BIEN/CENG 2310
MODELING FOR CHEMICAL AND BIOLOGICAL ENGINEERING

HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, FALL 2022
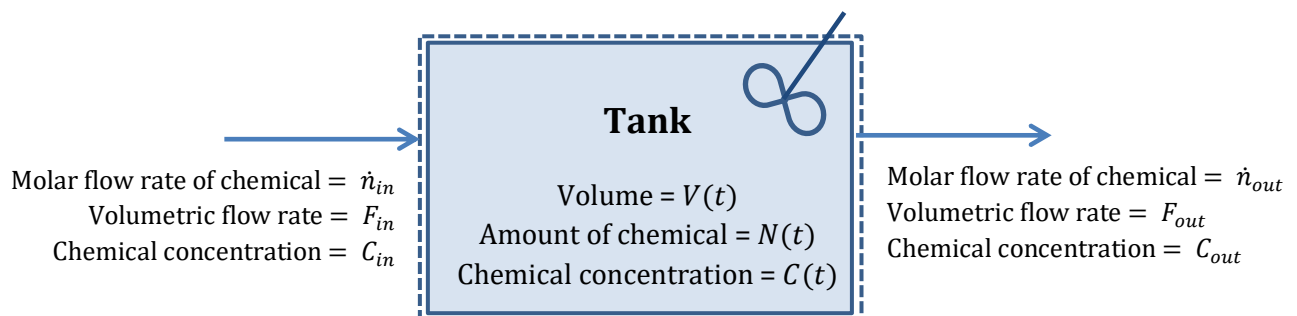
HOMEWORK #1 SOLUTION

1. A well-stirred tank, with an inlet and an outlet, is initially filled with water. The inlet stream contains a chemical dissolved in water at a fixed concentration.

    (a) Assuming the volume of the solution in the tank, V, is constant throughout the process, write down an ODE for the concentration of the chemical in the tank, $C(t)$. List any assumptions and define any parameters.

    (b) Solve the ODE in Part (b) analytically, and also simulate it using the Euler Method in Excel. Overlay the numerical solution over the analytical one.

    (c) Generalize your model for $C(t)$ to the case where V is not constant.

    (d) Solve the ODE in part (c) analytically, and also simulate it using the Euler Method in Excel. Overlay the numerical solution over the analytical one.

SOLUTION

PART (a)

We define the content of the well-stirred tank as our system:

Molar flow rate of chemical $= \dot{n}_{in}$
Volumetric flow rate $= F_{in}$
Chemical concentration $= C_{in}$

Tank
Volume $= V(t)$
Amount of chemical $= N(t)$
Chemical concentration $= C(t)$

Molar flow rate of chemical $= \dot{n}_{out}$
Volumetric flow rate $= F_{out}$
Chemical concentration $= C_{out}$

The following symbols are used:

t = Time elapsed since beginning of process [=] s
$N(t)$ = Amount of chemical in the tank [=] mol
$\dot{n}_{in}$ = Molar flow rate of the chemical into the tank [=] mol/s
$\dot{n}_{out}$ = Molar flow rate of the chemical into the tank [=] mol/s
$V(t)$ = Volume of the tank content, as a function of time [=] m³
$C(t)$ = Concentration of the chemical in the tank, as a function of time [=] mol/m³
F_{in} = Volumetric flow rate of the inlet stream [=] m³/s
F_{out} = Volumetric flow rate of the outlet stream [=] m³/s
C_{in} = Concentration of the chemical in the inlet stream [=] mol/m³
C_{out} = Concentration of the chemical in the outlet stream [=] mol/m³
C_0 = Initial concentration of the chemical in the tank [=] mol/m³

The following assumptions are made:

1. The tank is well-mixed, so the concentration of the chemical is uniform everywhere.

2. The volume of the solution in the tank, V, is constant (for Parts (a) and (b)).

3. The chemical solution is dilute. Its density is that of water's and does not depend on the concentration of the chemical, both in the tank and in the streams.

