Distributed Bayesian Filter Under Dynamically Changing Interaction Topologies

Chang Liu*
Department of Mechanical Engineering
University of California, Berkeley
Berkeley, CA 94720
Email: changliu@berkeley.edu

Shengbo Eben Li†
Department of Automotive Engineering
Tsinghua University
Beijing, China 100084
Email: lisb04@gmail.com

J. Karl Hedrick
Department of Mechanical Engineering
University of California, Berkeley
Berkeley, CA 94720
Email: khedrick@me.berkeley.edu

This paper presents a distributed Bayesian filtering (DBF) method for a network of multiple unmanned ground vehicles (UGVs) under dynamically changing interaction topologies. The information exchange among UGVs relies on a measurement dissemination scheme, called Latest-In-and-Full-Out (FIFO) protocol. Different from statistics dissemination approaches that transmit posterior distributions or likelihood functions, each UGV under FIFOonly sends a buffer that contains latest available measurements to neighboring nodes, which significantly reduces the transmission burden between each pair of UGVs to scale linearly with the size of the network. Under the condition that the union of undirected switching topologies is connected frequently enough, FIFOcan disseminate observations over the network within finite time. The FIFO-based DBF algorithm is then derived to estimate individual probability density function (PDF) for target localization in a static environment. The consistency of this algorithm is proved that each individual estimate of target position converges in probability to the true target position. The effectiveness of this method is demonstrated by comparing with consensus-based distributed filters and the centralized filter in simulations.

1 INTRODUCTION

Unmanned ground vehicles (UGV) that operate without on-board operators have been used for many applications that are inconvenient, dangerous, or impossible to human. Distributed estimation using a group of networked UGVs has been applied to collectively infer status of complex environment, such as intruder detection [1] and object tracking [2]. Several techniques have been developed for distributed estimation, including distributed linear Kalman filters (DKF) [3], distributed extended Kalman filters [4] and distributed particle filters [5], etc. The most generic filtering scheme is distributed Bayesian filters (DBF), which can be applied for nonlinear systems with arbitrary noise distributions [6,7]. This paper focuses on a communication-efficient DBF for networked UGVs.

The interaction topology plays a central role on the design of DBF, of which two types are widely investigated in literature: fusion center (FC) and neighborhood (NB). In the former, local statistics estimated by each agent is transmitted to a single FC, where a global posterior distribution is calculated at each filtering cycle [8,9]. In the latter, each agent individually executes distributed estimation and the agreement of local estimates is achieved by certain consensus strategies [10–12]. In general, the NB-based distributed filters are more suitable in practice since they do not require a fusion center with powerful computation capability and are more robust to changes in network topology and link failures. So far, the NB-based approaches have two mainstream schemes according to the transmitted data among agents, i.e., *statistics dissemination* (SD) and *measurement dissemination* (MD). In the SD scheme, each agent exchanges statistics such as posterior distributions and likelihood functions within neighboring nodes [13]. In the MD scheme, instead of exchanging statistics, each agent sends its observations to neighboring nodes.

Statistics dissemination scheme has gained increasing interest and been widely investigated during last decade. Madhavan et al. (2004) presented a distributed extended Kalman filter for nonlinear systems [4]. This filter was used to generate local terrain maps by using pose estimates to combine elevation gradient and vision-based depth with environmental features. Olfati-Saber (2005) proposed a dis-

*The first two authors have equally contributed to this research.
†Address all correspondence to this author.

tributed linear Kalman filter (DKF) for estimating states of linear systems with Gaussian process and measurement noise [3]. Each DKF used additional low-pass and band-pass consensus filters to compute the average of weighted measurements and inverse-covariance matrices. Gu (2007) proposed a distributed particle filter for Markovian target tracking over an undirected sensor network [5]. Gaussian mixture models (GMM) were adopted to approximate the posterior distribution from weighted particles and the parameters of GMM were exchanged via average consensus filter. Hlinka et al. (2012) proposed a distributed method for computing an approximation of the joint (all-sensors) likelihood function by means of weighted-linear-average consensus algorithm when local likelihood functions belong to the exponential family of distributions [14]. Saptarshi et al. (2014) presented a Bayesian consensus filter that uses logarithmic opinion pool for fusing posterior distributions of the tracked target [6]. Other examples can be found in [7] and [15].

Despite the popularity of statistics dissemination, exchanging statistics can consume high communication resources. One remedy is to approximate statistics with parametric models, e.g., Gaussian Mixture Model [16], which can reduce communication burden to a certain extent. However, such manipulation increases the computation burden of each agent and sacrifices filtering accuracy due to approximation. The measurement dissemination scheme is an alternative solution to address the issue of exchanging statistics. An early work on measurement dissemination was done by Coates et al. (2004), who used adaptive encoding of observations to minimize communication overhead [17]. Ribeiro et al. (2006) exchanged quantized observations along with error-variance limits considering more pragmatic signal models [18]. A recent work was conducted by Djuric et al. (2011), who proposed to broadcast raw measurements to other agents, and therefore each agent has a complete set of observations of other agents for executing particle filtering [19]. A shortcoming of aforementioned works is that their communication topologies are assumed to be a fixed and complete graph that every pair of distinct agents is constantly connected by a unique edge. In many real applications,the interaction topology may change dynamically due to unreliable links, external disturbances and/or range limits [20]. In such cases, dynamically changing topologies can cause random packet loss and variable transmission delay, thus decreasing the performance of distributed estimation, and even leading to inconsistency and non-consensus.

(**TODO:** cite Leung's paper, emphasize that it transmits pdfs.) (**TODO:** mention in my main contribution that my work on transmits measurements, no pdf.) The main contribution of the paper is that we present a measurement dissemination-based distributed Bayesian filtering (DBF) method for a group of networked UGVs with dynamically changing interaction topologies. The measurement dissemination scheme uses the so-called Latest-In-and-Full-Out (FIFO) protocol, under which each UGV is only allowed to broadcast observations to its neighbors by using single-hopping. Individual Bayesian filter is implemented locally by each UGV after exchanging observations using FIFO. Under the condition that the union of undirected switching topologies is connected frequently enough, two properties are achieved: (1) FIFOcan disseminate observations over the network within finite time; (2) FIFO-based DBF guarantees the consistency of estimation that each individual estimate of target position converges in probability to the true target position as the number of observations tends to infinity. The main benefit of using FIFOis on the reduction of communication burden, with the transmission data volume scaling linearly with the size of the UGV network.

