BIEN/CENG 2310

MODELING FOR CHEMICAL AND BIOLOGICAL ENGINEERING

HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, FALL 2022

HOMEWORK #3 SOLUTION

1. The Lotka-Volterra model for an ecosystem consisting of a predator and a prey is as follows:

$$\begin{cases} \dfrac{dR}{dt} = \alpha R - \beta RF \quad ; \quad R(t=0) = R_0 \\ \dfrac{dF}{dt} = \gamma RF - \delta F \quad ; \quad F(t=0) = F_0 \end{cases}$$

(a) Using MATLAB's `ode45` function, simulate this system of ODEs and plot the two populations R and F, versus time t from $t = 0$ to $t = t_f$. Allow the users to specify the parameters, the initial populations, and the final time point t_f.

(b) Plot the phase space diagram F vs. R. Keeping the rate constants the same, vary the initial populations and overlay the plots for different initial populations. Can you find a combination of initial populations for which the populations will stay the same forever?

(c) In the basic Lotka-Volterra model above, it was assumed that the prey is not subject to resource constraints and follow a Malthusian growth model. Replace the birth term for the prey to relax this assumption, and use the logistic growth model instead. Modify the provided `lotkaVolterra.m` to simulate this new model. Your function definition should be:

```
Nf = lotkaVolterraLogistic_<LastName>_<FirstName>(param, tf)
```

where `param` is a vector containing the parameters $\alpha, \beta, \gamma, \delta, K, R_0, F_0$ in that order. Here, K is the carrying capacity of the prey, and α is the net birth rate of the prey at the limit of $R = 0$.

(d) Experiment with different parameter settings and observe the populations over long time. How does the general behavior differ from the basic Lotka-Volterra model? Locate any fixed point(s) for this system in your simulation, and verify that they are exactly as predicted by setting $dR/dt = dF/dt = 0$.

(e) Modify your program from Part (c) to add an event trigger to stop the simulation at t_f, or if/when the system reaches steady state, whichever is later. The program should still return the populations at t_f.

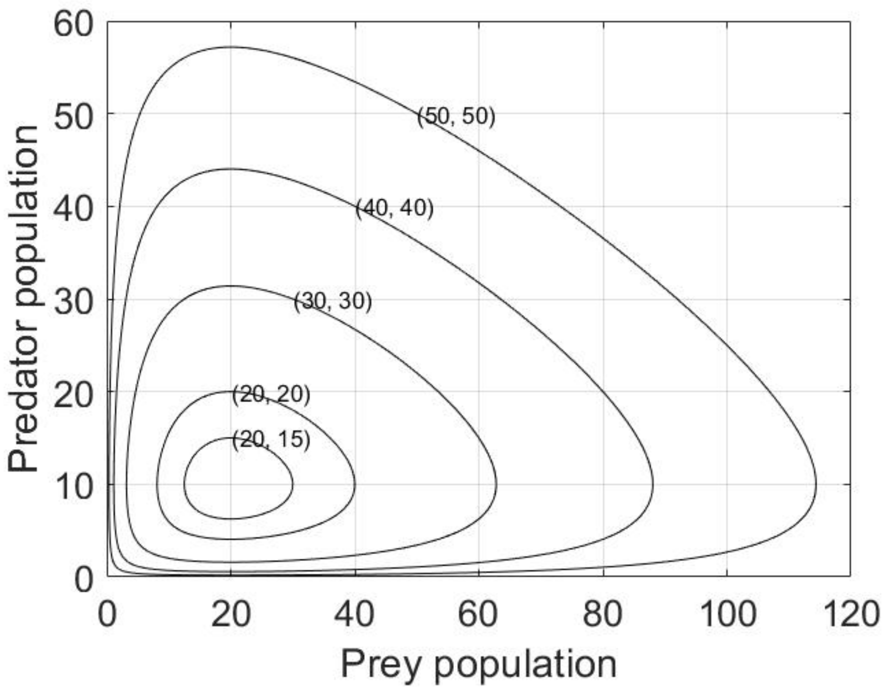
SOLUTION

PART (a)

Please see the attached `lotkaVolterra.m` for the MATLAB code to simulate this model.