4. The in-flow and out-flow rates are constant and do not vary with time.

5. The chemical is inert and does not react or decompose.

6. The tank does not contain the chemical initially, $C_0 = 0$.

By balancing the amount of the chemical, which is conserved, we can write the following ODE:

$$\frac{dN}{dt} = \dot{n}_{in} - \dot{n}_{out}$$

Expressing this in terms of volumes and concentrations:

$$\frac{d(CV)}{dt} = F_{in}C_{in} - F_{out}C_{out}$$

Since we assume that the density is constant, a mass balance over the system will imply that volume is balanced too. Since $V = V_0$ is constant, this means that the in-flow rate and the out-flow rate must also be balanced, $F_{in} = F_{out}$. The ODE simplifies to:

$$V_0 \frac{dC}{dt} = F_{in}(C_{in} - C_{out})$$

Now we apply our assumptions $C_{out} = C$ due to well-mixed tank, and we arrive at:

$$\frac{dC}{dt} = -\frac{F_{in}}{V_0}(C_{in} - C)$$

with the initial condition:

$$C(t = 0) = C_0 = 0$$

PART (b)

This is a separable ODE, and can be solved as follows:

$$\frac{dC}{dt} = -\frac{F_{in}}{V_0}(C_{in} - C)$$

$$\frac{dC}{C_{in} - C} = -\frac{F_{in}}{V_0}dt$$

Integrating from $t = 0$ to any arbitrary time point, with the corresponding integration limits for C:

$$\int_{C(0)}^{C(t)} \frac{dC}{C_{in} - C} = \int_0^t -\frac{F_{in}}{V_0}dt$$

$$\int_0^{C(t)} \frac{-d(C_{in} - C)}{C_{in} - C} = \int_0^t -\frac{F_{in}}{V_0}dt$$

$$\ln\left[\frac{C_{in} - C(t)}{C_{in} - 0}\right] = -\frac{F_{in}}{V_0}(t - 0)$$

$$\frac{C_{in} - C(t)}{C_{in}} = \exp\left(-\frac{F_{in}}{V_0}t\right)$$

$$C(t) = C_{in}\left[1 - \exp\left(-\frac{F_{in}}{V_0}t\right)\right]$$

See the attached Excel file "BIEN2310_HW1_Q1.xlsx" for the Euler Method simulation overlaid with the analytical solution.

PART (c)

If V is not constant, we have to account for how it changes with time. As mentioned above, since we assume that the density is constant, mass balance will imply "volume balance":

$$\frac{dV}{dt} = F_{in} - F_{out}$$

with initial condition $V(t = 0) = V_0$. This can be easily solved to yield:

$$V = V_0 + (F_{in} - F_{out})t$$

To generalize the model as required, we also have to relax the constraint that $F_{in} = F_{out}$. Thus, our ODE becomes:

$$\frac{d(CV)}{dt} = V\frac{dC}{dt} + C\frac{dV}{dt} = F_{in}C_{in} - F_{out}C$$

Note that we have to apply the product rule on the derivative, since V is also a function of t now. Plugging in the expressions for V and dV/dt, we have:

$$[V_0 + (F_{in} - F_{out})t]\frac{dC}{dt} + C(F_{in} - F_{out}) = F_{in}C_{in} - F_{out}C$$

$$\frac{dC}{dt} + \left[\frac{F_{in}}{V_0 + (F_{in} - F_{out})t}\right]C = \frac{F_{in}C_{in}}{V_0 + (F_{in} - F_{out})t}$$

This is a linear ODE. We can also re-arrange it as a separable ODE:

$$\frac{dC}{dt} = \left[\frac{F_{in}}{V_0 + (F_{in} - F_{out})t}\right](C_{in} - C)$$

$$\frac{dC}{C_{in} - C} = \left[\frac{F_{in}}{V_0 + (F_{in} - F_{out})t}\right]dt$$

By the way, with this change, to avoid unphysical situations, we will make an additional assumption: _that the volume cannot fall below zero_. In other words, if $F_{out} > F_{in}$, our model will only apply for $t \leq V_0/(F_{out} - F_{in})$.