The rest of this paper is organized as follows: the FIFOprotocol for dynamically changing interaction topologies is formulated in **??**; the FIFO-based DBF algorithm is described in Section 4, where the consistency of estimation is proved; simulation results are presented in Section 8 and Section 10 concludes the paper.

2 Problem Formulation

Consider a network of N UGVs in a bounded two-dimensional space S. The interaction topology can be dynamically changing due to limited communication range, varying team formation or link failure. Each UGV is equipped with a sensor for environmental perception. Due to the limit of communication range, each UGV can only exchange sensor observations with its neighbors. The Bayesian filter is run locally on each UGV based on its own and received observations via single-hopping to estimate the position of a static target in S.

2.1 Target and Sensor Model

The target motion takes a deterministic discrete-time model that can be described by

$$x^g_{k+1} = f(x^g_k, u^g_k), \tag{1}$$

where $x^g_k \in \mathbb{R}^2$ denotes the target position at time k; u^g_k represents the control input of the target.

Each UGV constantly measures the target position and the sensor measurement of i^{th} UGV can be described by a stochastic model:

$$z^i_k = g_i(x^g_k, w^i_k; y^i_k), \tag{2}$$

where w^i_k is the white measurement noise and $y^i_k = [x^i_k; \theta^i_k]$ represents the sensor state, consisting of the sensor position x^i_k and direction θ^i_k. g_i depends on the type of the sensor. Let $\mathcal{F}(y^i_k)$ denote the sensor field of view (FOV), g_i for several typical sensor types can be defined as follows:

Range-only sensors: when the target is within the sensor's FOV, the measurement only depends on the relative distance between the sensor and the target.

$$g_i(x^g_k, w^i_k; x^i_k) = \begin{cases} \|x^g_k - x^i_k\|_2 + w^i_k & \text{if}\, x^g_k \in \mathcal{F}(y^i_k), \\ \emptyset & \text{if}\, x^g_k \notin \mathcal{F}(y^i_k). \end{cases} \tag{3}$$

Bearing-only sensors: when the target is within the sensor's FOV, the measurement only depends on the relative bearing between the sensor and the target.

$$g_i(x_k^g, w_k^i; x_k^i) = \begin{cases} \measuredangle(x_k^g - x_k^i) + w_k^i & \text{if} x_k^g \in \mathcal{F}(y_k^i), \\ \emptyset & \text{if} x_k^g \notin \mathcal{F}(y_k^i). \end{cases} \quad (4)$$

A probabilistic sensor model that describes the conditional probability of a certain measurement given sensor and target state is a key component for Bayesian filtering. We define a likelihood function to represent the probability of the target being detected by a sensor:

$$p_{1,k}^i = P(z_k^i \neq \emptyset | x_k^g; x_k^i) \in [0,1], \; x_k^g \in S, \quad (5)$$

where x_k^i is the i^{th} sensor's position. Correspondingly, the likelihood function for no target being detected is:

$$p_{0,k}^i = P(z_k^i = \emptyset | x_k^g; x_k^i) = 1 - p_{1,k}^i. \quad (6)$$

The combination of Eq. (5) and Eq. (6) forms the probabilistic model for a sensor. If w_k^i is a zero-mean Gaussian white noise, then the probabilistic sensor model can be described as

$$\begin{cases} p_{1,k}^i \sim \mathcal{N}(\bar{z}_k^i, \Sigma_k^i) & \text{if} x_k^g \in \mathcal{F}(y_k^i) \\ p_{1,k}^i = 0 & \text{if} x_k^g \in \mathcal{F}(y_k^i), \end{cases} \quad (7)$$

where $\bar{z}_k^i$ is the nominal value of the measurement and equals $\|x_k^g - x_k^i\|$ and $\measuredangle(x_k^g - x_k^i)$ for range-only and bearing-only sensors, correspondingly.

Consequently,

$$p_{0,k}^i = \begin{cases} 0 & \text{if} x_k^g \in \mathcal{F}(y_k^i) \\ 1 & \text{if} x_k^g \in \mathcal{F}(y_k^i). \end{cases} \quad (8)$$

For the purpose of simplicity, we will not explicitly write x_k^i in the sensor model (Eq. (5) and (6)) for the rest of the paper.

Remark 1. *Given the knowledge of current target and UGV positions, current observation by each UGV can be considered conditionally independent from its own past observations and those by other UGVs [21].*

Remark 2. *The proposed FIFO protocol and the consistency property to be described in Section 4 are applicable for general sensors, not limited to the ones described in this section. In addition, they do not rely on the Gaussian noise assumption.*

2.2 Graphical Model of Interaction Topology

Consider a simple[1], undirected graph $G = (V, E)$ to represent the interaction topology of N networked UGVs, where

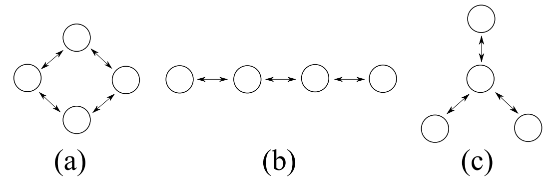

Fig. 1: Three types of topologies: (a) ring topology; (b) line topology; (c) star topology

$V = \{1, \ldots, N\}$ represents the index set of UGVs and $E = V \times V$ denotes the edge set. The *adjacency matrix* $M = [m_{ij}]$ of graph G describes the interaction topology:

$$m_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{if } (i,j) \notin E \end{cases},$$

where m_{ij} denotes the entity of adjacency matrix. The notation $m_{ij} = 1$ indicates that a communication link exists between i^{th} and j^{th} UGV and $m_{ij} = 0$ indicates no communication between them. Fig. 1 illustrates three types of typical topologies: ring [22], line [23], and star [24]. All of them are represented by simple and undirected graphs.

Let $\bar{G}$ denote the set of all possible simple and undirected graphs defined for the network of UGVs. It is easy to know that $\bar{G}$ has finite elements. The adjacency matrix associated with a graph $G_l \in \bar{G}$ is denoted as $M^l = [m_{ij}^l]$. Define the *union* of a collection of graphs $\{G_{i_1}, G_{i_2}, \ldots, G_{i_l}\} \subset \bar{G}$ as the undirected graph with nodes in V and edge set given by the union of edge sets of G_{i_j}, $j = 1 \ldots, l$. Such collection is defined to be *jointly connected* if the union of its members forms a connected graph.