PART (b)

As an example, with the rate constants $\alpha = 0.1, \beta = 0.01, \gamma = 0.005, \delta = 0.1$, the phase space diagram looks like this:



Each cycle shown results from one set of initial populations, indicated by the marked coordinates.

For this set of rate constants, if we set $R_0 = 20$ and $F_0 = 10$, the populations becomes stationary forever – no more cyclic behavior. This can be explained by the fact that if we plug in these initial populations as R and F, as well as the specified rate constants, we find that both derivatives are zero:

$$\left.\frac{dR}{dt}\right|_{(R_0,F_0)} = \alpha R_0 - \beta R_0 F_0 = 0.1(20) - 0.01(20)(10) = 0$$
$$\left.\frac{dF}{dt}\right|_{(R_0,F_0)} = \gamma R_0 F_0 - \delta F_0 = 0.005(20)(10) - 0.1(10) = 0$$

That is why the populations will stay the same.

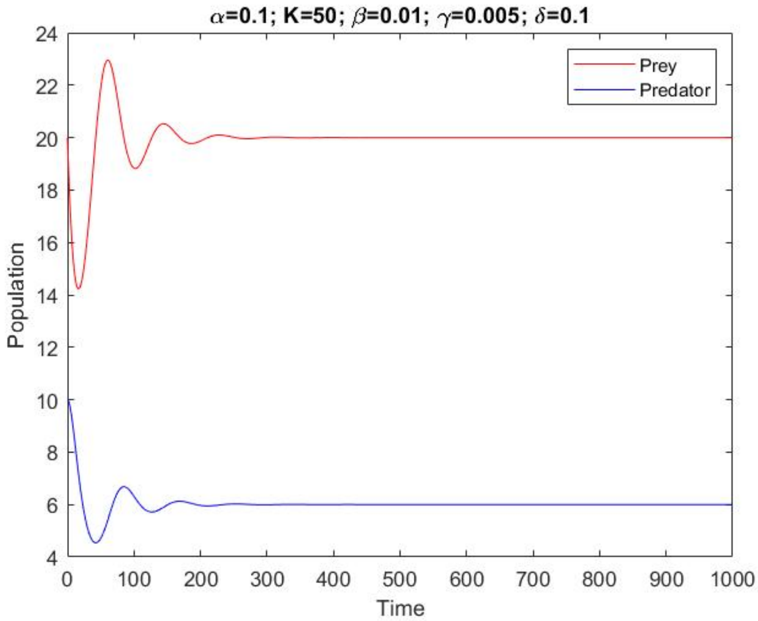
PART (c)

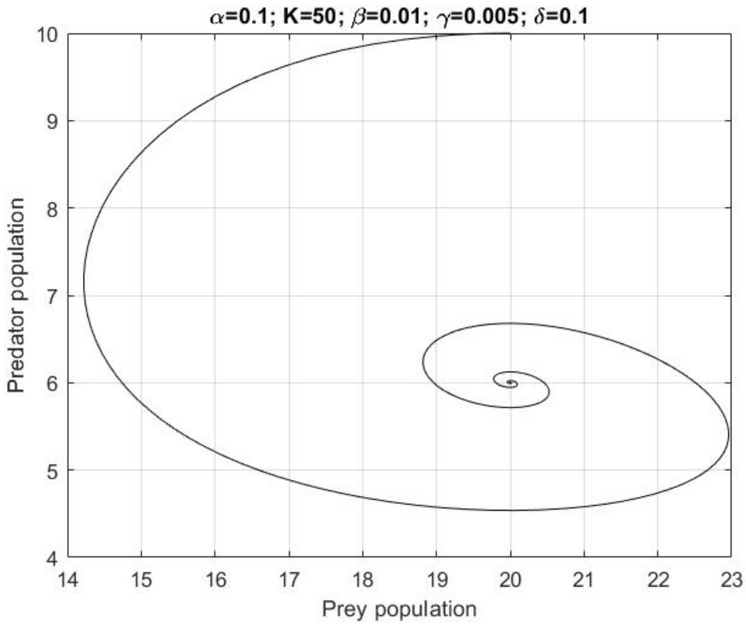
To account for resource constraints for the rabbits, we can change its birth term to incorporate a carrying capacity, K:

$$\frac{dR}{dt} = \alpha\left(1 - \frac{R}{K}\right)R - \beta RF$$

The other ODE for foxes, and the initial conditions, will be unchanged. The MATLAB program is slightly modified to simulate this model instead, and attached as `lotkaVolterraLogistic.m`

