# Distributed Bayesian Filters for Multi-Robot Network by Using Latest-In-and-Full-Out Exchange Strategy of Observations

Chang Liu[1], Shengbo Eben Li[2] and J. Karl Hedrick[3]

*Abstract*— **This paper presents a local exchange strategy of observations, i.e., Latest-In-and-Full-Out (LIFO), for distributed Bayesian filters (DBF) in a multi-robot network. Different from statistics dissemination approaches that transmit posterior distributions or likelihood functions, each robot under LIFO only receives the latest available measurements and then broadcasts full communication buffer to its neighborhood, which significantly reduces the transmission burden of each pair from the order of environmental size to that of robot number. Under the condition of fixed and undirected topology, LIFO can guarantee non-intermittent dissemination of all observations over the network within finite time. Two types of LIFO-based DBF algorithms are proposed to estimate individual posterior density function (PDF) for static and moving target, respectively. For the static target, each robot locally fuses the newly received observations while for the moving target, a triangular matrix of historical observations is stored and updated. The consistency of LIFO-based DBF is proved that individual PDF of each robot converges to the true target position when the number of observations tends to infinity. The effectiveness of this method is demonstrated by simulations. (TODO: find out what convergence is)**

## I. INTRODUCTION

Distributed filtering that focuses on using a group of networked robots to collectively infer environment status has been used for various applications, such as intruder detection [1], target tracking (**TODO:** change reference, more application-oriented) [2] and micro-environmental monitoring [3]. Several techniques have been developed for distributed filtering. For example, Olfati-Saber (2005) proposed a distributed Kalman filter (DKF) for estimating states of linear systems with Gaussian process and measurement noise [4]. (**TODO:** add a distributed EKF) Each DKF used low-pass and band-pass consensus filters for the average-censensus of weighted measurements and inverse-covariance matrices. Gu (2007) presented a distributed particle filter for target tracking over sensor networks [5]. Gaussian mixture model (GMM) was adopted to approximate the posterior distribution from weighted particles and the parameters of GMM was exchanged via average consensus filter. As a generic method for nonlinear system and arbitrary noise distribution, distributed Bayesian filter (DBF) has received increasing interest during past years [**?**], which is the focus of this study.

The design of distributed filtering algorithms is closely related to communication topology of multi-robot network, which can be classified into two types: fusion center-based and neighborhood-based. (**TODO:** check if FC method requires the all info before processing) (**TODO:** need to give one more level of details for FC. refer to Hlinka's paper.) Fusion center (FC)-based DBF has been a common structure for distributed filtering, in which local information collected by robots is transmitted (possibly via multi-hopping) to the fusion center for forming global estimation [6], [7]. FC-based DBF is efficient for estimation in that it can collectively utilize all robots' information and thus useful for applications that only require information at a single central unit, such as in environmental monitoring. (**TODO:** NB method is also verbose and not detailed.) Neighborhood(NB)-based DBF is another commonly adopted structure for distributed filtering. Instead of communicating with a fusion center, each robot only exchanges information with neighboring robots and forms local estimation of the environment state. NB-based DBF is advantageous over FC-based DBF in that no central unit is required, thus suitable for applications in which maintaining communication link between robots and center is challenging, such as in disaster situations. Besides, state estimation is locally conducted on each robot, which requires less computation power compared to that in the fusion center.

Thus far, most studies on NB-based DBF have mainly focused on the so-called *statistics dissemination* strategy that each robot actually exchanges posterior distributions or likelihood functions to neighboring robots for distributed estimation. (**TODO:** mention leader-based and consensus-based. 2 examples for leader-based and 3 examples for consensus-based.) For example, Sheng et al.[8] proposed a leader agent-based distributed particle filter (DPF) with Gaussian Mixer to track multiple moving targets. DPFs were run on a set of uncorrelated sensor cliques and the particles were approximated as GMMs, the parameters of which were then exchanged among cliques for global estimation of targets. A popular strategy for statistics dissemination is to use consensus-based approaches that all robots perform distributed filtering simultaneously, exchange statistics with neighbors and executes consensus algorithms, as proposed in [9], [10], [11], for fusion of statistics. For example, Saptarshi et al. [12] presented a Bayesian consensus filter (BCF) that uses logarithmic opinion pool for fusing posterior distri-

*The first two authors, C. Liu and S. Li, have equally contributed to this research. This work is supported by the Embedded Humans: Provably Correct Decision Making for Networks of Humans and Unmanned Systems project, a MURI project funded by the Office of Naval Research.

[1]Chang Liu is with the Vehicle Dynamics & Control Lab, Department of Mechanical Engineering, University of California, Berkeley, Berkeley, CA 94709, USA. Email: changliu@berkeley.edu

[2]Shengbo Eben Li is with the Department of Automotive Engineering, Tsinghua University, Beijing, 100084, China. He is currently working at Department of Mechanical Engineering, University of California, Berkeley as a visiting scholar. Email: lisb04@gmail.com

[3]J. Karl Hedrick is with the Vehicle Dynamics & Control Lab, Department of Mechanical Engineering, University of California, Berkeley, Berkeley, CA 94709, USA. Email: khedrick@me.berkeley.edu

butions of the tracked target among neighboring robots. A consensus-based distributed particle filter (DPF) is proposed by Julian et al. [13] for estimating environment state by exchanging and fusing posterior functions of the state among neighbors using a discrete-time linear consensus algorithm.