We define two concepts of neighborhood in a UGV network. The *direct neighborhood* of i^{th} UGV under topology G_l is defined as $\mathcal{N}_i(G_l) = \{ j | m_{ij}^l = 1, \, j \in \{1, \ldots, N\} \}$. All UGVs in $\mathcal{N}_i(G_l)$ can directly exchange information with i^{th} UGV via single-hopping. In addition to direct neighborhood, another set called *available neighborhood* is defined as $\mathcal{Q}_i(G_l)$, which contains indices of UGVs whose observations can be received by the i^{th} UGV given a specific observation exchange protocol and the interaction topology G_l. Note that in general $\mathcal{N}_i(G_l) \subseteq \mathcal{Q}_i(G_l)$.

2.3 Distributed Bayesian Filter for Multiple UGVs

The generic distributed Bayesian filter (DBF) is introduced in this section. Let $X_k \in S$ be the random variable that represents the position of the target at time k. The probability density function (PDF) of X_k, called *individual PDF*, of i^{th} UGV is then represented by $P_{pdf}^i(X_k | \mathbf{z}_{1:k}^i)$, where $\mathbf{z}_{1:k}^i$ denotes the set of measurements by i^{th} UGV and by UGVs in $\mathcal{Q}_i$, that have been received by i^{th} UGV until time k. The initial individual PDF, $P_{pdf}^i(X_0)$, is constructed given all available prior information including past experience and environment knowledge. It is necessary to initialize the individual PDF such that the probability density of true target position is nonzero, i.e., $P_{pdf}^i(X_0 = x_0^g) \neq 0$.

[1] An undirected graph $G = (V, E)$ is *simple* if it has no self-loops or repeated edges, i.e., $(i, j) \in E$, only if $i \neq j$ and E only contains distinct elements. A graph is *connected* when there is a path between every pair of vertices in V.

Under the framework of DBF, the individual PDF is recursively estimated by two steps: the prediction step and the updating step.

2.3.1 Prediction

At time k, the prior individual PDF $P^i_{pdf}(X_{k-1}|\mathbf{z}^i_{1:k-1})$ is first predicted forward by using the Chapman-Kolmogorov equation:

$$P^i_{pdf}(X_k|\mathbf{z}^i_{1:k-1}) = \int\limits_{X_{k-1}\in S} P(X_k|X_{k-1})P^i_{pdf}(X_{k-1}|\mathbf{z}^i_{1:k-1})dX_{k-1}, \tag{9}$$

where $P(X_k|X_{k-1})$ represents the state transition probability of the target, based on the Markovian motion model (Eq. (1)). For the deterministic motion model, the state transition probability is simplified to be

$$P(X_k = c_k|X_{k-1} = c_{k-1}) = \begin{cases} 1 & \text{if} \quad c_k = f(c_{k-1}, u^g_{k-1}) \\ 0 & \text{otherwise} \end{cases}. \tag{10}$$

2.3.2 Updating

The i^{th} individual PDF is then updated by Bayes' theorem using the set of newly received measurements at time k, i.e., $\mathbf{z}^i_k$:

$$P^i_{pdf}(X_k|\mathbf{z}^i_{1:k}) = K_i P^i_{pdf}(X_k|\mathbf{z}^i_{1:k-1})P(\mathbf{z}^i_k|X_k), \tag{11}$$

where $P(\mathbf{z}^i_k|X_k)$ comes from the sensor model and K_i is a normalization factor, given by:

$$K_i = \left[\int\limits_{X_k\in S} P^i_{pdf}(X_k|\mathbf{z}^i_{1:k-1})P(\mathbf{z}^i_k|X_k)dX_k\right]^{-1}.$$

3 Full-In-and-Full-Out (FIFO) Protocol

This study proposes a Full-In-and-Full-Out (FIFO) protocol for observation exchange. In our previous work, we proposed a Latest-In-and-Full-Out (LIFO) protocol and can be used for time-invariant topologies. FIFO is suitable for time-varying topologies. Let $Y^i_{\mathcal{K}} = \left\{\left[x^i_k, z^i_k\right] | k \in \mathcal{K}\right\}$ be the set of state-measurement pairs of robot i, where $\mathcal{K}$ is an index set of time steps. Under FIFO, each UGV contains a communication buffer (CB) to store a subset of measurements and the corresponding states of all UGVs:

$$B^i_k = \left[Y^1_{\mathcal{K}^{i,1}_k}, \ldots, Y^N_{\mathcal{K}^{i,N}_k}\right],$$

where B^i_k is the CB of i^{th} robot at time k and $Y^j_{\mathcal{K}^{i,j}_k}$ represents the set of robot j's measurements at time steps in $\mathcal{K}^{i,j}_k$ that are stored in robot i's CB. Note that under FIFO and certain conditions (Proposition 1) of interaction topologies, $Q_i = \{1, \ldots, N\} \setminus \{i\}$, i.e. each robot can know the measurements from all other robots. This will be proved in Corollary 1. Let $G_k \in \bar{G}$ represent the interaction topology at time k. The **FIFO protocol** is stated in Algorithm 1. Note that in the Updating Step, the algorithm uses Algorithm 3, which we will introduce in Section 5. For the purpose of clarity, we ignore this operation and do not trim CB at this stage.

Algorithm 1 FIFO Protocol

(1) Initialization.

The CB of i^{th} UGV is initialized at $k = 0$:

$$B^i_0 = \left[Y^1_{\mathcal{K}^{i,1}_0}, \ldots, Y^N_{\mathcal{K}^{i,N}_0}\right], \text{ where } Y^j_{\mathcal{K}^{i,j}_0} = \left\{\left[x^j_0, \varnothing\right]\right\}.$$

(2) At time k^{th} $(k \geq 1)$ for i^{th} UGV:

(2.1) Receiving Step.

The i^{th} UGV receives all CBs of its direct neighborhood $\mathcal{N}_i(G_{k-1})$. The received CBs are totally $|\mathcal{N}_i(G_{k-1})|$ groups, each of which corresponds to the $(k-1)^{\text{th}}$ step CB of a UGV in $\mathcal{N}_i(G_{k-1})$. The received CB from l^{th} UGV is

$$B^l_{k-1} = \left[Y^1_{\mathcal{K}^{l,1}_{k-1}}, \ldots, Y^N_{\mathcal{K}^{l,N}_{k-1}}\right], \; l \in \mathcal{N}_i(G_{k-1})$$

(2.2) Observation Step.

