BIEN/CENG 2310

MODELING FOR CHEMICAL AND BIOLOGICAL ENGINEERING

HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, FALL 2022

EXAMINATION 1

INSTRUCTIONS:

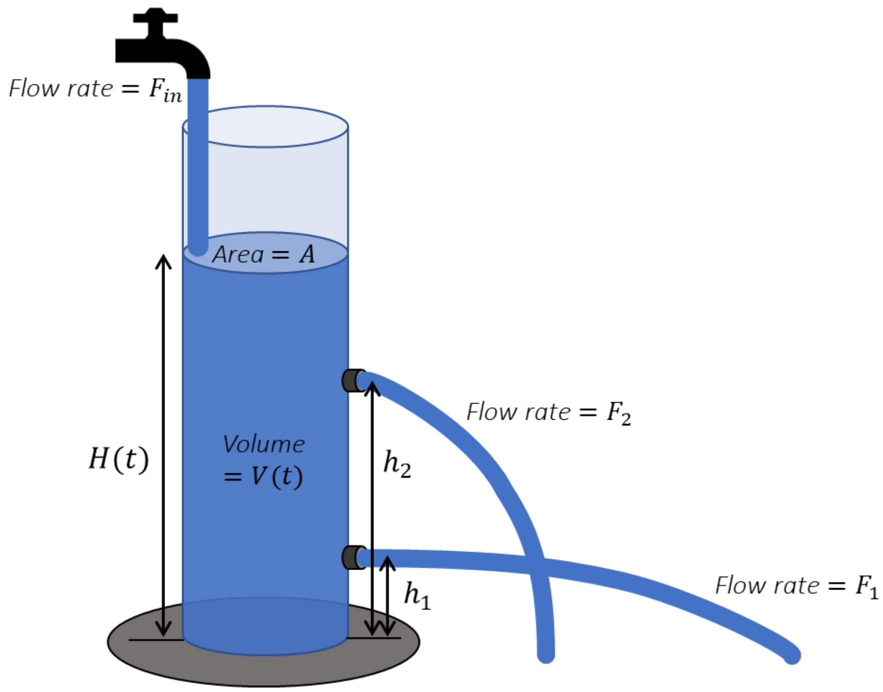
There is only **1** question in this exam. You may use any course materials (including those on Canvas) and use MATLAB through a desktop computer in the classroom. You may NOT communicate with anyone during the exam by any means, or access websites other than Canvas or the official MATLAB documentation pages at mathworks.com.

For Parts (a) and (b), please write your answers legibly in the space provided.

For Part (c), submit the MATLAB program to Canvas as instructed. You must start from the template `fountain.m`, and rename it to `fountain_<8-digit-Student-ID>.m` where `<8-digit-Student-ID>` is your HKUST Student ID number. You must only add your code to the section marked with:

```
% Add code here
```

The following is a simple apparatus that illustrates the concept of hydrostatic pressure:



Comparing the two water jets out of the cylinder, the lower one shoots farther because the water is ejected at higher speed. This is due to the higher hydrostatic pressure at the lower hole at height h_1. In fact, the *volumetric* flow rate out of the lower hole, F_1, is proportional to the hydrostatic pressure at the lower hole:

$$F_1 = \begin{cases} k\rho g(H - h_1) & if\ H > h_1 \\ 0 & if\ H \leq h_1 \end{cases}$$

where ρ is the density of water, g is the acceleration due to gravity, and k is a proportionality constant. Note that if the water level is not high enough to reach the hole, there is no flow out of that hole. Similarly, the volumetric flow rate out of the higher hole at height h_2 ($h_2 > h_1$) is:

$$F_2 = \begin{cases} k\rho g(H - h_2) & if\ H > h_2 \\ 0 & if\ H \leq h_2 \end{cases}$$

The cylinder is continuously filled with a stream of water, at a volumetric flow rate of F_{in}. Initially, there is no water in the cylinder.

You are asked to model the height of the water column in the cylinder, $H(t)$, as a function of time.

(a) **(24%)** By using a suitable balance equation, write down an ODE for $H(t)$, with an initial condition. Your ODE should only contain the variables t and H, and any parameters that do not change with time. State any assumption(s). *Hint: Your ODE should have 3 cases:* $H \leq h_1, h_1 < H \leq h_2,$ *and* $h_2 < H$.

SOLUTION

Assumptions:

- The density of water is constant and does not change with height or time.
- There is no evaporation from the cylinder.
- The cylinder is tall enough that there is no overflow.
- The volumetric flow rate into the tank, F_{in}, is constant.

Defining the water inside the tank as a control volume system, we write the mass balance of water:

$$\frac{dm}{dt} = \rho F_{in} - \rho F_1 - \rho F_2$$

where m is the mass of water in the system, ρ is the density (mass/volume) of water, which we assumed to be a constant. Next, we replace the m with the volume, which can be calculated by HA, where H is the height and A is the cross-sectional area of the cylinder:

$$\frac{d(\rho HA)}{dt} = \rho F_{in} - \rho F_1 - \rho F_2$$

$$\rho A \frac{dH}{dt} = \rho F_{in} - \rho F_1 - \rho F_2$$

$$\frac{dH}{dt} = \frac{F_{in} - F_1 - F_2}{A}$$

Now we can use the expression for the volumetric flow rates of the water jets given in the question, but we have to consider 3 cases:

<u>**Case 1:**</u> $H \leq h_1$. In this case, the water level does not reach the lower hole, so there is no outflow, and our ODE becomes:

$$\frac{dH}{dt} = \frac{F_{in}}{A}$$

<u>**Case 2:**</u> $h_1 < H \leq h_2$. In this case, the water level is higher than the lower hole and does not reach the higher hole, so there is only one water jet, and our ODE becomes:

$$\frac{dH}{dt} = \frac{F_{in} - k\rho g(H - h_1)}{A}$$

<u>**Case 3:**</u> $h_2 < H$. In this case, the water level is higher than both holes, and we have two water jets. The ODE becomes:

$$\frac{dH}{dt} = \frac{F_{in} - k\rho g(H - h_1) - k\rho g(H - h_2)}{A}$$

In summary, the ODE can be written:

$$\boldsymbol{\frac{dH}{dt} = \begin{cases} \frac{F_{in}}{A} & if\ H \leq h_1 \\ \frac{F_{in} - k\rho g(H - h_1)}{A} & if\ h_1 < H \leq h_2 \\ \frac{F_{in} - k\rho g(H - h_1) - k\rho g(H - h_2)}{A} & if\ h_2 < H \end{cases}}$$

And our initial condition is:

$$\boldsymbol{H(t = 0) = 0}$$

Grading: 6% for any 2 reasonable assumptions that were not already mentioned in the question; 3% for writing down a balance equation, and knowing to balance mass, not volume; 3% for expressing $m = \rho HA$; *9% for the final ODE (3% for each of the 3 cases), 3% for including the initial condition.*

(b) **(28%)** Without solving the ODEs, find all the fixed point(s) of this system. Specify the range of F_{in} for each fixed point to be possible. *Hint: You should consider 3 cases separately:* $H \leq h_1, h_1 < H \leq h_2,$ *and* $h_2 < H$.

SOLUTION

To find the fixed point(s), we evaluate them at the state H^* and set the ODE to zero:

$$\left.\frac{dH}{dt}\right|_{H^*} = 0$$

We will consider the 3 cases separately:

<u>Case 1:</u> $H \leq h_1$

$$\left.\frac{dH}{dt}\right|_{H^*} = \frac{F_{in}}{A} = 0$$

This can only be true if $F_{in} = 0$. If there no input at all, the water level will stay at the initial value. Hence the fixed point is:

Fixed Point 1: $\boldsymbol{H^* = 0}$**, possible if** $\boldsymbol{F_{in} = 0}$

<u>Case 2:</u> $h_1 < H \leq h_2$

$$\left.\frac{dH}{dt}\right|_{H^*} = \frac{F_{in} - k\rho g(H^* - h_1)}{A} = 0$$

$$H^* = h_1 + \frac{F_{in}}{k\rho g}$$

To be consistent, we require that $h_1 < H^* \leq h_2$, which implies:

$$h_1 < h_1 + \frac{F_{in}}{k\rho g} \leq h_2$$

$$0 < \frac{F_{in}}{k\rho g} \leq h_2 - h_1$$

$$0 < F_{in} \leq k\rho g(h_2 - h_1)$$

Therefore the fixed point is:

Fixed Point 2: $\boldsymbol{H^* = h_1 + \frac{F_{in}}{k\rho g}}$ **, possible if** $\boldsymbol{0 < F_{in} \leq k\rho g(h_2 - h_1)}$

<u>Case 3:</u> $h_2 < H$

$$\left.\frac{dH}{dt}\right|_{H^*} = \frac{F_{in} - k\rho g(H^* - h_1) - k\rho g(H^* - h_2)}{A} = 0$$

$$F_{in} - k\rho g(2H^* - h_1 - h_2) = 0$$

$$H^* = \frac{1}{2}\left(h_1 + h_2 + \frac{F_{in}}{k\rho g}\right)$$

To be consistent, we require that $h_2 < H^*$, which implies:

$$h_2 < \frac{1}{2}\left(h_1 + h_2 + \frac{F_{in}}{k\rho g}\right)$$

$$2h_2 < h_1 + h_2 + \frac{F_{in}}{k\rho g}$$

$$k\rho g(h_2 - h_1) < F_{in}$$

Therefore the fixed point is:

Fixed Point 3: $\boldsymbol{H^* = \frac{1}{2}\left(h_1 + h_2 + \frac{F_{in}}{k\rho g}\right)}$ **, possible if** $\boldsymbol{k\rho g(h_2 - h_1) < F_{in}}$

In summary, there are 3 possible fixed points. Which one the system will reach depends on the input flow rate.

Grading: 4% for knowing to set $\left.\frac{dH}{dt}\right|_{H^*} = 0$*; 12% (4% each) for the fixed point expression for* H^**; 12% (4% each) for the range of* F_{in} *for the fixed point to be possible.*

(c) **(48%)** Write a MATLAB program to simulate the ODE you have in Part (a), allowing the user to specify F_{in}, h_1 and h_2. The program should stop when it reaches steady state, and should plot H versus t in one plot, and F_1 and F_2 versus t (two curves in the same plot) in another, and output the value of H at the end of the simulation.

You must start with the provided template `fountain.m`, and rename it to `fountain_<8-digit-Student-ID>.m` where `<8-digit-Student-ID>` is your HKUST Student ID number.

The parameters k, ρ, g and A (cross-sectional area of cylinder) are fixed in the provided template. Do not change them.

You may add your code only in the section marked with:

```
% ----------------------------------
% Add code here



% ----------------------------------
```

SOLUTION

The working MATLAB program `fountain.m` is attached.

Grading: 3% for renaming the file and function name; 6% for calling ode45 correctly; 15% for setting up the function correctly to compute dH/dt (using `if` *statements or the* `max(0, x)` *construct); 6% for plotting the H vs t plot, including adding axis labels and title; 15% for plotting the F_1 and F_2 vs t plot, including calculating them by element-wise operation, adding axis labels, legends and title; 3% for setting the return value* `Hss`*.*