PART (d)

It is easier to solve the separable ODE. However, just to show how to solve a linear ODE, we will try the harder way. We first define the function $p(t) = \frac{F_{in}}{V_0 + (F_{in} - F_{out})t}$, and we can determine the integrating factor:

$$\mu(t) = \exp\int p(t)dt$$

$$\mu(t) = \exp\int \frac{F_{in}}{V_0 + (F_{in} - F_{out})t}dt$$

$$\mu(t) = \exp\left[\frac{F_{in}}{F_{in} - F_{out}}\int \frac{d[V_0 + (F_{in} - F_{out})t]}{V_0 + (F_{in} - F_{out})t}\right]$$

$$\mu(t) = \exp\left\{\frac{F_{in}}{F_{in} - F_{out}}\ln[V_0 + (F_{in} - F_{out})t]\right\}$$

$$\mu(t) = [V_0 + (F_{in} - F_{out})t]^{\frac{F_{in}}{F_{in} - F_{out}}}$$

As we learned, we now multiply the ODE throughout by this integrating factor, and then we can combine the terms on the left-hand side:

$$\mu(t)\frac{dC}{dt} + \mu(t)p(t)C = \mu(t)p(t)C_{in}$$

$$\frac{d}{dt}[\mu(t)C] = \mu(t)p(t)C_{in}$$

We can now integrate:

$$[\mu(t)C]_{(0,C_0)}^{(t,C)} = C_{in}\int_0^t \mu(t)p(t)\,dt$$

$$\mu(t)C - \mu(0)\,C_0 = C_{in}\int_0^t d\mu$$

Here, we are helped by the fact that the right-hand side of the linear ODE is just $p(t)C_{in}$, and by our definition of μ, we have:

$$d\mu = d\left[\exp\int p\,dt\right] = \left[\exp\int p\,dt\right]\left[d\int p\,dt\right] = \mu p\,dt$$

So the integration on the right-hand side is simplified. With $C_0 = 0$, we have:

$$\mu(t)C = C_{in}[\mu(t) - \mu(0)]$$

$$C = C_{in}\left[1 - \frac{\mu(0)}{\mu(t)}\right]$$

$$C(t) = C_{in}\left\{1 - \left[\frac{V_0}{V_0 + (F_{in} - F_{out})t}\right]^{\frac{F_{in}}{F_{in}-F_{out}}}\right\}$$

Again, this is with the additional constraint that if $F_{out} > F_{in}$, then this equation will only apply for $t \leq V_0/(F_{out} - F_{in})$. Beyond that point, there is no more liquid in the tank anyway. It becomes somewhat meaningless to refer to C, the concentration of chemical in the tank.

See the attached Excel file "BIEN2310_HW1_Q1.xlsx" for the Euler Method simulation overlaid with the analytical solution.

NOTES

You can get the same answer if you do not make the $d\mu = \mu p\, dt$ substitution above, and instead just plug in $\mu(t)$ and integrate the right-hand side:

$$C_{in}\int_0^t \mu(t)p(t)\, dt = C_{in}\int_0^t [V_0 + (F_{in} - F_{out})t]^{\frac{F_{in}}{F_{in}-F_{out}}}\left[\frac{F_{in}}{V_0 + (F_{in} - F_{out})t}\right]dt$$

$$= F_{in}C_{in}\int_0^t [V_0 + (F_{in} - F_{out})t]^{\frac{F_{in}}{F_{in}-F_{out}}-1}\, dt$$

$$= F_{in}C_{in}\int_0^t [V_0 + (F_{in} - F_{out})t]^{\frac{F_{out}}{F_{in}-F_{out}}}\, dt$$

$$= \frac{F_{in}C_{in}}{F_{in} - F_{out}}\int_0^t [V_0 + (F_{in} - F_{out})t]^{\frac{F_{out}}{F_{in}-F_{out}}}\, d[V_0 + (F_{in} - F_{out})t]$$

$$= \frac{F_{in}C_{in}}{F_{in} - F_{out}}\left\{[V_0 + (F_{in} - F_{out})t]^{\frac{F_{out}}{F_{in}-F_{out}}+1}\right\}_0^t$$

$$= \frac{F_{in}C_{in}}{F_{in} - F_{out}}\left\{[V_0 + (F_{in} - F_{out})t]^{\frac{F_{in}}{F_{in}-F_{out}}} - V_0^{\frac{F_{in}}{F_{in}-F_{out}}}\right\}$$