Despite the popularity of statistics dissemination-based approaches, exchanging posterior distributions or likelihood functions can consume high communication resources, which may be infeasible for applications in vast area or complex environment, such as marine search, seismological rescue, etc. This study focuses on the strategy of exchanging observations in the neighborhood of each robot, called the *measurement dissemination-based* strategy, for the purpose of achieving a consensus of the probability density function (PDF) of the tracked target. Some pioneering studies have been done on measurement dissemination strategies, in which raw or quantized observations are exchanged among robots. For example, Coates et al.[14] used adaptive encoding of observations to minimize communication overhead for tracking a manoeuvring object. Djuric et al. [15] proposed a decentralized particle filter for tracking targets. At each time instant, a subset of robots that are in proximity of the tracked targets share their observations for target position estimation. Another example can be found in [16], in which both observations and statistics were exchanged among sensors for distributed surveillance of the environment. In aforementioned works, either communication topology is assumed fully connected that each robot can broadcast observations to all other robots in single transmission step or the communication involves large amount of exchanged data.

This paper proposes a local-exchange-of-observation (LIFO) strategy for distributed Bayesian filters (DBF) for undirected and connected communication topology. Each robot only broadcasts observations to its neighbors and implements Bayesian filter locally after receiving observations transmitted from neighboring robots. The main benefit of LIFO is the reduction of communication burden, with the transmission data volume scaling linearly with the robot number, while statistics dissemination-based strategies suffer from the data volume on the order of environmental size. The proposed LIFO-DBF has following properties: (1) For an undirected and connected network with fixed topology, LIFO guarantees the global dissemination of all robots' observations among the network via multi-hopping, with each robot non-intermittently receiving (delayed) observations of all other robots via local communication. (2) The corresponding DBF ensures consistency of the estimation of states, which refers to the agreement between robots' estimates of target position and the true position of the target. Moreover, consistency implies the consensus of robots' target PDFs. In this study, formal proofs of the consistency of LIFO-DBF and consensus of robots' target PDFs are provided.

The rest of this paper is organized as follows: The distributed SAT of target is formulated in Section II. The LIFO-DBF algorithm is described in Section III, followed by the proof of consistency and consensus in Section IV. Simulation results of LIFO-DBF and conclusions are presented in Section V and Section VI, respectively.

## II. PROBLEM FORMULATION

Consider a network of $N$ robots in a bounded two-dimensional space $S$. Each robot is equipped with a binary sensor for environmental perception. Due to the limit of communication range, each robot can only exchange observations with its neighbors. The Bayesian filter is run locally on each robot based on its own and received observations.

### A. Probabilistic Model of Binary Sensor

The goal of distributed Bayesian filter is to estimate the true target position by using a network of binary sensors. The binary sensor only gives two types of observation: 1 if the target is detected, and 0 if no target is detected. The observation of $i^{\text{th}}$ sensor at $k^{\text{th}}$ time step is denoted as $z_k^i$. The likelihood function that the target is detected is:

$$P(z_k^i = 1 | x_k^T; x_k^R) \in [0, 1]\,,\ x_k^T \in S \tag{1}$$

where $x_k^T$ denotes the target positions; $x_k^R$ is the robot position. Correspondingly, the likelihood function that no target is detected is:

$$P(z_k^i = 0 | x_k^T; x_k^R) = 1 - P(z_k^i = 1 | x_k^T; x_k^R) \tag{2}$$

The combination of Eq. (1) and Eq. (2) becomes a binary sensor model parameterized by $x_k^T$ and $x_k^R$. For the purpose of simplicity, we will not explicitly write $x_k^R$ for the rest of the paper. The commonly used likelihood functions for binary sensor include Gaussian function [17], [18] and step function [19].

***Remark 1:*** Given the knowledge of current target position, current observation of each robot is conditionally independent from both its own past observations and those of other robots.

***Remark 2:*** This study is applicable for both homogeneous and heterogeneous binary sensors. A homogeneous model can simplify the analysis of completeness, while the heterogeneous model is more close to real sensing characteristics. In addition, it also works for other types of sensors, such as laser scanners [20] and cameras [21].

### B. Graphical model of communication topology

The robot network is always assumed to be connected. Under this assumption, consider an undirected and fixed graph $G = (V, E)$, where $V = \{1, \ldots, N\}$ represents the index set of robots and $E = V \times V$ denotes the edge set. The adjacency matrix $M = [m_{i,j}]$ describes the communication topology of $G$:

$$m_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{if } (i,j) \notin E \end{cases},$$

where $m_{ij}$ denotes the entity of adjacency matrix. The notation $m_{ij} = 1$ indicates that a communication link exists between $i^{\text{th}}$ and $j^{\text{th}}$ robot and $m_{ij} = 0$ indicates no communication between them.

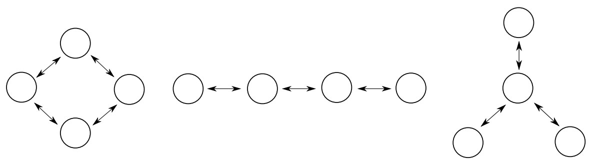



Fig. 1: Three types of topologies: (a)circular topology; (b)serial topology; (c)star topology

The *direct neighborhood* of $i^{\text{th}}$ robot is defined as $\mathcal{N}_i = \{j|m_{ij} = 1, \forall j \in \{1, \ldots, N\}\}$. All the robots in $\mathcal{N}_i$ can directly exchange information with $i^{\text{th}}$ robot. In addition to direct neighborhood, another set called *available neighborhood* is defined as $\mathcal{Q}_i$, which contains indices of robots whose observations are received by the $i^{\text{th}}$ robot given a specific observation exchange algorithm. Note that in general $\mathcal{N}_i \subseteq \mathcal{Q}_i$, but when only single-hopping is allowed, $\mathcal{N}_i = \mathcal{Q}_i$. Fig. 1 illustrates three types of typical topologies: circular [22], serial [23], and star [8]. All of them are undirected and connected topologies.

