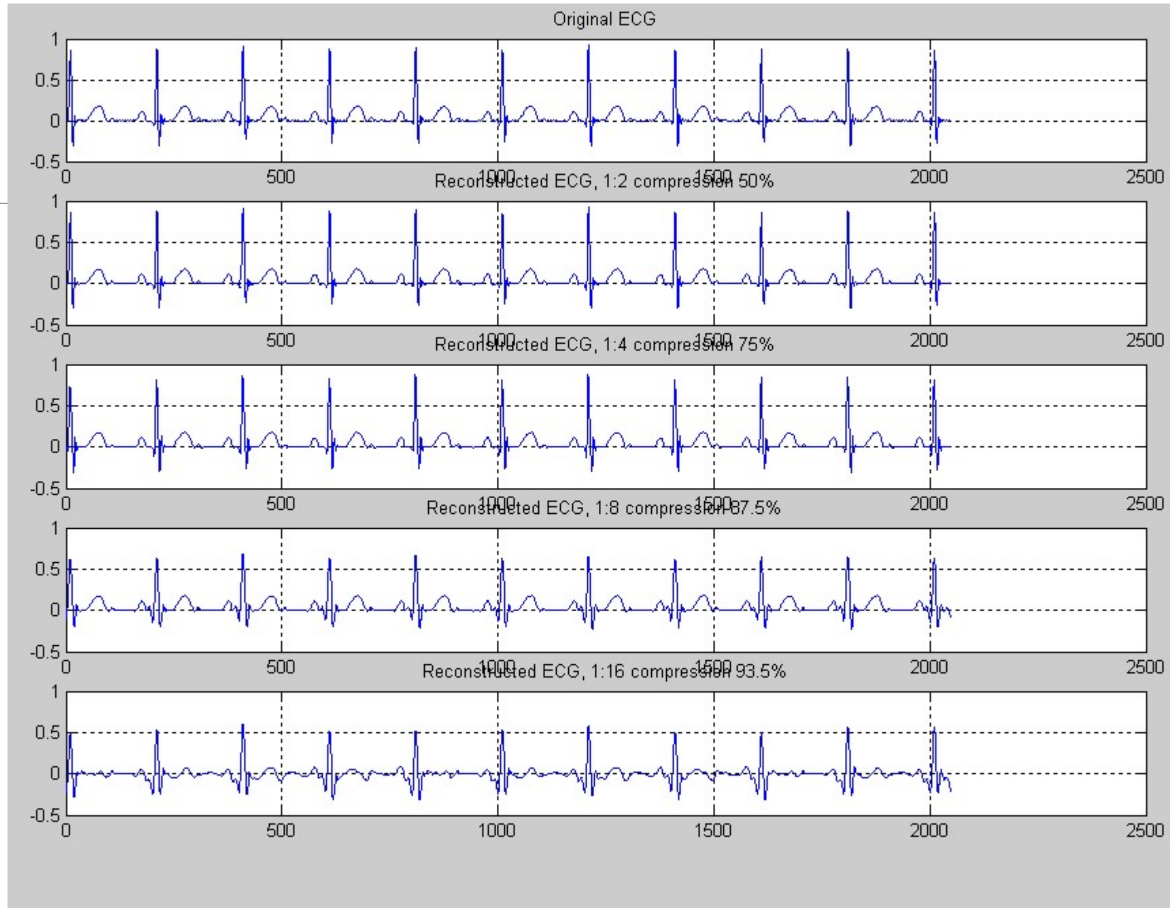
$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \cdot e^{j\omega t} dt$$







An ultrasonic A-scan using 1.5 MHz transducer, sampled at 10 MHz


Original ECG
Reconstructed ECG, 1:2 compression 50%
Reconstructed ECG, 1:4 compression 75%
Reconstructed ECG, 1:8 compression 87.5%
Reconstructed ECG, 1:16 compression 93.5%
1
0.5
0
-0.5
0
500
1000
1500
2000
2500