Distributed Bayesian Filters for Multi-Robot Network by Using Latest-In-and-Full-Out Exchange Protocol of Observations

Chang Liu[1], Shengbo Eben Li[2] and J. Karl Hedrick[3]

Abstract— **This paper presents a measurement dissemination-based distributed Bayesian filtering (DBF) method for a multi-robot network, utilizing the Latest-In-and-Full-Out (LIFO) local exchange protocol of observations. Different from statistics dissemination approaches that transmit posterior distributions or likelihood functions, each robot under LIFO only exchanges with neighboring robots a full communication buffer consisting of latest available measurements, which significantly reduces the transmission burden between each pair of robots from the order of environmental size to that of robot number. Under the condition of fixed and undirected topology, LIFO can guarantee non-intermittent dissemination of all observations over the network within finite time. Two types of LIFO-based DBF algorithms are then derived to estimate individual posterior density function (PDF) for static and moving target, respectively. For the static target, each robot locally fuses the newly received observations while for the moving target, a set of historical observations is stored and updated. The consistency of LIFO-based DBF is proved that estimated target position converges in probability to the true target position when the number of observations tends to infinity. The effectiveness of this method is demonstrated by simulations of target localization.**

I. INTRODUCTION

Distributed filtering that focuses on using a group of networked robots to collectively infer environment status has been used for various applications, such as intruder detection [?], pedestrian tracking [?] and micro-environmental monitoring [?]. Several techniques have been developed for distributed filtering. For example, Olfati-Saber (2005) proposed a distributed Kalman filter (DKF) for estimating states of linear systems with Gaussian process and measurement noise [?]. Each DKF used additional low-pass and band-pass consensus filters to compute the average of weighted measurements and inverse-covariance matrices. Madhavan et al. (2004) presented a distributed extended Kalman filter for nonlinear systems [?]. This filter was used to generate local terrain maps by using pose estimates to combine elevation gradient and vision-based depth with environmental features. Gu (2007) proposed a distributed particle filter for Markovian target tracking over an undirected sensor network [?]. Gaussian mixture models (GMM) were adopted to approximate the posterior distribution from weighted particles and the parameters of GMM were exchanged via average consensus filter. As a generic filtering scheme for nonlinear systems and arbitrary noise distributions, distributed Bayesian filters (DBF) have received increasing interest during past years [?], [?], which is the focus of this study.

The design of distributed filtering algorithms is closely related to communication topology of multi-robot network, which can be classified into two types: fusion center (FC)-based and neighborhood (NB)-based. In FC-based approaches, each robot uses a filter to estimate local statistics of environment status based on its own measurement. The local statistics is then transmitted (possibly via multi-hopping) to a single FC, where a global posterior distribution (or statistical moments in DKF [?]) is calculated at each filtering cycle after receiving all local information [?], [?]. In NB-based approaches, a set of robots execute distributed filters to estimate individual posterior distribution. Consensus of individual estimates is achieved by solely communicating statistics and/or observations within local neighbors of these robots. The NB-based methods have become popular in recent years since such approaches do not require complex routing protocols or global knowledge of the network and therefore are robust to changes in network topology and to link failures.

So far, most studies on NB-based distributed filtering have mainly focused on the so-called *statistics dissemination* strategy that each robot actually exchanges statistics, including posterior distributions and likelihood functions, with neighboring robots [?]. This strategy can be further categorized into two types: leader-based and consensus-based. In the former, statistics is sequentially passed and updated along a path formed by active robots, called leaders. Only leaders perform filtering based on its own measurement and received measurements from local neighbors. For example, Sheng et al. (2005) proposed a multiple leader-based distributed particle filter with Gaussian Mixer for target tracking [?]. Sensors are grouped into multiple uncorrelated cliques, in each of which a leader is assigned to perform particle filtering and the particle information is then exchanged among leaders. In consensus-based distributed filters, every robot diffuses statistics among neighbors, via which global agreement of the statistics is achieved by using consensus protocols [?], [?], [?]. For example, Hlinka et al. (2012) proposed a distributed method for computing an approximation of the joint (all-sensors) likelihood function by means of weighted-linear-average consensus algorithm when local likelihood functions belong to the exponential family of distributions

*The first two authors, C. Liu and S. Li, have equally contributed to this research.

[1]Chang Liu is with the Vehicle Dynamics & Control Lab, Department of Mechanical Engineering, University of California, Berkeley, Berkeley, CA 94709, USA. Email: `changliu@berkeley.edu`

[2]Shengbo Eben Li is with the Department of Automotive Engineering, Tsinghua University, Beijing, 100084, China. He is currently working at Department of Mechanical Engineering, University of California, Berkeley as a visiting scholar. Email: `lisb04@gmail.com`

[3]J. Karl Hedrick is with the Vehicle Dynamics & Control Lab, Department of Mechanical Engineering, University of California, Berkeley, Berkeley, CA 94709, USA. Email: `khedrick@me.berkeley.edu`

[?]. Saptarshi et al. (2014) presented a Bayesian consensus filter that uses logarithmic opinion pool for fusing posterior distributions of the tracked target [?]. Other examples can be found in [?], [?].

Despite the popularity of statistics dissemination strategy, exchanging statistics can consume high communication resources. Approximating statistics with parametric models, such as Gaussian Mixture Models [?], can significantly reduce communication burden. However, such manipulation increases the computation burden for each robot and sacrifices accuracy of filtering due to the approximation. One promising remedy is to disseminate measurement instead of statistics among neighbors, which, however, has not been fully exploited. One pioneering work was done by Coates et al. (2004), who used adaptive encoding of observations to minimize communication overhead [?]. Ribeiro et al. (2006) exchanged quantized observations along with error-variance limits considering more pragmatic signal models [?]. A recent work was conducted by Djuric et al. (2011), who proposed to broadcast raw measurements to other agents, and therefore each robot has a complete set of observations of other robots for executing particle filtering [?]. A shortcoming of aforementioned works is that their communication topologies are assumed to be a complete graph that every pair of distinct robots is directly connected by a unique edge, which is not always feasible in reality.

