Response to Reviewers' Comments

We would like to thank the editor and reviewers for their valuable comments and suggestions. The comments and suggestions have significantly helped improve the quality of this manuscript. All the reviewers' comments are fully addressed in the new manuscript. Especially, we have clarified the novelty and technical contributions of this work. We have also improved the theorem proofs to make them more rigorous and easier to understand. Besides, we have fixed typos and clarified several concepts that were confusing in the previous submission. The revisions are highlighted by the texts in the orange color in the revised manuscript. The following paragraphs detail our responses to each reviewer's comments.

Reviewer #1

1. **The simulation results are provided and some comparison results are provided. But this part should be improved to justify the claims made in the introduction. As the target distribution is not known in advance, the proposed Bayesian filter is also an approximation. How good it is when compared with parameterized particle filter?**

Response:

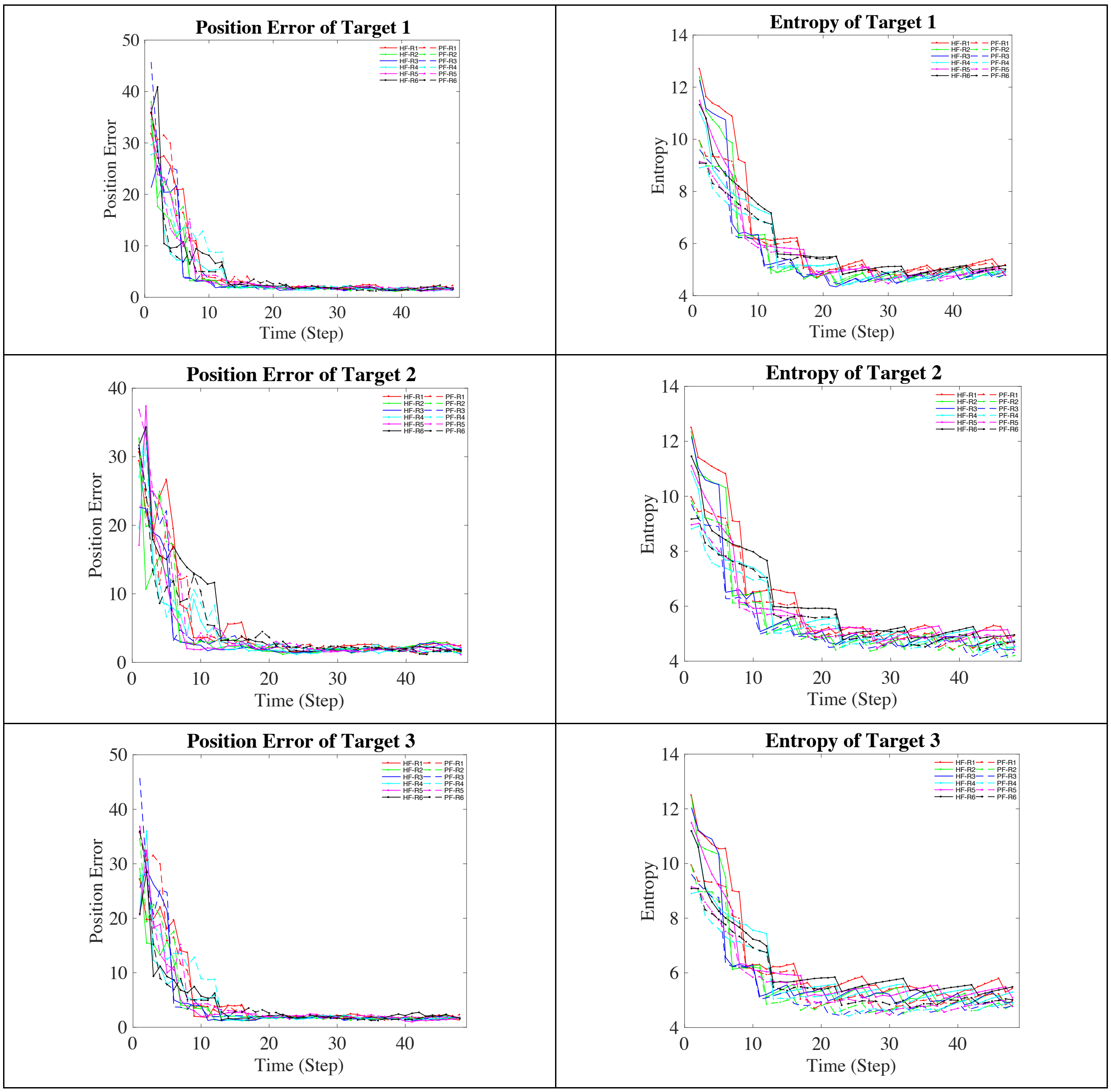
The authors would like to thank the reviewer for the comment. In general, the Bayesian filter is an optimal and generic framework for nonlinear filtering[1]. Many popular filtering approaches, such as the Kalman filter and particle filter, can be derived from the Bayesian filter. It is correct that the target distribution is unknown in advance. However, the estimation results from the Bayesian filter will rely less and less on the initial target distribution as more sensor measurements are fused, which is the consistency property proved in Section 5. Therefore, though the target distribution is unknown in advance, we can still use the Bayesian filter to optimally fuse sensor measurements to make estimation.

