# Complimentary Material to "A Guaranteed Cost Approach to Robust Model Predictive Control of Uncertain Linear Systems"

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**Abstract**

This complimentary material discusses the independence of $K_k$ and $\widetilde{K}$ controllers used at Equation (45) of the paper and presents the proof of Lemma 3.

## 1 Independence of controllers

Consider the system

$$\bar{x}_{k+1} = F\bar{x}_k + Gu_k + Hw_k, \tag{1}$$

and its deterministic ($w_k \equiv 0$) counterpart

$$x_{k+1} = Fx_k + Gu_k. \tag{2}$$

Consider a nil-potent controller $\widetilde{K}$ and let $u_k = -\widetilde{K}\bar{x}_k + p_k$, then the relation between the systems can be expressed as

$$\bar{x}_k = x_k + \sum_{i=0}^{k-1}(F - G\widetilde{K})^{k-i-1}Hw_i \tag{3}$$

and the closed loop deterministic system is

$$x_{k+1} = (F - G\widetilde{K})x_k + Gp_k. \tag{4}$$

Note that the control law $\widetilde{K}$ is used w.r.t. the uncertain state $\bar{x}_k$ in Equation (3).

Consider now the Guaranteed Cost Controller $K_k$ at timestep $k$ and let $p_k = -(K_k - \widetilde{K})x_k + v_k$, then the closed loop deterministic system becomes

$$\begin{aligned} x_{k+1} &= (F - G\widetilde{K})x_k - G(K_k - \widetilde{K})x_k + Gv_k \\ &= (F - GK_k)x_k + Gv_k. \end{aligned} \tag{5}$$

It is possible to observe that the independence of controller guaranteed cost controller $K_k$ and nil-potent controller $\widetilde{K}$ comes from the fact that the nil-potent controller is used to stabilize the uncertain system dynamics while the guaranteed cost controller is used to stabilize the deterministic system.

## 2 Proof of Lemma 3

**Lemma 3.** *Let* $\phi_k(x, v) = ||\widetilde{E}_{1,k}x + E_2 v||_2$ *and* $\rho_{i,k} = ||\widetilde{E}_{1,k}\widetilde{F}^i H||_2$. *Then,*

$$||w_k|| \leq \phi_k(\bar{x}_k, v_k) \leq \phi_k(x_k, v_k) + \sum_{i=0}^{k-1} c(k, i)\phi_i(x_i, v_i) \tag{6}$$

*where*

$$\forall i < k.\ c(k, i) = \rho_{k-i-1,k} + \sum_{j=0}^{k-i-2} \rho_{j,k} c(k - j - 1, i) \tag{7}$$

*Proof.* Consider the case $k = 1$ and $\phi_0(\bar{x}_0, v_0) = \phi_0(x_0, v_0)$, then

$$\begin{aligned} \phi_1(\bar{x}_1, v_1) &= ||\widetilde{E}_{1,1}\bar{x}_1 + E_2 v_1||_2 \\ &= ||\widetilde{E}_{1,1}x_1 + E_2 v_1 + \widetilde{E}_{1,1}Hw_0||_2 \\ &\leq \phi_1(x_1, v_1) + ||\widetilde{E}_{1,1}Hw_0||_2 \\ &\leq \phi_1(x_1, v_1) + ||\widetilde{E}_{1,1}H||_2||w_0||_2 \\ &\leq \phi_1(x_1, v_1) + \rho_{0,1}\phi_0(x_0, v_0) \end{aligned} \tag{8}$$

therefore, (6) is valid for $k = 1$.

Assuming that $\forall i \in [0, k-1]$ (6) is holds, it is possible to represent $\phi_k(\bar{x}_k, v_k)$ as

$$\begin{aligned} \phi_k(\bar{x}_k, v_k) &= \left|\left|\widetilde{E}_{1,k}x_k + E_2 v_k + \sum_{i=0}^{k-1} \widetilde{E}_{1,k}\widetilde{F}^{k-i-1}Hw_i\right|\right|_2 \\ &\leq ||\widetilde{E}_{1,k}x_k + E_2 v_k||_2 + \sum_{i=0}^{k-1} ||\widetilde{E}_{1,k}\widetilde{F}^{k-i-1}Hw_i||_2 \\ &\leq \phi_k(x_k, v_k) + \sum_{i=0}^{k-1} \rho_{k-i-1,k}\phi_i(\bar{x}_i, v_i) \end{aligned} \tag{9}$$

Substituting (6) and (7) on (9) yields

$$\phi_k(\bar{x}_k, v_k) \leq \phi_k(x_k, v_k) + \sum_{i=0}^{k-1} \rho_{k-i-1,k} \left( \phi_i(x_i, v_i) + \sum_{j=0}^{i-1} c(i, j)\phi_j(x_j, v_j) \right), \tag{10}$$

which can be rearranged to

$$\begin{aligned} \phi_k(\bar{x}_k, v_k) \leq \phi_k(x_k, v_k) + \rho_{0,1}\phi_k(x_{k-1}, v_{k-1}) + \\ + \sum_{i=0}^{k-2} \left( \rho_{k-i-1,k} + \sum_{j=0}^{k-i-2} \rho_{j,k} c(k - j - 1, i) \right) \phi_i(x_i, v_i). \end{aligned} \tag{11}$$

Thus, substituting (7) to (11) results

$$\phi_k(\bar{x}_k, v_k) \leq \phi_k(x_k, v_k) + \sum_{i=0}^{k-1} c(k, i)\phi_i(x_i, v_i) \tag{12}$$

concluding this proof by induction. □