This paper extends existing works by introducing a Latest-In-and-Full-Out (LIFO) protocol into distributed Bayesian filters (DBF) for networked robots. Each robot is only allowed to broadcast observations to its neighbors by using single-hopping and then implements individual Bayesian filter locally after receiving transmitted observations. The main benefit of using LIFO is on the reduction of communication burden, with the transmission data volume scaling linearly with the robot number, while a statistics dissemination-based strategy can suffer from the order of environmental size. The proposed LIFO-based DBF has following properties: (1) For a fixed and undirected network, LIFO guarantees the global dissemination of observations over the network in a non-intermittent manner. (2) The corresponding DBF ensures consistency of estimated target position, which also implies the consensus of target PDFs.

The rest of this paper is organized as follows: The problem of distributed Bayesian filtering is formulated in Section II. The LIFO-based DBF algorithm is described in Section III, followed by the proof of consistency and consensus in Section IV. Simulation results are presented in Section V and Section VI concludes the paper.

II. PROBLEM FORMULATION

Consider a network of N robots in a bounded two-dimensional space S. Each robot is equipped with a binary sensor for environmental perception. Due to the limit of communication range, each robot can only exchange observations with its neighbors. The Bayesian filter is run locally on each robot based on its own and received observations via single-hopping to estimate true position of target.

A. Probabilistic Model of Binary Sensor

The binary sensor only gives two types of observation: 1 if the target is detected, and 0 if no target is detected. The observation of i^{th} sensor at k^{th} time step is denoted as z_k^i. The likelihood function that the target is detected is:

$$P(z_k^i = 1 | x_k^T; x_k^{R,i}) \in [0, 1]\,,\ x_k^T \in S \tag{1}$$

where x_k^T denotes the target position; $x_k^{R,i}$ is the robot position. Correspondingly, the likelihood function that no target is detected is:

$$P(z_k^i = 0 | x_k^T; x_k^{R,i}) = 1 - P(z_k^i = 1 | x_k^T; x_k^{R,i}) \tag{2}$$

The combination of Eq. (1) and Eq. (2) forms a binary sensor model parameterized by x_k^T and $x_k^{R,i}$. For the purpose of simplicity, we will not explicitly write $x_k^{R,i}$ when no confusion may occur. The commonly used likelihood functions for binary sensor include Gaussian function [?] and step function [?].

Remark 1: Given the knowledge of current target and robot positions, current observation of each robot is conditionally independent from its own past observations and those of other robots.

Remark 2: The proposed LIFO protocol to be described in Section III is applicable for both homogeneous and heterogeneous binary sensors. A homogeneous model can simplify the analysis of completeness, while a heterogeneous model is more close to real sensing characteristics. In addition, it also works for other types of sensors, such as laser scanners and cameras.

B. Graphical Model of Communication Topology

The robot network is always assumed to be connected, i.e., there exists a path, either direct or indirect, between every pair of robots. Under this assumption, consider an undirected and fixed graph $G = (V, E)$, where $V = \{1, \dots, N\}$ represents the index set of robots and $E = V \times V$ denotes the edge set. The adjacency matrix $M = [m_{ij}]$ describes the communication topology of G:

$$m_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{if } (i,j) \notin E \end{cases},$$

where m_{ij} denotes the entity of adjacency matrix. The notation $m_{ij} = 1$ indicates that a communication link exists between i^{th} and j^{th} robot and $m_{ij} = 0$ indicates no communication between them.

The *direct neighborhood* of i^{th} robot is defined as $\mathcal{N}_i = \{j | m_{ij} = 1, \forall j \in \{1, \dots, N\}\}$. All the robots in $\mathcal{N}_i$ can directly exchange information with i^{th} robot. In addition to direct neighborhood, another set called *available neighborhood* is defined as $\mathcal{Q}_i$, which contains indices of robots whose observations can be received by the i^{th} robot given a specific observation exchange protocol. Note that in general $\mathcal{N}_i \subseteq \mathcal{Q}_i$. Fig. 1 illustrates three types of typical topologies: ring [?], line [?], and star [?]. All of them are undirected and connected topologies.

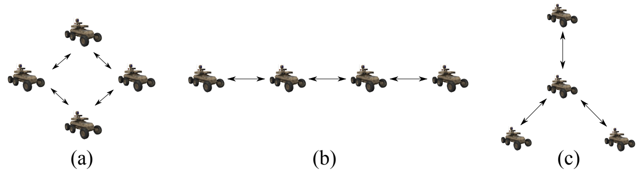


Fig. 1: Three types of topologies: (a) ring topology; (b) line topology; (c) star topology

C. Distributed Bayesian Filter for Multiple Robots

The generic distributed Bayesian filter (DBF) is introduced in this section. Each robot has its individual estimation of posterior density function (PDF) of target position, called *individual PDF*. The individual PDF of i^{th} robot at time k is defined as $P^i_{pdf}(x^T_k|\mathbf{z}^i_{1:k})$, where $\mathbf{z}^i_{1:k}$ denotes the set of observations by i^{th} robot and by robots in $\mathcal{Q}_i$ that are transmitted to i^{th} robot by time k. The individual PDF is initialized as $P^i_{pdf}(x^T_0|\mathbf{z}^i_0) = P(x^T_0)$, given all available prior information including past experience and environmental knowledge. Under the framework of DBF, the individual PDF is recursively estimated by two steps, i.e., prediction step and updating step, based on observations of i^{th} robot and robots in $\mathcal{Q}_i$.