There are different practical ways to implement the Bayesian filter. The particle filter is a popular one. The other one is the histogram filter, which is used in this manuscript for the simulation. The histogram filter is easy to implement and can keep track of the probability mass over the whole field. The particle filter, on the contrary, is advantageous when the field is very large, which means keeping track of the probability mass of the whole field is computationally heavy. The particle filter focuses the computational resource (particles) to the regions of high probability, thus reducing the computational burden. However, it can cause

[1] S. Särkkä. "Bayesian filtering and smoothing". Vol. 3. Cambridge University Press, 2013.

inaccuracy since the regions of low probability mass are likely to be deprived of particles[2]. In this simulation the field is of medium size, therefore we use the histogram filter to implement the Bayesian filter.

We would like to point out that it will not cause much difference whether we use the histogram filter or the particle filter to implement the Bayesian filter in our simulation. To illustrate this point, we implement the particle filter here and compare it with the histogram filter. The figures below show the results. It can be noticed that, the particle filter implementation achieves very similar performance as the histogram filter that is used in our paper. Therefore we keep our original simulation results in the manuscript.



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[2] S. Thrun, W. Burgard, and D. Fox. "Probabilistic robotics". MIT press, 2005.

Fig. 1 Results of FIFO-DBF implemented using the histogram filter (solid lines) and using the particle filter (dashed lines). In the legends, the 'HF' represents the histogram filter and the 'PF' represents the particle filter. 'R1'-'R6' mean that the lines are associated with the 1^{st}-6^{th} UGVs.

However, thanks to this comment, we realize that we did not clearly explain the implementation of Bayesian filter in the manuscript. So we have added the following description in Section 6 to help readers better understand our choice of the implementation method:

"There exists different methods to implement a Bayesian filter, including the histogram filter and the particle filter [5]. The histogram filter is easy to implement and can keep track of the probability mass over the whole field, but can be computationally heavy for large fields. The particle filter, on the contrary, is advantageous when the field is very large, but can introduce inaccuracy due to particle deprivation [5]. We use both methods to implement the Bayesian filter in the simulation, and their results are very similar. For the purpose of clarity, we only include the results from the histogram filter here."

We also realize that we did not clearly describe the benchmark methods in the simulation, i.e., the consensus-based filter (CbDF) and the centralized filter (CF). Therefore, it was not straightforward to understand the benefits of using FIFO-DBF compared to traditional approaches. We add the following sentences in Section 6 to make these points easier to understand:

"We compare FIFO-DBF with two commonly adopted approaches in multi-agent filtering: the consensus-based filter (CbDF) [33] and the centralized filter (CF) [34]. The CbDF requires UGVs to continually exchange their individual PDFs with neighbors, computing the average of its own and the received PDFs. Multiple rounds of communication and averaging are needed at each step to ensure the convergence of UGVs' individual PDFs. The CF assumes a central unit that can constantly receive and fuse all UGVs' latest measurements into a single PDF."

"Besides, CbDF requires multiple rounds of exchanging individual PDFs, which incurs much higher communication burden than FIFO-DBF at each time step. Therefore, FIFO-DBF is more preferable than CbDF."

2. **The paper organization is good. The presentation is ok, but there are some typos which should be corrected.**

Response:

Thank you for the suggestion. We have carefully checked the manuscript and fixed the typos. Please refer to texts in the orange color in the revised manuscript.