The i^{th} UGV updates $Y^i_{\mathcal{K}^{i,i}_k}$ by its own state-measurement pair at current step:

$$Y^i_{\mathcal{K}^{i,i}_k} = Y^i_{\mathcal{K}^{i,i}_{k-1}} \cup \left\{x^i_k, z^i_k\right\}.$$

(2.3) Updating Step.

The i^{th} UGV updates other elements of its own CB, i.e., $Y^j_{\mathcal{K}^{i,j}_k}$ $(j \neq i)$, by merging with all received CBs:

For all $j \neq i$,

$$Y^j_{\mathcal{K}^{i,j}_k} = Y^j_{\mathcal{K}^{i,i}_{k-1}} \cup Y^j_{\mathcal{K}^{l,j}_{k-1}}, \forall\, l \in \mathcal{N}_i(G_{k-1}).$$

Trim the measurements in CB using the track list. See Algorithm 3.

(2.4) Sending Step:

The i^{th} UGV broadcasts its updated CB to all of its neighbors defined in $\mathcal{N}_i(G_k)$.

(3) $k \leftarrow k + 1$ until stop

Fig. 2 illustrates the FIFO cycles of a network of 3 UGVs with switching line topologies. There are two types of topologies: under the first one only UGV 1 and UGV 2 can directly communicate and under second one only UGV 2 and UGV 3 can directly communicate. Several facts can be noticed in Fig. 2: (1) the two topologies are jointly connected within each time intervals $[0,3)$, $[3,5)$, $[5,7)$; (2) **(TODO:** may need to change**)** CBs of all UGVs are filled within 5 steps; (3) after being filled, each CB keeps updated every finite time steps, which means each UGV receives new observations of other UGVs with finite delay. Extending these facts to a network of N UGVs, we have the following proposition:

Proposition 1. *Consider a network of N UGVs with switching interaction topologies. If the following two conditions are satisfied: (1) there exists an infinite sequence of time intervals $[k_m, k_{m+1})$, $m = 1, 2, \ldots$, starting at $k_1 = 0$ and are contiguous, nonempty and uniformly bounded; (2) the union of graphs across each such interval is jointly connected, then arbitrary pair of UGVs can exchange observations under FIFO. In addition, the delay between each pair of UGVs is no greater than $(N-1)T_u$, where $T_u = \sup_{m=1,2,\ldots} (k_{m+1} - k_m)T$ is the upper bound of interval lengths.*

Proof. Consider the transmission from an arbitrary UGV i to an arbitrary one j. Since each robot will receive neighbors' CBs and send the merged one to its neighbors in next step, UGV i can transmit its information to j if and only if there is a path between them, i.e.,

$$\exists n \in \mathbb{N}_+, \text{ s.t. } A^n_{i,j} > 0.$$

Since the union of graphs across time interval $[k_1, k_2)$ is jointly connected, i^{th} UGV can directly communicate with at least one another UGV at a time instance, i.e., $\exists l_1 \in V, t_1 \in [k_1, k_2)$ such that $i \in \mathcal{N}_{l_1}(G_{t_1})$. Therefore, at least one UGV other than i^{th} UGV has received and received the CB from i^{th} UGV by k_2. If $l_1 = j$, then we have proved the exchange of CBs between i and j. If $l_1 \neq j$, we consider time interval $[k_2, k_3)$. Using similar derivation, it can be shown that all $N-1$ UGVs, except i^{th} UGV, will receive and store the state-measurement pair from i no later by k_N. Therefore, the transmission delay between an arbitrary pair of UGVs is no greater than $(N-1)T_u$.

Corollary 1. *With the same network condition in Proposition 1, all elements in CB under FIFO become filled within finite time. This implies $Q_i = \{1, \ldots, N\} \setminus \{i\}$. Additionally, each element keeps updated every finite period of time.*

Proof. According to Proposition 1, the transmission delay between an arbitrary pair of UGVs is no greater than $(N-1)T_u$. (**TODO:** this argument seems incorrect.) Therefore, CBs of all UGVs becomes filled when $k \geq (N-1)T_u$. In addition, each element in CBs gets updated every finite period of time that is no greater than $(N-1)T_u$.

4 Distributed Bayesian Filter via Full-In-and-Full-Out Protocol

This section derives the LIFO-DBF for localizing a target. At time k, each UGV maintains an two PDFs, the stored PDF and the individual PDF. The stored PDF is used to represent the PDF that is updated from the initial PDf and all

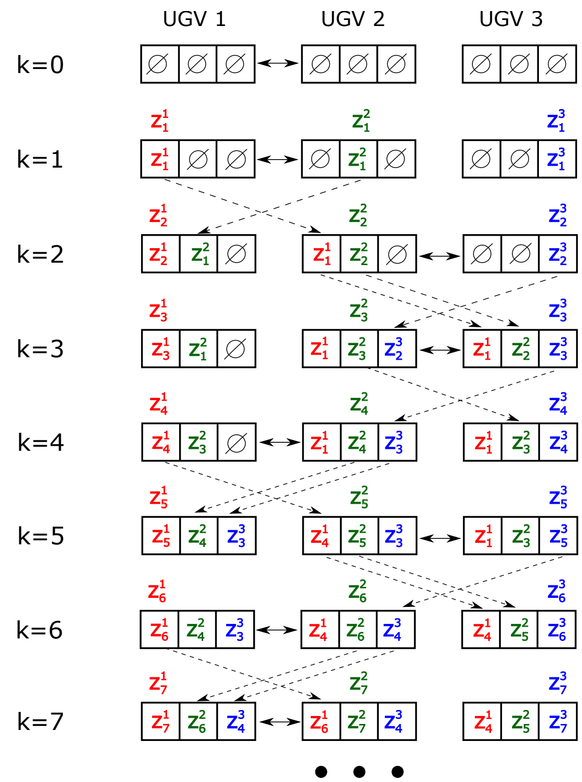

Fig. 2: Example of FIFOwith three UGVs using switching line interaction topologies. The double-headed arrow represents a communication link between two UGVs.

robots' measurements. The i^{th} individual PDF is then computed from the stored PDF and the measurements in CB by running the Bayesian filter (Eq. (9) and (11)).

The **LIFO-DBF algorithm** is stated in Algorithm 2. Note that we use Ω^i_ξ $(\xi = 1, \ldots, N)$ to denote the index set of UGVs whose measurement at time $(k - N + \xi)$ is stored in i^{th} UGV's CB, i.e. $\Omega^i_\xi = \left\{ j \in Q_i \bigcup \{i\} \,|\, z^j_{k-N+\xi} \text{ is in the record set} \right\}$. Fig. 3 illustrates the LIFO-DBF procedure for the 1[st] UGV as an example [2].