1) Prediction: At time k, the prior individual PDF $P^i_{pdf}(x^T_{k-1}|\mathbf{z}^i_{1:k-1})$ is first predicted forward by using the Chapman-Kolmogorov equation:

$$P^i_{pdf}(x^T_k|\mathbf{z}^i_{1:k-1}) = \int P(x^T_k|x^T_{k-1})P^i_{pdf}(x^T_{k-1}|\mathbf{z}^i_{1:k-1})dx^T_{k-1} \tag{3}$$

where $P(x^T_k|x^T_{k-1})$ is a Markov motion model of the target, independent of robot states. This model describes the state transition probability of the target from a prior state x^T_{k-1} to posterior state x^T_k. Note that the target is static in many search applications, such as the indoor search for stationary objects [**?**]. For a static target, its Markov motion model is simplified to be

$$P(x^T_k|x^T_{k-1}) = \begin{cases} 1 & \text{if } x^T_k = x^T_{k-1} \\ 0 & \text{if } x^T_k \neq x^T_{k-1} \end{cases}.$$

2) Updating: The i^{th} individual PDF is then updated by Bayes' formula using the set of newly received observations at time k, $\mathbf{z}^i_k$:

$$P^i_{pdf}(x^T_k|\mathbf{z}^i_{1:k}) = K_i P^i_{pdf}(x^T_k|\mathbf{z}^i_{1:k-1})P(\mathbf{z}^i_k|x^T_k) \tag{4}$$

where K_i is a normalization factor, given by:

$$K_i = 1/\int P^i_{pdf}(x^T_k|\mathbf{z}^i_{1:k-1})P(\mathbf{z}^i_k|x^T_k)dx^T_k$$

and $P^i_{pdf}(x^T_k|\mathbf{z}^i_{1:k})$ is called posterior individual PDF; $P(z^i_k|x^T_k)$ is the likelihood function of observation z^i_k, described in Eq. (1) and Eq. (2).

III. Distributed Bayesian Filter via Latest-In-and-Full-Out Protocol

This study proposes a Latest-In-and-Full-Out (LIFO) protocol for observation exchange and derives two corresponding distributed Bayesian filtering (DBF) algorithms, shorted as LIFO-DBF. The data communication in LIFO uses synchronized step as the execution of DBF. In each step, LIFO only allows single-hopping communication within the direct neighborhood, but is able to broadcast observations of each robot to any other agents after a finite number of steps. The individual PDF is forward predicted and updated in DBF after each LIFO cycle. The theoretical analysis show that LIFO-DBF can ensure the consistency and consensus of distributed estimation while requiring much less communication burden than statistics dissemination-based methods.

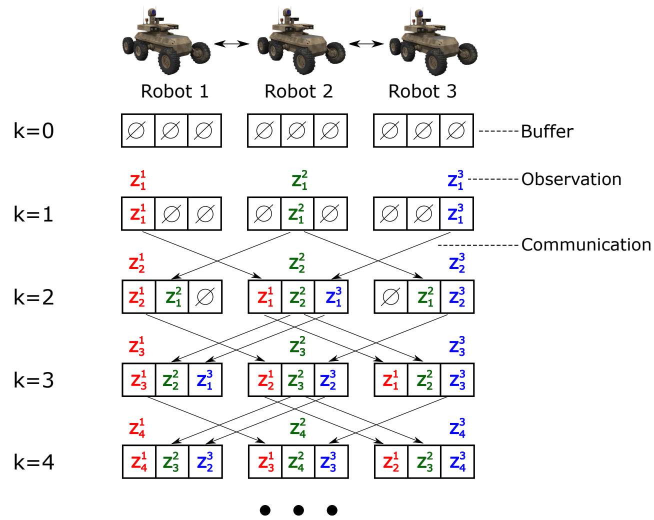


Fig. 2: Example of LIFO with three robots using line communication topology

A. Latest-In-and-Full-Out (LIFO) Protocol

Under LIFO, each robot contains a communication buffer (CB) to store its latest knowledge of observations of all robots:

$$\mathbf{z}^{CB,i}_k = \left[z^1_{k^i_1}, \dots, z^N_{k^i_N}\right]$$

where $z^j_{k^i_j}$ represents the observation made by j^{th} robot at time k^i_j. Note that under LIFO, $\mathcal{Q}_i = \{1, \dots, N\} \setminus \{i\}$, which will be proved in Corollary 1. At time k, $z^j_{k^i_j}$ is received and stored in i^{th} robot CB, in which k^i_j is the latest observation time of j^{th} robot available to i^{th} robot. Due to the communication delay, $k^i_j < k, \forall j \neq i$ and $k^i_i = k$ always holds. The **LIFO protocol** is stated in Algorithm 1.

Fig. 2 illustrates the LIFO cycles with 3 robots using a line topology. Two facts can be noticed in Fig. 2: (1) all robot CBs are filled within 3 steps, which means under LIFO each robot has a maximum delay of 2 steps for receiving observations from other robots; (2) after filled, CBs are updated non-intermittently, which means each robot continuously receives new observations of other robots. Extending the two facts to a network of N robots, we have the following proposition:

Proposition 1: For a fixed and undirected network of N robots, LIFO uses the shortest path(s) between i^{th} and j^{th} robot to exchange observation, the length of which is the delay $\tau_{i,j}$ between these two robots.