The population dynamics are quite different from the basic Lotka-Volterra model. Instead of the perfect cyclical behavior we have seen before, the populations oscillate at first with decreasing amplitude over time, and finally settle into a fixed point. For example, the plots for the parameters $\alpha = 0.1, K = 100, \beta = 0.01, \gamma = 0.005, \delta = 0.1, R_0 = 20$ and $F_0 = 10$ are shown below:





To find possible fixed points, we set both derivatives to zero:

$$\left.\frac{dR}{dt}\right|_{(R^*,F^*)} = \alpha\left(1-\frac{R^*}{K}\right)R^* - \beta R^* F^* = 0$$

$$\left.\frac{dF}{dt}\right|_{(R^*,F^*)} = \gamma R^* F^* - \delta F^* = 0$$

The second equation implies:

$$F^* = 0 \qquad or \qquad R^* = \frac{\delta}{\gamma}$$

If $F^* = 0$, the first equation simplifies to:

$$\alpha\left(1-\frac{R^*}{K}\right)R^* = 0$$

which implies:

$$R^* = 0 \qquad or \qquad R^* = K$$

Hence, we have two possible fixed points if there are no foxes:

$$\textbf{Fixed Point 1:} \quad (\boldsymbol{R^* = 0, F^* = 0})$$

$$\textbf{Fixed Point 2:} \quad (\boldsymbol{R^* = K, F^* = 0})$$

Considering the other possibility and plugging $R^* = \frac{\delta}{\gamma}$ into the first equation, we get:

$$F^* = \frac{\alpha}{\beta}\left(1-\frac{\delta}{\gamma K}\right)$$

Thus, we found one fixed point with both foxes and rabbits co-existing indefinitely:

$$\textbf{Fixed Point 3:} \quad \left(\boldsymbol{R^* = \frac{\delta}{\gamma}, F^* = \frac{\alpha}{\beta}\left(1-\frac{\delta}{\gamma K}\right)}\right)$$

This fixed point should be realistic provided that $\delta < \gamma K$ (to ensure positive F^*).

In our example above, we can compute the fixed point to be:

$$F^* = \frac{0.1}{0.01}\left[1 - \frac{0.1}{0.005(50)}\right] = 6 \ , \qquad R^* = \frac{\delta}{\gamma} = \frac{0.1}{0.005} = 20$$

which is exactly where our system ends up in.

NOTES

The following is completely out of scope for this course, and will only be covered in a more advanced math course in differential equations. But in case you are wondering, why didn't the basic Lotka-Volterra model also settles into a fixed point? Why does it oscillate around it, and only when the initial populations start exactly at the fixed point, that we will end up there?

This has to do with the *kind* of fixed point that a system possesses. In the basic Lotka-Volterra model, the fixed point:

$$\left.\frac{dR}{dt}\right|_{(R^*,F^*)} = \alpha R^* - \beta R^* F^* = 0 \quad \Rightarrow \quad F^* = \frac{\alpha}{\beta}$$

$$\left.\frac{dF}{dt}\right|_{(R^*,F^*)} = \gamma R^* F^* - \delta F^* = 0 \quad \Rightarrow \quad R^* = \frac{\delta}{\gamma}$$

is actually an oscillatory one. For a two-dimensional case, it is possible to have stable, unstable, and oscillatory fixed points, plus some other possibilities. In the modified one (with logistic growth for the prey), the fixed point we found is the stable kind. This can be determined by looking at the eigenvalues of the Jacobian matrix evaluated at the fixed point.

However, although you have not learned any of the mathematical theories (and perhaps never will), being able to simulate such a model with MATLAB is a nice substitute. You can experiment with the parameters and see for yourself what would happen.

