McMaster University | Integrated Biomedical Engineering & Health Sciences Program

IBEHS - 4QZ3
Modelling of Biological Systems

Lecture 9

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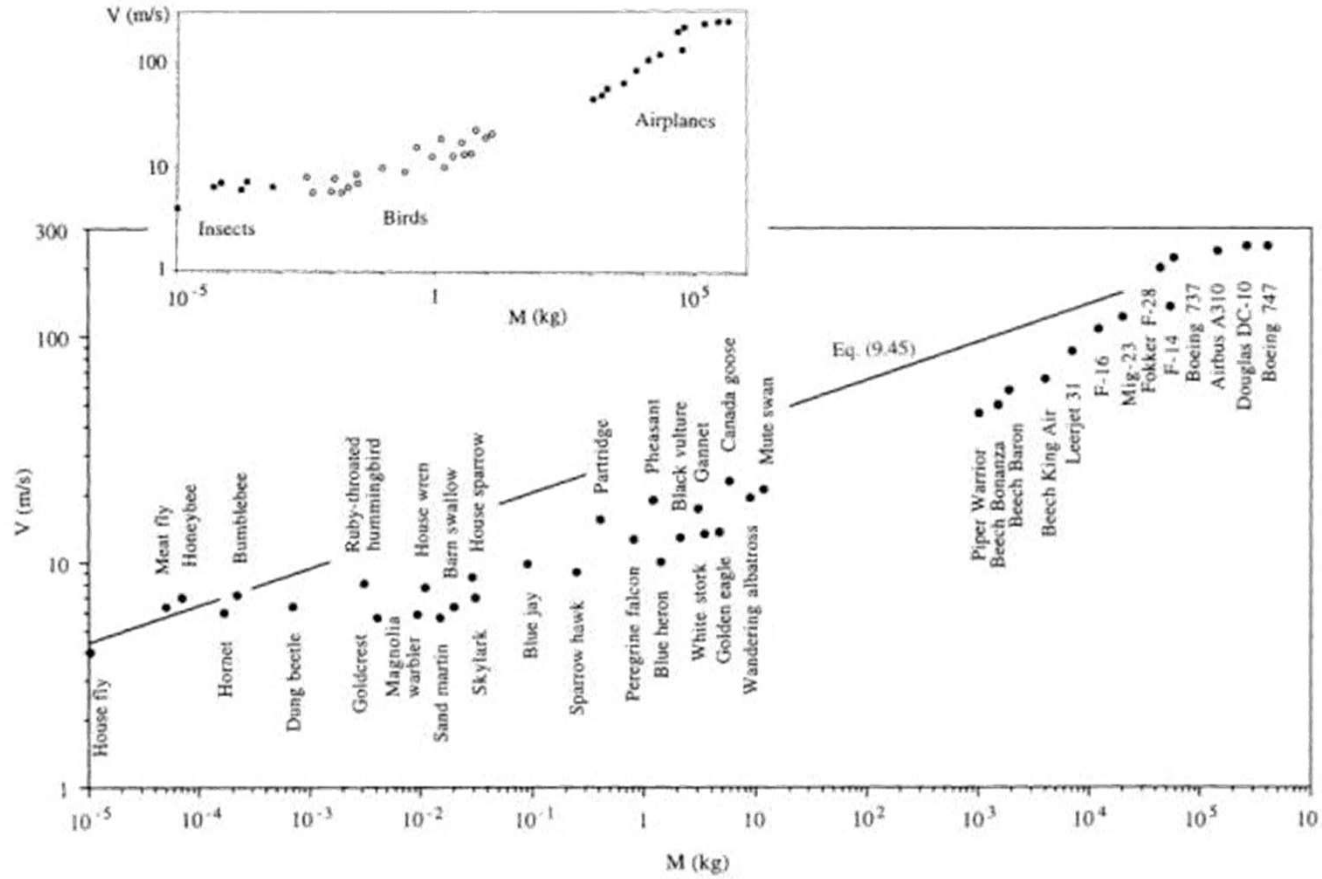
Todays Aims…

Scaling Allometry

Complexity of Biology

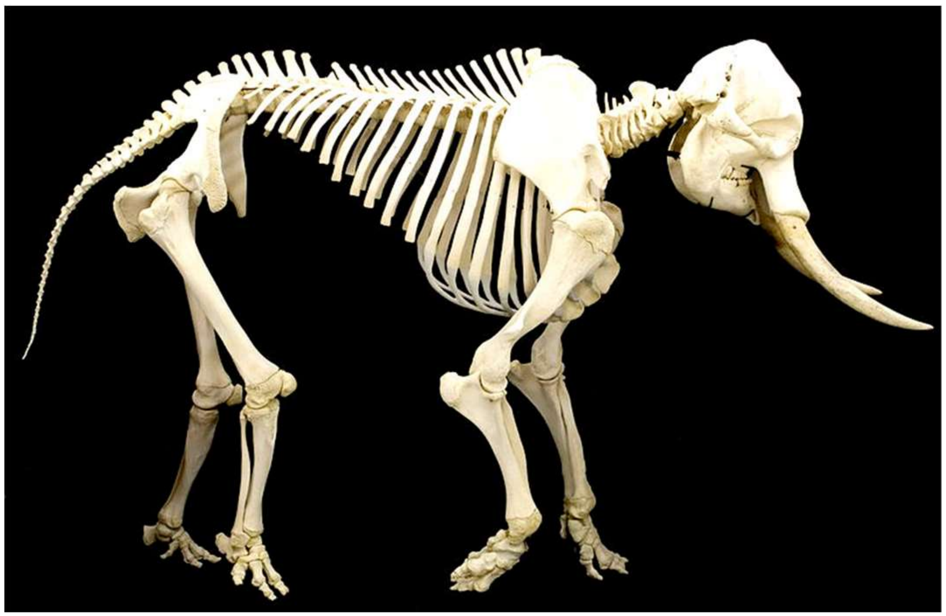
Chaos and Fractals

Power Laws in Biology: Science of Allometry



http://en.wikipedia.org/wiki/Allometry

Bone thickness related to size



Rhinoceros Beetle (*Dynastes hercules*)

Kleiber's law

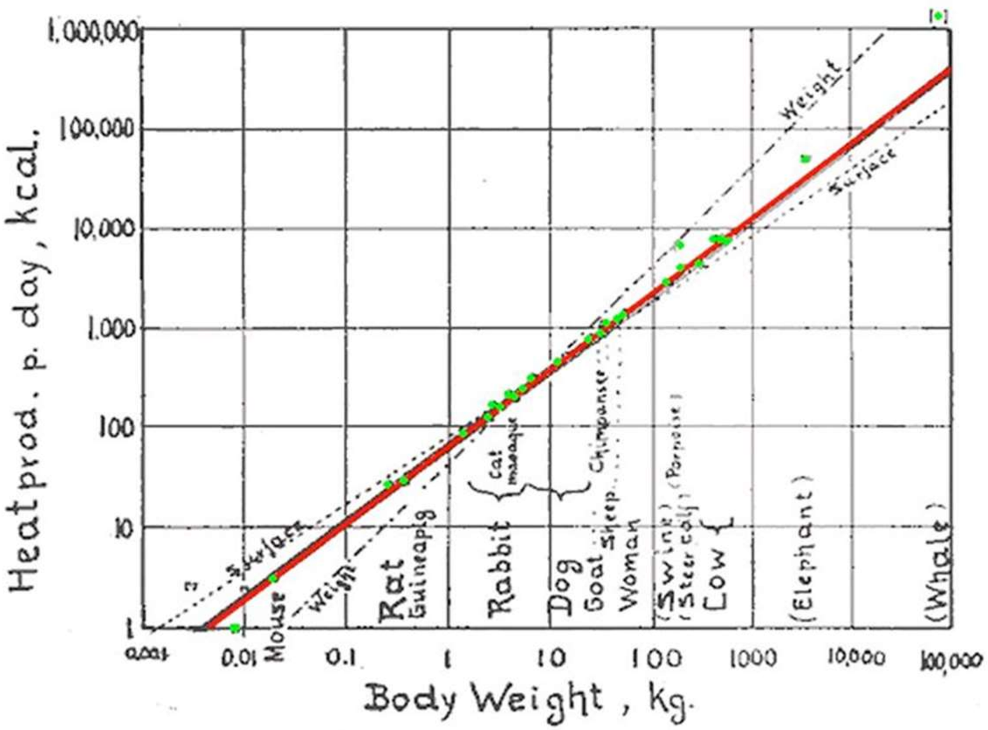


Fig. 1. Log. metabol. rate/log body weight

If q_0 = metabolic rate, and M=mass, then Kleiber's law states that:

$q_0 = 70 M^{3/4}$

Thus a cat (mass=100x mouse), will have a metabolism ~ 31x greater

http://en.wikipedia.org/wiki/Kleiber%27s_law

Power Laws

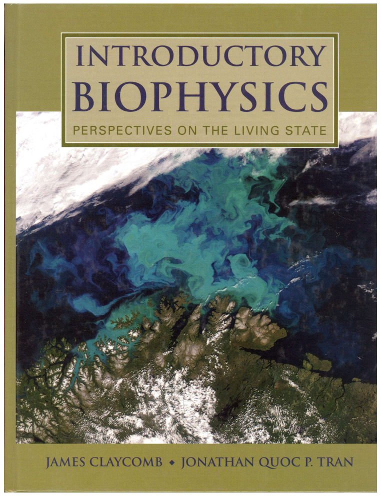
Many lower laws in biology follow the same form

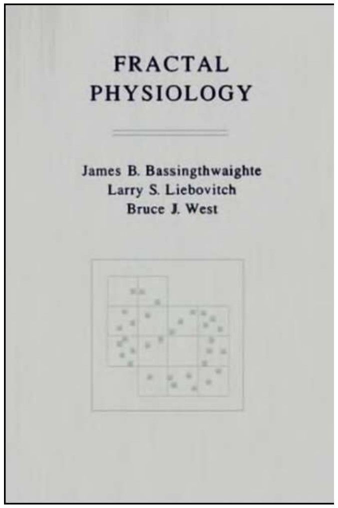
$$y = kx^a$$

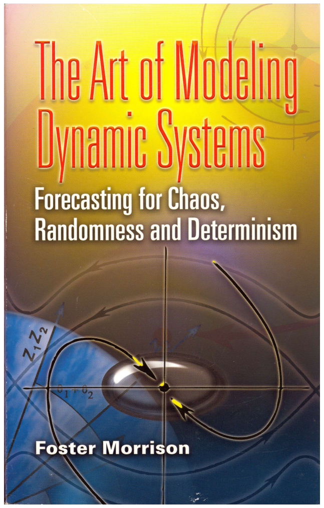
$$\log(y) = a \cdot \log(x) + \log(k)$$

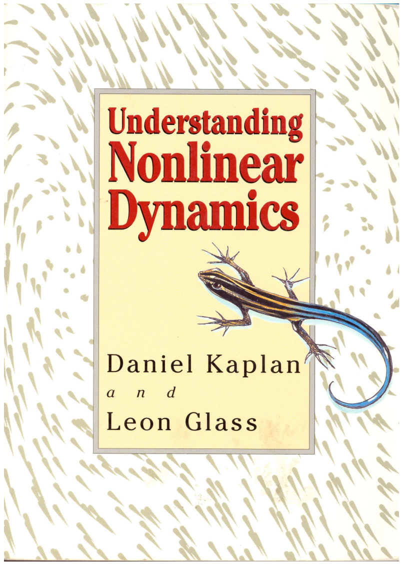
Observable, *y*	Power Law exponent, ***a***
Genome Length	$\frac{1}{4}$
Concentration of RNA	$-\frac{1}{4}$
Total Mitochondrial Mass Relative to Body Mass	$-\frac{1}{4}$
Basal Metabolic Rate	$\frac{3}{4}$
Heart Rate	$-\frac{1}{4}$
Life Span	$\frac{1}{4}$
Radii of Aorta and Tree Trunks	$\frac{3}{8}$

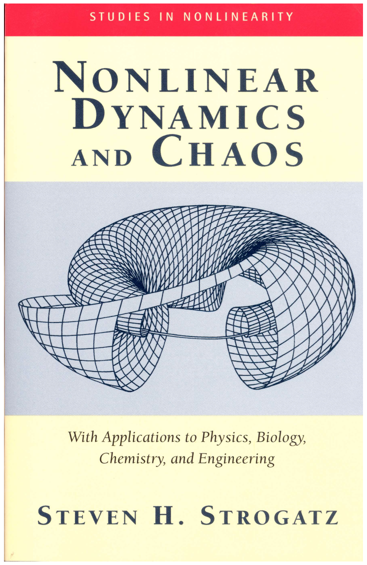
Unfortunately most things in biology don’t follow nice linear models…


INTRODUCTORY
BIOPHYSICS
PERSPECTIVES ON THE LIVING STATE
JAMES CLAYCOMB • JONATHAN QUOC P. TRAN


FRACTAL
PHYSIOLOGY
James B. Bassingthwaighte
Larry S. Liebovitch
Bruce J. West


The Art of Modeling
Dynamic Systems
Forecasting for Chaos,
Randomness and Determinism
Foster Morrison


Understanding
Nonlinear
Dynamics
Daniel Kaplan
and
Leon Glass


STUDIES IN NONLINEARITY
NONLINEAR
DYNAMICS
AND CHAOS
With Applications to Physics, Biology,
Chemistry, and Engineering
STEVEN H. STROGATZ

Why nonlinear processing?

The real world is nonlinear
- Often we can make linear assumptions, but we lose information that way.

Nonlinear analysis can provide greater information regarding **dynamics** of the measured system
- Dynamics refers to the governing rules of the system from which the signal originated

Why nonlinear processing?

- All other analyses have come up short
  - Not a very good reason
- We suspect our signal comes from a nonlinear system and we want to know more about it
  - A better reason
- We know our system is nonlinear and we want to differentiate measurements from different system conditions
  - Even better reason

Nonlinear Vs. Linear

Nonlinear signals come from nonlinear systems

Nonlinear systems have governing equations with nonlinear terms

- Sine, exponentials, logarithms, etc…

The input-output relationships do not follow super-position

A hallmark of a nonlinear system is that small changes can be amplified, or vice versa

Simple Nonlinear System

- $y = x^2$
  - Here y and x are variables representing the output and input of a signal
- Inputs of a and b to this system result in a^2 and b^2
- But an input of $(a + b)$ results in an output of

A More Complicated Nonlinear System

A damped driven pendulum can be modeled as:

Where θ and ω are the angular position and velocity respectively

B is the damping constant and k is the driving force

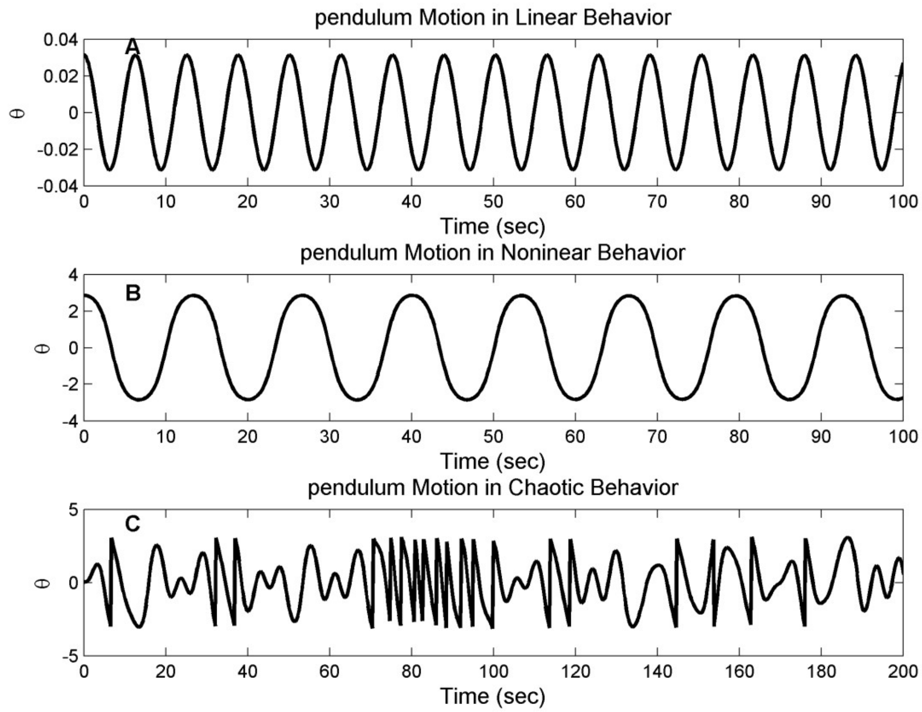
G is gravity

L is the string length

This system has altered modes of behaviour depending on what values b and k take, and the initial conditions