Algorithm 1 LIFO Protocol

(1) Initialization: The CB of i^{th} robot is initialized when $k=0$:

$$z^j_{k^i_j}=\varnothing,\ k^i_j=0,\ j=1,\ldots,N$$

(2) At k^{th} step for i^{th} robot :

(2.1) Receiving Step:

The i^{th} robot receives all CBs of its direct neighborhood $\mathcal{N}_i$. The received CBs are totally $|\mathcal{N}_i|$ groups, each of which corresponds to the (k-1)-step CB of a robot in $\mathcal{N}_i$. The received CB from l^{th} ($l\in\mathcal{N}_i$) robot is denoted as

$$\mathbf{z}^{CB,l}_{k-1}=\left[z^1_{(k-1)^l_1},\ldots,z^N_{(k-1)^l_N}\right],\ l\in\mathcal{N}_i$$

(2.2) Observation Step:

The i^{th} robot updates $z^j_{k^i_j}$ ($j=i$) by its own observation at current step:

$$z^j_{k^i_j}=z^i_k,\ k^i_j=k,\ \text{if}\ j=i.$$

(2.3) Comparison Step:

The i^{th} robot updates other elements of its own CB, i.e., $z^j_{k^i_j}$ ($j\neq i$), by selecting the latest information among all received CBs from $\mathcal{N}_i$. For all $j\neq i$,

$$l_{\text{latest}}=\underset{l\in\mathcal{N}_i,\ i}{\operatorname{argmax}}\left\{(k-1)^i_j,(k-1)^l_j\right\}$$
$$z^j_{k^i_j}=z^j_{(k-1)^{l_{\text{latest}}}_j},\ k^i_j=(k-1)^{l_{\text{latest}}}_j$$

(2.4) Sending Step:

The i^{th} robot broadcasts its updated CB to all of its neighbors defined in $\mathcal{N}_i$.

(3) $k\leftarrow k+1$ until stop

Proof: Without loss of generality, assume that there is a unique shortest path between i and j, denoted by $T^{j,i}_{n^*}=(v_1,\ldots,v_{n^*})$, with $v_1=j, v_{n^*}=i, v_{m+1}\in\mathcal{N}_{v_m}$. Then, the distance between i and j is $d(j,i)=n^*-1$. The following mathematical induction will prove Proposition 1.

Step (1): When $d(j,i)=1$, $j\in\mathcal{N}_i$ and j can directly send z^j_k to i. Then z^j_k is stored in i^{th} CB at time $k+1$, i.e. $\tau_{i,j}=1$. Proposition 1 holds for $d(j,i)=1$, $\forall i,j\in\{1,\ldots,N\}$.

Step (2): Suppose that Proposition 1 holds for $d(j,i)=s$, $s\geq 2$, $\forall i,j\in\{1,\ldots,N\}$. Then for $d(j,i)=s+1$, i.e., $n^*=s+2$, by the Bellman's principle of optimality, the path $T^{j,l}_{n^*-1}=(v_1,\ldots,v_{n^*-1})$ is a shortest path between j and l, where $v_{n^*-1}=l$ and $i\in\mathcal{N}_l$. The assumption that Proposition 1 holds for $d(j,i)=s$ implies that z^j_k is received and stored in l^{th} robot's CB at time $k+s$. Since $i\in\mathcal{N}_l$, i^{th} robot receives z^j_k at $k+s+1$. For any other path $T^{j,i}_n=(v_1,\ldots,v_n)$ with $n>n^*$, z^j_k cannot be received by i earlier than $k+s+1$. Therefore $\tau_{i,j}=s+1$. This proves the Proposition 1 for $d(j,i)=s+1$. ■

Corollary 1: For the same topology in Proposition 1, all elements in $\mathbf{z}^{CB,i}_k$ under LIFO become filled when $k\geq N$, i.e., $\mathcal{Q}_i=\{1,\ldots,N\}\setminus\{i\}$.

Proof: In a network of N robots, the maximal length of shortest paths is no greater than $N-1$. Based on Proposition 1, $\tau_{i,j}\leq N-1$ and thus all elements of $\mathbf{z}^{CB,i}_k$ become filled when $k\geq N$. ■

Corollary 2: For the same topology in Proposition 1, once all elements in $\mathbf{z}^{CB,i}_k$ are filled, the updating of each element is non-intermittent.

Proof: For a network with fixed topology, shortest path(s) between any pair of nodes are fixed. Therefore, based on Proposition 1, $\tau_{i,j}$ is constant and the updating of each element in $\mathbf{z}^{CB,i}_k$ is non-intermittent. ■

Remark 3: Compared to statistics dissemination, LIFO is generally more communication-efficient for distributed filtering. To be specific, consider an $M\times M$ grid environment with a network of N robots, the transmitted data of LIFO between each pair of robots are only the CB of each robot and the corresponding robot positions where observations were made, the length of which is $O(N)$. On the contrary, the length of transmitted data for a statistics dissemination approach that transmits unparameterized posterior distributions or likelihood functions is $O(M^2)$, which is in the order of environmental size. Since M is generally much larger than N in applications such as target localization, LIFO requires much less communication resources.