2. Two bacteria species A and B compete for nutrients in the same environment, leading to the following model for their populations, A and B, as functions of time, t:

$$\frac{dA}{dt} = r_A\left(1 - \frac{A + \beta B}{K}\right)A \quad ; \quad A(t=0) = A_0$$

$$\frac{dB}{dt} = r_B\left(1 - \frac{\alpha A + B}{K}\right)B \quad ; \quad B(t=0) = B_0$$

(a) Without solving the ODEs, find all the fixed point(s) of this system. Discuss the parameter settings at which the fixed point(s) are feasible. You can assume that all the parameters r_A, r_B, α, β and K are positive numbers.

(b) Write a MATLAB program to solve this system of ODEs and plot the populations over time $A(t)$ and $B(t)$ on the same graph, allowing the user to specify $r_A, r_B, \alpha, \beta, K, A_0, B_0$. The program should stop when it reaches steady state, or until $t = 10^4$, whichever is earlier, and return the final values for A and B. Your function definition should be:

```
[Af,Bf] = bacteriaCompetition_<LastName>_<FirstName>(param)
```

where `param` contains the parameters $r_A, r_B, \alpha, \beta, K, A_0, B_0$, in that order.

(c) Based on your results from Part (a), find a parameter setting for which both species will co-exist indefinitely, plot the populations versus time graphs using your program from Part (b), and verify that the observed steady-state populations A^* and B^* are indeed the fixed point(s) you predicted.

SOLUTION

PART (a)

To find the fixed point(s), we evaluate them at the state (A^*, B^*) and set both equations to zero:

$$\left.\frac{dA}{dt}\right|_{(A^*,B^*)} = r_A\left(1 - \frac{A^* + \beta B^*}{K}\right)A^* = 0$$

$$\left.\frac{dB}{dt}\right|_{(A^*,B^*)} = r_B\left(1 - \frac{\alpha A^* + B^*}{K}\right)B^* = 0$$

The first equation implies (with $r_A \neq 0$):

$$A^* = 0 \qquad or \qquad \left(1 - \frac{A^* + \beta B^*}{K}\right) = 0 \;\Rightarrow\; A^* = K - \beta B^*$$

Considering the first case, we plug $A^* = 0$ into the second equation:

$$r_B\left(1 - \frac{B^*}{K}\right)B^* = 0$$

which implies (with $r_B \neq 0$):

$$B^* = 0 \qquad or \qquad \left(1 - \frac{B^*}{K}\right) = 0 \quad \Rightarrow \quad B^* = K$$

Therefore, we found two fixed points:

$$\textbf{Fixed Point 1:} \quad (\boldsymbol{A^* = 0, B^* = 0})$$

$$\textbf{Fixed Point 2:} \quad (\boldsymbol{A^* = 0, B^* = K})$$

Now we consider the second case, and plug $A^* = K - \beta B^*$ into the second equation:

$$r_B \left[1 - \frac{\alpha(K - \beta B^*) + B^*}{K}\right] B^* = 0$$

which implies:

$$B^* = 0 \qquad or \qquad 1 - \frac{\alpha(K - \beta B^*) + B^*}{K} = 0 \quad \Rightarrow \quad B^* = K\left(\frac{1-\alpha}{1-\alpha\beta}\right)$$

With $B^* = 0$, $A^* = K - \beta B^* = K$, we found another fixed point:

$$\textbf{Fixed Point 3:} \quad (\boldsymbol{A^* = K, B^* = 0})$$

With $B^* = K\left(\frac{1-\alpha}{1-\alpha\beta}\right)$, $A^* = K - \beta B^* = K\left(1 - \beta\frac{1-\alpha}{1-\alpha\beta}\right) = K\left(\frac{1-\beta}{1-\alpha\beta}\right)$, and we found the only fixed point at which both species can potentially co-exist:

$$\textbf{Fixed Point 4:} \quad \left(\boldsymbol{A^* = K\left(\frac{1-\beta}{1-\alpha\beta}\right), B^* = K\left(\frac{1-\alpha}{1-\alpha\beta}\right)}\right)$$

Note that since A^* and B^* must be non-negative for this fixed point to be physical. In fact, if either of them, or both of them, is exactly zero, this fixed point would revert to the other three we already found. Therefore, Fixed Point 4 requires that:

$$K\left(\frac{1-\beta}{1-\alpha\beta}\right) > 0 \quad and \quad K\left(\frac{1-\alpha}{1-\alpha\beta}\right) > 0$$

If $(1-\alpha\beta)$ is positive, then we would require that $\beta < 1$ and $\alpha < 1$. On the contrary, if $(1-\alpha\beta)$ is negative, then we would require that $\beta > 1$ and $\alpha > 1$.