This is NOT linear due to sine terms

Pendulum in 3 modes of behavior

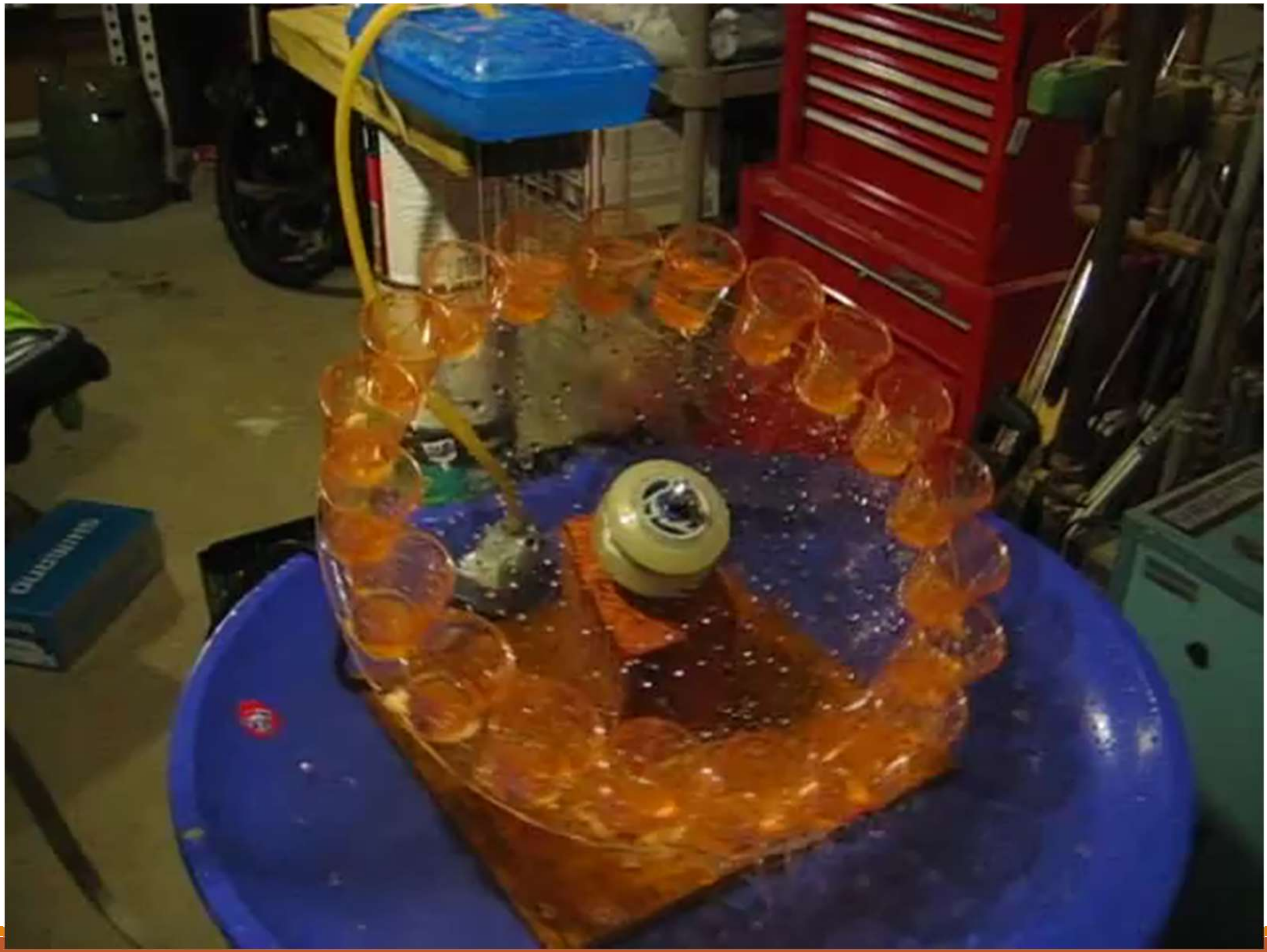


Initial conditions and parameters alter the pendulum behavior

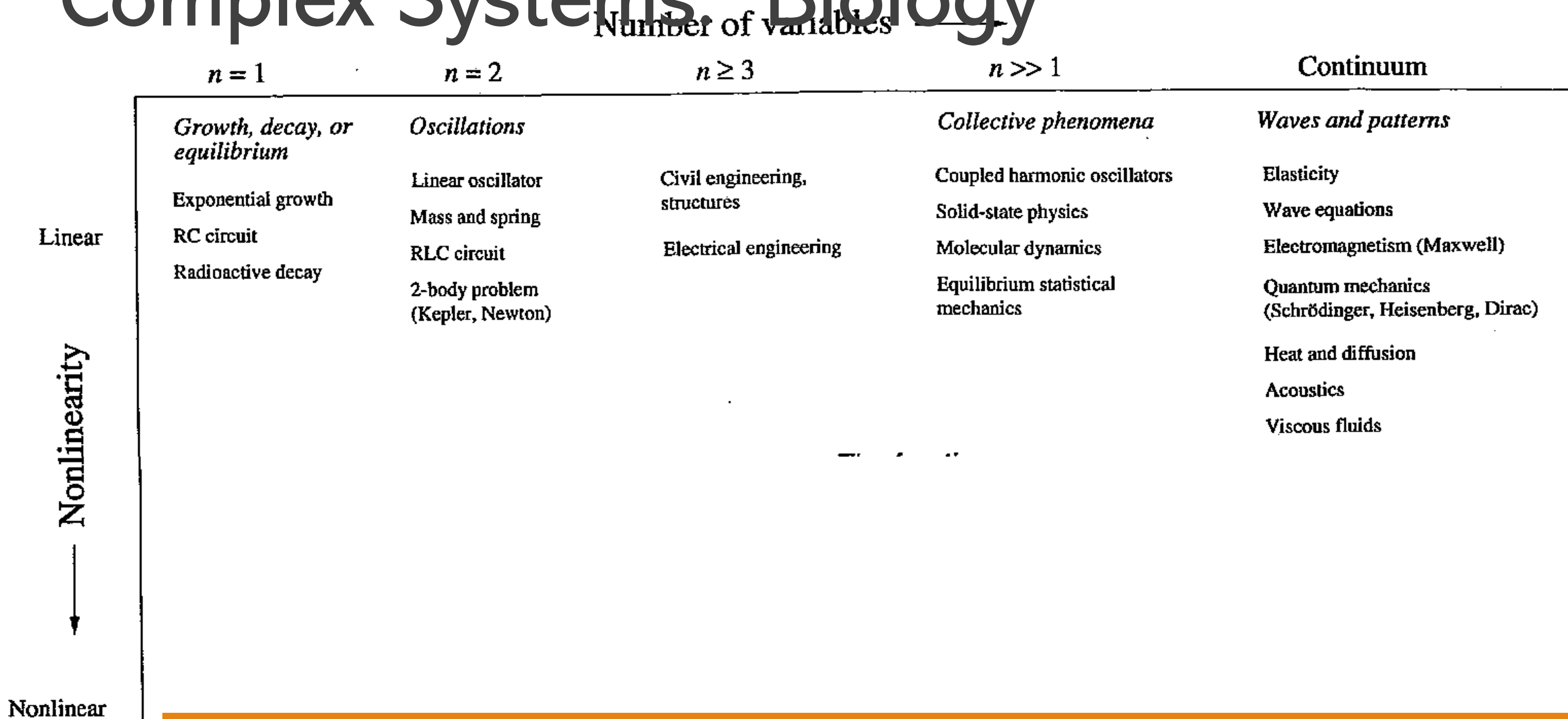
Initial θ (rads)	Force Parameter k	Behavior
π/100	0	Linear
π/1.1	0	Nonlinear (but still regular)
π/100	0.75	Chaotic


MOVIECLIPS.com

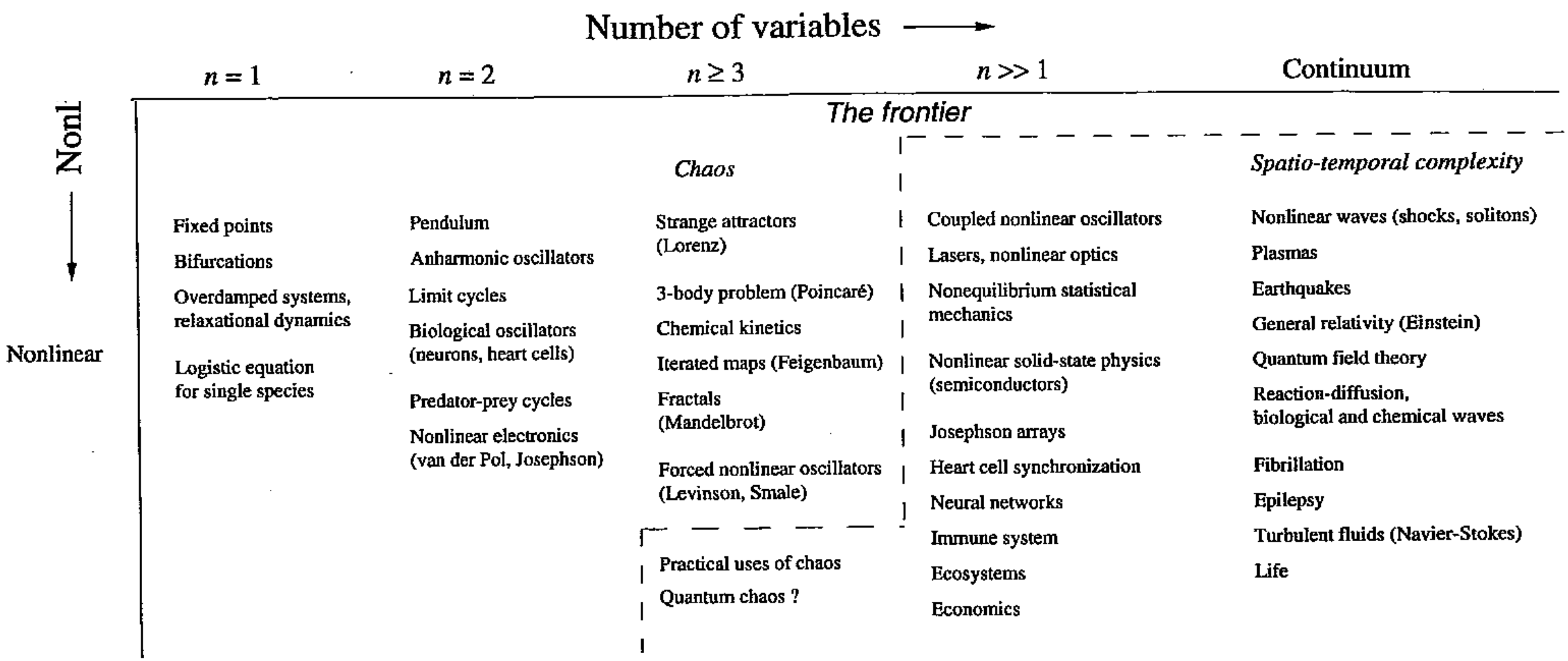
http://www.youtube.com/watch?v=HH2jPq9g6CI

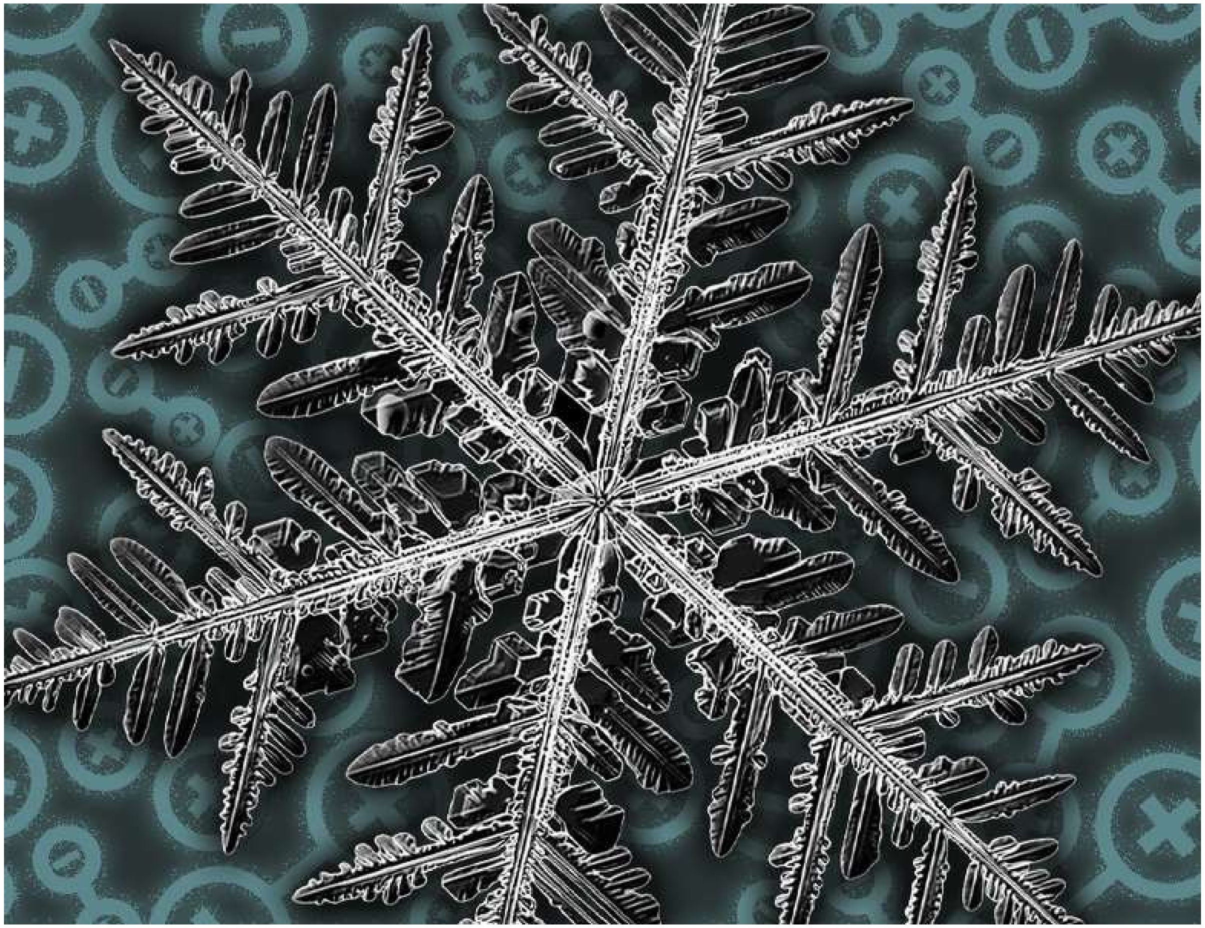


Complex Systems: Biology



Complex Systems: Biology





Fractals

a never-ending pattern.

infinitely complex patterns

repeating a process over and over in an ongoing feedback loop.

fractals are images of dynamic systems

Enter the Fractal Dimension ....

Assigns a number that assesses the extent of "chaotic" or oscillatory behavior.

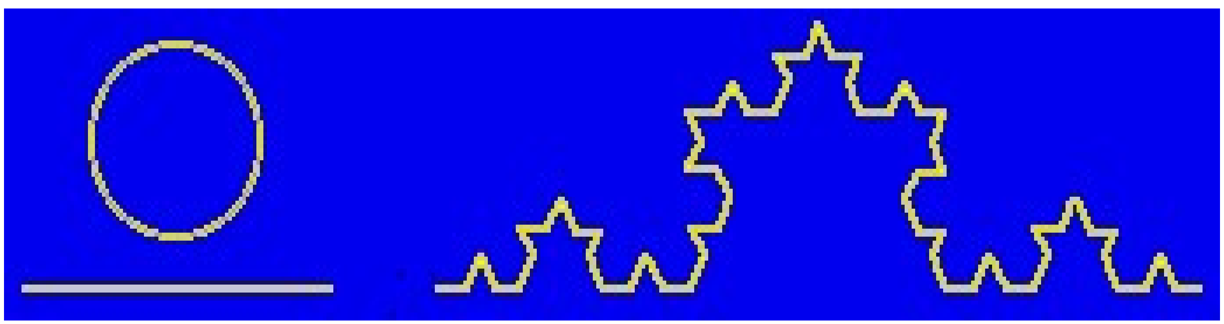
Measure of complexity

Measure of self similarity

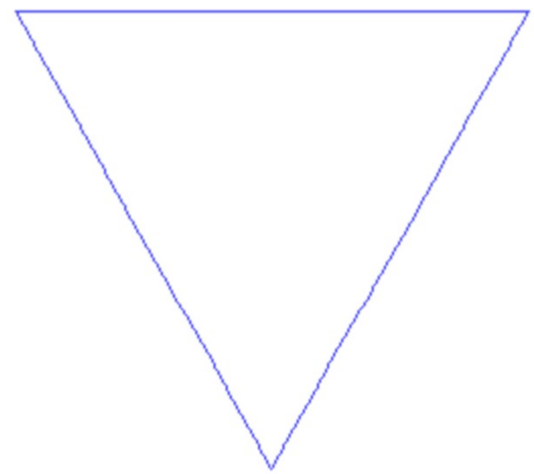
Self-similar across different scales.

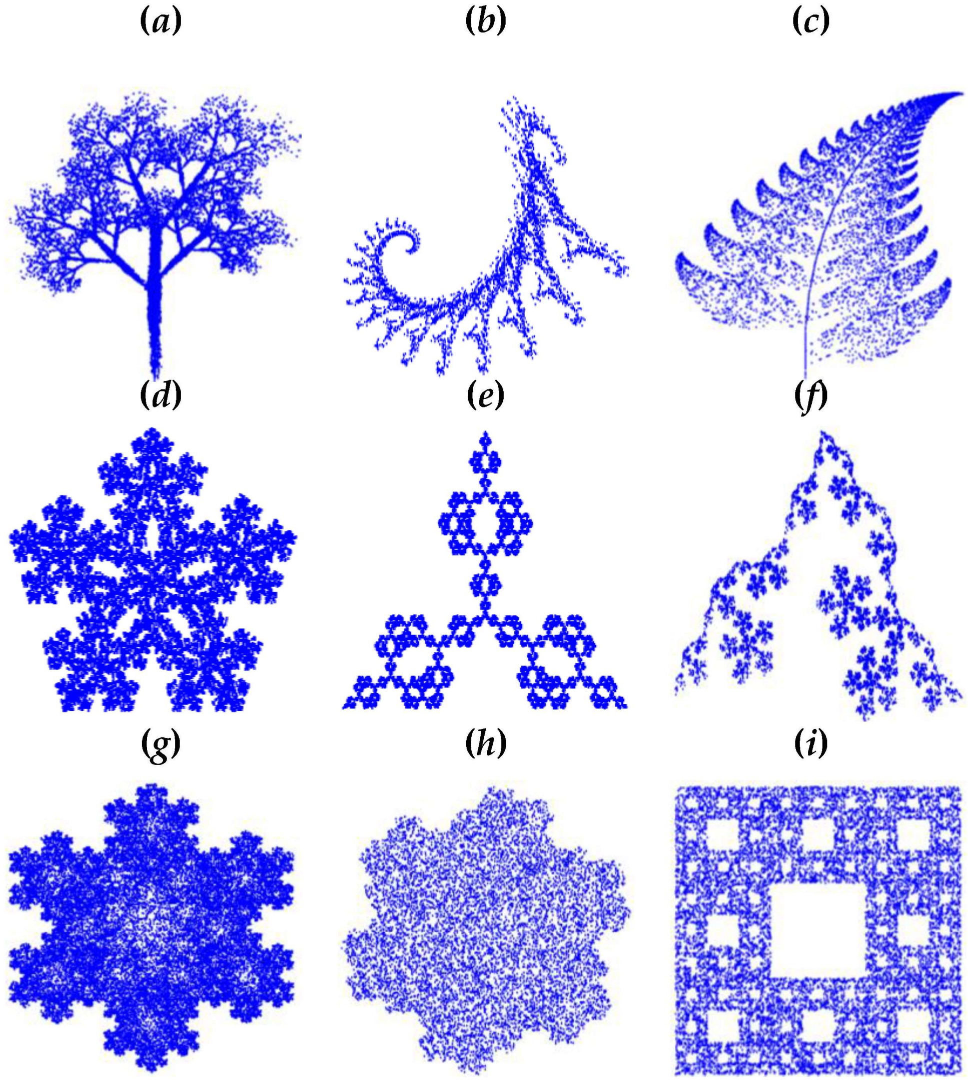
1.0

1.26



http://www.rad.washington.edu/exhibits/fractal.html




(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
(i)

Fractally generated world.
e.g. minecraft

Fractals in biology

- Fractal structures provide an advantageous architecture for living organisms seeking to cover a large area while conserving the amount of building material.

- Examples are branching networks in circulatory systems, lungs, nerves, and plant structures.

- Barnsley's fern is an example of a mathematical fractal that resembles natural plant structures. The fern is generated by the iterative transformation

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} e_j \\ f_j \end{pmatrix}$$

where the coefficients

$$\begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \text{ and } \begin{pmatrix} e_j \\ f_j \end{pmatrix}$$

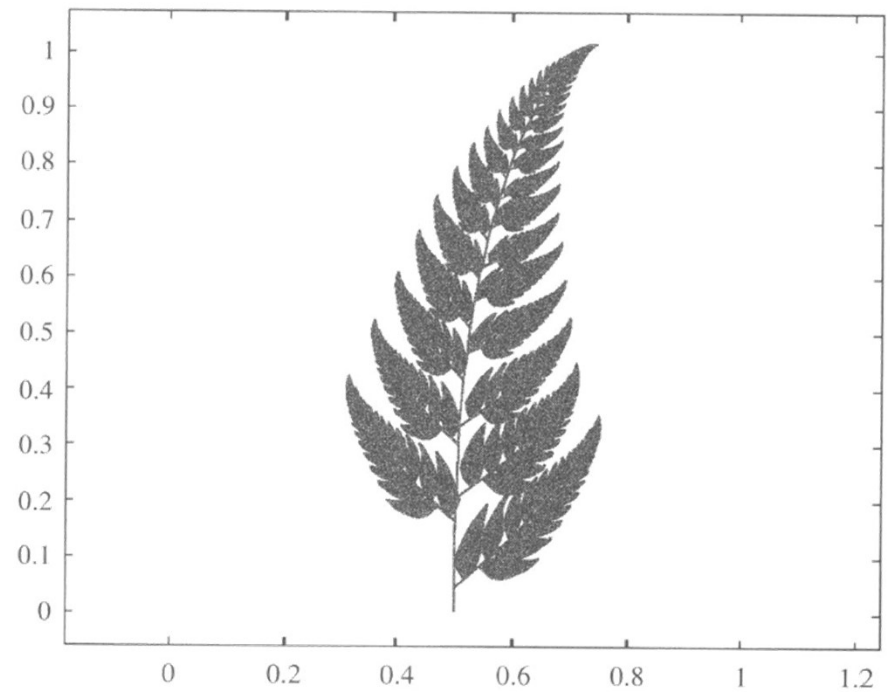
are chosen with probability P_j resulting in a rotation and translation of the vector (x_n, y_n) to (x_{n+1}, y_{n+1})

Fractal Ferns

This can be re-written in succinct notation as:

$$\mathbf{X}_{n+1} = A_j \mathbf{X}_n + B_j$$

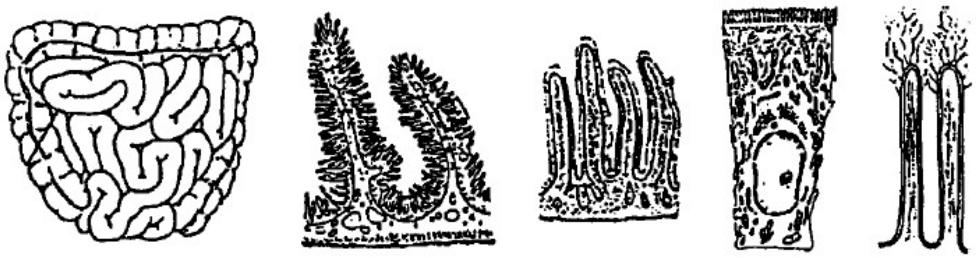
where Aj are rotation matrices and Bj are translation vectors

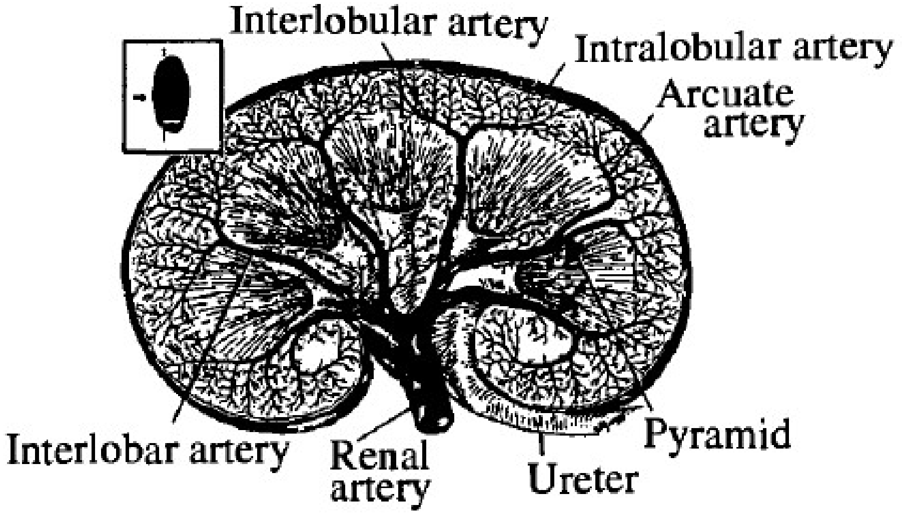


Fractal Ferns

The ultimate fractal vegetable (Wired Magazine) - natural representation of the Fibonacci or golden spiral, a logarithmic spiral where every quarter turn is farther from the origin by a factor of ϕ, the golden ratio.

https://en.wikipedia.org/wiki/Romanesco_broccoli





Fractals in the human body

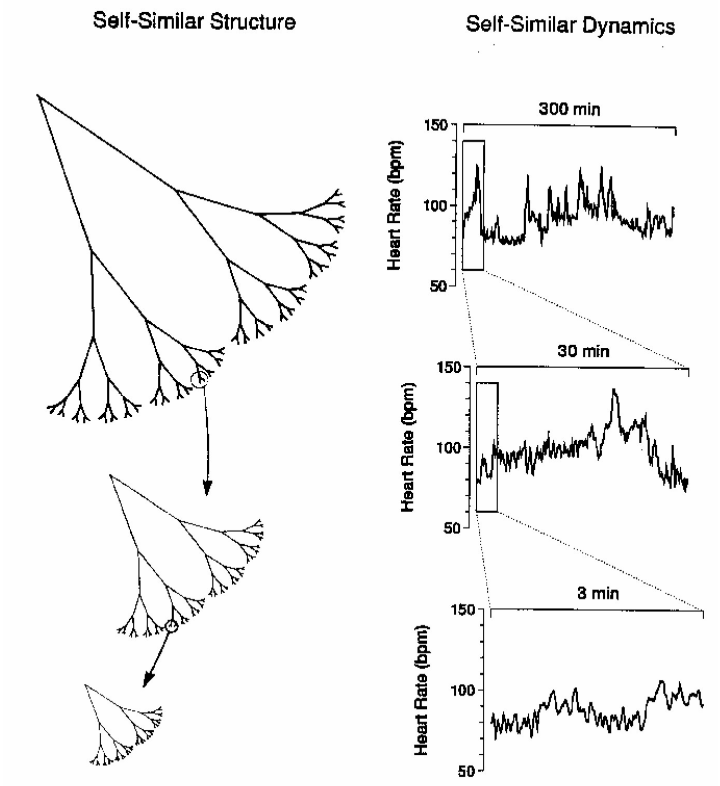
Fractals can be Visualized in Space and Time <u>or</u> <u>BOTH</u>

Fractal Dimension

Assigns a number between 1 - 1.5 that assesses the extent of "chaotic" or oscillatory behaviour.