Reviewer #2

1. **The paper is technically sounds. However, most of the technical points presented are not author's original idea or a direct consequence of work presented in this paper. For example, theorem 1 and corollary 1 is not a consequence of the work presented in this paper and is direct result from graph theory. Similarly, theorems 2 and 3 are of little trivial and can't be suggested as major findings under proposed work.**

Response:

The authors would like to thank the reviewer for the comment. The main motivation of this work is developing a distributed Bayesian filter (DBF) that relies on sensor measurement dissemination among UGVs under dynamically changing interaction topologies. We propose the Full-In-and-Full-Out (FIFO) protocol and the FIFO-based DBF (FIFO-DBF) to achieve this goal, which composes the main technical contributions of this work. The theorems presented in this work are derived from the FIFO protocol and FIFO-DBF, and they theoretically analyze the performance of FIFO-DBF. We explain these theorems in more details below.

First, Theorem 1 and Corollary 1 are the consequences of FIFO. In fact, if we follow the traditional method that each UGV only sends the current sensor measurement to neighboring UGVs without the use of FIFO, it can happen that two UGVs may never exchange their sensor measurements, even if there exists a path connecting them. It is only by using FIFO to make each UGV send all the received sensor measurements to neighboring UGVs that Theorem 1 holds. Besides, by using Theorem 1 and Corollary 1, we are able to prove the consistency of FIFO-DBF (Theorems 4 and 5). This is because the use of the law of large numbers to prove these two theorems relies on the condition that the sensor measurements of all UGVs are constantly received (with a finite delay) by each UGV. Theorem 1 and Corollary 1 guarantee that this condition hold.

The previous version of the paper has not clearly presented this point. So we add Remark 1 in Section 3 in the revised manuscript:
"Theorem 1 and Corollary 1 are the consequences of FIFO and the use of the communication buffer (CB). In fact, if we use the traditional methods that each UGV only sends the current sensor measurement to neighboring UGVs without the use of CB, it can happen that two UGVs may never exchange their sensor measurements, even there exists a path connecting them. The condition of frequently jointly strongly connectedness is also crucial for guaranteeing the consistency of the FIFO-based distributed Bayesian filter, as shown in Section 5."

Second, Theorems 2, 3 and Corollary 2 analyze the trimming of communication buffers (CBs) in the FIFO and its effects on the result of FIFO-DBF. To be specific, Theorem 2 guarantees that using the track list can reduce the communication burden while not affecting the result of FIFO-DBF. Theorem 3 and Corollary 2 quantitatively analyze the communication burden that FIFO-DBF incurs. These theorems and corollary justify the use of track list, which in turn makes the proposed FIFO-DBF practically useful, since the communication burden of FIFO is proved to be upper bounded.

To clarify this point, we have included the following sentences in Section 4.2 in the revised manuscript:
"The following theorem describes when CBs get trimmed, and it provides an upper bound of the communication burden that FIFO-DBF will incur. A detailed complexity analysis of FIFO-DBF is presented in Section 4.3."

2. **The simulation results are good but however, they cant be considered to be the main results of this paper and thus the technical part of the paper needs to be revised for better chances of acceptance.**

Response:

Thank you for the comments. We realize that we did not clarify our technical contributions in the previous submission. Here we would like to explain these contributions in more details:

First, the proposed FIFO protocol for measurement exchange is a novel design for distributed filtering. Previous works usually assume a fixed, strongly connected interaction topology. We propose FIFO so that the measurement dissemination-based distributed filtering can be applied to a class of dynamically changing networks while avoiding the commonly encountered out-of-sequence measurement issue.

Second, we emphasize the importance of the *frequently jointly strongly connectedness* condition of the dynamically changing interaction topology and prove that such class of network topologies can ensure the dissemination of UGVs' all sensor measurements in the network, and guarantee the consistency of FIFO-DBF. Though the counterpart concept, called the frequently jointly connected undirected networks, was proposed in some early work[3] for the network consensus problem, it is discovered in this work that the extension of such network condition to directed networks is critical for the measurement dissemination-based distributed filtering. We also analyze the communication burden incurred by FIFO-DBF under this network condition.

Third, FIFO-DBF is different from the traditionally used DBF in two ways: (1) The design of the track list

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[3] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," IEEE Transactions on automatic control, vol. 48, no. 6, pp. 988–1001, 2003.

is novel and crucial to making FIFO-DBF practically applicable. In fact, the track list significantly reduces the communication burden by keeping a finite size of the communication buffer (CB), proved by Theorem 3 and Corollary 2. In addition, the track list ensures that trimming CBs does not affect the filtering result of FIFO-DBF (Theorem 2). (2) FIFO-DBF uses the so-called *stored PDF* (defined in Section 4.1) that is obtained by fusing the sensor measurements of all UGVs up to a certain time *t*. The use of the stored PDF significantly reduces both the computational complexity of FIFO-DBF and the local storage of sensor measurements.