B. Algorithm of LIFO-DBF for Static Target

This section derives the LIFO-DBF algorithm for localizing a static target. Each robot stores last-step individual PDF, i.e., $P^i_{pdf}(x^T|\mathbf{z}^i_{1:k-1})$. According to Corollary 2, $\mathbf{z}^i_k=\mathbf{z}^{CB,i}_k$ and $\mathbf{z}^i_{1:k}=\mathbf{z}^{CB,i}_{1:k}=\left[z^1_{1:k^i_1},\ldots,z^N_{1:k^i_N}\right]$. The assumption of static target can simplify the Bayesian filter as the prediction step becomes unnecessary. Therefore, the i^{th} individual PDF is only updated by

$$\begin{aligned}P^i_{pdf}(x^T|\mathbf{z}^i_{1:k})&=K_iP^i_{pdf}(x^T_k|\mathbf{z}^i_{1:k-1})P(\mathbf{z}^i_k|x^T_k)\\&=K_iP^i_{pdf}(x^T|\mathbf{z}^i_{1:k-1})\prod_{j=1}^{N}P(z^j_{k^i_j}|x^T)\end{aligned}\quad(5)$$

where

$$K_i=1/\int P^i_{pdf}(x^T|\mathbf{z}^i_{1:k-1})\prod_{j=1}^{N}P(z^j_{k^i_j}|x^T)dx^T$$

C. Algorithm of LIFO-DBF for Moving Target

This section derives the LIFO-DBF for localizing a moving target. Instead of storing last-step PDF, at time k each robot maintains an individual PDF of time $(k-N)$ and a collection of historical observations, called the *record set*, from time $(k-N+1)$ to k. The i^{th} individual PDF is then alternatively predicted and updated by using aforementioned Bayesian filter (Eq. (3) and Eq. (4)) from $(k-N)$ to k. Fig. 3 illustrates the LIFO-DBF procedure for the 1[st] robot as an example. With a line topology, the record set of 1[st] robot is shown as a triangle.

Let Ω^i_ξ ($\xi=1,\ldots,N$) denote the index set of robots whose observation at time $(k-N+\xi)$ is stored in i^{th} robot's record set. The **LIFO-DBF algorithm** for moving target is then stated in Algorithm 2.

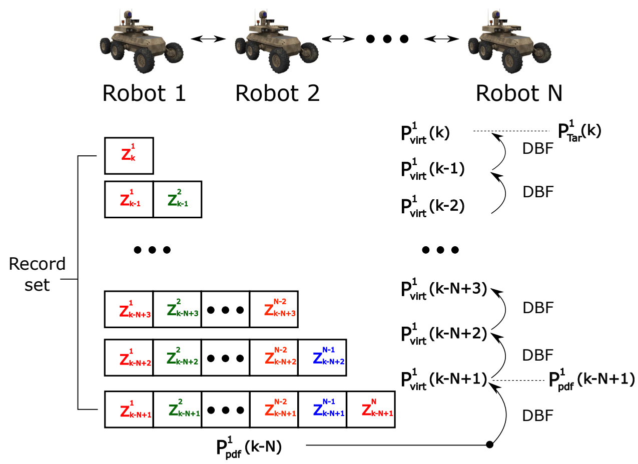


Fig. 3: Example of LIFO-DBF for 1st robot at time k. Networked robots take a line topology, shown in the top. The stored individual PDF is $P^1_{pdf}(k-N)$. The robot first calculates $P^1_{virt}(k-N+1)$ using DBF and stores it as $P^1_{pdf}(k-N+1)$. Repeating DBF until obtaining $P^1_{pdf}(k)$, which is then used as the target PDF estimation of 1st robot at time k. In this example, $\Omega^1_\xi = \{1, 2, \ldots, N+1-\xi\}$, $\xi = 1, \ldots, N$.

Remark 4: For the static target, each robot only needs current step CB to update individual PDFs. Therefore, besides storing individual PDFs, only current-step CB is stored in robot memory and all historical CBs can be discarded, which means that the size of occupied memory is $O(N)$. On the contrary, for the moving target, each robot needs to store a triangular matrix of historical observation with size of $O(N^2)$ and an individual PDF with size $O(M^2)$, which means that the size of occupied memory in each robot is $O(M^2 + N^2)$.

IV. Proof of Consistency and Consensus

This section proves consistency and consensus of LIFO-DBF. Only proofs for localizing static target are presented, including static robots and moving robots. The proof of LIFO-DBF for moving target is similar to that of static target by considering the dynamic model of the target, but with more complicated algebraic manipulation.

A. Proof for static robots

Considering S is finite and x^{T^*} is the true location of target, the consistency of LIFO-DBF for static robots is stated as follows:

Theorem 1: When robots are static, the estimated target position converges to the true position of target in probability using LIFO-DBF, i.e.,

$$\lim_{k\to\infty} P(x^T = x^{T^*}|\mathbf{z}^i_{1:k}) = 1, \ i = 1, \ldots, N.$$

Proof: Considering the conditional independence of observations given $x^T \in S$, the batch form of DBF at k^{th} step is:

$$P^i_{pdf}(x^T|\mathbf{z}^i_{1:k}) = P^i_{pdf}(x^T|z^1_{1:k^i_1}, \ldots, z^N_{1:k^i_N})$$
$$= \frac{P^i_{pdf}(x^T)\prod_{j=1}^{N}\prod_{l=1}^{k^i_j} P(z^j_l|x^T)}{\sum_{x^T\in S} P^i_{pdf}(x^T)\prod_{j=1}^{N}\prod_{l=1}^{k^i_j} P(z^j_l|x^T)},$$

where P^i_{pdf} is i^{th} initial individual PDF. It is known from Corollary 1 and Corollary 2 that $k - N < k^i_j \le k$.