1.0 → Uniform, correlated and distinctly periodic signal.

1.5 → random noise, approaching 'chaos'.



Goldberger et. al. (2000), physionet.org

Fractal Dimension of Norway

The Fractal Dimension of
the Norwegian coastline= 1.52

Fractal Dimension (d)

- The fractal dimension can be thought of as a measure of geometric complexity.

If we measure the length d of a one-dimensional curve by placing rulers end to end, then we will require a number of rulers N that is inversely proportional to the length of the rulers L since $d = N \times L$.

Fractal Dimension (d)

- If we halve the length of the rulers L → L/2, then we must double their number N → 2N to cover the same d.

- A plot of N vs. 1/L will be a straight line with slope equal to one, the same as the dimensionality of the curve.

Fractal Dimension (d)

- For a square of side L, the number of boxes of length $\boldsymbol{\varepsilon}$ = L/2 is given by $L^2/\boldsymbol{\varepsilon}^2$ = 4 so that d = 2.

- similarly use d = 1 for a line and d = 3 for a cube.

L	L/2	L/3

Box counting (Spatial domain)

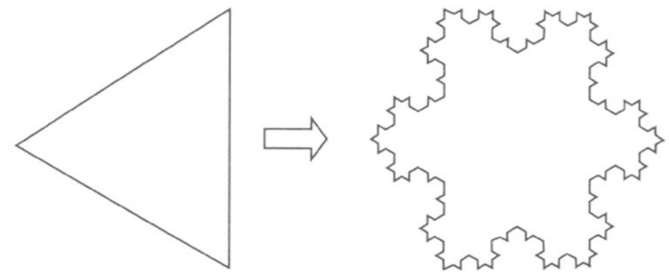
Simple method to describe fractal dimension

Determine length of a one dimensional curve by placing boxes (or rulers) end to end

In traditional geometry a measuring stick 1/3 the originals size will give a total length 3 times as many “sticks” long

If one measures the area of a square then measures again with a box of side length 1/3 the size of the original, one will find 9 times as many squares as with the first measure. Holds in 2 dimensions with boxes.

Koch Snowflake



Koch Snowflake after numerous iterations

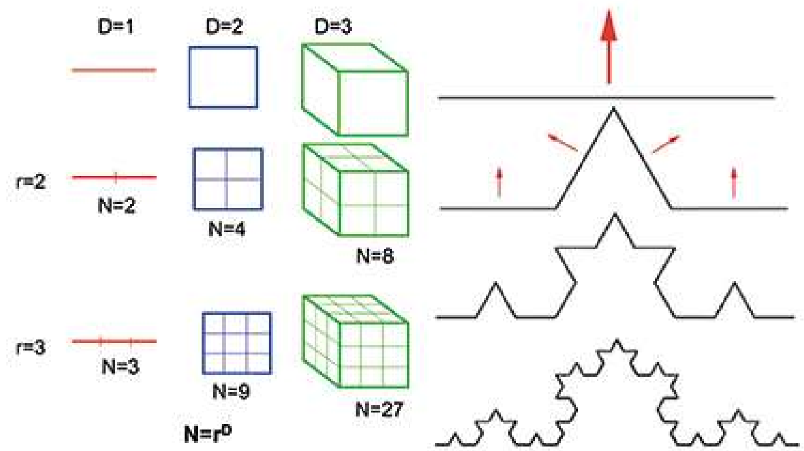
$$d_c = \lim_{\varepsilon \to 0} \frac{\ln N}{\ln (1/\varepsilon)} = \lim_{n \to \infty} \frac{\log 2^n}{\log L + \log 3^n} = \frac{\log 2}{\log 3}$$

Fractal Dimension (d)

- Instead of rulers, imagine covering the curve with boxes.

- Then count the # of boxes required to span the curve as a function of box size.

- The capacity dimension (d) is calculated by covering the curve with d-dimensional boxes where the # of boxes of length **ε** is given by:



Fractal Dimension (d)

- An equation for the capacity dimension is obtained by taking the logarithm of both sides of :

Which can be rearranged to:

Fractal dimension

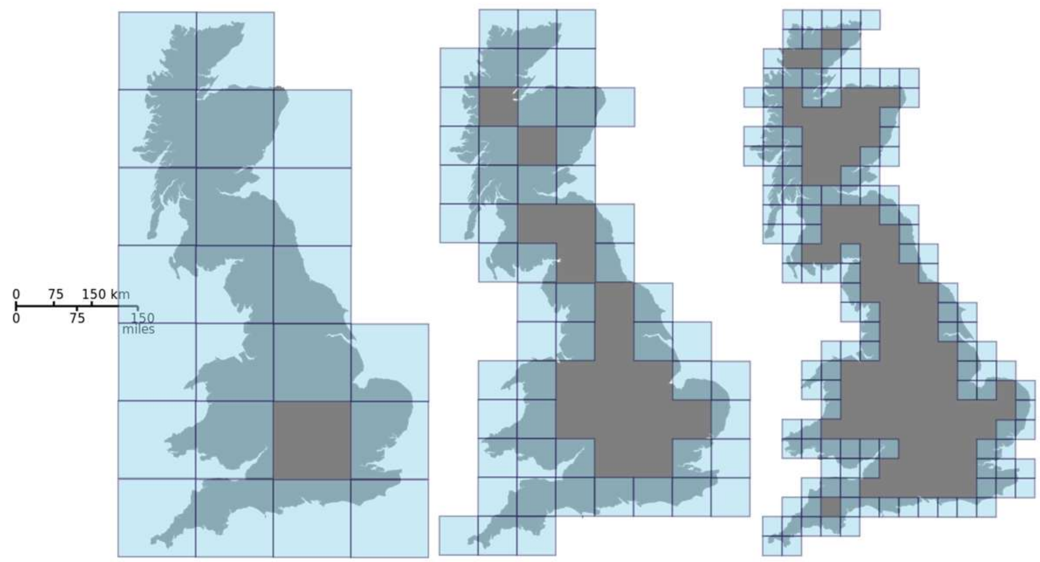
$$d = \frac{\log N(L)}{\log(1/L)}$$

Once again:

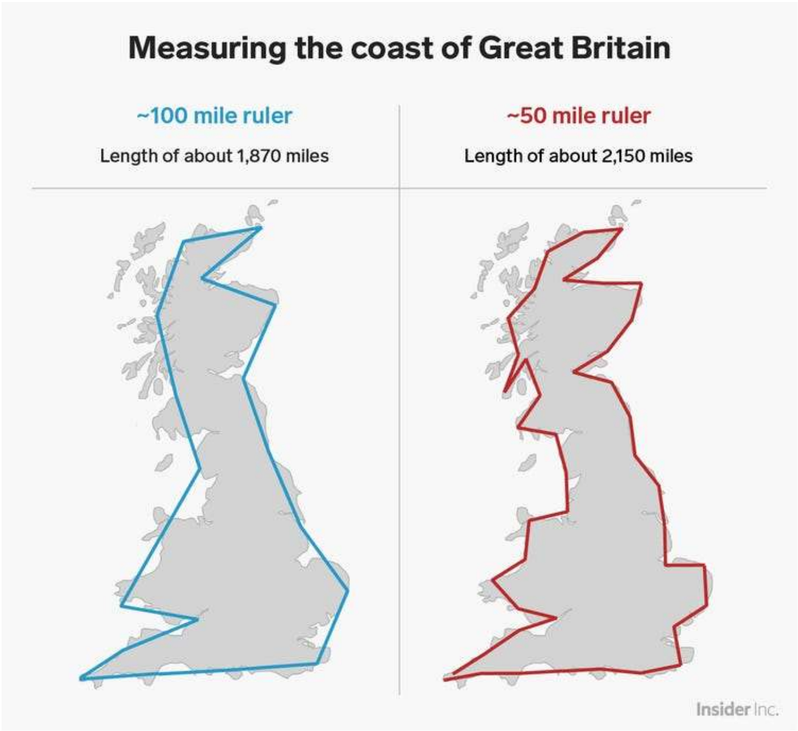
N is number of measuring devices

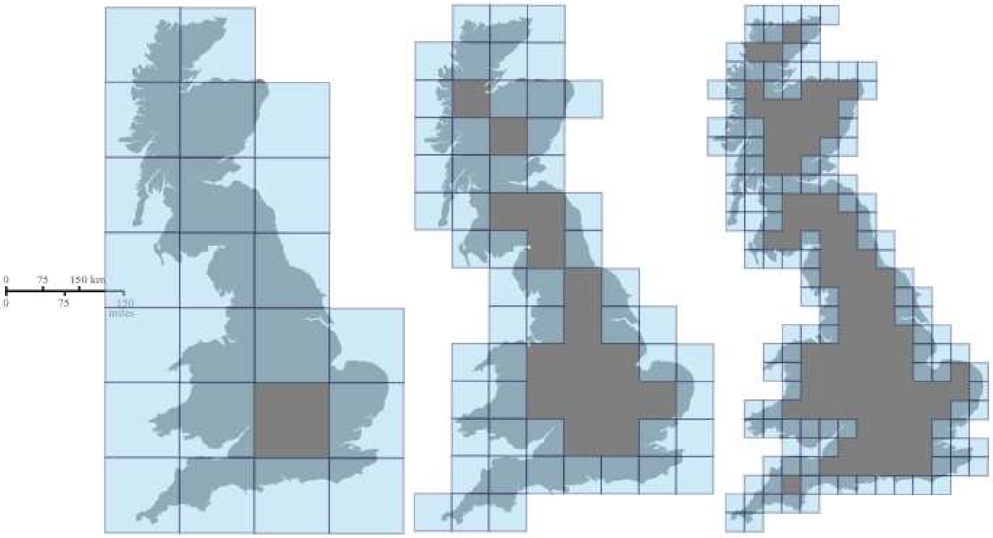
L is scaling factor

D is fractal dimension

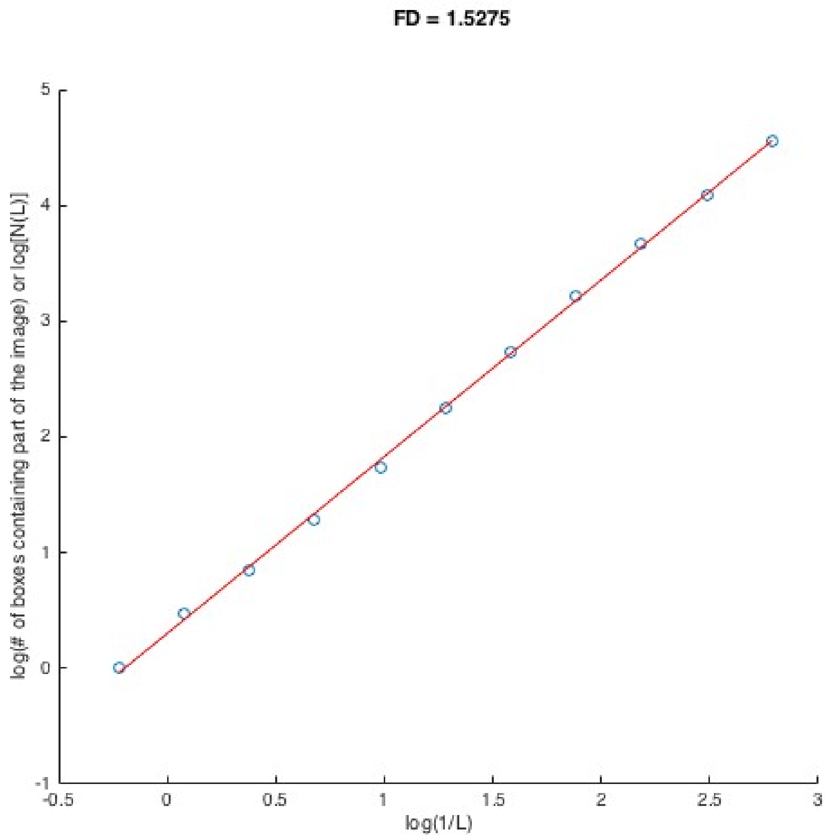


Fractal Dimension of Great Britain





Box-counting dimension of the coast of Britain



Boxcounting

Find the number of boxes you can use for an m x n image

- # of boxes should be equal to 2i (might have to crop or zeropad image)
- # of scaling factors that can be used will be equal to i

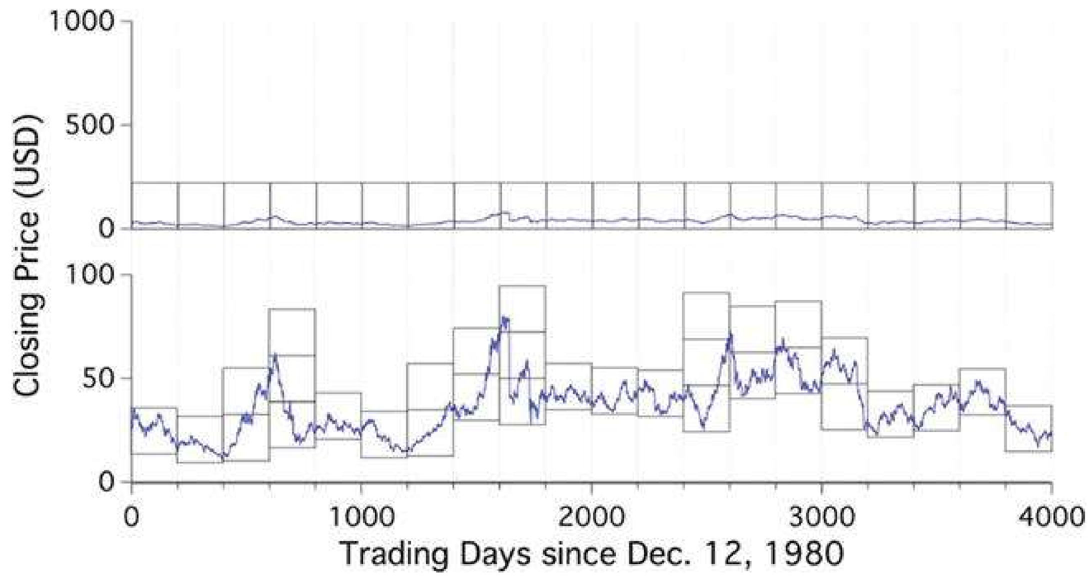
log (# of boxes including edges of image) vs log (# of boxes in grid = 2^i)

Find slope using polyfit function

Fractal Dimension (Time Domain)

Assigns a number between 1 -2 that assesses the extent of "chaotic" or oscillatory behaviour.

- 1.0 - Uniform, correlated and distinctly periodic signal.
- 1.5 - random noise, approaching 'chaos'.



Relative Dispersion (time domain)

Can estimate F.D. using relative dispersion

Empirically determined relation in 1 dimension (ex. time series):

$$RD(m)/RD(m_0) = (m/m_0)^{1-D}$$

Where,

- RD is the relative dispersion (standard deviation divided by mean)
- m is the sample size
- m_0 is the reference sample size

Fractal Dimension: RD Calculation

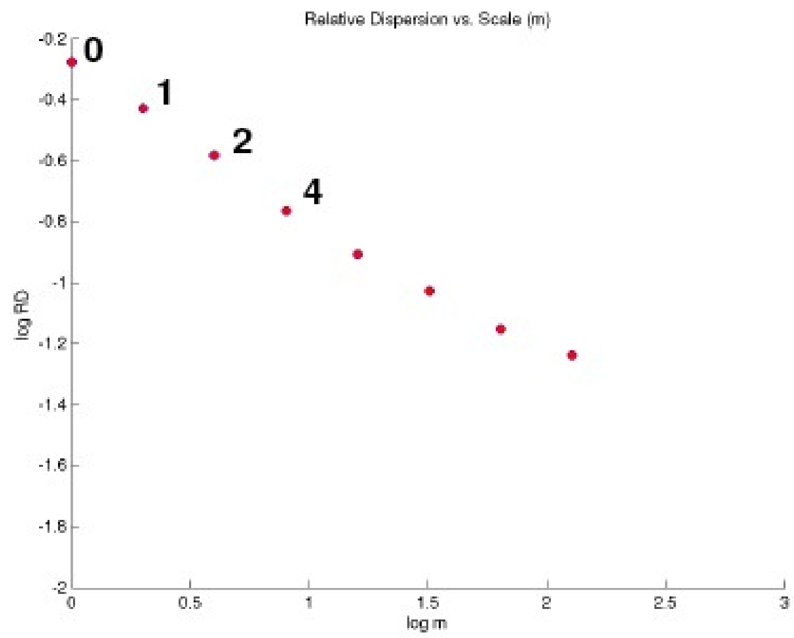
- Look at temporal data over a “time-scale”m, calculate mean and standard deviation
- i.e. for 2048 points it would look like this:

m,scale	RD=SD/mean
1	X_1
2	X_2
4	X_4
1024	X_{1024}

slope log(RD) vs. log (m)

slope = H – 1: H, Hurst Exp.

slope = 1 – D: D, Fractal Dim



Relative dispersion

1) Calculate RD for bin size of m0 samples (RD = SD/mean)

2) Rinse and repeat until m0= length of signal

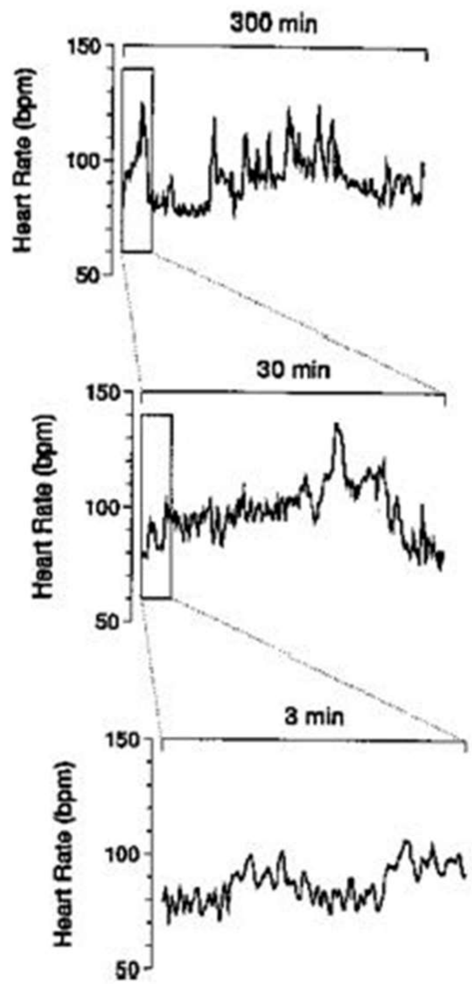
3) Now with the array of RD values, plot on a log(m) vs log(RD) plot and determine the slope

D = 1 – slope

H = slope -1 (Hurst Exponent)

[dim_rd, hurst_rd] = fractal_RD (sig , 1);

The values of the Hurst exponent vary between 0 and 1, with higher values indicating a smoother trend, less volatility, and less roughness.