We revise the Introduction section to clarify the contributions of this work. We include the paragraph here for your reference:

"This paper proposes a distributed Bayesian filtering (DBF) method that only uses measurement dissemination for a group of networked UGVs with dynamically changing interaction topologies. In our previous work [27], we have proposed a Latest-In-and-Full-Out (LIFO) protocol for measurement exchange and developed a corresponding DBF algorithm. However, it only applies to static targets with simple binary sensor model. In this work, we substantially extend the previous work and make the following contributions:

(1) We introduce a new protocol called the Full-In-and-Full-Out (FIFO) that allows each UGV to broadcast a history of measurements to its neighbors via single hopping, enabling the tracking of moving targets using general nonlinear sensor models under time-varying topologies.

(2) We propose the frequently jointly strongly connectedness condition of the interaction topology and show that, under this condition, FIFO can disseminate UGVs' measurements over the network within a finite time.

(3) We develop a FIFO-based distributed Bayesian filter (FIFO-DBF) for each UGV to implement locally, which can avoid the OOSM issue. A track list is designed to reduce the computational complexity of FIFO-DBF and the communication burden.

(4) We prove the consistency of FIFO-DBF: each UGV's estimate of target position converges in probability to the true target position asymptotically if the interaction topologies are frequently jointly strongly connected."

We also add the following sentences to explain our contributions in other parts of the manuscript, including:

"This study proposes a Full-In-and-Full-Out (FIFO) protocol for measurement exchange in dynamically changing interaction topologies. The use of FIFO allows us to apply measurement dissemination-based distributed filters to time-varying networks" in Section 3.

"A UGV's TL stores the information of this UGV's reception of all UGVs' measurements, and is used for trimming old state-measurement pairs in the CB to reduce the communication burden" in section 3.

"The counterpart definition for undirected graphs is given in [11]" in footnote 2.

3. **The crux of authors Bayesian filter is determined by equation 3-4. As this forms the main technical background of the method presented in the paper, it is advised to present a better explanation of the equation 4, ie., when does it hold—only if the measurements are iid? If that is the case, then this should be explicitly stated and should be an important assumption of the proposed work.**

Response:

Thank you for the suggestion. We have further explained equation 4 and added the assumption that makes equation 4 hold in Section 4 of the revised manuscript. We include the newly added paragraph here for your reference:

"Here we have utilized the commonly adopted assumption [17, 20, 31] in the distributed filtering literature that the sensor measurement of each UGV at current time is conditionally independent from its own previous measurements and the measurements of other UGVs given the target and the UGV's current position. This assumption allows us to simplify $P\left(Z_k^i \middle| X_k, Z_{1:k-1}^i\right)$ *as* $P\left(Z_k^i \middle| X_k\right)$ *in Eq. (4a) and factorize* $P\left(Z_k^i \middle| X_k\right)$ *as* $\prod_{z_k^j \in Z_k^i} P\left(z_k^j \middle| X_k\right)$ *in Eq. (4b)."*

4. **The main result of the paper is theorem 4 and 5 which present consistency of the proposed algorithm. However, they need better explanation—please expand on how the law of large numbers yields equation 8a (It is also advised to expand equation 6 i.e., how to obtain the batch form of the DBF at kth step).**

Response:

Thank you for this important comment. The batch form can be derived by recursively applying the updating step of the Bayesian filter (equation 4). We have added the derivation steps in the manuscript and we include them here for your review:

"The DBF can be transformed into the batch form by recursively applying Eq. 4 from k to the initial time 1 (back in time):

$$
\begin{aligned}
P^i_{pdf}(X|Z^i_{1:k}) &= K_i P^i_{pdf}(X|Z^i_{1:k-1}) \prod_{z^j_k \in Z^i_k} P(z^j_k|X) \\
&= K_i P^i_{pdf}(X|Z^i_{1:k-2}) \prod_{z^j_{k-1} \in Z^i_{k-1}} P(z^j_{k-1}|X) \prod_{z^j_k \in Z^i_k} P(z^j_k|X) \\
&= \dots \\
&= K_i P^i_{pdf}(X) \prod_{z^j_1 \in Z^i_1} P(z^j_1|X) \cdots \prod_{z^j_k \in Z^i_k} P(z^j_k|X) \\
&= K_i P^i_{pdf}(X) \prod_{j=1}^{N} \prod_{t \in K^{i,j}_k} P(z^j_t|X)
\end{aligned}
$$
.