Incidentally, only the case where $\beta < 1$ and $\alpha < 1$ will lead to a *stable* fixed point. This would not be obvious from looking at the ODEs, but we would find out readily by simulating the model.

PART (b)

The working MATLAB program `bacteriaCompetition.m` is attached.

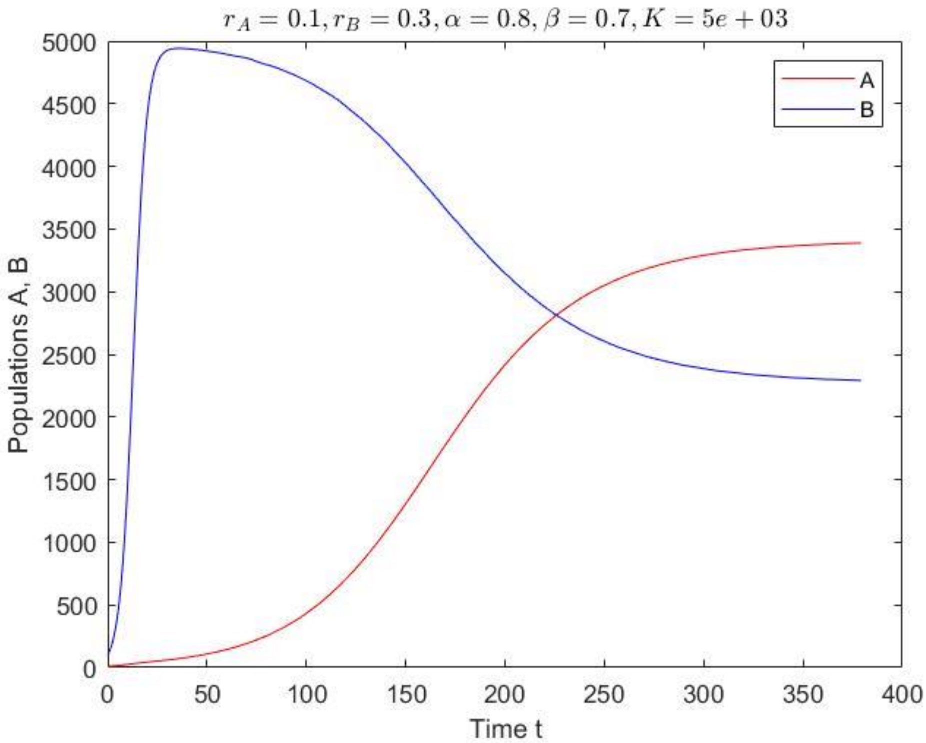
PART (c)

Based on our analysis in Part (a), for both species to co-exist indefinitely, we must reach Fixed Point 4, which requires that:

$$K\left(\frac{1-\beta}{1-\alpha\beta}\right) > 0 \quad and \quad K\left(\frac{1-\alpha}{1-\alpha\beta}\right) > 0$$

As explained, this would require either that $\beta < 1$ and $\alpha < 1$ or that $\beta > 1$ and $\alpha > 1$, and it would not matter what we set the other parameters r_A, r_B, K, A_0, B_0. By trying the model we would find that only the case where $\beta < 1$ and $\alpha < 1$ will give us the *stable* fixed point we need.

An example is below for $r_A = 0.1, r_B = 0.3, \alpha = 0.8, \beta = 0.7, K = 5000, A_0 = 10, B_0 = 100$:



For this parameter setting, the steady-state populations are approximately $A^* = 3470$ and $B^* = 2270$, as predicted by the fixed point expressions we found in Part (a).