Power Spectral Method (frequency domain)

Find the power spectrum of the time series, $|A|^2$

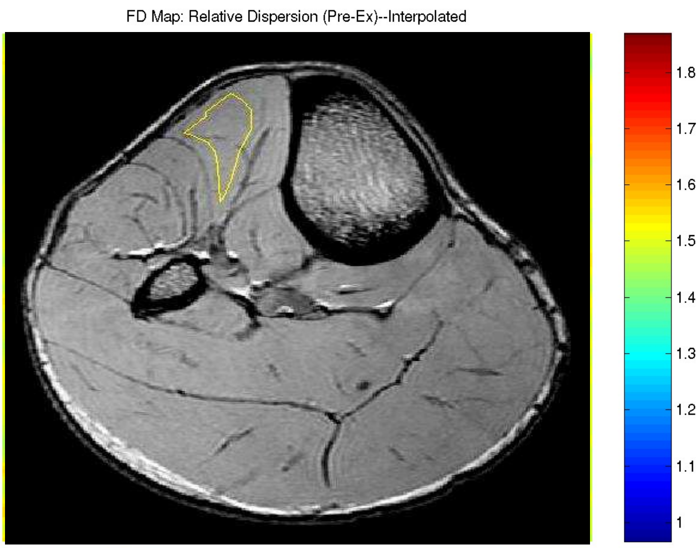
Plot log(f) vs log($|A|^2$) and estimate the slope r of the best-fit line

$\log|A|^2 = -r * \log f$

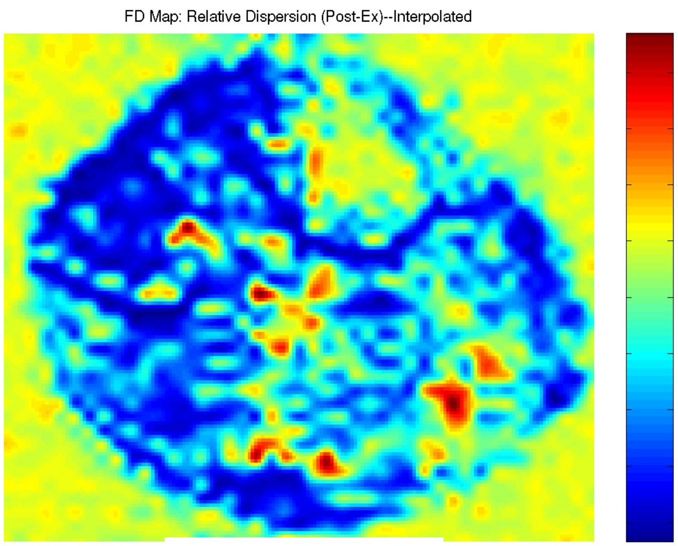
From the relation D = 2 – (r+1)/2

[dim_ps, hurst_ps] = fractal_PS (sig , 2)

Relative Dispersion



FD map (**PRE**)



FD Map (**POST**)

Bin/average BOLD data over time

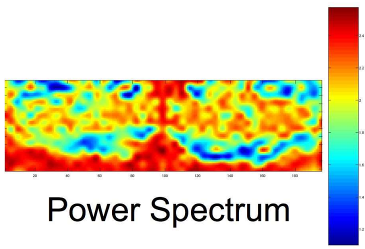
NOTE

- noise characteristics stay the <u>same</u> pre vs. post exercise.

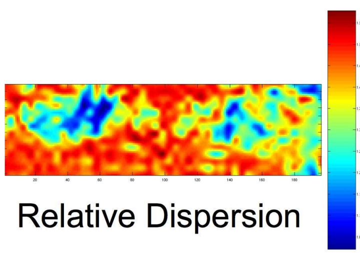
Noseworthy MD, Davis AD, Elzibak E. (2010) Sem. Musculoskel. Radiol. 14:257-268.

Visualizing the Effect of a [Therapeutic] Swedish Massage

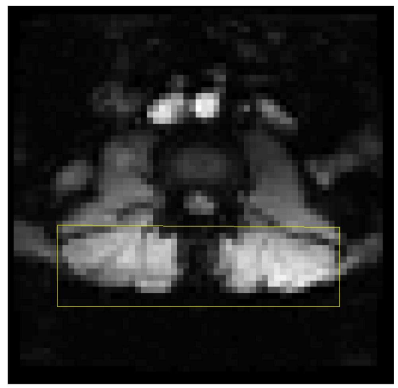
PRE



Power Spectrum

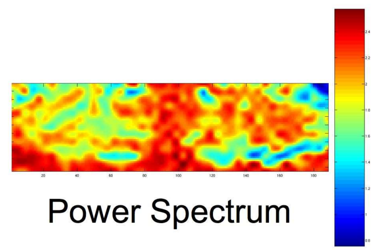


Relative Dispersion

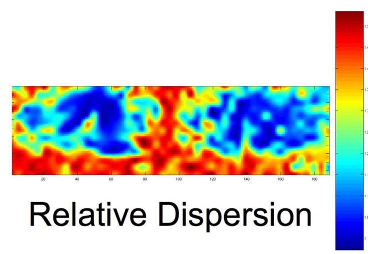


Axial Image
- box shows area of analysis

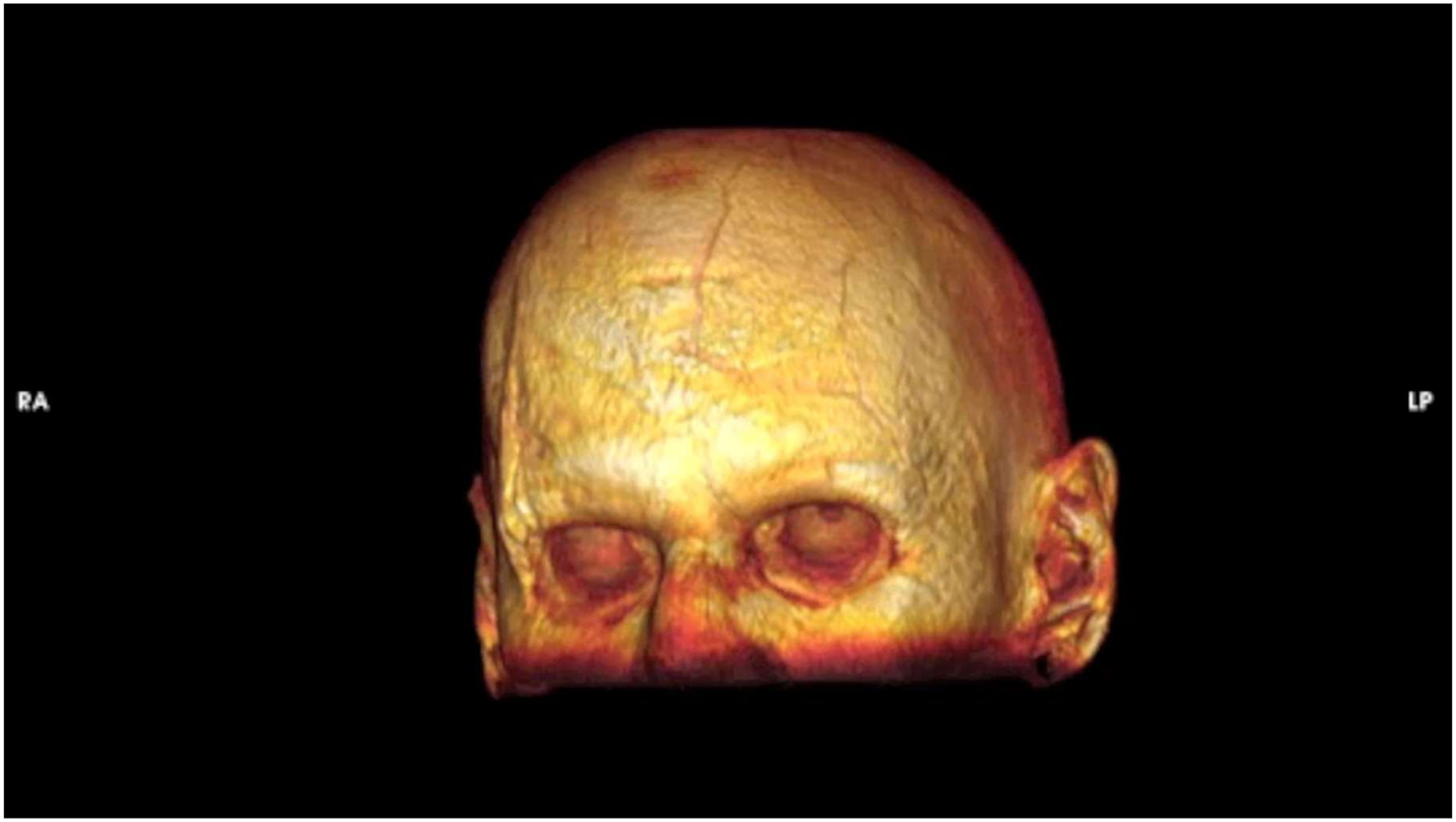
POST



Power Spectrum

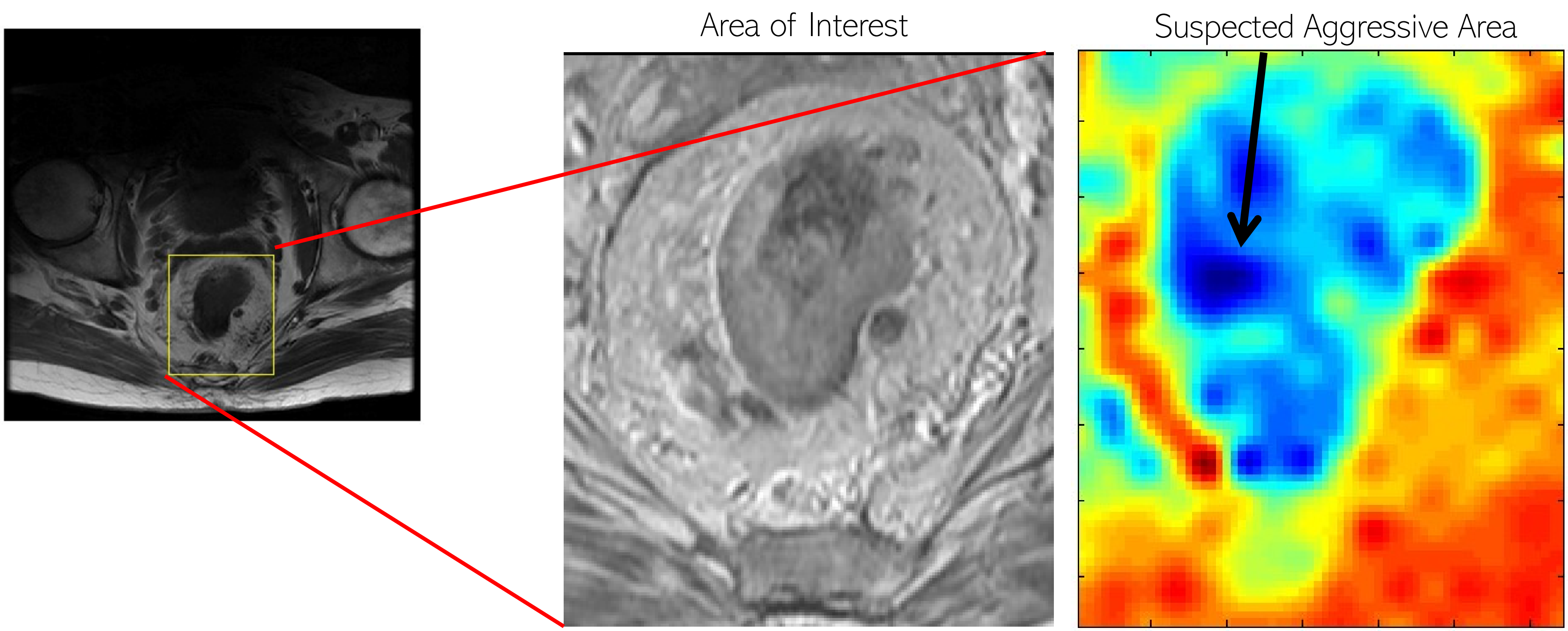


Relative Dispersion



FD signal mapping: Alzheimer's disease
Warsi, Molloy, Noseworthy (2012) MAGMA. 25:335-344.

Fractal Dimension (FD) Mapping: Oncology



Fractals and Chaos

- The studies of fractals and chaos are linked in many ways. They share common ideas and methods of analysis. Each field is used within the other.

- a phase space set can be formed from a time series. When this is fractal the system that generated the time series is chaotic.

- The fractal dimension of the phase space set tells us the minimum number of independent variables needed to generate the time series.

- Chaotic systems can be designed that generate a phase space set of a given fractal form.

Fractal vs Chaos

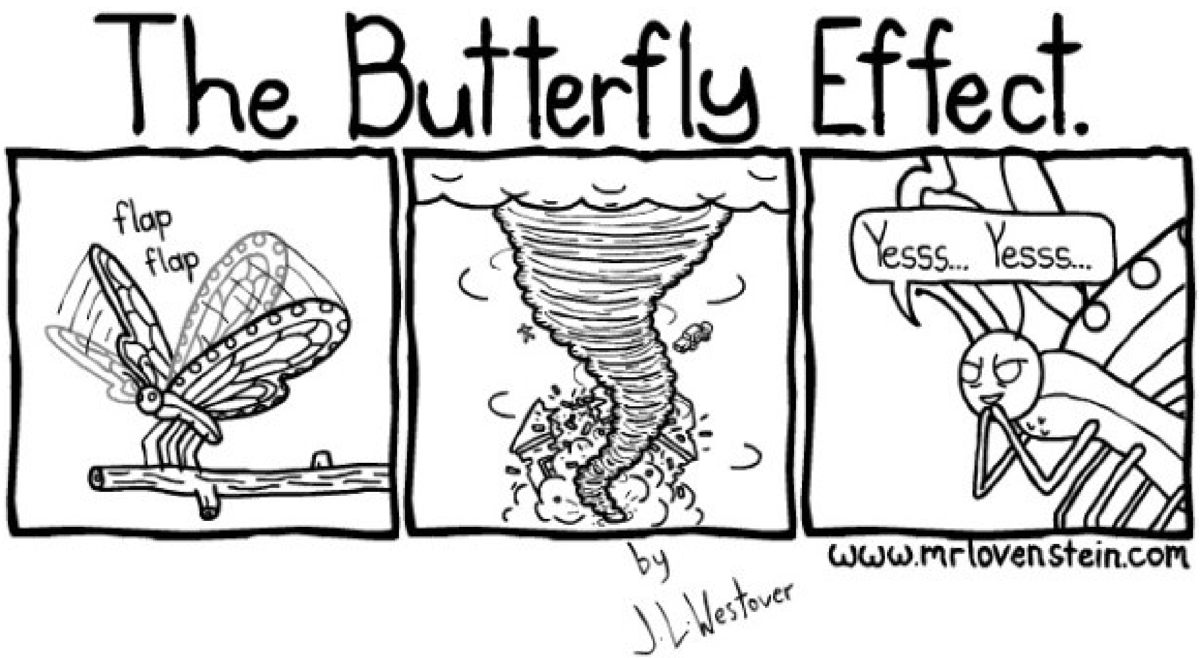
It is important to remember that the objects and processes studied by fractals and chaos are essentially different.

Fractals are objects or processes whose small pieces resemble the whole.

The goal of fractal analysis is to determine if experimental data contain self-similar features and if so, to use fractal methods to characterize the data set.

Chaos means that the output of a deterministic nonlinear system is so complex that it mimics random behavior.

The goal of chaos analysis is to determine if experimental data are due to a deterministic process, and if so, to determine the mathematical form of that process.



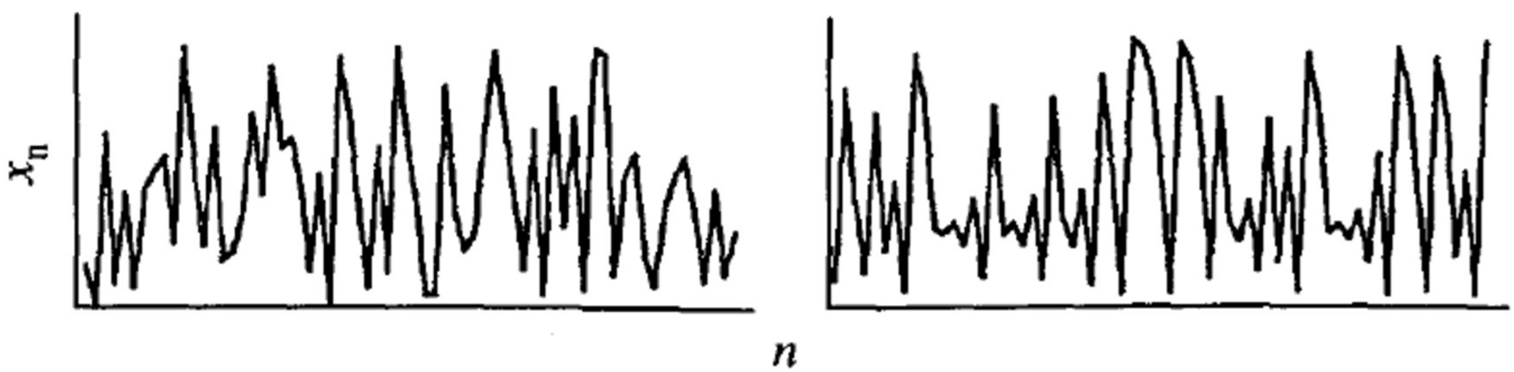
Properties of Chaos

- Exponential Divergence
  - Divergence is the phenomenon of trajectories of a system that begin with similar initial conditions ending up with very different trajectories.
  - This is the opposite of convergence, in which systems tend towards the same value over long periods of time
  - While non, chaotic systems may show divergence, only chaotic systems have trajectories that diverge exponentially.

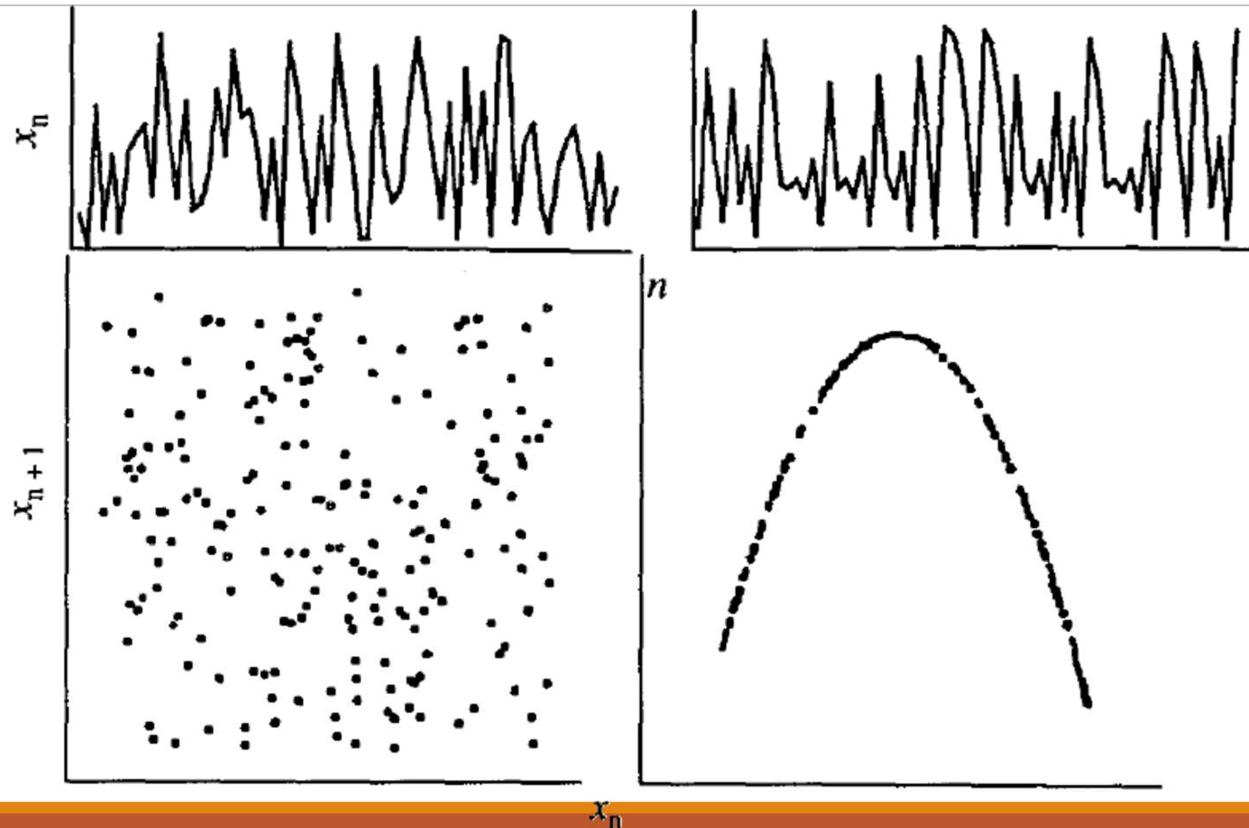
Properties of Chaotic Phenomena

Consider 2 time series. They have approximately the same statistical properties (i.e. similar means and variances)

- both look random



Plot X_n vs X_{n+1}



Chaos

- some systems are deterministic but the output is so complex that it mimics random behaviour

- jargon word is “chaos” which is unfortunate as day-to-day usage this means disordered!