### C. Distributed Bayesian Filter for Multiple Robots

The generic distributed Bayesian filter (DBF) is introduced in this section, which was also stated in [24] and [25]. Each robot has its individual estimation of posterior density function (PDF) of target position, called *individual PDF*. The $i^{\text{th}}$ individual PDF at time $k$ is defined as $P^i_{pdf}(x^T_k|z^i_{1:k}, z^{\mathcal{Q}_i}_{1:k})$. The individual PDF is initialized by the prior function $P^i_{pdf}(x^T_0|z^i_0, z^{\mathcal{Q}_i}_0) = P(x^T_0)$, given all available prior information including past experience and environmental knowledge. Under the framework of DBF, the individual PDF is recursively estimated by two steps, i.e., prediction step and updating step, based on the observations of $i^{\text{th}}$ robot and robots in $\mathcal{Q}_i$.

*1) Prediction:* At time $k$, the prior individual PDF $P^i_{pdf}(x^T_{k-1}|z^i_{1:k-1}, z^{\mathcal{Q}_i}_{1:k-1})$ is first predicted forward by using the Chapman-Kolmogorov equation:

$$\begin{aligned} &P^i_{pdf}(x^T_k|z^i_{1:k-1}, z^{\mathcal{Q}_i}_{1:k-1}) \\ &= \int P(x^T_k|x^T_{k-1})P^i_{pdf}(x^T_{k-1}|z^i_{1:k-1}, z^{\mathcal{Q}_i}_{1:k-1})dx^T_{k-1} \end{aligned} \tag{3}$$

where $P(x^T_k|x^T_{k-1})$ is a Markov motion model of the target, independent of robot states. This model describes the state transition probability of the target from a prior state $x^T_{k-1}$ to posterior state $x^T_k$. Note that the target is static in many search applications, such as the indoor search for stationary objects [26]. For a static target, its Markov motion model is simplified to be

$$P(x^T_k|x^T_{k-1}) = \begin{cases} 1 & \text{if } x^T_k = x^T_{k-1} \\ 0 & \text{if } x^T_k \neq x^T_{k-1} \end{cases}.$$

*2) Updating:* At time $k$, the $i^{\text{th}}$ individual PDF is then updated by Bayes' formula using the latest available observations at time $k$:

$$\begin{aligned} &P^i_{pdf}(x^T_k|z^i_{1:k}, z^{\mathcal{Q}_i}_{1:k}) \\ &= K_i P^i_{pdf}(x^T_k|z^i_{1:k-1}, z^{\mathcal{Q}_i}_{1:k-1})P(z^i_k|x^T_k)\prod_{j\in\mathcal{Q}_i} P(z^j_k|x^T_k) \end{aligned} \tag{4}$$

where $K_i$ is a normalization factor, given by:

$$K_i = 1/\int P^i_{pdf}(x^T_k|z^i_{1:k-1}, z^{\mathcal{Q}_i}_{1:k-1})P(z^i_k|x^T_k)\prod_{j\in\mathcal{Q}_i} P(z^j_k|x^T_k)dx^T_k$$

where $P^i_{pdf}(x^T_k|z^i_{1:k}, z^{\mathcal{Q}_i}_{1:k})$ is called posterior individual PDF; $P(z^i_k|x^T_k)$ is the likelihood function of observation $z^i_k$, described in Eq. (1) and Eq. (2).

## III. Distributed Bayesian Filter via Local Exchange of Observations

This study proposes a Latest-In-and-Full-Out (LIFO) strategy for observation exchange and derives two corresponding distributed Bayesian filtering (DBF) algorithms, shorted as LIFO-DBF. The data communication in LIFO uses synchronized step as the execution of DBF. In each step, LIFO only allows single-hopping communication within the neighborhood, but is able to broadcast observations of each robot to any other agents after a finite number of steps. The individual PDF is forward predicted and updated in DBF after each LIFO cycle. The theoretical analysis show that LIFO-DBF can ensure the consistency and consensus of distributed estimation while requiring much less communication burden than any statistics dissemination-based methods.

### A. Strategy for Latest-In-and-Full-Out (LIFO)

Under LIFO, each robot contains a communication buffer (CB) to store its latest knowledge of the observations of all robots:

$$\mathbf{z}^i_k = \left[z^1_{k^i_1}, \ldots, z^N_{k^i_N}\right]$$

where $z^j_{k^i_j}$ represents the observation made by $j^{\text{th}}$ robot at time $k^i_j$. Note that under LIFO, $\mathcal{Q}_i = \{1, \ldots, N\}$, which will be proved in Corollary 1. At time k, $z^j_{k^i_j}$ is received and stored in $i^{\text{th}}$ robot CB, in which $k^i_j$ is the latest observation time of $j^{\text{th}}$ robot available to $i^{\text{th}}$ robot. Due to the communication delay, $k^i_j < k, \forall j \neq i$ always holds in practice.

The **LIFO strategy** is stated as follows:

(1) Initialization: The buffer of $i^{\text{th}}$ robot is initialized when $k = 0$:

$$z^j_{k^i_j} = \emptyset,\ k^i_j = 0,\ j = 1:N$$

(2) At $k^{\text{th}}$ step for $i^{\text{th}}$ robot :

(2.1) Receiving Step:

The $i^{\text{th}}$ robot receives all CBs of its neighboring robots. The received CBs are totally $|\mathcal{N}_i|$ groups, each of which corresponding to the (k-1)-step CB of a robot in $\mathcal{N}_i$. The received CB from $l^{\text{th}}$ ($l \in \mathcal{N}_i$) robot is denoted as

$$\mathbf{z}^l_{k-1} = \left[z^1_{(k-1)^l_1}, \ldots, z^N_{(k-1)^l_N}\right],\ l \in \mathcal{N}_i$$