5 Track List for Reducing Communication Overhead

The size of CBs can keep increasing as measurements cumulate over time. To reduce the communication of unnecessary measurements in the CBs, we trim the measurements that have already been fused into the state estimation (stored PDF), as the Updating Step in Algorithm 1 shows. To avoid trimming the useful measurement, each robot keeps a *track list*, $P^i_k = \left[\mathbf{p}^1_{k^{i,1}}, \ldots, \mathbf{p}^N_{k^{i,N}} \right]^T$, which is a binary list that represents the robot's knowledge of the oldest measurements (in terms of the measurement time) in all robots' CBs. $\mathbf{p}^j_{t_j} = \left[p^{j,l}_{t_j}, l \in \{1, \ldots, N\} \right]$ is a binary vector with size N.

[2] Due to the space limit, in this figure we use $P^i_{pdf}(k)$, $P^i_{pdf}(k-N)$ and $P^i_{pdf}(k-N+1)$ to represent $P^i_{pdf}(X|\mathbf{z}^i_{1:k})$ $P^i_{pdf}(X|\mathbf{z}^i_{1:k-N})$ and $P^i_{pdf}(X|\mathbf{z}^i_{1:k-N+1})$, respectively.

Algorithm 2 LIFO-DBF Algorithm for Moving Target

For i^{th} UGV at k^{th} step:
After updating CB by Algorithm 1,
(1) Initialize a *virtual PDF* by assigning the stored individual PDF to it:

$$P^i_{virt}(X_{k-N}) = P^i_{stored},$$

where the stored individual PDF is for time $(k-N)$:

$$P^i_{stored} = P^i_{pdf}(X_{k-N}|z^1_{1:k-N},\ldots,z^N_{1:k-N}).$$

(2) For $\xi = 1$ to N, iteratively repeat two steps of Bayesian filtering:
(2.1) Prediction

$$P^{pre}_{virt}(X_{k-N+\xi}) = \int_S P(X_{k-N+\xi}|X_{k-N+\xi-1})P^i_{virt}(X_{k-N+\xi-1})dX_{k-N+\xi-1}.$$

(2.2) Updating

$$P^i_{virt}(X_{k-N+\xi}) = K_\xi P^{pre}_{virt}(X_{k-N+\xi}) \prod_{j\in\Omega^i_\xi} P(z^j_{k-N+\xi}|X_{k-N+\xi}).$$

$$K_\xi = \left[\int_S P^{pre}_{virt}(X_{k-N+\xi}) \prod_{j\in\Omega^i_\xi} P(z^j_{k-N+\xi}|X_{k-N+\xi})dX_{k-N+\xi}\right]^{-1}.$$

(2.3) When $\xi = 1$, if $z^j_{k-N+1} \neq \emptyset$ for $\forall j \in \{1,\ldots,N\}$, then the virtual PDF is equivalent to the individual PDF for time $(k-N+1)$. Store it to replace the old PDF:

$$P^i_{stored} = P^i_{virt}(X_{k-N+1}),$$

where

$$P^i_{virt}(X_{k-N+1}) = P^i_{pdf}(X_{k-N+1}|z^1_{1:k-N+1},\ldots,z^N_{1:k-N+1}).$$

(3) Individual PDF of i^{th} UGV at time k is $P^i_{pdf}(X_k|\mathbf{z}^i_{1:k}) = P^i_{virt}(X_k)$.

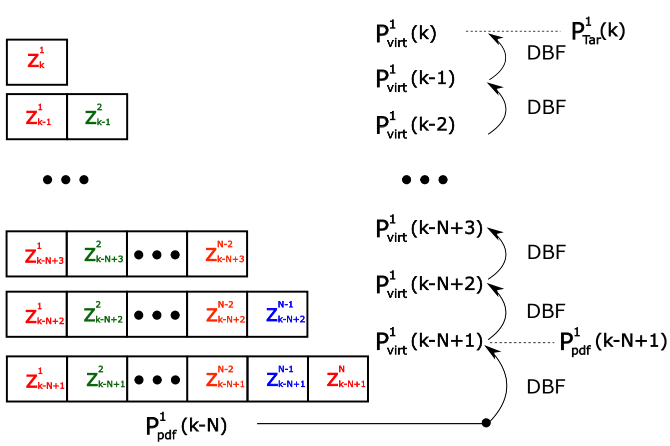

Fig. 3: Example of LIFO-DBF for 1^{st} UGV at time k. Networked UGVs take a line topology. The stored individual PDF is represented by $P^1_{pdf}(k-N)$. The UGV first calculates $P^1_{virt}(k-N+1)$, defined in Algorithm 2, and then stores it as $P^1_{pdf}(k-N+1)$. Repeating DBF until obtaining $P^1_{pdf}(k)$. In this example, $\Omega^1_\xi = \{1,2,\ldots,N+1-\xi\}$, $\xi = 1,\ldots,N$.

Each element, $p^{j,l}_t$, equals 1 if the i^{th} robot knows that the $[x^l_t, z^l_t]$ is in the j^{th} robot's CB, and equals 0 if the i^{th} robot cannot determine whether $[x^l_t, z^l_t]$ is in the j^{th} robot's CB. So it could happen that $[x^l_t, z^l_t]$ is in the j^{th} CB but robot i does not know this and thus $p^{j,l}_t = 0$. When all elements of P^i_t are 1's, it means that the robot i can be sure that all robots have the measurements of time t from all other robots and can thus fuse the measurements of time t in its own CB to update the stored PDF. Track lists are also exchanged when robots communicate and trim the corresponding measurements in CB. Algorithm 3 describes the exchange and updating of the track lists.

The use of track lists can ensure that all information are used while avoiding overuse or OOSM. It also guarantees that fusing the measurements and trimming the CBs will not lose any information; they are encoded into the stored PDF. The following theorem proves this property.

Theorem 1. *Each robot will utilize all robots' measurements sequentially. The measurements trimmed from the CB are already fused in the stored PDF.*

Proof. The design of track lists ensure that, measurements of certain times get trimmed only when all robots' CB have or have had them. Therefore, it causes no loss to trim theses measurements.

Since the track list is used to reduce the size of CBs, it is necessary to understand how often the CBs will get trimmed. The following theorem gives the result.