Of course, you can also get the same answer by using the "separable ODE" method:

$$\int_{C(0)}^{C(t)} \frac{dC}{C_{in} - C} = \int_0^t \left[\frac{F_{in}}{V_0 + (F_{in} - F_{out})t}\right]dt$$

$$-\ln\left[\frac{C_{in} - C(t)}{C_{in} - C(0)}\right] = \frac{F_{in}}{F_{in} - F_{out}}\ln\left[\frac{V_0 + (F_{in} - F_{out})t}{V_0}\right]$$

$$\frac{C_{in} - C(t)}{C_{in}} = \left[\frac{V_0 + (F_{in} - F_{out})t}{V_0}\right]^{\frac{-F_{in}}{F_{in}-F_{out}}}$$

$$C(t) = C_{in} \left\{ 1 - \left[\frac{V_0}{V_0 + (F_{in} - F_{out})t} \right]^{\frac{F_{in}}{F_{in} - F_{out}}} \right\}$$

As a sanity check, we should get back the result in Part (b) if we set $F_{in} = F_{out}$ in Part (d)'s result. To see that is the case, we need to know the identity:

$$\exp(x) = \lim_{n \to \infty} \left(1 + \frac{x}{n} \right)^n$$

In Part (d), we derived:

$$C(t) = C_{in} \left\{ 1 - \left[\frac{V_0}{V_0 + (F_{in} - F_{out})t} \right]^{\frac{F_{in}}{F_{in} - F_{out}}} \right\}$$

Let $n = \frac{F_{in}}{F_{in} - F_{out}}$. Consider the limit for the term in the above expression:

$$Y = \lim_{n \to \infty} \left[\frac{V_0}{V_0 + (F_{in} - F_{out})t} \right]^n$$

$$= \lim_{n \to \infty} \left[1 + \frac{-(F_{in} - F_{out})t}{V_0 + (F_{in} - F_{out})t} \right]^n$$

$$= \lim_{n \to \infty} \left[1 + \frac{\frac{-(F_{in} - F_{out})t}{V_0 + (F_{in} - F_{out})t} \left(\frac{F_{in}}{F_{in} - F_{out}} \right)}{\frac{F_{in}}{F_{in} - F_{out}}} \right]^n$$

$$= \lim_{n \to \infty} \left[1 + \frac{\frac{-F_{in}t}{V_0 + (F_{in} - F_{out})t}}{n} \right]^n$$

Now, let $x = \frac{-F_{in}t}{V_0 + (F_{in} - F_{out})t}$. With $F_{in} = F_{out}$, we seek the limit of this expression when $n \to \infty$, which is given by the identity above:

$$Y = \lim_{n \to \infty} \left(1 + \frac{x}{n} \right)^n = \exp(x)$$

$$Y = \exp \left[\frac{-F_{in}t}{V_0 + (F_{in} - F_{out})t} \right]$$

Therefore, as $F_{in} = F_{out}$, which implies $n \to \infty$, our solution in Part (d) becomes:

$$C(t) = C_{in}(1 - Y) = C(t) = C_{in} \left[1 - \exp \left(\frac{-F_{in}t}{V_0 + (F_{in} - F_{out})t} \right) \right] = C_{in} \left[1 - \exp \left(\frac{-F_{in}t}{V_0} \right) \right]$$

which is the same as our expression in Part (b).

2. One convenient way to prevent moth larvae from ruining our clothing is the use of mothballs, which slowly release a chemical vapor (typically 1,2-dicholorbenzene) in our wardrobes to repel the pests. The mechanism is that under ambient conditions, the solid repellent of the mothball sublimes to produce the vapor, and as time goes on, the mothball becomes smaller and smaller. The rate of sublimation (in g/month) is proportional to the surface area of the mothball in contact with air.
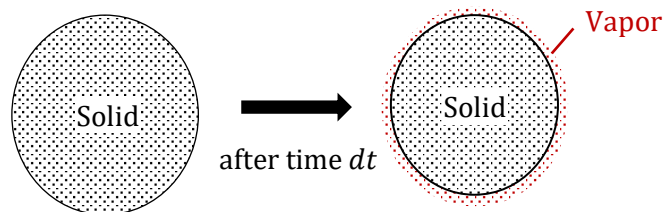
(a) Assuming the mothball is a perfect sphere throughout the process, model the radius of the mothball as a function of time, $R(t)$, as an ODE with an appropriate initial condition. Be explicit about your assumptions, and list and define all your parameters.

(b) Solve the ODE in Part (a) analytically, and provide an expression for $s(t)$, the rate of release of the vapor from one mothball.