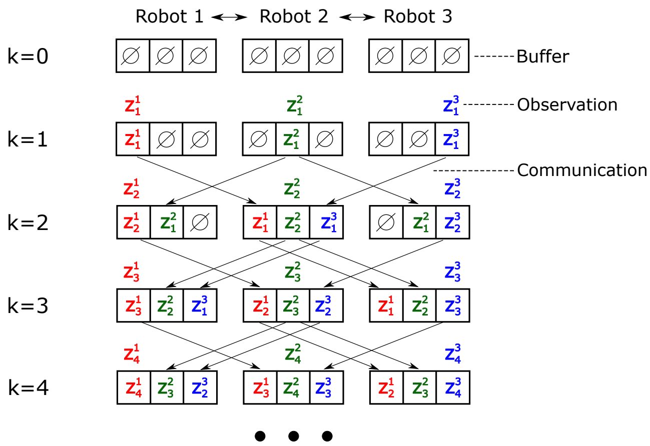



Fig. 2: Example of LIFO with three robots using serial communication topology

(2.2) Observation Step:

The $i^{\text{th}}$ robot updates $z^j_{k^i_j}$ $(j = i)$ by its own observation at current step:

$$z^j_{k^i_j} = z^i_k,\ k^i_j = k,\ \text{if } j = i.$$

(2.3) Comparison Step:

The $i^{\text{th}}$ robot updates other elements of its own CB, i.e., $z^j_{k^i_j}$ $(j \neq i)$, by selecting the latest information among all received CBs from $\mathcal{N}_i$. For all $j \neq i$,

$$l_{\text{latest}} = \underset{l \in \mathcal{N}_i, i}{\text{argmax}} \left\{ (k-1)^i_j, (k-1)^l_j \right\}$$
$$z^j_{k^i_j} = z^j_{(k-1)^{l_{\text{latest}}}_j},\ k^i_j = (k-1)^{l_{\text{latest}}}_j$$

(2.4) Sending Step:

The $i^{\text{th}}$ robot broadcasts its updated CB to all of its neighbors defined in $\mathcal{N}_i$.

(3) $k \leftarrow k + 1$ until stop ■

Fig. 2 illustrates the LIFO cycles with 3 robots using a serial topology. Two facts can be noticed in Fig. 2: (1) all robot CBs are filled within 3 steps, which means under LIFO each robot has a maximum delay of 2 steps for receiving observations from other robots; (2) after filled, the updating of CBs are non-intermittent, which means each robot continuously receives new observations of other robots. Extending the two facts to a network of $N$ robots, we have the following proposition:

(**TODO:** bold proposition, thm, ...)

*Proposition 1:* For a fixed and undirected network of $N$ robots, LIFO uses the shortest path(s) between $i^{\text{th}}$ and $j^{\text{th}}$ robot to exchange observation, the length of which equals the delay $\tau_{i,j}$ between them.

*Proof:* Without loss of generality, assume that there is a unique shortest path between $i$ and $j$, denoted by $T^{j,i}_{n^*} = (v_1, \ldots, v_{n^*})$, with $v_1 = j, v_{n^*} = i, v_{m+1} \in \mathcal{N}_{v_m}$. Then, the distance between $i$ and $j$ is $d(j,i) = n^* - 1$. The following mathematical induction will prove Proposition 1.

Step (1): When $d(j,i) = 1$, $j \in \mathcal{N}_i$ and $j$ can directly send $z^j_k$ to $i$. Then $z^j_k$ is stored in $i^{\text{th}}$ CB at time $k+1$, i.e. $\tau_{i,j} = 1$. Proposition 1 holds for $d(j,i) = 1,\ i,j \in \{1, \ldots, N\}$.

Step (2): Suppose that Proposition 1 holds for $d(j,i) = s,\ s \geq 2$. Then for $d(j,i) = s+1$, i.e., $n^* = s+2$, by the Bellman's principle of optimality, the path $T^{j,l}_{n^*-1} = (v_1, \ldots, v_{n^*-1})$ is a shortest path between $j$ and $l$, where $v_{n^*-1} = l$ and $i \in \mathcal{N}_l$. The assumption that Proposition 1 holds for $d(j,i) = s$ implies that $z^j_k$ is received and stored in $l^{\text{th}}$ robot's CB at time $k+s$. Since $i \in \mathcal{N}_i$, $i^{\text{th}}$ robot receives $z^j_k$ at $k+s+1$. For any other path $T^{j,i}_n = (v_1, \ldots, v_n)$ with $n > n^*$, $z^j_k$ cannot be received by $i$ earlier than $k+s+1$. Therefore $\tau_{i,j} = s+1$. This proves the Proposition 1 for $d(j,i) = s+1$. ■

(**TODO:** add empyt line)

*Corollary 1:* For the same topology in Proposition 1, all elements in $\mathbf{z}^i_k$ under LIFO become filled when $k \geq N$.

*Proof:* In a network of $N$ robots, the maximal length of shortest paths is no greater than $N-1$. Based on Proposition 1, $\tau_{i,j} \leq N-1$ and thus all elements of $\mathbf{z}^i_k$ become filled when $k \geq N$. ■

*Corollary 2:* For the same topology in Proposition 1, once all elements in $\mathbf{z}^i_k$ are filled, the updating of each element is non-intermittent.

*Proof:* For a network with fixed topology, shortest path(s) between any nodes are fixed. Therefore, based on Proposition 1, $\tau_{i,j}$ is constant and the updating of each element in $\mathbf{z}^i_k$ is non-intermittent. ■

***Remark 3:*** Compared to statistics dissemination, LIFO is more communication-efficient for distributed filtering. To be specific, consider an $M \times M$ grid environment with a network of $N$ robots, the transmitted data of LIFO between each pair of robots are only the CB of each robot, the length of which is $O(N)$. On the contrary, the length of transmitted data for a statistics dissemination approach is $O(M^2)$, which is the size of the environment. Since $M$ is generally much larger than $N$ in target search and tracking, LIFO requires much less communication resources.