Comparing $P^i_{pdf}(x^T|\mathbf{z}^i_{1:k})$ with $P^i_{pdf}(x^{T^*}|\mathbf{z}^i_{1:k})$ yields

$$\frac{P^i_{pdf}(x^T|\mathbf{z}^i_{1:k})}{P^i_{pdf}(x^{T^*}|\mathbf{z}^i_{1:k})} = \frac{P^i_{pdf}(x^T)\prod_{j=1}^{N}\prod_{l=1}^{k^i_j} P(z^j_l|x^T)}{P^i_{pdf}(x^{T^*})\prod_{j=1}^{N}\prod_{l=1}^{k^i_j} P(z^j_l|x^{T^*})} \tag{6}$$

Take the logarithm of Eq. (6) and average it over k steps:

$$\frac{1}{k}\ln\frac{P^i_{pdf}(x^T|\mathbf{z}^i_{1:k})}{P^i_{pdf}(x^{T^*}|\mathbf{z}^i_{1:k})} = \frac{1}{k}\ln\frac{P^i_{pdf}(x^T)}{P^i_{pdf}(x^{T^*})} + \sum_{j=1}^{N}\frac{1}{k}\sum_{l=1}^{k^i_j}\ln\frac{P(z^j_l|x^T)}{P(z^j_l|x^{T^*})}. \tag{7}$$

Since $P^i_{pdf}(x^T)$ and $P^i_{pdf}(x^{T^*})$ are bounded, then

$$\lim_{k\to\infty}\frac{1}{k}\ln\frac{P^i_{pdf}(x^T)}{P^i_{pdf}(x^{T^*})} = 0. \tag{8}$$

Algorithm 2 LIFO-DBF Algorithm

For i^{th} robot at k^{th} step:
After updating CB by Algorithm 1,
(1) The stored individual PDF for time $(k-N)$ is:

$$P^i_{pdf}(x^T_{k-N}|z^1_{1:k-N}, \ldots, z^N_{1:k-N})$$

(2) Initialize a virtual PDF by assigning the individual PDF to it:

$$P^i_{virt}(x^T_{k-N}) = P^i_{pdf}(x^T_{k-N}|z^1_{1:k-N}, \ldots, z^N_{1:k-N})$$

(3) From $\xi = 1$ to N, repeat two steps of Bayesian filtering:
(3.1) Prediction

$$P^{pre}_{virt}(x^T_{k-N+\xi})$$
$$= \int P(x^T_{k-N+\xi}|x^T_{k-N+\xi-1})P^i_{virt}(x^T_{k-N+\xi-1})dx^T_{k-N+\xi-1}$$

(3.2) Updating

$$P^i_{virt}(x^T_{k-N+\xi}) = K_\xi P^{pre}_{virt}(x^T_{k-N+\xi})\prod_{j\in\Omega^i_\xi} P(z^j_{k-N+\xi}|x^T_{k-N+\xi})$$

$$K_\xi = 1/\int P^{pre}_{virt}(x^T_{k-N+\xi})\prod_{j\in\Omega^i_\xi} P(z^j_{k-N+\xi}|x^T_{k-N+\xi})dx^T_{k-N+\xi}$$

(3.3) When $\xi = 1$, store the virtual PDF as the individual PDF for time $(k-N+1)$

$$P^i_{pdf}(x^T_{k-N+1}|z^1_{1:k-N+1}, \ldots, z^N_{1:k-N+1}) = P^i_{virt}(x^T_{k-N+1}).$$

(4) Individual PDF of i^{th} robot at time k is $P^i_{pdf}(x^T_k|\mathbf{z}^i_{1:k}) = P^i_{virt}(x^T_k)$.

The binary observations subject to Bernoulli distribution $B(1, p_j)$, yielding

$$P(z_l^j|x^T) = p_j^{z_l^j}(1-p_j)^{1-z_l^j}$$

where $p_j = P(z_l^j = 1|x^T)$. Utilizing the facts: (1) z_l^j are conditionally independent samples from $B(1, p_j^*)$ and (2) $k - N < k_j^i \le k$, the law of large numbers yields

$$\frac{1}{k}\sum_{l=1}^{k_j^i} z_l^j \xrightarrow{P} p_j^*, \quad \frac{1}{k}(k_j^i - \sum_{l=1}^{k_j^i} z_l^j) \xrightarrow{P} 1 - p_j^*$$

where $p_j^* = P(z_l^j = 1|x^{T^*})$ and "$\xrightarrow{P}$" denotes "convergence in probability". Then,

$$\frac{1}{k}\sum_{l=1}^{k_j} \ln \frac{P(z_l^j|x^T)}{P(z_l^j|x^{T^*})} \xrightarrow{P} p_j^* \ln \frac{p_j}{p_j^*} + (1-p_j^*)\ln\frac{1-p_j}{1-p_j^*} \quad (9)$$

Note that the right-hand side of Eq. (9) achieves maximum value 0 if and only if $p_j = p_j^*$. Define

$$c(x^T) = \sum_{j=1}^{N} p_j^* \ln \frac{p_j}{p_j^*} + (1-p_j^*)\ln\frac{1-p_j}{1-p_j^*}.$$

Considering Eq. (8) and Eq. (9), the limit of Eq. (7) is

$$\frac{1}{k}\ln\frac{P_{pdf}^i(x^T|\mathbf{z}_{1:k}^i)}{P_{pdf}^i(x^{T^*}|\mathbf{z}_{1:k}^i)} \xrightarrow{P} c(x^T) \quad (10)$$

It follows from Eq. (10) that

$$\frac{P_{pdf}^i(x^T|\mathbf{z}_{1:k}^i)}{P_{pdf}^i(x^{T^*}|\mathbf{z}_{1:k}^i)e^{c(x^T)k}} \xrightarrow{P} 1 \quad (11)$$

Define the set $\bar{X}^T = S \setminus \{x^{T^*}\}$ and $c_M = \max_{x^T \in \bar{X}^T} c(x^T)$. Then $c_M < 0$. Summing Eq. (11) over $\bar{X}^T$ yields

$$\frac{\sum_{x^T\in\bar{X}^T} P_{pdf}^i(x^T|\mathbf{z}_{1:k}^i)e^{[c_M - c(x^T)]k}}{P_{pdf}^i(x^{T^*}|\mathbf{z}_{1:k}^i)e^{c_M k}} \xrightarrow{P} |\bar{X}^T| \quad (12)$$

where $|\bar{X}^T|$ denotes the cardinality of $\bar{X}^T$.