Theorem 2. *The robot i can trim the measurements of time t at time $k_i = t + \max_{l,j}\{d_{lj} + d_{ji}\}$, where d_{lj} is the shortest path distance from robot l to j. Let $t^k_{l,j} = t + d_{lj}$ be the time that robot l talks to j. Let d_{ji} is the shortest path distance from robot j to i, starting at time $t^k_{l,j}$. In other words, after j gets message from l at time $t^k_{l,j}$, it takes d_{ji} steps for j to let i know that j has got the i's information of time k.*

Proof. (**TODO:** need to modify this proof.) Mathematically speaking,

$$\exists n_1, n_2 \in \mathbb{N}_+, \text{ s.t. } M^{n_1}_{l,j} > 0 \text{ and } M^{n_2}_{j,i} > 0.$$
$$\text{Besides, } n = n_1 + n_2.$$

From Theorem 2, we can get the following corollaries about how often the CB gets trimmed.

Corollary 2. *The first time to trim i^{th} CB occurs at $k_i = 1 + \max_{l,j}\{d_{lj} + d_{ji}\}$.*

Proof. The proof is straightforward by setting $t = 1$ in the proof of Theorem 2.

Corollary 3. *The time between two consecutive trimming of i^{th} CB is*

$$\Delta k_i = \begin{cases} 1 + n'_i - n_i & \text{if } n'_i \geq n_i \\ 1 & \text{if } n'_i < n_i \end{cases},$$

where $n_i = \max_{l,j}\left\{d_{lj}+d_{ji}\right\}$ *starting from* t *and* $n_i' = \max_{l,j}\left\{d_{lj}+d_{ji}\right\}$ *starting from* $t+1$.

Proof. Let $k_i = t+1+n_i$ and $k_i' = t+n_i'$ be the two consecutive trimming times, where t is the time of measurement that will be trimmed from the CB. Due to the rule of trimming, earlier measurements are always trimmed before that of later measurements. This guarantees that $\Delta k \geq 1$. If $n_i' \geq n_i$, then it takes longer time for later measurements to be trimmed and thus $\Delta k = 1+n_i'-n_i$.

Algorithm 3 Exchanging and Updating the Track List

(1) Initialization. The CB of i^{th} UGV is initialized when $k=0$:

$$P_0^i = \mathbf{0},\ \text{i.e. } p_0^{j,l} = 0, \forall j,l \in \{1\ldots,N\}.$$

(2) At k^{th} $(k \geq 1)$ step for i^{th} UGV:

(2.1) Receiving Step.

The i^{th} UGV receives all track lists of its direct neighborhood $\mathcal{N}_i(G_{k-1})$. The received track lists from l^{th} $(l \in \mathcal{N}_i(G[k-1]))$ UGV is $P_{t_l}^l$.

(2.2) Updating Track List.

The i^{th} UGV updates its own track list using all the received track lists:

For each $j \in \{1\ldots,N\}$,

if $k^{i,j} > k^{i,l}$, keep current $\mathbf{p}^j_{k^{i,j}}$;

if $k^{i,j} = k^{i,l}$, $\mathbf{p}^j_{k^{i,j}} = \mathbf{p}^j_{k^{i,j}} \vee \mathbf{p}^j_{k^{l,j}}$;

if $k^{i,j} < k^{i,l}$, $\mathbf{p}^j_{k^{i,j}} = \mathbf{p}^j_{k^{l,j}}$ and $k^{i,j} = k^{i,l}$.

(2.3) Trim CB.

Trim the CB based on the updated track lists. Find the smallest time in the track list: $k_m = \min\left\{k^{i,1},\ldots,k^{i,N}\right\}$. If all elements with the same time tag in P_k^i are 1's, then

1. set all these items in the track list to be 0;
2. remove all corresponding measurements in i^{th} CB;
3. increase all time tags equivalent to k_m by 1.
4. update the track list with items in current CB.

(2.4) Sending Step. The i^{th} UGV broadcasts its updated track list to all of its neighbors defined in $\mathcal{N}_i(G_k)$.

(3) $k \leftarrow k+1$ until stop

Using the track list to trim CBs, we can guarantee no information loss under than network condition in Proposition 1, which is stated in the following theorem:

Theorem 3. *Consider a frequently connected network of N UGVs, arbitrary pairs of UGVs can exchange observations under FIFO. By using track lists to trim CBs, no loss of information happens. The maximum size of CBs transmitted by UGVs is* $O(NT_m)$, *where* $T_m = \max_{l,j,i}\left\{d_{lj}+d_{ji}\right\}$. *Besides,* $T_m \leq NT_u$, *where* $T_u = \sup_{m=1,2,\ldots}(k_{m+1}-k_m)T$.

Proof. The proof is straightforward by considering Proposition 1, Theorems 1 and 2.

Complexity of FIFO-DBF

Compared to statistics dissemination, FIFO is generally more communication-efficient for distributed filtering. To be specific, consider a $D \times D$ grid environment with a network of N UGVs, the transmitted data of FIFObetween each pair of UGVs are only the CB of each UGV and the corresponding UGV positions where observations were made, the length of which is $O(N)$. On the contrary, the length of transmitted data for a statistics dissemination approach that transmits unparameterized posterior distributions or likelihood functions is $O(D^2)$, which is in the order of environmental size. Since D is generally much larger than N in applications such as target localization, FIFOrequires much less communication resources.

It is worth noting that, for the static target, each UGV only needs current-step CB to update individual PDFs. Therefore, besides storing its own individual PDF of size $O(M^2)$, only current-step CB of size $O(N)$ is stored in an UGV's memory and all previous CBs can be discarded, which means that the size of needed memory is $O(N+M^2)$. On the contrary, for the moving target, each UGV needs to store a set of measurement history of size $O(N^2)$ and an individual PDF of size $O(M^2)$. Therefore the size of the needed memory for each UGV is $O(M^2+N^2)$. This is generally larger than that of statistics dissemination-based methods, the memory of which is $O(M^2)$. Besides, additional computation power is needed for LIFO-DBF compared to statistics dissemination-based methods. Therefore, LIFO-DBF sacrifices storage space and computation resource for reducing communication burden. This is actually desirable for real applications as local memory of vehicles is usually abundant compared to the limited bandwidth for communication.

6 Proof of Consistency

This section proves the consistency of the maximum a posteriori (MAP) estimator of LIFO-DBF under unbiased sensors (sensors without offset). An estimator of a state is said to be consistent if it converges in probability to the true value of the state [**?**]. Consistency is an important metric for stochastic filtering approaches [**?**] and it differs from the concept of consensus; consensus implies that the estimation results of all sensors converge to a same value, while consistency not only implies achieving consensus asymptotically, but also requires the estimated value converge to the true value.