## *B. Algorithm of LIFO-DBF for Static Target*

This section derives the LIFO-DBF algorithm for localizing a static target. Each robot stores last-step individual PDF, i.e., $P^i_{pdf}(x^T|\mathbf{z}^i_{1:k-1})$. The assumption of static target can simplify the Bayesian filter as the prediction step becomes unnecessary. Therefore, the $i^{\text{th}}$ individual PDF is only updated by

$$P^i_{pdf}(x^T|\mathbf{z}^i_{1:k}) = K_i P^i_{pdf}(x^T|\mathbf{z}^i_{1:k-1}) \prod_{j=1}^{N} P(z^j_{k^i_j}|x^T) \quad (5)$$

where

$$K_i = 1 \Big/ \int P^i_{pdf}(x^T|\mathbf{z}^i_{1:k-1}) \prod_{j=1}^{N} P(z^j_{k^i_j}|x^T) dx^T$$

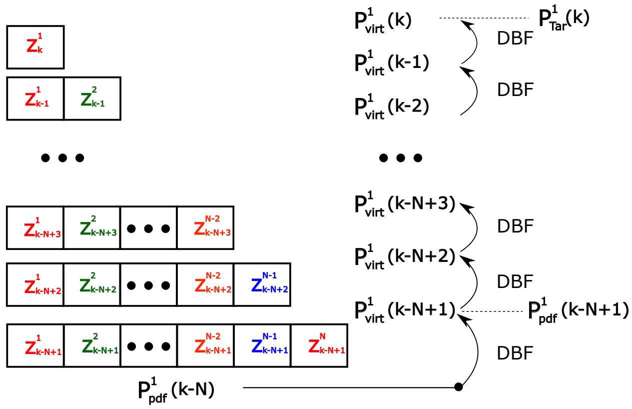



Fig. 3: Example of LIFO-DBF for 1st robot at time $k$. The current individual PDF is $P^1_{pdf}(x^T_{k-N}|z^1_{1:k-N},\dots,z^N_{1:k-N})$, denoted as $P^1_{pdf}(k-N)$ in the figure. The robot first calculates $P^1_{virt}(k-N+1)$ using DBF and stores it as $P^1_{pdf}(k-N+1)$. Repeating DBF until obtaining $P^1_{pdf}(k)$, which is then used as the target PDF estimation of 1st robot at time $k$.

### C. Algorithm of LIFO-DBF for Moving target

This section derives the LIFO-DBF for localizing a moving target. Instead of storing last-step PDF, each robot maintains an individual PDF of time $(k-N)$ and a triangular matrix of historical observations from time $(k-N+1)$ to current time $k$. The $i^{\text{th}}$ individual PDF is then alternatively predicted and updated by using aforementioned Bayesian filter (Eq. (3) and Eq. (4)) from $(k-N)$ to $k$. Fig. 3 illustrates the LIFO-DBF procedure for the 1st robot as an example. The generic **LIFO-DBF algorithm** for moving target is stated as follows:

For $i^{\text{th}}$ robot at $k^{\text{th}}$ step:

**(1)** The stored individual PDF for time $(k-N)$ is:

$$P^i_{pdf}(x^T_{k-N}|z^1_{1:k-N},\dots,z^N_{1:k-N})$$

**(2)** Initialize a virtual PDF by assigning the individual PDF to it:

$$P^i_{virt}(x^T_{k-N}) = P^i_{pdf}(x^T_{k-N}|z^1_{1:k-N},\dots,z^N_{1:k-N})$$

**(3)** From $\xi = 1$ to $N$, repeat two steps of Bayesian filtering:

(3.1) Prediction

$$P^{pre}_{virt}(x^T_{k-N+\xi}) = \int P(x^T_{k-N+\xi}|x^T_{k-N+\xi-1})P^i_{virt}(x^T_{k-N+\xi-1})dx^T_{k-N+\xi-1}$$

(3.2) Updating

$$P^i_{virt}(x^T_{k-N+\xi}) = K_\xi P^{pre}_{virt}(x^T_{k-N+\xi}) \prod_{j\in\Omega^i_\xi} P(z^j_{k-N+\xi}|x^T_{k-N+\xi})$$

$$K_\xi = 1/\int P^{pre}_{virt}(x^T_{k-N+\xi}) \prod_{j\in\Omega^i_\xi} P(z^j_{k-N+\xi}|x^T_{k-N+\xi})dx^T_{k-N+\xi}$$

**(4)** Store the first-step virtual PDF as the individual PDF for time $(k-N+1)$

$$P^i_{pdf}(x^T_{k-N+1}|z^1_{1:k-N+1},\dots,z^N_{1:k-N+1}) = P^i_{virt}(x^T_{k-N+1}).$$

■

Note that $\Omega^i_\xi$ denotes the index set of robots whose observation at time $(k-N+\xi)$ is stored in $i^{\text{th}}$ robot's CB. The individual PDF of $i^{\text{th}}$ robot at time $k$ is $P^i_{virt}(x^T_k)$.

***Remark 4:*** For the static target, each robot only needs current step CB to update individual PDFs. Therefore, except storing individual PDFs, all historical CBs can be discarded and only current-step CB is stored in robot memory, the length of which is $O(N)$. On the contrary, for the moving target, each robot needs to store a triangular matrix of history observation (except current step CB) with size of $O(N^2)$ and an individual PDF with size $O(M^2)$, which means that the size of occupied memory in each robot is $O(M^2+N^2)$.

## IV. Proof of Consistency and Consensus

This section proves consistency and consensus of LIFO-DBF. Only proofs for localizing static target using static robots and moving robots are presented. The proof of LIFO-DBF for moving target is similar to that of static target by considering the dynamic model of the target, but with more complicated algebraic manipulation.

Assume that $S$ is finite and $x^{T^*}$ is the true location. Define an *equivalent-location* set $X^T_{eq} \subseteq S$ such that

$$X^T_{eq} = \left\{x^T \in S|P(z_k|x^T) = P(z_k|x^{T^*}),\ \forall z_k \in \{0,1\}\right\},$$

i.e., $x^T$ gives the same the observation likelihood as $x^{T^*}$ for given robot positions. Since $X^T$ is finite, $X^T_{eq}$ is also finite.