Since $c_M < 0$, $P_{pdf}^i(x^{T^*}|\mathbf{z}_{1:k}^i)e^{c_M k} \longrightarrow 0$, Eq. (12) implies

$$\sum_{x^T\in\bar{X}^T} P_{pdf}^i(x^T|\mathbf{z}_{1:k}^i)e^{[c_M - c(x^T)]k} \xrightarrow{P} 0 \quad (13)$$

Utilizing the relation

$$0 \le P_{pdf}^i(x^T|\mathbf{z}_{1:k}^i) \le P_{pdf}^i(x^T|\mathbf{z}_{1:k}^i)e^{[c_M - c(x^T)]k},$$

it can be derived from Eq. (13) that

$$\sum_{x^T\in\bar{X}^T} P_{pdf}^i(x^T|\mathbf{z}_{1:k}^i) \xrightarrow{P} 0$$

Therefore,

$$\lim_{k\to\infty} P(x^T = x^{T^*}|\mathbf{z}_{1:k}^i) = 1 - \lim_{k\to\infty}\sum_{x^T\in\bar{X}^T} P_{pdf}^i(x^T|\mathbf{z}_{1:k}^i) = 1$$

■

B. *Proof for moving robots*

The difficulty of consistency proof for moving robots lies in the fact that each robot makes observations at multiple positions. Here, the main idea is to classify robot observation positions into two disjoint sets: *infinite-observation spots* that contains positions where a robot makes infinite observations, and *finite-observation spots* that contains positions where the robot makes finite observations. Before stating main theorem, the following lemma is introduced.

Lemma 1: For a set of robots moving within a collection of finite positions, each robot has at least one position where infinite observations are made as k tends to infinity.

Proof: Let $n_j^{i,k}$ denote the times that i^{th} robot visits j^{th} position up to time k. Then, $\sum_j n_j^{i,k} = k$. It is straightforward to see that $\exists n_j^{i,k}$, such that $n_j^{i,k} \to \infty$, as $k \to \infty$. ■

Theorem 2: If robots move within a collection of finite positions, the estimated target position converges to the true position of target in probability using LIFO-DBF, i.e.,

$$\lim_{k\to\infty} P(x^T = x^{T^*}|\mathbf{z}_{1:k}^i) = 1, \ i = 1, \dots, N$$

Proof: Similar to Theorem 1, the batch form of DBF at k^{th} step is

$$\frac{P_{pdf}^i(x^T|\mathbf{z}_{1:k}^i)}{P_{pdf}^i(x^{T^*}|\mathbf{z}_{1:k}^i)} = \frac{P_{pdf}^i(x^T)\prod_{j=1}^{N}\prod_{l=1}^{k_j^i} P(z_l^j|x^T; x_l^R)}{P_{pdf}^i(x^{T^*})\prod_{j=1}^{N}\prod_{l=1}^{k_j^i} P(z_l^j|x^{T^*}; x_l^R)} \quad (14)$$

The only difference from Eq. (6) is that $P(z_l^j|x^T; x_l^R)$ in Eq. (14) varies as the robot moves. For each robot, there exists at least one position where infinite observations are made as $k \to \infty$, according to Lemma 1. All positions can be classified into finite-observation spots and infinite-observation spots. For the former, by referring to Eq. (10) in proof of Theorem 1, it is easy to know that their contribution to Eq. (14) is zero when $k \to \infty$. Therefore, Eq. (14) can be reduced to only consider infinite-observation spots, which is similar to proof of Theorem 1. ■

Remark 5: Consistency implies that all individual PDFs converge to the same distribution (true target PDF), thus the consensus is also guaranteed. It must be noted that traditional statistics dissemination-based methods only ensure consensus of individual PDFs [?], [?]. To the best knowledge of authors, there is no proof of consistency on estimated target position.

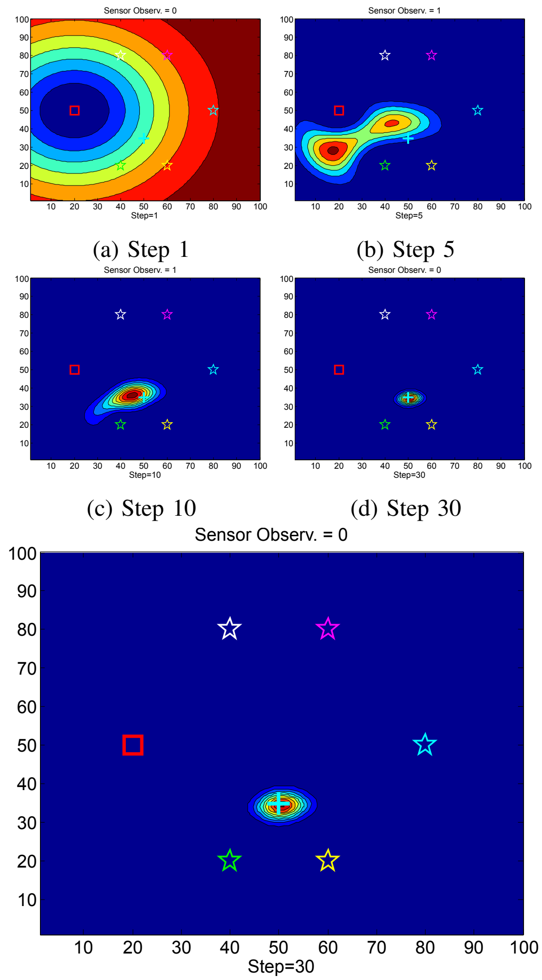
V. Simulation

This section simulates two scenarios of target localization to demonstrate the effectiveness of LIFO-BDF. In all scenarios, six robots are utilized and each robot is equipped with a binary sensor. All sensors are modeled with identical Gaussian functions [?]:

$$P(z_k^i = 1|x_k^T; x_k^{R,i}) = e^{-\frac{1}{2}(x_k^T - x_k^{R,i})^T\Sigma^{-1}(x_k^T - x_k^{R,i})} \quad (15a)$$

$$P(z_k^i = 0|x_k^T; x_k^{R,i}) = 1 - P(z_k^i = 1|x_k^T; x_k^{R,i}). \quad (15b)$$