The last step is obtained using the relations $B^i_k = [Y^1_{K^{i1}_k}, \dots, Y^N_{K^{iN}_k}]$ and $Z^i_{1:k}$ is the set of all measurements in B^i_k.*"*

The equation 7 (it was the equation 8 in the previous submission) uses the law of large numbers. Notice that the sensor measurement z^j_t is a random variable with the distribution $P(\cdot\,|x^g)$. Based on Theorem 1 and Corollary 1, sensor measurements from all UGVs can be constantly received by each UGV, thus the law of large numbers can apply to obtain the asymptotic value of the summation, as shown in equation (7a). We have added the following sentences in the revised manuscript to help clarify equation 7:

"

Note that the sensor measurement is a random variable drawn from the underlying distribution associated with the sensor model, i.e., $z^j_t \sim P(\cdot\,|x^g), j \in V$. *Therefore* $\ln \frac{P(z^j_t|x)}{P(z^j_t|x^g)}$ *is a random variable associated with* z^j_t. *Due to the finite delay of measurement arrival (Corollary 1), i.e.,* $k - NT_u \le |K^{ij}_k| \le k$, *where* $|\cdot|$ *is the set cardinality, we can use the law of large numbers to study the asymptotic behavior of the series in Eq. (7):*

$$
\begin{aligned}
\frac{1}{k} \sum_{t \in K^{i,j}_k} \ln \frac{P(z^j_t|x)}{P(z^j_t|x^g)} &\xrightarrow{P} \mathbb{E}_{z^j_t} \left[\frac{P(z^j_t|x)}{P(z^j_t|x^g)} \right] \\
&= \int_{z^j_t} P(z^j_t|x^g) \frac{P(z^j_t|x)}{P(z^j_t|x^g)} dz^j_t \\
&= -D_{KL}\left(P(z^j_t|x) \| P(z^j_t|x^g) \right).
\end{aligned}
$$

"

5. **The proof of theorem 5 is rather ad-hoc and needs more explanation and needs to be precise.**

Response:

We appreciate the reviewer for this comment. In the previous submission we gave an informal derivation. In this revised version we rigorously prove Theorem 5. Especially, we have added the following paragraph:

“

The only difference from Eq. (5) is that $P(\cdot|x;x_l^j)$ *varies as the* j^{th} *UGV moves, since* x_l^j *changes over time. Similar to Eq. (6), we obtain*

$$\frac{1}{k}\ln\frac{P^i_{pdf}(x|Z^i_{1:k})}{P^i_{pdf}(x^g|Z^i_{1:k})} = \frac{1}{k}\ln\frac{P^i_{pdf}(x)}{P^i_{pdf}(x^g)} + \sum_{j=1}^{N}\frac{1}{k}\sum_{t\in K_k^{i,j},x_t^j\in S_F}\ln\frac{P(z_t^j|x;x_t^j)}{P(z_t^j|x^g;x_t^j)} + \sum_{j=1}^{N}\frac{1}{k}\sum_{t\in K_k^{i,j},x_t^j\in S_I}\ln\frac{P(z_t^j|x;x_t^j)}{P(z_t^j|x^g;x_t^j)},$$

where the second summand corresponds to the sensor positions that are in the finite-measurement spots set, and the third summand corresponds to the positions in the infinite-measurement spots set. By referring to Eq. (7), it is straightforward to know

$$\sum_{j=1}^{N}\frac{1}{k}\sum_{t\in K_k^{i,j},x_t^j\in S_F}\ln\frac{P(z_t^j|x;x_t^j)}{P(z_t^j|x^g;x_t^j)} \xrightarrow{P} 0,$$

$$\sum_{j=1}^{N}\frac{1}{k}\sum_{t\in K_k^{i,j},x_t^j\in S_I}\ln\frac{P(z_t^j|x;x_t^j)}{P(z_t^j|x^g;x_t^j)} \xrightarrow{P} \mathbb{E}_{z_t^j}\left[\frac{P(z_t^j|x)}{P(z_t^j|x^g)}\right],$$

since only finitely many observations associated with sensor positions in S_F *are obtained but infinitely many observations associated with sensor positions in* S_I *are received. The rest of the proof is similar to that of Theorem 4.”*