- in mathematical systems chaos means ordered (but complex)

- need to differentiate between “chaos” and “noise”

Defining Properties of Chaos

1) Aperiodic
   - system is never repeated, ever

2) Bounded
   - system stays within a finite range and does not approach $\pm\infty$

3) Deterministic
   - in principle, x0 can be used to calculate all future values of xt

4) Sensitivity to initial conditions
   - two points that are initially close at t=0 will drift apart as time proceeds

1. A chaotic system is a deterministic dynamical system

(i.e. values of the variables that describe the system in the future are determined by the present values.

- e.g. third-order, single-variable equation for a nonlinear damped spring with sinusoidal forcing (Duffing equation):

$$\frac{d^2x}{dt^2}+\gamma\frac{dx}{dt}+\alpha x+\beta x^3=\mathrm{B}\cos(\omega t)$$

CONSTANTS

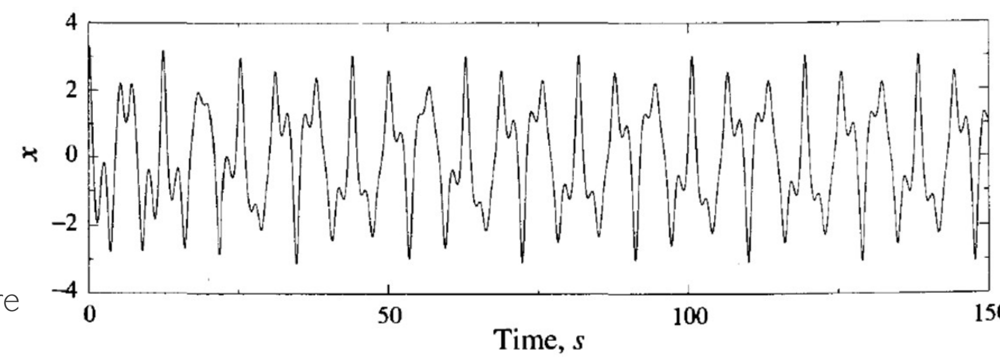
γ = 0.05: magnitude of damping (i.e. Friction)

α = 1: magnitude of restoring force

β =1: Amount of non-linearity in restoring force. If β=0 describes a damped and driven simple harmonic oscillator.

B=7.5: controls amplitude of periodic driving force. If γ=0 there is no driving force

ω =1: frequency of periodic driving force



2. A chaotic system is described by either difference or differential equations

- A difference equation has values where variables at the next time step are a function of their current values.

- Values of variables computed at discrete steps (i.e. logistic equations are one-variable difference equations that can demonstrate chaotic behavior).

- A differential equation, has values of variables changing continuously in time. The values of future variables depend on current values and the derivatives of their current values.

- A chaotic system can consist of a single equation with one variable if it is discrete, or a set of coupled equations with more than three variables if it is continuous (e.g. Duffing equation)

3. A chaotic system has sensitivity to initial conditions.

- values of the variables after a given time depend on their initial values.

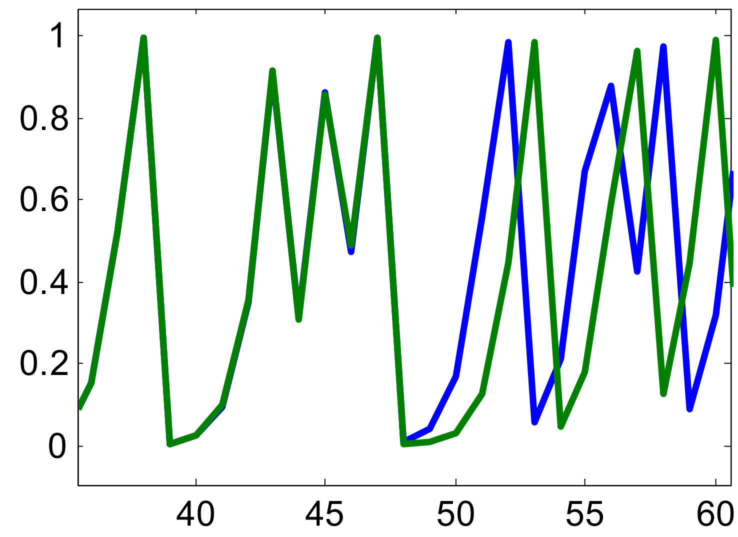
- very small changes in these initial values produce very large changes in later values. i.e. if initial values at time t=0 were $x1\,(t = 0)$ and $x2\,(t = 0)$, then after a time t:

where λ is called the Lyapunov exponent. Thus, the difference in the values diverges exponentially quickly in time.

- A system displaying this sensitivity to initial conditions is said to be "chaotic."

Sensitivity to Initial Conditions

- Since the divergence is exponential, a plot of the log divergence should give a straight line
- The slope of the line gives an estimate of the Lyapunov exponent, a measurement of how quickly the divergence happens



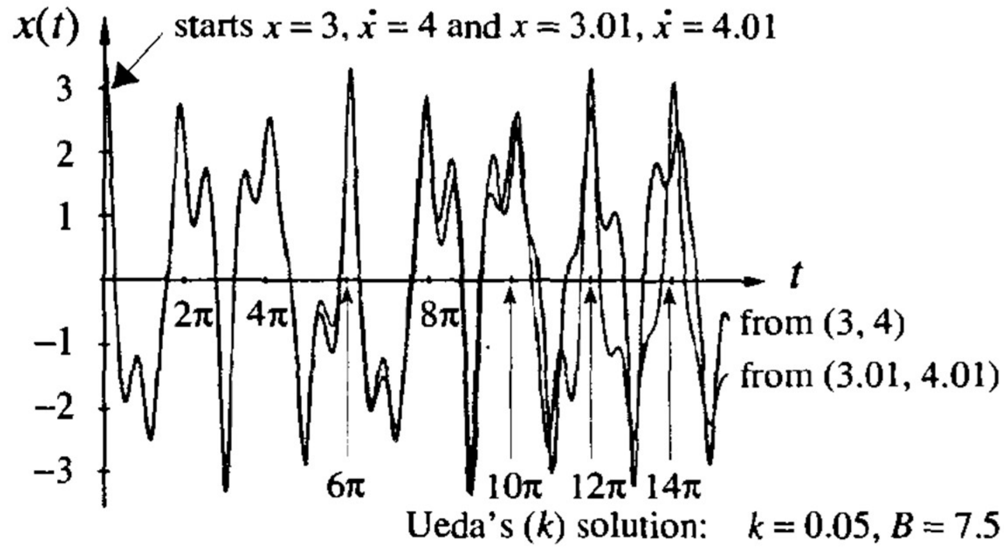
Sensitivity to Initial Conditions

We can only specify values of initial conditions to finite precision

- accuracy is continually lost.
- we cannot predict exactly their values over a long time.

Therefore:

- although the chaotic system is fully deterministic,
  it is not predictable in the long run, because of the sensitivity to initial conditions.



4. The values of the variables are not predictable in the long run.

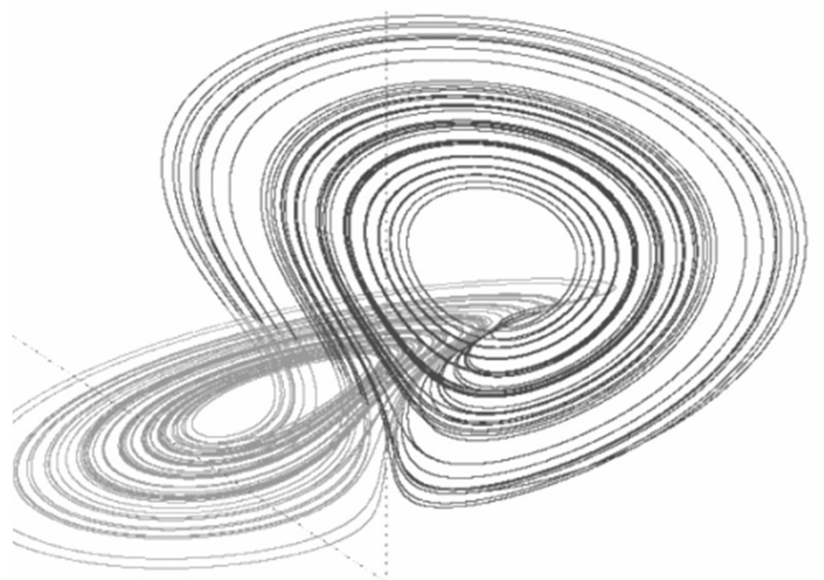
- Because of the sensitivity to initial conditions, values computed [as time goes by] will diverge ever further from their true values based on their exact initial values.

e.g. if values of initial conditions are specified with 5 digits, in time the accuracy of their values will fall to 4, then 3, then 2, digits.

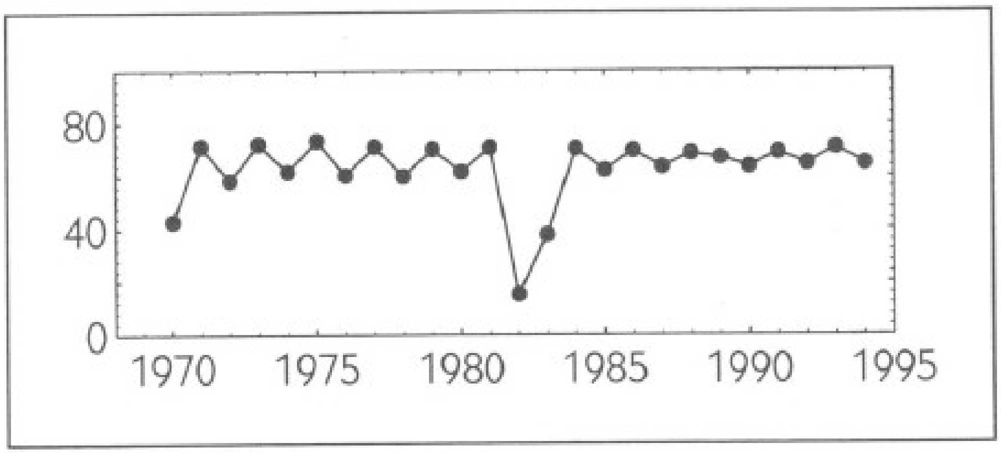
5. The values of the variables do not take on all possible values.

- In the long run, although the values of the variables seem to fluctuate widely, they do not take on all combinations of values.

- This restricted set of possible values is called the attractor.



Finite Difference Equations



$$N_{t+1} = f(N_t)$$

Equations of this form, which relate values at discrete times (e.g., each May), are called finite-difference equations.

$$N_1 = RN_0,$$

$$N_2 = RN_1 = R^2 N_0,$$

$$N_3 = RN_2 = R^2 N_1 = R^3 N_0,$$

$$\vdots$$

$$N_t = R^t N_0$$

Verification:

$$N_{t+1} = R^{t+1} N_0 = RR^t N_0 = RN_t$$

Example: Different Constants

Figure 1.2
The solution to
$N_{t+1} = 0.90N_t$.

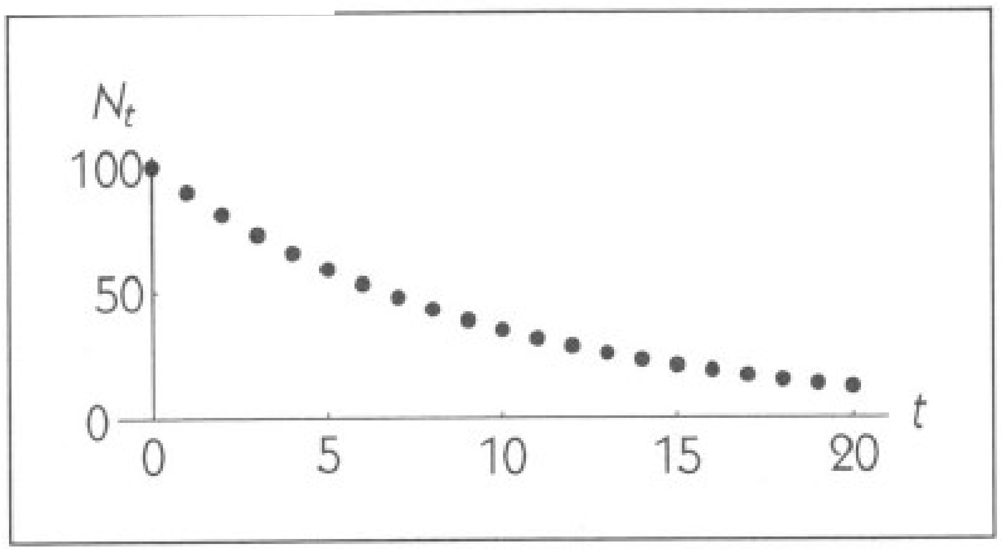
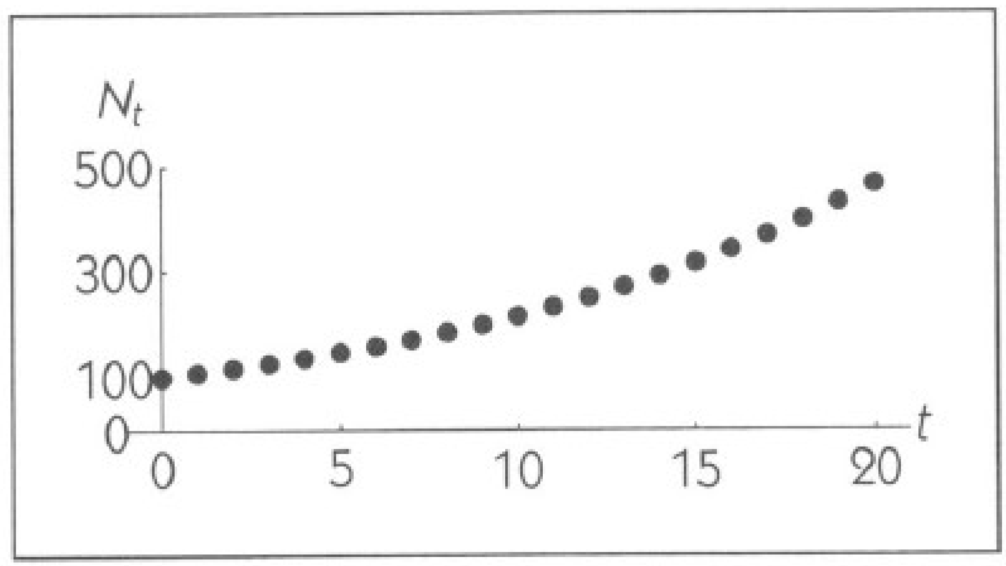


Figure 1.3
The solution to
$N_{t+1} = 1.08N_t$.



Example: Different Constants

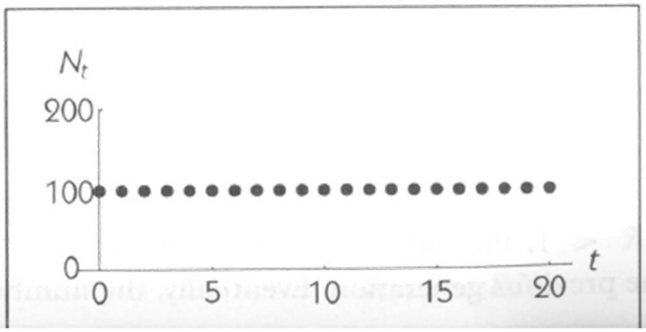


Figure 1.4 The solution to $N_{t+1} = 1.00N_t$.

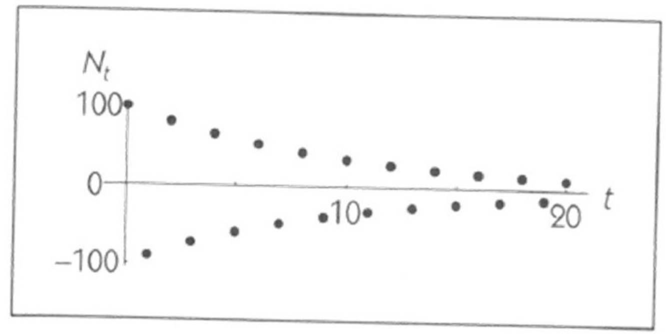


Figure 1.5 The solution to $N_{t+1} = -0.90N_t$.

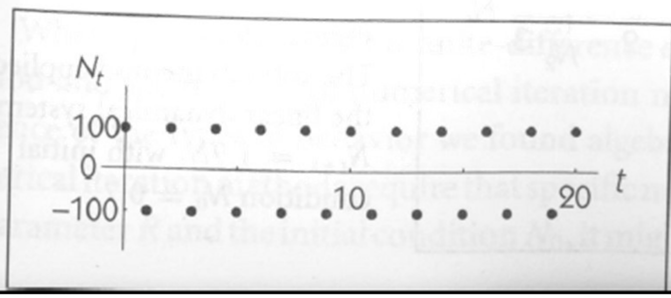


Figure 1.7 The solution to $N_{t+1} = -1.00N_t$.

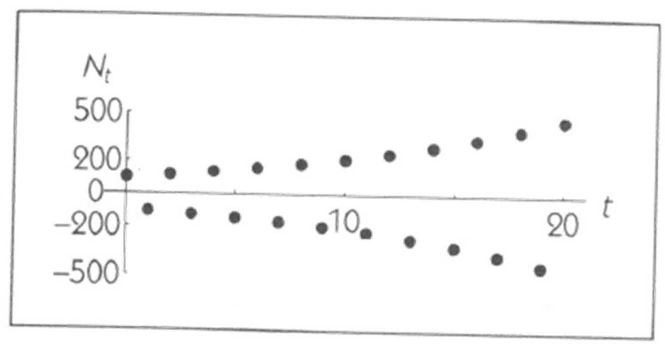


Figure 1.6 The solution to $N_{t+1} = -1.08N_t$.

Diverging

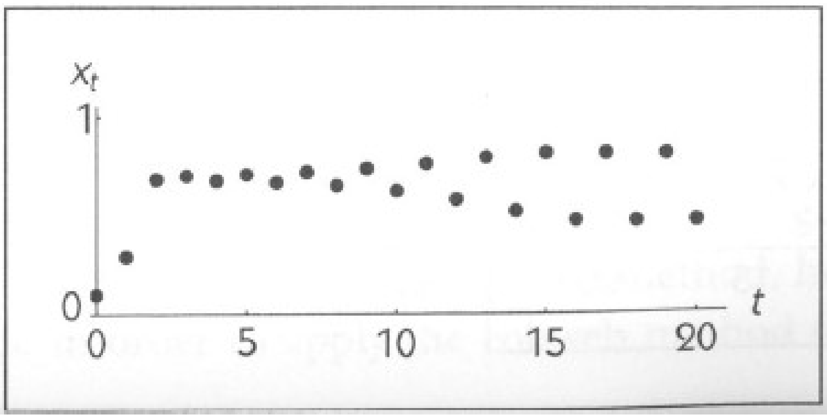


Figure 1.13
The solution to
$x_{t+1} = 3.3(1 - x_t)x_t$.

Approaching Chaos

Figure 1.14
The solution to
$x_{t+1} = 3.52(1 - x_t)x_t$.