The reason to introduce equivalent-location set is that ghost target might exist in some special robot arrangement and sensor types. For example, for undirected binary sensors that are linearly arranged, a ghost target can exist at the mirror position of the true target. When sensors are overlapped at a single point, ghost targets can exist on a circle that contains the true target. In theory, DBF cannot rule out ghost targets in such cases and prior knowledge is needed for further clarification. This study only proves the convergence to the equivalent-location set rather to the true location.

### A. Proof for static robots

The consistency of LIFO-DBF for static robots is stated as follows:

*Theorem 1:* For static robots, each individual PDF converges to $X^T_{eq}$ using LIFO-DBF when the number of observations tends to infinity, i.e.

$$P^i_{pdf}(x^T \in X^T_{eq}|\mathbf{z}^i_{1:k}) \longrightarrow 1,\ k\to\infty$$

where $\mathbf{z}^i_{1:k} = \left(z^1_{1:k_1},\dots,z^N_{1:k_N}\right)$.

*Proof:* Considering the conditional independence of observations $z^j_k$ for given $x^T \in S$, the batch form of DBF at $k^{\text{th}}$ step is:

$$P^i_{pdf}(x^T|\mathbf{z}^i_{1:k}) = P^i_{pdf}(x^T|z^1_{1:k_1},\dots,z^N_{1:k_N}) = \frac{P^i_{pdf}(x^T)\prod_{j=1}^{N}\prod_{l=1}^{k_j}P(z^j_l|x^T)}{\sum_{x^T\in S}\prod_{j=1}^{N}\prod_{l=1}^{k_j}P(z^j_l|x^T)},$$

where $P^i_{pdf}$ is $i^{\text{th}}$ robot's initial individual PDF. It is known from Corollary 1 and Corollary 2 that $k - N < k_j \leq k$.

Compare $P^i_{pdf}(x^T|\mathbf{z}^i_{1:k})$ with $P^i_{pdf}(x^{T^*}|\mathbf{z}^i_{1:k})$:

$$\frac{P^i_{pdf}(x^T|\mathbf{z}^i_{1:k})}{P^i_{pdf}(x^{T^*}|\mathbf{z}^i_{1:k})} = \frac{P^i_{pdf}(x^T)\prod_{j=1}^{N}\prod_{l=1}^{k_j}P(z^j_l|x^T)}{P^i_{pdf}(x^{T^*})\prod_{j=1}^{N}\prod_{l=1}^{k_j}P(z^j_l|x^{T^*})} \tag{6}$$

Take the logarithm of Eq. (6) and average it over k steps:

$$\frac{1}{k}\ln\frac{P^i_{pdf}(x^T|\mathbf{z}^i_{1:k})}{P^i_{pdf}(x^{T^*}|\mathbf{z}^i_{1:k})} = \frac{1}{k}\ln\frac{P^i_{pdf}(x^T)}{P^i_{pdf}(x^{T^*})} + \sum_{j=1}^{N}\frac{1}{k}\sum_{l=1}^{k_j}\ln\frac{P(z^j_l|x^T)}{P(z^j_l|x^{T^*})} \tag{7}$$

Since $P^i_{pdf}(x^T)$ and $P^i_{pdf}(x^{T^*})$ are bounded,

$$\lim_{k\to\infty}\frac{1}{k}\ln\frac{P^i_{pdf}(x^T)}{P^i_{pdf}(x^{T^*})} \longrightarrow 0 \tag{8}$$

The binary observations subject to Bernoulli distribution $B(1, p_j)$, yielding

$$P^i_{pdf}(z^j_k|x^T) = p_j^{z^j_k}(1-p_j)^{1-z^j_k}$$

where $p_j = P(z^j_k = 1|x^T)$. Utilizing the facts: (1) $z^j_l$ are conditionally independent samples from $B(1, p^*_j)$ and (2) $k - N < k_j \leq k$, the law of large numbers yields

$$\lim_{k\to\infty}\frac{1}{k}\sum_{l=1}^{k_j} z^j_l = p^*_j, \quad \lim_{k\to\infty}\frac{1}{k}(k_j - \sum_{l=1}^{k_j} z^j_l) = 1 - p^*_j$$

where $p^*_j = P(z^j_k = 1|x^{T^*})$. Then,

$$\lim_{k\to\infty}\frac{1}{k}\sum_{l=1}^{k_j}\ln\frac{P(z^j_l|x^T)}{P(z^j_l|x^{T^*})} = p^*_j\ln\frac{p_j}{p^*_j} + (1-p^*_j)\ln\frac{1-p_j}{1-p^*_j} \tag{9}$$

Note that the right-hand side of Eq. (9) achieves maximum if and only if $p_j = p^*_j$. Considering Eq. (8) and Eq. (9), the limit of Eq. (7) is:

$$\lim_{k\to\infty}\frac{1}{k}\ln\frac{P^i_{pdf}(x^T|\mathbf{z}^i_{1:k})}{P^i_{pdf}(x^{T^*}|\mathbf{z}^i_{1:k})} = \sum_{j=1}^{N} p^*_j\ln\frac{p_j}{p^*_j} + (1-p^*_j)\ln\frac{1-p_j}{1-p^*_j} \tag{10}$$

It is known from Eq. (10) that

1) When $p_j \neq p^*_j$, that is $x^T \notin X^T_{eq}$,

$$\lim_{k\to\infty}\frac{1}{k}\ln\frac{P^i_{pdf}(x^T|\mathbf{z}^i_{1:k})}{P^i_{pdf}(x^{T^*}|\mathbf{z}^i_{1:k})} < 0, \text{ thus}$$
$$\lim_{k\to\infty}\frac{P^i_{pdf}(x^T|\mathbf{z}^i_{1:k})}{P^i_{pdf}(x^{T^*}|\mathbf{z}^i_{1:k})} = 0$$