We first prove the consistency for static UGVs. The consistency for moving UGVs is subsequently proved. For simplicity and clarity, we assume S is a finite set (e.g. a finely discretized field).

6.1 Static UGVs

The consistency of LIFO-DBF for static UGVs is stated as follows:

Theorem 4. *Assume the UGVs are static and the sensors are unbiased. If the network of N UGVs satisfies the condition of interaction topology in proposition 1, then the MAP estimator of target position converges in probability to the true position of the target using LIFO-DBF, i.e.,*

$$\lim_{k\to\infty} P(X_k^{MAP} = x_k^g|\mathbf{z}_{1:k}^i) = 1,\ i = 1,\ldots,N,$$

where

$$X_k^{MAP} = \arg\max_X P_{pdf}^i(X_k|\mathbf{z}_{1:k}^i).$$

Proof. For the purpose of clarity, define the time set of i^{th} UGV, $\mathscr{K}_{j,k}^i$, $j \in \{1,\ldots,N\}$, that contains the time steps of measurement by j^{th} UGV that are contained in B_k^i. According to Theorem 3, it is known that the cardinality of $\mathscr{K}_{j,k}^i$ has following property: $k-(N-1)T_u < |\mathscr{K}_{j,k}^i| \le k$. Considering the conditional independence of measurements given $x_k^g \in S$, the batch form of DBF at k^{th} step is

$$P_{pdf}^i(X_k|B_k^i) = P_{pdf}^i(X_k|z_{1:k_1^i}^1,\ldots,z_{1:k_N^i}^N)$$
$$= \frac{P_{pdf}^i(X_0)\prod_{j=1}^N\prod_{l=1}^{k_j^i} P(z_l^j|X_l)P(X_l|X_{l-1})}{\sum_{X_0,\ldots,X_k\in S} P_{pdf}^i(X_0)\prod_{j=1}^N\prod_{l=1}^{k_j^i} P(z_l^j|X_l)P(X_l|X_{l-1})}.$$

Comparing $P_{pdf}^i(x^T|B_k^i)$ with $P_{pdf}^i(x^{T^*}|B_k^i)$ yields

$$\frac{P_{pdf}^i(X_k = x_k|B_k^i)}{P_{pdf}^i(X_k = x_k^g|B_k^i)} = \frac{P_{pdf}^i(x_0)\prod_{j=1}^N\prod_{l\in\mathscr{K}_{j,k}^i} P(z_l^j|x_l)}{P_{pdf}^i(x_0^g)\prod_{j=1}^N\prod_{l\in\mathscr{K}_{j,k}^i} P(z_l^j|x_l^g)}. \tag{13}$$

Take the logarithm of Eq. (13) and average it over k steps:

$$\frac{1}{k}\ln\frac{P_{pdf}^i(X_k = x_k|B_k^i)}{P_{pdf}^i(X_k = x_k^g|B_k^i)} = \frac{1}{k}\ln\frac{P_{pdf}^i(x_0)}{P_{pdf}^i(x_0^g)} + \sum_{j=1}^N \frac{1}{k}\sum_{l\in\mathscr{K}_{j,k}^i}\ln\frac{P(z_l^j|x_k)}{P(z_l^j|x_k^g)}. \tag{14}$$

Since $P_{pdf}^i(x_0)$ and $P_{pdf}^i(x_0^g)$ are bounded, then

$$\lim_{k\to\infty}\frac{1}{k}\ln\frac{P_{pdf}^i(x_0)}{P_{pdf}^i(x_0^g)} = 0. \tag{15}$$

Utilizing the facts: (1) z_l^j are conditionally independent samples from $P(z_l^j|x_l^g)$ and (2) $k-(N-1)T_u < |\mathscr{K}_{j,k}^i| \le k$, the law of large numbers yields

$$\frac{1}{k}\sum_{l=1}^{k_j^i}\ln\frac{P(z_l^j|x_l)}{P(z_l^j|x_l^g)} \xrightarrow{P} \mathbb{E}_{z_l^j}\left[\frac{P(z_l^j|x_l)}{P(z_l^j|x_l^g)}\right] \tag{16a}$$
$$= \int_{z_l^j} P(z_l^j|x_l^g)\frac{P(z_l^j|x_l)}{P(z_l^j|x_l^g)}dz_l^j \tag{16b}$$
$$= -D_{KL}\left(P(z_l^j|x_l)\|P(z_l^j|x_l^g)\right), \tag{16c}$$

where "$\xrightarrow{P}$" represents "convergence in probability" and $D_{KL}(P_1\|P_2)$ denotes the Kullback-Leibler (KL) divergence between two probability distribution P_1 and P_2. KL divergence has the property that

$$\forall P_1, P_2,\ D_{KL}(P_1\|P_2) \le 0$$
$$\text{equality holds iff } P_1 = P_2.$$

This leads to the following conclusion:

$$\frac{1}{k}\sum_{l=1}^{k_j^i}\ln\frac{P(z_l^j|x_l)}{P(z_l^j|x_l^g)} < 0,\quad x_l \ne x_l^g \tag{18a}$$
$$\frac{1}{k}\sum_{l=1}^{k_j^i}\ln\frac{P(z_l^j|x_l)}{P(z_l^j|x_l^g)} = 0,\quad x_l = x_l^g. \tag{18b}$$

Then by considering the limiting case of Eq. (14), we can get:

$$\lim_{k\to\infty}\frac{1}{k}\ln\frac{P_{pdf}^i(x_l|B_k^i)}{P_{pdf}^i(x_l^g|B_k^i)} < 0,\quad x_l \ne x_l^g \tag{19a}$$
$$\lim_{k\to\infty}\frac{1}{k}\ln\frac{P_{pdf}^i(x_l|B_k^i)}{P_{pdf}^i(x_l^g|B_k^i)} = 0,\quad x_l = x_l^g. \tag{19b}$$

Eq. (19a) and (19b) imply that

$$\frac{P_{pdf}^i(x_l|B_k^i)}{P_{pdf}^i(x_l^g|B_k^i)} \xrightarrow{P} 0,\quad x_l \ne x_l^g \tag{20a}$$
$$\frac{P_{pdf}^i(x_l|B_k^i)}{P_{pdf}^i(x_l^g|B_k^i)} \xrightarrow{P} 1,\quad x_l = x_l^g. \tag{20b}$$

Therefore,

$$\lim_{k\to\infty} P(X_k^{MAP} = x_k^g|B_k^i) = 1.$$

6.2 Moving UGVs

This subsection considers the case of using moving UGVs to localize a target, either static or moving. The difficulty of consistency proof for this case lies in the fact that each UGV makes measurements at multiple positions. Here, the main idea of the proof is to classify UGV measurement positions into two disjoint sets: *infinite-measurement spots* that contain positions where a UGV visits infinitely many times as time tends to infinity, and *finite-measurement spots* that contain positions where the UGV visits finitely many times (i.e., the UGV does not visit again after a finite time period). Before stating the main theorem, the following lemma is introduced.