(c) Suppose we put n mothball in a wardrobe of volume V_w, and we are interested in the concentration of the chemical vapor $L(t)$ in the wardrobe over a long period of many months. The wardrobe is not air-tight, and the air inside is continuously exchanged with fresh air outside. This can be modeled by an air stream going through the wardrobe with some small constant flow rate F (in L/month). Create a model in Excel to predict $L(t)$, allowing the user to specify all the relevant parameters. *(No need to solve this equation analytically.)*

PART (a)

We desire a model of the radius of a mothball as a function of time, $R(t)$. We first define the following quantities relevant to the problem:

t = Time elapsed since the mothball begins to sublime [=] month
$m(t)$ = Mass of the mothball as a function of time [=] g
$R(t)$ = Radius of mothball, as a function of time [=] cm
$V(t)$ = Volume of the mothball, as a function of time [=] cm^3
$A(t)$ = Surface area of the mothball, as a function of time [=] cm^2
$s(t)$ = Rate of sublimation, also the rate of vapor release, as a function of time [=] g/month
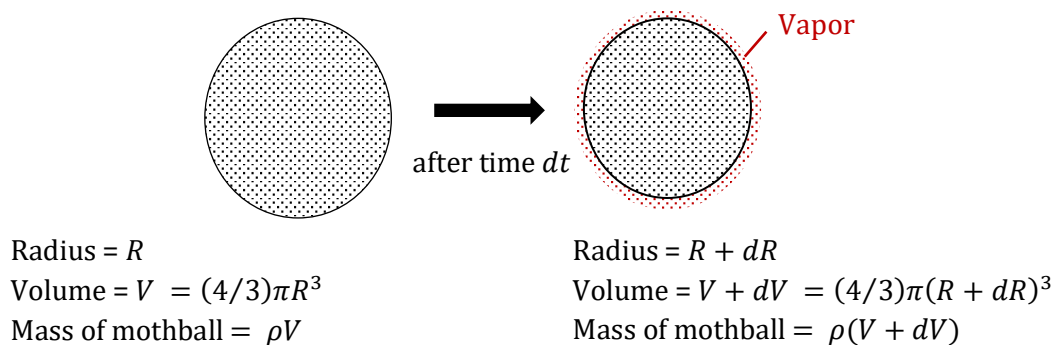ρ = Density of the mothball [=] g cm^{-3}
q = Proportionality constant between release rate and surface area [=] g/(t cm^2)
R_0 = Initial radius of the mothball [=] cm

We make the following assumptions

1. The rate of sublimation is proportional to the exposed surface area.

2. The mothball is spherical at all times.

3. The density of the mothball is constant.

The following schematic shows the subliming mothball:

Radius = R
Volume = $V = (4/3)\pi R^3$
Mass of mothball = ρV

Radius = $R + dR$
Volume = $V + dV = (4/3)\pi(R + dR)^3$
Mass of mothball = $\rho(V + dV)$

Here dR and dV are negative for our process. If we treat the mothball as a control volume system, that is, the boundary of our system is the interface between the solid mothball and the air around it, we can write a balance equation based on conservative of mass:

$$\frac{dm}{dt} = -s(t)$$

since there is no "in" term, and the "out" term should be equal to the rate of sublimation. By Assumption 1, we can write:

$$s(t) = qA(t)$$

where q is some positive proportionality constant. Thus, with $m = \rho V$, we have:

$$\frac{d(\rho V)}{dt} = -qA$$

We want an ODE in terms of R, so we apply the chain rule to change variable from V to R:

$$\frac{dV}{dt} = -\frac{q}{\rho}A$$

$$\left(\frac{dV}{dR}\right)\left(\frac{dR}{dt}\right) = -\frac{q}{\rho}A$$

$$\frac{d}{dR}\left(\frac{4}{3}\pi R^3\right)\left(\frac{dR}{dt}\right) = -\frac{q}{\rho}(4\pi R^2)$$

$$(4\pi R^2)\left(\frac{dR}{dt}\right) = -\frac{q}{\rho}(4\pi R^2)$$

$$\frac{dR}{dt} = -\frac{q}{\rho}$$

with the initial condition:

$$R(t = 0) = R_0$$

PART (b)

This ODE of $R(t)$ is separable:

$$dR = -\frac{q}{\rho}dt$$

which we can solve easily to yield:

$$R(t) = R_0 - \frac{q}{\rho}t$$

If we try to plot this solution, we will notice that R can become negative at large enough t, which does not make sense physically. When R falls to zero, it means that the mothball has completely sublimed, and after that time point, R should simply stay at zero forever. To complete our solution, we need to use our engineering sense to constrain the value of R to non-negative values:

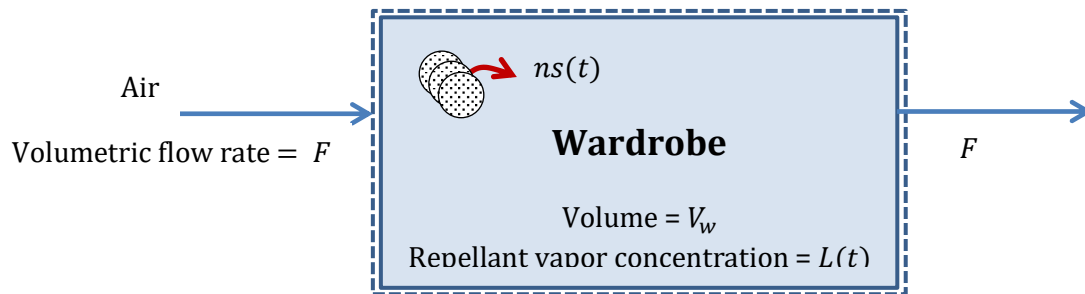$$R(t) = \begin{cases} R_0 - \dfrac{q}{\rho}t & if\ t \leq \rho R_0/q \\ 0 & otherwise \end{cases}$$

The release rate is given by:

$$s(t) = qA(t) = q(4\pi)R^2$$

$$s(t) = \begin{cases} q(4\pi)\left(R_0 - \dfrac{q}{\rho}t\right)^2 & if\ t \leq \rho R_0/q \\ 0 & otherwise \end{cases}$$

PART (c)

The following is a schematic of the system we are modeling, the wardrobe:

We define a few more symbols:

$L(t)$ is the concentration of the repellant as a function of time [=] g/L
F = volumetric flow rate of air passing through the wardrobe [=] L/month
n = Number of mothballs we put into the wardrobe at the beginning (dimensionless)
V_w = Volume of wardrobe [=] L

Here we make three more assumptions:

1. The wardrobe is well-mixed, such that the concentration of the repellant is the same everywhere in the wardrobe. This also implies that the air leaving the wardrobe has the same concentration L.

2. We put n *identical* mothballs into the wardrobe initially, and the total rate of vapor release at time t is simply given by $ns(t)$.

3. The initial amount of repellant in the wardrobe is zero, and the incoming air contains no repellant.

The mass balance of the drug is given by the following ODE:

$$\frac{d(V_w L)}{dt} = ns(t) - FL$$

$$\frac{dL}{dt} = \frac{ns(t) - FL}{V_w}$$

Note that we have to be careful to not allow R to become negative. Our ODE should be:

$$\frac{dL}{dt} = \begin{cases} \frac{n}{V_w} q(4\pi)\left(R_0 - \frac{q}{\rho}t\right)^2 - \frac{F}{V_w}L & for \quad t \le \rho R_0 / q \\ -\frac{F}{V_w}L & for \quad t > \rho R_0 / q \end{cases}$$

with the initial condition:

$$L(t = 0) = 0$$

Please see the attached Excel file "BIEN2310_HW1_Q2.xlsx" for the numerical solution.

For your curiosity, this ODE in Part (d) can also be solved analytically, since it is linear.

$$\frac{dL}{dt} + \frac{F}{V_w} L = \frac{n}{V_w} q(4\pi) \left(R_0 - \frac{q}{\rho} t \right)^2$$

Note that this is only up to the point when the mothball sublimes completely, which is when $t = \rho R_0 / q$, which we will call τ. At $t > \tau$, there is no more mothball chemical released to the wardrobe, and our ODE becomes:

$$\frac{dL}{dt} + \frac{F}{V_w} L = 0 \quad for \ \ t > \tau$$

which is just an exponential decay. Solving this with the value of L at t at $t = \tau$ as the initial value, we should find:

$$L = L(\tau) \exp \left[-\frac{F}{V_w} (t - \tau) \right] \quad for \ \ t > \tau$$