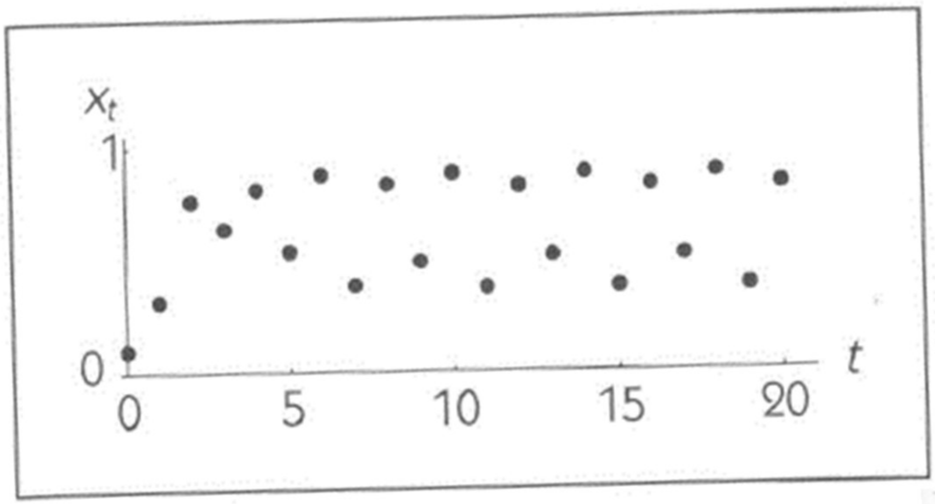
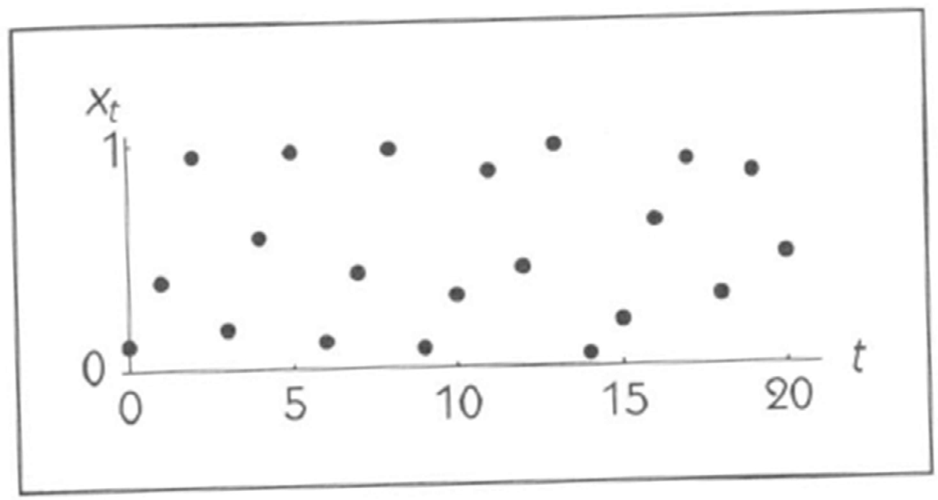


Figure 1.15
The solution to
$x_{t+1} = 4(1 - x_t)x_t$.



Aperiodic Behaviour

– now we have reached chaos

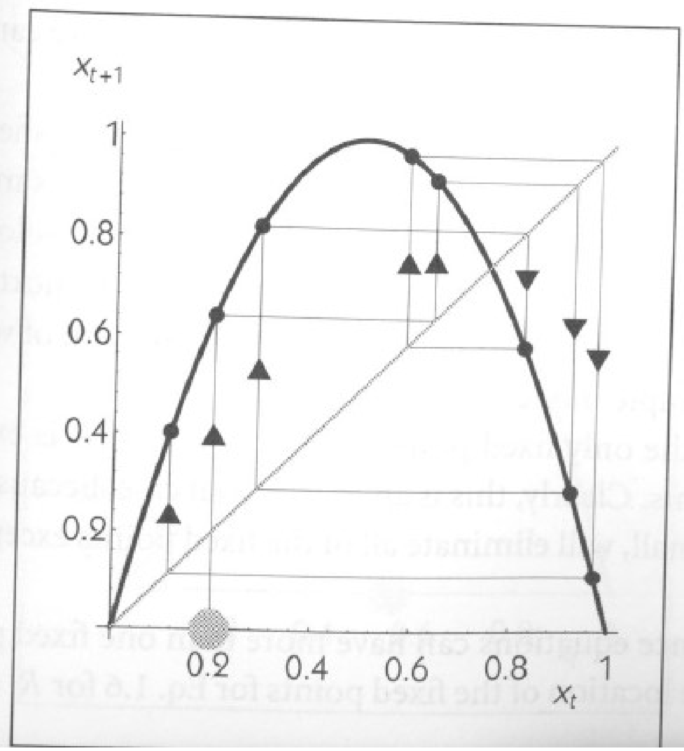


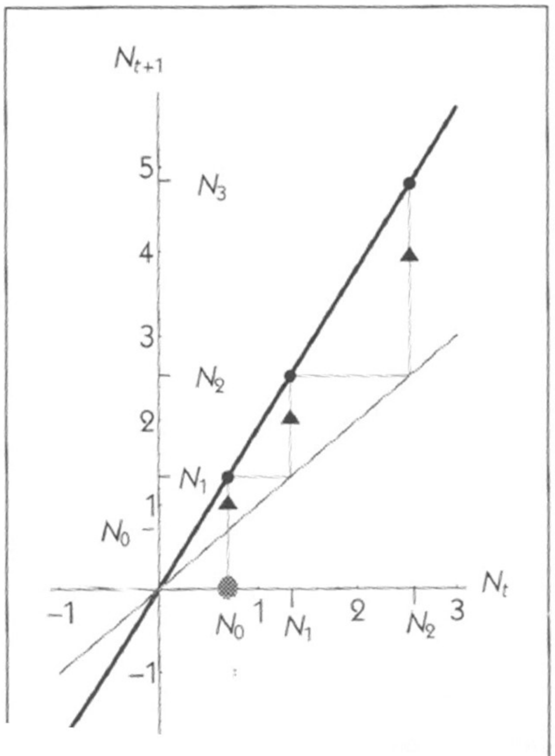
Figure 1.16 Cobweb iteration of $x_{t+1} = 4(1 - x_t)x_t$.

Methods of Iteration: Numerical

$N_{t+1}=0.9N_t$ with $N_0=100$

$$
\begin{aligned}
N_0 &= 100, \\
N_1 &= f(N_0) = 0.9 \times 100 = 90, \\
N_2 &= f(N_1) = 0.9 \times 90 = 81, \\
N_3 &= f(N_2) = 0.9 \times 81 = 72.9, \\
&\vdots
\end{aligned}
$$

Methods of Iteration: Cobweb Plot



$N_{t+1}=1.9N_t$ with $N_0=0.7$

Nonlinear Finite-Difference Equations

In the linear equation, R is the number in each generation t

- to make the number of subsequent generations decrease as Nt gets larger, make the growth rate a function of Nt

- A simple solution is the function (R – bNt)

- b governs how growth rate decreases as the total gets larger

- R is the growth rate when the population is very, very small.

This assumption that the number is (R - bNt) gives a new finite-difference equation:

Nonlinear Finite-Difference Equations

Here there are 2 parameters, R and b, that can vary independently.

- However, a simple change of variables shows that there is only one parameter that affects the dynamics:

- Define a new variable xt=bNt/R which is just a way of scaling the total number by b/R

- Substituting xt and xt+1 results in:

- A solution cannot generally be found using algebra. Hence, numerical iteration and the cobweb method, is needed to find solutions.

Steady States and their Stability

A simple type of dynamical behavior is when the system stays at a steady state. A steady state is a state of the system that remains fixed, that is, where:

Steady states in finite-difference equations are associated with the mathematical concept of a fixed point.

Fixed Points

There are 3 important questions to ask about fixed points in finite difference equations:

1. Are there any fixed points?

2. If the initial condition happens to be near a fixed point, will subsequent iterates approach the fixed point? If subsequent iterates approach the fixed point, we say the fixed point is locally stable.

3. Will the system approach a given fixed point regardless of the initial conditions? If the fixed point is approached for all initial conditions, we say that the fixed point is globally stable.

Finding Fixed Points

- A fixed point of a function $f(xt)$ is a value xt^* that satisfies $xt^*=f(xt^*)$

- From the graph of $xt+1= f(xt)$ it is easy to locate fixed points: These are the points where the graph intersects the line $xt+1=xt$

- Or, can solve the equation $xt=f(xt)$

- For a linear finite-difference equation, xt^* is a fixed point if it satisfies $xt^*=Rxt^*$

- One solution is always $xt^* =0$ (i.e. the origin is a fixed point for a linear system).

Finding Fixed Points - Example

- The solution xt = 0 is the only fixed point, unless R = 1

- If R is exactly 1, then all points are fixed points

- Clearly, this defines an exceptional case, because any change in R will eliminate all fixed points (except the one at the origin)

- non-linear finite difference equations can have >1 fixed points.

$$x_{t+1} = x_t$$

Cobweb Plot

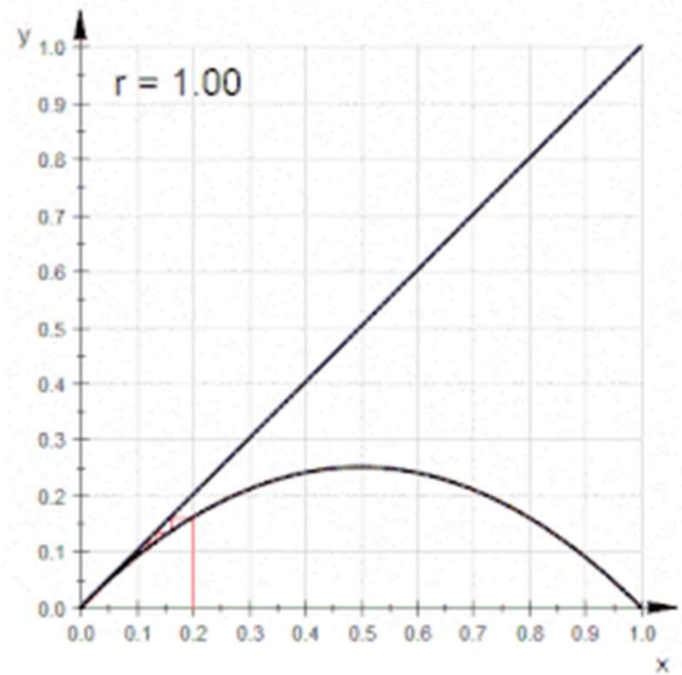
used investigate the qualitative behaviour of 1D iterated functions

possible to infer the long-term status of an initial condition under repeated application

Assess Stability

a stable fixed point → inward spiral

unstable fixed point →outward spiral

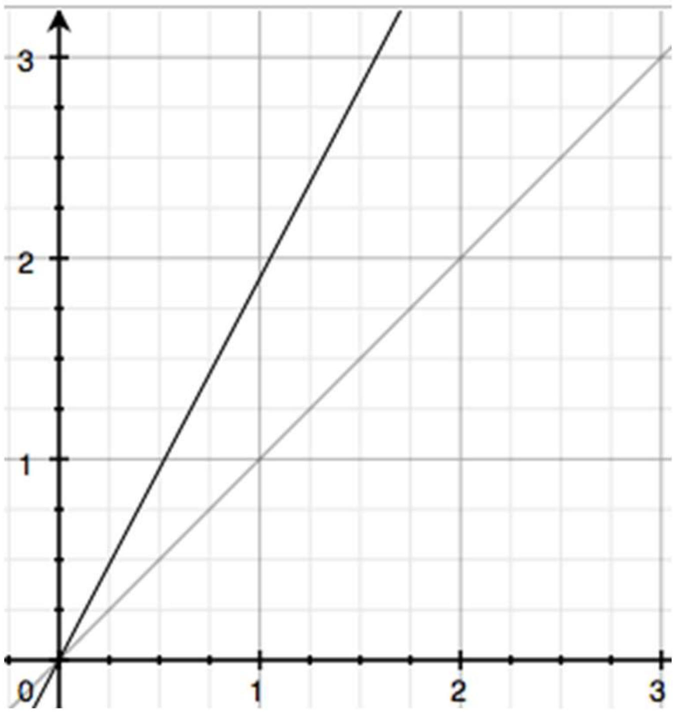


The Cobweb Method

- a graphical method for iterating finite-difference equations

e.g. linear system of: $N_{t+1}=RN_t$

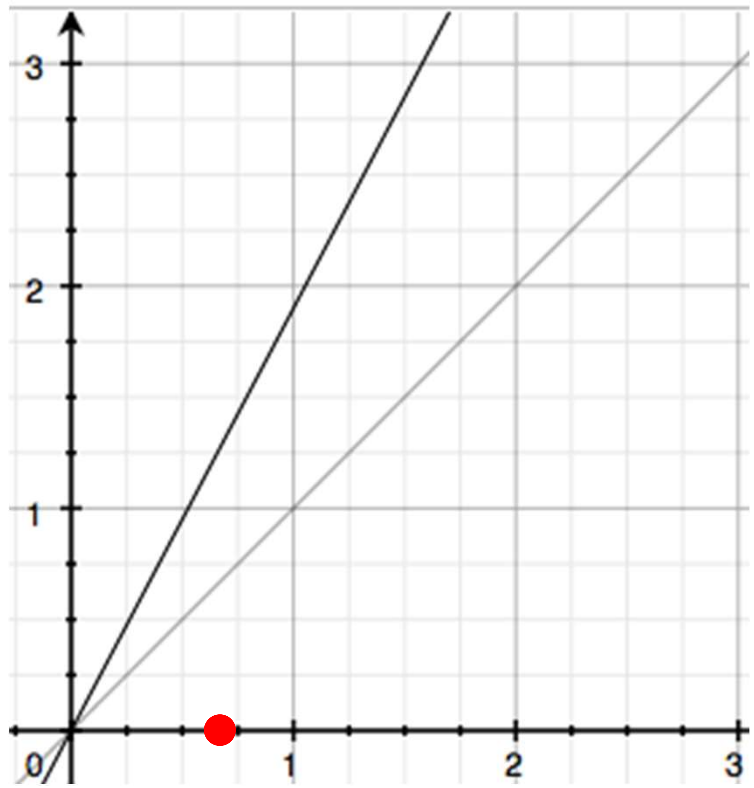
1. Graph the function. Here it is $f(N_t)=RN_t$. Pick a specific value for R (e.g. 1.9), so that the finite difference equation is $N_{t+1}=1.9N_t$



The Cobweb Method

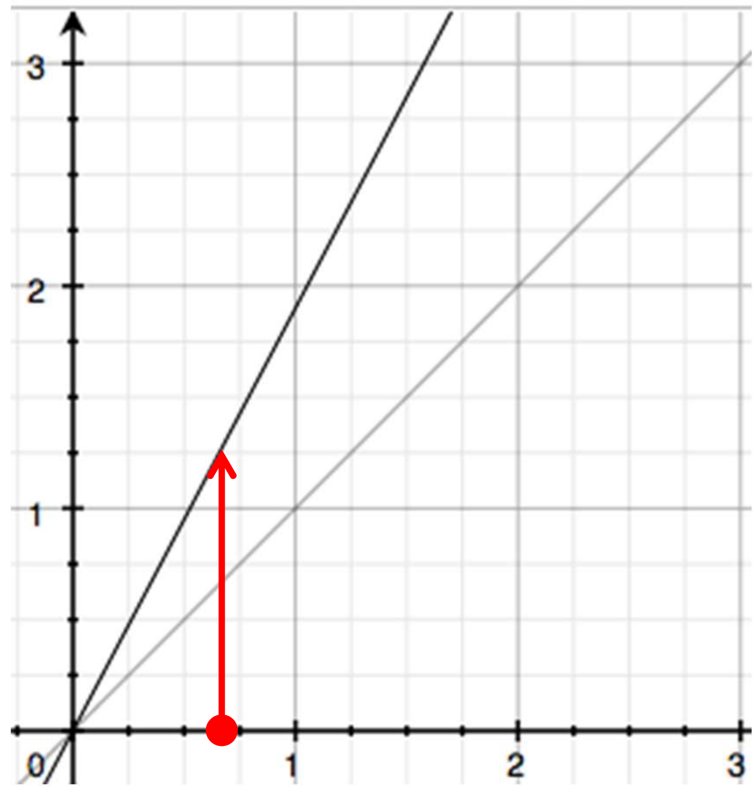
2. Pick an initial condition N0)

In this example = 0.7, shown as the red dot on the x-axis below



The Cobweb Method

3. Draw a vertical line from N0 on the x-axis up to the function. The position where this vertical line hits the function (shown as a solid dot at the end of the arrow) tells us the value of N1



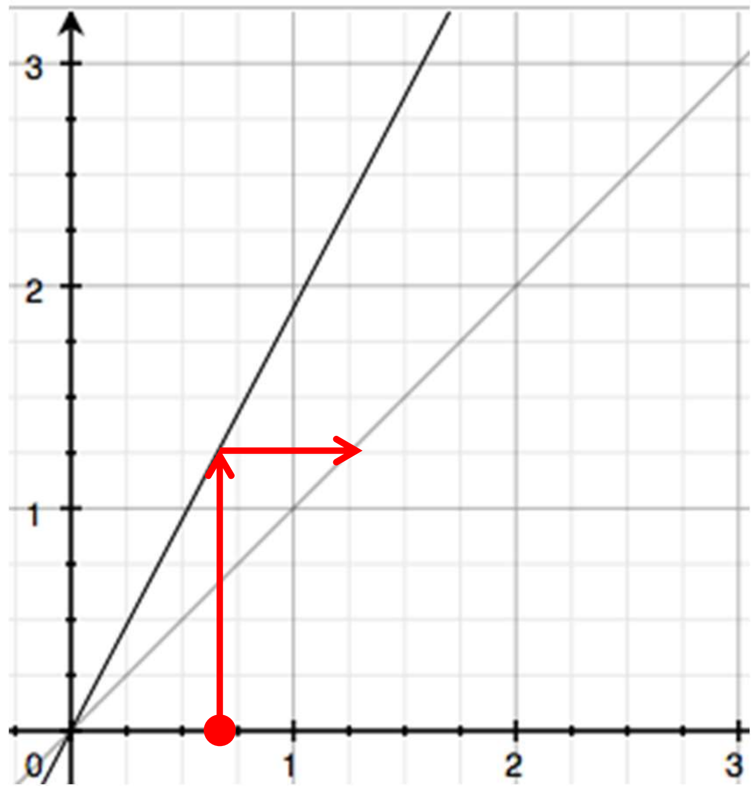
The Cobweb Method

4. Take this value of N1, plot it again on the x-axis, and again draw a vertical line to find the value of N2.

Simple shortcut:

- Draw a horizontal line to the Nt+1=Nt line.

- The place where the horizontal line intersects the 45-degree line is the point from which to draw the next vertical line to find N2

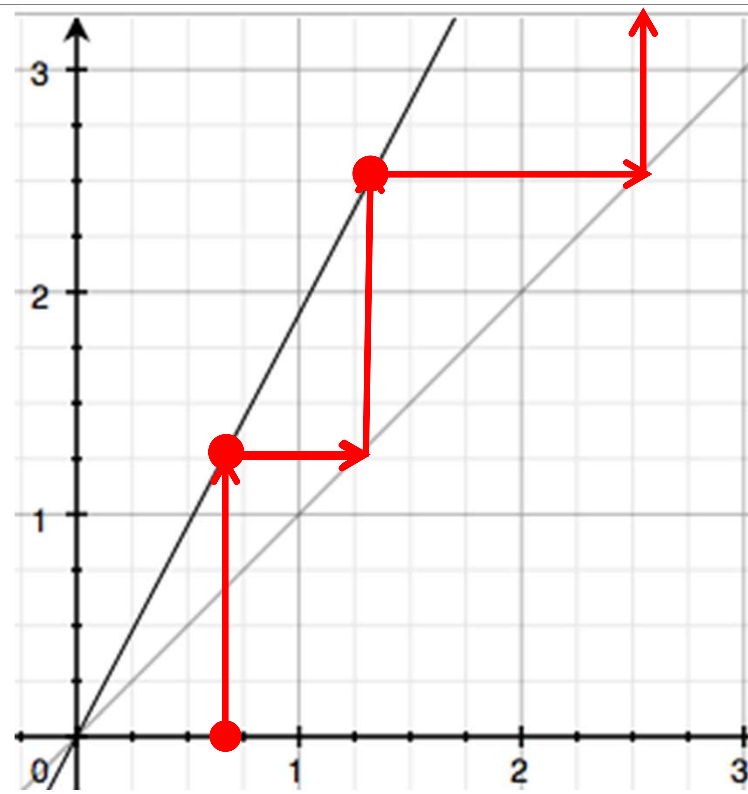


The Cobweb Method

5. To find N3, N4, etc., repeat the process of drawing vertical lines to the function and horizontal lines to the line of Nt+1 = Nt

- The result of iterating Nt+1 = 1.9Nt is growth towards ∞

- This is consistent with the algebraic solution found previously for R > 1



Cobweb Method - Nonlinear

In order to apply the cobweb method to we first must draw a graph of the function:

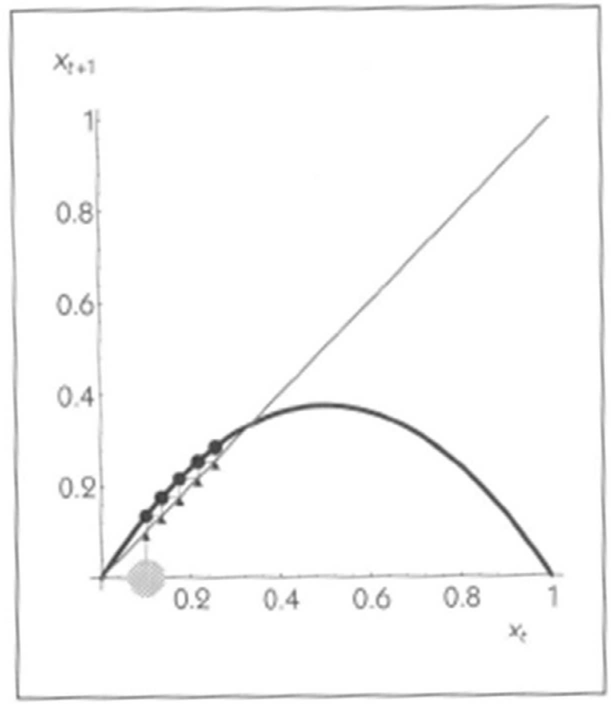
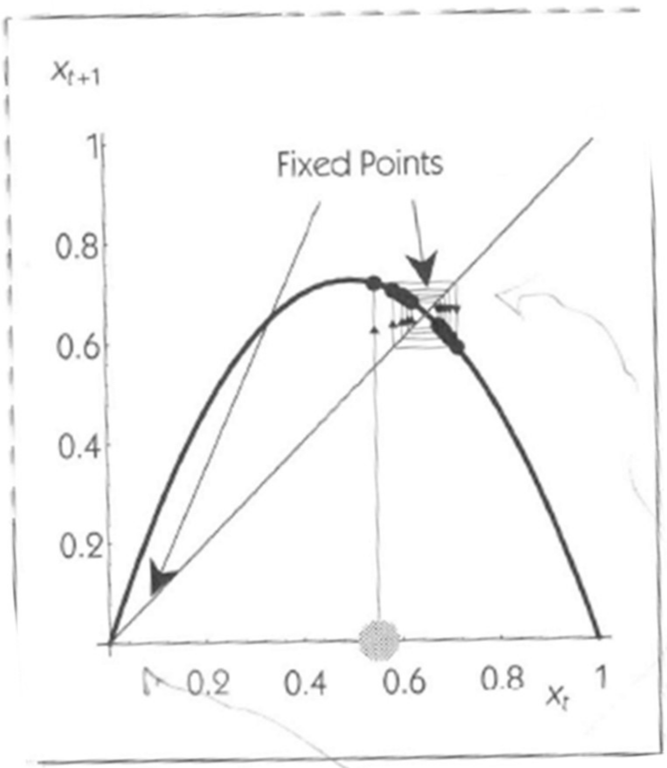
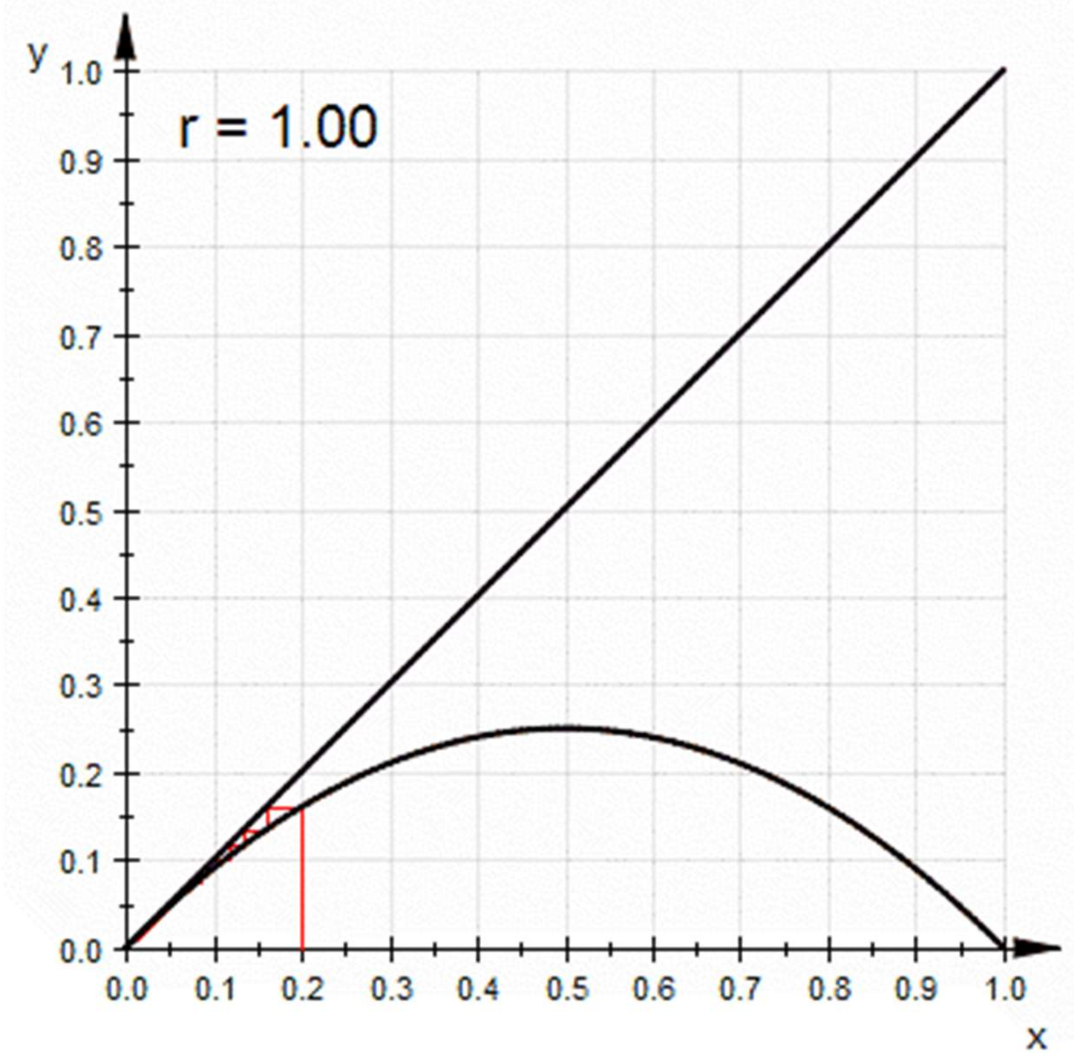


Figure 1.9
Cobweb iteration of $x_{t+1} = 1.5(1 - x_t)x_t$.

Finding Fixed Points



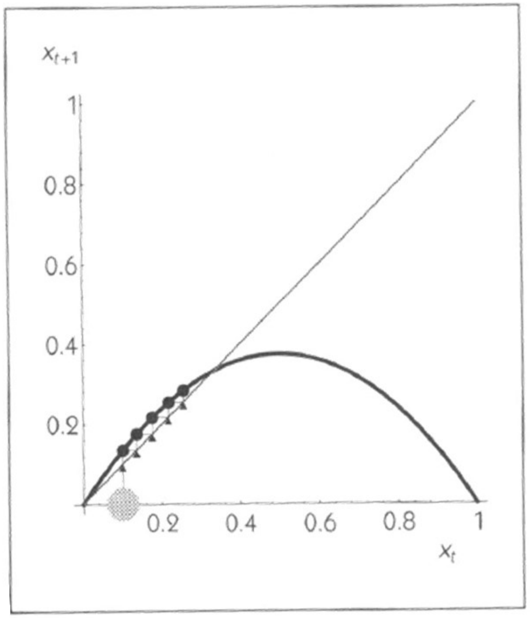
$$x_{t+1} = 2.9\left(1 - x_t\right)x_t$$



y
r = 1.00
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
x

Non-linear Finite-Difference Equations

$$N_{t+1} = (R - bN_t)N_t = RN_t - bN_t^2$$



Steady State

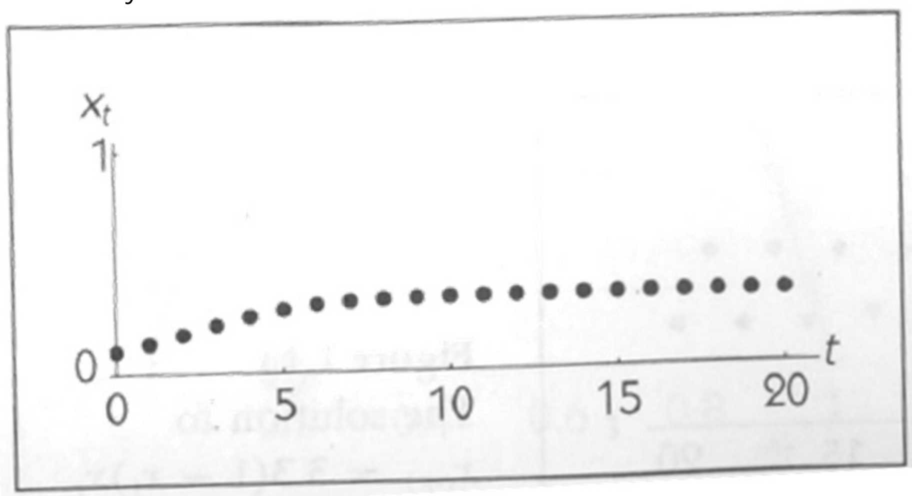
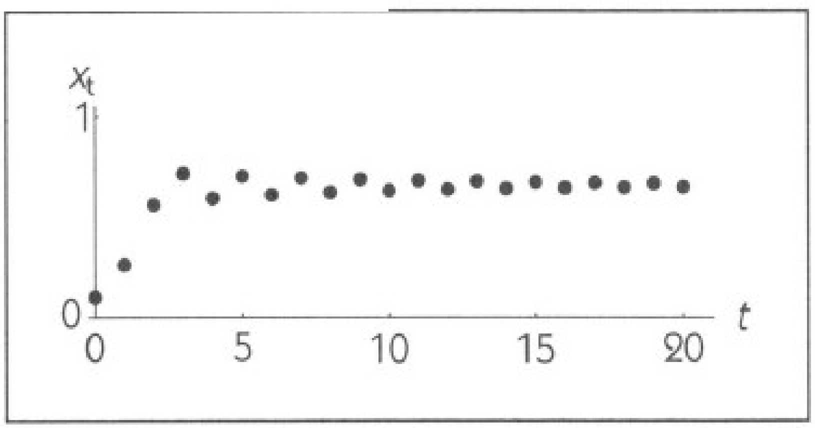


Figure 1.10
The solution to
$x_{t+1} = 1.5(1 - x_t)x_t$

Periodic Cycles

Figure 1.11
The solution to
$x_{t+1} = 2.9(1 - x_t)x_t$.



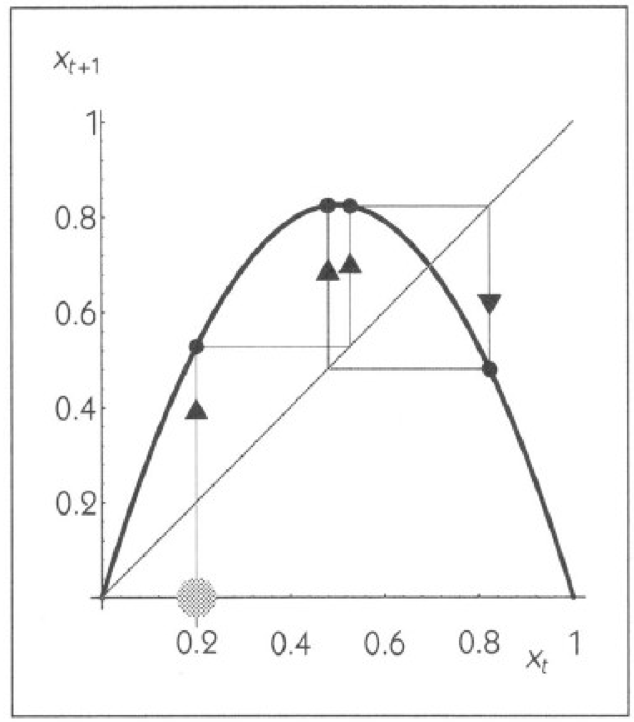


Figure 1.12
Cobweb iteration of
$x_{t+1} = 3.3(1 - x_t)x_t$.

Example

- Cells reproduce by division

- One way to regulate the rate of reproduction of cells is by regulating mitosis.

- There is biochemical evidence that there are compounds, called chalones, that are tissue-specific inhibitors of mitosis

- For simplicity, assume that the generations of cells are distinct and that the number of cells in each generation is given by Nt

Example

- Following the same logic as before, assume that for each cell in generation t, there are R cells in generation t+1. (If every cell divided in half every time step, then R would equal 2.)

- The finite difference equation describing this situation is the linear equation $N_{t+1} = RN_t$, which leads either to exponential growth or to decay to zero.

- A possible role of chalones is to make R depend on the number of cells.

- Assume that the amount of chalones produced is proportional to the number of cells. The more chalone there is, the greater the inhibitory effect on mitosis

Example

The biochemical action of chalones is to bind to a protein involved in mitosis, rendering the protein inactive.

Binding of molecules to proteins is often modeled by a Hill function

We will assume that $n \geq 2$ and $R=2$, $\theta=5$, and $n=3$.

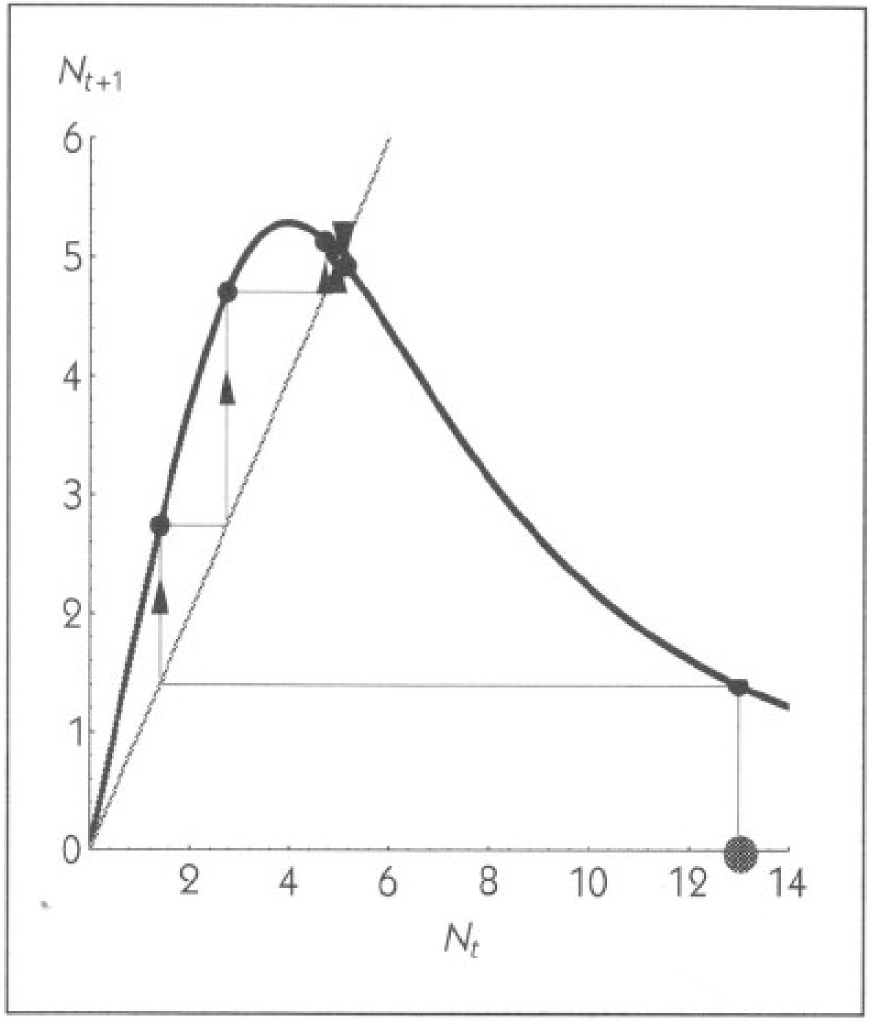
Find the fixed points of this system and determine their stability

Example

1. To determine the fixed points we solve the equation

$$N^{\star} = \frac{RN^{\star}}{1 + \left(\frac{N^{\star}}{\theta}\right)^{n}}$$

These are the only fixed points. There are also imaginary solutions that can be ignored in this case because we are only concerned with biologically meaningful solutions, and the number of cells in each generation must be real numbers


N_{t+1}
6
5
4
3
2
1
0
2
4
6
8
10
12
14
N_t

Example – Fixed point stability

2. To determine the stability of the fixed points you have to computer the slope at the fixed points. If we take the derivative we find:

3. From the above equation we find that the slope at the fixed point $N^* = 0$ is just R.

- If $R > 1$, the fixed point at the origin is always unstable. (To be a plausible model of the regulation of cell reproduction, we must have $R > 1$.

- Otherwise, the population would always fall to zero even in the complete absence of the mitosis-inhibiting chalones.)

Example – Fixed point stability

The slope at the fixed point $N^* = \theta(R-1)^{\frac{1}{n}}$ is

$$\left.\frac{df}{dN_t}\right|_{N^*} = 1 + n\left(\frac{1}{R} - 1\right).$$

For R = 2, the fixed point will be unstable when n > 4 and stable otherwise.

- Local stability tells us whether the fixed point is approached if the initial condition is sufficiently close to the fixed point.

- can be assessed simply by looking at the slope of the function at the fixed point.

A more difficult-question is whether a locally stable fixed point is globally stable.

Fixed Point Stability

For linear finite-difference equations a locally stable fixed point is also globally stable:

i.e. regardless of initial conditions, the iterates will eventually reach the locally stable point (i.e., the origin) from any initial condition.

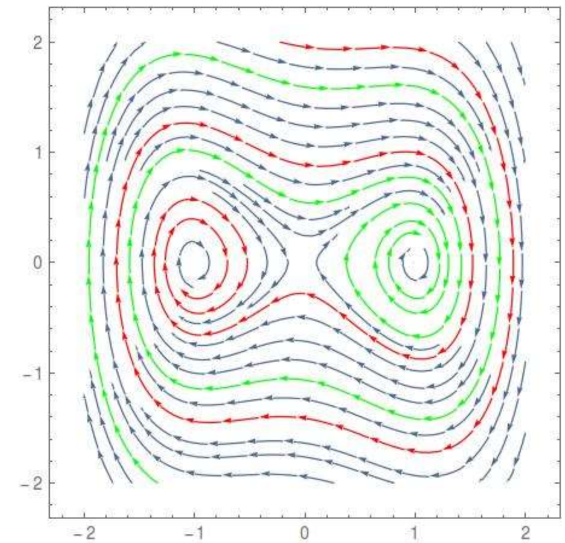
Stability and dynamics at fixed points

- $|df/dX| > 1$: Unstable (doesn't converge at fixed point)
- $0 < df/dX < 1$: Stable, monotonically approach to fixed point (converges)
- $-1 < df/dX < 0$: Stable, oscillatory approach to fixed point (converges)

Fixed Point Stability

- For nonlinear finite-difference equations, there can be more than one fixed point. AND when multiple fixed points are present, none of the fixed points can be globally stable.

The set of initial conditions that eventually leads to a fixed point is called the basin of attraction of the fixed point.



Often, the basin of attraction for fixed points in nonlinear systems can have a very complicated geometry

If multiple fixed points are locally stable we say there is <u>multistability.</u>

The Period-Doubling Route to Chaos

We have seen that the simple finite-difference equation:

$$N_{t+1} = (R - bN_t)N_t = RN_t - bN_t^2$$

can display various qualitative types of behavior for different values of R:

1) steady states; 2) periodic cycles of different lengths; 3) chaos.

The change from one form of qualitative behavior to another as a parameter is changed is called a bifurcation.

An important goal in studying nonlinear finite-difference equations is to understand the bifurcations that can occur as a parameter is changed!

Feigenbaum

- For $3.0000 < R < 3.4495$, there is a stable cycle of period 2.
- For $3.4495 < R < 3.5441$, there is a stable cycle of period 4.
- For $3.5441 < R < 3.5644$, there is a stable cycle of period 8.
- For $3.5644 < R < 3.5688$, there is a stable cycle of period 16.
- As R is increased closer to 3.570, there are stable cycles of period 2^n, where the period of the cycles increases as 3.570 is approached.
- For values of $R > 3.570$, there are narrow ranges of periodic solutions as well as aperiodic behavior.

Feigenbaum's Number

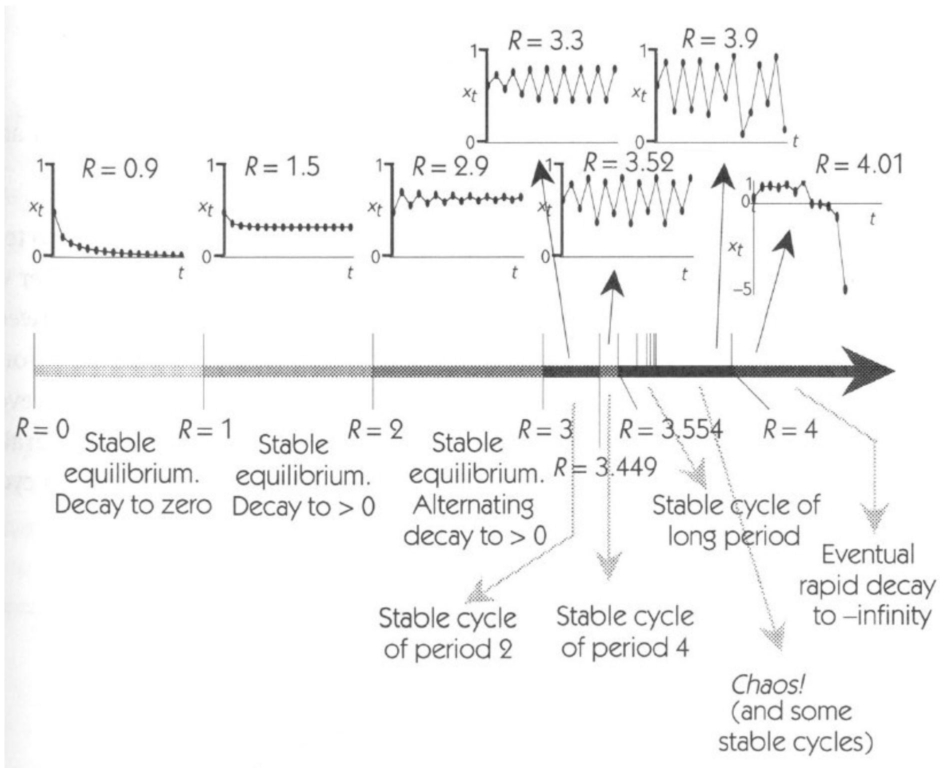
This illustrates a sequence of period-doubling bifurcations at R = 3.0000, R = 3.4495, R = 3.5441, R = 3.5644, with additional period-doubling bifurcations as R increases.