2) When $p_j = p^*_j$, that is $x^T \in X^T_{eq}$,

$$\lim_{k\to\infty}\frac{1}{k}\ln\frac{P^i_{pdf}(x^T|\mathbf{z}^i_{1:k})}{P^i_{pdf}(x^{T^*}|\mathbf{z}^i_{1:k})} = 0, \text{ thus}$$
$$\lim_{k\to\infty}\frac{P^i_{pdf}(x^T|\mathbf{z}^i_{1:k})}{P^i_{pdf}(x^{T^*}|\mathbf{z}^i_{1:k})} = 1$$

■

### B. *Proof for moving robots*

The difficulty of consistency proof for moving robots lies in the fact that each robot observes at multiple positions. The main idea is to classify robot observation positions into two disjoint sets: *infinite-observation spots* that contains positions where a robot makes infinite observations, and *finite-observation spots* that contains positions where the robot makes finite observations. Before stating main theorem, the following lemma is introduced.

*Lemma 1:* For a set of robots with a collection of finite positions, each robot has at least one position where has infinite observations as $k$ tends to infinity.

*Proof:* Let $n^{i,k}_j$ denote the times that $i^{\text{th}}$ robot visits $j^{\text{th}}$ position up to time $k$. Then, $\sum_j n^{i,k}_j = k$. It is straightforward to see that $\exists n^{i,k}_j$, such that $n^{i,k}_j \to \infty$, as $k \to \infty$. ■

*Theorem 2:* Under the condition of moving robots, each individual PDF by LIFO-DBF converges to $X^T_{eq}$ when the number of observations tends to infinity, i.e.,

$$P^i_{pdf}(x^T \in X^T_{eq}|\mathbf{z}^i_{1:k}) \longrightarrow 1, \; k \to \infty$$

where $\mathbf{z}^i_{1:k} = (z^1_{1:k_1}, \ldots, z^N_{1:k_N})$.

*Proof:* Similar to Theorem 1, the batch form of DBF at $k^{\text{th}}$ step is:

$$\frac{P^i_{pdf}(x^T|\mathbf{z}^i_{1:k})}{P^i_{pdf}(x^{T^*}|\mathbf{z}^i_{1:k})} = \frac{P^i_{pdf}(x^T)\prod_{j=1}^{N}\prod_{l=1}^{k_j}P(z^j_l|x^T; x^R_l)}{P^i_{pdf}(x^{T^*})\prod_{j=1}^{N}\prod_{l=1}^{k_j}P(z^j_l|x^{T^*}; x^R_l)} \tag{11}$$

The only difference from Eq. (6) is that $P(z^j_l|x^T; x^R_l)$ in Eq. (11) depends on $x^R_l$. For each robot, there exists at least one position from which infinite observations is made as $k \to \infty$, according to Lemma 1. All positions can be classified into finite-observation spots and infinite-observation spots. For the former, it is straightforward to know that their contribution to Eq. (10) is zero. Therefore, the proof of Eq. (11) can be reduced by only considering infinite-observation spots, which is similar to Theorem 1. ■

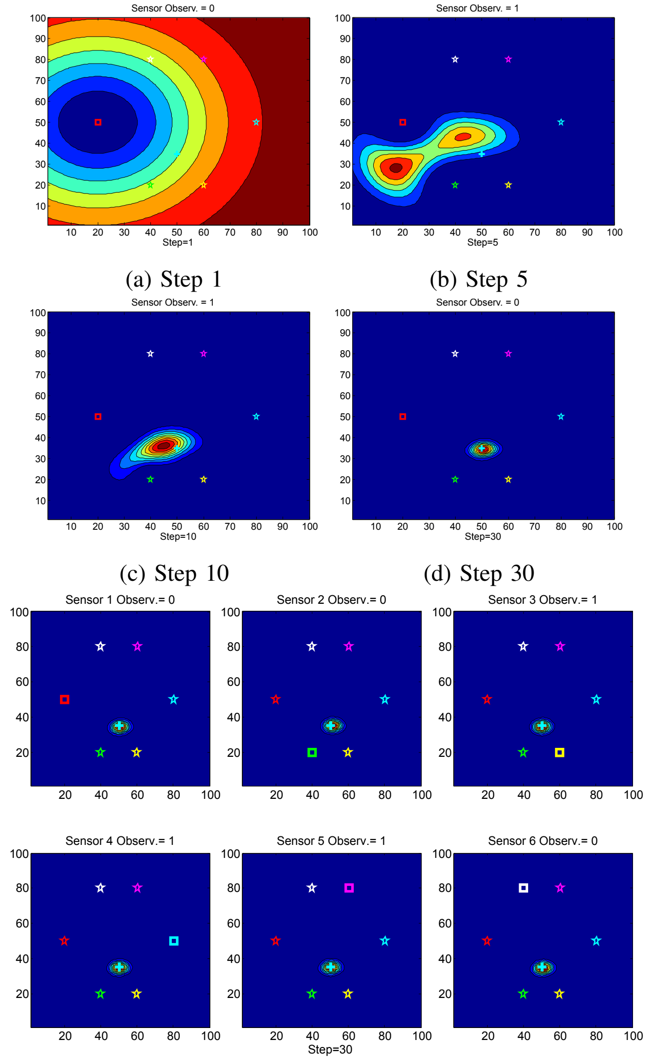
***Remark 5:*** (**TODO:** what convergence is?) Under LIFO-DBF, consistency implies that all individual PDFs converge to the same $X^T_{eq}$, thus the consensus is guaranteed. It must be noted that traditional statistics dissemination-based methods only ensure consensus of individual PDFs. To the best knowledge of authors, there is no proof of consistency of individual PDFs (**TODO:** add reference).