Lemma 1. *For a set of UGVs moving within a collection of finite positions, each UGV has at least one position where infinite measurements are made as k tends to infinity.*

Proof. Let $n_s^{i,k}$ denote the times that i^{th} UGV visits s^{th} position up to time k. Then, $\sum_{s\in S} n_s^{i,k} = k$. It is straightforward to see that $\exists n_s^{i,k}$, such that $n_s^{i,k} \to \infty$, as $k \to \infty$.

Theorem 5. *Assume UGVs move within a collection of finite positions and sensors are unbiased, then the MAP estimator of target position converges in probability to the true position of the target using LIFO-DBF, i.e.,*

$$\lim_{k\to\infty} P(X_k^{MAP} = x_k^g | B_k^i) = 1,\ i = 1, \ldots, N.$$

Proof. The batch form of DBF at k^{th} step is

$$\frac{P^i_{pdf}(X_k = x_k | B_k^i)}{P^i_{pdf}(X_k = x_k^g | B_k^i)} = \frac{P^i_{pdf}(x_0) \prod_{j=1}^{N} \prod_{l\in\mathscr{K}^i_{j,k}} P(z_l^j | x; x_l^j)}{P^i_{pdf}(x_0^g) \prod_{j=1}^{N} \prod_{l\in\mathscr{K}^i_{j,k}} P(z_l^j | x^g; x_l^j)}. \tag{21}$$

The only difference from Eq. (13) is that $P(z_l^j|x;x_l^j)$ in Eq. (21) varies as the UGV moves. For each UGV, there exists at least one position where infinite measurements are made as $k \to \infty$, according to Lemma 1. For the finite-measurement spots, by referring to Eq. (16), it is easy to know that their contribution to Eq. (14) diminishes when $k \to \infty$. Therefore, proof using Eq. (21) can be reduced to only considering infinite-measurement spots and the rest of proof is similar to that of Theorem 4.

Remark 3. *The assumption of unbiased sensors are important for the consistency of the estimator. In fact, with unknown non-zero bias, the distribution of z_l^j differs from $P(z_l^j|x^g)$, which invalidates the derivation in Eq. (16) and the consistency proof. This assumption also makes intuitive sense. In the extreme case, if each sensor has a very large unknown measurement offset, then the estimated target position of each sensor (without communicating with other sensors) will be very different from each other's. Therefore, no common target position can be correctly obtained when they fuse measurements.*

7 Discussion

7.1 Approximate algorithm

to be filled.

7.2 Changing number of robots

to be filled.

8 Simulation

An example result is shown in Figure 4. Will generate better simulation later.

9 Experiment

to be filled.

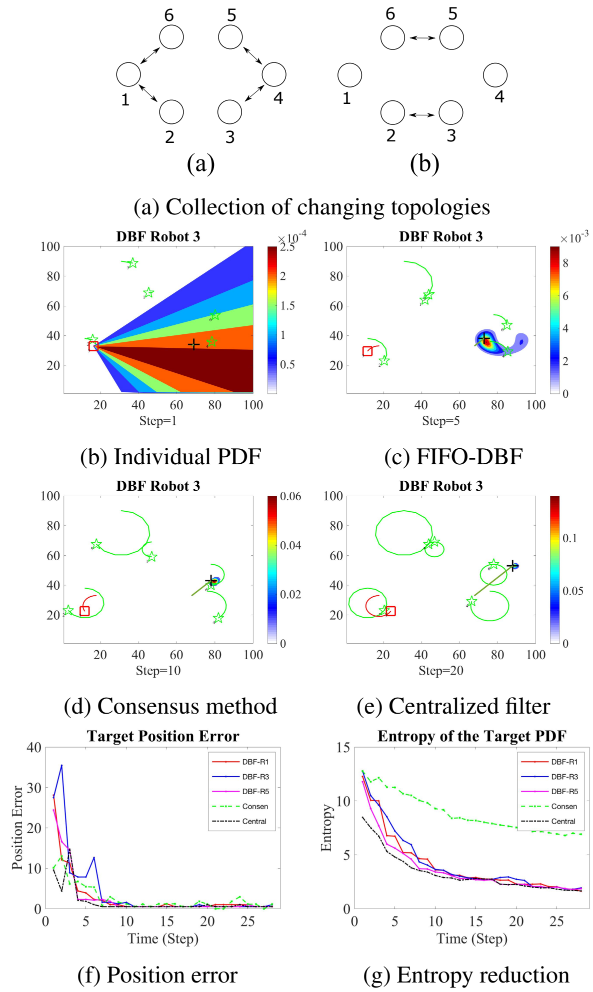

Fig. 4: First scenario: (a) two types of topologies; (b) individual PDF of the 3rd UGV after initial observation; (c)-(e) PDFs at the end of simulation using different filters; (f) average position estimation errors; (g) average entropy of PDF. In last two figures, metrics are based on the PDFs of the 1st, 3rd and 5th UGV using FIFO-DBF, the common PDF using CbDF and using CF.

10 Conclusion

This paper presents a measurement dissemination-based distributed Bayesian filtering (DBF) method for a network of multiple unmanned ground vehicles (UGVs) under dynamically changing interaction topologies. The information exchange among UGVs relies on the Full-In-and-Full-Out (FIFO) protocol, under which UGVs exchange full communication buffers with neighbors and can significantly reduce the transmission burden between each pair of UGVs to scale linearly with the network size. Under the condition that the union of the switching topologies is connected frequently enough, FIFOcan disseminate observations over the network within finite time. The FIFO-based DBF algorithm is then derived to estimate individual probability density function (PDF) for target localization. The consistency of this algorithm is proved by utilizing the law of large numbers, ensuring that each individual estimate of target position converges in probability to the true value. Simulations comparing FIFO-DBF with consensus-based distributed filters (CbDF) and the centralized filter (CF) show that FIFO-DBF achieves similar performance as CF and superior performance over CbDF while requiring less communication resource.

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