- This transition from the stable periodic cycles to the chaotic behavior at R = 3.570 is called the period-doubling route to chaos.

$$\lim_{n \to \infty} \frac{\Delta_n}{\Delta_{2n}} = 4.6692\ldots.$$

The constant, 4.6692 ... is now called Feigenbaum's number, appearing not only in the simple theoretical models but also in other theoretical models and in experimental systems in which there is a period-doubling route to chaos.

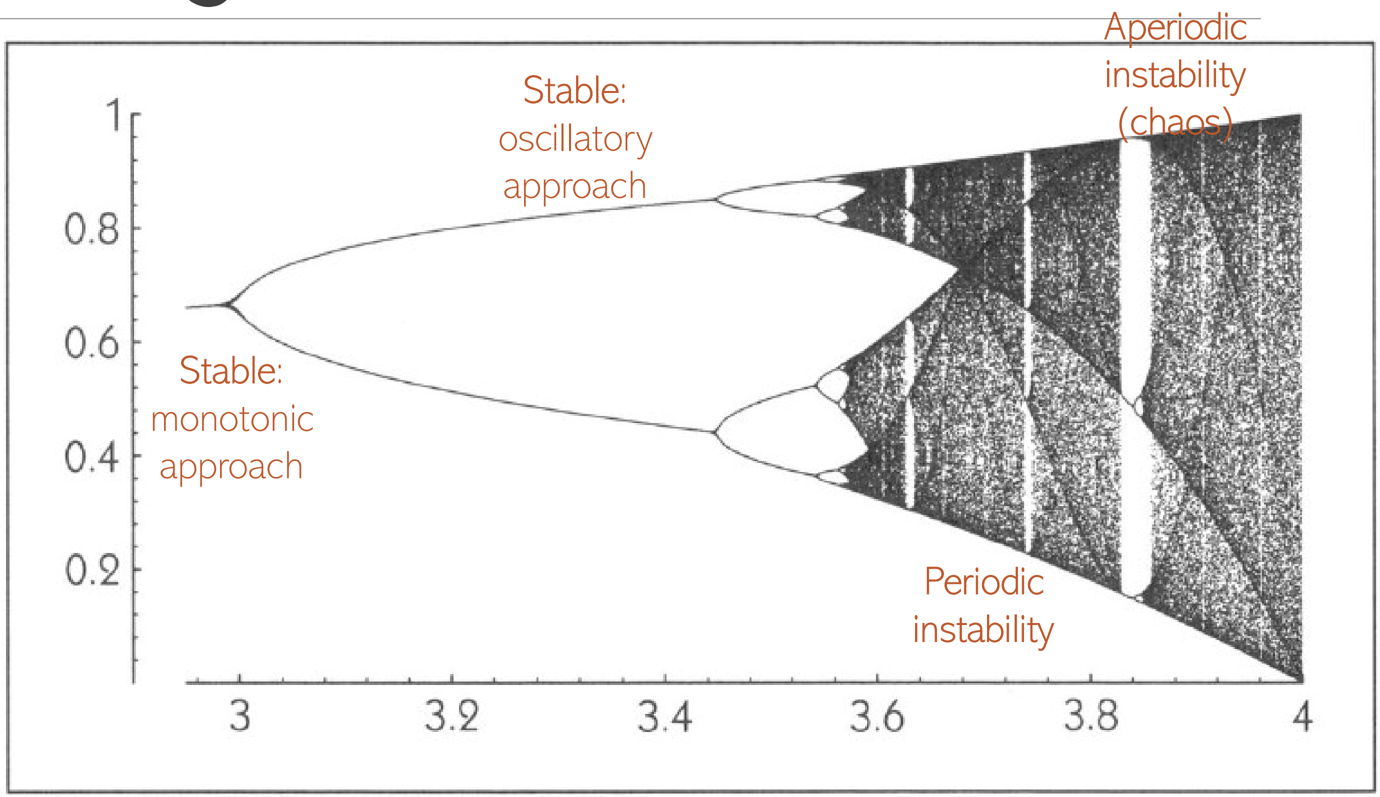
Chaotic Behaviour



Bifurcation Diagram

try matlab code:
feigen(0.5,100,100);

A bifurcation diagram of nonlinear finite difference equation with asymptotic values of Xt plotted vs. R



Non-linear Dynamics

Self-organized criticality (SOC)

- refers to a type of dynamical behavior exhibited by systems with many interacting degrees of freedom where minor fluctuations lead to system rearrangements without characteristic size or time scale.

- rearrangements are referred to as avalanches with power laws describing their size, lifetime, and power spectrum.

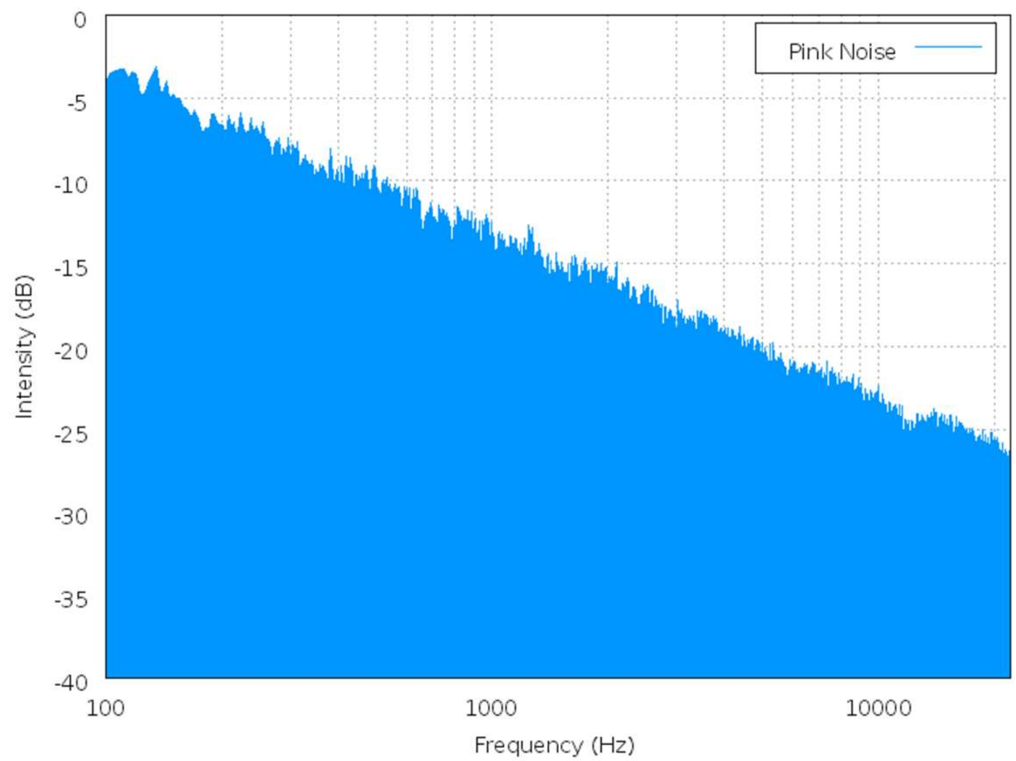
SOC-like behavior is often associated with the formation of fractals in nature

Non Linear Dynamics

- Seen in biological systems, is present in heart beat rhythms, neural activity, mental states (modeling in psychology) and the statistics of DNA sequences.

- Also describes the statistical structure of many natural images (images from the natural environment).

- SOC model was developed by Bak, Tang, and Wiesenfeld (1987), as a possible explanation of noise spectra with $1/f^{\alpha}$ frequency dependence



http://en.wikipedia.org/wiki/File:Pink_noise_spectrum.png

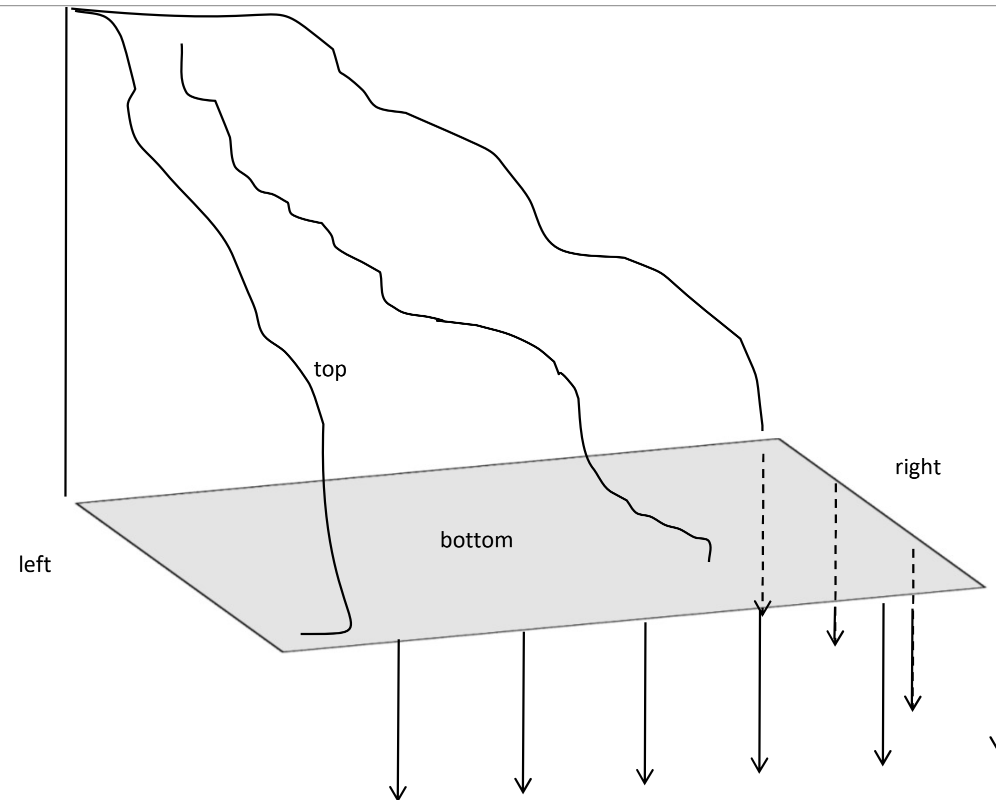
Non Linear Dynamics

- In the classic SOC model, sand grains are numerically added to a rectangular lattice building up the slope of a pile.

- When the sand pile is critically poised, the addition of a single sand grain can result in a domino effect where anywhere from a few grains to a large fraction of the sand pile may be shifted.

- SOC systems evolve into a critical state exhibiting avalanches of all sizes without the requirement of finely tuned parameters.

Sandpile Example



Sandpile Example

- The SOC sand pile model consists of a two-dimensional grid with an integer number zij specified at each grid location (i,j).

- The 'idealized sand' is stacked randomly at each site until the slope at a given site exceeds a threshold value (i.e. zij>zthreshold ; e.g. zij= 4 and zthreshold= 3)

- Sand is then redistributed to the nearest neighbors as such:
  - 1) first decrease the slope of the site (i,j) by four: zij → (zij – 4)
  - 2) The slope of the four nearest neighbors is then increased by one:

$$z_{i\pm1,j} \rightarrow z_{i\pm1,j} + 1$$

$$z_{i,j\pm1} \rightarrow z_{i,j\pm1} + 1$$

Sandpile Example

- This may cause one or more neighboring sites to exceed the threshold, resulting in further rearrangements until all the sites are less than zthreshold.

- The size of the avalanche is given by the total number N sites toppled

- Avalanche duration is given by the total number of time steps the avalanche propagates.

- Here the resulting power law in avalanche size distribution is:

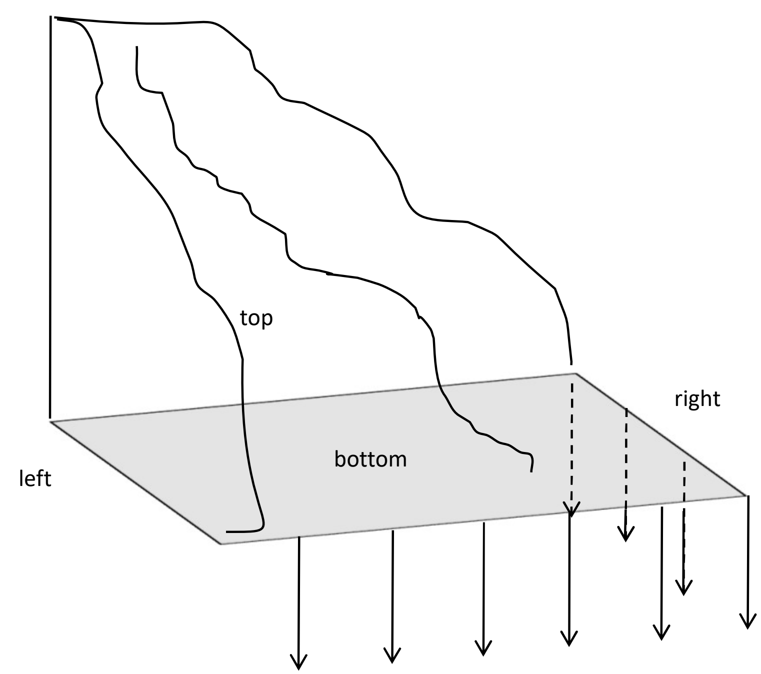
$$N(s) \propto s^{-\tau} \text{ with } \tau \approx 1.1$$

Boundary conditions of the sand pile model

1) Top and left edges are constrained to be zero corresponding to a zero slope of the sand pile at it apex.

2) If a site exceeds threshold on the bottom or the right boundary, then 4 is subtracted from every site on these two boundaries.

3) This simulates sand falling off the edge as indicated by the arrows in the figure.



matlab code: sandpile.m

- size of grid matrix is given by siz (e.g. [20 20]) and nrsteps is a scalar positive integer defining the number of time steps of the model run.

- Each time step a sandgrain is added to a random location on the grid.

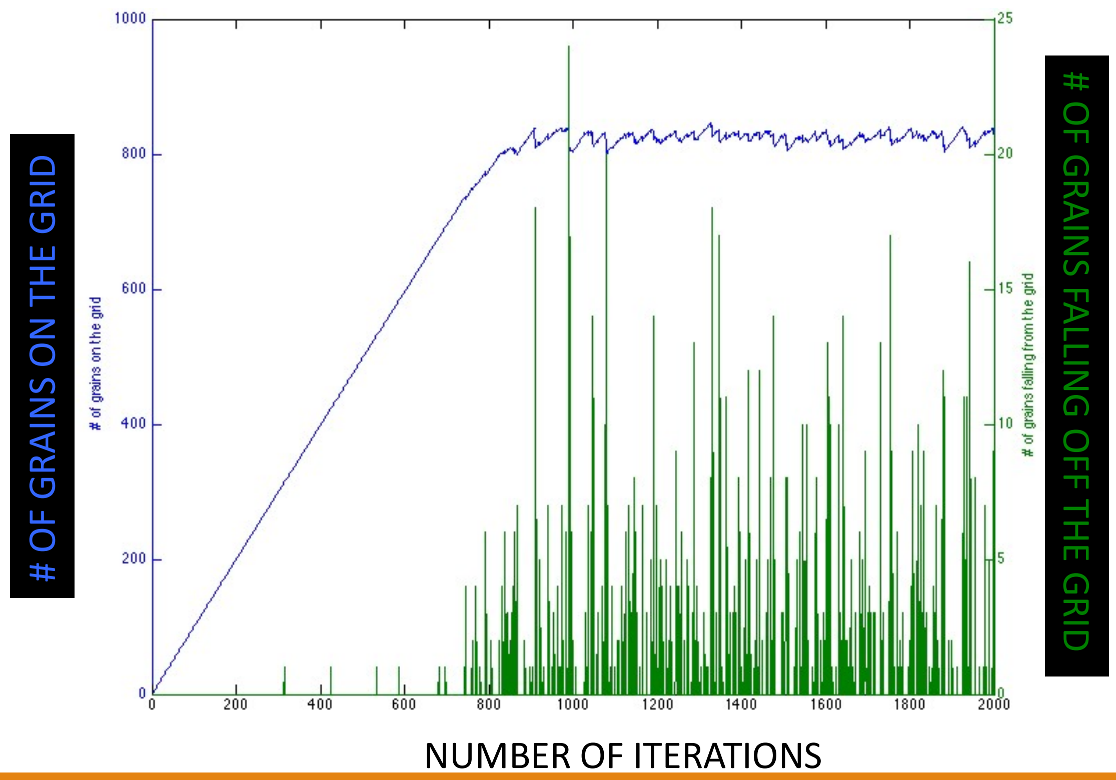
- When the critical number of grains in a cell exceeds 3 all grains in the cell are turned to the 4 neighbors (von Neumann Neighborhood).

This avalanche may prograde and trigger even more avalanches.

- After a while the sandpile comes to a state of

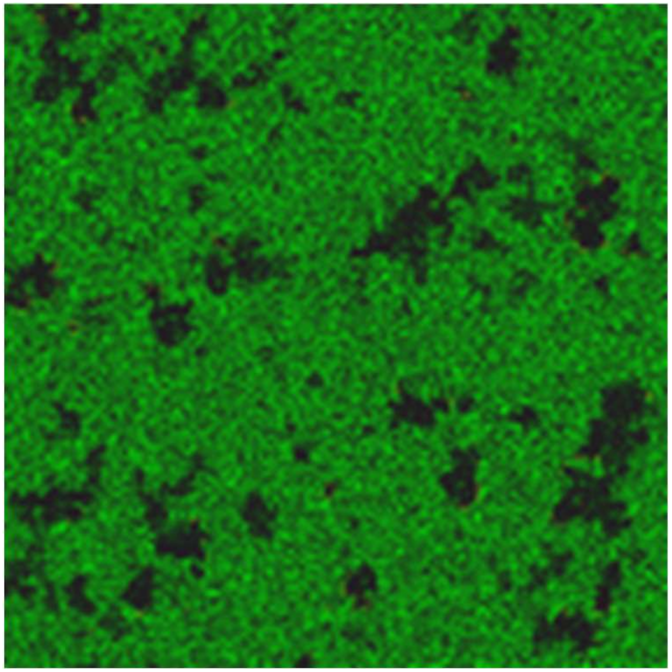
self-organized criticality (SOC).

Matlab code: sandpile.m



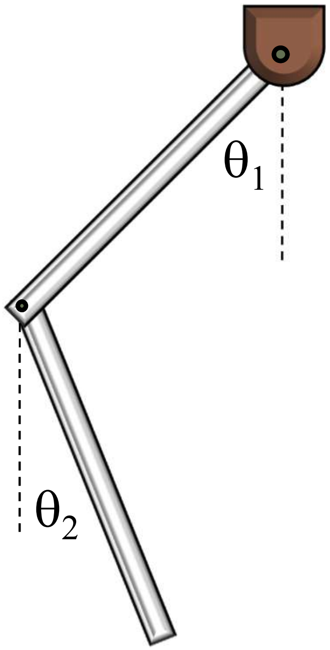
Forest Fire Model

1. A burning cell turns into an empty cell
2. A tree will burn if at least one neighbor is burning
3. A tree ignites with probability f even if no neighbor is burning
4. An empty space fills with a tree with probability p



https://en.wikipedia.org/wiki/Forest-fire_model

Chaotic Systems



Simple double pendulum consisting of two connected rods that are free to pivot at their upper ends.

This system will exhibit chaotic oscillations when set in motion with sufficiently large initial values of q1 and q2.

- Chaotic systems are often described by systems of nonlinear differential equations that do not have analytical solutions.

- Physical processes that are purely random (e.g. radioactive decay) are not chaotic.

- In biology systems, complex oscillations can occur in cellular metabolism, population dynamics, heart rhythms, nerve impulses.

Check out matlab code: double_pendulum_init.m

Chaos final notes

- Chaos occurs in systems whose time evolution is described by nonlinear differential equations.

- BUT: Nonlinearity does not necessarily imply chaotic behavior!

- Chaos never occurs in linear systems or systems with an analytical solution.