## V. SIMULATION

This section simulates three scenarios of target localization to demonstrate the effectiveness of LIFO-BDF. In all scenarios, six robots are utilized and each robot is equipped with a binary sensor. All sensors are modeled with identical Gaussian functions:

$$P(z = 1|x^T; x^R) = \exp\left\{-\frac{1}{2}(x^T - x^R)^T\Sigma^{-1}(x^T - x^R)\right\} \tag{12a}$$

$$P(z = 0|x^T; x^R) = 1 - P(z = 1|x^T; x^R). \tag{12b}$$




(a) Step 1

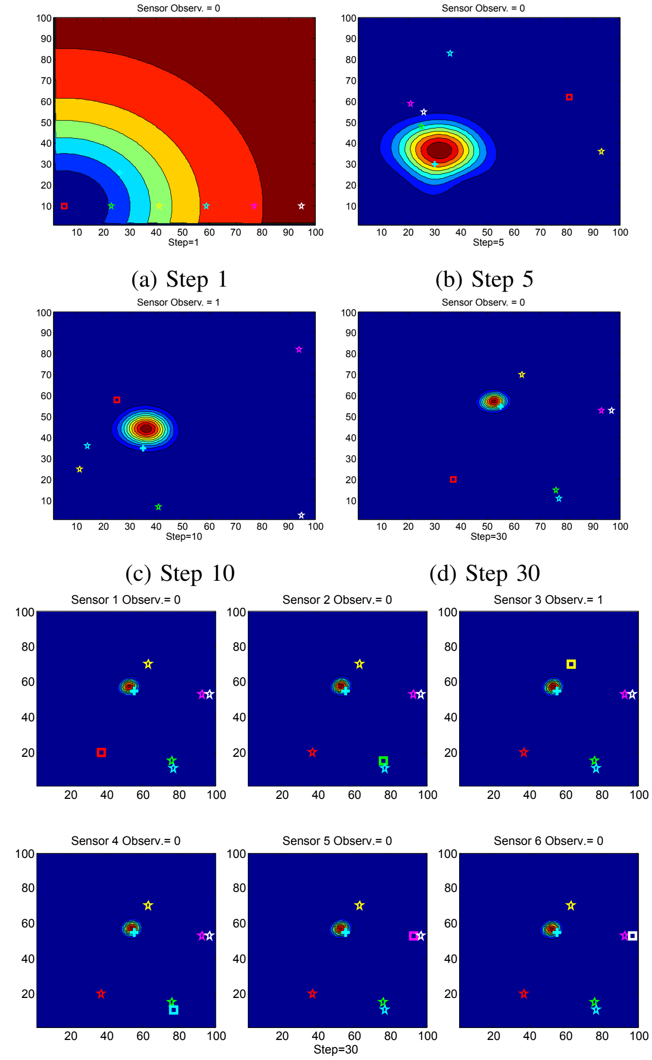
(b) Step 5

(c) Step 10

(d) Step 30

(e) Step 30

Fig. 4: (a)-(d): The 1st robot's individual PDFs at different time; (e) All robots' individual PDFs at time 30.(**TODO:** revise symbols) □ denotes the current robot and ⋆ represent other robots. + stands for the target.




(a) Step 1

(b) Step 5

(c) Step 10

(d) Step 30

(e) Step 30

Fig. 5: (a)-(d): The 1st robot's individual PDFs at different time; (e) All robots' individual PDFs at time 30.

The first scenario consists of six static robots and single static target. The second scenario subsequently deals with six moving robots for localizing a single static target. Robot positions are randomly generated at each time step. The third scenario contains six moving sensors and one moving target. A single-integrator dynamics is used for the target motion model.

### A. *Static Robots, Static Target*

The positions of six static robots are shown as stars in Fig. 4. The LIFO-DBF for static target is implemented on each robot for target position estimation. Fig. 4 shows the estimation results of the static target. After the initial observation, each robot forms a circular individual PDF, centered at its own position. The circular PDF happens because the Gaussian sensor model (Eq. (12)) only depends on the distance between robot and target. As more observations are received, the posterior individual PDF concentrates (**TODO:** concentrate or converge?) on the true location of the target (Fig. 4d), which accords with the consistency of LIFO-DBF.

### B. *Moving Robots, Moving Target*

The six robots move within the field to estimate the target position. In this work, the robot positions are randomly generated at each time in order to demonstrate the effectiveness of LIFO-DBF approach. The target dynamics is given by a single-integrator model:

$$x^T(k+1) = x^T(k) + v\Delta T$$

where $v$ is the constant velocity of the target; $\Delta T$ is the sampling time.

The LIFO-DBF described in Section III-C is utilized for target localization. Fig. 5 shows the estimation results of the moving target. It is interesting to notice that the posterior individual PDFs concentrate to the true target location at each time, even when the target constantly moves.

## VI. CONCLUSION

This paper presents the Latest-In-and-Full-Out (LIFO) strategy for measurement dissemination-based distributed Bayesian filters (DBF) in a multi-robot network. Different from statistics dissemination approaches that transmit posterior distributions or likelihood functions, each robot under LIFO only receives the latest available measurements and then broadcasts full communication buffer to its neighborhood, which significantly reduces the transmission burden of each pair from the order of environmental size to that of robot number. (**TODO:** expand this part) Under the condition of fixed and undirected topology, LIFO can guarantee non-intermittent dissemination of all observations over the network within finite time. Two types of LIFO-based DBF algorithms are proposed to estimate individual PDF for static and moving target, respectively. For the static target, each robot locally fuses the newly received observations while for the moving target, a triangular matrix of historical observations is stored and updated. (**TODO:** expand this part) The consistency of LIFO-based DBF is proved that individual PDF of each robot converges to the true target position when the number of observations tends to infinity. (**TODO:** find out what convergence is)

Future work includes extensions to other types of sensors and switching topology. Other types of sensors may have biased observations and subject to non-Bernoulli distribution, which complicates the design and analysis of LIFO-based Bayesian filters. The switching topology, including package loss, can lead to unpredictable delay and intermittent transmission, which may affect the consistency and consensus of individual PDFs.

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