The first scenario consists of six static robots and a single static target. The second scenario subsequently deals with six

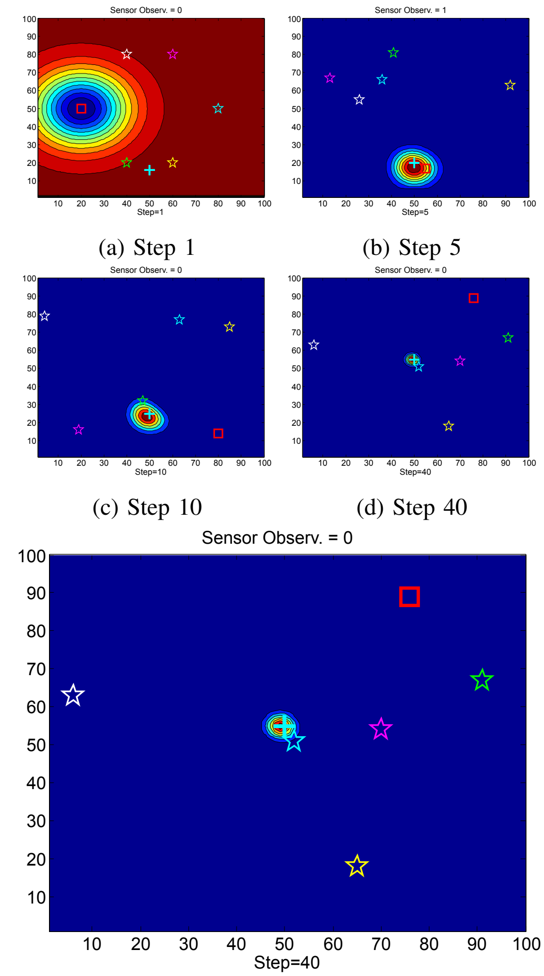


(a) Step 1 (b) Step 5

(c) Step 10 (d) Step 30

(e) Step 30

Fig. 4: (a)-(d) The 1st robot's individual PDFs at different times; (e) All robots' individual PDFs at time 30. The square denotes the current robot and stars represent other robots. The cross stands for the target.



(a) Step 1 (b) Step 5

(c) Step 10 (d) Step 40

(e) Step 40

Fig. 5: (a)-(d) The 1st robot's individual PDFs at different times; (e) All robots' individual PDFs at time 40.

moving robots for localizing a moving target. Robot positions are randomly generated at each time step. A single-integrator dynamics is used as the target motion model.

A. Static Robots, Static Target

The positions of six static robots are shown as stars and square in Fig. 4. The LIFO-DBF for static target is implemented on each robot for target localization. The networked robots use a ring communication topology that each robot can communicate with two fixed neighbors. Fig. 4 shows the estimation results of the static target. After the initial observation, each robot forms a circular individual PDF, centered at its own position. The circular PDF happens because the Gaussian sensor model (Eq. (15)) only depends on the distance between robot and target. As more observations are used, the posterior individual PDF concentrates to the true location of the target (Fig. 4d), which accords with the consistency of LIFO-DBF.

B. Moving Robots, Moving Target

The six robots move within the field to estimate the target position, using the same topology and neighborhood as in Section V-A. In this work, robot positions are randomly generated at each time. Readers interested in the motion planning of robots for effective target localization can refer to [**?**]. The target dynamics is given by a single-integrator model: $x^T(k+1) = x^T(k) + v\Delta T$, where v is the constant velocity of the target; ΔT is the sampling time.

The LIFO-DBF described in Section III-C is utilized for target localization. Fig. 5 shows the estimation results of the moving target. It is interesting to notice that the posterior individual PDFs concentrate to the true target location at each time, even when the target constantly moves.

VI. CONCLUSION

This paper presents a measurement dissemination-based distributed Bayesian filtering (DBF) method for a multi-robot network, utilizing the Latest-In-and-Full-Out (LIFO) protocol for observation exchange. Different from statistics dissemination approaches that transmit posterior distributions or likelihood functions, each robot under LIFO only exchanges with neighboring robots a full communication buffer consisting of latest available measurements, which significantly reduces the transmission burden between each pair of robots from the order of environmental size to that of robot number. Under the condition of fixed and undirected topology, LIFO can guarantee non-intermittent dissemination of all observations over the network within finite time via local communication among direct neighborhood. It is worth noting that LIFO is a general measurement exchange protocol and applicable to various sorts of sensors. Two types of LIFO-based DBF algorithms are proposed to estimate individual PDF for static and moving target, respectively. For the static target, each robot locally fuses the newly received observations while for the moving target, a record set of historical observations is stored and updated. The consistency of LIFO-based DBF is proved by utilizing the law of large numbers, which ensures that estimated target position converges in probability to the true target position when the number of observations tends to infinity.

Future work includes how to handle other types of sensors and switching topology. Other types of sensors may have biased observations and subject to non-Bernoulli distribution, which complicates the design and analysis of LIFO-based Bayesian filters. The switching topology, including package loss, can lead to unpredictable delay and intermittent transmission, which may affect the consistency and consensus of LIFO-DBF.

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