Returning to our ODE, and defining $h = \frac{4\pi n q}{V_w}$, and $k = \frac{F}{V_w}$ to make it easier to read, we proceed as follows:

$$e^{kt} \frac{dL}{dt} + e^{kt} kL = h \left(R_0 - \frac{q}{\rho} t \right)^2 e^{kt}$$

$$\frac{d}{dt} (e^{kt} L) = h \left(R_0 - \frac{q}{\rho} t \right)^2 e^{kt}$$

$$[e^{kt} L]_{(0,0)}^{(t,L)} = \int_0^t h \left(R_0 - \frac{q}{\rho} t \right)^2 e^{kt} \ dt$$

This integral needs two steps of "integration by parts":

$$e^{kt} L = \int_0^t \frac{h}{k} \left(R_0 - \frac{q}{\rho} t \right)^2 d(e^{kt})$$

$$\frac{k e^{kt}}{h} L = \left[\left(R_0 - \frac{q}{\rho} t \right)^2 e^{kt} \right]_0^t - \int_0^t -\frac{2q}{\rho} \left(R_0 - \frac{q}{\rho} t \right) e^{kt} \ dt$$

$$\frac{k e^{kt}}{h} L = \left[\left(R_0 - \frac{q}{\rho} t \right)^2 e^{kt} - R_0^2 \right] + \frac{2q}{\rho} \int_0^t \left(R_0 - \frac{q}{\rho} t \right) e^{kt} \ dt$$

$$\frac{k e^{kt}}{h} L = \left[\left(R_0 - \frac{q}{\rho} t \right)^2 e^{kt} - R_0^2 \right] + \frac{2q}{\rho k} \int_0^t \left(R_0 - \frac{q}{\rho} t \right) d(e^{kt})$$

$$\frac{k e^{kt}}{h} L = \left[\left(R_0 - \frac{q}{\rho} t \right)^2 e^{kt} - R_0^2 \right] + \frac{2q}{\rho k} \left[\left(R_0 - \frac{q}{\rho} t \right) e^{kt} \right]_0^t - \frac{2q}{\rho k} \int_0^t -\frac{q}{\rho} e^{kt} \ dt$$

$$\frac{k e^{kt}}{h} L = \left[\left(R_0 - \frac{q}{\rho} t \right)^2 e^{kt} - R_0^2 \right] + \frac{2q}{\rho k} \left[\left(R_0 - \frac{q}{\rho} t \right) e^{kt} - R_0 \right] + 2 \left(\frac{q}{\rho k} \right)^2 [e^{kt}]_0^t$$

$$\frac{k e^{kt}}{h} L = \left[\left(R_0 - \frac{q}{\rho} t \right)^2 e^{kt} - R_0^2 \right] + \frac{2q}{\rho k} \left[\left(R_0 - \frac{q}{\rho} t \right) e^{kt} - R_0 \right] + 2 \left(\frac{q}{\rho k} \right)^2 (e^{kt} - 1)$$

Our final answer is:

$$L(t) = \frac{h}{k}\left\{\left[\left(R_0 - \frac{q}{\rho}t\right)^2 + \frac{2q}{\rho k}\left(R_0 - \frac{q}{\rho}t\right) + 2\left(\frac{q}{\rho k}\right)^2\right] - \left[R_0{}^2 + \frac{2q}{\rho k}R_0 + 2\left(\frac{q}{\rho k}\right)^2\right]e^{-kt}\right\}$$

which is only valid for $t \leq \tau$!

To complete the solution, let's find $L(\tau)$ using this result:

$$L\left(t = \frac{\rho R_0}{q}\right) = \frac{h}{k}\left\{2\left(\frac{q}{\rho k}\right)^2 - \left[R_0{}^2 + \frac{2q}{\rho k}R_0 + 2\left(\frac{q}{\rho k}\right)^2\right]e^{-\frac{k\rho R_0}{q}}\right\}$$

Hence, for $t > \tau$, the function is:

$$L = \frac{h}{k}\left\{2\left(\frac{q}{\rho k}\right)^2 - \left[R_0{}^2 + \frac{2q}{\rho k}R_0 + 2\left(\frac{q}{\rho k}\right)^2\right]e^{-\frac{k\rho R_0}{q}}\right\}\exp\left[-\frac{F}{V_w}(t - \tau)\right] \quad for \ t > \tau$$

It is indeed quite a mess, even for a relatively simple problem. I hope now you appreciate why we have to learn solving ODEs numerically!